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HETEROCLINIC TRAVELLING WAVES FOR THE LATTICE SINE-GORDON EQUATION WITH LINEAR PAIR INTERACTION

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Abstract. The existence of travelling heteroclinic waves for the sine-Gordon lattice is proved for a linear interaction of neighbouring atoms. The asymptotic states are chosen such that the action functional is finite. The proof relies on a suitable concentration-compactness argument, which can be shown to hold even though the associated functional has no sub-additive structure.

1. **Introduction.** We consider the lattice sine-Gordon equation

$$\ddot{q}_k(t) = V'\left(q_{k+1}(t) - q_k(t)\right) - V'\left(q_k(t) - q_{k-1}(t)\right) - K\sin\left(q_k(t)\right), \qquad k \in \mathbb{Z}, \quad (1)$$

with a constant $K > 0$. Equation (1) describes the evolution of an infinite chain of atoms with elastic nearest neighbour interaction and an on-site potential, according to Newton's law. The interaction potential $V\colon \mathbb{R} \to \mathbb{R}$ takes as argument the discrete strain, which is given by the difference of the positions of the atoms $q_{k+1}(t) - q_k(t)$. In this article, we assume that V is a quadratic function $V(\varepsilon) := \frac{c_0^2}{2}\varepsilon^2$ with $c_0 > 0$ and seek a solution to (1) in the form of a travelling wave by setting $q_k(t) = u(k - ct)$ for $k \in \mathbb{Z}$. Then, a substitution into (1) yields immediately

$$c^2 u''(\tau) = c_0^2(u(\tau+1) - 2u(\tau) + u(\tau-1)) - K\sin(u(\tau)). \quad (2)$$

In the setting introduced in Section 2, Equation (2) can be seen to be the Euler-Lagrange equation of the action functional

$$J(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2}\left(u'(\tau)\right)^2 - \frac{c_0^2}{2}(u(\tau+1) - u(\tau))^2 + K(1 + \cos(u(\tau)))\right] \mathrm{d}\tau. \quad (3)$$

The action functional is the kinetic energy $\int_{\mathbb{R}} \frac{c^2}{2}\left(u'(\tau)\right)^2 \mathrm{d}\tau$ minus the potential energy, consisting of interaction part, $\int_{\mathbb{R}} \frac{c_0^2}{2}(u(\tau+1) - u(\tau))^2 \mathrm{d}\tau$, and on-site part, $\int_{\mathbb{R}} -K(1 + \cos(u(\tau))) \mathrm{d}\tau$. This specific choice of the on-site potential is made for simplicity of the presentation; however, all results in this paper can be generalised in a straightforward way to any non-negative, 2π-periodic C^1-function with zero set $\{(2k+1)\pi : k \in \mathbb{N}\}$ in place of $(1 + \cos(\cdot))$.

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We are interested in heteroclinic waves (that is, waves that connect two different asymptotic states at $\pm\infty$) for supersonic velocities $c > c_0$. Before stating the precise results, we give a brief overview of some related work.

Bates and Zhang [2] have shown that for a large class of similar models, homoclinic travelling waves exist for supersonic velocities. Their existence result also holds for long-range interaction, but the specialisation to nearest neighbour interaction covers the case

$$c^2 u''(\tau) = c_0^2(u(\tau+1) - 2u(\tau) + u(\tau-1)) + K\sin(u(\tau)). \tag{4}$$

The on-site potential energy can here be taken to be $\int_{\mathbb{R}} [K\cos(u(\tau)) - 1]\, \mathrm{d}\tau$. Bates and Zhang [2] consider homoclinic waves that have their asymptotic states in the maximum of the on-site potential. We study the analogous situation for heteroclinic waves. That is, we consider waves with asymptotic states in two different maxima of the on-site potential. For the choice $-K(1+\cos(u(\tau)))$ made above for the on-site potential, this leads to the boundary conditions

$$\lim_{\tau\to-\infty} u(\tau) = -\pi \qquad \text{and} \qquad \lim_{\tau\to+\infty} u(\tau) = +\pi. \tag{5}$$

The existence proof will rely on minimisation and a novel type of concentration-compactness. The main difficulties are: Firstly, the action functional, which is to be minimised, is highly nonconvex due to the periodicity of the on-site potential. The second challenge is a lack of compactness due to the infinite domain $\mathbb{R}$. We show in Section 4 that these difficulties can be overcome with a suitable variant of concentration-compactness [9]. This is not obvious, since the functional (3) is not subadditive. We show that a concentration-compactness result holds nevertheless. This argument relies on the fact that the lattice action functional (3) can be related to the Mortola-Modica functional [10], so that a crucial L^∞-*a-priori* bound can be inferred. This connection to the Mortola-Modica functional is made explicit in Section 3. Concentration-compactness arguments for lattice models were introduced by Friesecke and Wattis [6] (see also, e.g., [1]).

These two difficulties, namely a highly nonconvex functional and lack of compactness also persist for other boundary conditions, in particular

$$\lim_{\tau\to-\infty} u(\tau) = 0 \qquad \text{and} \qquad \lim_{\tau\to+\infty} u(\tau) = 2\pi, \tag{6}$$

that is, asymptotic states in the minima of the on-site potential (possibly to be understood in an averaged sense). These boundary conditions correspond to a moving dislocation in the Frenkel-Kontorova model [5]. The existence of periodic solutions and sliding solutions for the two-dimensional generalisation of the Frenkel-Kontorova model can be shown with topological and variational methods [4]. A survey over some related results can be found in the book by Pankov [11]. For the one-dimensional Frenkel-Kontorova model, there are existence results for heteroclinic waves with asymptotic states (6) for the special case of a piecewise quadratic on-site potential in the physics literature [8]. There, it is assumed that the solution satisfies the sign condition of the kind

$$u(\tau) < \pi \text{ for } \tau < 0 \qquad \text{and} \qquad u(\tau) > \pi \text{ for } \tau > 0. \tag{7}$$

Under this assumption, the analogue of the Euler-Lagrange equation (2) for piecewise on-site potential simplifies to an equation with a nonlinearity that depends only on τ, rather than $u(\tau)$. This simplified system is then solved by Fourier methods, where the solution is represented as a sum of Fourier components. The difficulty is to show that the solution satisfies the sign condition (7). Kresse and Truskinovsky [8] observe that this condition probably does not hold for a specific interval of subsonic velocities. A rigorous proof that the sign condition holds in some regime seems, at the time of writing, only to be available for the Fermi-Pasta-Ulam chain with piecewise quadratic pair interaction [12]. The extension of this result to more general potentials is an open problem.

2. **Main result.** We set $X := \big\{u \in H^1_{\text{loc}}(\mathbb{R}) : u' \in L^2(\mathbb{R})\big\}$ and remark that X is a Hilbert space when equipped with the inner product

$$\langle u, v\rangle_X := u(0)\, v(0) + \int_{\mathbb{R}} u'(\tau)\, v'(\tau)\ \mathrm{d}\tau.$$

Further, let us define

$$\mathcal{M}_{-\pi,\pi} := \{u \in X : u(-\infty) = -\pi, u(\infty) = \pi\}. \tag{8}$$

We are now in a position to formalise the connection of Equation (2) and the action functional $J\colon X \to \mathbb{R} \cup \{\infty\}$ given in (3).

Let $v_0\colon \mathbb{R} \to [-\pi, \pi]$ be a monotone function in $C^\infty(\mathbb{R})$ such that $v_0(\tau) = -\pi$ for $\tau < -1$ and $v_0(\tau) = \pi$ for $\tau > 1$ and define $\Psi\colon H^1(\mathbb{R}) \to \mathbb{R}$ by

$$\Psi(v) := J(v_0 + v).$$

It is not hard to see that $\Psi(v) < \infty$ for all $v \in H^1(\mathbb{R})$, and that, conversely, a minimiser u of J on $\mathcal{M}_{-\pi,\pi}$ can be written as $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$ (for details see [7]). Furthermore, Ψ is continuously differentiable on $H^1(\mathbb{R})$.

Lemma 2.1 (Euler-Lagrange equation and regularity)**.** *Suppose $v \in H^1(\mathbb{R})$ is a critical point of Ψ; set $u := v_0 + v \in \mathcal{M}_{-\pi,\pi} \subset X$. Then $u \in C^2(\mathbb{R})$ and u is a solution of (2) with boundary conditions (5).*

Proof. Every critical point $v \in H^1(\mathbb{R})$ of Ψ satisfies by definition $\langle \Psi'(v), h\rangle = 0$ for all $h \in H^1(\mathbb{R})$, that is,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \big[c^2 u'(\tau)\, h'(\tau) - c_0^2 (u(\tau+1) - u(\tau))(h(\tau+1) - h(\tau)) - K \sin(u(\tau))\, h(\tau)\big]\, \mathrm{d}\tau \\ &= \int_{\mathbb{R}} \big[c^2 u'(\tau)\, h'(\tau) + c_0^2 \left[u(\tau-1) - 2u(\tau) + u(\tau+1)\right] h(\tau) - K \sin(u(\tau)) h(\tau)\big]\, \mathrm{d}\tau. \end{aligned}$$

This means that u is a weak solution of (2). Applying a classical bootstrap argument, we find that $u \in C^2(\mathbb{R})$ is a strong solution of (2). □

Lemma 2.1 shows in particular that a minimiser of the variational problem

$$\text{minimise } J, \text{ as defined in (3), on } \mathcal{M}_{-\pi,\pi} \subset X \tag{9}$$

is a solution to (2) with boundary conditions (5).

We now formulate the existence result for (2), for sufficiently large supersonic wave speed and heteroclinic boundary conditions.

Theorem 2.2. *Let $c^2 > \frac{9}{8} c_0^2$. Then there exists a minimiser u_0 of J on $\mathcal{M}_{-\pi,\pi} \subset X$, that is, the variational problem (9) possesses a solution. This minimiser u_0 is a C^2-function which satisfies (2) and the asymptotic boundary condition (5).*

The proof of this Theorem will follow easily from Lemma 2.1 and the statements in Sections 3 and 4; it is given in Section 5.

3. **A-priori bound.** For more a compact notation, we introduce on X a difference operator A as $Au(z) := u(z+1) - u(z)$. Observe that for fixed $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$,

$$\int_{T_1}^{T_2} \left[Au(\tau)\right]^2 \,\mathrm{d}\tau \le \int_0^1 \int_{T_1+s}^{T_2+s} \left[u'(\tau)\right]^2 \,\mathrm{d}\tau\,\mathrm{d}s; \tag{10}$$

this follows with Jensen's inequality and Fubini's theorem,

$$\begin{aligned}\int_{T_1}^{T_2} \left[Au(\tau)\right]^2 \,\mathrm{d}\tau &= \int_{T_1}^{T_2} \left[\int_\tau^{\tau+1} u'(s)\,\mathrm{d}s\right]^2 \mathrm{d}\tau = \int_{T_1}^{T_2} \left[\int_0^1 u'(t+\tau)\,\mathrm{d}t\right]^2 \mathrm{d}\tau \\ &\le \int_{T_1}^{T_2} \int_0^1 \left[u'(t+\tau)\right]^2 \mathrm{d}t\,\mathrm{d}\tau = \int_0^1 \int_{T_1+t}^{T_2+t} \left[u'(\zeta)\right]^2 \mathrm{d}\zeta\,\mathrm{d}t.\end{aligned}$$

By the same argument, $\int_{\mathbb{R}} \left[Au(\tau)\right]^2 \mathrm{d}\tau \le \int_{\mathbb{R}} \left[u'(\tau)\right]^2 \mathrm{d}\tau$ (see also [13]). This implies

$$\begin{aligned}&\int_{\mathbb{R}} \left[\frac{c^2 - c_0^2}{2} \left(u'(\tau)\right)^2 + K(1+\cos(u(\tau)))\right] \mathrm{d}\tau \\ &\qquad \le J(u) \le \int_{\mathbb{R}} \left[\frac{c^2}{2} \left(u'(\tau)\right)^2 + K(1+\cos(u(\tau)))\right] \mathrm{d}\tau.\end{aligned}$$

for all $u \in X$. Modica and Mortola [10] have studied a very similar functional to those in this inequality. We quote a relevant result on the minimal values of such functionals from [3, Section 6.2].

Lemma 3.1. *For $\gamma > 0$, let $I_\gamma(u) := \int_{\mathbb{R}} \left[\gamma \left(u'(\tau)\right)^2 + K(1+\cos(u(\tau)))\right] \mathrm{d}\tau$. Then the minimum of I_γ on $\mathcal{M}_{-\pi,\pi}$ is attained and*

$$\min_{u\in\mathcal{M}_{-\pi,\pi}} I_\gamma(u) = \vartheta := 2\sqrt{\gamma K} \int_{-\pi}^{\pi} \sqrt{1+\cos(\xi)}\,\mathrm{d}\xi. \tag{11}$$

Moreover, with the same ϑ,

$$\inf_{T>0} \inf \left\{ \int_{-T}^{T} \left[\gamma \left(u'\right)^2 + K(1+\cos(u))\right] \mathrm{d}\tau : \begin{array}{l} u \in H^1(-T,T), \\ u(-T) = -\pi, u(T) = \pi \end{array} \right\} = \vartheta. \tag{12}$$

As an immediate consequence we get, by evaluating the integral in (11),

$$8\sqrt{\left(c^2 - c_0^2\right) K} \le \inf_{u\in\mathcal{M}_{-\pi,\pi}} J(u) \le 8c\sqrt{K}. \tag{13}$$

This inequality and (12) will serve as basis for the L^∞-*a-priori* bound in the next lemma.

Lemma 3.2. *Let $c^2 > c_0^2$. A global minimiser u_0 of J on $\mathcal{M}_{-\pi,\pi}$ satisfies*

$$\|u_0\|_{L^\infty(\mathbb{R})} < (2k+3)\pi,$$

where $k := \max\left\{\kappa \in \mathbb{N}_0 : (2\kappa+1) \le \sqrt{\frac{c^2}{c^2 - c_0^2}}\right\}$.

Proof. The proof relies on the fact that (12) and (13) provide, loosely speaking, an estimate for the "cost" for u_0 to traverse a height of 2π from one minimum of $\cos(\cdot)$ to the next. More precisely, we have

$$J(u_0) \ge I_{\frac{1}{2}(c^2-c_0^2)}(u_0) := \int_{\mathbb{R}} \left[\frac{c^2 - c_0^2}{2} \left[u_0'(\tau)\right]^2 + K\left[1 + \cos\left(u_0(\tau)\right)\right]\right] \mathrm{d}\tau. \tag{14}$$

Hence, (13) and $1+\cos(u) \geq 0$ show

$$\frac{c^2-c_0^2}{2}\left\|u_0'\right\|_{L^2(\mathbb{R})}^2 \leq J\left(u_0\right) \leq 8c\sqrt{K}. \tag{15}$$

Let $T_1, T_2 \in \mathbb{R}\cup\{\pm\infty\}$ with $T_1 < T_2$ be such that $u_0\left(T_1\right)=\nu\pi$ and $u_0\left(T_2\right)=(\nu+2)\pi$ for some odd integer ν. Then the contribution of the interval $[T_1, T_2]$ to the value $I_{\left(c^2-c_0^2\right)}\left(u_0\right)$ is, from (12),

$$\begin{aligned}
&\int_{T_1}^{T_2}\left[\frac{c^2-c_0^2}{2}\left[u_0'(\tau)\right]^2+K\left[1+\cos\left(u_0(\tau)\right)\right]\right]\mathrm{d}\tau\\
&\geq \inf_{\substack{u\in H^1_{\mathrm{loc}}(T_1,T_2)\,\cap\, C^0[T_1,T_2],\\ u(T_1)=\nu\pi,\ u(T_2)=(\nu+2)\pi}}\int_{T_1}^{T_2}\left[\frac{c^2-c_0^2}{2}\left[u'(\tau)\right]^2+K\left[1+\cos(u(\tau))\right]\right]\mathrm{d}\tau\\
&\geq 8\sqrt{\left(c^2-c_0^2\right)K}.
\end{aligned} \tag{16}$$

The boundary conditions (5) imply that the height 2π needs to be covered; any further increase in height of 2π has to be compensated by a decrease in height of 2π and vice versa. Hence there is an odd number of such increases or decreases. We write this odd number as $2\kappa+1$ with $\kappa\in\mathbb{N}$, so that $(\kappa+1)\cdot 2\pi \leq \left|\{u_0(z) : z\in\mathbb{R}\}\right| < (\kappa+2)\cdot 2\pi$; thus κ can be understood as a lower bound on the number of times that u_0 grows by full 2π in excess to the one time required by $u_0\in\mathcal{M}_{-\pi,\pi}$. Then (14) and (16) show that

$$J\left(u_0\right) \geq I_{\frac{1}{2}\left(c^2-c_0^2\right)}\left(u_0\right) \geq 8(2\kappa+1)\sqrt{\left(c^2-c_0^2\right)K}.$$

On the other hand, $J(u_0)$ is bounded by the Modica-Mortola bound (13). Therefore, using (15),

$$8(2\kappa+1)\sqrt{\left(c^2-c_0^2\right)K} \leq J\left(u_0\right) \leq 8c\sqrt{K} \tag{17}$$

so that

$$\kappa \leq k := \max\left\{\kappa\in\mathbb{N}_0 : (2\kappa+1) \leq \sqrt{\frac{c^2}{c^2-c_0^2}}\right\}.$$

Hence $(k+1)\cdot 2\pi \leq \left|\{u_0(z) : z\in\mathbb{R}\}\right| < (k+2)\cdot 2\pi$. Due to (5), $(-\pi,\pi)\subseteq\{u_0(z) : z\in\mathbb{R}\}$, so

$$\sup_{\tau\in\mathbb{R}}|u_0(\tau)| < (k+2)\cdot 2\pi - \pi = (2k+3)\pi;$$

note in particular that the inequality is strict. □

4. **Concentration-compactness.** The next step is to prove a variant of the concentration-compactness lemma of P.-L. Lions [9, Lemma I.1] that is adapted to our situation.

The setting in this classical paper [9] (see also [6]) is as follows. The general problem is to minimise a functional $E\colon U\to\mathbb{R}$ on a Banach space U subject to a constraint $L(u)=\lambda>0$. It is shown that, for fixed λ, that any minimising sequence is, up to a subsequence, either relatively compact, or vanishes, or splits into two or more parts which drift away arbitrarily distant from each other. Vanishing can usually be excluded quite easily. Setting

$$I_\lambda := \inf\{E(u) : u\in U, L(u)=\lambda\},$$

splitting cannot occur, heuristically speaking, if and only if

$$I_\lambda < I_\alpha + I_{\lambda-\alpha} \text{ for all } \alpha \in (0, \lambda).$$

In comparison to the classical setting, a major difference in the present paper is that the constraint $u(\pm\infty) = \pm\pi$ cannot be varied continuously. Hence it is impossible to consider the minimum value of the functional on level sets of the constraint as a function of a continuous parameter in the constraint. As a consequence of this, no meaningful analog to the above subadditivity inequality can be formulated. Instead, we will exclude splitting by means of the *a priori* bound from Lemma 3.2.

The most important difference in contrast to other variants of the concentration-compactness lemma is therefore in the alternative of splitting. The value of the functional J is split up between sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ (whose sum is essentially the original sequence $(u_n)_{n\in\mathbb{N}}$)—not the value of the constraint, as usual. On the other hand, the present lemma holds not just for minimising sequences $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$, but for all sequences for which the values of the functional converge.

The following proof will be formulated using symmetrised differences $u\left(\tau + \frac{1}{2}\right) - u\left(\tau - \frac{1}{2}\right)$, rather than $u(\tau+1) - u(\tau)$, in order to exploit the symmetry of the integration domains. It is clear that J, and hence the minimisation problem, remains unchanged because $\int_{\mathbb{R}} \left[Au(\tau)\right]^2 \,\mathrm{d}\tau = \int_{\mathbb{R}} \left[u\left(\tau+\frac{1}{2}\right) - u\left(\tau - \frac{1}{2}\right)\right]^2 \,\mathrm{d}\tau$.

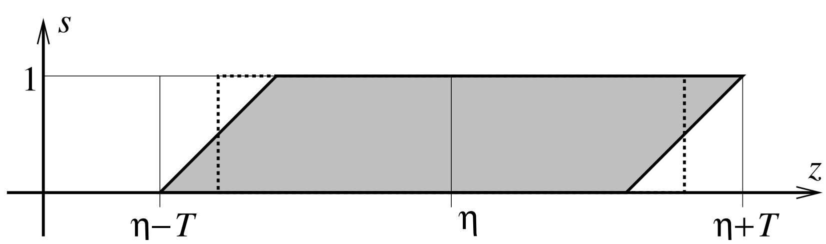

FIGURE 1. Domain of integration of $\int_0^1 \int_{\eta-T+s}^{\eta+T-1+s} \left[u'(\tau)\right]^2 \,\mathrm{d}\tau\,\mathrm{d}s$, which is up to a multiplicative factor the first term in the definition of $J_T(\cdot;\eta)$ in (18).

We introduce a truncated version of J. For parameters $T > 1$ and $\eta \in \mathbb{R}$, set

$$J_T(u;\eta) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\eta-T+\frac{1}{2}+s}^{\eta+T-\frac{1}{2}+s} \frac{c^2}{2} \left[u'(\tau)\right]^2 \,\mathrm{d}\tau\,\mathrm{d}s$$
$$- \int_{\eta-T+\frac{1}{2}}^{\eta+T-\frac{1}{2}} \frac{c_0^2}{2} \left[u\left(\tau+\tfrac{1}{2}\right) - u\left(\tau-\tfrac{1}{2}\right)\right]^2 \,\mathrm{d}\tau + \int_{\eta-T+\frac{1}{2}}^{\eta+T-\frac{1}{2}} K\left[1+\cos(u(\tau))\right] \,\mathrm{d}\tau \quad (18)$$

We point out that all integrals are taken over symmetric intervals around η which simplifies some estimates later in this proof. For use in Section 5, we mention that the second summand equals $\int_{\eta-T}^{\eta+T-1} \left[Au(\tau)\right] \,\mathrm{d}\tau$.

The domain of integration for the first term in the definition of J_T can be thought of as the shaded parallelogram shown in Figure 1, with the integrand being constant on vertical lines. This choice is motivated by the second term because

$$Au(\tau) = \int_{\tau}^{\tau+1} u'(s)\,\mathrm{d}s$$

shows that, roughly speaking, the second term could be interpreted as an integration over the same domain. This idea has already been suggested by (10).

Lemma 4.1 (Concentration-compactness)**.** *Let* $c^2 > c_0^2$ *and* $\theta \geq \inf J(u)|_{\mathcal{M}_{-\pi,\pi}}$, $\theta < \infty$. *Then every sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$ *with the property*

$$\lim_{n\to\infty} J(u_n) = \theta \tag{19}$$

possesses a subsequence, not relabelled and still denoted by $(u_n)_{n\in\mathbb{N}}$, *which satisfies one of the following three alternatives:*

(i) Tightness: *There is a sequence* $(\eta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *such that, for every* $\varepsilon > 0$, *there exists* $T_0 > 0$ *such that for all* $T > T_0$

$$J(u_n) - J_T(u_n; \eta_n) < \varepsilon \text{ for every } n \in \mathbb{N}.$$

(ii) Vanishing: *For all* $T > 0$,

$$\lim_{n\to\infty} \sup_{\eta\in\mathbb{R}} J_T(u_n; \eta) = 0. \tag{20}$$

(iii) Splitting: *There exists* $\varepsilon_1 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_1$, *there are* $f_n, g_n \in X$ *such that*

$$|u_n - (f_n + g_n - \pi)| \leq \varepsilon$$
$$|J(u_n) - (J(f_n) + J(g_n))| \leq \varepsilon, \qquad \lim_{n\to\infty} \operatorname{dist}(\operatorname{supp}(f_n'), \operatorname{supp}(g_n')) = \infty,$$
$$\lim_{n\to\infty} J(f_n) = \alpha, \qquad \lim_{n\to\infty} J(g_n) = \beta,$$

for some $0 < \alpha, \beta < \theta$. *(*$\pi$ *is needed in the first inequality to ensure* $J(f_n) < \infty$ *and* $J(g_n) < \infty$.*)*

The condition (19) is in particular satisfied if $(u_n)_{n\in\mathbb{N}}$ is a minimising sequence for J. We actually need Lemma 4.1 only for that case, in which $\theta = \inf J(u)|_{\mathcal{M}_{-\pi,\pi}}$.

Proof. The proof is given in four steps. First we introduce a concentration functional, discuss its properties (Step 1). The rest is concerned with the proof that the only alternative to cases (i) and (ii) is case (iii). Step 2 identifies the intervals which will become the support of f_n and g_n, respectively. Further estimates show the statements about the sequences f_n (Step 3) and g_n (Step 4).

Step 1. As in Lions' proof [9], a concentration function is introduced. Namely, given a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$ with (19) and a parameter $\eta \in \mathbb{R}$, define a sequence of functions $P_n(\,\cdot\,;\eta)\colon (0,\infty) \to \mathbb{R}$,

$$P_n(T;\eta) := J_{T+\frac{1}{2}}(u_n;\eta) \qquad \text{(compare Figure 1)} \tag{21}$$

the shift by $\frac{1}{2}$ from the definition of J_T saves some summands $\frac{1}{2}$ in subsequent estimates of this proof while the form of J_T is more useful in Section 5 below.

Note that, for every fixed $n \in \mathbb{N}$ and $\eta \in \mathbb{R}$, P_n is nondecreasing in T. Namely, for all $\varepsilon > 0$ and all $\eta \in \mathbb{R}$, (10) and $c > c_0$ show that the increment of the second integral in (21) is bounded by the increment of the first one,

$$\frac{c_0^2}{2}\int\limits_{\eta+T}^{\eta+T+\varepsilon}\left[u_n\left(\tau+\tfrac{1}{2}\right)-u_n\left(\tau-\tfrac{1}{2}\right)\right]^2\,\mathrm{d}\tau=\frac{c_0^2}{2}\left\|Au_n\right\|^2_{\left(\eta+T-\frac{1}{2},\eta+T-\frac{1}{2}+\varepsilon\right)}$$

$$\le\frac{c^2}{2}\int\limits_0^1\int\limits_{\eta+T-\frac{1}{2}+s}^{\eta+T-\frac{1}{2}+\varepsilon+s}\left[u_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s=\frac{c^2}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\int\limits_{\eta+T+s}^{\eta+T+\varepsilon+s}\left[u_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s,\quad(22)$$

and the very same estimate holds on $(\eta-T-\varepsilon,\eta-T)$. This implies $P_n(T+\varepsilon;\eta)\ge P_n(T;\eta)$ for all $T,\varepsilon>0$ because $1+\cos(u_n)$ is always non-negative.

Now we can define for each $n\in\mathbb{N}$ the concentration function

$$Q_n(T):=\sup_{\eta\in\mathbb{R}}P_n(T;\eta).\qquad(23)$$

As supremum of monotone and nondecreasing functions, Q_n enjoys the same properties. It is clear that Q_n is bounded on $(0,\infty)$ because, for each $n\in\mathbb{N}$,

$$\lim_{T\to\infty}Q_n(T)=J\left(u_n\right).$$

By assumption (19), $\left(J\left(u_n\right)\right)_{n\in\mathbb{N}}$ converges to θ and is therefore *a fortiori* bounded in $\mathbb{R}$; thus the sequence $\left(Q_n\right)_{n\in\mathbb{N}}$ is bounded from above in $L^\infty(0,\infty)$. Hence, by Helly's selection theorem (see, e.g., [14, Section 17.4]), a subsequence, not relabelled, converges pointwise almost everywhere to a monotone nondecreasing function $Q\colon(0,\infty)\to\mathbb{R}$ and

$$l:=\lim_{T\to\infty}Q(T)\in[0,\theta].\qquad(24)$$

Obviously, alternative (i) in the statement occurs for $l=\theta$, and alternative (ii), vanishing, occurs when $l=0$. What remains is to show that $0<l<\theta$ corresponds to alternative (iii), splitting.

Step 2. Let $\varepsilon>0$. By definition of l in (24), there exists $T_0\in\mathbb{R}$ such that $Q\left(T_0\right)\ge l-\frac{1}{3}\varepsilon$. Since $Q_n(T)\to Q(T)$ as $n\to\infty$ for almost every T, we may assume, possibly after increasing T_0, that $Q_n\left(T_0\right)\to Q(T_0)$. Thus, $Q_n\left(T_0\right)\ge l-\frac{2}{3}\varepsilon$, if we consider only large enough n. The definition (23) of Q_n implies that we can find $\eta_n\in\mathbb{R}$ such that for all large enough n

$$P_n\left(T_0;\eta_n\right)\ge l-\varepsilon.$$

We can also find a sequence $\left(T_n\right)_{n\in\mathbb{N}}$ with $T_n\to\infty$ as $n\to\infty$ (and in particular $T_n\gg T_0$ for all $n\in\mathbb{N}$) such that $Q_n\left(T_n\right)\le l+\varepsilon$; this follows from the facts that $Q_n(T)\to Q(T)$ as $n\to\infty$ for almost every T, and that $Q(T)\to l$ as $T\to\infty$, see (24). Since Q_n has been defined as supremum over P_n in (23), the sequence $\left(T_n\right)_{n\in\mathbb{N}}$ satisfies $P_n\left(T_n;\eta_n\right)\le l+\varepsilon$. As P_n is monotone and nondecreasing in T for each $n\in\mathbb{N}$,

$$\left|P_n\left(T;\eta_n\right)-l\right|\le\varepsilon\text{ for all }T\in\left[T_0,T_n\right].\qquad(25)$$

Now we are going to analyse the behaviour of $u_n(\tau)$ for $|\tau-\eta_n|\in[T_0,T_n]$; the goal is to show that there exist $k_n^\pm\in\mathbb{Z}$ such that

$$\begin{aligned}&\left|u_n(\tau)-\left(2k_n^+ +1\right)\pi\right|\le\delta(\varepsilon)\text{ for }\tau\in\left[\eta_n+T_0+\tfrac{1}{2},\eta_n+T_n-\tfrac{1}{2}\right]\text{ and}\\&\left|u_n(\tau)-\left(2k_n^- +1\right)\pi\right|\le\delta(\varepsilon)\text{ for }\tau\in\left[\eta_n-T_n+\tfrac{1}{2},\eta_n-T_0-\tfrac{1}{2}\right]\end{aligned}\qquad(26)$$

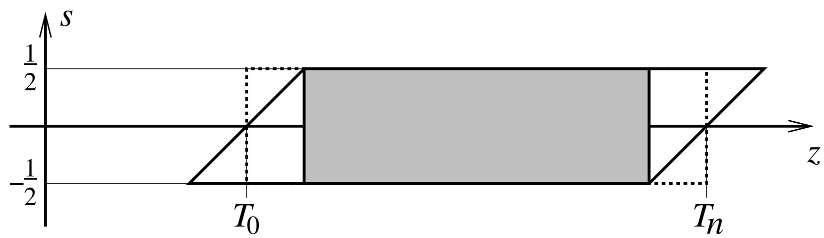

FIGURE 2. Domains of integration of the first term (parallelogram) and the second term (dashed rectangle) in (27) and of the integral in the following line (shaded rectangle). To prevent a complicated labelling of axes, the situation has been sketched for $\eta_n = 0$.

Starting with (25) and considering first only the interval $[\eta_n + T_0, \eta_n + T_n]$, we find, first arguing as for the derivation of the inequality in (22) and then shrinking the domain of integration in the second step (compare Figure 2),

$$\begin{aligned}2\varepsilon &\geq P_n\left(T_n;\eta_n\right) - P_n\left(T_0;\eta_n\right)\\ &\geq \frac{c^2-c_0^2}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\eta_n+T_0+s}^{\eta_n+T_n+s}\left[u_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s + K\int_{\eta_n+T_0}^{\eta_n+T_n}\left[1+\cos u_n(\tau)\right]\,\mathrm{d}\tau\\ &\geq \int_{\eta_n+T_0+\frac{1}{2}}^{\eta_n+T_n-\frac{1}{2}}\left[\frac{c^2-c_0^2}{2}\left[u_n'(\tau)\right]^2 + K\left[1+\cos(u_n(\tau))\right]\right]\,\mathrm{d}\tau \end{aligned}\tag{27}$$

and, for any T', $T'' \in \mathbb{R}$ with $T_0 + \frac{1}{2} \leq T' < T'' \leq T_n - \frac{1}{2}$, by shrinking the domain of integration again,

$$\geq \int_{\eta_n+T'}^{\eta_n+T''}\left[\frac{c^2-c_0^2}{2}\left[u_n'(\tau)\right]^2 + K\left[1+\cos(u_n(\tau))\right]\right]\,\mathrm{d}\tau,$$

and using the trivial estimate $x^2 + y^2 \geq 2xy$ and the change of variables $\xi = u_n(\tau)$,

$$\begin{aligned}&\geq 2\int_{\eta_n+T'}^{\eta_n+T''}\left[\sqrt{\frac{c^2-c_0^2}{2}}\,|u_n'(\tau)|\cdot\sqrt{K(1+\cos(u_n(\tau)))}\right]\,\mathrm{d}\tau\\ &= \sqrt{2K\left(c^2-c_0^2\right)}\left|\int_{u_n(\eta_n+T')}^{u_n\left(\eta_n+T''\right)}\sqrt{1+\cos(\xi)}\,\mathrm{d}\xi\right|.\end{aligned}$$

This shows that $|u_n\left(\eta_n + T'\right) - u_n\left(\eta_n + T''\right)| \leq \delta_1$, where $\delta_1 = \delta_1(\varepsilon)$ is given by the relation

$$\int_{\pi-\frac{\delta_1}{2}}^{\pi+\frac{\delta_1}{2}}\sqrt{1+\cos(x)}\,\mathrm{d}x = \frac{2\varepsilon}{\sqrt{2K\left(c^2-c_0^2\right)}}.$$

To see that this interval $\left\{u_n(\tau) : \tau \in \left[\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}\right]\right\}$ of length $\leq \delta_1$ is near an element of $\{(2k+1)\pi : k \in \mathbb{Z}\}$, observe that (27) implies also

$$2\varepsilon \geq K\left(T_n - T_0 - 1\right) \cdot \min_{\tau \in \left[\eta_n + T_0 + \frac{1}{2}, \eta_n + T_n - \frac{1}{2}\right]} \left[1 + \cos\left(u_n(\tau)\right)\right].$$

Suppose this minimum is attained at $\tau_{0,n}$. Then $\frac{2\varepsilon}{K(T_n - T_0 - 1)} \geq 1 + \cos\left(u_n\left(\tau_{0,n}\right)\right)$. As we may assume $\varepsilon \ll 1$ and $T_n \gg T_0$, there exists a k_n^+ such that

$$\left|u_n\left(\tau_{0,n}\right) - \left(2k_n^+ + 1\right)\pi\right| \leq \pi - \arccos\left(\frac{2\varepsilon}{K\left(T_n - T_0 - 1\right)} - 1\right) =: \delta_2$$

Hence, with $\delta = \delta(\varepsilon) = \delta_1 + \delta_2$,

$$\left|u_n(\tau) - \left(2k_n^+ + 1\right)\pi\right| \leq \delta \quad \text{for all} \quad \tau \in \left[\eta_n + T_0 + \tfrac{1}{2}, \eta_n + T_n - \tfrac{1}{2}\right],$$

and $\delta \to 0$ for $\varepsilon \to 0$. To establish (26), it suffices to observe that the same argument yields k_n^- with the corresponding property

$$\left|u_n(\tau) - \left(2k_n^- + 1\right)\pi\right| \leq \delta \quad \text{for all} \quad \tau \in \left[\eta_n - T_n + \tfrac{1}{2}, \eta_n - T_0 - \tfrac{1}{2}\right].$$

Step 3. Define

$$f_n(\tau) := \begin{cases} \left(2k_n^- + 1\right)\pi & \text{for } \tau < -T_0 - 2, \\ u_n\left(\eta_n + \tau\right) & \text{for } \tau \in \left[-T_0 - 1, T_0 + 1\right], \\ \left(2k_n^+ + 1\right)\pi & \text{for } \tau > T_0 + 2 \end{cases} \tag{28}$$

and interpolate linearly on $[-T_0 - 2, -T_0 - 1]$ and $[T_0 + 1, T_0 + 2]$. In analogy to P_n, we introduce, replacing u_n by f_n in (21), $\widetilde{P}_n \colon (0, \infty) \to \mathbb{R}$ by

$$\widetilde{P}_n(T) := \frac{c^2}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-T+s}^{T+s} \left[f_n'(\tau)\right]^2 \mathrm{d}\tau\, \mathrm{d}s - \frac{c_0^2}{2} \int_{-T}^{T} \left[f_n\left(\tau + \tfrac{1}{2}\right) - f_n\left(\tau - \tfrac{1}{2}\right)\right]^2 \mathrm{d}\tau + K \int_{-T}^{T} \left[1 + \cos\left(f_n(\tau)\right)\right] \mathrm{d}\tau. \tag{29}$$

For $|\tau| > T_0 + \frac{5}{2}$, each of the integrands vanishes because f_n is on $\{\tau \in \mathbb{R} : |\tau| > T_0 + 2\}$ by definition constant and equal to an odd multiple of π. Therefore

$$\widetilde{P}_n\left(T_0 + \tfrac{5}{2}\right) = \widetilde{P}_n(T) = \widetilde{P}_n(\infty) = J\left(f_n\right) \text{ for all } T > T_0 + \tfrac{5}{2}. \tag{30}$$

The goal is now to show that, up to a subsequence, $J\left(f_n\right) \to \alpha \in (0, \theta)$ for $n \to \infty$. To do so, we are going to estimate, with l from (24),

$$\left|\widetilde{P}_n\left(T_n\right) - l\right| = \left|\left[\widetilde{P}_n\left(T_n\right) - P_n\left(T_n; \eta_n\right)\right] - \left[P_n\left(T_n; \eta_n\right) - l\right]\right|$$

in terms of ε. We know already $\left|P_n\left(T_n; \eta_n\right) - l\right| \leq \varepsilon$ from (25), so we are left with $\left|\widetilde{P}_n\left(T_n\right) - P_n\left(T_n; \eta_n\right)\right|$. Note that $P_n\left(T_0 + \frac{1}{2}; \eta_n\right) = \widetilde{P}_n\left(T_0 + \frac{1}{2}\right)$ because, by definition, $f_n(\tau) = u_n\left(\eta_n + \tau\right)$ for all $|\tau| \leq T_0 + 1$. Thus the triangle inequality yields

$$\left|P_n\left(T_n; \eta_n\right) - \widetilde{P}_n\left(T_n\right)\right| \leq \left|P_n(T_n; \eta_n) - P_n(T_0 + \tfrac{1}{2}; \eta_n)\right| + \left|\widetilde{P}_n\left(T_0 + \tfrac{1}{2}\right) - \widetilde{P}_n\left(T_n\right)\right|. \tag{31}$$

It follows from (25) that the first term on the right-hand side of (31) can be estimated as

$$\left|P_n\left(T_n;\eta_n\right)-P_n\left(T_0+\tfrac{1}{2};\eta_n\right)\right|\leq 2\varepsilon. \tag{32}$$

As for the second term on the right-hand side of (31), observe that the domains of integration in $\widetilde{P}_n\left(T_0+\frac{1}{2}\right)$ and $\widetilde{P}_n\left(T_n\right)$ overlap so that, with $\mathcal{I}_n := \left(-T_n,-T_0-\frac{1}{2}\right)\cup\left(T_0+\frac{1}{2},T_n\right)$,

$$\begin{aligned}\left|\widetilde{P}_n\left(T_0+\tfrac{1}{2}\right)-\widetilde{P}_n\left(T_n\right)\right| &= \frac{c^2}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\int\limits_{\substack{\left(-T_n+s,-T_0-\frac{1}{2}+s\right)\\ \cup\left(T_0+\frac{1}{2}+s,T_n+s\right)}}\left[f_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s\\
&\quad-\frac{c_0^2}{2}\int\limits_{\mathcal{I}_n}\left[f_n\left(\tau+\tfrac{1}{2}\right)-f_n\left(\tau-\tfrac{1}{2}\right)\right]^2\,\mathrm{d}\tau+K\int\limits_{\mathcal{I}_n}\left[1+\cos\left(f_n(\tau)\right)\right]\,\mathrm{d}\tau\\
&\leq\frac{c^2}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\int\limits_{\substack{\left(-T_n+s,-T_0-\frac{1}{2}+s\right)\\ \cup\left(T_0+\frac{1}{2}+s,T_n+s\right)}}\left[f_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s+K\int\limits_{\mathcal{I}_n}\left[1+\cos\left(f_n(\tau)\right)\right]\,\mathrm{d}\tau\end{aligned} \tag{33}$$

because $\left[f_n\left(\tau+\frac{1}{2}\right)-f_n\left(\tau-\frac{1}{2}\right)\right]^2\geq 0$. We estimate the terms first on the intervals which lie in $\mathbb{R}^+$, and start with the first summand of (33).

Using $f'(\tau)=0$ for $\tau>T_0+2$, we obtain

$$\frac{c^2}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\int\limits_{T_0+\frac{1}{2}+s}^{T_n+s}\left[f_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s=\frac{c^2}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\int\limits_{T_0+\frac{1}{2}+s}^{T_0+\frac{5}{2}+s}\left[f_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s$$

(the domain of integration is here the largest parallelogram shown in Figure 3) and, splitting the integral at $z=T_0+1$ and employing $f_n'=u_n'\left(\eta_n+\cdot\right)$ on (T_0,T_0+1) (this is used on the left half of the shaded parallelogram in Figure 3) and $f_n'=0$ on (T_0+2,T_0+3) (thus no contribution comes from the rightmost triangle in Figure 3)

$$=\frac{c^2}{2}\left[\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\int\limits_{T_0+\frac{1}{2}+s}^{T_0+1}\left[u_n'\left(\eta_n+\tau\right)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s+\int\limits_{T_0+1}^{T_0+2}\left[f_n'(\tau)\right]^2\,\mathrm{d}\tau\right]$$

and, extending the first integral to the whole shaded parallelogram of Figure 3, while evaluating the second integral, in which $|f'|\leq\delta$ a.e. due to (26),

$$\leq\frac{c^2}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\int\limits_{\eta_n+T_0+\frac{1}{2}+s}^{\eta_n+T_0+\frac{3}{2}+s}\left[u_n'(\tau)\right]^2\,\mathrm{d}\tau\,\mathrm{d}s+\frac{c^2}{2}\cdot 1\cdot\delta^2$$

and, bounding the remaining integral by $\frac{2}{c^2-c_0^2}\left[P_n\left(T_0+\frac{3}{2};\eta_n\right)-P_n\left(T_0+\frac{1}{2};\eta_n\right)\right]$ by (21) and (22),

$$\leq\frac{c^2}{c^2-c_0^2}\left[P_n\left(T_0+\tfrac{3}{2};\eta_n\right)-P_n\left(T_0+\tfrac{1}{2};\eta_n\right)\right]+\tfrac{1}{2}\cdot c^2\cdot\delta^2$$

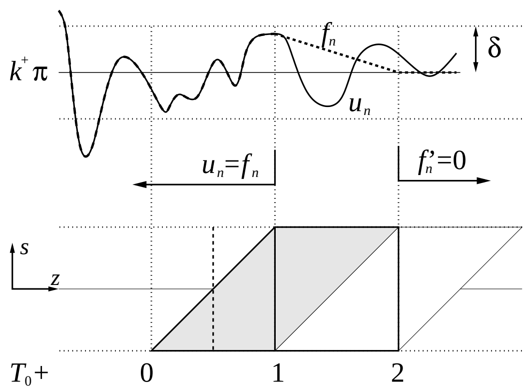

FIGURE 3. Illustration of the deduction of (34). The equality $u_n = f_n$ is to be understood modulo a translation in the argument of the two functions as described in the text.

and finally, using the monotonicity of P_n and (25),

$$\leq \frac{c^2}{c^2 - c_0^2} \cdot 2\varepsilon + \tfrac{1}{2} \cdot c^2 \cdot \delta^2 = c^2 \left(\frac{2\varepsilon}{c^2 - c_0^2} + \frac{\delta^2}{2} \right). \tag{34}$$

We continue estimating the right-hand side of (33) on $\left(T_0 + \frac{1}{2}, T_n\right)$ and find for the remaining third integral, from (26),

$$\left| K \int_{T_0+\frac{1}{2}}^{T_n} \left[1 + \cos\left(f_n(\tau)\right) \right] \mathrm{d}\tau \right| = K \int_{T_0+\frac{1}{2}}^{T_0+2} \left[1 + \cos\left(f_n(\tau)\right) \right] \mathrm{d}\tau \leq \tfrac{3}{2} K \left(1 - \cos(\delta)\right). \tag{35}$$

The very same estimates hold for the interval $\left(-T_n, -T_0 - \frac{1}{2}\right)$. Thus when combining (33), (34) and (35), we obtain

$$\left| \widetilde{P}_n\left(T_n\right) - \widetilde{P}_n\left(T_0 + \tfrac{1}{2}\right) \right| \leq c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos\delta). \tag{36}$$

Therefore, using $\widetilde{P}_n(T_n) = J(f_n)$ (from (30)) and inserting (32) and (36) into (31), we obtain

$$\begin{aligned} \left|P_n\left(T_n; \eta_n\right) - J\left(f_n\right)\right| &= \left| P_n\left(T_n; \eta_n\right) - \widetilde{P}_n\left(T_n\right) \right| \\ &\leq 2\varepsilon + c^2 \left(\frac{4\varepsilon}{c^2 - c_0^2} + \delta^2 \right) + 3K(1 - \cos\delta) =: \widetilde{\varepsilon}, \end{aligned} \tag{37}$$

hence, with (25),

$$|J(f_n) - l| \leq \left|J\left(f_n\right) - P_n\left(T_n; \eta_n\right)\right| + \left|P_n\left(T_n; \eta_n\right) - l\right| \leq \widetilde{\varepsilon} + \varepsilon.$$

Now choose $\varepsilon_0 > 0$ such that

$$l + 2\left(\widetilde{\varepsilon} + \varepsilon\right) < \theta \quad \text{and} \quad l - 2\left(\widetilde{\varepsilon} + \varepsilon\right) > 0 \quad \text{for all} \quad \varepsilon < \varepsilon_0; \tag{38}$$

this is possible because $\delta = \delta(\varepsilon) \to 0$ for $\varepsilon \to 0$ while c, c_0 and K are constants, and $0 < l < \theta$ by assumption. We will from now on assume that $\varepsilon < \varepsilon_0$.

With this choice, we find that there exists a subsequence (not relabelled) of $(f_n)_{n\in\mathbb{N}}$ for which the limit of $J(f_n)$ for $n\to\infty$ exists, and

$$\lim_{n\to\infty} J(f_n) = \alpha,$$

for some $\alpha\in\mathbb{R}$ with $|l-\alpha|\le\widetilde{\varepsilon}$ and, by choice of ε_0, $0<\alpha<\theta$.
Step 4. Introduce now, with $k^{\pm}$ as in (26),

$$g_n(\tau) := \begin{cases} u_n(\eta_n+\tau)-2k_n^-\pi & \text{for } \tau < -T_n+1, \\ \pi & \text{for } \tau\in[-T_n+2, T_n-2], \\ u_n(\eta_n+\tau)-2k_n^+\pi & \text{for } \tau > T_n-1, \end{cases} \tag{39}$$

and interpolate linearly on $[-T_n+1,-T_n+2]$ and $[T_n-2,T_n-1]$. Clearly

$$\operatorname{dist}\left(\operatorname{supp}\left(f_n'\right),\operatorname{supp}\left(g_n'\right)\right)\ge (T_n-T_0-4)\to\infty$$

for $n\to\infty$. By definition of f_n and g_n,

$$u_n\left(\eta_n+\tau\right) = f_n(\tau)+g_n(\tau)-\pi \quad \text{for} \quad |\tau|\in[0,T_0+1]\cup[T_n-1,\infty),$$

and for $\pm\tau\in[T_0+1,T_n-1]$ we have $f_n(\tau)+g_n(\tau)-\pi = (2k^{\pm}+1)\,\pi$, while (26) shows $|u_n-(2k^{\pm}+1)\,\pi|\le 2\varepsilon$.

We are now going to estimate $|J(u_n)-J(f_n)-J(g_n)|$. The last statement to be shown, $J(g_n)\to\beta\in(0,\theta)$ for $n\to\infty$, will then be an easy consequence of it. By the triangle inequality, (25) and (37),

$$\begin{aligned} |J(u_n)-J(f_n)-J(g_n)| &\le |J(u_n)-J(g_n)-P_n(T_n-1;\eta_n)| \\ &\quad + |P_n(T_n-1;\eta_n)-P_n(T_n;\eta_n)| \\ &\quad + |P_n(T_n;\eta_n)-J(f_n)| \\ &\le |J(u_n)-J(g_n)-P_n(T_n-1;\eta_n)| + 2\varepsilon+\widetilde{\varepsilon}. \end{aligned} \tag{40}$$

In a very similar manner to (33)–(36) (see also [7]), it is possible to show in analogy to (37) that

$$|J(u_n)-P(T_0;\eta_n)-J(g_n)| \le c^2\left(\frac{4\varepsilon}{c^2-c_0^2}+\delta^2\right)+3K(1-\cos\delta) = \widetilde{\varepsilon}-2\varepsilon, \tag{41}$$

with $\widetilde{\varepsilon}$ as in (37). This inequality implies together with (40)

$$|J(u_n)-J(f_n)-J(g_n)|\le\widetilde{\varepsilon}-2\varepsilon+2\varepsilon+\widetilde{\varepsilon} = 2\widetilde{\varepsilon},$$

thus for sufficiently large n

$$|J(g_n)-(\theta-\alpha)|\le 2\widetilde{\varepsilon}+\varepsilon,$$

and switching to a subsequence if necessary, we find that the limit

$$\lim J(g_n) =: \beta$$

exists, and that $0<\beta<\theta$, by choice of ε_0 in (38). This finishes the proof. □

5. **Proof of the main result.** We prove the existence of a heteroclinic wave by ruling out two of the three cases in the concentration-compactness Lemma 4.1, namely vanishing (possibility (ii)) and splitting (possibility (iii)).

Lemma 5.1 (No vanishing)**.** *Let $c^2>c_0^2$ and $(u_n)_{n\in\mathbb{N}}\subset\mathcal{M}_{-\pi,\pi}$ be a minimising sequence for J. Then $(u_n)_{n\in\mathbb{N}}$ does not vanish.*

Proof. The argument is by contradiction. Suppose that (20) holds. Fix $T \gg 1$, $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for all u_n with $n > N$

$$\sup_{\eta\in\mathbb{R}} \Bigg[\int_{\eta-T}^{\eta+T} \frac{c^2}{2} \left[u_n'(\tau)\right]^2 \,\mathrm{d}\tau - \int_{\eta-T}^{\eta+T-1} \frac{c_0^2}{2} \left[Au_n(\tau)\right]^2 \,\mathrm{d}\tau + \int_{\eta-T}^{\eta+T} K\left[1+\cos\left(u_n(\tau)\right)\right] \mathrm{d}\tau \Bigg] \leq \frac{\varepsilon}{T}.$$

Observe that for supersonic waves the quantity on the left-hand side is always non-negative since the first term dominates the second one by (10), and the on-site potential $1+\cos(\xi) \geq 0$ non-negative for every $\xi \in \mathbb{R}$. In particular,

$$\frac{c^2-c_0^2}{2} \left\|u'\right\|^2_{L^2(-T,T)} \leq \frac{c^2}{2} \left\|u'\right\|^2_{L^2(-T,T)} - \frac{c_0^2}{2} \left\|Au\right\|^2_{L^2(-T,T-1)} \leq \frac{\varepsilon}{T}$$

holds, hence $\left\|u'\right\|_{L^2(-T,T)} \leq \sqrt{\frac{4\varepsilon}{T(c^2-c_0^2)}}$. We claim that this implies

$$|u(T)-u(-T)| \leq \sqrt{\frac{8\varepsilon}{c^2-c_0^2}} =: \varepsilon_1. \tag{42}$$

To see this, consider the auxiliary variational problem

$$\text{Maximise } v(2T), \text{ subject to } v(0)=0,\ \left\|v'\right\|_{L^2(0,2T)} = \alpha \tag{43}$$

on $H^1(0,2T)$. Since

$$|v(2T)| = \left| \int_0^{2T} v'(s)\,\mathrm{d}s \right| \leq \left\|v'\right\|_{L^1(0,2T)} \leq \sqrt{2T} \left\|v'\right\|_{L^2(0,2T)},$$

where equality holds only for $v' = $ const., we conclude that the maximum must be attained by the linear functions $v(s) = \pm\frac{\alpha s}{\sqrt{2T}}$, which yields $v(2T) = \alpha\sqrt{2T}$ as maximum.

Since J is invariant under the transformation $\tau \mapsto \tau + \tau_0$ for arbitrary $\tau_0 \in \mathbb{R}$, we may assume without loss of generality that $u_n(0) = 0$ for all $n \in \mathbb{N}$. It follows

$$K \int_{-T}^{T} \left[1+\cos(u(\tau))\right] \mathrm{d}\tau \geq K\left(1+\cos\left(\varepsilon_1\right)\right) \cdot 2T.$$

This, however, is a contradiction to

$$\sup_{\eta\in\mathbb{R}} \int_{\eta-T}^{\eta+T} K\left[1+\cos(u(\tau))\right] \mathrm{d}\tau < \varepsilon$$

for ε small enough. □

Lemma 5.2 (No splitting)**.** *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{-\pi,\pi}$ be a minimising sequence for J. Then, for every c with $c^2 > \frac{9}{8}c_0^2$, $(u_n)_{n\in\mathbb{N}}$ does not split in the sense of Lemma 4.1 (iii).*

Proof. The condition $c^2 > \frac{9}{8}c_0^2$ ensures that $\sqrt{\frac{c^2}{c^2-c_0^2}} < 3$, thus $k \in \{0,1\}$ in the statement of Lemma 3.2, so any minimiser u_0 of $J|_{\mathcal{M}_{-\pi,\pi}}$ satisfies $\|u_0\|_{L^\infty(\mathbb{R})} < 3\pi$. By the same arguments as in the proof of Lemma 3.2, the same $L^\infty(\mathbb{R})$-bound applies to the members of a minimising sequence (possibly after dropping finitely many of them); indeed, if $\left(J(u_n) - \inf J|_{\mathcal{M}_{-\pi,\pi}}\right) < \delta$ then, in analogy to (17),

$$8(2\kappa+1)\sqrt{(c^2-c_0^2)K} \le J(u_n) \le 8c\sqrt{K} + \delta,$$

so if $\sqrt{\frac{c^2}{c^2-c_0^2}} < 3$, as guaranteed by assumption, then $\frac{8\sqrt{c^2K}+\delta}{8\sqrt{(c^2-c_0^2)K}} < 3$ for δ small enough.

Suppose now, for a contradiction, that $(u_n)_{n\in\mathbb{N}}$ splits. Let $\varepsilon > 0$ and choose, as in the proof of Lemma 4.1 (iii), f_n, g_n such that $|u_n - (f_n + g_n - \pi)| < \varepsilon$. Then the L^∞-bound $\|u_n\|_{L^\infty(\mathbb{R})} < 3\pi$ shows $k_n^\pm \in \{-1, 0\}$, see Equation (26). For given k_n^+, and k_n^-, we can immediately determine the values of $f_n(\infty) - f_n(-\infty)$ respectively $g_n(\infty) - g_n(-\infty)$ from (28) respectively (39); recall the boundary conditions $u_n(\infty) - u_n(-\infty) = 2\pi$. This straightforward calculation yields

$$\begin{aligned}
&f_n(\infty) - f_n(-\infty) = 0, && g_n(\infty) - g_n(-\infty) = 2\pi && (\text{if } k_n^+ = k_n^- \in \{-1,0\}),\\
\text{or}\quad &f_n(\infty) - f_n(-\infty) = 2\pi, && g_n(\infty) - g_n(-\infty) = 0 && (\text{if } k_n^+ = 0,\ k_n^- = -1),\\
\text{or}\quad &f_n(\infty) - f_n(-\infty) = -2\pi, && g_n(\infty) - g_n(-\infty) = 4\pi && (\text{if } k_n^+ = -1,\ k_n^- = 0).
\end{aligned}$$

Thus, let $\tilde{f}_n$ be defined by $\tilde{f}_n(\tau) := f_n(-\tau)$. Then for each $n \in \mathbb{N}$ one of the three functions f_n, g_n and $\tilde{f}_n$ belongs to $\mathcal{M}_{-\pi,\pi}$. Let this function be denoted by $\tilde{u}_n$. As both

$$\lim_{n\to\infty} J(f_n) < \inf_{u\in\mathcal{M}_{-\pi,\pi}} J(u) \qquad \text{and} \qquad \lim_{n\to\infty} J(g_n) < \inf_{u\in\mathcal{M}_{-\pi,\pi}} J(u),$$

by Lemma 4.1 (iii), there exists a subsequence of $(\tilde{u}_n)_{n\in\mathbb{N}}$, not relabelled, such that

$$\lim_{n\to\infty} J(\tilde{u}_n) < \inf_{u\in\mathcal{M}_{-\pi,\pi}} J(u) = \lim_{n\to\infty} J(u_n),$$

a contradiction to the fact that $(u_n)_{n\in\mathbb{N}}$ is a minimising sequence in $\mathcal{M}_{-\pi,\pi}$. □

Now we are in a position to prove the main theorem 2.2.

Proof. The Modica-Mortola bound (13) implies that J is bounded from below on X. Let $(u_n)_{n\in\mathbb{N}}$ be a minimising sequence in $M_{-\pi,\pi}$. We know from Lemma 4.1 that $(u_n)_{n\in\mathbb{N}}$ has to concentrate, or vanish, or split. Lemma 5.1 rules out vanishing, and Lemma 5.2 rules out splitting for $c > \frac{9}{8}c_0^2$. Thus $(u_n)_{n\in\mathbb{N}}$ must concentrate. Hence, for fixed $\varepsilon > 0$, it is possible to choose a sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $T_0 > 0$ such that

$$|J(u_n) - J_{T_0}(u_n; \eta_n)| < \varepsilon.$$

We write $w_n(\tau) := u_n(\eta_n + \tau)$. The sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in X, since $\|w_n'\|_{L^2(\mathbb{R})} = \|u_n'\|_{L^2(\mathbb{R})} \le \frac{2}{c^2-c_0^2} J(u_n)$ and $|w(0)| < 3\pi$ by (15) and Lemma 3.2. As X is a Hilbert space, there exists a subsequence, not relabelled, which converges weakly. On the interval $[-T_0, T_0]$, the weak convergence of $(w_n)_{n\in\mathbb{N}}$ implies strong convergence in $L^2(-T_0, T_0)$ and $C^0[-T_0, T_0]$ to some limit u. Hence for all $n > N$, N sufficiently large,

$$\left| \int_{-T_0}^{T_0-1} \left[(Aw_n(\tau))^2 - (Au(\tau))^2 \right] \mathrm{d}\tau \right| < \varepsilon$$

and

$$\left| \int_{-T_0}^{T_0} \left[\cos(w_n(\tau)) - \cos u(\tau) \right] \mathrm{d}\tau \right| < \varepsilon.$$

As the weak convergence implies $\|u'\|^2_{L^2(-T_0,T_0)} \leq \liminf_{n\in\mathbb{N}} \|w_n'\|^2_{L^2(-T_0,T_0)}$, we may conclude $J_{T_0}(u) \leq \liminf_{n\in\mathbb{N}} J_{T_0}(w_n; 0)$.

We now extend the domain of u inductively to $\mathbb{R}$. Take any monotone sequence $T_k \to \infty$ with $k \in \mathbb{N}_0$ and assume that u has already been defined as uniform limit of $(w_n)_{n\in\mathbb{N}}$ on the interval $[-T_k, T_k]$. As $(w_n)_{n\in\mathbb{N}}$ is still bounded in X we can again choose a subsequence, not relabelled, which converges uniformly in $C^0[-T_{k+1}, T_{k+1}]$ to some limit $\tilde{u}$, which, by construction, coincides with u on $[-T_k, T_k]$.

It follows that this function u on $\mathbb{R}$ satisfies the boundary conditions (5) and, with a constant $C = C(c, c_0, K)$,

$$\begin{aligned} J(u) = \lim_{T\to\infty} J_T(u;0) &\leq \lim_{T\to\infty} \liminf_{n\to\infty} J_T(w_n;0) \\ &\leq \lim_{T\to\infty} \lim_{n\to\infty} J(w_n) + C\varepsilon = \lim_{n\to\infty} J(u_n) + C\varepsilon; \end{aligned}$$

in particular $u' \in L^2(\mathbb{R})$, thus $u \in M_{-\pi,\pi}$. As ε was arbitrary, the previous inequality shows

$$J(u) \leq \liminf_{n\to\infty} J(u_n).$$

This means that u is a minimiser of J on $M_{-\pi,\pi}$, and the theorem follows from the fact that (2) is the Euler-Lagrange equation of J (in the sense of Lemma 2.1). $\square$

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REFERENCES

[1] G. Arioli and F. Gazzola, *Periodic motions of an infinite lattice of particles with nearest neighbor interaction*, Nonlinear Anal., **26** (1996), 1103–1114.

[2] P. W. Bates and C. Zhang, *Traveling pulses for the Klein-Gordon equation on a lattice or continuum with long-range interaction*, Discrete Contin. Dyn. Syst., **16** (2006), 235–252.

[3] A. Braides, "Γ-Convergence for Beginners," volume **22** of "Oxford Lecture Series in Mathematics and its Applications," Oxford University Press, Oxford, 2002.

[4] M. Fečkan and V. M. Rothos, *Travelling waves in Hamiltonian systems on 2D lattices with nearest neighbour interactions*, Nonlinearity, **20** (2007), 319–341.

[5] J. Frenkel and T. Kontorova, *On the theory of plastic deformation and twinning*, Acad. Sci. U.S.S.R. J. Phys., **1** (1939), 137–149.

[6] G. Friesecke and J. A. D. Wattis, *Existence theorem for solitary waves on lattices*, Comm. Math. Phys., **161** (1994), 391–418.

[7] C.-F. Kreiner, "A Variational Approach to Travelling Waves in Particle Chains with On-Site Potential," PhD thesis, Max Planck Institute for Mathematics in the Sciences / University of Leipzig, 2007.

[8] O. Kresse and L. Truskinovsky, *Mobility of lattice defects: discrete and continuum approaches*, J. Mech. Phys. Solids, **51** (2003), 1305–1332.

[9] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **1** (1984), 109–145.

[10] L. Modica and S. Mortola, *Un esempio di Γ^--convergenza*, Boll. Un. Mat. Ital. B (5), **14** (1977), 285–299.

[11] A. Pankov, "Travelling Waves and Periodic Oscillations in Fermi-Pasta-Ulam Lattices," Imperial College Press, London, 2005.

[12] H. Schwetlick and J. Zimmer, *Existence of dynamic phase transitions in a one-dimensional lattice model with piecewise quadratic interaction potential*, SIAM Journal on Mathematical Analysis **41** (2009), 1231–1271.

[13] D. Smets and M. Willem, *Solitary waves with prescribed speed on infinite lattices*, J. Funct. Anal., **149** (1997), 266–275.

[14] D. Williams, "Probability with Martingales," Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1991.

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