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Understanding Game Semantics through Coherence Spaces

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Abstract

Game Semantics has successfully provided fully abstract models for a variety of programming languages not possible using other denotational approaches. Although it is a flexible and accurate way to give semantics to a language, its underlying mathematics is awkward. For example, the proofs that strategies compose associatively and maintain properties imposed on them such as innocence are intricate and require a lot of attention. This work aims at beginning to provide a more elegant and uniform mathematical ground for Game Semantics. Our quest is to find mathematical entities that will retain the properties that make games an accurate way to give semantics to programs, yet that are simple and familiar to work with. Our main result is a full, faithful strong monoidal embedding of a category of games into a category of coherence spaces, where composition is simple composition of relations.

Keywords: Game Semantics, Coherence Spaces, relations, strategies

1   Introduction

Although Game Semantics is a flexible and accurate way to model the semantics of programming languages (see for example [3], [9], [1]), there is a vast proliferation of different categories of games, and very often there are basic structural facts (associativity, validity of composition) which are proved over and over again, with subtle differences each time. It therefore makes sense to attempt a study of the fundamental building blocks of game semantics, aimed at making the field more mathematically mature, with the hope that in future, one can concentrate on what is new or different when proposing a new category of games.

Harmer, Hyland and Mellies's work [8] can be seen as addressing a part of this question, focussing specifically on the exponentials and innocence of strategies. Our work takes a different approach, with the goal of explaining game semantics through the more familiar category of coherence spaces [7]. Motivated by the work of Hyland and Schalk [10], which presents a faithful functor from the category of games and deterministic strategies to the category of sets and relations, we ask if it is possible to provide a more elegant and uniform mathematical ground for Game Semantics.

In the category of games, a map $\sigma : A \multimap B$ is given by a strategy on a game $A \multimap B$, whose plays are interleavings of plays in A and in B. Hyland and Schalk's functor maps such a strategy to a relation between P_A and P_B (the set of plays of A and B) given by $\{(s \restriction_A, s \restriction_B) | s \in \sigma\}$. Faithfulness of this functor is somewhat surprising, because the functor eliminates the interleaving information, which seems to be the essence of a games model.

Though this functor is faithful, it is far from being full, and it does not preserve a lot of the structure of the category of games, for instance the monoidal structure. In this work we seek to improve on this situation by successively refining the codomain category. This is related to work by Hyland and Schalk [10] as discussed in further detail below.

The category of coherence spaces and linear maps (Coh) can be seen as a refinement of Rel by means of the coherence relation, whose purpose can be thought of as imposing determinacy on the naturally non-deterministic model of relations. We first show that Hyland and Schalk's functor lifts to a faithful functor from Games to Coh.

We then refine Coh by imposing an order relation, intended to mimic the prefix ordering on plays. We call this refined category $Pcoh$ and show that it posesses a monoidal structure akin to that of the category of Games. Our main result is that the Hyland-Schalk functor lifts to a fully faithful strong monoidal embedding into a certain subcategory of $Pcoh$.

1.1 Related Work

The work closest to ours is that of Hyland and Schalk [10], which provides a full and faithfully functor fromthe category of games and deterministic strategies to a category whose maps consists of relations, which may be seen as a generalised category of coherence spaces. However, the functor presented there is not strong monoidal, that is, the "interleaved parallel" monoidal operation characteristic of game semantics is lost. Our target category does possess such a monoidal structure, and we obtain a strong monoidal functor as a reult.

The work of [4,15,6] defines a "time-forgetting" operation on games which is not functorial but lax-functorial, and maps the game semantics of multiplicative-exponential linear logic onto its relational semantics. However, a somewhat different situation arises if one focuses on innocent strategies. Melliès's work [16,13,14] shows that innocent strategies are "relational", that is, can be characterized as relations between positions. Thus his work gives rise to a strong monoidal functor from a category of games and innocent strategies to Rel, which is in essence the same as the time-forgetting map.

Our interest is in the full range of potentially non-innocent strategies, because of their use in modelling imperative programming features. In this setting, the time-forgetting operation appears to correspond to the collapse from the game semantics of Idealized Algol [3] to Reddy's object-space semantics of interference-controlled Algol [17]. A proper analysis of the situation and its relationship with Melliès's results must be left for further work.

2 From Games to Relations

2.1 Preliminaries

We first define the category of games on which our work is based, and review Hyland and Schalk's faithful functor into Rel. We are not aware of a previously published proof of the faithfulness of this functor, so we provide one here.

Notation

Given sets X and Y, we denote by $Alt(X,Y)$ the set of sequences whose elements alternate between X and Y. Given a sequence $s \in Alt(X,Y)$ we write $s \restriction_X$ for the subsequence of s consisting only of elements of X and similarly for $s \restriction_Y$. We use $X+Y$ to denote the disjoint union of X and Y and X^* the set of sequences whose elements belong to X.

Given sequences s,t we write $|s|$ to denote the length of s. We use "$\sqsubseteq$" to denote the prefix order on sequences i.e. $t \sqsubseteq s$ if and only if there exists some sequence u such that $t \cdot u = s$. We write $t \sqsubseteq^{\text{even}} s$ if and only if $t \sqsubseteq s$ and $|t|$ is even and $t \sqsubseteq^{\text{odd}} s$ if and only if $t \sqsubseteq s$ and $|t|$ is odd.

Definition 2.1 A game, $A = (M_A, P_A)$, consists of a set $M_A = M_A^O + M_A^P$, called its set of moves, and a non-empty, prefix closed subset, P_A, of $Alt(M_A^O, M_A^P)$, called its set of plays, such that the first element of a sequence $s \in P_A$ belongs to M_A^O. We call the elements of M_A^O opponent moves and the elements of M_A^P player moves.

Remark 2.2 [Parity] Given a game A if $s \in P_A$ then, since the first element of s belongs to M_A^O and $s \in Alt(M_A^O, M_A^P)$, if s ends in an opponent move then $|s|$ is odd and if s ends in a player move then $|s|$ is even. This will be crucial in the work with coherence spaces.

Definition 2.3 A strategy σ on a game A consists of a non-empty set of even-length elements of P_A such that if $s \in \sigma$ and $t \sqsubseteq^{\text{even}} s$ then $t \in \sigma$.

Definition 2.4 Let E and F be coherence posets, define the coherence poset $E \multimap F$ as $(E \times F, \frown_{E \multimap F}, \preccurlyeq_{E \multimap F})$ where
$(e, f) \frown_{E \multimap F} (e', f')$ iff $e \frown_E e'$ implies that $f \frown_F f'$ and if $e \frown e'$ and $f \preccurlyeq f'$ then $e \preccurlyeq e'$.
and $(e, f) \preccurlyeq_{E \multimap F} (e', f')$ iff $e \preccurlyeq e'$ and $f \preccurlyeq f'$

Definition 2.5 A deterministic strategy σ on a game A is given by a strategy that satisfies *determinacy* i.e. if $sab, sac \in \sigma$, $|s|$ even, then $b = c$.

Definition 2.6 The game $A \multimap B$ is defined as

- $M_{A \multimap B}^O = M_B^O + M_A^P$
- $M_{A \multimap B}^P = M_B^P + M_A^O$
- $P_{A \multimap B} \subseteq Alt(M_{A \multimap B}^O, M_{A \multimap B}^P)$ consists of all sequences, s, such that $s \upharpoonright_A \in P_A$ and $s \upharpoonright_B \in P_B$.

Remark 2.7 Let $s \in P_{A \multimap B}$ and suppose x is the last move of s. Then if $x \in M_A^O$ or $x \in M_B^P$ then $|s|$ is even, and if $x \in M_A^P$ or $x \in M_B^O$ then $|s|$ is odd.

The category GAM has games as objects, and an arrow $A \to B$ is given by a deterministic strategy $\sigma : A \multimap B$. As usual, composition is given by parallel composition plus hiding and the identities given by copycat strategies (see, for example [12,9,2]).

Lemma 2.8 *Let $s \in P_{A \multimap B}$. If $|s|$ is odd then $|s \upharpoonright_A |$ is even and $|s \upharpoonright_B |$ is odd.*

From this lemma one can deduce that the only protagonist allowed to switch between components A and B in $A \multimap B$ is the player i.e. if $scx \in P_{A \multimap B}$ and $|s|$ is odd then if $c \in M_A$ (respectively M_B) then $x \in M_A$ (respectively M_B). This is called the *switching condition* [1].

Definition 2.9 Given games A and B, we define the game $A \otimes B$ as

- $M^O_{A \otimes B} = M^O_B + M^O_A$
- $M^P_{A \otimes B} = M^P_B + M^P_A$
- $P_{A \otimes B} \subseteq Alt(M^O_{A \otimes B}, M^P_{A \otimes B})$ consists of all sequences s such that $s \upharpoonright_A \in P_A$ and $s \upharpoonright_B \in P_B$.

Define $I_{GAM} := (\emptyset, \{\epsilon\})$. Let $\sigma : A \multimap B, \tau : C \multimap D$ be strategies and define

$$\sigma \otimes \tau := \{s \in P_{A \otimes C \multimap B \otimes D} | s \upharpoonright_{A,B} \in \sigma \text{ and } s \upharpoonright_{C,D} \in \tau\}$$

Proposition 2.10 $\otimes : GAM \times GAM \to GAM$ *equips GAM with a monoidal structure.*

Definition 2.11 The functor $grel : GAM \to Rel$ [10] is defined as follows: Its action on objects maps a game, A, to its set of plays, P_A; and, given games A and B, its action on morphisms maps a strategy $\sigma : A \multimap B$ to a relation $grel(\sigma) := \{(s \upharpoonright_A, s \upharpoonright_B) | s \in \sigma\} \subseteq P_A \times P_B$.

Observe that at first glance, this functor seems to destroy the interleaving information of plays. For example, consider a strategy $\sigma : A \multimap B$ with the plays:

A	$\multimap$	B	and	A	$\multimap$	B
		b_1				b_1
		b_2		a_1		
		b_3		a_2		
a_1						b_2
a_2						b_3
a_3				a_3		

Both these plays get mapped to $(a_1 a_2 a_3, b_1 b_2 b_3) \in grel(\sigma)$. However, prefix closure implies that $b_1 a_1, b_1 b_2 \in \sigma$ which breaks determinacy. It turns out that determinacy and prefix closure are enough to recover the interleaving information of plays; this will be crucial in proving that the functor is faithful. The following lemma formalizes this discussion.

Lemma 2.12 *Let $\sigma : A \multimap B$ be a deterministic strategy, with $p, p' \in \sigma$ player positions. If*

$$p \upharpoonright_A = p' \upharpoonright_A \text{ and } p \upharpoonright_B = p' \upharpoonright_B \tag{1}$$

then $p = p'$.

Proof. Let p and p' be as above. By the definition of $P_{A \multimap B}$, from $p \upharpoonright_B = p' \upharpoonright_B$ we know that p, p' have at least their initial moves in common, so we may take m to be the largest possible sequence of moves that p and p' have in common. If we assume, seeking a contradiction that $p \neq p'$, then (1) implies that $|p| = |p'|$

and it must be the case that there are sequences of moves s_1 and s_2 so that

$$p = ms_1 \text{ and } p' = ms_2 \text{ where } |s_1| = |s_2| \neq 0 \tag{2}$$

We will show that the moves from m exhaust p and exhaust p'. Suppose $|m|$ is odd, then we can write it as $m = m'a$ with a an opponent move. Substituting into (1) we have $p = m'ab_1s'_1$ and $p' = m'ab_2s'_2$ where $b_1s'_1 = s_1$ and $b_2s'_2 = s_2$ and $b_1 \neq b_2$. Because strategies are even-length prefix closed, it follows that $mab_1, mab_2 \in \sigma$ with $b_1 \neq b_2$ which contradicts determinacy. We conclude that m is of even length, and hence the first moves in s_1 and s_2 are done by opponent, by the switching condition they must be in the same component as a. By (1) $b_1 = b_2$. But by maximality of m, $b_1 \neq b_2$, so $s_1 = s_2 = \epsilon$ contradicting (2). We thus conclude that $p = p'$. □

The following lemma is an easy consequence of determinacy.

Lemma 2.13 *Let $\sigma : A \multimap B$ be a deterministic strategy. Let $p, p' \in \sigma$, $p \neq p'$. If $p \upharpoonright_A, p' \upharpoonright_A$ first differ at an opponent move (in A); then p, p' first differ in B.*

Proposition 2.14 *$grel : GAM \rightarrow Rel$ is faithful*

Proof. Let $\sigma, \tau : A \multimap B$ be deterministic strategies with $grel(\sigma) = grel(\tau)$. Let $p \in \sigma$ then there exists $p' \in \tau$ with

$$p \upharpoonright_A = p' \upharpoonright_A \quad \text{and} \quad p \upharpoonright_B = p' \upharpoonright_B \tag{3}$$

We will show that for all t such that $t \sqsubseteq^{even} p$, $t \sqsubseteq^{even} p'$:

Let $t \sqsubseteq^{even} p$. If $|t| = 0$ then $t = \epsilon$ and $t \sqsubseteq^{even} p'$. For the inductive step, suppose $|t| > 0$. Then $t = t'mn$ for some $t' \sqsubseteq^{even} p$; by induction, $t' \sqsubseteq^{even} p'$. Suppose $p' = t'm'n'$; we will show that $t'mn = t'm'n'$.

Since $|t|$ is even, by switching, m, m' lie in the same component. So, by (3) $m = m'$. Suppose $n \neq n'$. Now, (3) forces n, n' to belong to different components; suppose $n \in A$ and $n' \in B$ (the case when $n \in B$ and $n' \in A$ is dealt with symmetrically). Write $s = tmn$ and $s' = tmn'$. We know that $(s \upharpoonright_A, s \upharpoonright_B), (s' \upharpoonright_A, s' \upharpoonright_B) \in grel(\sigma)$ (since $grel(\sigma) = grel(\tau)$).

Hence, there exists some $s_1, s'_1 \in \sigma$ with

$$s_1 \upharpoonright_A = s \upharpoonright_A \quad s_1 \upharpoonright_B = s \upharpoonright_B \text{ and } s'_1 \upharpoonright_A = s' \upharpoonright_A \quad s'_1 \upharpoonright_B = s' \upharpoonright_B$$

Observe that $n \in A$ is an opponent move in A (since it is a player move in $A \multimap B$). Now, $s_1 \upharpoonright_A, s'_1 \upharpoonright_A$ first differ at an opponent move and by lemma 2.13, s_1, s'_1 first differ in B. Now, $s_1 \upharpoonright_B, s'_1 \upharpoonright_B$ first differ at $n' \in B$ which is a player move; but this contradicts determinacy of σ. Hence, $n = n'$ and we conclude that for every $t \sqsubseteq^{even} p$, $t \sqsubseteq^{even} p'$ as required. From this it follows

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that $p = p'$ and hence $\sigma \subseteq \tau$. A symmetric argument shows that $\tau \subseteq \sigma$ and hence $\sigma = \tau$ as required. □

Remark 2.15 The functor *grel* eliminates interleaving at the top-level only: the full detail of the plays in A and B is retained. This is in contrast with the "time-forgetting" operation studied in [4,15,6] which recursively removes interleavings in A and B.

3 From Games to Coherence Spaces

We move to the category *Coh* [7] which has as objects coherence spaces, and as morphisms relations subject to some constraints, and already we can see some game structure in this category. We establish that $grel : GAM \to Rel$ lifts to a faithful functor $gcoh : GAM \to Coh$.

Definition 3.1

- A *coherence relation* on a set E, denoted $\frown_E$, is a symmetric reflexive relation on E. We write $e_1 \smallfrown e_2$ if and only if $e_1 = e_2$ or $e_1 \not\smile e_2$.
- A *coherence space*, $(E, \frown_E)$, consists of a set, E, and a coherence relation $\frown_E$.
- A *configuration* of a coherence space E is a subset F of E so that $f_1 \frown f_2$ for every $f_1, f_2 \in F$.

Definition 3.2 We now describe a category, *Coh*. Its objects are given by coherence spaces. Given two coherence spaces E, F, an arrow $E \to F$ is given by a relation $\Gamma \subseteq E \times F$ such that for every $(e, f), (e', f') \in \Gamma$ if $e \frown_E e'$ then $f \frown_F f'$, and if $e \frown_E e'$ and $f = f'$ then $e = e'$. Composition and identity are as in the category of sets and relations, *Rel*.

Observe that the condition on the maps is equivalent to requiring that for all $(e, f), (e', f') \in \Gamma$ if $e \frown_E e'$ then $f \frown_F f'$ and if $f \asymp f'$ then $e \asymp e'$.

Definition 3.3 Let E and F be coherence spaces, we define the coherence space $E \multimap F$ as $(E \times F, \frown_{E \multimap F})$ where $(e, f) \frown_{E \multimap F} (e', f')$ if and only if $e \frown_E e' \Rightarrow (f \frown_F f'$ and $(f = f' \Rightarrow e = e'))$.

Observe that arrows $E \to F$ in the category *Coh* are given by configurations of $E \multimap F$.

Given a game, A with set of plays P_A, we can build a coherence space by defining $s \frown t$ if and only if the largest common prefix of s and t has even length or $s = t$. Then $(P_A, \frown)$ is a coherence space, which we denote $gcoh(A)$.

The following proposition shows that this notion of coherence precisely captures determinacy of strategies.

Proposition 3.4 *Given games A and B; a strategy $\sigma : A \multimap B$ is deterministic if and only if $grel(\sigma) : gcoh(A) \to gcoh(B)$ is a a map in Coh.*

Proof. Let A and B be games and suppose that $\sigma : A \multimap B$ is a deterministic strategy. Let $(s_1, s_2), (t_1, t_2) \in grel(\sigma)$ with $s_1 \smile t_1$. We know that there exists $s, t \in \sigma$ with $(s \upharpoonright_A, s \upharpoonright_B) = (s_1, s_2)$ and $(t \upharpoonright_A, t \upharpoonright_B) = (t_1, t_2)$. If $s = t$; then clearly $t_2 \smile s_2$ and $t_2 = s_2$ and $t_1 = s_1$.

Suppose that $s \neq t$; then by determinacy their first point of difference, $m \in M_{A \multimap B}$, must be an opponent move in $A \multimap B$ and hence either $m \in A$ and it is a player move or $m \in B$ and it is an opponent move. If $m \in A$ then $s_1 \not\smile t_1$, so it must be that $m \in B$; note that m is also the first point of difference between s_2 and t_2 and hence $s_2 \smile t_2$. Suppose $s_2 = t_2$; since we have just shown that s and t necessarily first differ in B, we must have $s_1 = t_1$ and therefore $grel(\sigma)$ is a configuration.

On the other hand, suppose that $grel(\sigma)$ is a configuration. Let $sab, sac \in \sigma$ where $|s|$ is even and $a, b, c \in M_{A \multimap B}$. We claim that $b = c$. Observe that b, c are player moves in $A \multimap B$ and hence each is either an opponent move in A or a player move in B.

- $b, c \in A$ then observe that $sab \upharpoonright_A = sa \upharpoonright_A \cdot b \quad sac \upharpoonright_A = sa \upharpoonright_A \cdot c$. If $b \neq c$; then s, t first differ at b (or c) which is an opponent move in A. Hence $sab \upharpoonright_A \smile sac \upharpoonright_A$. Now, $sab \upharpoonright_B = sac \upharpoonright_B$ so it must be, since $grel(\sigma)$ is a configuration, that $sab \upharpoonright_A = sac \upharpoonright_A$ and so $b = c$.

- $b, c \in B$. We have that $sab \upharpoonright_A = sa \upharpoonright_A = sac \upharpoonright_A$ and hence $sab \upharpoonright_A \smile sac \upharpoonright_A$ so $sab \upharpoonright_B \smile sac \upharpoonright_B$. Since b, c are player moves it follows that $|sa \upharpoonright_B|$ is odd and it must be that $b = c$ else $sab \upharpoonright_B \not\smile sab \upharpoonright_C$

- b, c lie in different components of $A \multimap B$; wlog $b \in B$, $c \in A$. Then we have $sab \upharpoonright_A = sa \upharpoonright_A$ and $sac \upharpoonright_A = sa \upharpoonright_A \cdot c$, c is an opponent move in A and so $|sa \upharpoonright_A|$ is even and hence $sab \upharpoonright_A \smile sac \upharpoonright_A$; this implies that $sab \upharpoonright_B \smile sac \upharpoonright_B$. Now, $sab \upharpoonright_B = sa \upharpoonright_B \cdot b$ and $sac \upharpoonright_B = sa \upharpoonright_B$ but b is a player move and so $|sa \upharpoonright_B|$ is odd which contradicts $sab \upharpoonright_B \smile sac \upharpoonright_B$ and hence this case does not happen.

$\square$

Corollary 3.5 *grel lifts to a faithful functor $gcoh : GAM \to Coh$.*

4   From Games to Coherence Posets

We now impose a little more game-like structure on Coh and get a new category which we call *Pcoh*, more specifically we impose an order relation, intended to mimic prefix ordering on games.

Definition 4.1 A coherence poset, $(E, \smile_E, \preccurlyeq_E)$, consists of a partially or-

dered set E with a least element $\bot \in E$ and a coherence relation $\frown_E$ on E that satisfies:

- $e \frown \bot$ for any $e \in E$.
- $e_1 \frown e_2, e_1 \prec e_2, e_2 \frown e_3 , e_2 \prec e_3$ imply that $e_1 \frown e_3$, for any $e_1, e_2, e_3 \in E$.

Definition 4.2 We now describe a category, *Pcoh*. Its objects are given by coherence posets. Given two coherence posets E and F, an arrow $E \to F$ is given by a relation $\Gamma \subseteq E \times F$ such that $(\bot, \bot) \in \Gamma$, and, for every $(e, f), (e', f') \in \Gamma$ if $e \frown_E e'$ then $f \frown_F f'$, and, if $e \frown_E e'$ and $f' \preccurlyeq_F f$ then $e' \preccurlyeq_E e$.
Composition is relational composition and, given a coherence poset E, the identity $id_E : E \to E$ is given by the identity relation.

Definition 4.3 Let E and F be coherence posets, define the coherence poset $E \multimap F$ as $(E \times F, \frown_{E \multimap F}, \preccurlyeq_{E \multimap F})$ where
$(e, f) \frown_{E \multimap F} (e', f')$ iff $e \frown_E e'$ implies that $f \frown_F f'$ and if $e \frown e'$ and $f \preccurlyeq f'$ then $e \preccurlyeq e'$.
and $(e, f) \preccurlyeq_{E \multimap F} (e', f')$ iff $e \preccurlyeq e'$ and $f \preccurlyeq f'$

Observe that maps $E \to F$ in the category *Pcoh* are given by configurations of the coherence poset $E \multimap F$.

Given a game A, we can build a coherence poset $(P_A, \frown, \preccurlyeq)$: P_A is the set of plays of A, with coherence relation as before, and $\preccurlyeq$ is the prefix ordering. The functor $gcoh : GAM \to Coh$ lifts to a faithful functor $gcoh : GAM \to Pcoh$. A game A gets mapped to a coherence poset as described above and a strategy σ gets mapped to a configuration $gcoh(\sigma)$.

Remark 4.4 We write $gcoh$ for both functors $gcoh : GAM \to Coh$ and $gcoh : GAM \to Pcoh$; this should cause no confusion as it will be clear from the context what the codomain category is.

Let $(F, \frown, \preccurlyeq)$ be a coherence poset; we write $t \prec s$ if $t \preccurlyeq s$ and $t \neq s$. And we write $t \prec^{\max} s$ whenever t is maximal in F such that $t \prec s$.

Definition 4.5 Let E, F be coherence posets and suppose that $\Gamma : E \to F$ is a configuration. We call Γ a *configuration with memory* if for every $(e, f) \in \Gamma$ if there exists some $f' \in F$ such that $f' \frown f$ and $f' \prec f$ then there exists a unique e' such that $e' \preccurlyeq e$, $e' \frown e$ and $(e', f') \in \Gamma$. And, if there exists some $e' \in E$ with $e' \asymp e$ and $e' \prec e$ then there exists a unique $f' \preccurlyeq f$ such that $f' \asymp f$ and $(e', f') \in \Gamma$.

Lemma 4.6 *There is a subcategory $Pcoh_m$ of $Pcoh$ where the objects of $Pcoh_m$ are those of $Pcoh$, and whose maps consist of configurations with memory.*

Now, the functor $gcoh : GAM \to Pcoh_m$ is such that we can recover prefix-closure of σ from $gcoh(\sigma)$. However, it is still not full: there is nothing corresponding to the switching condition, so given $(s,t) \in \Gamma : gcoh(A) \to gcoh(B)$ we cannot always interleave them in an alternating fashion to recover a play of $A \multimap B$. Further, Γ may contain some odd-length sequences. The condition below addresses both of these problems.

Remark 4.7 We now only deal with objects, X, of $Pcoh_m$ that satisfy for every $s \in X$, $\{s' | s' \prec s\}$ is finite.

Definition 4.8 Let E, F be coherence posets. A configuration $\Gamma : E \multimap F$ *satisfies switching* if for every $(e, f) \in \Gamma$, there exists some $e' \in E$ with $e' \prec^{max} e$ and $e \frown e'$ if and only if there exists some $f' \in F$ with $f' \prec^{max} f$ and $f' \frown f$.

Corollary 4.9 *We define a subcategory, $Pcoh_{m,s}$, of Pcoh whose arrows are given by configurations with memory that satisfy switching.*

Lemma 4.10 *Suppose $\Gamma : gcoh(A) \multimap gcoh(B)$ is a configuration that satisfies switching. Let $(s,t) \in \Gamma$, then $|s|$ is even if and only if $|t|$ is even.*

Proof. Follows from switching and the observation that whenever $s, t \in gcoh(C)$ for some C, $s \prec^{\max} t$ $s \frown t$ implies that $|s|$ is even and $|t|$ is odd, and $s \prec^{\max} t$ $s \asymp t$ implies that $|s|$ is odd and $|t|$ is even. $\square$

Theorem 4.11 *$gcoh : GAM \to Coh$ lifts to a fully faithful functor $gcoh_m : GAM \to Pcoh_{m,s}$*

Proof. Faithfulness has already been established, we proceed to demonstrating that the functor is full.

Let A, B be games and suppose that $\Gamma : gcoh(A) \multimap gcoh(B)$ is a configuration. We will now inductively define interleavings, u, of elements $(s,t) \in \Gamma$ and show that $u \in P_{A \multimap B}$. We denote the collection of all such u as σ. We map $(s,t) \in \Gamma$ to $u \in \sigma$ as follows:

Map $(\epsilon, \epsilon) \in \Gamma$ to $\epsilon \in \sigma$.

First observe that $(s, \epsilon) \notin \Gamma$ for any non-empty $s \in P_A$. So the cases are:

If $(\epsilon, b_1 b_2) \in \Gamma$, map $(\epsilon, b_1 b_2) \in \Gamma$ to $b_1 b_2 \in \sigma$.

If $(a_1, b_1) \in \Gamma$, map $(a_1, b_1) \in \Gamma$ to $b_1 a_1 \in \sigma$.

Then given $(s', t') \in \Gamma$ with $u \in \sigma$ already defined ($u \in P_{A \multimap B}$ such that $u \upharpoonright_A = s'$ and $u \upharpoonright_B = t'$), search for all elements $(s,t) \in \Gamma$ such that $(s', t') \prec^{\max} (s,t)$, map $(s,t) \in \Gamma$ to $v \in \sigma$ as follows:

Observe that if $|s'|, |t'|$ are both odd then s extends s' by at most two moves, and t extends and t' by at most one move. For if $t' \cdot b_i \prec t$, $t' \cdot b_1 \frown t$ and memory imply that there exists a unique $s'' \preccurlyeq s$ $s'' \frown s$ and $(s'', t' \cdot b_1) \in \Gamma$.

But then $(s', t') \prec (s'', t' \cdot b_1) \prec (s, t)$ contradicts $(s', t') \prec^{\max} (s, t)$. And if $s \succ s' \cdot a_i a_{i+1}$ then, since $s' \cdot a_i a_{i+1} \asymp s$ it follows that $(s' \cdot a_i a_{i+1}, t'') \in \Gamma$ for a unique t'' which again contradicts maximality.

If $(s, t) = (s' \cdot a_i a_{i+1}, t')$ then $v = u \cdot a_i a_{i+1}$.

If $(s, t) = (s' \cdot a_i, t' \cdot b_i)$ then $v = u \cdot a_i b_i$ if the last move of u is the last move of s' and $v = u \cdot b_i a_i$ if the last move of u is the last move of t'.

If $(s, t) = (s', t' \cdot b_i)$, or $(s, t) = (s' \cdot a_i a_{i+1}, t' \cdot b_i)$, or $(s, t) = (s' \cdot a_i, t')$ then this contradicts switching and hence these elements cannot exist in Γ.

If $|s'|, |t'|$ are both even then similarly to above, s extends s' by at most one move and t extends t' by at most two moves.

Moreover, observe that (s, t) cannot be of the form $(s' \cdot a_i, t' \cdot b_i b_{i+1})$, $(s', t' \cdot b'_i)$ or $(s, t) \neq (s' \cdot a_i, t')$ because Γ satisfies switching.

If $(s, t) = (s', t' \cdot b_i b_{i+1})$ then $v = u \cdot b_i b_{i+1}$.

If $(s, t) = (s' \cdot a_i, t' \cdot b_i)$ then $v = u \cdot b_i a_i$ if the last move of u equals the last move of t', and $v = u \cdot a_i b_i$ if the last move of u equals the last move of s'.

To see that these sequences give plays in $A \multimap B$ observe that the first element of a sequence defined as above is always an element of B. That the alternating condition is satisfied follows from switching, the fact that for any $(s, t) \in \Gamma$ s and t are plays and therefore alternating and the way we constructed the sequences.

It can be shown that the collection σ of all such plays forms a strategy: prefix closure follow from the memory condition, and determinacy by an argument similar to Lemma 3.4. □

4.1   Monoidal Structure

We now outline a monoidal structure on *Pcoh*. With this structure the functor $gcoh : GAM \to Pcoh$ is strong monoidal.

Remark 4.12 We index our sequences starting at 1, so that the elements of a sequence s of length n are $s_1, s_2, ..., s_n$. By convention we let s_0 denote $\perp$.

Definition 4.13 Let E, F be coherence posets. We define the coherence poset $E \otimes F$ as follows; the set $E \otimes F$ is given by all $s \in Alt(E \backslash \{\perp\}, F \backslash \{\perp\})$ that satisfy

$$s_{i-2} \prec_X s_i \forall i \in \{2, \ldots, |s|\} X = E, F$$
$$s_{i-2} \frown_X s_i \forall i \in \{2, \ldots, |s|\} X = E, F.$$

$\frown_{E \otimes F}, \preccurlyeq_{E \otimes F}$ are respectively defined as ([17]) $s \frown_{E \otimes F} t$ if and only if either s_1 and t_1 lie in different coherence posets, or for every $n \leq \min\{|s|, |t|\}$ $s_1 ... s_n = t_1 ... t_n$ implies that $s_{n+1} \frown t_{n+1}$.

$s \preccurlyeq_{E \otimes F} t$ if and only if $s \sqsubseteq t$ or $s = s_1 \ldots s_{i+1}$, $t = t_1 \ldots t_{i+n}$, $n \geq 1$ and $s_1 \ldots s_i = t_1 \ldots t_i$ and $s_{i+1} \preccurlyeq t_{i+1}$.

The idea behind the definition of the tensor is to mimic what happens in games as the following example demostrates. Consider the game N, with $M_N := \{q\} \cup \mathbb{N}$, on the left is a play in $N \otimes N$ and on the right the equivalent sequence in $gcoh(N) \otimes gcoh(N)$.

N	$\otimes$	N		$gcoh(N) \otimes gcoh(N)$
		q		$q1q2$
		1	$q7$	
		q		$q1q2q4$
		2	$q7q6$	
q				
7				
		q		
		4		
q				
6				

Observe that $q1\,q2 \prec_{gcoh(N)} q1\,q2\,q4$, since $q1\,q2 \sqsubset q1\,q2\,q4$ and $q1\,q2 \frown_{gcoh(N)} q1\,q2\,q4$, since $|q1\,q2|$ is even.

And similarly, $q7 \prec_{gcoh(N)} q7\,q6$ and $q7 \prec_{gcoh(N)} q7\,q6$.

So $s = (s_1)(s_2)(s_3)(s_4) = (q1\,q2)(q7)(q1\,q2\,q4)(q7\,q6) \in gcoh(N) \otimes gcoh(N)$.

Lemma 4.14 *We can extend $\otimes : Pcoh_{m,s} \times Pcoh_{m,s} \to Pcoh_{m,s}$ to a bifunctor*

The tensor unit is defined as $I := (\{\bot\}, id, id)$.
Given configurations $\Gamma : E \to F$ and $\Delta : G \to H$, we define $\Gamma \otimes \Delta$ as
$\{(s,t) \in E \otimes G \to F \otimes H | \forall i \leq |s| \exists j \leq |t|$ s. t.
if $s_i \in E$ then $t_j \in F$ and $(s_i, t_j) \in \Gamma$
if $s_i \in G$ then $t_j \in H$ and $(s_i, t_j) \in \Delta$
and
$\forall i' < i \exists j' \leq j$ such that
 if $s_{i'} \in E$ then $u_{j'} \in F$ and $(s_{i'}, u_{j'}) \in \Gamma$
if $s_{i'} \in G$ then $u_{j'} \in H$ and $(s_{i'}, u_{j'}) \in \Delta$ and
$\forall j' < j \exists i' \leq i$ such that
if $u_{j'} \in F$ then $s_{i'} \in E$ and $(s_{i'}, u_{j'}) \in \Gamma$
if $u_{j'} \in H$ then $s_{i'} \in G$ and $(s_{i'}, u_{j'}) \in \Delta$

$\forall j \leq |u| \exists i \leq |s|$ s. t.
if $t_j \in F$ then $s_i \in E$ and $(s_i, t_j) \in \Gamma$
if $t_j \in H$ then $s_i \in G$ and $(s_i, t_j) \in \Delta$
and
$\forall i' < i \exists j' \leq j$ such that
if $s_{i'} \in E$ then $u_{j'} \in F$ and $(s_{i'}, u_{j'}) \in \Gamma$
if $s_{i'} \in G$ then $u_{j'} \in H$ and $(s_{i'}, u_{j'}) \in \Delta$ and

$\forall j' < j \exists i' \leq i$ such that

if $u_{j'} \in F$ then $s_{i'} \in E$ and $(s_{i'}, u_{j'}) \in \Gamma$

if $u_{j'} \in H$ then $s_{i'} \in G$ and $(s_{i'}, u_{j'}) \in \Delta$}

Th following lemma gives an alternative characterization of the tensor operation, and assists in proving its functoriality.

Lemma 4.15 *Let $\Gamma : E \to F$ and $\Delta : G \to H$ be configurations. Then, $(s, t) \in \Gamma \otimes \Delta$ if and only if there exists a unique function $f : \{0, \ldots, |t|\} \to \{0, \ldots, |s|\}$ which is order-preserving and surjective such that $(s_{f(i)}, t_i) \in \Gamma$ or $(s_{f(i)}, t_i) \in \Delta$ for all $i \leq |t|$.*

Lemma 4.16 $\otimes : Pcoh_{m,s} \times Pcoh_{m,s} \to Pcoh_{m,s}$ *equips $Pcoh_{m,s}$ with a monoidal structure*

The majority of the work is in showing that coherent associativity isomorphisms exist; we give the definition here. Given coherence posets E, F and G, the isomorphism $\gamma : (E \otimes F) \otimes G \to E \otimes (F \otimes G)$ is given by the composition of two isomorphisms $\alpha : (E \otimes F) \otimes G \to E \otimes F \otimes G$ and $\beta : E \otimes F \otimes G \to E \otimes (F \otimes G)$, where $E \otimes F \otimes G \subseteq ((E + F + G) \backslash \bot)^*$ is the evident ternary version of the tensor.

Given a sequence, $s \in (E \otimes F) \otimes G$, for example $s = (e_1 f_1 e_2) g_1 (e_1 f_1 e_2 f_2) g_2$ the isomorphism α eliminates all repetition in $E \otimes F$ arriving at a sequence $s = e_1 f_1 e_2 g_1 f_2 g_2$. The isomorphism β rebrackets this sequence, repeating elements where necessary to obtain a sequence $t \in E \otimes (F \otimes G)$, in this case $t = e_1(f_1)e_2(f_1 g_1 f_2 g_2)$.

α has a straightforward inductive definition:

- $\alpha(\epsilon) = \epsilon$

- $\alpha(s) = s$ if $s = g$, $g \in G$ or $s = s_1 \ldots s_n \in E \otimes F$.

- $\alpha(s) = \alpha(s_1 \ldots s_i) \cdot s_{i+1}$ if $|s| = i + 1$, $s_{i+1} \in G$.

- $\alpha(s) = \alpha(s_1 \ldots s_i) \cdot s_{i+1} \backslash u$ if $|s| = i + 1$ $s_{i+1} \in E \otimes F$ where u is the largest common prefix of s_{i+1} and s_{i-1} if $i > 1$ and $u = \epsilon$ otherwise, and $s_{i+1} \backslash u$ denotes s_{i+1} with the prefix u deleted.

In the definition of β, care must be taken because, given for example $e_1 f_1 e_2 f_2 g_1 f_3 g_2$ we must be careful to produce a sequence $e_1(f_1)e_2(f_2 g_1 f_3 g_2)$ and not the sequence $e_1(f_1)e_2(f_1 f_2 g_1 f_3 g_2) \notin E \otimes (F \otimes G)$. For this purpose we define the operator $\sharp$ where given sequences t, t', $|t| = i$, $|t'| = n$

$t \sharp t' = t_1 \ldots t_i \cdot t'_1 \ldots t'_n$ if $t_i \not\preccurlyeq t'_1$ and

$t \sharp t' = t_1 \ldots t_{i-1} \cdot t'_1 \ldots t'_n$ if $t_i \preccurlyeq t'_1$.

We define $\beta : E \otimes F \otimes G \to E \otimes (F \otimes G)$, as, given $t \in E \otimes F \otimes G$, $t = t_1 \ldots t_i t_{i+1} \ldots t_{i+n}$ with $t_{i+1} \ldots t_{i+n} \notin E$, $t_i \in E$,

If $n = 0$, then $\beta(t) = (\beta(t_1 \ldots t_{i-1})) \cdot t_i$, else:

- if $i = 0$ then $\beta(t) = (t_1 \ldots t_n)$
- if $i = 1$ then $\beta(t) = t_1(t_2 \ldots t_{n+1})$
- if $i > 1$ then $\beta(t) = \beta(t_1 \ldots t_i) \cdot (\ \mathrm{last}(\beta(t_1 \ldots t_{i-1}))\sharp t_{i+1} \ldots t_{i+n}))$ where $\mathrm{last}(\beta(t_1 \ldots t_{i-1}))$ is the last element of $\beta(t_1 \ldots t_{i-1})$.

We will now define a natural isomorphism $\delta : gcoh(A) \otimes gcoh(B) \to gcoh(A \otimes B)$. δ is given by:

- $\delta(\epsilon) = \epsilon$
- $\delta(s) = s$ if $|s| = 1$
- $\delta(s \cdot s_i) = \delta(s) \cdot s_i \backslash s_{i-2}$.

We illustrate the definition by example. Let A and B be games and let $s_i, t_i, i \in \mathbb{N}$ be plays in A and B respectively The isomorphism $\delta : gcoh(A) \otimes gcoh(B) \to gcoh(A \otimes B)$ maps a sequence

$gcoh(A) \otimes gcoh(B)$	to a play	A	$\otimes$	B
s_1				t_1
s_1			s_1	
$t_1 t_2$				t_2
$s_1 s_2$			s_2	
$t_1 t_2 t_3$				t_3
$s_1 s_2 s_3$			s_3	

Straightforward verification that δ is a natural isomorphism satisfying appropriate coherence diagrams, together with Theorem 4.11, yields:

Theorem 4.17 $gcoh : GAM \to Pcoh_{m,s}$ *is a full and faithful strong monoidal functor.*

5 Future Work

As an extension to the current work we will analyse more of the categorical structure of $Pcoh$, including an investigation of the sequoidal structure (in Laird's terminology [11]) and the linear exponential comonad it induces on $Pcoh$. It would also be interesting to investigate those parts of $Pcoh$ which lie outside the image of $gcoh$. Perhaps this category gives us access to "game-like" objects which cannot readily be expressed as games, and which are useful in modelling logics or programming languages. For example, can we have game-like structures that do not always have an assigned initial move, and if so what could we model with such structures.

As indicated earlier, the connection with the "time-forgetting" map of Baillot *et al.* [5] should be studied more closely, particularly in the light of

Melliès's work on positionality and innocent strategies [16]. It is worth studying both how innocent strategies can be located in *Pcoh*, perhaps using the techniques of [8], and how Melliès's work can be extended to the full range of history-sensitive strategies. We believe that this will establish a deeper connection between games models and Reddy's object-spaces model [17], but this remains to be seen.

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