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SMALL PLANAR TRAVELING WAVES IN TWO DIMENSIONAL NETWORKS OF COUPLED OSCILLATORS

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ABSTRACT. The existence of several small planar traveling waves with arbitrary direction of propagation is shown for two-dimensional cubic networks of oscillators with linear nearest-neighbor coupling. The analysis is based on a spatial dynamics reformulation of the relevant advance-delay equation. Whereas important aspects of the analysis are discontinuous in the direction of travel, continuity of the traveling wave is shown with respect to the direction.

1. INTRODUCTION

In this short paper we analyze some aspects of traveling waves for the Hamiltonian dynamics of a 2d network of oscillators coupled via linear nearest-neighbor interaction. Our assumptions include the natural 2d analogons of the Klein-Gordon or Frenkel-Kontorova models. For various physical applications of these discrete systems see e.g. [16]. There is now an extensive literature on the existence and behavior of coherent modes in one-dimensional Hamiltonian lattice equations [3, 6, 7, 12, 10, 11, 15, 17]. Recently there has also been interest in the extension to higher dimension for 2d sine-Gordon models [4] and for some special directions in elastodynamics for 2d particle spring models [5]. For some numerical results in Schrödinger lattices see [8] or for a scalar 2d Fermi-Pasta-Ulam lattice see [1].

We will look for small traveling wave solutions, our analysis is relevant for a number of models including discrete Klein-Gordon as well as Frenkel-Kontorova models. Our main results are the existence of planar periodic and solitary waves. We study in particular the dependence of such solutions on their direction of propagation. In other discrete models – the analogon of scalar reaction-diffusion equations – a discontinuous dependence on the direction was discovered, see [2, 14]. Several properties like the spectrum of certain linearization of the traveling wave ansatz show similar discontinuities in our examples. However we manage to derive continuity with respect to the direction. The existence proof of the waves proceeds by the reduction to an effective 1d Hamiltonian, which accounts for the various coupling terms and then we use recent work on 1d Klein-Gordon lattice [12, 11]. This is based on a spatial dynamics formulation and a center manifold reduction, for which we carefully study the dependence on parameters to rule out higher singularities for the reduced dynamics.

We consider a two-dimensional network of coupled nonlinear oscillators, where the dynamics at each node are determined by

$$\ddot{x}_{ij} = \gamma(x_{i,j+1} + x_{i,j-1} + x_{i+1,j} + x_{i-1,j} - 4x_{ij}) - f(x_{ij}) \tag{1}$$

and $\gamma > 0$ is a coupling constant. The on-site potential energy V with $f = V'$ is an analytic function independent of position in the network with the property $V'(0) = 0$ and $V''(0) = 1$.

Key words and phrases. Lattice Dynamics, Infinite Dimensional Hamiltonian Dynamics, Normal Forms, Center Manifold Theory, Traveling Waves, Advance–Delay Equation.

We are interested in planar traveling wave solutions of the system (1). In particular, we use the ansatz

$$x_{ij}\left(\frac{\tilde{t}}{\tau}\right) = u\left(\frac{\tilde{t}}{\tau} - \begin{pmatrix} i \\ j \end{pmatrix}^T \cdot \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}\right). \tag{2}$$

Here the angle θ describes the direction of the planar wave and $1/\tau$ is its speed. After rescaling time $\tilde{t} = \tau t$ and setting $\xi = t - \binom{i}{j}^T \cdot \binom{\sin\theta}{\cos\theta}$ this transforms (1) into the advance-delay equation

$$\begin{aligned} u''(\xi) = \gamma\tau^2[u(\xi-\cos\theta) + u(\xi+\cos\theta) + \\ + u(\xi-\sin\theta) + u(\xi+\sin\theta) - 4u(\xi)] - \tau^2 f(u(\xi)). \end{aligned} \tag{3}$$

In order to solve this equation, we adopt methods of [11], where a corresponding problem in oscillator chains is analyzed. We will specify a setting, which allows us to rewrite (3) and interpret it as an evolution equation. These methods first require a detailed spectral analysis of the linear part of (3). Then we will find center manifolds of dimensions two and six. Finally, normal form theory is applied to discuss two different (γ, τ, θ) regimes. Namely the situation below the first bifurcation curve, when no bifurcation occurs and all small traveling waves on the center manifold are periodic, and the situation near the first bifurcation curve. The existence results in these two cases are formulated in theorems 7 and 8. In particular we prove the existence of traveling waves homoclinic to small periodic solutions.

2. Spatial dynamics formulation

We reformulate (3) as a dynamical system by interpreting the unbounded direction as time. Firstly, define the spaces

$$\begin{aligned} \mathbb{H} &= \mathbb{R}^2 \times C^0[-1;1] \\ \mathbb{D} &= \{(x,y,X)^T \in \mathbb{R}^2 \times C^1[-1;1] | X(0) = x\} \end{aligned}$$

and consider the equation

$$\partial_t U = L_{\gamma,\tau,\theta} U + M_\tau U, \tag{4}$$

for $U \in C^0(\mathbb{R},\mathbb{D}) \cap C^1(\mathbb{R},\mathbb{H})$ where

$$L_{\gamma,\tau,\theta} = \begin{pmatrix} 0 & 1 & 0 \\ -\tau^2(1+4\gamma) & 0 & \gamma\tau^2(\delta^{-\cos\theta} + \delta^{+\cos\theta} + \delta^{-\sin\theta} + \delta^{+\sin\theta}) \\ 0 & 0 & \partial_v \end{pmatrix}$$

and

$$M_\tau = -\tau^2 \begin{pmatrix} 0 \\ f(x) - x \\ 0 \end{pmatrix}$$

Observe that $L_{\gamma,\tau,\theta}$ maps $\mathbb{D}$ to $\mathbb{H}$ continuously. The operator $L_{\gamma,\tau,\theta} : \mathbb{H} \supset \mathbb{D} \to \mathbb{H}$ acting on $\mathbb{H}$ with domain $\mathbb{D}$ is closed and has a compact resolvent. We recall, that operators with these properties have a simple spectrum composed solely of isolated eigenvalues of finite multiplicities. Further, note that both $L_{\gamma,\tau,\theta}$ and M_τ anticommute with the reflection S defined by $S(x,y,X(v))^T = (x,-y,X(-v))^T$. This is the reversibility condition mentioned above.

We call $U = (x,y,X)^T \in C^0(\mathbb{R},\mathbb{D}) \cap C^1(\mathbb{R},\mathbb{H})$ a solution of (4) if $x \in C^2(\mathbb{R},\mathbb{R})$, $y = \dot{x}$ and $[X(.)](v) = x(.+v)$

3. Spectrum of the advance-delay operator

In order to determine the spectrum $\sigma(L_{\gamma,\tau,\theta})$ of $L_{\gamma,\tau,\theta}$ we have to solve the resolvent equation.

$$(\lambda \mathrm{id} - L_{\gamma,\tau,\theta})U = (f_0, f_1, F_2)^T \tag{5}$$

for $(f_0, f_1, F_2)^T \in \mathbb{H}$ and $U \in \mathbb{D}$. Because of $X(0) = x$ we can use the variation of constants formula to solve the third component of this system, which yields:

$$X(v) = e^{\lambda v}x - \int_0^v e^{\lambda(v-s)}F_2(s)\mathrm{d}s \tag{6}$$

Substituting in X and rearranging gives the first two components:

$$x = -[N(\lambda;\gamma,\tau,\theta)]^{-1}(\lambda f_0 + f_1 + \gamma\tau^2\tilde{f}_{\lambda,\theta}) \tag{7}$$

$$y = -[N(\lambda;\gamma,\tau,\theta)]^{-1}\left[(N(\lambda;\gamma,\tau,\theta) + \lambda^2)f_0 + \lambda f_1 + \lambda\gamma\tau^2\tilde{f}_{\lambda,\theta}\right] \tag{8}$$

where

$$N(\lambda;\gamma,\tau,\theta) = -\tau^2(1+4\gamma) - \lambda^2 + \gamma\tau^2(e^{-\lambda\cos\theta} + e^{+\lambda\cos\theta} + e^{-\lambda\sin\theta} + e^{+\lambda\sin\theta})$$

and

$$\tilde{f}_{\lambda,\theta} = \int_0^{\cos\theta} -e^{\lambda(\cos\theta - s)}F_2(s) + e^{-\lambda(\cos\theta-s)}F_2(-s)\mathrm{d}s + \int_0^{\sin\theta} -e^{\lambda(\sin\theta-s)}F_2(s) + e^{\lambda(\sin\theta-s)}F_2(-s)\mathrm{d}s$$

Then $U = (x, y, X)^T \in \mathbb{D}$ holds for all $(f_0, f_1, F) \in \mathbb{H}$, as long as $N(\lambda;\gamma,\tau,\theta) \neq 0$. So the spectrum of $L_{\gamma,\tau,\theta}$ consists entirely of the roots of $N(\lambda;\gamma,\tau,\theta)$. Note the following important properties:

From reversibility it follows, that $\sigma(L_{\gamma,\tau,\theta})$ is invariant under $\lambda \mapsto -\lambda$ and the fact that $L_{\gamma,\tau,\theta}$ is a real operator implies invariance under $\lambda \mapsto \overline{\lambda}$. It is thus sufficient to consider eigenvalues $\lambda = p + iq$ with nonnegative p and q.

By elementary trigonometric identities for $N(\lambda;\gamma,\tau,\theta)$ we find, that $\sigma(L_{\gamma,\tau,\theta}) = \sigma(L_{\gamma,\tau,-\theta}) = \sigma(L_{\gamma,\tau,-\theta+\frac{\pi}{2}})$, which allows us to restrict our attention to $0 \leq \theta \leq \frac{\pi}{4}$.

As in [11], continuity of N leads to the assertion, that for all parameter triples $(\gamma,\tau,\theta) \in \mathbb{R}_+^2 \times [0, \frac{\pi}{4}]$ there is a $p_0 > 0$, such that all $\lambda \in \sigma(L_{\gamma,\tau,\theta}) \setminus \sigma_0(L_{\gamma,\tau,\theta})$ satisfy $|\Re\lambda| \geq p_0$, where $\sigma_0(L_{\gamma,\tau,\theta})$ contains the purely imaginary eigenvalues of $L_{\gamma,\tau,\theta}$.

Moreover, by considering the equation $\Re N(\lambda;\gamma,\tau,\theta) = 0$ and applying properties of the hyperbolic cosine, we find, that for $\lambda = p + iq \in \sigma(L_{\gamma,\tau,\theta}) \setminus \sigma_0(L_{\gamma,\tau,\theta})$ the following inequality holds:

$$q \leq \tau + \sqrt{2(4\gamma\tau^2 + 8)}\cosh^2\left(\frac{p}{2}\right) \tag{9}$$

We can conclude from the Casorati-Weierstrass theorem, that $N(\lambda;\gamma,\tau,\theta)$ has countably many zeros, which proves together with the closedness of $L_{\gamma,\tau,\theta}$ and compactness of its resolvent, that $\sigma(L_{\gamma,\tau,\theta})$ consists of countably many eigenvalues with finite multiplicities. It can also be shown, that $\sigma(L_{\gamma,\tau,\theta})$ is not bisectorial.

Application of the center manifold theory of [19] requires checking certain properties of the purely imaginary spectrum of $\sigma_0(L_{\gamma,\tau,\theta})$. We are therefore interested in roots of

$$N(iq;\gamma,\tau,\theta) = 0 \tag{10}$$

where $q \in \mathbb{R}$ for single eigenvalues. We write $\frac{\mathrm{d}}{\mathrm{d}q} =\prime$ and require

$$N'(iq;\gamma,\tau,\theta) = 0 \tag{11}$$

for double eigenvalues and additionally

$$N''(iq;\gamma,\tau,\theta)=0 \tag{12}$$

for triple eigenvalues. For $\gamma\tau^2<1$ and any θ, the $\sigma_0(L_{\gamma,\tau,\theta})$ consists only of one pair of simple purely imaginary eigenvalues. To show this, we observe, that for $a=\gamma\tau^2$ (10) is equivalent to

$$\tau^2=\underbrace{q^2+2a[\cos(q\cdot\cos\theta)+\cos(q\cdot\sin\theta)-2]}_{=:f_{a,\theta}(q)}$$

Clearly, $f_{a,\theta}$ is an unbounded even function with $f_{a,\theta}(0)=0$ and $f'_{a,\theta}(q)>0$ for $a<1$ and $q>0$. In this case, the equation $\tau^2=f_{a,\theta}(q)$ has two solutions $\pm q_0$ for all $\tau>0$ and $\pm iq_0$ is a pair of simple eigenvalues, because τ^2 is a regular value of $f_{a,\theta}$.

We first give a description of the parameter values that lead to the existence of multiple imaginary eigenvalues:

Lemma 1. *Consider the set of points MEC (multiple imaginary eigenvalue curve) in parameter space $\Omega=\mathbb{R}^2_+\times[0,\frac{\pi}{4}]$, given by the parametrization $MEC:(q,\theta)\mapsto(\gamma_\theta(q),\tau_\theta(q),\theta)$, where*

$$\gamma_\theta(q) = \frac{q}{\tau_\theta^2}[\cos\theta\cdot\sin(q\cdot\cos\theta)+\sin\theta\cdot\sin(q\cdot\sin\theta)]^{-1} \tag{13}$$

$$\tau_\theta^2(q) = q^2-4q\cdot\frac{\sin^2\left(\frac{q}{2}\cdot\cos\theta\right)+\sin^2\left(\frac{q}{2}\cdot\sin\theta\right)}{\cos\theta\cdot\sin(q\cdot\cos\theta)+\sin\theta\cdot\sin(q\cdot\sin\theta)}. \tag{14}$$

We define the domain D of MEC by $D=\bigcup_{\theta\in[0,\frac{\pi}{4}]}D_\theta\times\{\theta\}$, where for each θ we denote $D_\theta=\{q|\gamma_\theta(q)>0 \text{ and } \tau^2_\theta(q)>0\}$. Then the following two statements hold:

1. *For parameter values $(\gamma,\tau,\theta)\in MEC$, the operator $L_{\gamma,\tau,\theta}$ has at least one pair of purely imaginary double eigenvalues.*
2. *Let $0\le\theta\le\frac{\pi}{4}$ and $q_*\in D_\theta$. Then: $\tau'_\theta(q_*)=\gamma'_\theta(q_*)=0$ if and only if $L_{\gamma_\theta(q_*),\tau_\theta(q_*),\theta}$ has a pair of purely imaginary triple eigenvalues.*

Proof. The assertions can be checked directly by substituting (13),(14) into (10),(11) and (12). □

For a given θ let Δ_θ denote the set of parameters in the (γ,τ)–quadrant under consideration where there exists exactly one pair of purely imaginary eigenvalues. This set is never empty, since $\{\gamma\tau^2<1\}\subset\Delta_\theta$ for all θ as shown above. Let us call the set $\Gamma_\theta:=\partial\Delta_0\setminus(\mathbb{R}^+_0\times\{0\}\cup\{0\}\times\mathbb{R}^+_0)$ the first bifurcation curve. Then on this curve the directional dependence does not introduce too many complications:

Lemma 2. *Let $0\le\theta_0\le\frac{\pi}{4}$ be given. Except for isolated exceptional points, the spectrum for parameter values $(\gamma,\tau)\in\Gamma_{\theta_0}$ can be described as follows:*

1. *L_{γ,τ,θ_0} has exactly one pair of simple eigenvalues and exactly one pair of double eigenvalues on the imaginary axis.*
2. *There is an $\epsilon>0$ such that either for all $\gamma<\sigma<\gamma+\epsilon$ or for all $\gamma-\epsilon<\sigma<\gamma$ the operator $L_{\sigma,\tau,\theta}$ has exactly three pairs of simple imaginary eigenvalues.*

A proof can be found in the appendix. A full analysis of the imaginary spectrum is elementary but tedious. In the following we will state the main properties and indicate their validity. Outlines of the proofs can be found in the appendix. It turns out that we have to distinguish two cases for $0\le\theta\le\frac{\pi}{4}$:

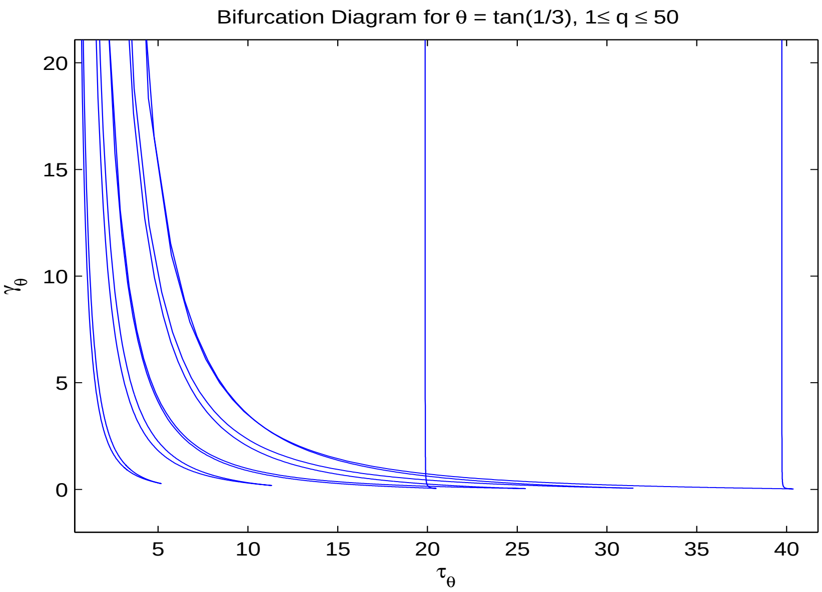

FIGURE 1. Bifurcation diagram in the case of rational $\tan\theta = \frac{1}{3}$ and $\frac{2\pi}{\alpha} \approx 19.1$. The bifurcation curves are given by (13),(14).

3.1. **Spectrum for rational directions.** Assume $\tan\theta = \frac{m}{n} \in \mathbb{Q}$, where $\gcd(m,n) = 1$ and $m \leq n$, i.e. there exists $\alpha > 0$ such that $\sin\theta = \alpha m$ and $\cos\theta = \alpha n$. For $r \in \mathbb{N}$ we denote $b_r^0 = r \cdot \frac{2\pi}{\alpha}$ and $J_r = [b_r^0, b_{r+1}^0]$. The function

$$G_\theta : q \mapsto \cos\theta \sin(q \cdot \cos\theta) + \sin\theta \sin(q \cdot \sin\theta) \tag{15}$$

has exactly $2n-1$ zeros $b_r^1 < \cdots < b_r^{2n-1}$ in the interior of J_r and the boundaries are zeros as well. The derivative of G_θ at each root is non-zero. We denote the intervals between those zeros by: $J_r^i = (b_r^i, b_r^{i+1})$, with $0 \leq i \leq 2n-1$ and $b_r^{2n} = b_{r+1}^0$. There is exactly one extremal value of G_θ in each J_r^i.
The function τ_θ^2 is continuously extendable by setting $\tau_\theta^2(b_r^0) = \left(b_r^0\right)^2$. Using the quadratic growth of the first term in τ_θ^2 and the periodicity of the fraction in the second term we can conclude from the behavior of $\tau_\theta^2(q)$ as q approaches the boundaries of one of the intervals J_r^{2i} $(1 \leq i \leq n-1)$ (namely: $\lim_{q \searrow b_r^{2i}} \tau_\theta^2(q) = \lim_{q \nearrow b_r^{2i+1}} \tau_\theta^2(q) = -\infty$) that there exists a $K > 0$, such that for all of these intervals to the right of K there exist unique (and maximal) open subintervals $(\tilde{b}_r^{2i}, \tilde{b}_r^{2i+1})$ in which τ_θ^2 is strictly positive and $\tau_\theta^2(\tilde{b}_r^{2i}) = \tau_\theta^2(\tilde{b}_r^{2i+1}) = 0$.
In each of these subintervals, τ_θ^2 contains a maximum. We call the position in J_r^{2i}, where this maximum is attained $b_{r,2i}^*$. Then it is easy to see that the values of these maxima tend to infinity as $b_{r,2i}^*$ goes to infinity. Furthermore the operator $L_{\gamma_\theta(b_{r,2i}^*),\tau_\theta(b_{r,2i}^*),\theta}$ has a pair of purely imaginary triple eigenvalues at $\pm ib_{r,2i}^*$.
We want to understand more closely the connection between extremal values of the functions γ_θ and τ_θ and the existence of purely imaginary multiple eigenvalues of $L_{\gamma,\tau,\theta}$. The following function turns out to be useful in this attempt:

$$h_{\tau,\theta} = 2\left[\sin^2\left(\frac{q}{2} \cdot \cos\theta\right) + \sin^2\left(\frac{q}{2} \cdot \sin\theta\right)\right] \cdot \left(q^2 - \tau^2\right)^{-1} \tag{16}$$

for $q > \tau$. Observe that, independently of whether or not $\tan\theta$ is rational or irrational, $\pm iq$ is an eigenvalue of $L_{\gamma,\tau,\theta}$ if and only if $h_{\tau,\theta}(q) = (2\gamma\tau^2)^{-1}$. For

$(q,\theta) \in D$, the pair $\pm iq$ are purely imaginary eigenvalues with multiplicity at least $k+1$, if and only if $h_{\tau,\theta}(q) = (2\gamma\tau^2)^{-1}$ and $h^{(i)}_{\tau,\theta}(q) = 0$ for $1 \leq i \leq k$. In the case of rational $\tan\theta$ the zeros of $h_{\tau,\theta}$ have the form $k \cdot \frac{2\pi}{\alpha}$ for $k \in \mathbb{N}$ such that $k \cdot \frac{2\pi}{\alpha} > \tau$. An important difference between the setting in [11] and the current discussion reveals itself in the structure of the function (16). In that work a similar function is introduced and it turns out that the only minima are the zeros of that corresponding function and that between two zeros there is exactly one maximum, the value of the maxima decreasing monotonously. This in turn accounts for the regular structure of the bifurcation diagram in their case. Here, however, $h_{\tau,\theta}$ has a much less regular structure, allowing for example for more than one local extremum between two adjacent zeros (in the rational direction case) leading for example to branches of MEC_θ having the γ–axis as both of their asymptotes. Moreover we cannot a priori exclude that a single branch shows self-intersections or multiple cusps. That said, it can be shown that branches which come from connected components of D_θ that lie far enough to the right cannot exhibit this kind of behavior.

Lemma 3. *Fix $0 \leq \theta_0 \leq \frac{\pi}{4}$, with $\tan\theta_0 = \frac{m}{n}$, $\gcd(m,n) = 1$ and $(\sin\theta_0, \cos\theta_0) = \alpha(m,n)$. For each fixed $\tau > 0$ choose $r_0 \in \mathbb{N}$ such that $(r_0 - 1) \cdot \frac{2\pi}{\alpha} \leq \tau < r_0 \cdot \frac{2\pi}{\alpha}$. There exists a strictly increasing sequence $\gamma^*_j(\tau)$ such that the eigenvalues of L_{γ,τ,θ_0} satisfy for all $\gamma > \gamma^*_j(\tau)$:*

1. *The intervals $\pm i \cdot]\tau, r_0 \cdot \frac{2\pi}{\alpha}[$ contain exactly one single eigenvalue and each of the intervals $\pm i \cdot \left(J_r + r_0 \cdot \frac{2\pi}{\alpha}\right)$ for $0 \leq r \leq j$ contain exactly two single eigenvalues.*
2. *The purely imaginary spectrum is bounded by the constant $\tau^2(2\gamma+1)$.*
3. *Number and multiplicity of the eigenvalues in the intervals $\pm i \cdot \left(J_r + r_0 \cdot \frac{2\pi}{\alpha}\right)$ stays constant with increasing j.*

A proof is given in the appendix.
Read figure 1 in the light of this lemma: For a given value of τ we consider (τ,γ) for values of γ increasing from zero. At certain values of γ we will cross the multiple imaginary eigenvalue curve MEC_θ. Components of MEC_θ with the γ axis as both of their asymptotes will be crossed an even number of times (and their contributions to the purely imaginary spectrum will "disappear" again), while components that have an asymptote of the form $\{k \cdot \frac{2\pi}{\alpha}\}$ may be crossed an odd number of times (their contributions may "stay")

3.2. **Spectrum for irrational directions.** Now, assume $\tan\theta \notin \mathbb{Q}$. We use similar techniques as in the rational case to show that the function G_θ defined in (15) has countably many roots on the positive reals which we denote by $(b_k)_{k\in\mathbb{N}}$, $b_0 = 0$. G_θ changes signs at each such root and the interval between to roots contains exactly one local extremum. The map $\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2; x \mapsto (x\alpha, x\beta) \mod 1$ has a dense image on the torus $[0,1]^2$ iff $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Hence the values of the local extrema of G_θ are dense in the interval $[-\cos\theta - \sin\theta, \cos\theta + \sin\theta]$.
The important difference to the rational case is given by the fact that now the function $q \mapsto \sin^2(q \cdot \cos\theta/2) + \sin^2(q \cdot \sin\theta/2)$ does not have any zeros except for the trivial root $q = 0$. This is mirrored in the fact that now $h_{\tau,\theta}(q)$ does not have any positive real roots as well as in the fact that $\tau^2_\theta(q)$ diverges for $q \to b_k$ for all k: $\lim_{q\to b^{\pm}_{2i}} \tau^2_\theta(q) = \mp\infty$ and $\lim_{q\to b^{\pm}_{2i+1}} \tau^2_\theta(q) = \pm\infty$. We can now state the main result for the irrational case:

Lemma 4. *Fix $0 \leq \theta_0 \leq \frac{\pi}{4}$ and $\tau > 0$. Now choose $k \in \mathbb{N}$ such that $b_{2k-1} \leq \tau < b_{2k}$. Then there exists a strictly increasing sequence $\gamma^*_j(\tau)$ such that the eigenvalues of $L_{\gamma_j,\tau,\theta_0}$ satisfy for all $\gamma > \gamma^*_j(\tau)$:*

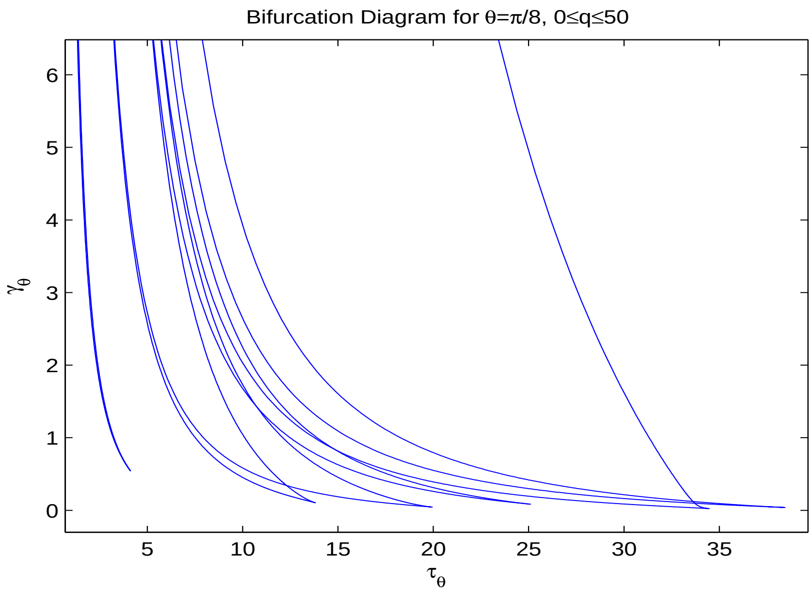

FIGURE 2. Bifurcation diagram in the case of irrational $\tan\theta$

(1) *For* $0 \le r \le j$ *the intervals* $\pm iJ^{2(k+r)}$ *do not contain any eigenvalues of* $L_{\gamma,\tau,\theta}$
(2) *The purely imaginary spectrum is bounded by the constant* $\tau^2(2\gamma+1)$.

The proof of this lemma follows the same idea as the proof of Lemma 2 – namely a discussion of the properties of $h_{\tau,\theta}$, the crucial difference being the absence of zeros of this function in the irrational case which leads to the absence of branches with nonzero asymptotes, as exemplified in figure 2. Here by moving upwards on the line (γ,τ) for a given $\tau > 0$ we cross all components an even number of times so that by choosing γ large enough we can make the contributions from the lower intersections to the purely imaginary spectrum disappear.
The next lemma gives continuity of the parts of the spectrum of $L_{\gamma,\tau,\theta}$:

Lemma 5. *Let* $K \subset \{(\gamma,\tau)|\ \gamma,\tau > 0\}$ *be open and conditionally compact with smooth boundary. Assume in addition that no component of* MEC_{θ_0} *has a tangency with* $K \times \{\theta_0\}$*, so that all components of* MEC_{θ_0} *intersect* $\partial K \times \{\theta_0\}$ *transversally and all intersections of* $\partial K \times \{\theta_0\}$ *with* MEC_{θ_0} *are due to components of* MEC_{θ_0} *that intersect the interior of* $K \times \{\theta_0\}$*. Assume further that* $\partial K \times \{\theta_0\}$ *does not contain a cusp of the curve* MEC_{θ_0}*. Then for a sequence of angles* $\theta_n \to \theta_0$ *we have uniform convergence of* $MEC_{\theta_n} \cap K \times \{\theta_n\} \to MEC_{\theta_0} \cap K \times \{\theta_0\}$ *in the following sense: There is a finite union of intervals on the positive real line which contains the preimage of* K *under* $D_{\theta_0} : q \mapsto \left(\tau^2_{\theta_0}(q), \gamma_{\theta_0}(q)\right)$ *such that for* q *in this set we have* $MEC_{\theta_n}(q) \to MEC_{\theta_0}(q)$ *uniformly.*

This is again proved in the appendix.

4. Center manifolds and traveling wave solutions

For finding solutions of (4) in the region where only one pair of single eigenvalues exists and in the region around the first bifurcation curve a standard center manifold approach can be used as in [11]. To illustrate this we will briefly state some of the intermediate results. The method consists of the following steps:

(1) We compute the projection operator P_c from $\mathbb{H}$ on the central eigenspace of $L_{\gamma,\tau,\theta}$ and show that it commutes with $L_{\gamma,\tau,\theta}$. Denote $P_h := \mathrm{id} - P_c$.
(2) Then we project the linear part of (4) on the hyperbolic eigen-space and show that there is $\alpha_0 > 0$ such that for all $0 \leq \eta \leq \alpha_0$ the solution operator $K_h(\eta)$ for the equation $\partial_t x - P_h L_{\gamma,\tau,\theta} x = f$ exists and is bounded as an operator from $BC^\eta(\mathbb{R}, P_h\mathbb{H})$ to $BC^\eta(\mathbb{R}, P_h\mathbb{D})$, where

$$BC^\eta(\mathbb{R}, E) = \left\{ f \in C(\mathbb{R}, E) \,\middle|\, \|f\|_{BC^\eta} := \sup_{t\in\mathbb{R}} e^{-\eta t}\|f(t)\|_E < \infty \right\}.$$

We verify that the norm of $K_h(\eta)$ is bounded by a continuous function of η. For the derivation of K_h Fourier theory is used.
(3) The conditions above justify the application of the center manifold reduction theorem from [19]. Then it is possible to consider the reduced equation

$$\dot{x}_c = L^c_{\gamma,\tau,\theta} x_c + P_c f(x_c + \psi(x_c)) \tag{17}$$

and use normal form theory to continue solutions of the linearized equation into the full equation.

In our case the projectors for the parameter region below the first bifurcation curve are given by

Lemma 6. *Let $0 \leq \theta_0 \leq 2\pi$ and (γ, τ, θ) below the first bifurcation curve in the plane $\theta = \theta_0$. The purely imaginary part of the spectrum is given by $\sigma_0 L_{\gamma,\tau} = \pm i q_1$ and the spectral projector $P_1 : \mathbb{H} \to \mathbb{H}$ is calculated to equal:*

$$(P_1 U)_0 = \frac{a_1(U)}{N_1} \tag{18}$$

$$(P_1 U)_1 = \frac{q}{N_1} \cdot b_1(U) \tag{19}$$

$$(P_1 U)_2 = \frac{1}{N_1} \cdot [\cos qv \cdot a_1(U) + \sin qv \cdot b_1(U)] \tag{20}$$

where we have defined:

$$\begin{aligned}
N_1 &= q_1 - \gamma\tau^2(cos\theta \cdot \sin(q_1 \cos\theta) + \sin\theta \cdot \sin(q_1 \sin\theta)) \\
a_1(U) &= q_1 f_0 - \gamma\tau^2 \sigma^1_\theta(U) \\
b_1(U) &= f_1 - \gamma\tau^2 \rho^1_\theta(U) \\
\sigma^1_\theta(U) &= \int_0^{\cos\theta} \sin q_1(\cos\theta - s) \cdot [F_2(s) + F_2(-s)]\mathrm{d}s \\
&\quad + \int_0^{\sin\theta} \sin q_1(\sin\theta - s) \cdot [F_2(s) + F_2(-s)]\mathrm{d}s \\
\rho^1_\theta(U) &= \int_0^{\cos\theta} \cos q_1(\cos\theta - s) \cdot [F_2(s) - F_2(-s)]\mathrm{d}s \\
&\quad + \int_0^{\sin\theta} \cos q_1(\sin\theta - s) \cdot [F_2(s) - F_2(-s)]\mathrm{d}s
\end{aligned}$$

Note that the eigenspace $\mathbb{H}_c := P_1(\mathbb{H})$ has the eigenbasis $\zeta_1 = (1, iq_1, e^{iq_1 v})^T$ and $\overline{\zeta}_1$. One can also see that $\mathbb{D}_c := P_1(\mathbb{D}) = \mathbb{H}_c \subset \mathbb{D}$ holds. It can be shown analogously to [11] that $K_h f = (\tilde{x}, \tilde{y}, \tilde{X})$ is a solution to the hyperbolic part of our evolution equation, where:

$$\tilde{x}_h(t) = -[H(.;\gamma,\tau,\theta) * f](t)$$

$$\tilde{y}_h(t) = \frac{d}{dt}\tilde{x}_h(t)$$

$$\tilde{X}_h(t,v) = \tilde{x}_h(t+v) + \frac{1}{N_1}\int_0^v f(t+v-s)\sin(q_1 s)ds$$

and H is the inverse Fourier transform of the function

$$\hat{H} = N^{-1}(ik;\gamma,\tau,\theta) - q_1 \cdot N_1^{-1}(k^2-q^2)^{-1}$$

Given the current eigenvalue structure of L_{γ,τ,θ_0} it follows from the Devaney-Lyapunov theorem applied to the projected equation (onto the two dimensional center manifold) that there is a neighborhood of the origin such that the set of solutions that never leave this neighborhood is a one parameter family of periodic orbits periodic orbits around the origin on the center manifold. Hence we can state the following theorem:

Theorem 7. *Let B_0 be the parameter region where only one pair of purely imaginary eigenvalues of L_{γ,τ,θ_0} exists. Then for $(\gamma,\tau,\theta_0) \in B_0$ the equation (1) has a one parameter family of small periodic waves of the form (2) with u periodic. This family is continuous with respect to $(\gamma,\tau,\theta_0) \in B_0$.*

Proof. The existence for fixed (γ,τ,θ_0) follows from the above analysis. Then we observe that the purely imaginary spectrum is locally continuous in (γ,τ) for fixed θ_0. Then by considering a neighborhood of (γ,τ) of the form $(\gamma-\delta,\gamma+\delta)\times(\tau-\delta,\tau+\delta)$ for δ appropriately, such that no cusp of MEC_{θ_0} lies on the boundary of the neighborhood. Now it follows from the analysis of curves in MEC_{θ_0} in lemma 1, that MEC_{θ_0} intersects the boundary transversally and we can apply lemma 5 to obtain continuity with respect to θ_0 too, as long as the parameters are in B_0.
Furthermore the center manifold and the normal form depend continuously on the parameters as long as the spectrum depends continuously on them. Then we observe, that we can apply the Devaney-Lyapunov theorem in some neighborhood V of origin on the center manifold such that there are only periodic solutions in V for all parameters in a neighborhood of (γ,τ,θ_0) in parameter space. This yields the continuous dependence of the profiles u by continuity of the flow on the center manifold with respect to initial conditions and parameters. This completes the proof. □

Suppose now that the parameters are near the first bifurcation curve Γ_θ that induces exactly one pair of simple $(\pm iq_1)$ and exactly one pair of double $(\pm iq_0)$ eigenvalues. In order to carry out the following analysis we will exclude neighborhoods of cusps and neighborhoods of parameter values that induce eigenvalue pairs whose q_1 and q_0 satisfy $q_1/q_0 = r/s \in \mathbb{Q}$ with $r+s<5$.
Then we follow the same basic approach, the computations are a little more involved, but the structure of every term still closely resembles the calculations in [11]. In particular, it is still possible to compute the projectors onto the six dimensional center eigenspace, which is spanned by ζ_1, $\overline{\zeta}_1$, $\zeta_0 := (1, iq_0, e^{iq_0 v})^T$, $\tilde{\zeta}_0 := (0,1,ve^{iq_0 v})^T$, $\overline{\zeta}_0$,$\overline{\tilde{\zeta}}_0$, $\overline{\zeta}_1$. The first step from here is again to solve the hyperbolic projection of (4) using Fourier transform techniques and then show certain bounds to establish applicability of the center manifold reduction theorem from [19]. This time normal forms from [9][Theorem I.7] are used for the center projected equations, $\dot{x}_c = P_c F(x_c + \psi(x_c))$ with $F \in C^k(\mathbb{R}^n,\mathbb{R}^n)$ and $F\circ R = -R\circ F$ for a (non–identical) involution R on $\mathbb{R}^n$, there exists a (vector) polynomial Φ of degree k such that for $x_c =: \tilde{x} + \Phi(\tilde{x})$ we have $\dot{\tilde{x}} = L_0\tilde{x} + P(\tilde{x}) + o(\|x\|^k)$ for a

polynomial P of degree k, which can be chosen to either commute or anticommute with the restriction of R onto the center eigenspace.

Before we compute the normal form for parameters close to Γ_θ let us acknowledge that the normal form for the single pair of imaginary eigenvalues has the form:

$$\dot{\tilde{x}} = L_0\tilde{x} + \phi(A\overline{A})\tilde{x} + o(\|\tilde{x}\|^k)$$

where ϕ is a polynomial and $\tilde{x} = A\zeta_1 + \overline{A\zeta}_1$, since we're only interested in real solutions. Note, however, that the application of the Devaney Lyapunov theorem, and hence the result of theorem seven, does not depend on the specifics of this normal form.

We will now compute the normal form in the case that we were originally interested in discussing (parameters close to Γ_θ) after separating the different regimes in the form

$$\begin{aligned} I_{r/s} =& \{(\gamma,\tau,\theta) \in \tilde{\Gamma}_0; \left|\frac{q_1(\gamma,\tau,\theta)}{q_0(\gamma,\tau,\theta)} - \frac{r}{s}\right| < \epsilon_{r,s}, \\ & \text{and } \frac{q_1(\gamma,\tau,\theta)}{q_0(\gamma,\tau,\theta)} = \frac{r'}{s'} \text{ implies } r'+s' \geq r+s\}. \end{aligned}$$

The normal form will carry over locally by continuity of normal forms. Let $(A, B, C, \overline{A}, \overline{B}, \overline{C})$ denote the coefficients of an element of the center eigenspace with respect to the basis $(\zeta_0, \tilde{\zeta}_0, \zeta_1, \overline{\zeta}_0, \overline{\tilde{\zeta}}_0, \overline{\zeta}_1)$. As we are only looking for real solutions, the last three components are the complex conjugates of the first three. Then a lengthy computation exactly along the lines of appendix 2 in [11] leads us to the following representation of the three components of P:

$$\begin{aligned} P_0 =& iA[M_0(u_1,u_2,u_4) + u_5M_1(u_1,u_2,u_4,u_5) + \overline{u}_5M_2(u_1,u_2,u_4,\overline{u}_5)] \\ &+ i\overline{A}^{r-1}C^sM_3(u_2,u_4,\overline{u}_5) \\ P_1 =& iB[M_0(u_1,u_2,u_4) + u_5M_1(u_1,u_2,u_4,u_5) + \overline{u}_5M_2(u_1,u_2,u_4,\overline{u}_5)] \\ &+ A[N_0(u_1,u_2,u_4) + u_5N_1(u_1,u_2,u_4,u_5) + \overline{u}_5N_2(u_1,u_2,u_4,\overline{u}_5)] \\ &+ i\overline{A}^{r-2}\overline{B}C^sM_3(u_2,u_4,\overline{u}_5) + \overline{A}^{r-1}C^sN_3(u_2,u_4,\overline{u}_5) \\ P_2 =& iC[R_0(u_1,u_2,u_4) + u_5R_1(u_1,u_2,u_4,u_5) + \overline{u}_5R_2(u_1,u_2,u_4,\overline{u}_5)] \\ &+ i\overline{C}^{s-1}A^rR_3(u_1,u_2,u_5) \end{aligned}$$

This is exactly what is obtained in [11], where one can also find the definitions of the u_i as functions of the A, B, C. For the normal form analysis we go along the following lines: Look at some of the lower order coefficients of M_0, N_0 and R_0. The zero order coefficients can be computed by acknowledging that the eigenvalues of the linearization of the reduced equation and their conjugates have to be the same as the imaginary eigenvalues of the full equations. In particular for $N_0(u_1,u_2,u_4) = b_1(\gamma,\tau,\theta) + b_2(\gamma,\tau,\theta)u_1 + \dots$ we find that $b_1 \equiv 0$ on Γ_θ, $b_1 > 0$ below that curve. We determine the value

$$b_2(\gamma,\tau,\theta) = \frac{\tau^2((f''(0))^2K(q_0;\gamma,\tau,\theta)/2 + (f''(0))^2 + f'''(0)/2)}{1 - \gamma\tau^2[\cos^2\theta\cdot\cos(q_0\cdot\cos\theta) + \sin^2\theta\cdot\cos(q_0\cdot\sin\theta)]}$$

where $K(q;\gamma,\tau,\theta) = \tau^2N^{-1}(2iq;\gamma,\tau,\theta)$. As in [11] we obtain the existence of a family of pairs of reversible solutions of (3), that are homoclinic to periodic solutions of exponentially small amplitude and observe that all coefficients are continuous with respect to the parameters (γ,τ,θ).

Theorem 8. *For given θ and parameters (γ,τ) close to Γ_θ, excluding exceptional points (cusps, intersections, strong resonances) the system (4) reduces to a six dimensional reversible vector field, whose small bounded solutions correspond to* *small traveling waves of (1). For $b_2 < 0$ there exist solitary waves to periodic solutions in (1) of the form (2) for parameters near non-degenerate triple eigenvalue points, whose profile u is locally continuous with respect to γ, τ, θ.*

Proof. The existence of these solitary waves follows from the above normal form analysis, i.e. near each point in parameter space on MEC_{θ_0} we find an open set of parameters in the (γ, τ) plane, such that there are these solitary waves on the center manifold.
Dependence of the reduction and the normal form on parameters is continuous with lemma 5 as in the proof of theorem 7. For every set of parameters, we find a neighborhood such that the same normal form can be used. The continuity of the profile with respect to the parameters then follows from continuous dependence on parameters in the construction of [13] for parameters in this neighborhood. □

Appendix

Here we will state the ideas necessary to prove the lemmas in the above text. Before we start this let us go into a few details concerning the function (16): Zeros of the first derivative of $h_{\tau,\theta}$ exist only within the intervals J_r^{2i} and it can be shown that for all $\tau \geq 0$ there exists a value $Q(\tau) > \tau$ with the property, that for all J_r^{2i} to the right of this value, there are exactly two such zeros, the first one corresponding to a local minimum and the second one corresponding to a local maximum.
By noting $\frac{\mathrm{d}}{\mathrm{d}\tau} h'_{\tau,\theta}(q) > 0$ we can then also prove that for every interval J_r^{2i} that contains exactly one minimum and exactly one maximum of $h_{\tau,\theta}$, there is a $\tau_{r,2i}$ with the property: For all $\tau < \tilde{\tau} < \tau_{r,2i}$ the function $h_{\tilde{\tau},\theta}$ has exactly one minimum and exactly one maximum in J_r^{2i}. For $\tilde{\tau} = \tau_{r,2i}$ the function $h_{\tilde{\tau},\theta}$ has an inflection point in J_r^{2i} and for all $\tilde{\tau} > \tau_{r,2i}$ the function $h_{\tilde{\tau},\theta}$ is strictly decreasing in J_r^{2i}.

Proof of Lemma 2. Fix a θ_0, we need to show that on the boundary of the region where only a pair of single simple eigenvalues exists, parameter values that imply a pair of eigenvalues of order higher than two (like cusps) or those that imply more than one pair of double eigenvalues (like intersections of two branches) are isolated. Let us deal with the latter case first, i.e. assume in particular that

$$h_{\tau_0,\theta_0}(q_1^*) = h_{\tau_0,\theta_0}(q_2^*) = \frac{1}{2\gamma_0\tau_0^2}$$

and $\frac{\partial^2 h_{\theta_0}}{\partial q^2}(\tau_0, q_i^*) \neq 0$, where $\pm i q_1^*$ and $\pm i q_2^*$ (without loss assume $q_1 < q_2$) denote the two pairs of double eigenvalues of $L_{\gamma_0,\tau_0,\theta_0}$ We want to prove, that for small enough perturbations of τ the values of $h_{\tau,\theta}$ evaluated at the extremal values $q_i^*(\tau)$ cannot be the same. Note first that

$$\frac{d}{d\tau} h_{\tau_0,\theta_0}(q_1) - \frac{d}{d\tau} h_{\tau_0,\theta_0}(q_2) > 0$$

as $q_1 < q_2$. Then we can apply the implicit function theorem to continue the extremal value q_i^* for small changes in τ. Explicitly we obtain

$$[q_i^*]'(\tau_0) = \left[\frac{\partial^2 h_\theta}{\partial q^2}(\tau_0, q_i^*)\right]^{-1} \cdot \frac{\partial^2 h_\theta}{\partial q \partial \tau}(\tau_0, q_i^*) \tag{21}$$

Taylor expanding the term for τ near τ_0

$$h_\theta(\tau, q_1^*(\tau)) - h_\theta(\tau, q_2^*(\tau))$$

and using in (21) proves the claim in this case.
It remains to be shown that points on MEC_θ with eigenvalues of order higher than two are isolated: Suppose there is a finite accumulation point of points with eigenvalues of order three and higher parameterized by q_n. As for (γ, τ) bounded,

q_n is bounded by lemma 3 and 5, there exists a subsequence q_{n_k} converging with $\tau'_\theta(q_{n_k}) = \gamma'_\theta(q_{n_k}) = 0$ to some q_* with $\tau'_\theta(q_*) = \gamma'_\theta(q_*) = 0$. As $\tau'_\theta(.)$ and $\gamma'_\theta(.)$ are analytic functions, this would imply that they are constantly zero, a clear contradiction to the form of (13) and (14).
As the last case we have to show that intersection points of the various branches in the (γ, τ)–plane cannot accumulate towards a cusp point. Suppose there exists a sequence of such intersections with eigenvalues $(q_n^1)_{n\in\mathbb{N}}$ and $(q_n^2)_{n\in\mathbb{N}}$. For values of (γ, τ) in a compact subset the eigenvalues have to lie in a compact subset of the real numbers as well, so that we can pick subsequences and limits q_*^1 and q_*^2. By choosing further subsequences we can ensure that the sequences are monotone and the convex hulls of these subsequences in $\mathbb{R}$ are disjoint. Due to analyticity and by choosing further subsequences we can assume γ_{θ_0} to be monotonous on the intervals given by these convex hulls. Thus we can solve in analytic fashion for γ and obtain functions $q^1(\gamma)$ and $q^2(\gamma)$. Then by renaming the last subsequence we have $\gamma_{\theta_0}(q_n^1) = \gamma_{\theta_0}(q_n^2)$ and $\tau^2_{\theta_0}(q_n^1) = \tau^2_{\theta_0}(q_n^2)$. Hence $\tau^2_{\theta_0}(q^1(\gamma)) = \tau^2_{\theta_0}(q^2(\gamma))$ for infinitely many values of γ, so that by analyticity the curves $\tau^2_{\theta_0}(q^1(\cdot)) = \tau^2_{\theta_0}(q^2(\cdot))$. This contradicts the first part of the proof, because that part states that intersections of the curves are transversal. Altogether this implies lemma 2. □

Proof of Lemma 3. The idea is identical to the one used in [11], but its application is more subtle due to the rich structure of $h_{\tau,\theta}$. In particular for given τ and θ we consider the graph of $h_{\tau,\theta}$ and let γ increase until we find a value $\gamma_j^*(\tau)$ big enough, such that $(2\gamma_j^*(\tau)\tau^2)^{-1}$ is smaller than the values of all the minima (or points of inflection) of $h_{\tau,\theta_0}(q)$ in the interval $(\tau, (r_0 + j + 1) \cdot \frac{2\pi}{\alpha}]$. As $h_{\tau,\theta_0}(q) = 0$ at the boundaries of $J_r + r \cdot \frac{2\pi}{\alpha}$, we can always find the required simple eigenvalues for all $\gamma > \gamma_j^*(\tau)$. On the other hand, because $h_{\tau,\theta}$ decays to zero as its argument goes to infinity, there cannot be infinitely many values of q such that $h_{\tau,\theta_0}(q) = \frac{1}{2\gamma(\tau)\tau^2}$ holds. This implies the bound in part (b). For the last statement, note that zeros of h_{τ,θ_0} are independent of γ. Since the sequence $\gamma_j^*(\tau)$ is increasing and the possible intersections in the intervals $J_r + r_0 \cdot \frac{2\pi}{\alpha}$ are already below any zeros of the derivative of $h_{\tau,\theta}$, we cannot introduce new eigenvalues or higher orders. □

Proof of Lemma 4. We again consider the graph of $h_{\tau,\theta}$ and let γ increase until we find a value $\gamma_j^*(\tau)$ big enough, such that $(2\gamma_j^*(\tau)\tau^2)^{-1}$ is smaller than the values of all the minima of $h_{\tau,\theta_0}(q)$ in the interval $(\tau, (r_0 + j + 1) \cdot \frac{2\pi}{\alpha}]$. But as $h_{\tau,\theta_0}(q) > 0$ for all points in $J_r + r \cdot \frac{2\pi}{\alpha}$, there are no intersections with the graph left in these intervals. The estimates on the magnitude of the imaginary part follow as above directly from $h_{\tau,\theta_0}(q) = \frac{1}{2\gamma(\tau)\tau^2}$ as above.

Proof of Lemma 5. First we see from the discussions of the rational and irrational cases that at its zeros $\tilde{b}_k(\theta_n)$ the function $\tau^2_{\theta_n}$ has a non-zero derivative with respect to q for all θ_n, θ_0. Thus we can locally write each $\tilde{b}_k$ as a function of θ, and hence, for small perturbations in the angle, continue zeros of $\tau^2_{\theta_0}$ to zeros of $\tau^2_{\theta_n}$ (i.e. the boundary points of the domain D_{θ_n}).
Let M_n be the subset of $\mathbb{R}$ given by the inverse image of $\tilde{K}_n := MEC_{\theta_n} \cap K \times \{\theta_n\}$ under the parametrization defined by (13) and (14). Each M_n is bounded and contains a finite number of components (by continuity of $(\gamma_{\theta_n}, \tau_{\theta_n}, \theta_n)$ on D_{θ_n} and the fact that for each $l > 0$ the number of intervals contained in $D_{\theta_n} \cap [0, l]$ is finite - see the paragraph before lemma 2). In all, each M_n consists of a finite union of open intervals which we will call $I_k^n =]q_k^{-,n}, q_k^{+,n}[$ where $0 \le k \le k_n$. By the transversality conditions stated in the lemma we can continue the boundary points of M_0 to boundary points of M_n for small perturbations of the angle.

These continuation properties imply that there has to be a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the number of connected components of M_n is equal to the number of connected components of M_0 and then $q_k^{\pm,n} \to q_k^{\pm,0}$ as well as $\tilde{b}_k(\theta_n) \to \tilde{b}_k(\theta_0)$. Now, let $\mathrm{dist}(\partial D_{\theta_0}, \partial M_0) =: \epsilon$ and choose n_1 big enough such that all (of the finitely many) $\tilde{b}_k(\theta_n)$ are $\frac{\epsilon}{3}$–close to their respective $\tilde{b}_k(\theta_0)$, so that $I_k^n \subset I_k^0 \pm \frac{\epsilon}{3}$. Then clearly $I_k^0 \pm \frac{\epsilon}{3} \subset D_{\theta_n}$ for all $n \geq n_1$ which means that the parametrization is defined for all q contained in those intervals. Now note that the sequence of continuous functions $(\gamma_{\theta_n}, \tau_{\theta_n})$ converges pointwise on the compact region $\bigcup_k I_k^0 \pm \frac{\epsilon}{3}$ to the continuous function $(\gamma_{\theta_n}, \tau_{\theta_n})$, hence we also have uniform convergence in this region.□

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