Citation for published version:
Galaktionov, VA 2008, 'On higher-order viscosity approximations of odd-order nonlinear PDEs', Journal of Engineering Mathematics, vol. 60, no. 2, pp. 173-208. https://doi.org/10.1007/s10665-007-9146-6

DOI:
10.1007/s10665-007-9146-6

Publication date:
2008

ON HIGHER-ORDER VISCOSITY APPROXIMATIONS OF ODD-ORDER NONLINEAR PDES

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Abstract. Some aspects of vanishing viscosity ($\varepsilon \to 0^+$) approximations of discontinues solutions of odd-order nonlinear PDEs are discussed. The first problem concerns entropy solutions of the classic first-order conservation law (Euler's equation)

$$u_t + uu_x = 0 \quad (\text{or } \; u_t + u^2 u_x = 0), \tag{0.1}$$

which are approximated by solutions $u_\varepsilon(x,t)$ of the higher-order parabolic equation

$$u_t + uu_x = \varepsilon(-1)^{m+1} D_x^{2m} u, \quad D_x = \partial/\partial x, \quad \text{with integer } \; m \ge 2. \tag{0.2}$$

Unlike the classic case $m = 1$ (Burgers' equation), which is the cornerstone of modern theory of entropy solutions, direct higher-order approximations of many known entropy conditions and inequalities are not possible. Using the concept of proper solutions from extended semigroup theory, we show that (0.2) and other types of approximations via $2m$th-order linear or quasilinear operators correctly describe the solutions of two basic Riemann's problems for (0.1) with initial data

$$S_\mp(x) = \mp \operatorname{sign} x,$$

corresponding to the shock (S_-) and rarefaction (S_+) waves respectively.

The second model is taken from nonlinear dispersion theory with the parabolic approximation

$$u_t - (uu_x)_{xx} = \varepsilon(-1)^{m+1} D_x^{2m} u, \quad \text{with } \; m \ge 2. \tag{0.3}$$

We establish similar evolution properties of ε-approximations of stationary shocks $S_\pm(x)$ posed for (0.3). Special "integrable" quasilinear odd-order PDEs are known to admit non-smooth compacton or peakon-type solutions (e.g., the Rosenau-Hyman and FFCM equations), while for more general non-integrable PDEs such results are unknown. It is shown that the shock $S_-(x)$ for (0.3) is obtained as $\varepsilon \to 0$ by an ODE approximation and also via blow-up self-similar solutions focusing as $t \to T^-$. For $S_+(x)$, the corresponding smooth rarefaction similarity solution is indicated that explains the collapse of this non-entropy shock wave.

A survey on entropy-viscosity methods developed in the last fifty years is included.

Key phrases: Conservation laws, entropy solutions, vanishing viscosity, higher-order parabolic operators.

Dedicated to the memory of Professor S.N. Kruzhkov

Date: April 11, 2007.
1991 *Mathematics Subject Classification.* 35K55, 35K65.
Research supported by RTN network HPRN-CT-2002-00274.

1. Introduction: Euler's equation and others

1.1. **Odd-order quasilinear PDEs are special in general theory and applications.** Odd-order partial differential equations (PDEs) always played a special role in general theory and in applications. Originated from fluid and gas dynamics (e.g., Euler's equations from eighteenth century), these equations deeply penetrated into many crucial applications and created several famous and sometimes isolated mathematical areas of PDE theory.

It is key that such nonlinear PDEs are able to describe singularity phenomena that are not available in other classes of even-order (say, reaction-diffusion type) equations or in quasilinear wave equations. Actually, the singularity effects such as appearance and propagation of *shock waves* and other strong or weak *discontinuities* are the main features that these PDEs were oriented to describe and to be good at.

Mathematically speaking, such shocks and other singularities were the origin of many fundamental theoretical difficulties. Unlike many classes of even-order elliptic, parabolic, and hyperbolic PDEs, odd-order equations do not contain a mechanism of smoothing of solutions, which is called the *interior regularity* of solutions (obviously, this would destroy shock waves). Since for complicated PDEs of such type, there exists a variety of shocks with different local behaviour and local laws of propagation (generally called *Rankine–Hugoniot conditions*), the correct choice of good *"entropy"* solutions, which makes the problem well-posed, becomes a principal difficult question.

In the twenties century, the necessity of entropy definitions of correct solutions has been recognized since 1940's, and first results were due to Burgers' and Hopf, who first develop "viscosity" approximation approach to Euler's equation (a scalar conservation law). The idea of the viscosity approximation consists in adding to the main odd-order operator a higher-order diffusion-like term with vanishing viscosity parameter $\varepsilon > 0$. This moves the PDE into the better class of even-order equations that makes possible to construct a unique approximate solution. The main mathematical difficulty then appears in the singular limit $\varepsilon \to 0$, which leads to a number of difficult mathematical problems.

The main ideas of viscosity and entropy theory of scalar first-order 1D conservation laws are associated with such names as Oleinik, Lax, Gel'fand, Glimm, Kruzhkov, and others, who developed complete existence and uniqueness theory for such first-order PDEs in the 1950's and 60's. This research initiated a large amount of further study and further discoveries in the theory of conservation laws and hyperbolic systems that are reflected in a number of monographs to be cited.

It must be noted that the main crucial results have been obtained by using a classical diffusion viscosity approximation via the Laplace operator . Mathematically, this gives a lot of advantages of the analysis, since the Laplacian, as mathematicians say, has the sign, i.e., it is a negative operator in natural topology, and preserves this sign in many related nonlinear mathematical manipulations. This is a crucial and an exceptional property of the Laplacian that make it so popular and widely used in PDE theory. In other words, the resulting ε-regularized equations obey the Maximum Principle (MP), which is a key ingredient of modern theory of parabolic equations. Actually, it is not exaggeration to say that precisely these special properties of regularization via Laplacian made it possible to create such a mathematically perfect viscosity-entropy theory of conservation laws.

For higher-odd-order PDEs, a natural viscosity approximation leads to higher-order regularized problems without such good features, since any iteration of the Laplacian (higher-order diffusion) loses its "global" negativity, to say nothing about the MP. This implies several, not that pleasant, conclusions underlying that many previous approaches and results achieved for first and lower-order equations and systems fail in principle. Moreover, fundamentally new mathematical ideas and methodology are necessary to overcome such difficulties.

Using a mixture of analytical, formal, and numerical methods, we will explain the main difficulties of analysis of such odd-order PDEs. We also present a number of rather positive (but not general and exhaustive, which possibly are non-existent for such PDEs) conclusions on viscosity-entropy analysis of the solutions. We include a survey based on a detailed list of mathematical references on these subjects for those Readers who are more interested in purely mathematical issues of PDE theory.

1.2. **Preliminary survey: on some known models and results.** Thus, we consider the questions occurring in higher-order approximations (the *vanishing viscosity method*) of nonlinear odd-order PDEs. To illustrate their rather special features, we begin with the classic scalar *conservation law* or *Euler's equation*

$$u_t + uu_x = 0 \quad \text{in } Q = \mathbf{R} \times \mathbf{R}_+, \quad u(x,0) = u_0(x) \quad \text{in } \mathbf{R}, \tag{1.1}$$

with bounded measurable initial data u_0. This equation originated from gas-dynamics played a key role in general theory of discontinuous entropy solutions of conservation laws developed in the 1950's; see books [1, 2, 3]. Among others, the well-established method to define the unique entropy solutions of the Cauchy problem (1.1) is to consider its viscosity approximations via regular (analytic) solutions of uniformly parabolic Burgers' equation with parameter $\varepsilon > 0$,

$$u_\varepsilon : \quad u_t + uu_x = \varepsilon u_{xx}, \tag{1.2}$$

with the same data u_0. The solvability of (1.2) and existence-uniqueness of u_ε are straightforward by standard parabolic theory and the Maximum Principle (MP). Then the entropy solutions is obtained by the limit

$$u(x,t) = \lim_{\varepsilon \to 0^+} u_\varepsilon(x,t), \tag{1.3}$$

which is proved to exist. This methodology goes back to Hopf (1950); see the above monographs for main results and a detailed historical survey.

Our FIRST HIGHER-ORDER MODEL occurs when we approximate entropy solutions of (1.1) via higher-order viscosity that leads to $2m$th-order *extended Burgers' equation* ($m \geq 2$ is arbitrary integer, $D_x = \frac{\partial}{\partial x}$)

$$u_\varepsilon : \quad \boxed{u_t + uu_x = \varepsilon(-1)^{m+1} D_x^{2m} u,} \quad u(x,0) = u_{0\varepsilon}(x), \tag{1.4}$$

with the positive viscosity parameter $\varepsilon \to 0^+$. On the right-hand side, we see mth iteration of the 1D Laplacian $-D_x^2 = -\frac{\partial^2}{\partial x^2} > 0$ (a positive operator; see below), taking with the minus sign. Overall, this gives again a negative diffusion-like operator in (1.4). The second-order ($m = 1$) vanishing viscosity method coincides with (1.2). It turns out that passing to the limit $\varepsilon \to 0$ in (1.4) for $m \ge 2$ is uncomparably much more difficult than in the classic approximation manner (1.2).

One can note that the higher-order approximation (1.4) of Euler's equation (1.1) is not needed once the simpler second-order one (1.2) serves extremely well, which, of course, that is correct. In our analysis, the first problem (1.4) of higher-order approximation becomes a basic mathematical model for revealing main difficulties and features of such ε-regularization to be applied to other more complicated odd-order PDEs. In addition, the problem (1.4) has other independent applications and motivations; see a discussion below.

Beyond the entropy theory and more general aspects of extended semigroup theory, singular perturbation problems such as (1.4) have other remarkable applications. For instance, higher-order viscosity terms occur via Grad's method in Chapman–Enskog expansions for hydrodynamics, where the viscosity part, being put into our hyperbolic equation, gives

$$u_t + uu_x = \textstyle\sum_{n=0}^{\infty} \varepsilon^{2n+1} \Delta^n (\mu_n \Delta u) = \varepsilon\big(\mu_0 \Delta u + \varepsilon^2 \mu_1 \Delta^2 u + ...\big),$$

where $\varepsilon > 0$ is essentially the Knudsen number; see details in Rosenau's regularization approach, [4]. In a full model, truncating such series at $n = 0$ leads to the Navier–Stokes equations, while $n = 1$ is associated with the Burnett equations (ill-posed since $\mu_1 > 0$, so a backward parabolic equation occurs), etc. Several aspects of the fourth-order approximations occurring in mechanical and physical applications and, in particular, in stability theory of finite-difference schemes (third-order methods) were attracted significant interest and have been studied in the literature on numerical study of the so-called *postshock oscillations*; see references in [5, 6, 7]. There exists a large literature in gas and aerodynamics on the influence of viscosity and heat conduction processes on the structure of shocks in compressed flows. This leads to hard higher-order nonlinear systems; see Zel'dovich–Raiser [8].

The PDE (1.4) for $m = 2$ (here $\varepsilon = 1$ by scaling)

$$u_t + uu_x = -u_{xxxx}, \tag{1.5}$$

which is a version of the *Kuramoto–Sivashinsky equation*, has the independent interest and, in particular, occurs as a model for Bunsen burner, [9]. Several important results on existence, uniqueness, and asymptotic stability of the non-monotone *viscosity shock profile* (VSP) for (1.5) have been proved since the 1970s; see [10, 11, 12, 13], [14]–[17], where further references on application can be found (the existence of the VSP is known for any $m \ge 2$; see further comments below).

Returning to general viscosity approximation (1.4), a crucial result was obtained recently by Tadmor [18], who showed that L^2 solutions $\{u_\varepsilon\}$, converge in L^p, with $p < \infty$, to the entropy solutions, under the assumption that they are uniformly bounded in L^∞. It seems that the last assumption is also difficult to prove without a detailed analysis of shock layers occurring as $\varepsilon \to 0$. Tadmor's proof uses Tartar-Murat compensation compactness theory and interesting crucial spectral ideas.

Concerning other types of regularization of the hyperbolic equation (1.1), for third-order operators leading, in particular, to the *Korteweg-de Vries equation*

$$u_t + uu_x = \varepsilon u_{xxx}, \tag{1.6}$$

approximation of entropy solutions with shocks are known to be impossible; see general conclusions of [19] concerning ODEs and detailed PDE analysis in [20]; see also Lax's survey [21]. In the case of a small dispersion perturbation of Burgers' equation

$$u_t + uu_x = \varepsilon u_{xx} + \delta(\varepsilon) u_{xxx}, \tag{1.7}$$

for $\delta(\varepsilon) = o(\varepsilon^2)$ as $\varepsilon \to 0$, the solutions converge to entropy ones of (1.1); see [22], [23], and more references in [24]. Other types of quasilinear p-Laplacian approximations with the right-hand side

$$\varepsilon(|u_x|u_x) + A\varepsilon u_{xxx}$$

can lead to *nonclassical shocks* (not satisfying Oleinik's entropy condition), [25, 26, 6].

A similar analysis has been performed for higher-order viscosity approximations

$$u_t + uu_x = \varepsilon u_{xx} - \delta(\varepsilon) u_{xxxx}; \tag{1.8}$$

see [27] and [28], where a more general diffusion term $\delta D_x^{4n+2} u$ was considered.

As another more recent example of successful viscosity approximation for odd-order PDEs, we refer to the *Fuchssteiner–Fokas–Camassa–Holm equation*,

$$u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} \quad \text{in } \mathbf{R} \times \mathbf{R}, \tag{1.9}$$

which arises as an asymptotic model describing the wave dynamics at the free surface of fluids under gravity. General existence-uniqueness theory of non-smooth *peakon* solutions, possessing discontinuous derivative u_x, was developed with the essential use of viscosity approximations in a number of papers including [29] (the regularization term εu_{xxxxt} is used), [30], [31] (Kato's semigroup approach), and [32] (parabolic regularization εu_{xx} in the equivalent integral equation obtained by application to $(I - D_x^2)^{-1}$ to (1.9), which recovered an analogy of Oleinik's entropy condition [33] to be discussed below).

In general, higher-order semilinear parabolic equations occur in several areas of applications, and their qualitative mathematical theory is the important popular subject; see monographs [34, 35, 36]. The questions of $2m$th-order approximation of odd-order evolution equations are related to difficult problems of smooth regularization of semigroups of discontinuous solutions and construction of discontinuous extended semigroups occurring in the study of singularity formation phenomena in PDEs; see [37, Ch. 6,7] and [38, Ch. 3-5] for further references .

1.3. **Other models and layout of the paper.** As a first step, we intend to describe some asymptotic properties of solutions $u_\varepsilon(x,t)$ of (1.4) for small $\varepsilon \to 0$, in order to understand why these violate all classical entropy inequalities (actually, many of such aspects are a well-known matter). We put into Section 2 a short survey describing classical local (pointwise) and nonlocal entropy conditions for the hyperbolic equation (1.1), as well as Gel'fand's ODE admissibility concept (the *G-admissibility*) of solutions. Indeed, it has been known for a long time that parabolic approximations with $m \ge 2$ of conservation laws are not good for using BV-spaces of functions of bounded variation due to the oscillatory character of the kernels of fundamental solutions of $2m$th-order parabolic operators. This affects the total variation of solutions and leads to other unpleasant features.

Section 3 is devoted to the well-posedness of the Cauchy problem for (1.4), which is also a non-trivial matter for such higher-order parabolic flows. In Section 4 we concentrate on detailed discussion why for $m \ge 2$ the regularized solutions $\{u_\varepsilon(x,t)\}$ of (1.4) do not approximate as $\varepsilon \to 0$ known local entropy conditions for solutions $u(x,t)$ of (1.1). This happens due to the *discontinuity* of total variation of $u_\varepsilon(x,t)$ at $\varepsilon = 0$ in approximating of entropy shocks. We show that, for $m \ge 2$, there exists a *variation deficiency* denoted by dV_m that is determined via the *viscosity shock profile* having finite total variation. It turns out that

$$\text{the variation deficiency} \quad dV_m = 0 \quad \text{for} \quad m = 1 \quad \text{only.}$$

Actually, this made it possible to approximate local entropy inequalities in classical viscosity-entropy theory. It is curious that, for Riemann's problems, the total variation remains bounded, so Helly's theorem for functions of bounded variation can be applied similar to the classic case $m = 1$. Unfortunately, other unsolvable difficulties arise.

As a first step, as a preamble to other approximation problems, we easily demonstrate that, for any $m \ge 2$, parabolic approximations correctly describe entropy solutions of Riemann's problems for (1.1) with initial data

$$S_\mp(x) = \mp \operatorname{sign} x, \tag{1.10}$$

which lead to the *shock* and *rarefaction* waves respectively. We prove that $S_-(x)$ is a *proper* solution, i.e., is obtainable by higher-order approximations, while $S_+(x)$ is not and, as initial data, lead to standard self-similar rarefaction wave.

The results on $2m$th-order approximation of arbitrary entropy solutions generates the two key asymptotic (large-time behaviour) problems for the corresponding rescaled $2m$th-order parabolic equations (which can be solved for some equations (1.4)):

(I) asymptotic stability of the viscosity shock profile (Section 6), and

(II) asymptotic stability of the rarefaction profile (Section 8).

We also discuss other types of quasilinear $2m$th-order approximations of entropy solutions, including quasilinear parabolic or even thin film-type regularizations

$$\begin{aligned} u_t + uu_x &= -\varepsilon(1+u^2)u_{xxxx}, \\ u_t + uu_x &= -\varepsilon u^2\, u_{xxxx}, \\ u_t + uu_x &= -\varepsilon\big(u^2 u_{xxx}\big)_x, \end{aligned} \tag{1.11}$$

for which applications of known methods become very difficult or even not possible. We check, relying on straightforward numerical evidence, that, for all types of regularization in (1.11), $S_-(x)$ remains a proper shock. As another version, we also briefly discuss similar properties of the regularized cubic equation

$$u_t + u^2 u_x = -\varepsilon u_{xxxx}, \tag{1.12}$$

with initial data $H(-x)$, where H is the Heaviside function. We show that there exists an approximating sequence $\{u_\varepsilon\}$ such that, in L^1,

$$u_\varepsilon(x,t) \to H\left(\tfrac{1}{3}\,t - x\right) \quad \text{as } \varepsilon \to 0^+.$$

Other related approximation problems are studied in Section 7. In particular we consider parabolic approximations of odd-order PDEs including our SECOND MODEL

$$\boxed{u_t - (uu_x)_{xx} = -\varepsilon u_{xxxx}} \quad \left(\text{or } \ ... = \varepsilon u_{xxxxxxx}, \ \text{etc.}\right), \tag{1.13}$$

which for $\varepsilon = 0$ occurs as a nonlinear dispersion model in the pattern formation in liquid drops. Such equations are known to admit solutions with finite interfaces and singularities. For instance Rosenau-Hyman's equation [39]

$$u_t - (uu_{xx})_x = uu_x$$

possesses non-smooth compactly supported solutions with discontinuous derivatives $D_x^k u$ for any $k \geq 2$. See various exact solutions and related mathematical aspects of such odd-order PDEs in [38, Ch. 4]. Entropy theory for odd-order equations such as (1.13) seems to be nonexistent and, most probably, entropy-like characterization cannot be represented as an explicit inequality (or a differential inclusion) for classes of such PDEs.

It is remarkable that the *stationary* shock $S_-(x)$ turns out to be proper for (1.13) with $\varepsilon = 0$, i.e., approachable by regular solutions $\{u_\varepsilon\}$ of (1.13). For $S_+(x)$, existence of the rarefaction self-similar solution is studied numerically (the corresponding ODEs are not easy at all). On the contrary, for initial data $S_-(x)$ given in (1.10), it is proved that such a rarefaction similarity solution describing evolution collapse does not exist. This confirms that $S_-(x)$ does exist as a proper (entropy) standing shock wave.

Including into this paper several accompanying rigorous and formal asymptotic results, we indent to show that even in simpler models (1.11) or in the more general PDEs like (1.13) or related others, one cannot expect powerful compensation compactness techniques to be applied, so that convergence of $\{u_\varepsilon\}$ as $\varepsilon \to 0$ towards entropy solutions needs a delicate asymptotic analysis of corresponding singularity formation phenomena (shock layers), and that this is an unavoidable difficulty. In this and other related approximation problems connected with general extended semigroup theory of nonlinear degenerate or singular higher-order PDEs, the questions of existence, uniqueness, and asymptotic behaviour of limit proper solutions cannot be studied separately, are indivisible, and cannot be tackled in a sufficient generality by known traditional unified techniques borrowed from classic theory.

2. Entropy conditions and Gel'fand's ODE admissibility concept

2.1. **Entropy inequalities.** It is known from the 1950's that the Cauchy problem for general single conservation laws admits a unique entropy solution. We refer to first complete results by Oleinik, who introduced entropy conditions in 1D and proved existence and uniqueness results (see survey [33]) and by Kruzhkov [40], who developed general non-local theory of entropy solutions in $\mathbf{R}^N$. In the general case, one of Oleinik's local entropy condition has the form [33, p. 106]

$$\tfrac{u(x_1,t)-u(x_2,t)}{x_1-x_2} \le K(x_1,x_2,t) \quad \text{for all } \ x_1,x_2 \in \mathbf{R}, \ t>0, \tag{2.1}$$

where K is a continuous function for $t>0$. Oleinik's local condition E (Entropy) introduced in [41], for the model equation (1.1) with convex function $\varphi(u)=\frac{1}{2}\,u^2$ corresponds to the well-known principle of non-increasing entropy from gas dynamics,

$$u(x^+,t) \le u(x^-,t) \quad \text{in } \ Q, \tag{2.2}$$

with strict inequality on lines of discontinuity, [33, p. 101].

Kruzhkov's entropy condition on solution $u \in L^\infty(Q)$ [40] is the nonlocal inequality

$$|u-k|_t + \tfrac{1}{2}\left[\mathrm{sign}(u-k)(u^2-k^2)\right]_x \le 0 \quad \text{in } \ \mathcal{D}'(Q) \quad \text{for any } \ k \in \mathbf{R}. \tag{2.3}$$

This inequality is understood in the sense of distributions meaning that the sign $\le$ is preserved after multiplying the inequality by any smooth compactly supported cut-off function $\varphi \in C_0^\infty$ and $\varphi \ge 0$ and integrating by parts. Oleinik's and Kruzhkov's approaches coincide in the 1D geometry.

It was known beginning with the first rigorous results by Hopf [42] (previous ones were due to Burgers [43]) that entropy solutions can be obtained by the vanishing viscosity method, i.e., as the limit (1.3) of a sequence of classical solutions $\{u_\varepsilon\}$ of the Cauchy problem for *Burgers' equation* (1.2) with the same initial data. The convergence in (1.3) takes place in $L^1(\mathbf{R})$ for $t>0$ and is pointwise at any point of continuity of $u(x,t)$. Approximations of the initial data can be included, where

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x) \to u_0 \quad \text{as} \quad \varepsilon \to 0 \ \text{ in } \quad L^1. \tag{2.4}$$

See the comparison theorem in [40], and [33] for survey of results on general 1D hyperbolic equations. We refer to well known Smoller's book [3] and recent monographs by Dafermos [2] and Bressan [1] for more detailed information.

The following consequence of the parabolic approximation is of principle importance for the theory. Let $E'(u)$ be a monotone C^1-approximation of the sign-function $\mathrm{sign}(u-k)$ with a fixed $k \in \mathbf{R}$, i.e., $E(u)$ is an approximation of $|u-k|$. Multiplying equation (1.2) by $E'(u)\chi$ with a nonnegative test function $\chi \in C_0^1(Q)$ and integrating over Q yields

$$\begin{aligned} &-\textstyle\iint [E(u)\chi_t + F(u)\chi_x]\,\mathrm{d}x\,\mathrm{d}t \\ &= -\,\varepsilon \textstyle\iint E''(u)(u_x)^2\chi\,\mathrm{d}x\,\mathrm{d}t + \varepsilon \iint E(u)\chi_{xx}\,\mathrm{d}x\,\mathrm{d}t \equiv J_1(\varepsilon), \end{aligned}$$

where $F(u)=\int uE'(u)\mathrm{d}u$. The first integral on the right hand side in non-positive, while the second one is of order $O(\varepsilon)$ on uniformly bounded regularized solutions u_ε. Passing to the limit $\varepsilon \to 0$ yields that the limit solution obtained by (1.3) satisfies the nonlocal *Kruzhkov-Lax entropy inequality* (see [44] for hyperbolic equations and [45] for systems)

$$E(u)_t + F(u)_x \le 0 \quad \text{in } \ \mathcal{D}'(Q). \tag{2.5}$$

For single conservation laws in $\mathbf{R}^N$, (2.5) being true for any convex C^2-function $E : \mathbf{R} \to \mathbf{R}$, gives an equivalent to (2.3) definition of unique entropy solutions; see [44], [40, p. 241] and [46]. Note that this is related to a parabolic version of Kato's inequality [47]: if $u, f \in L^1_{\rm loc}(Q)$, then (see [48, p. 75])

$$u_t - \Delta u = f \;\; \text{in} \;\; \mathcal{D}'(Q) \;\Longrightarrow\; |u|_t - \Delta |u| \le f \;\; \text{in} \;\; \mathcal{D}'(Q). \tag{2.6}$$

2.2. **ODE-admissible approximations in the sense of Gel'fand.** It was well understood in the theory of entropy solutions that a crucial principle is the correct description of propagation of *shock-waves*, which are discontinuous travelling waves (TWs) satisfying (1.1) in the weak sense,

$$u(x,t) = S(\eta), \quad \eta = x - \lambda t, \tag{2.7}$$

where λ is the TW speed and $S(\eta)$ is a step function. Using obvious scaling and translational invariance of the equation, we set $\lambda = 0$. Assuming that the discontinuity is located at $x = 0$, by the *Rankine-Hugoniot condition*

$$\lambda = \tfrac{1}{2}\,[S(0^+) + S(0^-)], \tag{2.8}$$

this corresponds to two initial functions with the following entropy solutions of (1.1) (*Riemann's problems*):

$$S_-(x) = -\mathrm{sign}(x) \;\Longrightarrow\; u(x,t) = S_-(x) \;\; \text{for} \;\; t > 0, \tag{2.9}$$

and

$$S_+(x) = \;\mathrm{sign}(x) \;\Longrightarrow\; u_+(x,t) = \begin{cases} S_+(x) & \text{for} \;\; |x| \ge t, \\ \frac{x}{t} & \text{for} \;\; |x| \le t. \end{cases} \tag{2.10}$$

The first discontinuous TW $S_-(x)$ (called a *standing shock-wave* in gas-dynamics) is the entropy one. $S_+(x)$ is not entropy and the continuous for $t > 0$ solution $u_+(x,t)$ in (2.10) (the *rarefaction wave*) describes collapse of this initial singularity.

Consider now a higher-order approximation of the conservation law, where the regularizing sequence $\{u_\varepsilon\}$ is given by the Cauchy problem for the $2m$th-order uniformly parabolic equations (1.4) of arbitrary order $2m \ge 4$. Note that (1.4) is invariant under a two-parametric group of scalings and translations, so that if $u(x,t)$ is a solution, then

$$\mathcal{T}_{\alpha\beta} u(x,t) = \beta^{2m-1}[u(\beta x + \beta^{2m}\alpha t, \beta^{2m} t) - \alpha] \tag{2.11}$$

is also a solution for any constants α, $\beta \in \mathbf{R}$.

The approximating operator on the right-hand side of (1.4) is called *admissible* (or ODE-admissible to be distinguished from the PDE-one to be introduced later on) if equation (1.4) admits a TW approximating the entropy one $S_-(x)$ as $\varepsilon \to 0$ in a reasonable topology. The concept of admissible approximations (the G-admissibility, in what follows) was introduced by Gel'fand in [49] and was developed on the basis of TW-solutions of hyperbolic equations and systems; see [49, Sect. 2 and 8].

In view of the invariance (2.11), we again put $\lambda = 0$ (though there exists other types of solutions with $\lambda \neq 0$; see below). From (1.4) we then obtain the ODE for the *viscosity* *shock profile* (VSP) f_- corresponding to the entropy shock-wave $S_-(x)$. It is a sufficiently smooth stationary solution of (1.4)

$$u_\varepsilon(x) = f_-(y), \quad y = x/\varepsilon^\alpha, \quad \text{where} \quad \alpha = \tfrac{1}{2m-1}. \tag{2.12}$$

Here, f_- solves the following ODE problem:

$$(-1)^{m+1} f^{(2m)} = ff' \quad \text{in } \mathbf{R}, \quad f(-\infty) = 1, \;\; f(+\infty) = -1. \tag{2.13}$$

The family of solutions (2.12) describes formation of the singular *shock layer* as $\varepsilon \to 0$ in the ODE. For $m = 1$, (2.13) is solved explicitly to give the unique (up to translation) *monotone decreasing* VSP

$$f_-(y) = \tfrac{1-\mathrm{e}^y}{1+\mathrm{e}^y} = \tanh \tfrac{y}{2}. \tag{2.14}$$

The VSP f_- of (2.13) exists for any $m \ge 2$; see [10], [13] and [17]. For the fourth-order approximation $m = 2$ it is known to be unique [14] and stable in a weighted Sobolev space [11, 12].

Thus the higher-order approximations (1.4) for any $m \ge 1$ are G-admissible in this ODE (TW) sense.

3. Well-posedness of higher-order approximations: first L^∞ bound

The problem of $2m$th-order approximations of first-order PDEs seems was less studied in the literature. Higher-order parabolic equations of the type (1.4) are well-posed and admit unique smooth classical solutions local in time [34, 35]. For $m = 1$, global existence and the uniform bound $|u(x,t)| \le \sup |u_0|$ follow from the Maximum Principle. Such global existence results for higher-order semilinear parabolic equations with lower-order nonlinear perturbations are known in classes of sufficiently small initial data; see [50, 51, 52, 53]. Estimates in Sobolev spaces of solutions of PDEs (1.4) can be found in [28]. For $m = 2$, global existence is established in [11, 12] via stability analysis of the VSP (i.e., for initial data sufficiently close to f_-).

Thus, it is principal to confirm that, for any $m \ge 2$, solutions of (1.4) are global in time and cannot blow-up in the L^∞-norm. We consider the Cauchy problem (1.4) with initial data satisfying

$$|u_{0\varepsilon}| \le C, \quad \|u_{0\varepsilon}\|_2 \le C, \tag{3.1}$$

where $C > 0$ denotes different constants depending on ε. By approximation of L^2 initial data via compactly supported one, we may assume that solutions have fast exponential decay as $x \to \infty$. Multiplying equation (1.4) by u and integrating over $\mathbf{R}$ gives

$$\tfrac{1}{2}\,\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 = -\varepsilon \int |D_x^m u|^2 \le 0, \tag{3.2}$$

from which comes the first uniform bound on the solution

$$\|u(t)\|_2 \le \|u_{0\varepsilon}\|_2 \le C \quad \text{for all } \; t > 0. \tag{3.3}$$

Proposition 3.1. *Let $m \ge 2$ and* (3.1) *hold. For a fixed $\varepsilon > 0$, the solution $u_\varepsilon(x,t)$ of* (1.4) *is uniformly bounded in $\mathbf{R} \times \mathbf{R}_+$.*

Proof is rather technical and is postponed until Appendix A.

4. On key differences in approximations for $m = 1$ and $m \geq 2$

4.1. **Non-monotonicity of the VSP and the variation deficiency.** We now describe a crucial non-monotonicity property of the VSP for $m \geq 2$, which directly prohibits any parabolic approximations of local entropy conditions. Denote by $|f_-|_{TV}$ the *total variation* of $f_-(y)$ on $\mathbf{R}$. We introduce the variation deficiency dV_m of f_- as follows.

Proposition 4.1. *For any $m \geq 2$, the VSP f_- given by* (2.13) *has bounded variation,*

$$|f_-|_{TV} > 2 = |S_-|_{TV} \implies dV_m \equiv |f_-|_{TV} - |S_-|_{TV} > 0. \tag{4.1}$$

Proof. As $y \to \infty$, the linearized ODE (2.13) has the form

$$(-1)^{m+1} f^{(2m)} = -f',$$

so that the exponential decaying behaviour is determined by functions

$$f(y) \sim \mathrm{e}^{\mu y}$$

with the characteristic equation

$$(-1)^{m+1} \mu^{2m-1} = -1;$$

see [55, Ch. 3]. For any $m \geq 2$, solutions are oscillatory at $y = +\infty$, i.e., the characteristic number μ with the maximal $\operatorname{Re}\mu < 0$ is such that $\operatorname{Im}\mu \neq 0$. This implies (4.1). $\square$

The variation deficiency (4.1) shows that a finite discontinuity of variation occurs for $\varepsilon = 0^+$ at shocks of entropy solutions (though, in order to justify this, one needs the asymptotic stability of the VSP; see Section 6). Note that dV_m vanishes for the second-order approximation $m = 1$ only and actually, this lies in the heart of parabolic approximations of local entropy inequalities in classic entropy theory. We will conclude that for $m \geq 2$ this is not possible.

In Figures 1 and 2 we show the stationary shock together with moving TWs,

$$-\lambda f' + f f' = (-1)^{m+1} f^{(2m)}. \tag{4.2}$$

Figure 3 for $m = 7$ illustrates the fact that, for large m's, the total variation of TW profiles with $\lambda \neq 0$ (and essential deviation from the odd structure for $\lambda = 0$) can increase dramatically via high oscillations near the shock. All these satisfy

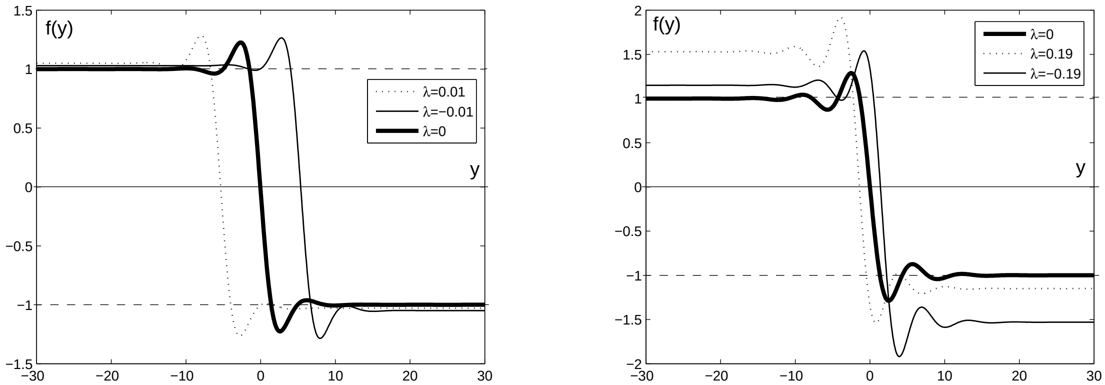
$$(-1)^{m+1} f^{(2m-1)} = -\lambda f + \tfrac{1}{2} f^2 + C, \tag{4.3}$$

obtained from (4.2) on integration. The constant $C = C(\lambda, f_\pm)$ depends on the limit values

$$f_\pm = \lim_{y \to \pm\infty} f(y).$$

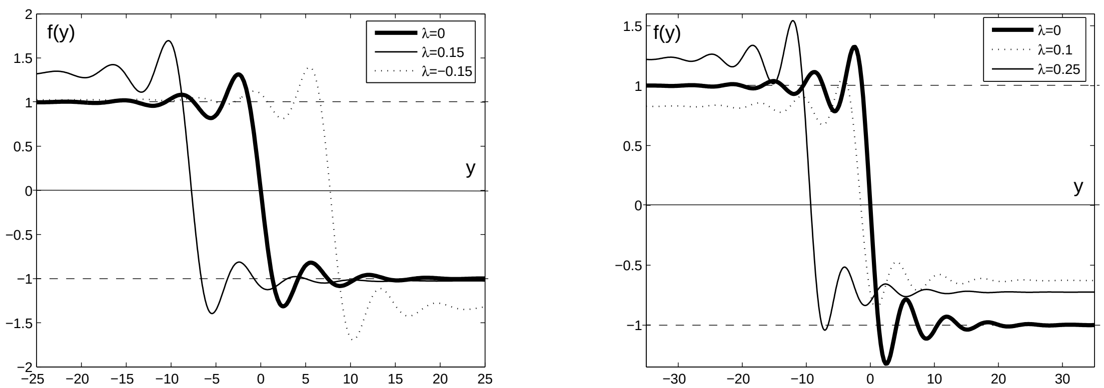
It follows from (4.3) that $f_\pm = \mp 1$ for $\lambda = 0$ only, and then $C(0, \mp 1) = -\frac{1}{2}$.

For all $m = 2, 3, 4, 5, 6, 7$, the applied iterative numerical method detects a fast (exponential) convergence to the unique stable VSP. Recall that, rigorously, stability is known for $m = 2$ only, [15, 11, 12]. For comparison, Figures 4 and 5 shows the character of non-monotonicity of the VSP's ($\lambda = 0$), which can be associated with typical oscillating and sign-changing properties of the fundamental solutions of higher-order parabolic operators, [34]. We complete this discussion by stating an open problem.



(a) $f^{(4)} = \lambda f' - ff'$ (b) $f^{(6)} = -\lambda f' + ff'$

FIGURE 1. The VSP and TW solutions of (4.2) for $m = 2$ (a) and $m = 3$ (b).



(a) $f^{(8)} = -\lambda f' + ff'$ (b) $f^{(10)} = -\lambda f' + ff'$

FIGURE 2. The VSP and TW solutions of (4.2) for $m = 4$ (a) and $m = 5$ (b).

Conjecture 4.1. *For any $m \geq 3$, there exists a unique exponentially stable VSP.*

Thus, for $m \geq 2$, the *total variation diminishing* (TVD) property of entropy solutions,

$$|u(t)|_{TV} \leq |u_0|_{TV} \equiv \|u_0'\|_{M^1} \quad \text{for all } \; t > 0 \tag{4.4}$$

(M^1 is the space of bounded Radon measures), is violated. This property remains valid for approximations with $m = 1$ and admits further extensions; see [56].

Next, we consider straightforward consequences of Proposition 4.1 prohibiting approximation of other entropy conditions.

A RELATION TO ORDER DEFICIENCY. Here we observe a phenomenon similar to the *order deficiency* [53] that can be expressed in terms of the following constant:

$$D_* = \textstyle\int |F| > 1 \quad \text{for all} \quad m \geq 2,$$

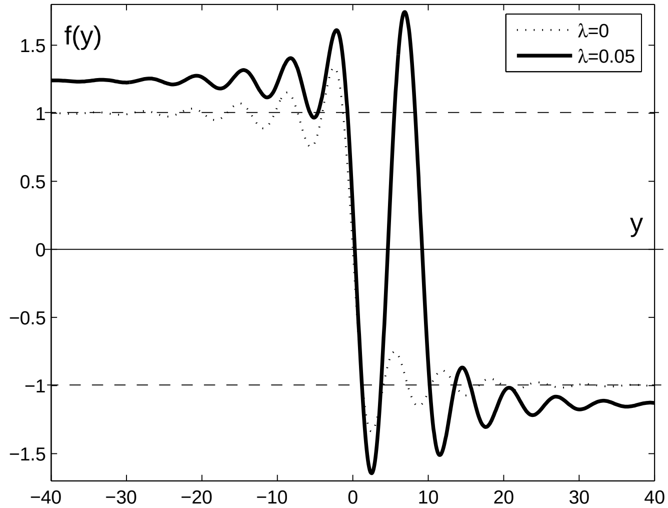


FIGURE 3. The VSP and the TW profiles for $m = 7$ can have different oscillatory properties and total variations.

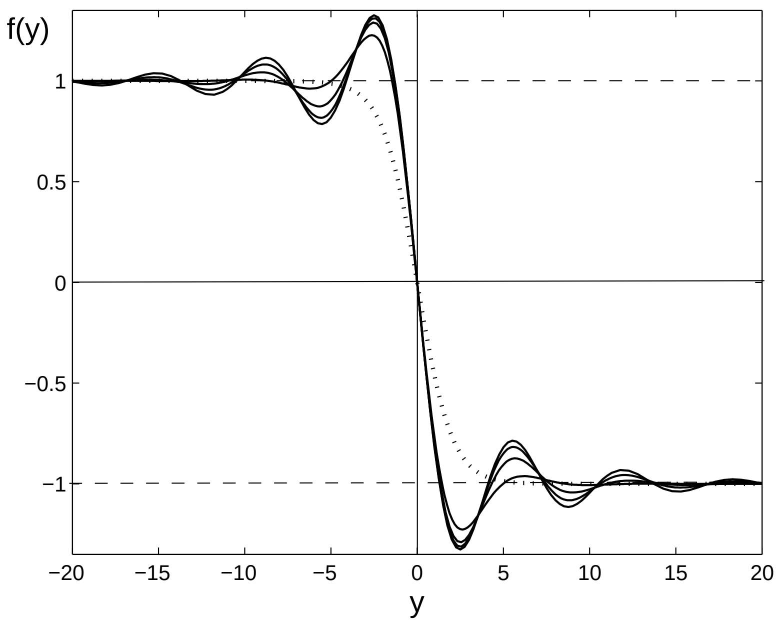


FIGURE 4. VSPs $f_{-}(y)$ for $m = 1$ (the monotone dotted line), 2, 3, 4, 5, 6 and 7.

that measures a "degree" of violation of order-preserving properties of semigroups induced by higher-order parabolic operators $\partial/\partial t + (-\Delta)^m$ (being order-preserving for $m = 1$ only, where $F > 0$ and hence $D_* = 1$). We easily show that the same order deficiency is responsible for the finite increase of total variation in the higher-order linear parabolic flows.

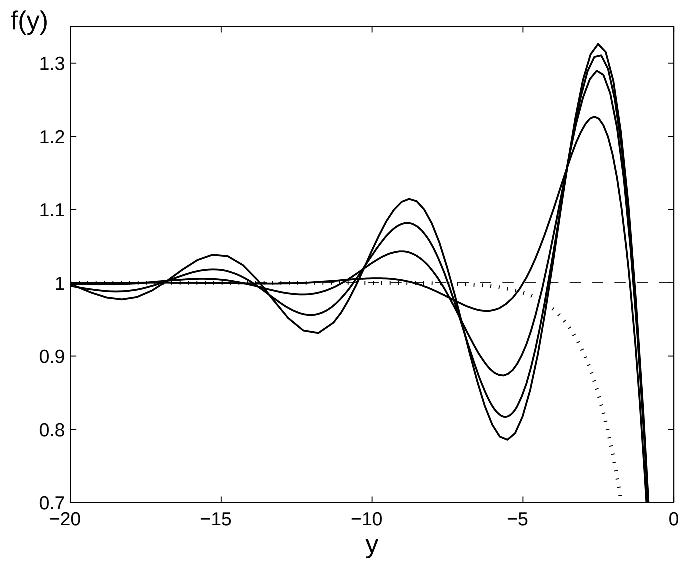


FIGURE 5. The enlarged left branches of VSPs $f_-(y)$ from Figure 4 for $m = 1$ (the monotone dotted line), 2, 3, 4, 5, 6 and 7. Clearly, the oscillations and total variations of $f_-(y)$ increase with m.

Proposition 4.2. *Let $m \ge 2$ and $u(x,t)$ satisfy the Cauchy problem*

$$u_t = \varepsilon(-1)^{m+1} D_x^{2m} u \quad \text{in } \mathbf{R} \times \mathbf{R}_+, \quad u(x,0) = u_0(x) \quad \text{in } \mathbf{R}.$$

Then: (i) the following estimate holds:

$$|u(t)|_{TV} \le D_* |u_0|_{TV} \quad \text{for } t > 0, \quad \text{with the constant } D_* = \textstyle\int |F| > 1, \tag{4.5}$$

and (ii) estimate (4.5) *is sharp for bounded data $u_0 \in L^\infty$.*

For the proof, see Appendix B.

Therefore the main difficulty in higher-order parabolic approximations is not establishing the compactness of the family $\{u_\varepsilon\}$ and using Helly's theorem for functions of bounded variation; cf. its systematic applications in [33] for $m = 1$. It is crucial that, in this case, the convergence $u_\varepsilon \to u$ to the entropy solutions assumes an extra hard asymptotic analysis and this cannot be avoided in estimating of the total variation of solutions to (1.4).

ON CUBIC EQUATION. Consider briefly the regularized PDE (1.12). For $\varepsilon = 0$, the Rankine-Hugoniot condition takes the form

$$\lambda = \tfrac{1}{3}\,[S^2(0^+) + S^2(0^-) + S(0^+)S(0^-)] \quad \big(= \tfrac{1}{3} \text{ for } S(0^-) = 1,\ S(0^+) = 0\big).$$

After scaling out the parameter ε, the TW profiles solve the ODE

$$-\lambda f' + f^2 f' = -f^{(4)}. \tag{4.6}$$

In Figure 6 we show three such profiles including the one (the bold line) corresponding as $\varepsilon \to 0$ to step-like initial data

$$u_0(x) = H(-x), \tag{4.7}$$

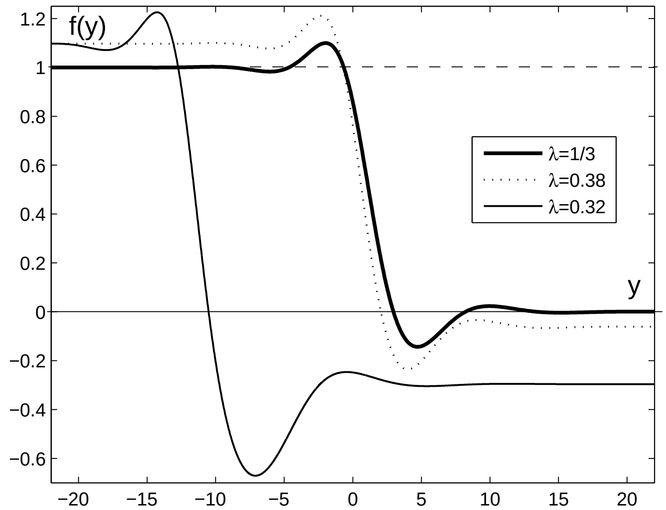


FIGURE 6. TW solutions of the ODE (4.6) for various λ's.

with $\lambda = \frac{1}{3}$, where $H(-x)$ is the reflected Heaviside function

$$H(-x) = \begin{cases} 1 & \text{for} \quad x \le 0, \\ 0 & \text{for} \quad x \ge 0. \end{cases}$$

Hence,

$$f\big(y/\varepsilon^{\frac{1}{3}}\big) \to u_-(x,t) = H\big(\tfrac{1}{3}\,t - x\big) \quad \text{as } \varepsilon \to 0.$$

For initial data $H(x)$, we have the rarefaction solution (limit $\varepsilon \to 0$ is not studied)

$$u_+(x,t) = \begin{cases} H(x) & \text{for } x \le 0,\ x \ge t; \\ \sqrt{\frac{x}{t}} & \text{for } 0 \le x \le t. \end{cases}$$

We observe typical oscillatory behaviour of viscosity shock waves. In what follows, we return to quadratic nonlinearities, though most of results and conclusions can be extended to cubic models.

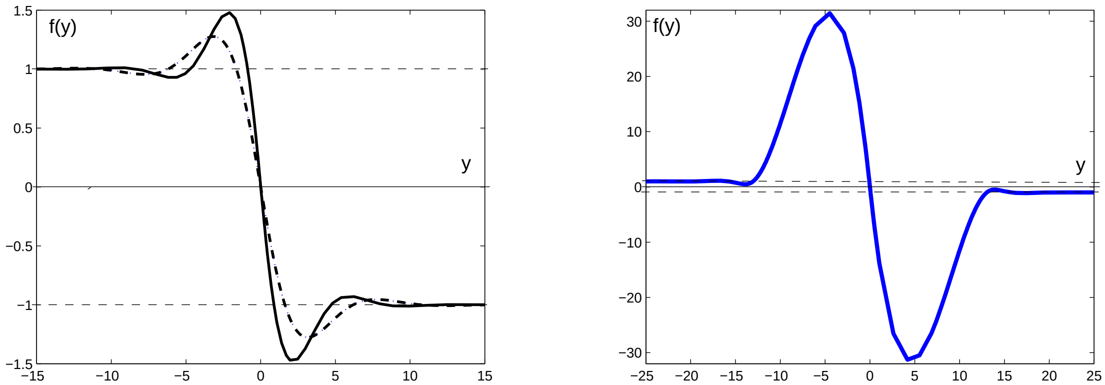
VSPS FOR QUASILINEAR AND THIN FILM-TYPE REQULARIZATIONS. The VSPs corresponding to $S_-(x)$ for PDEs (1.11) are presented in Figure 7. Notice a huge defect of variation $dV_2 \sim 120$ of the profile on (b) corresponding to the approximation via the "non-fully divergent" thin film operator in the last PDE in (1.11). Here we have used the regularization in the degenerate differential term by replacing

$$f^2 \mapsto \delta + f^2$$

with $\delta = 10^{-4}$, so decreasing δ increases dV_2, which reaches ~ 160 for $\delta = 5 \cdot 10^{-5}$.

4.2. **Regularized solutions do not satisfy Oleinik's upper gradient bound.** In the second-order approximation (1.2), it is known that for general hyperbolic equations, it is sufficient to choose

$$K(x,t) = \frac{C}{t}$$



(a) (b) $((10^{-4}+f^2)f''')' = -ff'$

FIGURE 7. The VSPs for rescaled PDEs (1.4). In (a), we present $(1+f^2)f^{(4)} = -ff'$ (dash line) and $f^2 f^{(4)} = -ff'$ (solid line).

in the entropy inequality (2.1); see [33, p. 145]. This follows from the Maximum Principle for (1.2), since the derivative $w = u_{\varepsilon x}$ satisfies the parabolic equation

$$w_t = \varepsilon w_{xx} - u w_x - w^2 \tag{4.8}$$

possessing the explicit solution

$$w_*(t) = \tfrac{1}{t} \quad \text{for } \ t > 0, \quad w_*(0^+) = +\infty. \tag{4.9}$$

Therefore, as a straightforward consequence, by comparison of solutions to (4.8) one obtains the following upper gradient bound for *arbitrary* initial data (including both shocks $u_0 = S_\pm(x)$ where for S_+ translations in time are performed):

$$u_{\varepsilon x} \le \tfrac{1}{t} \quad \text{in } \ Q. \tag{4.10}$$

This makes it possible to get in the limit (1.3) the entropy solutions satisfying (2.1).

Let now $m > 1$ in (1.4). Then similarly we get for $w = u_{\varepsilon x}$ the equation

$$w_t = \varepsilon(-1)^{m+1} D_x^{2m} w - u w_x - w^2 \tag{4.11}$$

possessing the same explicit solution (4.9) though the Maximum Principle does not apply and (4.10) does not follow. Anyway, the negative quadratic term $-w^2$ on the right-hand side of (4.11) stays the same and suggests to assume that $K(x,t) = \frac{C}{t}$ with some $C \gg 1$ possibly depending, in view of (2.1), on $u(x,t)$. Just in case, we write down such a suggestion in the general form: for $\varepsilon \approx 0^+$,

$$u_{\varepsilon x} \le K(x,t) \quad \text{uniformly in } \ Q, \tag{4.12}$$

assuming that K is bounded for $t > 0$. We now easily prove that this is not the case, and hence uniform estimates (4.10) or (4.12) are associated with the Maximum Principle for the second-order PDEs such as (1.2) only.

Proposition 4.3. *For $m \geq 2$,* (i) (4.12) *does not hold with any function $K(x,t)$ uniformly bounded in $x \in \mathbf{R}$ for $t > 0$, and* (ii) *the same is true for the discrete relation* (2.1).

Proof. (i) For approximation (2.12) as $\varepsilon \to 0$,

$$u_\varepsilon(x) = f_-(y) \to S_-(x) \quad \text{in } L^1(\mathbf{R}) \text{ and a.e.}, \quad y = x/\varepsilon^\alpha, \ \alpha = \tfrac{1}{2m-1}, \tag{4.13}$$

(4.12) implies that for any fixed $t > 0$,

$$f_-'(y) \leq \varepsilon^{2m-1} K(x,t) \to 0 \quad \text{as } \varepsilon \to 0, \tag{4.14}$$

which leads to $f_-'(y) \leq 0$ contradicting Proposition 4.1.

(ii) Let, for definiteness, $f_-(y)$ be oscillating as $y \to -\infty$. Taking the family (2.12) and using the fact that

$$\delta_0 = f_-(y_1) - f_-(y_2) > 0 \quad \text{for some } y_2 < y_1 < 0, \tag{4.15}$$

we have

$$\tfrac{u_\varepsilon(x_1)-u_\varepsilon(x_2)}{x_1-x_2} = \tfrac{\delta_0}{(y_1-y_2)\varepsilon^\alpha} \to +\infty \quad \text{as} \quad \varepsilon \to 0,$$

i.e., (2.1) does not holds on the family $\{u_\varepsilon\}$ approximating the entropy shock-wave S_-. $\square$

Obviously, there is no way to improve such a "bad" property of higher-order approximations, for instance, by neglecting the uniformity of (2.1), i.e., assuming that u_ε satisfies (2.1) for any $|x_1 - x_2| \geq C(\varepsilon) \to 0$ with $C(\varepsilon) \gg \varepsilon^\alpha$ as $\varepsilon \to 0$ (so that at $\varepsilon = 0^+$ we arrive at (2.1)). Indeed, taking as above $x_2 = y_2\varepsilon^\alpha$, where $y_2 < 0$ is the point of the absolute maximum of $f_-(y)$, and $x_1 \sim x_2 - C(\varepsilon)$, we still obtain the divergence

$$\tfrac{u_\varepsilon(x_1)-u_\varepsilon(x_2)}{x_1-x_2} \geq \tfrac{\delta_0}{2C(\varepsilon)} \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

In the given TW-approximation of $S_-(x)$, the approximating sequence satisfies

$$\sup_x u_{\varepsilon x}(x,t) \to +\infty \quad \text{as } \varepsilon \to 0 \tag{4.16}$$

and, moreover, we will introduce a strong evidence of the fact that (4.16) is a generic property of higher-order approximation of any entropy shocks. Therefore, any family $\{u_\varepsilon\}$ converging to a discontinuous entropy solution cannot approximate the entropy condition (2.1) in the sense of (4.12), which is directed to delete shocks S_+ from the entropy class. On the other hand, if $K(x,t)$ is not bounded for $t > 0$, e.g.,

$$K(x,t) = \tfrac{C}{|x|}$$

(this leads to a reasonable estimate of $u_{\varepsilon x}$), then estimate (2.1) does not exclude the non-entropy solution $S_+(x)$ either.

4.3. **Regularized solutions do not approximate Oleinik's condition E.** It follows from (4.15) that for arbitrarily small $\varepsilon > 0$, there exists the point $\bar{x} = \frac{1}{2}(y_1 + y_2)\varepsilon^\alpha$ and $h = \frac{1}{2}(y_2 - y_1)\varepsilon^\alpha$ such that for a constant $\delta_0 > 0$, there holds

$$u_\varepsilon(\bar{x} + h, t) \geq u_\varepsilon(\bar{x} - h, t) + \delta_0, \quad \text{where } h = O(\varepsilon^\alpha). \tag{4.17}$$

In this sense, due to non-monotonicity of the VSP, regularized solutions $\{u_\varepsilon\}$ do not approximate the condition E (2.2) as $\varepsilon \to 0$.

As a next corollary of Proposition 4.1, bounded variation of $u_\varepsilon(x,t)$ in x for $t > 0$ (and hence suitable compactness of the family $\{u_\varepsilon\}$) cannot be proved via local entropy conditions (2.1) or (2.2). For $m = 1$, this is a classical approach for 1D problems; see applications of the theory of functions with bounded variation and Helly's theorem throughout Sections 2–4 in [33]. One can expect that u_ε has uniformly bounded variation as $\varepsilon \to 0$, but this cannot be established in such a straightforward way as for $m = 1$.

It seems that for establishing of compactness of $\{u_\varepsilon\}$ for $m \geq 2$ for equations in 1D, the analysis in the class $BV(\mathbf{R}^N)$ of functions of bounded variations (see [3, Ch. 16] and [1]) or estimates for compactness in $L^1(\mathbf{R}^N)$ [40] can be useful, which are powerful tools in solving hyperbolic equations in $\mathbf{R}^N$. It is worth mentioning that both approaches are based on the Maximum Principle ideas. For instance, main estimates in [40, pp. 232-237] use comparison barrier techniques and do not extend to higher-order equations.

The only case where (4.14) does not lead to a contradiction, is $m = 1$, when the VSP (2.14) is monotone, a property to be associated with the Maximum Principle for the second-order ODE (2.13). In Section 7, we introduce higher-order models with monotone VSP's, which is important for their asymptotic stability.

4.4. **Direct approximation of nonlocal entropy inequality is impossible, problem E''–E''''.** Let us finally show that special "geometry" of the VSP affects also parabolic approximations of the nonlocal entropy inequality (2.5), though not in such a direct way as the local ones do above.

The derivation of the entropy inequality (2.5) from (2.5) is associated with the Maximum Principle for the second-order parabolic equations. One can see that (2.5) cannot be obtained in such a way if $m > 1$. For instance, let $m = 2$. Multiplying (1.4) by $E'(u)\chi$ and integrating by parts yields the following right-hand side in (2.5):

$$\begin{aligned} J_2(\varepsilon) = &- \varepsilon \iint \left[E''(u)(u_{xx})^2 - \tfrac{1}{3}E''''(u)(u_x)^4\right] \chi \, \mathrm{d}x \, \mathrm{d}t \\ &+ \varepsilon \iint \left[\tfrac{4}{3}E'''(u)(u_x)^3\chi_x + 2E''(u)(u_x)^2\chi_{xx} - E(u)\chi_{xxxx}\right] \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{4.18}$$

Consider the first integral in (4.18) depending on χ only, while the second one contains x-derivatives of χ which hence may be assumed to vanish on any open subset inside $\operatorname{supp}\chi$. We observe here two terms, the first positive one with $E'' \geq 0$ and the second one depending on E'''', which can have any sign (actually, if $E(u)$ is sufficiently close to $|u-k|$ then E'''' changes sign). We will show below that multiplicative, interpolation inequalities comparing the two terms do not help for coefficients given by sufficiently arbitrary smooth convex $E(u)$. Taking into account the above rescaled variable $y = x/\varepsilon^{\frac{1}{3}}$ (assuming shock to be put at $x = 0$), we have that both terms in the integral are of the same order $O(\varepsilon^{-\frac{4}{3}})$, i.e., even this precise structure of the singular shock layer is not enough to guarantee the necessary sign. Therefore, in order to get the entropy condition (2.5) directly, firstly, it is necessary to discuss the following problem E''–E'''': *is there a sufficiently wide set of smooth functions $E(u)$ satisfying*

$$E''(u) \ge 0 \quad and \quad E''''(u) \le 0 \quad in \ \mathbf{R}? \tag{4.19}$$

Obviously, such non-trivial bounded E's do not exist ($E''(u)$ is sufficiently smooth, non-negative, concave in $\mathbf{R}$, hence $E'' \equiv$ const). More involved E''–E''''–$E^{(6)}$–... unsolvable problems occur for $m = 3, 4, ...$ This expresses the fact that Kato's inequality (2.6) (or multiplication by sign) does not admit extension to higher-order operators $\partial t/\partial + (-\Delta)^m$.

4.5. **Interpolation inequalities do not guarantee the sign.** These aspects are discussed in Appendix C.

5. Parabolic approximation of shock-waves $S_\pm(x)$

5.1. **Proper and improper solutions.** We now begin more general analysis of the admissibility of various higher-order approximations of odd-order equations in the PDE sense. As a key example, we continue to study the Cauchy problem (1.4) and will use the following definition, which includes standard properties of weak (generalized) solutions of conservation laws; see [33] and [3, Ch. 15].

Definition 5.1. We say that a weak solution $u(x,t)$ of the conservation law (1.1) is *m-proper*, iff there exists a bounded sequence of initial data $\{u_{0\varepsilon}\} \to u_0$ in L^1 as $\varepsilon \to 0$ such that the family of classical solutions $\{u_\varepsilon(x,t)\}$ of the $2m$th-order parabolic problems (1.4) satisfies

$$u_\varepsilon(x,t) \to u(x,t) \quad \text{in } \ L^1 \ \text{ for any } \ t \ge 0. \tag{5.1}$$

$u(x,t)$ is *proper* if it is m-proper for any $m \ge 2$.

Classical theory for $m = 1$ says that

$$u(x,t) \quad \text{is 1-proper} \quad \Longleftrightarrow \ u(x,t) \text{ is entropy.} \tag{5.2}$$

According to the definition, proper solutions $u(x,t)$ are only those which can be obtained by arbitrary $2m$th-order parabolic approximations (1.4). Sometimes we will omit "m" from the "m-proper", if no confusion is likely. Keeping "m" can be key for other higher-order models with different viscosity approximation, for which the convergence results for any $m \ge 2$ are more difficult. The definition includes "approximation" of initial data. Indeed, once convergence (5.1) is established for fixed data u_{0n}, so $u_{\varepsilon n} \to u_n$ as $\varepsilon \to 0$, convergence $u_{\varepsilon n} \to u$ relative to both ε, n follows for arbitrary L^1-approximation of data $u_{0n} \to u_0$ as $n \to \infty$ by the triangle inequality,

$$\|u_{\varepsilon n}(t) - u(t)\|_1 \le \|u_{\varepsilon n}(t) - u_n(t)\|_1 + \|u_n(t) - u(t)\|_1, \tag{5.3}$$

since $u_n \to u$ in view of comparison theorems for entropy solutions, [33, 40].

The concept of proper solutions plays an important role in the theory of nonlinear singular parabolic equations creating finite-time singularities like blow-up, extinction, or quenching, where regular approximations (truncation of nonlinearities) make it possible to construct unique, maximal or minimal, extensions of solutions beyond singularity time; see [58] and earlier references therein. Another area, where approximation approaches are important, is concerned nonlinear evolution equations with singular initial data, e.g, with measures as initial conditions. Then weak solutions can cease to exist; see pioneering results in Brezis–Friedman [48]. In this cases approximation of singular data is of principal importance and approximation of equations is not necessary. Such extended semigroups constructed by approximation can be essentially *discontinuous* in any weak sense or in the sense of measures, and other concepts of solutions (demanding more detailed information on solutions properties) often do not apply. For instance, the positive approximation of nonnegative initial data, $u_{0\varepsilon}(x) = u_0(x) + \varepsilon$, with $\varepsilon > 0$, in constructing weak solutions $u = \lim u_\varepsilon$ of degenerate filtration equations

$$u_t = (\varphi(u,x))_{xx}, \quad \varphi'_u(0,x) = 0 \quad (\varphi(u,x) = u^2 \text{ for the porous medium equation})$$

is rather folklore after the seminal paper [59].

Clearly, such proper solutions concept is not necessary for the conservation laws, where classical entropy solution theory applies. It will be used below simply to test the concept and identify specific asymptotic properties to be treated later on. For a class of higher-order problems including (1.13) to be studied in Section 7, where entropy theory is not available, the concept of approximation becomes key.

Let $u_\varepsilon(x,t)$ in $Q_+ = \mathbf{R}_+ \times \mathbf{R}_+$ be odd in x, so satisfy the anti-symmetry conditions

$$D_x^k u(0,t) = 0 \quad \text{for } t > 0, \quad k = 0, 2, ..., 2m-2. \tag{5.4}$$

As the next step, we use in (1.4) the scaling

$$u_\varepsilon(x,t) = U_\varepsilon(y,\tau), \quad y = x/\varepsilon^\alpha, \;\; \tau = t/\varepsilon^\alpha, \quad \text{with exponent } \alpha = \tfrac{1}{2m-1}, \tag{5.5}$$

where $U = U_\varepsilon$ solves a uniformly parabolic equation of the form

$$U_\tau + UU_y = (-1)^{m+1} D_y^{2m} U, \quad \text{with initial data } U_\varepsilon(y,0) = U_{0\varepsilon}(y) \equiv u_{0\varepsilon}(y\varepsilon^\alpha). \tag{5.6}$$

The scaling (5.5) establishes as $\varepsilon \to 0$ a “parabolic zoom” for weak solutions of the conservation law in a shrinking neighbourhood of any point (x_0, t_0) in the $\{x,t\}$-plane (by replacing $x \to x - x_0$ and $t \to t - t_0$ in (5.5)). Therefore, it is key to describe the character of “smeared” shocks created by parabolic approximations. As we have seen, scaling (5.5) deletes the small parameter ε from the equation and U solves the uniformly parabolic equation (5.6). Global solvability follows from Proposition 3.1.

For any $m \geq 1$, equation (5.6) has the explicit linear solution

$$\bar U(y,\tau) = \tfrac{y}{\tau} \quad \big(\bar u_\varepsilon(x,t) = \tfrac{x}{t}\big) \text{ in } Q, \tag{5.7}$$

which occurs in the entropy rarefaction solution (2.10). Later on, the asymptotic stability of this rarefaction profile will be of crucial importance in our analysis.

The next two conclusions are elementary and apply to other models in Section 4.1.

Proposition 5.1. *The entropy shock wave $S_-(x)$ is proper.*

Proof. Let f be a solution of (2.13) with any $m \geq 2$. Then, since the convergence $f_-(y) \to \pm 1$ as $y \to \mp\infty$ given by the ODE (2.13) is exponential [55], the following holds:

$$u_\varepsilon(x) = f_-(x/\varepsilon^\alpha) \to S_-(x) \quad \text{as } \varepsilon \to 0 \tag{5.8}$$

in L^1, pointwise and uniformly on sets $\{|y| \geq c\}$ with any $c > 0$. $\square$

On the other hand, it is easy to see that a VSP f_+ corresponding to the non-entropy solution $S_+(x)$ does not exist. The proof applies to the ODEs corresponding to all three models in (1.4).

Proposition 5.2. *The problem for* f_+,

$$(-1)^{m+1} f^{(2m)} = f f' \quad \text{in } \mathbf{R}, \quad f(-\infty) = -1, \;\; f(+\infty) = 1, \tag{5.9}$$

does not have a solution.

Proof. Integrating the equation once yields

$$(-1)^{m+1} f^{(2m-1)} = \tfrac{1}{2}(f^2 - 1).$$

Multiplying by f' and integrating over $\mathbf{R}$ again by using exponential decay of the derivative $f^{(2m-2)}(y)$ as $y \to \pm\infty$, we get the contradiction

$$\textstyle\int (f^{(m)})^2 = -\frac{2}{3}. \quad \square$$

Nonexistence of f_+ is of a general nature and holds for various types of quasilinear divergent parabolic approximations. For instance, if instead of (1.4) we consider a parabolic regularization via the quasilinear p-Laplacian operator (gradient-dependent diffusivity coefficients are natural in regularization of conservation laws; see [49])

$$u_t + u u_x = \varepsilon (-1)^{m+1} D_x^m \big(|D_x^m u|^{p-2} D_x^m u \big), \quad p > 1, \tag{5.10}$$

then the corresponding "non-entropy" VSP f_+ given by

$$u_\varepsilon = f_+(y), \quad y = x/\varepsilon^\alpha, \quad \text{where} \quad \alpha = \tfrac{1}{mp-1},$$

is a weak solution of the ODE

$$(-1)^{m+1} (|f^{(m)}|^{p-2} f^{(m)})^{(m)} = f f', \quad f(-\infty) = -1, \;\; f(+\infty) = 1. \tag{5.11}$$

Integrating once yields

$$(-1)^{m+1} (|f^{(m)}|^{p-2} f^{(m)})^{(m-1)} = \tfrac{1}{2}(f^2 - 1)$$

and multiplying by f' and integrating over $\mathbf{R}$ leads to the same contradiction

$$\textstyle\int |f^{(m)}|^p = -\frac{2}{3}.$$

It seems that no reasonable divergent elliptic operators on the left-hand side of (5.11) can produce a heteroclinic connection $-1 \to 1$ in the corresponding ODE. For such approximation operators, this can be done only by taking negative parameters $\varepsilon < 0$ (then f_+ becomes f_-) creating ill-posed parabolic equations backward in time.

Nonexistence of the VSP does not trivially imply that $S_+(x)$ is not proper, i.e., cannot be obtained by parabolic approximations. In this sense, the case $m = 1$ is exceptional since the proof is straightforward by comparison with the exact solution (5.7). Indeed, if u_ε is an approximation, then $u_\varepsilon(x,t) \le \frac{x}{t}$ in Q_+. Hence, $u_\varepsilon(x,t)$ cannot stabilize to $S_+(x)$ as $\varepsilon \to 0$. For $m>1$, where the semigroup induced by equation (5.6) is not order-preserving, we cannot use comparison, and the result is based on a Lyapunov-type analysis that easily extends to the two first PDEs in (1.4).

Proposition 5.3. *$S_+(x)$ is not a proper solution.*

Proof. Without loss of generality, we assume that $U_\varepsilon(x,t) \to 1$ as $y \to +\infty$ sufficiently fast (e.g., exponentially, which happens if $U_{0\varepsilon}(y)=1$ for $y \gg 1$, following from the exponential decay of the fundamental solution of the parabolic operator [34]). Then all integrations below make sense. Multiplying equation (5.6) by U and integrating over $\mathbf{R}_+$ yields a Lyapunov function that is monotone decreasing on evolution orbits,

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\Phi(U)(\tau) \equiv \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}\tau}\left[\int_0^\infty (U^2-1)\,\mathrm{d}y\right] = -\tfrac{1}{3} - \int_0^\infty \left(D_y^m U\right)^2 \mathrm{d}y \le -\tfrac{1}{3}. \tag{5.12}$$

Therefore,

$$\Phi(U)(\tau) \le -\tfrac{\tau}{3} + \Phi(U_{0\varepsilon}) \quad \text{for} \quad \tau>0.$$

Using the rescaled variables given in (5.5), we have that, for any $t>0$,

$$\int \left[u_\varepsilon^2(x,t)-1\right]\mathrm{d}x \le -\tfrac{2t}{3} + 2\Phi(u_{0\varepsilon}). \tag{5.13}$$

Passing to the limit $\varepsilon \to 0$ and using that $u_\varepsilon \to S_+$ in L^1 (then $\Phi(u_{0\varepsilon}) \to 0$), we obtain a contradiction in the inequality (5.13). The analysis applies to the Cauchy problem in Q without the anti-symmetry conditions (5.4). $\square$

5.2. **On the asymptotics of singular layer for S_-.** This is much more difficult, and only local asymptotic estimates are known rigorously. Concerning formal asymptotic expansions as $\varepsilon \to 0$, notice that even the case $m=1$ is rather difficult to justify; see Il'in [60, Ch. 6]. For instance, it is worth mentioning that the appearance of the logarithmic expansion terms $\ln\varepsilon$ in [60, p. 235] is justified by multiple reductions of Burgers'-like equations to the heat equation. Obviously, this is not possible for $m>1$.

We discuss the structure of a connecting orbit $S_- \mapsto f$, $m \ge 2$, for the problem (5.6) with initial data $S_-(y)$. In the main Region I, with $\tau \ll 1$, we use the rescaled variables

$$\eta = y/\tau^{\frac{1}{2m}}, \quad s=\ln\tau \to -\infty \;\text{ as }\; \tau \to 0,$$

so that the PDE for $U=U(\eta,s)$ contains an exponentially small perturbation,

$$U_s = \mathbf{A}U - \mathrm{e}^{\frac{3s}{4}} UU_\eta \;\text{ as }\; s \to -\infty. \tag{5.14}$$

Here

$$\mathbf{A} = (-1)^{m+1}D_\eta^{2m} + \tfrac{1}{2m}\,\eta D_\eta \tag{5.15}$$

is a linear non self-adjoint operator. It has the discrete spectrum $\left\{-\frac{k}{2m}<0,\, k\ge 1\right\}$ in a weighted L^2-space L^2_ρ, with the weight

$$\rho = \mathrm{e}^{a|\eta|^\alpha}, \quad \alpha = \tfrac{2m}{2m-1},$$

where $a > 0$ is small enough, with complete and closed set of eigenfunctions $\{\psi_k\}$, compact resolvent, etc.; see [52] and Appendix D, where the adjoint operator $\mathbf{A}^*$ with eigenfunctions $\{\psi_k^*\}$ are studied, for greater detail. Therefore, passing to the limit $s \to -\infty$ in (5.14), we have that such a solution satisfies in L^1

$$U(\eta, s) \to \theta(\eta) \quad \text{as } \ s \to -\infty, \tag{5.16}$$

so $U(\eta, s)$ stabilizes to a stationary solution $\theta(\eta)$,

$$\mathbf{A}\theta \equiv (-1)^{m+1}\theta^{(2m)} + \tfrac{1}{2m}\,\eta\theta' = 0 \quad \text{in } \mathbf{R}, \quad \theta(\pm\infty) = \mp 1. \tag{5.17}$$

Since $\theta(\eta)$ satisfies the poly-harmonic equation

$$U_\tau = (-1)^{m+1} D_y^{2m} U$$

with initial data $S_-(y)$, we have

$$\theta(\eta) = 1 - 2 \textstyle\int_{-\infty}^{\eta} F(\zeta)\,\mathrm{d}\zeta, \quad \text{where } F \text{ is the kernel in (A.1).}$$

Figure 8 shows functions θ for $m = 2$ and 3, together with the corresponding VSP profiles, i.e., stationary solutions of (5.6). It is remarkable that θ and f have very similar shapes. Therefore, further evolution consists of a slight deformation of these shapes to match the stationary profiles in Region III, for $t \gg 1$. Beforehand, in the intermediate Region II, with $\tau = O(1)$ ($s = O(1)$), we can use equation (5.14), where we apply the approximation (5.16) in the nonlinear term to get, by convolution with the fundamental solution (A.1) (here $\varepsilon = 1$ and $x, t \mapsto \eta, s$),

$$U(y, \tau) \approx \theta(\eta) - \tfrac{1}{2} \textstyle\int_0^\tau (\tau - \rho)^{-\frac{1}{m}}\,\mathrm{d}\rho \int_{-\infty}^{\infty} F'\big(\frac{y-z}{(\tau-\rho)^{1/2m}}\big)\theta^2\big(\frac{z}{\rho^{1/2m}}\big)\,\mathrm{d}z, \ \ \eta = \frac{y}{\tau^{1/2m}}\,. \tag{5.18}$$

The first function $\theta(\eta)$ describes the above "parabolic" smoothing of the shock for small τ, while the second opposite term prevents further collapse of the step behaviour and thus describes the intermediate stage of attracting to the VSP $f_-(y)$. For deriving matching conditions between Regions I–II and II–III containing in (5.18) and describing the connecting heteroclinic orbit, spectral properties of linearized operators in Regions I (see Appendix D) and III (see next Section 6) are key. Namely, in Region I, the perturbed behaviour is as follows:

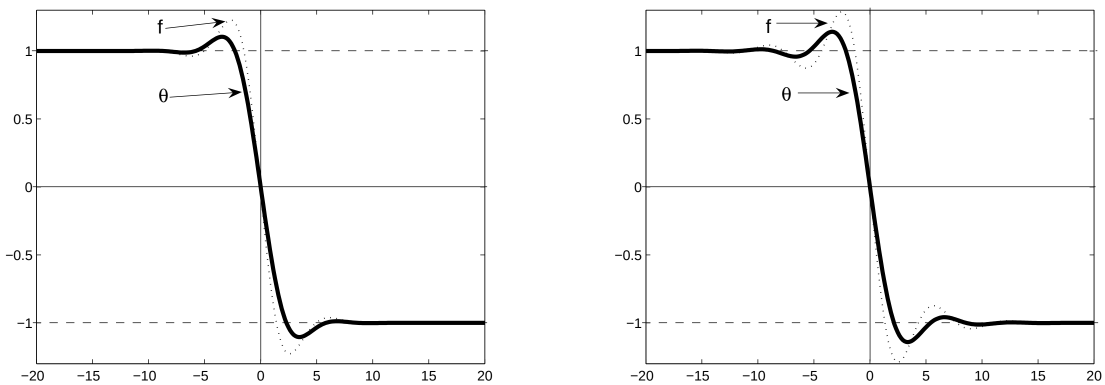
$$U(y, \tau) = \theta(\eta) + \mathrm{e}^{\frac{3s}{4}}\big(\mathbf{A} - \tfrac{3}{4}I\big)^{-1}\theta\theta' + ... \quad \text{as } \ s \to -\infty \quad \big(\tfrac{3}{4} \notin \sigma(\mathbf{A})\big). \tag{5.19}$$

In Region III, we have

$$U(y, \tau) = f_-(y) + C\mathrm{e}^{\lambda_1 \tau}\hat{\psi}_1(y) + ... \quad \text{as } \ \tau \to +\infty, \tag{5.20}$$

where $\lambda_1 < 0$ is the first negative eigenvalue with eigenfunction $\hat{\psi}_1$ of the linearized (about f_-) operator $\mathbf{N}_{2m}$ to be introduced in (6.2). We expect that (5.19), (5.20) determine a heteroclinic connection $S_- \mapsto f$ (in the rescaled variables, it is $\theta \mapsto f$). Rigorously, problems of connecting orbits remain open for all semilinear higher-order parabolic PDEs, since Sturm's First Theorem on zero sets applies for $m = 1$ only.

On a formal connection for $m = 2$ by averaging method. This approach is discussed in Appendix E.



(a) $m=2$ (b) $m=3$

FIGURE 8. Solutions θ of the ODE (5.17) for $m=2$ (a) and $m=3$ (b). VSPs f solving (5.13), $\lambda=0$, are given for comparison.

6. Stability of the VSP and entropy inequalities. Asymptotic stability of the rarefaction profile

In this section, we deal the first asymptotic problem. In view of negative conclusions of Section 4, we claim that, in general, the convergence (5.1) to entropy solutions cannot be proved without deep understanding of the corresponding asymptotic problems. The approximation problem for $m \geq 2$ is thus an example, where the existence of a solutions (as the limit of $\{u_\varepsilon\}$) cannot be separated from corresponding parabolic asymptotic theory. As usual in scaling techniques, due to variables (5.5), the limit $\varepsilon \to 0^+$ for $u_\varepsilon(x,t)$ in a natural sense is equivalent to $\tau \to +\infty$ for $U(y,\tau)$.

6.1. **On generic formation of the shock layer: stability of the VSP.** We now discuss conditions under which the VSP satisfying (2.13) describes the generic formation of the shock layer in the convergence (5.1) to the entropy shock $S_-(x)$. This means that f is the asymptotically stable stationary solution of the rescaled equation (5.6), so we perform the standard linearization by setting

$$U(y,\tau) = f_-(y) + Y(y,\tau), \quad \text{where } Y \text{ solves} \tag{6.1}$$

$$Y_\tau = \mathbf{N}_{2m} Y + \mathbf{D}(Y) \quad \text{with } \mathbf{N}_{2m} = (-1)^{m+1} D_y^{2m} - f D_y, \tag{6.2}$$

$\mathbf{D}(Y) = YY_y$ being the quadratic perturbation. By the principle of the linearized stability (see e.g. [63, Ch. 9]), one needs to study the spectral problem

$$\mathbf{N}_{2m}\psi = \lambda\psi, \tag{6.3}$$

where by classical ODE theory [55], $\psi(y)$ is assumed to have exponential decay as $|y| \to \infty$. Multiplying equation (6.3) by $\bar{\psi}$ in $L^2(\mathbf{R})$ and the conjugated one by ψ yields

$$(\mathrm{Re}\,\lambda)\,\|\psi\|_2^2 = -\|\psi^{(m)}\|_2^2 + \tfrac{1}{2}\int f'(y)|\psi(y)|^2\,\mathrm{d}y. \tag{6.4}$$

We thus observe another "bad" consequence of the VSP $f_-(y)$ being non-monotone: if $f'_-(y)$ changes sign, then (6.4) does not directly imply the necessary stability condition

$$\operatorname{Re}\lambda < 0 \quad \text{for} \;\; \lambda \in \sigma(\mathbf{N}_{2m}), \tag{6.5}$$

unlike the only case $m = 1$, where $f'_- < 0$ by (2.14) and (6.5) follows from (6.4). Nevertheless, since $f_-(y)$ must be "effectively" decreasing as a heteroclinic connection $1 \to -1$, one can expect that (6.5) remains true for such $f_-(y)$. This is proved in [15] for $m = 2$ (the proof is partially computational; for an analytic proof of linear stability, see [12]). As a result, the operator $\mathbf{N}_4$ was shown [11] to be sectorial with the spectrum satisfying ($m = 2$)

$$\sigma(\mathbf{N}_{2m}) \subset \{\operatorname{Re}\lambda \le -k\} \quad \text{with a constant} \;\; k > 0, \tag{6.6}$$

in the weighted Sobolev space $H^3_\rho(\mathbf{R})$ with the exponential weight $\rho(y) = \cosh(\mu y)$, where $\mu > 0$ is a small constant. This guarantees the exponential decay of the semigroup $\|\mathrm{e}^{\mathbf{N}_{2m}\tau}\|_{\mathcal{L}} \le C\mathrm{e}^{-k\tau}$ in the space of linear maps $\mathcal{L}(H^3_\rho, H^3_\rho)$, and hence the exponential stability of the VSP by the principle of linearized stability; see [63, Ch. 9].

It is natural to expect that such stability results are true for arbitrary $m > 2$. Namely, the eigenvalue problem (6.3) for the ODE operator $\mathbf{N}_{2m}$ in the weighted space $L^2_\rho(\mathbf{R})$ of odd functions satisfying (5.4) with the dense domain $H^{2m}_\rho(\mathbf{R})$, satisfies (6.5) and (6.6). Nevertheless, even a computational proof, which can be done for $m = 3$ and 4 by rather standard codes, is expected to get more and more involved for larger m. Once (6.5) is proved, the theory of sectorial operators [35, 63] and interpolation inequalities apply to guarantee the exponential stability of VSP's. In its turn, this will imply that entropy conditions (Section 2) cannot be approximated in the viscosity sense for any initial data. Furthermore, then the variation deficiency (4.1) scaled according to (2.11) for moving shocks actually describes locally the "jump" of total variation at $\varepsilon = 0$.

6.2. **Asymptotic stability of the rarefaction profile.** This analysis also features interesting related spectral properties of the linear operators involved and is performed in Appendix D.

7. On other higher-order models: shocks and approximations

7.1. **Preliminary properties of odd-order models.** In this section, we describe similarities with higher-odd-order equations from nonlinear dispersion theory (cf. [39] for $m = 2$ and [66] for $m = 4$; see also [38, Ch. 4] for further models and references)

$$\boxed{u_t + (-1)^{m-1} D_x^{2m-2}(uu_x) = 0 \quad \text{in} \;\; Q,} \quad u(x,0) = u_0(x) \quad \text{in} \;\; \mathbf{R}, \tag{7.1}$$

where $m = 1$ gives the conservation law (1.1). As above, we concentrate on evolution properties of solutions corresponding to initial data $S_\pm(x)$, (1.10). This analysis is a first step towards understanding general weak solutions of such PDEs. Entropy-like theories for (7.1) for any $m \ge 2$ are still not known, so that we will rely on our proper (extended semigroup) concept of solutions and necessary numerical ODE results.

(i) Similarity shock and rarefaction waves: a PDE admissibility. Consider first the following blow-up (as $t \to T^- > 0$) similarity solution of (7.1):

$$u_S(x,t) = g(z), \;\; \text{where } z = x/(T-t)^{\frac{1}{2m-1}} \quad \text{and } g \text{ solves the ODE} \tag{7.2}$$

$$(-1)^{m-1}(gg')^{(2m-2)} + \tfrac{1}{2m-1}\, g'z = 0 \quad \text{in } \mathbf{R}, \quad g(\pm\infty) = \mp 1. \tag{7.3}$$

Assuming, as for $m = 1$, that $g(z)$ is odd and that $g(z) > 0$ for $z < 0$ (to be checked numerically), we set $G(z) = g^2(z)$ to obtain a semilinear ODE with m anti-symmetry conditions at the origin,

$$G^{(2m-1)} = \tfrac{(-1)^m}{(2m-1)}\, \tfrac{G'z}{\sqrt{G}} \;\; \text{for } z<0, \quad G(-\infty) = 1, \; G(0) = ... = G^{(2m-2)}(0) = 0. \tag{7.4}$$

Such ($2m$-1)th-order ODE problems are not easy for analytical study. In this framework, it is crucial that the asymptotic bundle of solutions as $z \to -\infty$, where $G(z) \to 1$, is given by the linear ODE

$$G^{(2m-1)} = \tfrac{(-1)^m}{2m-1} G'z$$

and has dimension m. Therefore, this is sufficient to match *precisely* m conditions at the origin in (7.4). The oscillatory character of solutions for $z \ll -1$ (see below) confirms that this is possible. Uniqueness of such a matching for $m \ge 2$ remains an open problem.

Numerically, we obtain a clear evidence on existence and uniqueness of such smooth solutions g and we state

Conjecture 7.1. *For any $m \ge 2$, there exists a unique stable solution $G(z)$ of* (7.4).

In Figure 9 we present the anti-symmetric profiles $G(z) > 0$ for $z < 0$ in cases $m = 2$ (a) and $m = 3$ (b). We observe that, on the left-hand side, the similarity profiles $G(z)$ are strongly oscillatory. For $m = 2$, i.e., for the third-order PDE (7.1), this corresponds to the behaviour of the Airy function as $z \to -\infty$,

$$G(z) \sim 1 + c\mathrm{Ai}(z) \sim 1 + c|z|^{-\frac{1}{4}} \cos\left(a_0|z|^{\frac{3}{2}} + c_0\right) \;\; \text{where} \;\; a_0 = \tfrac{2}{9}\sqrt{3}. \tag{7.5}$$

Indeed, as $x \to -\infty$ and hence $u \to 1$, (7.3), with $m = 2$, becomes asymptotically linear with the fundamental solution

$$u_t = u_{xxx} \quad \Longrightarrow \quad b(x,t) = t^{-\frac{1}{3}} \mathrm{Ai}\big(\tfrac{x}{t^{1/3}}\big).$$

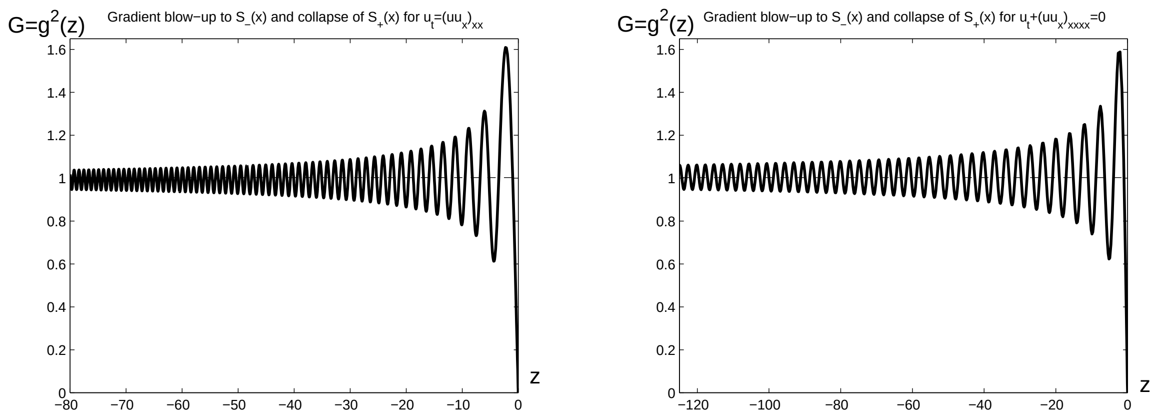
In particular, the asymptotics (7.5) implies that the total variation of any solution of (7.4) (and $u_S(x,t)$ for any $t < T$) is *infinite*. It is easy to check, deriving asymptotics similar to (7.5), that the same holds for any $m \ge 2$. This is in striking contrast with solutions for $m = 1$, i.e., of (1.1), where finite variation approaches are key. In view of conditions at $\pm\infty$ in (7.3), for such $g(z)$,

$$u_S(x,t) \to S_-(x) \quad \text{as} \;\; t \to T^- \tag{7.6}$$

for any $x \in \mathbf{R}$, uniformly in $\mathbf{R} \setminus (\delta, \delta)$, $\delta > 0$ small, and in $L^p_{\mathrm{loc}}(\mathbf{R})$ for $p \in [1, \infty)$.

Using the reflection symmetry $u \mapsto -u$, $t \mapsto -t$ of PDEs (7.3), we also conclude that the same similarity solutions defined for $t > 0$,

$$u_S(x,t) = g(z), \;\; \text{with } z = x/t^{\frac{1}{2m-1}}, \tag{7.7}$$



(a) third-order, $m = 2$ (b) fifth-order, $m = 3$

FIGURE 9. The shock wave similarity profile $G(z) = g^2(z)$ satisfying the ODE (7.4).

describe collapse as $t \to 0^+$ of the non-entropy shock $S_+(x) = \operatorname{sign} x$, posed as initial data. Then (7.7) plays the role of the *rarefaction wave* for higher-order "conservation laws" such as (7.1). This means that $S_+(x)$ is not an entropy shock.

Thus, the above similarity solution (7.2) of (7.3) describe formation of the "entropy" shock $S_-(x)$ from a sufficiently smooth initial data, while the rarefaction wave (7.7) is responsible for smooth collapse of the initial singularity $S_+(x)$. This means that these two Riemann's problems admit the same treatment for any order $m \ge 1$.

(ii) QUARTIC NONLINEARITY. To confirm a generic character of such shocks $S_-(x)$, we consider similar PDEs with fourth-degree nonlinearity

$$\boxed{u_t + (-1)^{m-1} D_x^{2m-2}(u^3 u_x) = 0,}$$

where the blow-up solutions has the same form (7.2), while g and $G = g^4$ solve

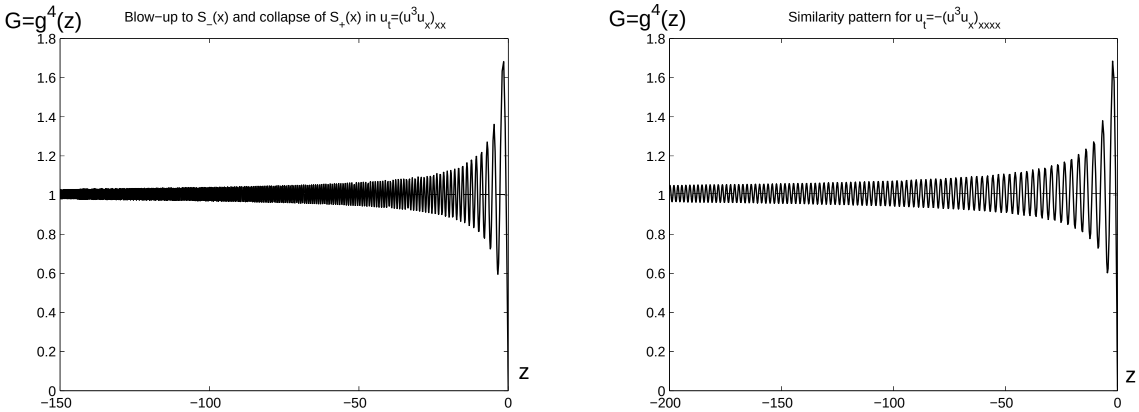
$$(-1)^{m-1}(g^3 g')^{(2m-2)} + \tfrac{1}{2m-1}\, g' z = 0 \implies G^{(2m-1)} = \tfrac{(-1)^m}{(2m-1)}\, \tfrac{G' z}{G^{3/4}} \tag{7.8}$$

with the same boundary conditions. Shown in Figure 10 are numerical oscillatory profiles $G(z)$, $z < 0$, for $m = 2$ (a) and $m = 3$ (b), for which (7.6) holds.

(iii) ON NON-OSCILLATORY FBP. PDEs (7.1) for $m \ge 2$ admit a natural free-boundary setting with typical "zero contact angle" conditions at levels $\{u = \pm 1\}$. Then, for above $S_\pm$–Riemann's problems, similarity solutions have finite interfaces and are not oscillatory nearby. This assumes studying the ODE (7.3) with the same number m of conditions at a free interface position $z = a < 0$

$$g(a) = 1, \quad g'(a) = ... = g^{(m-1)}(a) = 0. \tag{7.9}$$

By the anti-symmetry conditions at $z = 0$ in (7.4) we have an $(m-1)$-dimensional bundle for $z \approx 0^-$ that is enough to match m conditions (7.9) with a free parameter a. Numerics show existence and uniqueness of such a similarity FBP solution $g(z)$ of (7.3), (7.9) for



(a) third-order, $m = 2$ (b) fifth-order, $m = 3$

FIGURE 10. The shock wave similarity profile $G(z) = g^4(z)$ of the ODE (7.8).

$m = 2$ and 3. In particular, global similarity solutions (7.7) represent the rarefaction waves.

Figure 11 shows the RP (the boldface line) for $m = 2$ and $m = 3$, obtained numerically by shooting in the ODE (7.4) for $G = g^2$. Here, we have performed reflection $z \mapsto -z$ for convenience, so we are restricted to the semi-interval $(0, a)$, with the interface position $a > 0$, putting the free-boundary conditions

$$G(0) = G''(0) = 0 \;\text{ for }\; m = 2, \;\text{ and }\; ... = G^{(4)}(0) = 0 \;\text{ for }\; m = 3.$$

We extend the solution into $(-a, 0)$ by $-G(-z)$.

For $m = 2$, (7.3) with $z \mapsto -z$ this leads to a third-order equation,

$$(g^2)''' + \tfrac{2}{3}\, g'z = 0,$$

which is invariant under a group of scalings. The change of variables

$$\xi = \ln z, \quad g = \mathrm{e}^{3\xi}\varphi(\xi), \quad \text{and} \quad P(\varphi) = \varphi'$$

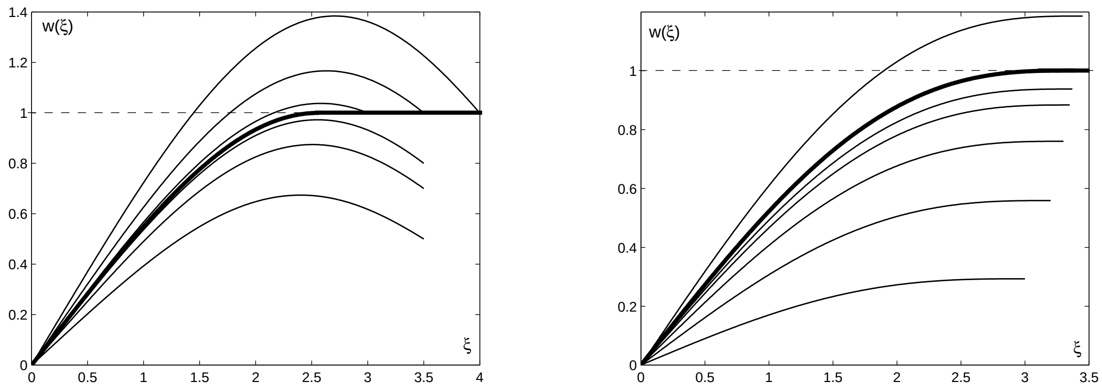
reduces it to a second-order ODE for P, which can be studied to guarantee existence of a suitable RP. For $m \geq 3$, a shooting-type argument is suitable (but indeed difficult) to prove existence, while the origin of uniqueness remains obscure and open.

(iv) $2m$TH-ORDER PARABOLIC APPROXIMATION. A natural regularization of (7.1) is

$$u_t + (-1)^{m-1} D_x^{2m-2}(uu_x) = \varepsilon(-1)^{m+1} D_x^{2m} u, \qquad u(x, 0) = u_{0\varepsilon}(x), \tag{7.10}$$

where $u_{0\varepsilon} \to u_0$ as $\varepsilon \to 0^+$ in a suitable (L^p) topology. The well-posedness of such approximations in the sense of Proposition 3.1 becomes much more delicate problem, which is not studied here. The corresponding rescalings are

$$u(x, t) = U\big(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon^{2m-1}}\big) \implies U_\tau + (-1)^{m-1} D_y^{2m-2}(UU_y) = (-1)^{m+1} D_y^{2m} U. \tag{7.11}$$



(a) $m = 2$, $a \approx 2.565$ (b) $m = 3$, $a \approx 3.392$

FIGURE 11. Shooting the RP (the boldface line) satisfying (7.4), $z \mapsto -z$, and free-boundary conditions (7.9) for $m = 2$ (a) and $m = 3$ (b).

It is important that the PDE (7.11) does not admit moving TWs as heteroclinic connections $1 \mapsto -1$. Indeed, for $\lambda \neq 0$, integrating once, we obtain the ODE

$$-\lambda f + (-1)^{m-1}\tfrac{1}{2}\,(f^2)^{(2m-2)} = (-1)^{m+1} f^{(2m-1)} + C, \tag{7.12}$$

where C is a fixed constant. Clearly, for any $m > 1$, (7.12) cannot possess smooth solutions satisfying $f(\pm\infty) = \mp 1$ unless $\lambda = 0$. This negative result suggests that the original PDE (7.1) does not admit *moving* discontinuous solutions of the $S_{\pm}$-type (but indeed there exist others of different shapes). Therefore we need to study approximation properties of these standing shocks with $\lambda = 0$ only.

(v) MONOTONE VISCOSITY SHOCK PROFILE. The VSP corresponding to the proper (see below) shock-wave $S_-(x)$ as a stationary solution of (7.10) has the form

$$u_\varepsilon(x) = f_-(y), \quad \text{with} \quad y = x/\varepsilon,$$

satisfying

$$f^{(2m)} = \tfrac{1}{2}(f^2)^{(2m-1)}, \; f(\pm\infty) = \mp 1 \implies f' = \tfrac{1}{2}(f^2 - 1). \tag{7.13}$$

Hence the unique VSP has the form (2.14). Indeed, as we have seen, the monotonicity of the VSP is an essential positive feature of this higher-order model. Recall that this non-oscillatory stationary states of (7.10) are quite special, and general solutions of this PDE must be highly oscillatory about ± 1 for any $m \geq 2$, as the fundamental solution (A.1) guarantees.

7.2. **Proper and improper shock-waves.** Similar to Section 5, we say that $u(x,t)$ is a *proper* solution (we do not use "m-proper" by obvious reason) of the Cauchy problem (7.1) if there exists a sequence of initial data $u_{0\varepsilon} \to u_0$ such that the solutions of parabolic problems (7.10) satisfies (5.1) (at least in H^{-m}). Let us study the evolution properties of the shock-waves $S_{\pm}(x)$.

Proposition 7.1. (i) $S_-(x)$ *is a proper solution, and* (ii) $S_+(x)$ *is not.*

Proof. (i) We have that convergence (5.8) with the VSP (2.14) holds a.e. (ii) The VSP f_+ corresponding to $S_+(x)$, i.e., a solution of the ODE satisfying $f(\pm\infty) = \pm 1$, does not exist. Consider the equation (7.11) in $Q_+ = \mathbf{R}_+ \times \mathbf{R}_+$ with conditions (5.4). Assuming that $u_{0\varepsilon}(x) \to 1$ as $x \to \infty$ exponentially fast (then the same holds for solutions $u_\varepsilon(x,t)$ following from the integral equation), we apply to equation (7.11) operator $(-D_y^2)^{1-m}$ naturally defined via integrating equation $2m-2$ times and integrate again over (y,∞). Next, multiplying by $U-1$ in L^2, we arrive at a Lyapunov function (cf. (5.12))

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}\tau} \textstyle\int_0^\infty \big[(D_y)^{-m}(U-1)\big]^2 = -\tfrac{1}{3} - \textstyle\int_0^\infty (U_y)^2 \le -\tfrac{1}{3}. \tag{7.14}$$

Integrating and rescaling this identity, similarly to the proof of Proposition 5.3, we have that u_ε cannot converge to S_+ as $\varepsilon \to 0$. $\square$

7.3. **For $m = 2$ the VSP is stable.** Let us prove that for $m = 2$ the monotonicity of the VSP (2.14) guarantees the necessary condition (6.5) of its stability. The linearization (6.1) yields the quadratically perturbed equation (6.2) with the linear operator

$$\mathbf{N}_4 Y = -Y^{(4)} + (fY)'''. \tag{7.15}$$

Solving the eigenvalue problem (6.3) in a space of exponentially decaying functions (hence from L^2) and setting $\psi = \phi'''$, we arrive at the eigenvalue equation

$$-\phi^{(4)} + f\phi''' = \lambda\phi, \quad \phi \in H^4.$$

Multiplying this equation by $\bar{\phi}''$ in L^2 and the conjugate one by ϕ'', after integration by parts one obtains

$$(\mathrm{Re}\,\lambda) \textstyle\int |\phi'|^2 = -\int |\phi'''|^2 + \tfrac{1}{2}\int f'|\phi''|^2.$$

It follows that in suitable classes of even or odd functions, (6.5) holds. By the interpolation inequalities, this implies that (7.15) is a sectorial operator in a weighted L^2-space (see [11]) and the exponential stability of the VSP follows.

This means that convergence (5.8) describes the generic formation of the shock layers in fourth-order parabolic approximations of such shock-waves as weak solutions of the third-order equation (7.1).

7.4. **On higher-order approximations: existence and stability of VSP.** Equation (7.1) admits various parabolic approximations of different orders. For instance, consider its $(2m+2)$th-order approximation

$$u_t + (-1)^{m-1} D_x^{2m-2}(uu_x) = \varepsilon(-1)^m D_x^{2m+2} u, \tag{7.16}$$

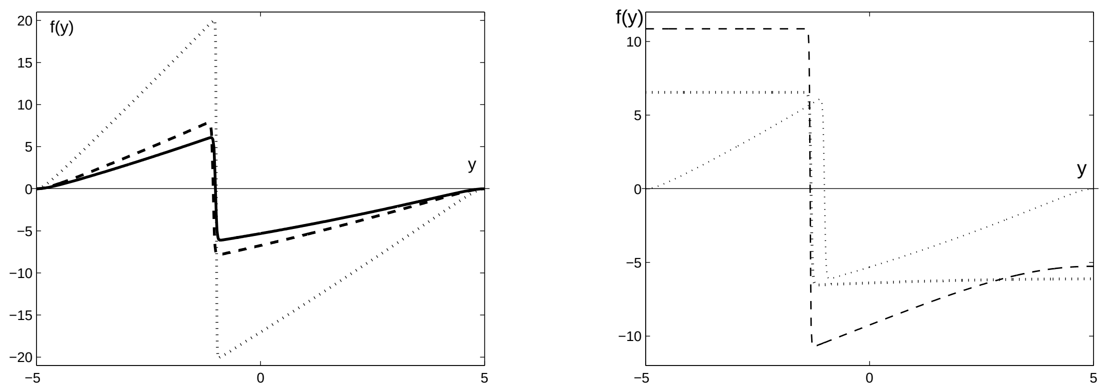
with rescaled variables

$$u(x,t) = U(y,\tau), \quad y = x/\varepsilon^{\frac{1}{3}}, \quad \tau = t/\varepsilon^{\frac{2m-1}{3}},$$

where U solves the parabolic PDE

$$U_\tau + (-1)^{m+1} D_y^{2m-2}(UU_y) = (-1)^m D_y^{2m+2} U. \tag{7.17}$$

Then the VSP $f_-(y)$ for the shock wave $S_-(x)$ is the same as for extended Burgers' equation (1.4) with $m = 2$ and is uniquely determined by the ODE problem (2.13). The



(a) $\varepsilon \approx 0.1,\ \lambda = -\frac{1}{4}$ (b) $\varepsilon = 0.1,\ \lambda \approx -\frac{1}{4}$

FIGURE 12. Varying the solutions of (7.20) with $\lambda \approx -\frac{1}{4}$ and $\varepsilon \approx 0.1$.

stability analysis is based on the results from [15] and [11, 12]. The characterization of shock-waves, Proposition 7.1, remains unchanged.

For instance, for $m = 2$ we have the sixth-order parabolic PDE

$$u_t = (uu_x)_{xx} + \varepsilon u_{xxxxxx} \quad \Longrightarrow \quad u(x,t) = U(y,\tau),\ y = x/\varepsilon^{\frac{1}{3}},\ \tau = t/\varepsilon, \tag{7.18}$$

where U solves the rescaled ε-independent parabolic equation

$$U_\tau = (UU_y)_{yy} + U_{yyyyyy}.$$

For the VSP with $\lambda = 0$, $U(y) = f_-(y)$, we have the ODE

$$(ff')'' + f^{(6)} = 0.$$

Hence, on integration, we obtain another ODE, which is more difficult than that in (7.13) and appeared earlier (see (2.13) for $m = 2$)

$$f''' = \tfrac{1}{2}\,(1 - f^2). \tag{7.19}$$

The VSP solving (7.19) is given in Figure 1(a) (the boldface line).

7.5. **On moving proper shocks.** The moving TWs $u(x,t) = f(x - \lambda t)$ with $\lambda \neq 0$ of equation (7.18) satisfy the ODE with the Rankine-Hugoniot condition ($\varepsilon = 0$)

$$-\lambda f' = (ff')'' + \varepsilon f^{(6)} \quad \Longrightarrow \quad \lambda = \tfrac{[(ff')'']}{[f']}. \tag{7.20}$$

The passage to the limit in (7.20) as $\varepsilon \to 0$ to describe proper shocks is more difficult and falls out of the scope of this paper. In Figure 12 we present a few types of shocks, showing that these discontinuous solutions can be approximated by ε-viscosity.

7.6. **A quasilinear ε-approximation.** Such an example is treated in Appendix F.

8. Conclusions

We have discussed two main problems of higher-order viscosity approximations of nonlinear degenerate odd-order partial differential equations (PDEs). As is known from classic theory, unlike parabolic or elliptic (even-order) PDEs, such odd-order equations do not exhibit internal regularity and therefore may admit essentially discontinuous solutions as the intrinsic property of the models. The principal question is how to distinguish the so-called *entropy* shock waves from the non-entropy shocks that are collapsed and evolve to smoother *rarefaction* waves.

Nowadays, there exists fully developed theory of the first-order PDEs called the *conservation laws* with the classic representative, such as the *Euler equation* originated from gas-dynamics

$$u_t + uu_x = 0. \tag{8.1}$$

Its discontinuous shocks are well known for more than a century, and a complete theory was created in the 1950s. One crucial conclusion is that the entropy solutions are those, which can be obtained in the limit $\varepsilon \to 0^+$ of smooth analytic solutions of the uniformly parabolic *Burgers equation*

$$u_t + uu_x = \varepsilon u_{xx}. \tag{8.2}$$

First problem. We discussed the possibility of higher-order approximations of entropy solutions via the analytic semigroup generated by the *extended Burgers equation*

$$u_t + uu_x = \varepsilon(-1)^{m+1} D_x^{2m} u \quad \text{with } \; m \geq 2. \tag{8.3}$$

Despite the violation of order-preserving, comparison and discontinuity of total variation (characterized by the *variation deficiency*, Section 4), our conclusion, though not being completely rigorously proved, is positive: such an approximation makes sense. We have showed this using both the ODE (the G-admissibility of shocks in Gel'fand's sense, Section 2) and sometimes the PDE approximations (Section 5). In particular, we have proved that the non-entropy shocks $S_+(x)$ (actually evolving to rarefaction waves) cannot be obtained via any parabolic approximations (Section 5).

Second Problem. Higher-order parabolic approximations begin to play a key role for third-order nonlinear PDEs such as

$$u_t = (uu_x)_{xx} \equiv \tfrac{1}{2}(u^2)_{xxx}, \tag{8.4}$$

which are associated with a number of important applications in nonlinear dispersion theory. From its fully divergent version it follows that both shocks

$$S_\pm(x) = \mp \operatorname{sign} x$$

are indeed week solutions since on both, $S_\pm^2(x) \equiv 1$, so that the PDE admits a standard multiplication by a test function and integration by parts. There is no a concept of entropy solutions for equation (8.4), so we have used approximation approach to reveal entropy (proper) and non-entropy shocks. The natural approximation of (8.4) leads to the fourth-order parabolic PDE

$$u_t - (uu_x)_{xx} = -\varepsilon u_{xxxx}. \tag{8.5}$$

Again, using various concepts of approximation, we have shown that $S_- = -\operatorname{sign} x$ is the proper stationary entropy shock, while $S_+ = \operatorname{sign} x$ is not (Section 7). Numerically, we constructed smooth similarity solutions of (8.4) describing both the finite-time formation of the proper shock and the collapse of the non-proper rarefaction wave.

General results and conclusions of this paper make it possible to justify that approximation (viscosity-like) techniques, which are well-known to be efficient in parabolic and Hamilton-Jacobi theory, can be also applied to higher odd-order nonlinear PDEs. The mathematics of such entropy approximations then becomes much more difficult than in the first-order theory and leads to a number of open problems that are indicated throughout the paper.

Acknowledgement. The author thank K. Khanin for posing the question on fourth-order approximation of conservation laws, and P. Plotnikov for useful discussions. The author thanks S. Manuilovich for drawing his attention to earlier papers on gas dynamics, and J. Williams for helping to perform numerical calculations of non-monotone VSP's.

References

[1] A. Bressan, Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy Problem, Oxford Univ. Press, Oxford, 2000.

[2] C. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer-Verlag, Berlin, 1999.

[3] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1983.

[4] P. Rosenau, *Extending hydrodynamics via the regularization of the Chapman-Enskog expansions*, Phys. Lett. A, **40** (1989), 7193–7196.

[5] M. Arora and P. Roe, *On postshock oscillations due to shock capturing schemes in unsteady flows*, J. Comput. Physics, **130** (1997), 25–40.

[6] P.G. LeFloch, J.M. Mercier, and C. Rohde, *Fully discrete, entropy conservative schemes of arbitrary*, SIAM J. Numer. Anal., **40** (2002), 1968–1992.

[7] Y. Striba and R. Donat, *A numerical study of postshock oscillations in slowly moving shock waves*, Comp. Math. Appl., **46** (2003), 719–739.

[8] Ya.B. Zel'dovich and Yu.P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Vols. I and II, Academic Press, New York, 1966, 1967.

[9] G. Sivashinsky, *Nonlinear analysis of hydrodynamic instability in laminar flames*, Acta Astronautica, **4** (1977), 1117–1206.

[10] C. Conley, Isolated Invariant Sets and the Morse Index, Conf. Bd. Math. Sci., No. 38, AMS, Providence, RI, 1978.

[11] S. Engelberg, *The stability of the viscosity shoch profiles of the Burgers' equation with a fourth order viscosity*, Comm. Partial Differ. Equat., **21** (1996), 889–922.

[12] S. Engelberg, *An analytical proof of the linear stability of the viscosity shoch profile of the Burgers equation with fourth order viscosity*, SIAM J. Math. Anal., **30** (1999), 927–936.

[13] N. Koppel and L. Howard, *Bifurcations and trajectories joining critical points*, Adv. in Math., **18** (1975), 306–358.

[14] C.K. McCord, *Uniqueness of connecting orbits in the equation* $Y^{(3)} = Y^2 - 1$, J. Math. Anal. Appl., **114** (1986), 584–592.

[15] D. Michelson, *Stability of the Bunsen flame profiles in the Kuramoto-Sivashinsky equation*, SIAM J. Math. Anal., **27** (1996), 765–781.

[16] F. Murat, *Compacité por compensation*, *Ann. Scuola Norm. Sup. Pisa Sci. Math.,* **5** (1978), 489–507; **8** (1981), 69–102.

[17] M.S. Mock, *On fourth-order dissipation and single conservation laws*, Comm. Pure Appl. Math., **29** (1976), 383–388.

[18] E. Tadmor, *Burgers' equations with vanishing hyper-viscosity*, Comm. Math. Sci., **2** (2004), 317–324.

[19] C. Conley and J.A. Smoller, *Shoch waves as limits of progressive wave solutions of higher order equations*, Comm. Pure Appl. Math., **24** (1971), 459–472.

[20] P.D. Lax and C.D Livermore, *The small dispertion limit for the KdV equation I - III*, Comm. Pure Appl. Math., **36** (1983), 253–290, 571–594, 809–829.

[21] P.D. Lax, *The zero dispersion limit, a deterministic analogue of turbulence*, Comm. Pure Appl. Math., **44** (1991), 1047–1056.

[22] M.E. Schonbek, *Convergence of solutions to nonlinear dispersive equations*, Comm. Part. Differ. Equat., **7** (1982), 959–1000.

[23] P.G. LeFloch and R. Natalini, *Conservation laws with vanishing nonlinearbdiffusion and dispertion*, Nonl. Anal., **36** (1999), 213–130.

[24] C.I. Kondo and P.G. Lefloch, *Zero diffusion-dispersive limits for scalar conservation laws*, SIAM J. Math. Anal., **33** (2002), 1320–1329.

[25] D. Amadori, P. Baiti, P.G. LeFloch, and B. Piccoli, *Nonclassical shocks and the Cauchy problem for nonconvex conservation laws*, J. Differ. Equat., **151** (1999), 345–372.

[26] B.T. Hayes and P.G. LeFloch, *Nonclassical shocks and kinetic relations: scalar conservation laws*, Arch. Rat. Mech. Anal., **139** (1997), 1–56.

[27] B. Casis, *Compensated compactness applied to a singular perturbed fourth order PDE*, Comm. Partial Differ. Equat., **14** (1989), 619–634.

[28] H. Zhao, C. Zhu, and Z. Yu, *Existence and convergence of solutions to a singularly perturbed higher order partially differential equations*, Nonl. Anal., TMA, **24** (1995), 1435–1455.

[29] Yi.A. Li and P.J. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equations*, J. Differ. Equat., **162** (2000), 27–63.

[30] A. Constantin and J. Escher, *Well-posedness, global existence, and blow-up phenomena for a periodic quasi-linear hyperbolic equation*, Commun. Pure Appl. Math., **51** (1998), 475–504.

[31] A. Constantin and L. Molinet, *Orbital stability of solitary waves for a shallow water equation*, Physica D, **157** (2001), 75–89.

[32] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Commun. Pure Appl. Math., **53** (2000), 1411–1433.

[33] O.A. Oleinik, *Discontinuous solutions of non-linear differential equations*, Uspehi Mat. Nauk., **12** (1957), 3–73; Amer. Math. Soc. Transl. (2), **26** (1963), 95–172.

[34] S.D. Eidelman, Parabolic Systems, North-Holland Publ. Comp., Amsterdam/London, 1969.

[35] A. Friedman, Partial Differential Equations, Robert E. Krieger Publ. Comp., Malabar, 1983.

[36] L.A. Peletier and W.C. Troy, Spatial Patterns. Higher Order Models in Physics and Mechanics, Birkhäuser, Boston/Berlin, 2001.

[37] V.A. Galaktionov, Geometric Sturmian Theory of Nonlinear Parabolic Equations and Applications, Chapman & Hall/CRC, Boca Raton, Florida, 2004.

[38] V.A. Galaktionov and S.R. Svirshchevskii, Exact Solution and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics, Chapman & Hall/CRC, Boca Raton, Florida, 2007.

[39] P. Rosenau and J.M. Hyman, *Compactons: solitons with finite wavelength*, Phys. Rev. Lett., **70** (1993), 564–567.

[40] S.N. Kruzhkov, *First-order quasilinear equations in several independent variables*, Math. USSR Sbornik, **10** (1970), 217–243.

[41] O.A. Oleinik, *Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation*, Uspehi Mat. Nauk., **14** (1959), 165–170; Amer. Math. Soc. Transl. (2), **33** (1963), 285–290.

[42] E. Hopf, *The partial differential equaiton $u_t + uu_x = \mu u_{xx}$*, Comm. Pure Appl. Math., **3** (1950), 201–230.

[43] I.M. Burgers, *A mathematical model illustrating the theory of turbulence*, Adv. in Mech., **1** (1948), 171–199.

[44] S.N. Kruzhkov, *Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications,* Math. Notes, **6** (1969), 517–523.

[45] P.D. Lax, *Schock waves and entropy*, In: Contr. Nonl. Funct. Anal., E.A. Zarantonello, Ed., Acad. Press, New York, 1971.

[46] Ph. Bénilan and S.N. Kruzkov, *Conservation laws with continuous flux functions*, NoDEA, **3** (1996), 395–419.

[47] T. Kato, *Shrödinger operators with singular potentials*, Israel J. Math., **13** (1972), 135–148.

[48] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. pures appl., **62** (1983), 73–97.

[49] I.M. Gel'fand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. (2), **29** (1963), 295–381.

[50] W. Baoxiang, *The Cauchy problem for critical and subcritical semilinear parabolic equations in L^r* (I), Nonliner Anal., TMA, **48** (2002), 747–764.

[51] S. Cui, *Local and global existence of solutions to semilinear parabolic initial value problems*, Nonlinear Anal., TMA **43** (2001), 293–323.

[52] Yu.V. Egorov, V.A. Galaktionov, V.A. Kondratiev, and S.I. Pohozaev, *Asymptotic behaviour of global solutions to higher-order semilinear parabolic equations in the supercritical range*, Adv. Differ. Equat., **9** (2004), 1009–1038.

[53] V.A. Galaktionov and S.I. Pohozaev, *Existence and blow-up for higher-order semilinear parabolic equations: majorizing order-preserving operators*, Indiana Univ. Math. J., **51** (2002), 1321–1338.

[54] V.A. Galaktionov and J.F. Williams, *On very singular similarity solutions of a higher-order semilinear parabolic equation*, Nonlinearity, **17** (2004), 1075–1099.

[55] E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, Inc., New York/London, 1955.

[56] B. Pethame and M. Westdickenberg, *Total oscillation diminishing property for scalar conservation laws*, Numer. Anal., **100** (2005), 331–349.

[57] M.E. Taylor, Partial Differential Equations III. Nonlinear Equations, Springer, New York, 1996.

[58] V.A. Galaktionov and J.L. Vazquez, *Continuation of blow-up solutions of nonlinear heat equations in several space dimensions,* Comm. Pure Appl. Math., **50** (1997), 1–68.

[59] O.A. Oleinik, A.S. Kalashnikov, and Chzhou Yui-Lin', *The Cauchy problem and boundary-value problems for equations of unsteady filtration type*, Izv. Akad. Nauk SSSR, Ser. Mat., **22**, No. 5 (1958), 667–704.

[60] A.M. Il'in, Matching of Asymptotic Expansions of Solutions of Boundary Value Problems, Amer. Math. Soc., Providence, R.I., 1992.

[61] G.G. Elenin, S.P. Kurdyumov, and A.A. Samarskii, *Non-stationary dissipative structures in a nonlinear heat-conducting medium*, USSR Comput. Math. Math. Phys., **23** (1983), 80–86.

[62] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, and A.P. Mikhailov, Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter & Co., Berlin, 1995.

[63] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel/Berlin, 1995.

[64] M.S. Birman and M.Z. Solomjak, Spectral Theory of Self-Adjoint Operators in Hilbert Space, D. Reidel, Dordrecht/Tokyo, 1987.

[65] V.A. Galaktionov, *On a spectrum of blow-up patterns for a higher-order semilinear parabolic equations*, Proc. Royal Soc. London A, **457** (2001), 1–21.

[66] P. Rosenau and D. Levy, *Compactons in a class of nonlinearly quintic equations*, Phys. Lett. A, **252** (1999), 297–306.
[67] R. DiPerna, *Measured-valued solutions to conservation laws*, Arch. Ration. Mech. Anal., **88** (1985), 223–270.
[68] L. Tartar, *Compensated compactness and applications to PDEs*, Research Notes in Math., Vol. **39**, Pitman Publ. Inc., New York, 1979.

Appendix A: Proof of Proposition 3.1

Proof. Step 1: L^∞-estimate on $[0,T]$. Consider the fundamental solution of the linear operator $\partial/\partial t + \varepsilon(-1)^m D_x^{2m}$,

$$b_\varepsilon(x,t) = (\varepsilon t)^{-\frac{1}{2m}} F\big(x/(\varepsilon t)^{\frac{1}{2m}}\big), \tag{A.1}$$

with the exponentially decaying rescaled kernel F; see [34] and applications to global existence in [52]. Writing down $uu_x = \frac{1}{2}(u^2)_x$ in the equivalent integral equation

$$u(t) = b_\varepsilon(t) * u_{0\varepsilon} - \tfrac{1}{2}\textstyle\int_0^t b_\varepsilon(t-s) * (u^2)_x(s)\,\mathrm{d}s, \tag{A.2}$$

using the Hölder inequality in the first term and integrating by parts in the last one yields

$$|u(t)| \le \sup|u_{0\varepsilon}|\ \|F\|_1 + \tfrac{1}{2}\textstyle\int_0^t \sup_x |b_{\varepsilon x}(t-s)|\,\|u(s)\|_2^2\,\mathrm{d}s \le C\big(1 + T^{\frac{m-1}{m}}\big),$$

where we have used the estimate

$$|b_{\varepsilon x}(x,t)| = (\varepsilon t)^{-\frac{1}{m}}\big|F'\big(x/(\varepsilon t)^{\frac{1}{2m}}\big)\big| \le C t^{-\frac{1}{m}}.$$

Step 2: uniform L^∞-estimate. This leads to a more delicate scaling analysis, and it seems that such an estimate cannot be obtained from the integral equation (A.2) just by embedding and interpolation inequalities or weighted Gronwall's type techniques. We use a modification of the rescaling technique in [54, Prop. 2.1], assuming for contradiction that there exist sequences $\{t_k\} \to \infty$, $\{x_k\} \subset \mathbf{R}$, and $\{C_k\} \to +\infty$ such that

$$\sup_{\mathbf{R}^N \times [0,t_k]} u(x,t) = u(x_k,t_k) = C_k. \tag{A.3}$$

Then we perform the scaling

$$u_k(x,t) = u(x_k + x, t_k + t) = C_k v_k(y,s), \quad x = a_k y, \quad t = a_k^{2m} s. \tag{A.4}$$

where $\{a_k\}$ is such that the L^2 norm is preserved, i.e.,

$$\|u_k\|_2 = \|v_k\|_2 \quad \Longrightarrow \quad a_k = C_k^{-2}. \tag{A.5}$$

Substituting (A.4) into equation (1.4) yields that v_k satisfies

$$(v_k)_s = -\varepsilon(-D_y)^m v_k - \delta_k v_k (v_k)_y \quad \text{in} \quad \mathbf{R} \times \mathbf{R}_+, \quad \text{where} \tag{A.6}$$

$$\delta_k = a_k^{2m-1} C_k = C_k^{3-4m} \to 0 \quad \text{as} \ \ k \to \infty. \tag{A.7}$$

Fixing s_0 large enough and setting $w_k(s) = v_k(s - s_0)$, we have that

$$|w_k(s)| \le 1 \quad \text{on} \ \ (0,s_0), \quad \|w_k\|_2 \le C, \tag{A.8}$$

are bounded classical solutions of the uniformly parabolic equations (A.6), so that, by parabolic regularity theory (see e.g., [34, 35]), we may assume that $w_k(s) \to w(s)$ as $k \to \infty$ uniformly on compact subsets, where w solves

$$w_s = -\varepsilon(-D_y)^m w \quad \text{for} \ \ s > 0, \quad w(0) = w_0, \tag{A.9}$$

with $\|w_0\|_\infty \le 1$ and $\|w_0\|_2 \le C$. By the Hölder inequality it follows that

$$\|w(s_0)\|_\infty \le (\varepsilon s_0)^{-\frac{1}{4m}} \|F\|_2 \|w_0\|_2 \ll 1,$$

if s_0 is large enough. Hence, the same holds for $\|\bar v_k(s_0)\|_\infty$ for $k \gg 1$ and there occurs a contradiction with the assumption $\|w_k(s_0)\|_\infty = 1$. $\square$

Appendix B: Proof of Proposition 4.2

Proof. (i) It follows from the convolution $u(t) = b_\varepsilon(t) * u_0$ (see (A.1)) that

$$\begin{aligned} |u(t)|_{TV} &= \int |u_x(x,t)| \, \mathrm{d}x \\ &\le (\varepsilon t)^{-\frac{1}{2m}} \int\int \left|F\big(z/(\varepsilon t)^{\frac{1}{2m}}\big)\right| |u_0'(x-z)| \, \mathrm{d}x \, \mathrm{d}z \le D_* |u_0|_{TV}. \end{aligned}$$

(ii) To show that the estimate is sharp, we take the step-like initial data

$$u_0(x) = \begin{cases} 1 & \text{for} \quad x < 0; \\ 0 & \text{for} \quad x \ge 0, \end{cases}$$

so that $|u_0|_{TV} = 1$. Then the solution

$$u(x,t) = \int_{x/t^{\frac{1}{2m}}}^\infty F(z) \, \mathrm{d}z$$

satisfies (4.5) of the form

$$|u(t)|_{TV} = D_* \equiv D_* |u_0|_{TV} \quad \text{for all} \quad t > 0,$$

i.e., the equality sign is achieved. $\square$

Appendix C: Interpolation inequalities do not guarantee the sign

Consider the first integral given in (4.18) with $\chi \equiv 1$,

$$P_2 \equiv -\int E''(u_{xx})^2 \, \mathrm{d}x + \tfrac{1}{3} \int E''''(u_x)^4 \, \mathrm{d}x \equiv -P_{21} + \tfrac{1}{3} P_{22}, \tag{C.1}$$

assuming that sufficiently smooth solutions $u = u_\varepsilon(x,t)$ have fast (exponential) decay as $x \to \infty$. Let us estimate the second positive term via a simple integration by parts

$$P_{22} = \int E''''(u_x)^3 u_x = -\int u E^{(5)}(u_x)^4 - 3\int u E''''(u_x)^2 u_{xx},$$

and using the Hölder inequality (recall that $E(u)$ is convex)

$$\begin{aligned} \int (E'''' + uE^{(5)})(u_x)^4 &\le 3 \int (\sqrt{E''}|u_{xx}|) \big[|uE''''|(u_x)^2/\sqrt{E''}\,\big] \\ &\le 3 \big[\int E''(u_{xx})^2\big]^{\frac{1}{2}} \big[\int (uE'''')^2 (u_x)^4/E''\big]^{\frac{1}{2}}. \end{aligned} \tag{C.2}$$

In order to derive a suitable comparison of the two terms on the right-hand side of (C.1), we first impose the following conditions on functions $E'(u)$:

$$E''''(u) + uE^{(5)}(u) \ge C_1 E''''(u), \quad \big[uE''''(u)\big]^2/E''(u) \le C_2 E''''(u), \quad u \in \mathbf{R}, \tag{C.3}$$

where C_1 and C_2 are some positive constants. Then (C.2) implies

$$\int E''''(u_x)^4 \le C_3 \int E''(u_{xx})^2, \quad \text{with} \;\; C_3 = \tfrac{9C_2}{C_1^2}, \tag{C.4}$$

and hence by (C.1)

$$P_2 \le \big(\tfrac{3C_2}{C_1^2} - 1\big) \int E''(u_{xx})^2 \le 0 \quad \text{if} \;\; C_1^2 \ge 3C_2. \tag{C.5}$$

The second condition in (C.3) assumes that $E''''(u) \ge 0$ in $\mathbf{R}$, which is not true for $E(u) \approx \operatorname{sign} u$. Replacing E'''' by $|E''''|$ on the right-hand sides of (C.3) (or other optimizations) does not extend the resulting estimate like (C.5) to the necessary sufficiently wide class of functions $E(u)$. For the typical power functions

$$E'(u) = |u|^{2k} u, \quad \text{with a } \; k > 1, \tag{C.6}$$

(C.1) reads

$$P_2 = -(2k+1) \int |u|^{2k} (u_{xx})^2 + \tfrac{2}{3}\,(2k-1)k(2k+1) \int |u|^{2k-2}(u_x)^4. \tag{C.7}$$

Integrating by parts and using the Holder inequality as in (C.2) yields

$$\int |u|^{2k-2}(u_x)^4 \le \tfrac{9}{(2k-1)^2} \int |u|^{2k}(u_{xx})^2,$$

and we arrive at the following estimate (cf. (C.5)):

$$P_2 \le C_* \int |u|^{2k}(u_{xx})^2, \tag{C.8}$$

where $C_* = \frac{(1+4k)(2k+1)}{2k-1} > 0$ for all $k > \frac{1}{2}$. Thus, we cannot get the necessary sign $P_2 \le 0$ on particular functions (C.6) for large k. In fact, this shows that the Nash–Moser technique for second-order parabolic equations (see [57, p. 344]) do not apply to the fourth-order operators with $m = 2$ and to other higher-order ones. Indeed, the iterative nature of the technique with the eventual limit $k \to \infty$ assumes certain order-preserving properties via the Maximum Principle (available for $m = 1$ only), so that the inequality $C_* \le 0$ for all large k cannot be achieved in principle via optimization of constants in the interpolation and embedding inequalities.

Appendix D: Asymptotic stability of the rarefaction profile

We now consider the second asymptotic problem (not of less importance) of the stability of the rarefaction wave occurring for initial data $U_0(y) = S_+(y)$ in the Cauchy problem (5.6). It is convenient to introduce new self-similar rescaled variables

$$U = (1+\tau)^{-\frac{2m-1}{2m}} \theta, \quad \xi = y/(1+\tau)^{\frac{1}{2m}}, \quad s = \ln(1+\tau) : \mathbf{R}_+ \to \mathbf{R}_+. \tag{D.1}$$

Then the rescaled solution $\theta = \theta(\xi, s)$ solves the autonomous equation

$$\theta_s = (-1)^{m+1} D_\xi^{2m} \theta - \theta \theta_\xi + \mu\, \theta_\xi \xi + (2m-1)\mu\theta, \quad \mu = \tfrac{1}{2m}, \tag{D.2}$$

with the same initial data. Equation (D.2) has the explicit stationary solution

$$\bar\theta(\xi) = \xi \quad \text{in } \; \mathbf{R}. \tag{D.3}$$

Obviously, (D.1) shows that it is precisely the solution (5.7), so that we refer to (D.3) as the rarefaction profile (RP) defined in $\mathbf{R}$. We prove that the RP is asymptotically stable. The linearization $\theta = \xi + Y$ yields the perturbed equation

$$Y_s = \mathbf{A} Y - Y Y_\xi \quad \text{with } \; \mathbf{A} = (-1)^{m+1} D_\xi^{2m} - (2m-1)\mu\, \xi \tfrac{\mathrm{d}}{\mathrm{d}\xi} - \mu\, I. \tag{D.4}$$

Setting $\xi = c\eta$ with $c^{2m} = \frac{1}{2m-1}$ gives

$$\mathbf{A} = (2m-1)\mathbf{B}^* - \tfrac{1}{2m} I.$$

The corresponding linear elliptic operator

$$\mathbf{B}^* = -(-\Delta_\eta)^m - \mu\, \eta \cdot \nabla_\eta \quad \text{in} \quad \mathbf{R}^N$$

is known to have discrete spectrum $\sigma(\mathbf{B}^*) = \{-\frac{l}{2m},\, l = 0,1,2,...\}$ [52]. Note that the second-order case $m = 1$ is classic and exceptional, where

$$\mathbf{B}^* \equiv \tfrac{1}{\rho^*}\, \nabla \cdot (\rho^* \nabla)$$

with the weight $\rho^*(y) = \mathrm{e}^{-|y|^2/4}$, is self-adjoint in $L^2_{\rho^*}(\mathbf{R}^N)$ with the domain $\mathcal{D}(\mathbf{B}^*) = H^2_{\rho^*}(\mathbf{R}^N)$, and a discrete spectrum. The eigenfunctions are Hermite polynomials that form an orthonormal basis in $L^2_{\rho^*}(\mathbf{R}^N)$, and classical Hilbert-Schmidt theory applies [64].

We describe the spectral properties of the linearized operator in (D.4) which is not self-adjoint for $m > 1$. We consider $\mathbf{A}$ given in (D.4) in the weighted space $L^2_{\rho^*}(\mathbf{R}_+)$ of odd functions with the exponentially decaying weight function

$$\rho^*(y) = \mathrm{e}^{-a|y|^\beta} > 0, \quad \beta = \tfrac{2m}{2m-1}, \tag{D.5}$$

where $a > 0$ is a sufficiently small constant. The following holds [52].

Lemma D.1. $\mathbf{A} : H^{2m}_{\rho^*}(\mathbf{R}_+) \to L^2_{\rho^*}(\mathbf{R}_+)$ *is a bounded linear operator with the discrete spectrum*

$$\sigma(\mathbf{A}) = \{\lambda_l = -\tfrac{1+(2m-1)l}{2m},\;\; l = 1,3,5,...\}, \tag{D.6}$$

and the eigenfunction set $\{\psi_l(\xi)\}$ (*lth-order polynomials*) *is complete in* $L^2_{\rho^*}(\mathbf{R}_+)$.

As we have mentioned, for $m = 1$, these are well-known properties of the separable Hermite polynomials generated by a self-adjoint Sturm-Liouville problem [64]. In view of the principle of linearized stability [63, Ch. 9], we have that the RP is asymptotically stable in $L^2_{\rho^*}(\mathbf{R}_+)$ and moreover, since the real spectrum is uniformly bounded from the imaginary axis, we have the exponential convergence of the order $O(\mathrm{e}^{-s}) = O(\tau^{-1})$ as $\tau \to \infty$. Since the weight (D.5) is exponentially decaying at infinity, the stability conclusion is true for a wide class of initial data.

Thus, the rarefaction solution (5.7) exhibits the exponential asymptotic stability for parabolic approximations of any order. This explains once more why the non-entropy shocks of type S_+ cannot occur in the evolution, cf. Proposition 5.3. For the Cauchy problem (5.6) with bounded initial data $U_{0\varepsilon} \sim S_+$, the stable RP (D.3) also plays a role, but the convergence as $\varepsilon \to 0$ is again a hard asymptotic problem, which includes a delicate matching-type analysis.

Note that the linear operator $\mathbf{B}^*$ occurs in the study of blow-up solutions of a completely different reaction-diffusion equation

$$u_t = -(-\Delta)^m u + |u|^p \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R} \quad (p > 1);$$

see [65]. The analysis of its global solutions in the supercritical Fujita range $p > 1 + \frac{2m}{N}$ [52] is based on spectral properties of the adjoint operator

$$\mathbf{B} = -(-\Delta_\eta)^m + \tfrac{1}{2m}\, \eta \cdot \nabla_\eta + \tfrac{N}{2m}\, I.$$

Appendix E: On a formal connection for $m = 2$ by averaging method

A simple formal, but rather practical and sharp, approach to matching is as follows, where we use the idea of Elenin's averaging method [61]; see more details in [62, p. 201]. For instance, comparing profiles for $m = 2$ in Figure 8(b), we have that

$$f_-(y) \approx A\theta(\tfrac{y}{a}), \quad \text{with } A = \tfrac{43}{39} = 1.103..., \;\; a = \tfrac{8}{10} = 0.25. \tag{E.1}$$

Then the global evolution of $U(y,\tau)$ can be expressed as follows:

$$
\begin{aligned}
& U(y,\tau) \approx \psi(\tau)\theta(\zeta), \quad \zeta = \tfrac{x}{\varphi(\tau)}, \quad \text{where} \\
& \psi(\tau) \to 1, \;\; \varphi(\tau) \sim \tau^{\frac{1}{4}} \quad \text{as } \tau \to 0 \quad \text{(Region I)}; \\
& \psi(\tau) \to A, \;\; \varphi(\tau) \to a \quad \text{as } \tau \to +\infty \quad \text{(Region III)}.
\end{aligned}
\tag{E.2}
$$

To derive a dynamical system describing the evolution of $\{\varphi(\tau), \psi(\tau)\}$, we take two identities obtained from (5.6), $m=2$, via multiplying by U_{yy} and U_{yyyy} in $L^2(\mathbf{R})$,

$$
\begin{cases}
-\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} \int (U_y)^2 - \frac{1}{2} \int (U_y)^3 = \int (U_{yyy})^2, \\
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} \int (U_{yy})^2 + \frac{1}{2} \int U U_y U_{yyyy} = - \int (U_{yyyy})^2.
\end{cases}
\tag{E.3}
$$

Substituting into (E.3) the representation of $U(y,\tau)$ from (E.2) yields the following ODE system for $\{\varphi, \psi\}$:

$$
\begin{cases}
-\frac{a_1}{2} \left(\frac{\psi^2}{\varphi}\right)' + \frac{b_1}{2} \frac{\psi^3}{\varphi^2} = c_1 \frac{\psi^2}{\varphi^5}, \\
\frac{a_2}{2} \left(\frac{\psi^2}{\varphi^3}\right)' + b_2 \frac{\psi^3}{\varphi^4} = -c_2 \frac{\psi^2}{\varphi^7},
\end{cases}
\tag{E.4}
$$

where the positive constant coefficients are given by

$$
\begin{aligned}
& a_1 = \int (\theta')^2, \;\; b_1 = -\int (\theta')^3, \;\; c_1 = \int (\theta''')^2, \\
& a_2 = \int (\theta'')^2, \;\; b_2 = \int \theta\theta'\theta^{(4)} = \tfrac{1}{4} \int \zeta \theta (\theta')^2, \;\; c_2 = \int (\theta^{(4)})^2.
\end{aligned}
\tag{E.5}
$$

It is easy to reduce (E.4) to a standard dynamical system

$$
\begin{cases}
\varphi' = -\mu_1 \psi + \nu_1 \frac{1}{\varphi^3}, \\
\psi' = \mu_2 \frac{\psi^2}{\varphi} - \nu_2 \frac{\psi}{\varphi^4},
\end{cases}
\tag{E.6}
$$

with the following positive parameters:

$$
\mu_1 = \tfrac{1}{2}\left(\tfrac{b_1}{a_1} - \tfrac{2b_2}{a_2}\right), \;\; \nu_1 = \tfrac{c_2}{a_2} - \tfrac{c_1}{a_1}, \;\; \mu_2 = \tfrac{1}{4}\left(\tfrac{3b_1}{a_1} + \tfrac{2b_2}{a_2}\right), \;\; \nu_2 = \tfrac{1}{2}\left(\tfrac{3c_1}{a_1} - \tfrac{c_2}{a_2}\right). \tag{E.7}
$$

Then (E.6) has the necessary equilibrium point (a, A) given in (E.1) provided that

$$
\tfrac{\nu_1}{\mu_1} = \tfrac{\nu_2}{\mu_2} \quad \Longleftrightarrow \quad \tfrac{2b_1c_2}{a_1a_2} = \tfrac{c_1}{a_1}\left(\tfrac{3b_1}{a_1} - \tfrac{2b_2}{a_2}\right) > 0. \tag{E.8}
$$

In this case, (E.6) gives an approximate description of the evolution in the transitional Region II. We claim that the dynamical system (E.4) can be put in a rigorous approximate framework. The same construction applies to any $m \ge 2$.

Appendix F: On a quasilinear approximation

As a final example, we show that even quasilinear degenerate approximations of S_- can preserve the main features of parabolic regularization. Consider the following approximation of (7.1) via the p-Laplacian operator as in (5.10):

$$
u_t + (-1)^{m-1} D_x^{2m-2}(u u_x) = \varepsilon (-1)^{m+1} D_x^m (|D_x^m u|^{p-2} D_x^m), \quad p > 1, \tag{F.1}
$$

where

$$
u_\varepsilon(x,t) = U_\varepsilon(y,\tau), \quad y = x/\varepsilon^\alpha, \quad \tau = t/\varepsilon^{(2m-1)\alpha}, \quad \text{and} \quad \alpha = \tfrac{1}{1+m(p-2)}.
$$

Let $m=2$. Then the entropy VSP f_- satisfies the ODE

$$
f f' = |f''|^{p-2} f'', \quad \text{with} \quad f(\pm\infty) = \mp 1
$$

(one can see that the non-entropy VSP f_+ does not exist). For $y>0$, we have $f<0$, $f' \le 0$ and $f'' \ge 0$, and setting $-f' = R \ge 0$ yields

$$f'' \equiv RR_f = (-f)^{\frac{1}{p-1}} R^{\frac{1}{p-1}}.$$

If $p \in (1, \frac{3}{2}]$, integrating once yields that a solution satisfying $R(-1)=0$ does not exist, i.e., approximation (F.1) in not admissible. For $p > \frac{3}{2}$, from the equation

$$R = -f' = a_0 \big[1 - (-f)^{\frac{p}{p-1}}\big]^{\frac{p-1}{2p-3}}, \quad a_0 = \big(\tfrac{2p-3}{p}\big)^{\frac{p-1}{2p-3}}, \tag{F.2}$$

one obtains the unique VSP $f = f_-(y)$ from the quadrature

$$\textstyle\int_0^{-f} \big(1 - z^{\frac{p}{p-1}}\big)^{-\frac{p-1}{2p-3}}\, {\mathrm d}z = a_0 y, \quad y>0. \tag{F.3}$$

Hence, for $p \in (\frac{3}{2}, 2]$, $f_-(y)$ is strictly monotone decreasing in $\mathbf{R}$ and is a C^∞ function as in the linear case $p=2$. For $p>2$ it has finite regularity at the interface, where $f_-(y_0) = -1$ at

$$y_0 = \tfrac{p-1}{a_0 p}\, B\big(\tfrac{p-1}{2p-3}, \tfrac{p-1}{p}\big),$$

B being Euler's Beta function. Though for $p>2$ the VSP is strictly decreasing on $I_0 = (-y_0, y_0)$, the stability analysis and other related questions on such approximations become more involved. Indeed, linearization (6.1) leads to a singular ODE operator $\mathbf{N}_{2m}$ on I_0 in the equation (6.2). The functional setting becomes more complicated (the weight function ρ is expected to be unbounded at the singular end-points $y = \pm y_0$), and a delicate matching procedure extending the stability analysis beyond interval I_0 should be performed. Such quasilinear approximations are not well-posed (e.g., uniqueness of solutions is not well-understood in general), though keep some typical features of semilinear parabolic regularizations.

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