Travelling wave solutions for the discrete sine-Gordon equation with nonlinear pair interaction

Carl-Friedrich Kreiner [a,* 1], Johannes Zimmer [b]

[a] *Mathematisches Forschungsinstitut Oberwolfach, 77709 Oberwolfach, Germany*

[b] *Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom*

Abstract

The focus of study is the nonlinear discrete sine-Gordon equation, where the nonlinearity refers to a nonlinear interaction of neighbouring atoms. The existence of travelling heteroclinic, homoclinic and periodic waves is shown. The asymptotic states are chosen such that the action functional is finite. The proofs employ variational methods, in particular a suitable concentration-compactness lemma combined with direct minimisation and mountain pass arguments.

Key words: Nonlinear Klein-Gordon lattice, travelling waves, concentration compactness, calculus of variations
MSc.: 37K60, 74J30, 34A34, 49J35

1 Introduction

This article is concerned with travelling waves in the *discrete sine-Gordon equation*

$$\ddot{q}_k(t) = V'\left(q_{k+1}(t) - q_k(t)\right) - V'\left(q_k(t) - q_{k-1}(t)\right) - K\sin\left(q_k(t)\right), \ k \in \mathbb{Z}, \quad (1)$$

[1] Present address: Lehrstuhl C für Mathematik, RWTH Aachen, Templergraben 55, 52056 Aachen, Germany

* Corresponding author. Tel.: +49 (0)241 80 94540, Fax: +49 (0)241 80 92390

Email addresses: kreiner at mathC.rwth-aachen.de (Carl-Friedrich Kreiner), zimmer at maths.bath.ac.uk (Johannes Zimmer).

URL: http://www.maths.bath.ac.uk/~zimmer/ (Johannes Zimmer).

with a constant $K > 0$. Equation (1) describes the evolution of an infinite chain of atoms with elastic nearest neighbour interaction and an on-site potential, according to Newton's law. The argument of the interaction potential $V\colon \mathbb{R} \to \mathbb{R}$ is the discrete strain $q_{k+1}(t) - q_k(t)$. In an earlier work [7], we assumed that V is a quadratic function $V(\varepsilon) := \frac{c_0^2}{2}\varepsilon^2$ with $c_0 > 0$; here, we consider an anharmonic interaction, that is, $V(\varepsilon) \neq \frac{c_0^2}{2}\varepsilon^2$. We are interested in *travelling wave solutions* to (1), that is, solutions of the form $q_k(t) = u(k - ct)$ for all $k \in \mathbb{Z}$, where $u\colon \mathbb{R} \to \mathbb{R}$ is the *wave profile* and $c > 0$ is the *wave speed*. For this ansatz, (1) becomes

$$c^2 u''(\tau) = V'\left(u(\tau+1) - u(\tau)\right) - V'\left(u(\tau) - u(\tau-1)\right) - K\sin(u(\tau)). \quad (2)$$

In a suitable setting, Equation (2) is the Euler-Lagrange equation of the action functional

$$J(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2}\left(u'(\tau)\right)^2 - V(u(\tau+1) - u(\tau)) + K(1 + \cos(u(\tau)))\right] \mathrm{d}\tau. \quad (3)$$

Here, $-K(1 + \cos(u(\tau)))$ is the *on-site potential (density)*.

The specific form of the on-site potential is not crucial for the results presented here. In fact, they will, with obvious modifications, also hold for any non-negative 2π-periodic $W^{1,\infty}$-function with zero set $\{(2k+1)\pi : k \in \mathbb{N}\}$ instead of $(1 + \cos(\cdot))$.

In this article, we only consider *supersonic* waves, that is, restrict the analysis to wave speeds $c > V''(0)$. Under suitable conditions on the interaction potential V, we show the existence of three types of solutions:

- heteroclinic travelling waves: $\lim\limits_{z\to-\infty} u(z) = -\pi$ and $\lim\limits_{z\to+\infty} u(z) = \pi$, (4)
- homoclinic travelling waves: $\lim\limits_{z\to-\infty} u(z) = \lim\limits_{z\to+\infty} u(z) = \pi$,
- periodic travelling waves: $u(z) = u(z+T)$ for some $T > 0$ and for every $z \in \mathbb{R}$.

The first part generalises an existence result for supersonic heteroclinic waves in [7] to the case of nonlinear interaction.

The results for homoclinic waves presented here are related to those of Bates and Zhang [3]. They have, among other results, shown the existence of supersonic travelling waves for

$$c^2 u''(\tau) = c_0^2(u(\tau+1) - 2u(\tau) + u(\tau-1)) + K\sin(u(\tau)). \quad (5)$$

Bates and Zhang [3] consider homoclinic waves that have their asymptotic states in the maximum of the on-site potential, which can here be taken to be

$K\cos(u(\tau))-1$. Employing entirely different methods, we study the analogous situation with nonlinear interaction and thus achieve a complementary result.

In addition, we prove the existence of periodic solutions. This result is related to work on periodic solutions with nonlinear interaction, but without on-site potential [1,2]. The interest in periodic solutions can be explained with the desire to analyse the (non-)ergodicity of a system; see the discussion in [1], also regarding the (non-)equipartitioning of energy of the Fermi-Pasta-Ulam experiment (nonlinear interaction without on-site potential).

Other choices of boundary conditions and their physical interpretations are discussed in [7].

2 Heteroclinic travelling waves

In this section, we prove the existence of heteroclinic waves for (2) with boundary conditions (4). The solution will be found as a minimiser of a penalised variant of the action functional (3). The penalisation is necessary since the action functional is, unlike in the case of linear interaction, not bounded from below.

We introduce the function-analytic setting. Let us define the space

$$X := \left\{u \in H^1_{\mathrm{loc}}(\mathbb{R}) : u' \in L^2(\mathbb{R})\right\};$$

when equipped with the inner product $\langle u, v\rangle_X := u(0)\,v(0) + \displaystyle\int_{\mathbb{R}} u'(\tau)\,v'(\tau)\,\mathrm{d}\tau$, it becomes a Hilbert space. Further, we set

$$\mathcal{M}_{-\pi,\pi} := \{u \in X : u(-\infty) = -\pi, u(\infty) = \pi\}. \tag{6}$$

Throughout this section, the following assumptions are made.

Assumption 2.1 (i) $V \in C^1(\mathbb{R})$, $V(0) = 0$, *and* $V(x) \geq 0$ *for all* $x \in \mathbb{R}$.
(ii) *The interaction potential is growing at infinity,*

$$\lim_{|x|\to\infty} V(x) = \infty.$$

(iii) *(Super-)quadratic growth at* 0*:* $\lim_{x\to 0}\left|\frac{V(x)}{x^2}\right|$ *exists and is finite.*
(iv) *The wave speed satisfies*

$$c^2 > c_1^2 := 2\sup_{|x|<6\pi}\left|\frac{V(x)}{x^2}\right|.$$

The main result of this section is as follows.

Theorem 2.2 *Let Assumption 2.1 be satisfied and suppose that c is large enough to ensure $\delta < \pi$ for δ given by*

$$\delta := \frac{4c_1^2}{c^2 - c_1^2 + c\sqrt{(c^2 - c_1^2)}}. \tag{7}$$

Then a solution $u \in C^2(\mathbb{R})$ of (2) exists with boundary conditions (4).

Assumption 2.1 allows for interaction potentials V which grow superquadratically at infinity, i.e., $\lim_{x\to\pm\infty} x^{-2} \cdot V(x) = \infty$; for such potentials the action functional J from (3) is unbounded from below (and from above). In the next subsection, we gather some general properties of J and introduce a penalised functional that agrees with J on a suitable neighbourhood of $0 \in X$ which includes a relevant part of $\mathcal{M}_{-\pi,\pi}$. It is then shown that a global minimiser of the penalised functional, if it exists, lies in the interior of this neighbourhood so that it is necessarily a local minimiser of J as well. The last subsection establishes the existence of such a global minimiser of the penalised functional, which is the solution claimed in Theorem 2.2.

2.1 Auxiliary statements

For more a compact notation, we introduce on X a difference operator A by $Au(\tau) := u(\tau + 1) - u(\tau)$. It is easy to see (e.g., [12, Proposition 1]) that

$$\max\left\{\|Au\|_{L^2(\mathbb{R})}, \|Au\|_{L^\infty(\mathbb{R})}\right\} \leq \|u'\|_{L^2(\mathbb{R})}. \tag{8}$$

The action functional (3) can then be rewritten as

$$J(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2}\Big(u'(\tau)\Big)^2 - V(Au(\tau)) + K\Big(1 + \cos(u(\tau))\Big)\right] \mathrm{d}\tau. \tag{9}$$

We now give the precise connection between the action functional and (2) by showing that the latter is, in a suitable sense, the Euler-Lagrange equation associated with (9).

Lemma 2.3 *Let $v_0 \colon \mathbb{R} \to [-\pi, \pi]$ be a monotone function in $C^\infty(\mathbb{R})$ such that $v_0(\tau) = -\pi$ for $\tau < -1$ and $v_0(\tau) = \pi$ for $\tau > 1$. Define $\Psi \colon H^1(\mathbb{R}) \to \mathbb{R}$ by*

$$\Psi(v) := J(v_0 + v)$$

and suppose that Assumption 2.1 is satisfied. Then the following holds:

(i) $\Psi(v) < \infty$ *for all* $v \in H^1(\mathbb{R})$*, or, equivalently,* $J(u) < \infty$ *for all* u *of the form* $u = v_0 + v$ *for some* $v \in H^1(\mathbb{R})$*.*

(ii) $J(u) = \infty$ *for all* $u \in \mathcal{M}_{-\pi,\pi}$ *which are* not *of the form* $u = v_0 + v$ *for any* $v \in H^1(\mathbb{R})$*. In particular, a minimiser* u *of* J *on* $\mathcal{M}_{-\pi,\pi}$ *can be written as* $u = v_0 + v$ *for some* $v \in H^1(\mathbb{R})$*.*

(iii) Ψ *is continuously differentiable on* $H^1(\mathbb{R})$*.*

(iv) *Let* $v \in H^1(\mathbb{R})$ *be a critical point of* Ψ *and set* $u := v_0 + v$*. Then* $u, v \in C^2(\mathbb{R})$ *and* u *is a solution of* (2) *with boundary conditions* (4)*.*

The proof is straightforward and thus omitted here (see [6,7] for details).

The penalisation is defined as follows. Let F be a non-negative function in $C^\infty(\mathbb{R})$ such that

$$\begin{aligned} &F(x) = 0 && \text{for all } |x| \le \tfrac{5}{2}\pi, \\ &F(x) \ge 4\int_0^{2x} |V'(\xi)|\,\mathrm{d}\xi \text{ and } F(x) \ge 2K && \text{for all } |x| \ge 3\pi, \\ &\tfrac{1}{2} \le 1 + \cos(x) + \tfrac{1}{2K}F(x) && \text{for all } |x| \in \left(\tfrac{5}{2}\pi, 3\pi\right). \end{aligned} \tag{10}$$

The existence of such a function is immediate. In particular, for all $\lambda > 0$,

$$1 + \cos(x) + \lambda F(x) = 0 \text{ if and only if } |x| = \pi. \tag{11}$$

We then define the *penalised functional* $J_P \colon X \to \mathbb{R} \cup \{\infty\}$ by

$$J_P(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2}\left(u'(\tau)\right)^2 - V(Au(\tau)) + K\left(1 + \cos(u(\tau))\right) + F(u(\tau)) \right] \mathrm{d}\tau. \tag{12}$$

Obviously $J_P(u) = J(u)$ for all $u \in X$ with $\|u\|_{L^\infty(\mathbb{R})} \le \frac{5}{2}\pi$.

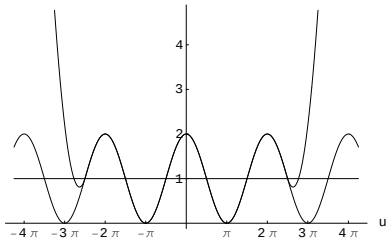

Fig. 1. Graphs of $1 + \cos(u)$ and $1 + \cos(u) + \frac{1}{K}F(u)$, the on-site potentials in the definitions of J and J_P, respectively. Here $K = 1$.

To simplify the notation, we denote the monotonised interaction potential of (10) by $\widetilde{V}$, that is, $\widetilde{V}(x) := |\int_0^x |V'(\xi)|\,\mathrm{d}\xi|$. Then (10) implies for all $|x| \ge 3\pi$

$$V(2x) \le \widetilde{V}(2x) \le \tfrac{1}{4} \cdot F(x) \tag{13}$$

and in particular, by Assumption 2.1 (ii), $F(x) \to \infty$ for $x \to \pm\infty$.

Lemma 2.4 *Let $T, \alpha > 0$ be given. The solutions of the variational problem*

$$\text{minimise } \int_0^T \Big(u'(s)\Big)^2 \,\mathrm{d}s, \text{ subject to } \|u\|_{L^\infty(0,T)} = 1,\ u(0) = u(T) = 0$$

on $H^1(0,T)$ are the piecewise affine function $u_+(s) = \frac{2}{T}\min\{s, T-s\}$ and its negative $u_-(s) := -u_+(s)$. The value of the minimum is $\|u_\pm\|^2_{L^2(0,T)} = \frac{4}{T}$.

Again, the proof is immediate (see [6]).

2.2 A-priori bounds

We start with an auxiliary statement, which is taken from [4, Section 6.2] (see also [10]).

Lemma 2.5 *Let $W \in C^1(\mathbb{R})$ be such that $W(\pm\pi) = 0$ and $W(\xi) > 0$ for $|\xi| < \pi$ and set $I(u) := \int_{\mathbb{R}} \Big[(u'(\tau))^2 + W(u(\tau))\Big] \,\mathrm{d}\tau$. Then the minimum of I on $\mathcal{M}_{-\pi,\pi}$ is attained and*

$$\min_{u\in\mathcal{M}_{-\pi,\pi}} I(u) = \vartheta := 2\int_{-\pi}^{\pi} \sqrt{W(\xi)}\,\mathrm{d}\xi. \tag{14}$$

Moreover, with the same ϑ,

$$\inf_{T>0}\ \inf\left\{ \int_{-T}^{T} \Big[(u'(\tau))^2 + W(u(\tau))\Big] \,\mathrm{d}\tau : \begin{array}{c} u \in H^1(-T,T), \\ u(-T) = -\pi, u(T) = \pi \end{array} \right\} = \vartheta. \tag{15}$$

Lemma 2.6 (Bounds for J_P) *Suppose that Assumption 2.1 is satisfied. Then*

(i) *For all $u \in X$ and J_P as defined in (12),*

$$J_P(u) \geq \int_{\mathbb{R}} \left[\frac{c^2 - c_1^2}{2} \Big(u'(\tau)\Big)^2 + K\Big(1 + \cos(u(\tau))\Big) + \frac{1}{2}F(u(\tau)) \right] \mathrm{d}\tau. \tag{16}$$

(ii) *The functional J_P is bounded from below on $\mathcal{M}_{-\pi,\pi} \subset X$, as defined in (6). Indeed, J_P satisfies*

$$b^- := 8\sqrt{(c^2 - c_1^2)K} < \inf_{u\in\mathcal{M}_{-\pi,\pi}} J_P(u) < b^+ := 8c\sqrt{K}. \tag{17}$$

Proof: (i) Since $|Au(\tau)| \le |u(\tau+1)| + |u(\tau)| \le 2\max\left\{|u(\tau+1)|, |u(\tau)|\right\}$, we find for every $k > 0$

$$\begin{aligned}\left\{\tau \in \mathbb{R} : |Au(\tau)| > k\right\} &\subseteq \left\{\tau \in \mathbb{R} : \max\{|u(\tau+1)|, |u(\tau)|\} > \tfrac{k}{2}\right\} \\ &\subseteq \left\{\tau \in \mathbb{R} : |u(\tau)| > \tfrac{k}{2}\right\} \cup \left\{\tau \in \mathbb{R} : |u(\tau+1)| > \tfrac{k}{2}\right\}.\end{aligned}$$

Therefore, using (13) and the fact that the monotonised interaction potential $\widetilde{V}$ is monotone on $(-\infty, 0)$ and $(0, \infty)$,

$$\begin{aligned}&\int\limits_{\{\tau\in\mathbb{R} \,:\, |Au(\tau)|>6\pi\}} V(Au(\tau))\,\mathrm{d}\tau \le \int\limits_{\{\tau\in\mathbb{R} \,:\, |Au(\tau)|>6\pi\}} \widetilde{V}(Au(\tau))\,\mathrm{d}\tau \\ &\le \int\limits_{\{\tau\in\mathbb{R} \,:\, |Au(\tau)|>6\pi\}} \widetilde{V}\left(2\max\{|u(\tau)|, |u(\tau+1)|\}\right)\,\mathrm{d}\tau \\ &\le \int\limits_{\{\tau\in\mathbb{R} \,:\, \max\{|u(\tau+1)|,|u(\tau)|\}>3\pi\}} \tfrac{1}{4}\cdot F\left(\max\{|u(\tau)|, |u(\tau+1)|\}\right)\,\mathrm{d}\tau \\ &\le 2\int\limits_{\{\tau\in\mathbb{R} \,:\, |u(\tau)|>3\pi\}} \tfrac{1}{4}\cdot F(u(\tau))\,\mathrm{d}\tau \quad \le \quad \frac{1}{2}\int\limits_{\mathbb{R}} F(u(\tau))\,\mathrm{d}\tau. \qquad (18)\end{aligned}$$

Employing c_1 from Assumption 2.1, we obtain

$$\begin{aligned}\int\limits_{\{\tau\in\mathbb{R} \,:\, |Au(\tau)|\le 6\pi\}} V(Au(\tau))\,\mathrm{d}\tau &\le \int\limits_{\{\tau\in\mathbb{R} \,:\, |Au(\tau)|\le 6\pi\}} \tfrac{1}{2}\cdot c_1^2\left(Au(\tau)\right)^2\,\mathrm{d}\tau \\ &\le \int\limits_{\mathbb{R}} \tfrac{1}{2}\cdot c_1^2\left(Au(\tau)\right)^2\,\mathrm{d}\tau.\end{aligned}$$

Thus we obtain with (18) for all $u \in X$

$$\begin{aligned}J_P(u) &\ge \int\limits_{\mathbb{R}} \left[\frac{c^2}{2}\left(u'(\tau)\right)^2 - \frac{c_1^2}{2}\left(Au(\tau)\right)^2 + K\left(1+\cos(u(\tau))\right) + F(u(\tau))\right]\,\mathrm{d}\tau \\ &\quad - \int\limits_{\{\tau\in\mathbb{R} \,:\, |Au(\tau)|>6\pi\}} V(Au(\tau))\,\mathrm{d}\tau \\ &\ge \int_{\mathbb{R}} \left[\frac{c^2-c_1^2}{2}\left(u'(\tau)\right)^2 + K\left(1+\cos(u(\tau))\right) + \frac{1}{2}F(u(\tau))\right]\,\mathrm{d}\tau.\end{aligned}$$

(ii) Lemma 2.5 can be applied to

$$I_1(u) = \frac{c^2-c_1^2}{2}\int_{\mathbb{R}} \left[\left(u'(\tau)\right)^2 + W_1(u(\tau))\right]\,\mathrm{d}\tau,$$

with $W_1(\xi) := \frac{2K}{c^2-c_1^2}\left[(1+\cos(\xi)) + \frac{1}{2K}F(\xi)\right]$, so that (16) and (14) show

$$\begin{aligned}\inf_{u\in\mathcal{M}_{-\pi,\pi}} J_P(u) &\geq \frac{c^2-c_1^2}{2}\cdot 2\left|\int_{-\pi}^{\pi}\sqrt{W_1(\xi)}\,\mathrm{d}\xi\right| \\ &= \sqrt{2\left(c^2-c_1^2\right)K}\left|\int_{-\pi}^{\pi}\sqrt{1+\cos(\xi)}\,\mathrm{d}\xi\right| = 8\sqrt{\left(c^2-c_1^2\right)K} =: b^-.\end{aligned}$$

F does not contribute to the integral because $F(x) = 0$ for $|x| \leq \pi$.

On the other hand, using $V \geq 0$, we can estimate J_P for all $u \in X$ by

$$J_P(u) \leq \frac{c^2}{2}\int_{\mathbb{R}}\left[\left(u'(\tau)\right)^2 + \frac{2}{c^2}\Big(K(1+\cos(u(\tau))) + F(u(\tau))\Big)\right]\mathrm{d}\tau.$$

Lemma 2.5 can be applied to

$$I_2(u) = \frac{c^2}{2}\int_{\mathbb{R}}\left[\Big(u'(\tau)\Big)^2 + W_2(u(\tau))\right]\mathrm{d}\tau,$$

now with $W_2(\xi) := \frac{2K}{c^2}\left[(1+\cos(\xi)) + \frac{1}{K}F(\xi)\right]$, in order to obtain

$$\inf_{u\in\mathcal{M}_{-\pi,\pi}} J_P(u) \leq \frac{c^2}{2}\cdot 2\left|\int_{-\pi}^{\pi}\sqrt{W_2(\xi)}\,\mathrm{d}\xi\right| = 8c\sqrt{K} =: b^+. \qquad \square$$

In the next statement we will use that $b^+ - b^-$ is small for $c \gg c_1$. Indeed,

$$b^+ - b^- = 8\sqrt{K}\left(c - \sqrt{c^2-c_1^2}\right) = 8\sqrt{K}\frac{c^2-(c^2-c_1^2)}{c+\sqrt{c^2-c_1^2}} = \frac{8c_1^2\sqrt{K}}{c+\sqrt{c^2-c_1^2}}. \quad (19)$$

Lemma 2.7 (L^∞ bound for minimisers of J_P) *Suppose Assumption 2.1 is satisfied. If $u_P \in \mathcal{M}_{-\pi,\pi}$ minimises J_P on $\mathcal{M}_{-\pi,\pi}$ then*

$$\|u_P\|_{L^\infty(\mathbb{R})} \leq \frac{3}{2}\pi + \delta.$$

In particular, if c is large enough to ensure $\delta < \pi$, then $\|u_P\|_{L^\infty(\mathbb{R})} < \frac{5}{2}\pi$.

Proof: Define $\mathcal{S} := \{\tau \in \mathbb{R} : |u_P(\tau)| \geq \frac{3}{2}\pi\}$. If $|\mathcal{S}| = 0$ then the statement is obvious. We thus suppose $|\mathcal{S}| > 0$. Let $T_1, T_2 \in \mathbb{R}\cup\{\pm\infty\}$ with $T_1 < T_2$ be such that $u_P(T_1) = -\pi$, $u_P(T_2) = \pi$ and $|u_P(\tau)| \leq \pi$ for all $\tau \in [T_1, T_2]$; then in particular $F(u_P(\tau)) = 0$ for all $\tau \in [T_1, T_2]$. Then, using (15), it follows

that

$$\begin{aligned}\sigma &:= \int_{T_1}^{T_2} \left[\frac{c^2 - c_1^2}{2} \left(u_P'(\tau) \right)^2 + K \left(1 + \cos\left(u_P(\tau) \right) \right) + \frac{1}{2} F\left(u_P(\tau) \right) \right] \mathrm{d}\tau \\ &\geq \inf_{\substack{u \in H^1_{\mathrm{loc}}(T_1,T_2) \,\cap\, C^0[T_1,T_2], \\ u(T_1) = -\pi,\ u(T_2) = \pi}} \int_{T_1}^{T_2} \left[\frac{c^2 - c_1^2}{2} \left(u'(\tau) \right)^2 + K \left(1 + \cos(u(\tau)) \right) + 0 \right] \mathrm{d}\tau \\ &\geq b^-.\end{aligned}$$

Employing (17), (16), the preceding estimate and $[T_1, T_2] \cap \mathcal{S} = \emptyset$,

$$\begin{aligned}b^+ &\geq J_P\left(u_P \right) \\ &\geq \int_{\mathbb{R}} \left[\frac{c^2 - c_1^2}{2} \left(u_P'(\tau) \right)^2 + K \left(1 + \cos\left(u_P(\tau) \right) \right) + \frac{1}{2} F\left(u_P(\tau) \right) \right] \mathrm{d}\tau \\ &= \sigma + \int_{\mathbb{R} \setminus [T_1,T_2]} \left[\frac{c^2 - c_1^2}{2} \left(u_P'(\tau) \right)^2 + K \left(1 + \cos\left(u_P(\tau) \right) \right) + \frac{1}{2} F\left(u_P(\tau) \right) \right] \mathrm{d}\tau \\ &\geq b^- + \int_{\mathcal{S}} \left[\frac{c^2 - c_1^2}{2} \left(u_P'(\tau) \right)^2 + K \left(1 + \cos\left(u_P(\tau) \right) \right) + \frac{1}{2} F\left(u_P(\tau) \right) \right] \mathrm{d}\tau.\end{aligned}$$

By definition of F in (10) and of $\mathcal{S}$, $(1 + \cos(u_P(\tau)) + \frac{1}{2K} F(u_P(\tau)) \geq \frac{1}{2}$ for all $\tau \in \mathcal{S}$ (see Figure 1), therefore

$$\int_{\mathcal{S}} \left[K \left(1 + \cos\left(u_P(\tau) \right) \right) + \frac{1}{2} F\left(u_P(\tau) \right) \right] \mathrm{d}\tau \geq \frac{K}{2} \left| \mathcal{S} \right|.$$

Lemma 2.4 shows (see Figure 2),

$$\int_{\mathcal{S}} \left(u_P'(\tau) \right)^2 \mathrm{d}\tau \geq 2 \cdot \frac{|\mathcal{S}|}{2} \left[\frac{\|u_P\|_{L^\infty(\mathbb{R})} - \frac{3}{2}\pi}{\frac{|\mathcal{S}|}{2}} \right]^2 = \frac{4}{|\mathcal{S}|} \left(\|u_P\|_{L^\infty(\mathbb{R})} - \tfrac{3}{2}\pi \right)^2.$$

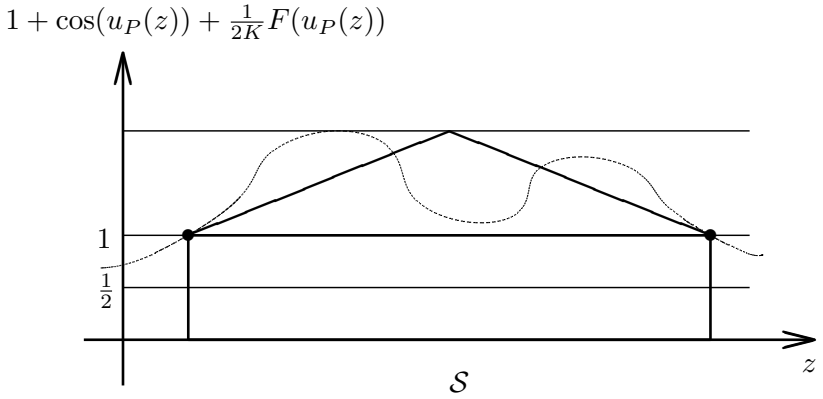

Fig. 2. Illustration of the proof of Lemma 2.7 for the case of connected $\mathcal{S}$.

This includes the case that $\mathcal{S}$ consists of several connected components; indeed, let $\widetilde{\mathcal{S}} = [T_3, T_4]$ be a connected component of $\mathcal{S}$ such that $|u_P(\tau)| = \|u_P\|_{L^\infty(\mathbb{R})}$ for some $\tau \in [T_3, T_4]$ and apply Lemma 2.4 on $\widetilde{\mathcal{S}}$; the claim follows from $\left|\widetilde{\mathcal{S}}\right| \leq |\mathcal{S}|$.

Combining the last three inequalities, we obtain, using in the third line the trivial estimate $x^2 + y^2 \geq 2xy$,

$$\begin{aligned} b^+ - b^- &\geq \int_{\mathcal{S}} \left[\frac{c^2 - c_1^2}{2} \left(u_P'(\tau)\right)^2 + K\left(1 + \cos\left(u_P(\tau)\right)\right) + \frac{1}{2} F\left(u_P(\tau)\right) \right] \mathrm{d}\tau \\ &\geq \frac{c^2 - c_1^2}{2} \frac{4}{|\mathcal{S}|} \left(\|u_P\|_{L^\infty(\mathbb{R})} - \tfrac{3}{2}\pi \right)^2 + \frac{K}{2} |\mathcal{S}| \\ &\geq 2\sqrt{\frac{2\left(c^2 - c_1^2\right)}{|\mathcal{S}|} \left(\|u_P\|_{L^\infty(\mathbb{R})} - \tfrac{3}{2}\pi \right)^2} \cdot \sqrt{\tfrac{1}{2} K |\mathcal{S}|} \\ &= 2\sqrt{K\left(c^2 - c_1^2\right)} \left(\|u_P\|_{L^\infty(\mathbb{R})} - \tfrac{3}{2}\pi \right). \end{aligned}$$

Thus

$$\|u_P\|_{L^\infty(\mathbb{R})} \leq \tfrac{3}{2}\pi + \frac{b^+ - b^-}{2\sqrt{K\left(c^2 - c_1^2\right)}} = \tfrac{3}{2}\pi + \delta,$$

where, employing the expression for $b^+ - b^-$ from (19),

$$\delta := \frac{b^+ - b^-}{2\sqrt{K\left(c^2 - c_1^2\right)}} = \frac{8c_1^2\sqrt{K}}{\left(c + \sqrt{c^2 - c_1^2}\right) 2\sqrt{K\left(c^2 - c_1^2\right)}} = \frac{4c_1^2}{c^2 - c_1^2 + c\sqrt{\left(c^2 - c_1^2\right)}},$$

as defined in (7). □

2.3 Existence proof

The proof relies on an argument in the spirit of concentration-compactness. For given parameters $T > 1$ and $\eta \in \mathbb{R}$, we thus introduce a truncated version of J_P,

$$\begin{aligned} J_{P,T}(u;\eta) := \int_0^1 \int_{\eta - T + s}^{\eta + T - 1 + s} \frac{c^2}{2} \left(u'(\tau)\right)^2 \mathrm{d}\tau\, \mathrm{d}s - \int_{\eta - T}^{\eta + T - 1} V\left(Au(\tau)\right)^2 \mathrm{d}\tau \\ + \int_{\eta - T + \frac{1}{2}}^{\eta + T - \frac{1}{2}} \left[K(1 + \cos(u(\tau))) + F(u(\tau)) \right] \mathrm{d}\tau. \quad (20) \end{aligned}$$

A corresponding concentration-compactness statement is given in Lemma A.1 in the Appendix.

Lemma 2.8 *Under the assumptions of Lemma A.1, J_P possesses a minimiser on $\mathcal{M}_{-\pi,\pi}$.*

Proof: By Lemma 2.6, J_P is bounded from below on $\mathcal{M}_{-\pi,\pi}$. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$ be a minimising sequence. Lemma A.1 implies that a subsequence of $(u_n)_{n\in\mathbb{N}}$, not relabelled, is tight, or vanishes, or splits.

Splitting cannot occur; indeed, as $f_n, g_n \in X$ and $J_P(f_n), J_P(g_n) < \infty$, the analogue statement of Lemma 2.3 (with J replaced by J_P) shows $f_n(\pm\infty) \in \{\pm\pi\}$ and $g_n(\pm\infty) \in \{\pm\pi\}$. Since $f_n + g_n - \pi \in \mathcal{M}_{-\pi,\pi}$, either $f_n(-\infty) = f_n(\infty)$ or $g_n(-\infty) = g_n(\infty)$, but not both. Define $\widetilde{u}_n := g_n$ in the first case and $\widetilde{u}_n := f_n$ in the second case. Then $(\widetilde{u}_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$, and Lemma A.1 (iii) implies that, possibly after passing to a subsequence,

$$\lim_{n\to\infty} J_P(\widetilde{u}_n) < \inf_{u\in\mathcal{M}_{-\pi,\pi}} J_P(u) = \lim_{n\to\infty} J_P(u_n),$$

contradicting the assumption that $(u_n)_{n\in\mathbb{N}}$ is a minimising sequence.

Vanishing cannot occur either; the proof of [7, Lemma 5.1] carries over verbatim.

Hence, for fixed $\varepsilon > 0$, it is possible to choose a sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $T_0 > 0$ such that

$$\left| J_P(u_n) - J_{P,T_0}(u_n;\eta_n) \right| < \varepsilon. \tag{21}$$

We write $w_n(\tau) = u_n(\eta_n + \tau)$. The sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in X because, by (16), $\|w_n'\|_{L^2(\mathbb{R})} = \|u_n'\|_{L^2(\mathbb{R})} \le \frac{2}{c^2-c_1^2} J(u_n)$, and $|w_n(0)| \le \frac{3}{2}\pi + \delta$ by Lemma 2.7.

It follows that $(w_n)_{n\in\mathbb{N}}$ contains a subsequence, not relabelled, which converges weakly to some limit $u \in X$. On $[-T_0, T_0]$, the convergence is uniform, and $\|u'\|_{L^2(-T_0,T_0)} \le \liminf \|w_n'\|_{L^2(-T_0,T_0)}$. Since $V(u)$, $(1+\cos(u))$ and $F(u)$ are $C^1(\mathbb{R})$ and therefore Lipschitz continuous for $|u| \le \frac{3}{2}\pi + \delta$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$

$$\left| \left(J_P(u) - \tfrac{c^2}{2}\|u'\|_{L^2(\mathbb{R})} \right) - \left(J_{P,T_0}(w_n) - \tfrac{c^2}{2}\|u'\|_{L^2(-T_0,T_0)} \right) \right| \le \varepsilon.$$

Since this holds also for all $T > T_0$, it follows with help of Lemma 2.3 that $u \in \mathcal{M}_{-\pi,\pi}$. Moreover, as $T \mapsto J_{P,T}(w_n;0)$ is non-decreasing for each $n \in \mathbb{N}$, we have $J_{P,T}(w_n;0) \le J_P(w_n)$, hence

$$\begin{aligned} J_P(u) = \lim_{T\to\infty} J_{P,T}(u;0) &\le \lim_{T\to\infty} \liminf_{n\to\infty} J_{P,T}(w_n;0) \\ &\le \lim_{T\to\infty} \lim_{n\to\infty} J_P(w_n) = \lim_{n\to\infty} J_P(w_n) = \lim_{n\to\infty} J_P(u_n). \end{aligned}$$

This means that u is a minimiser of J_P on $\mathcal{M}_{-\pi,\pi}$. $\square$

We now come to the proof of Theorem 2.2.

Proof of Theorem 2.2: The assumptions imply, by Lemma 2.8, that the penalised functional J_P possesses a minimiser u_∞ on $\mathcal{M}_{-\pi,\pi}$. With v_0 and Ψ as in Lemma 2.3 and Ψ_P defined analogously to Ψ, it is equivalent to say that $v_\infty := u_\infty - v_0$ minimises Ψ_P on $H^1(\mathbb{R})$. As the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous and $\|u_\infty\|_{L^\infty(\mathbb{R})} = \|v_0 + v_\infty\|_{L^\infty(\mathbb{R})} < \frac{5}{2}\pi$ (Lemma 2.7), we have $\|v_0 + v\|_{L^\infty(\mathbb{R})} < \frac{5}{2}\pi$ for all v in a neighbourhood $\mathcal{V} \subset H^1(\mathbb{R})$ of v_∞. Now for all $v \in \mathcal{V}$,

$$\Psi(v) = J(v_0 + v) = J_P(v_0 + v) = \Psi_P(v)$$

by the remark after (12), so v_∞ is minimises Ψ as well as Ψ_P on $\mathcal{V} \subset H^1(\mathbb{R})$. In particular, v_∞ is a local minimiser of Ψ on $H^1(\mathbb{R})$, and thus a critical point of Ψ. Hence $u_\infty = v_0 + v_\infty$ is, by Lemma 2.3 (iv), a solution of (2) with boundary conditions (4). □

3 Homoclinic travelling waves

We now consider (2) with homoclinic boundary conditions, that is,

$$\lim_{\tau\to-\infty} u(\tau) = \lim_{\tau\to+\infty} u(\tau) = \pi,$$

or equivalently, using $\sin(u+\pi) = -\sin(u)$,

$$\left.\begin{aligned} c^2u''(z) = V'(u(z+1)-u(z)) - V'(u(z)-u(z-1)) + K\sin(u(z)),\\ \lim_{\tau\to-\infty} u(\tau) = \lim_{\tau\to+\infty} u(\tau) = 0. \end{aligned}\right\} \quad (22)$$

The assumptions in this section are

$$\left.\begin{aligned} c > c_0 \geq 0,\\ V(x) = \tfrac{1}{2}c_0^2x^2 + W(x), \text{ where } W(x) = \beta|x|^\alpha, \beta > 0, \alpha \geq 3. \end{aligned}\right\} \quad (23)$$

At the expense of additional technicalities, the results can be generalised to W with $0 \leq \alpha W(x) \leq xW'(x)$. This is essentially the assumption in [5,12] (there without on-site potential). For details see [6]. Also, it is possible to weaken the growth assumption (23) so that it holds only for $|x| \leq \frac{\pi}{2}$. This is explained in the remark preceding Lemma 3.5. For simplicity, however, we assume here that (23) is valid for every $x \in \mathbb{R}$.

The action functional for (22) is $J\colon H^1(\mathbb{R}) \to \mathbb{R}$,

$$J(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2}\Big(u'(\tau)\Big)^2 - V(Au(\tau)) + K(1-\cos(u(\tau)))\right] \mathrm{d}\tau. \quad (24)$$

For some results we will need to consider an auxiliary functional. We set

$f(u) := \frac{1}{2}\left(\max\{0, (|u| - \frac{\pi}{2})\}\right)^2$ and define

$$\widetilde{J}(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2} \left(u'(\tau)\right)^2 - V(Au(\tau)) + K\left(1 - \cos(u(\tau)) + f(u(\tau))\right)\right] \mathrm{d}\tau.$$

Obviously $J(u) = \widetilde{J}(u)$ for all u with $\|u\|_{L^\infty(\mathbb{R})} \leq \frac{\pi}{2}$. Furthermore, $f \in C^1(\mathbb{R})$, $1 - \cos(u) + f(u) \leq \frac{1}{2}u^2$, and there exists a constant $\kappa > 0$, depending only on α, such that for all $u \in \mathbb{R}$

$$\kappa u^2 \leq 1 - \cos(u) + \tfrac{1}{\alpha} u \sin(u) + f(u) - \tfrac{1}{\alpha} u f'(u). \tag{25}$$

Lemma 3.1 *Let* (23) *be satisfied. Then* $J(u) < \infty$ *and* $\widetilde{J}(u) < \infty$ *for all* $u \in H^1(\mathbb{R})$. J *and* $\widetilde{J}$ *are continuously differentiable on* $H^1(\mathbb{R})$, *and the Fréchet derivative of* J *is*

$$\langle J'(u), \varphi\rangle = \int_{\mathbb{R}} \left[c^2\, u'(\tau)\, \varphi'(\tau) - V'(Au(\tau))\,(A\varphi)(\tau) + K \sin(u(\tau))\, \varphi(\tau)\right] \mathrm{d}\tau.$$

If $u_0 \in H^1(\mathbb{R})$ *is a critical point of* J *then* $u_0 \in C^2(\mathbb{R})$, *and* u_0 *is a solution of* (22).

Again, the proof is straightforward (see [6]).

Lemma 3.2 (Mountain pass geometry of J and $\widetilde{J}$) *Let* (23) *hold. Then there exist* $r > 0$ *and* $e \in H^1(\mathbb{R})$ *with* $\|e\|_{H^1(\mathbb{R})} > r$ *such that*

$$\inf_{\|u\|_{H^1(\mathbb{R})} = r} J(u) > J(0) \geq J(e).$$

The same holds for $\widetilde{J}$.

Proof: Let $\varepsilon > 0$ such that $0 < \frac{1}{2}(c^2 - c_0^2 - 2\varepsilon)$ and choose $r \in \left(0, \frac{\pi}{2}\right)$ small enough such that $|W(x)| \leq \varepsilon x^2$ for all $x \leq r$. Using (8) and the estimate $1 - \cos(u) \geq \frac{1}{4}u^2$ which holds for $|u| < \frac{\pi}{2}$, we obtain for every u with $\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})} \leq r < \frac{\pi}{2}$

$$\begin{aligned} J(u) &\geq \int_{\mathbb{R}} \left[\frac{c^2}{2} \left(u'(\tau)\right)^2 - \frac{c_0^2}{2} |Au(\tau)|^2 - \varepsilon |Au(\tau)|^2 + K\left(1 - \cos(u(\tau))\right)\right] \mathrm{d}\tau \\ &\geq \tfrac{1}{2}\left(c^2 - c_0^2 - 2\varepsilon\right) \|u'\|_{L^2(\mathbb{R})}^2 + \tfrac{1}{4} K \|u\|_{L^2(\mathbb{R})}^2 \\ &\geq \min\left\{\tfrac{1}{2}\left(c^2 - c_0^2 - 2\varepsilon\right), \tfrac{1}{4}K\right\} \cdot \|u\|_{H^1(\mathbb{R})}^2, \end{aligned}$$

hence for $\|u\|_{H^1(\mathbb{R})}^2 = r^2$

$$J(u) \geq r^2 \cdot \min\left\{\tfrac{1}{2}(c^2 - c_0^2 - 2\varepsilon), \tfrac{1}{4}K\right\} =: m > 0. \tag{26}$$

To find $e \in H^1(\mathbb{R})$ with $\|e\| > r$ and $J(e) \leq J(0)$, fix an arbitrary $u_0 \in H^1(\mathbb{R})$. Then for all $\lambda \geq 0$, due to $1 - \cos(u) \leq \frac{1}{2}u^2$,

$$J(\lambda u_0) \leq \tfrac{1}{2}c^2\lambda^2 \|u_0'\|^2_{L^2(\mathbb{R})} - \beta\lambda^\alpha \|Au_0\|^\alpha_{L^\alpha(\mathbb{R})} + \tfrac{1}{2}K\lambda^2 \|u_0\|^2_{L^2(\mathbb{R})},$$

hence, due to $\alpha > 2$ and $\beta > 0$ and the special form of W,

$$\lim_{\lambda\to\infty} J(\lambda u_0) = -\infty.$$

This establishes the claim for J. The proof for $\widetilde{J}$ is almost identical. □

Lemma 3.3 (Palais-Smale sequence) *Let* (23) *hold. There exists a Palais-Smale sequence, that is, a sequence* $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ *such that for* $n \to \infty$

$$J(u_n) \to d \quad \text{and} \quad \|J'(u_n)\|_{\mathcal{L}(H^1(\mathbb{R}),\mathbb{R})} \to 0,$$

for some

$$d \in [m, M] \tag{27}$$

with constants $m, M \in \mathbb{R}$ *that can be determined explicitly. The same holds for* $\widetilde{J}$*, with the same constants* M*,* m*.*

Proof: The existence of a Palais-Smale sequence follows from the Mountain Pass Theorem [13, Theorem 1.15], which can be applied as a consequence of Lemma 3.2. The lower bound m can be taken as in (26). For the upper bound M we can fix some function $u_0 \in H^1(\mathbb{R})$ and set $M := \max_{\lambda\in(0,\infty)} J(\lambda_0 u_0)$. The proof for $\widetilde{J}$ is again almost identical. □

Lemma 3.4 (Boundedness of Palais-Smale sequences) *Suppose that assumption* (23) *holds and suppose further that there exists* $\varepsilon_0 > 0$ *such that the parameters satisfy*

$$\sqrt{\frac{2M}{\min\left\{\frac{\alpha-2}{2\alpha}(c^2 - c_0^2), \kappa K\right\}}} < \frac{\pi}{2} - \varepsilon_0, \tag{28}$$

with M *as in Lemma 3.3 and* κ *as in* (25)*. Then the following holds. A Palais-Smale sequence* $(u_n)_{n\in\mathbb{N}}$ *for* J *with the properties stated in Lemma 3.3 is bounded in* $H^1(\mathbb{R})$*, and*

$$\|u_n\|_{L^\infty(\mathbb{R})} \leq \frac{\pi}{2} - \varepsilon_0.$$

for all $n \in \mathbb{N}$ *large enough. The same holds for* $\widetilde{J}$*.*

Proof: We consider $\widetilde{J}$ first. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence with the properties stated in Lemma 3.3. Choose $\delta > 0$ and let $n_0 \in \mathbb{N}$ be large enough such that

$$\widetilde{J}(u_n) < M + \delta \text{ and } \frac{1}{\alpha}\left\|\langle \widetilde{J}'(u_n), u_n\rangle\right\| \leq \delta \|u_n\|_{H^1(\mathbb{R})}$$

for all $n \geq n_0$.

Then for all $n > n_0$, estimating the first summand in the second line with (8) and (23), the second summand with $W(x) \leq \frac{1}{\alpha} x W'(x)$, and for the third summand with (25) (see [12,11] for a similar argument),

$$\begin{aligned}
&(M+\delta) + \delta \left\|u_n\right\|_{H^1(\mathbb{R})} \geq \widetilde{J}\left(u_n\right) - \frac{1}{\alpha}\langle \widetilde{J}'\left(u_n\right), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\alpha}\right)\left(c^2 \left\|u_n'\right\|_{L^2(\mathbb{R})}^2 - c_0^2 \left\|Au_n\right\|_{L^2(\mathbb{R})}^2\right) \\
&\quad + \int_{\mathbb{R}} \left[-W\left(Au_n\right) + \tfrac{1}{\alpha} Au_n W'\left(Au_n\right)\right] \mathrm{d}\tau \\
&\quad + K \int_{\mathbb{R}} \left[1 - \cos(u) - \frac{1}{\alpha} u \sin(u) + f(u) - \frac{1}{\alpha} u f'(u)\right] \mathrm{d}\tau \\
&\geq \frac{\alpha - 2}{2\alpha}\left(c^2 - c_0^2\right) \left\|u_n'\right\|_{L^2(\mathbb{R})}^2 + 0 + \kappa K \left\|u_n\right\|_{L^2(\mathbb{R})}^2 \\
&\geq \min\left\{\tfrac{\alpha-2}{2\alpha}\left(c^2 - c_0^2\right), \kappa K\right\} \cdot \left\|u_n\right\|_{H^1(\mathbb{R})}^2,
\end{aligned}$$

thus

$$0 \leq (M+\delta) + \delta \left\|u_n\right\|_{H^1(\mathbb{R})} - \min\left\{\tfrac{\alpha-2}{2\alpha}\left(c^2 - c_0^2\right), \kappa K\right\} \left\|u_n\right\|_{H^1(\mathbb{R})}^2.$$

This implies by direct calculation

$$\begin{aligned}
\left\|u_n\right\|_{H^1(\mathbb{R})} \leq \frac{1}{2 \cdot \min\left\{\frac{\alpha-2}{2\alpha}\left(c^2 - c_0^2\right), \kappa K\right\}} \cdot \\
\cdot \left(\delta + \sqrt{\delta^2 + 4(M+\delta)\, \min\left\{\tfrac{\alpha-2}{2\alpha}\left(c^2 - c_0^2\right), \kappa K\right\}}\right).
\end{aligned}$$

Therefore, since $\sqrt{\delta^2 + \xi^2} \leq |\delta| + |\xi|$,

$$\left\|u_n\right\|_{H^1(\mathbb{R})} \leq \sqrt{\frac{M+\delta}{\min\left\{\frac{\alpha-2}{2\alpha}\left(c^2 - c_0^2\right), \kappa K\right\}}} + 2 \cdot \frac{\delta}{2 \min\left\{\frac{\alpha-2}{2\alpha}\left(c^2 - c_0^2\right), \kappa K\right\}}.$$

By assumption (28), the right-hand side is bounded by $\frac{\pi}{2} - \varepsilon_0$ for δ small enough. This shows for all $n \geq n_0$

$$\left\|u_n\right\|_{L^\infty(\mathbb{R})} \leq \left\|u_n\right\|_{H^1(\mathbb{R})} \leq \tfrac{\pi}{2} - \varepsilon_0,$$

as claimed. The statement for J is an immediate consequence of the statement for $\widetilde{J}$ since $J(u) = \widetilde{J}(u)$ for all $u \in H^1(\mathbb{R})$ with $\|u\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}$. $\square$

We remark that the parameters $(c, c_0, \alpha, \beta, K)$ can be chosen such that there exists an $\varepsilon_0 > 0$ with the property that (28) holds. In other words, the assumptions of Lemma 3.4 can be satisfied.

Lemma 3.4 shows that the growth of the interaction potential at infinity as required in (23) is not essential to the argument. Namely, a critical point

$u \in B := \left\{u \in H_1(\mathbb{R}) : \|u\|_{H^1(\mathbb{R})} < \frac{\pi}{2}\right\}$ of J is also a critical point of any functional that agrees with J on B. Now if V is an interaction potential satisfying (23) for all $x \in \mathbb{R}$ and if V_1 is a potential with $V_1(x) = V(x)$ for $|x| \leq \frac{\pi}{2}$ only then, by (8), $V_1(Au(\tau)) = V(Au(\tau))$ for all $u \in B$ and all $\tau \in \mathbb{R}$; thus the action functionals with V_1 and V as interaction potentials agree on B.

Lemma 3.5 *Under the assumptions of Lemma 3.4, a Palais-Smale sequence $(u_n)_{n\in\mathbb{N}}$ for J with the properties of Lemma 3.3 does not converge to zero in measure.*

Proof: By Lemma 3.4, $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R})$, hence $(Au_n)_{n\in\mathbb{N}}$ is also bounded in $H^1(\mathbb{R})$ and in $L^\infty(\mathbb{R})$; more precisely, by (8),

$$\max\left\{\|Au_n\|_{L^2(\mathbb{R})}, \|Au_n\|_{L^\infty(\mathbb{R})}\right\} \leq \sup_{n\in\mathbb{N}} \|u_n\|_{H^1(\mathbb{R})} \leq C_1 := \frac{\pi}{2} - \varepsilon_0.$$

By assumption, $\frac{1}{2}W'(x)x - W(x) = \left(\frac{\alpha}{2} - 1\right)\beta\,|x|^\alpha$, so $x^{-2}\left(\frac{1}{2}W'(x)x - W(x)\right)$ $\to 0$ for $x \to 0$. Likewise, $u^{-2}\left(1 - \cos(u) - \frac{1}{2}u\sin(u)\right) \to 0$. Hence there exists $C_2 > 0$ such that

$$\sup_{|x|\leq C_1} \frac{\frac{1}{2}W'(x)x - W(x)}{x^2} \leq C_2 \quad \text{and} \quad \sup_{|u|\leq C_1} \frac{1 - \cos(u) - \frac{1}{2}u\sin(u)}{u^2} \leq C_2.$$

Given $\varepsilon > 0$, there exists, for the same reason (see [6,12]), $\delta > 0$ such that for all $|x| < \delta$

$$\left|\tfrac{1}{2}W'(x)x - W(x)\right| \leq \varepsilon x^2 \quad \text{and} \quad \left|1 - \cos(x) - \tfrac{1}{2}x\sin(x)\right| \leq \varepsilon x^2.$$

Then, due to $\frac{1}{2}W'(x)x - W(x) = \left(\frac{\alpha}{2} - 1\right)\beta\,|x|^\alpha \geq 0$,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} \left[\tfrac{1}{2}W'\left(Au_n\right)Au_n - W\left(Au_n\right)\right] \mathrm{d}\tau \\ &\leq \left|\{\tau \in \mathbb{R} : |Au_n(\tau)| > \delta\}\right| \cdot C_2 \|Au_n\|^2_{L^\infty(\mathbb{R})} + \varepsilon \|Au_n\|^2_{L^2(\mathbb{R})} \\ &\leq C_1^2 \left(\left|\{\tau \in \mathbb{R} : |Au_n(\tau)| > \delta\}\right| \cdot C_2 + \varepsilon\right) \end{aligned}$$

and, due to $\|u_n\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}$,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} \left[1 - \cos(u_n) - \tfrac{1}{2}u_n \sin\left(u_n\right)\right] \mathrm{d}\tau \\ &\leq \left|\{\tau \in \mathbb{R} : |u_n(\tau)| > \delta\}\right| \cdot C_2 \|u_n\|^2_{L^\infty(\mathbb{R})} + \varepsilon \|u_n\|^2_{L^2(\mathbb{R})} \\ &\leq C_1^2 \left(\left|\{\tau \in \mathbb{R} : |u_n(\tau)| > \delta\}\right| \cdot C_2 + \varepsilon\right). \end{aligned}$$

For all n large enough, $\left|\langle J'\left(u_n\right), u_n\rangle\right| < \frac{m}{4}$ and $J\left(u_n\right) - \frac{3}{4}m > 0$. Therefore for ε small enough, we find with Lemma 3.1 (using the preceding estimates and

observing that all quadratic terms cancel)

$$\begin{aligned}
0 < \frac{m}{2} &\leq J\left(u_n\right) - \tfrac{1}{2}\langle J'\left(u_n\right), u_n\rangle \\
&= \int_{\mathbb{R}} \left[\tfrac{1}{2}W'(Au_n)Au_n - W(Au_n)\right] \mathrm{d}\tau + K\int_{\mathbb{R}} \left[1 - \cos(u_n) - \tfrac{1}{2}u_n \sin(u_n)\right] \mathrm{d}\tau \\
&\leq C_1^2 \left(\left|\{\tau \in \mathbb{R} : |Au_n(\tau)| > \delta\}\right| \cdot C_2 + \varepsilon\right) \\
&\quad + KC_1^2 \left(\left|\{\tau \in \mathbb{R} : |u_n(\tau)| > \delta\}\right| \cdot C_2 + \varepsilon\right) \\
&= \left(\left|\{\tau \in \mathbb{R} : |Au_n(\tau)| > \delta\}\right| + \left|\{\tau \in \mathbb{R} : |u_n(\tau)| > \delta\}\right|\right) C_1^2 C_2 K \\
&\quad + (1+K)\varepsilon C_1^2.
\end{aligned}$$

If u_n converges to zero in measure, then $Au_n \to 0$ in measure as well (see the proof of Lemma 2.6). In this case, the right-hand side can be made arbitrarily small. But m from (26) is a positive constant, so convergence to zero in measure would lead to a contradiction. □

Theorem 3.6 *Let* (23) *be satisfied and suppose the parameters are such that there exists* $\varepsilon_0 > 0$ *such that the parameters satisfy* (28)*, that is,*

$$\sqrt{\frac{2M}{\min\left\{\frac{\alpha-2}{2\alpha}\left(c^2 - c_0^2\right), \kappa K\right\}}} < \tfrac{\pi}{2} - \varepsilon_0,$$

with M *as in Lemma 3.3 and* κ *as in* (25)*. Then* (22) *possesses a nonconstant solution* $u \in C^2(\mathbb{R})$*.*

Proof: Lemma 3.3 provides a Palais-Smale sequence which is bounded by Lemma 3.4 and does not converge to zero in measure by Lemma 3.5. Hence, by the Lieb-Brezis Lemma [8, Lemma 6], there exist a subsequence of $(u_n)_{n\in\mathbb{N}}$ (not relabelled) and a sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that weakly in $H^1(\mathbb{R})$

$$w_n := u_n\left(\,\cdot\, + \eta_n\right) \rightharpoonup u \neq 0.$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ be a test function. Weak convergence $w_n \rightharpoonup u$ in $H^1(\mathbb{R})$ implies weak convergence $w_n \rightharpoonup u$ in $L^2(\mathbb{R})$, therefore

$$\int_{\mathbb{R}} c^2 w_n'(\tau)\varphi'(\tau)\,\mathrm{d}\tau \longrightarrow \int_{\mathbb{R}} c^2\, u'(\tau)\varphi'(\tau)\,\mathrm{d}\tau.$$

The convergence $Aw_n \to Au$ is strong in $C^0(\mathrm{supp}(A\varphi))$, and as V' is uniformly continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the maximal range of w_n by Lemma 3.4), so the second term of J converges as well,

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}} V'\left(Aw_n(\tau)\right) A\varphi(\tau)\,\mathrm{d}\tau &= \lim_{n\to\infty} \int_{\mathrm{supp}(A\varphi)} V'\left(Aw_n(\tau)\right)\, A\varphi(\tau)\,\mathrm{d}\tau \\
&= \int_{\mathrm{supp}(A\varphi)} V'(Au(\tau))\, A\varphi(\tau)\,\mathrm{d}\tau = \int_{\mathbb{R}} V'(Au(\tau))\, A\varphi(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

The argument for the term from the on-site potential is similar and yields

$$\lim_{n\to\infty}\int_{\mathbb{R}}\sin\left(w_n(\tau)\right)\,\varphi(\tau)\,\mathrm{d}\tau=\int_{\mathbb{R}}\sin(u(\tau))\,\varphi(\tau)\,\mathrm{d}\tau.$$

Altogether for all $\varphi\in C_0^\infty(\mathbb{R})$, $\langle J'\left(w_n\right),\varphi\rangle\to\langle J'(u),\varphi\rangle$ for $n\to\infty$, so

$$\begin{aligned}\left|\langle J'(u),\varphi\rangle\right|&=\lim_{n\to\infty}\left|\langle J'(w_n),\varphi\rangle\right|=\lim_{n\to\infty}\left|\langle J'\left(u_n(\,\cdot+\eta_n)\right),\varphi\rangle\right|\\&=\lim_{n\to\infty}\left|\langle J'\left(u_n\right),\varphi\left(\,\cdot-\eta_n\right)\rangle\right|\le\lim_{n\to\infty}\|J'(u_n)\|_{\mathcal{L}(H^1(\mathbb{R}),\mathbb{R})}\;\|\varphi\|_{H^1(\mathbb{R})}=0,\end{aligned}$$

thus $\langle J'(u),\varphi\rangle=0$ for all $\varphi\in C_0^\infty(\mathbb{R})$. By density, this shows that u is a critical point of J. The theorem follows now directly from Lemma 3.1. □

4 Periodic travelling waves

This section considers travelling wave solutions for a system as in (1) with a periodicity condition. Specifically, we study

$$\begin{aligned}&\ddot{q}_k(t)=V'\left(q_{k+1}(t)-q_k(t)\right)-V'\left(q_k(t)-q_{k-1}(t)\right)+K\sin\left(q_k(t)\right),\\&q_k(t)=q_{k+N}(t),\qquad k\in\mathbb{Z},\end{aligned}$$

for some fixed $N\in\mathbb{N}$. Inserting the travelling wave ansatz $q_k(t)=u(ct+k)$ and setting $T:=\frac{N}{c}$, we obtain

$$\left.\begin{aligned}c^2u''(\tau)=V'\left(u(\tau+1)-u(\tau)\right)-V'\left(u(\tau)-u(\tau-1)\right)+K\sin(u(\tau)),\\u(\tau)=u(\tau+T)\qquad\text{for every }\tau\in\mathbb{R}.\end{aligned}\right\}\tag{29}$$

Assumption 4.1 *Let $c>c_0\ge 0$ and $T>3$. Suppose further $V(x)=\frac{1}{2}c_0^2x^2+W(x)$, where $W\in C^1(\mathbb{R})$ is even, $W\not\equiv 0$ and $0\le\alpha W(x)\le xW'(x)$ for all $x\in\mathbb{R}$ and some $\alpha>2$.*

The following existence result can be stated for periodic solutions.

Theorem 4.2 *Let Assumption 4.1 be satisfied and suppose that $d<2KT$ for d as defined in (32) below. Then (29) possesses a nonconstant periodic solution $u\in H^1_{\mathrm{per}}(0,T)$ with period T and $J(u)=d$.*

The solution is constructed as critical point of $J\colon H^1_{\mathrm{per}}(0,T)\to\mathbb{R}$,

$$J(u):=\int_0^T\left[\frac{c^2}{2}\Big(u'(\tau)\Big)^2-V(Au(\tau))+K\Big(1-\cos(u(\tau))\Big)\right]\mathrm{d}\tau.\tag{30}$$

Similarly to Lemma 3.1 and Lemma 3.2, it is shown that J is continuously differentiable, that its Euler-Lagrange equation is indeed given by (29), and that J possesses the mountain pass geometry. The proof of Theorem 4.2 turns out to be much easier than the proof of Theorem 3.6 because the present functional satisfies a Palais-Smale condition.

Lemma 4.3 *Under the assumptions of Theorem 4.2, every Palais-Smale sequence for J, that is, every sequence $(u_n)_{n\in\mathbb{N}} \subset H^1_{\mathrm{per}}(0,T)$ with*

$$J(u_n) \to d \in \mathbb{R} \quad \textit{and} \quad \|J'(u_n)\|_{\mathcal{L}(H^1_{\mathrm{per}}(0,T),\mathbb{R})} \to 0,$$

contains a strongly converging subsequence.

Proof: The proof of Lemma 3.4 shows with a few obvious modifications (see [6]) that $(u_n)_{n\in\mathbb{N}}$ is bounded. Since $H^1_{\mathrm{per}}(0,T)$ is reflexive a weakly converging subsequence of $(u_n)_{n\in\mathbb{N}}$ exists, not relabelled, such that $u_n \rightharpoonup u \in H^1_{\mathrm{per}}(0,T)$. By standard embedding theorems for Sobolev spaces, this convergence is strong in $L^2(0,T)$ and in $C^0[0,T]$.

Since $\|J'(u_n)\|_{\mathcal{L}(H^1_{\mathrm{per}}(0,T),\mathbb{R})} \to 0$ by assumption, we have $\langle J'(u_n), u\rangle \to 0$ for $n \to \infty$ and, as $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1_{\mathrm{per}}(0,T)$, also $\langle J'(u_n), u_n\rangle \to 0$. The weak convergence $u_n \rightharpoonup u$ implies $\langle J'(u), u_n - u\rangle \to 0$ because $J'(u)$ is a continuous linear functional on $H^1_{\mathrm{per}}(0,T)$. Thus altogether

$$\Big\langle J'(u_n) - J'(u), u_n - u \Big\rangle \to 0. \tag{31}$$

As $V \in C^1(\mathbb{R})$, V is Lipschitz-continuous on compact intervals. Therefore,

$$\begin{aligned} 0 &\le \left| \int_0^T \Big[V(Au_n) - V(Au)\Big] \Big[Au_n - Au\Big] \,\mathrm{d}\tau \right| \\ &\le C \int_0^T \Big[Au_n(\tau) - Au(\tau)\Big]^2 \,\mathrm{d}\tau \le 2C \|u_n - u\|^2_{L^2(0,T)} \longrightarrow 0, \end{aligned}$$

where $C > 0$ is a constant. Similarly, the strong convergence $u_n \to u$ in $L^2(0,T)$ implies $\int_0^T \Big[\sin(u_n(\tau)) - \sin(u(\tau))\Big] \Big[u_n(\tau) - u(\tau)\Big] \,\mathrm{d}\tau \to 0$. Now

$$\begin{aligned} \Big\langle J'(u_n) - J'(u), u_n - u \Big\rangle = \frac{c^2}{2} \int_0^T \Big[u_n' - u'\Big]^2 \,\mathrm{d}\tau - \int_0^T \Big[V(Au_n) - V(Au)\Big] \\ \cdot \Big[Au_n - Au\Big] \,\mathrm{d}\tau + K \int_0^T \Big[\sin(u_n(\tau)) - \sin(u(\tau))\Big] \Big[u_n(\tau) - u(\tau)\Big] \,\mathrm{d}\tau, \end{aligned}$$

and all terms on the right-hand side except $\int_0^T [u_n' - u']^2 \,\mathrm{d}\tau$ have been shown to converge to 0 for $n \to \infty$. We conclude then with (31) that

$$\|u_n' - u'\|^2_{L^2(0,T)} = \int_0^T \Big[u_n' - u'\Big]^2 \,\mathrm{d}\tau \;\longrightarrow\; 0,$$

i.e., $u_n' \to u'$ in $L^2(0,T)$. This shows strong convergence in $H^1_{\mathrm{per}}(0,T)$. $\square$

Proof of Theorem 4.2: The existence of a critical point $u_0 \in H^1_{\mathrm{per}}(0,T)$ of J follows from the Mountain Pass Theorem [13, Theorem 1.17], which is applicable because J possesses the mountain pass geometry (cf. the remark after (30)) and satisfies a Palais-Smale condition (Lemma 4.3). According to this theorem, the critical value of J is given by

$$J(u_0) = d := \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} J(\gamma(s)), \tag{32}$$

where $\Gamma := \{\gamma \in C^0([0,1], H^1(0,T)) : \gamma(0) = 0, \gamma(1) = e\}$, with e as in Lemma 3.2. The condition on d ensures that this solution is not constant. Indeed, a constant solution u to (29) satisfies necessarily $\sin(u) \equiv 0$, hence $u \equiv k\pi$ for some $k \in \mathbb{Z}$. If k is even then $J(k\pi) = 0$ but $d > 0$. If k is odd then $J(k\pi) = 2KT$ but $d < 2KT$ by assumption. $\square$

Acknowledgements

CFK was funded by the MPI for Mathematics in the Sciences, its IMPRS, a DAAD short research grant (D/05/44843), the DFG Priority Program 1095, the University of Bath and an Oberwolfach Leibniz Fellowship. JZ gratefully acknowledges the financial support of the Deutsche Forschungsgemeinschaft through an Emmy Noether grant (Zi 751/1-1), and the EPSRC through an Advanced Research Fellowship (GR/S99037/1).

A Concentration-compactness

The following lemma is in the spirit of Lions' well-known concentration-compactness principle [9]. However, as the constraint $u(\infty) - u(-\infty) = 2\pi$ cannot be varied continuously, there is no meaningful analogue to the subadditivity inequality as required in the classic setting. Accordingly, the third alternative (splitting) refers to splitting the value of the functional, not of the constraint. For a discussion see [6,7].

Lemma A.1 (Concentration-compactness) *Let Assumption 2.1 be satisfied and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$ be a minimising sequence for J_P as in (12) in $\mathcal{M}_{-\pi,\pi}$. Suppose c is large enough to ensure $\delta < \pi$ for δ as defined in (7).*

Then a subsequence exists, still denoted by $(u_n)_{n\in\mathbb{N}}$, which satisfies one of the following three alternatives:

(i) Tightness: *There is a sequence* $(\eta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *such that for all small enough* $\varepsilon > 0$ *there exists* $T > 0$ *such that, with* $J_{P,T}$ *from* (20),

$$|J_P(u_n) - J_{P,T}(u_n;\eta_n)| < \varepsilon \text{ for every } n \in \mathbb{N}.$$

(ii) Vanishing: *For all* $T > 0$,

$$\lim_{n\to\infty} \sup_{\eta\in\mathbb{R}} J_{P,T}(u_n;\eta) = 0.$$

(iii) Splitting: *There exists* $\varepsilon_1 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_1$, *there are* $f_n, g_n \in X$ *such that*

$$\begin{gathered} |u_n - (f_n + g_n - \pi)| \le \varepsilon \\ |J_P(u_n) - (J_P(f_n) + J_P(g_n))| \le \varepsilon, \quad \lim_{n\to\infty} \operatorname{dist}(\operatorname{supp}(f_n'), \operatorname{supp}(g_n')) = \infty, \\ \lim_{n\to\infty} J_P(f_n) = \alpha, \quad \lim_{n\to\infty} J_P(g_n) = \beta, \end{gathered}$$

for some $0 < \alpha, \beta < \inf J_P|_{\mathcal{M}_{-\pi,\pi}}$. *($\pi$ is needed in the first inequality to ensure* $J_P(f_n) < \infty$ *and* $J_P(g_n) < \infty$.*)*

The proof is almost verbatim the same as the one given in [7] for a similar statement. We thus omit the proof and refer to [6,7] for the details. We remark that the statement in Lemma A.1 is weaker than that of the concentration-compactness statement in [7]; indeed, here the statement holds only for minimising sequences, while the statement in [7] requires only a uniform bound for $(J(u_n))_{n\in\mathbb{N}}$.

References

[1] G. Arioli, F. Gazzola, S. Terracini, Multibump periodic motions of an infinite lattice of particles, Math. Z. 223 (4) (1996) 627–642.

[2] G. Arioli, A. Szulkin, Periodic motions of an infinite lattice of particles: the strongly indefinite case, Ann. Sci. Math. Québec 22 (2) (1998) 97–119, dedicated to the memory of Gilles Fournier (Sherbrooke, PQ, 1997).

[3] P. W. Bates, C. Zhang, Traveling pulses for the Klein-Gordon equation on a lattice or continuum with long-range interaction, Discrete Contin. Dyn. Syst. 16 (1) (2006) 235–252.

[4] A. Braides, Γ-convergence for beginners, vol. 22 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2002.

[5] G. Friesecke, J. A. D. Wattis, Existence theorem for solitary waves on lattices, Comm. Math. Phys. 161 (2) (1994) 391–418.

[6] C.-F. Kreiner, A variational approach to travelling waves in particle chains with on-site potential, Ph.D. thesis, Max Planck Institute for Mathematics in the Sciences / University of Leipzig (2007).

[7] C.-F. Kreiner, J. Zimmer, Heteroclinic travelling waves for the lattice sine-Gordon equation with linear pair interaction, submitted (2008).

[8] E. H. Lieb, On the lowest eigenvalue of the Laplacian for the intersection of two domains, Invent. Math. 74 (3) (1983) 441–448.

[9] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (2) (1984) 109–145.

[10] L. Modica, S. Mortola, Un esempio di Γ^--convergenza, Boll. Un. Mat. Ital. B (5) 14 (1) (1977) 285–299.

[11] H. Schwetlick, J. Zimmer, Solitary waves for nonconvex FPU lattices, J. Nonlinear Sci. 17 (1) (2007) 1–12.

[12] D. Smets, M. Willem, Solitary waves with prescribed speed on infinite lattices, J. Funct. Anal. 149 (1) (1997) 266–275.

[13] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston Inc., Boston, MA, 1996.