Smooth rationally connected threefolds contain all smooth curves

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It is very easy to see that every smooth projective curve can be embedded in $\mathbb{P}^3$. Eisenbud and Harris asked whether the same is true if $\mathbb{P}^3$ is replaced by an arbitrary smooth rational projective 3-fold X and Eisenbud suggested starting with the case where X is toric. In that case the answer is yes, and one can see that in a very explicit way, as was done in my preprint [Sa2].

In response to [Sa2], it was pointed out by János Kollár that much more is true: the property of containing every curve sufficiently often actually characterises rationally connected 3-folds over the complex numbers. In fact, this is already implicit in the work of Kollár and others on rational curves in algebraic varieties, but had apparently not been directly noticed.

The purpose of this note is to explain these facts. In the first part I follow Kollár's hints and show how to assemble a proof of the characterisation of rationally connected 3-folds (Theorem 1.8). In the second part, which is a shortened version of [Sa2], I show explicitly (Theorem 2.5) how to construct an embedding of a given curve into a given smooth projective toric 3-fold by toric methods.

Acknowledgements: Much of this paper is really due to other people. David Eisenbud asked me the question and drew Kollár's attention to my partial solution. Dan Ryder listened patiently to me while I tried to answer the toric version. The toric case uses ideas developed long ago in conversation with Tadao Oda. Most importantly, János Kollár pointed out in a series of increasingly simple emails how to obtain better results, and then allowed me to use his ideas. I thank all of them, and also the several people who, by asking me about [Sa2], encouraged me to write this version.

1 Rationally connected varieties

In this section X is a smooth projective variety over an algebraically closed field.

1.1 RC and SRC

We recall some standard definitions from [Ko2] and [AK].

Definition 1.1 *[Ko2, IV.3.2.3] X is* separably rationally connected, *abbreviated* SRC, *if there exists a variety W and a morphism $u\colon \mathbb{P}^1 \times W \to X$ such that*

$$u^{(2)}\colon (\mathbb{P}^1 \times W) \times_W (\mathbb{P}^1 \times W) \cong \mathbb{P}^1 \times \mathbb{P}^1 \times W \longrightarrow X \times X$$

is dominant.

In other words, X is SRC if for all $x_1,\ x_2$ in some Zariski-dense subset of X we can find $w \in W$ and $P_1,\ P_2 \in \mathbb{P}^1$ such that $u(P_i, w) = x_i$ for $i = 1,\ 2$.

Definition 1.2 *[Ko2, IV.3.2.2] X is* rationally connected, *abbreviated* RC, *if there exists a variety W, a proper family $U \to W$ whose fibres are irreducible rational curves, and a morphism $u\colon U \to X$ such that*

$$u^{(2)}\colon U \times_W U \longrightarrow X \times X$$

is dominant.

Clearly SRC $\implies$ RC, and the converse is also true in characteristic zero ([Ko2, IV.3.3.1]).

Definition 1.3 *[AK, Definition 8] A morphism $f\colon \mathbb{P}^1 \to X$ is said to be* very free *if f^*T_X is an ample vector bundle.*

Recall that a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ is ample if and only if $\mathcal{E} = \bigoplus \mathcal{O}(a_j)$ with all $a_j > 0$.

Lemma 1.4 *[Ko2, IV.3.9] If X is a smooth projective SRC variety then there exists a very free map $g_0\colon \mathbb{P}^1 \to X$.*

1.2 Maps from curves

We can use Lemma 1.4 to obtain results about maps from curves to SRC varieties.

Lemma 1.5 *If X is a smooth projective SRC variety, then for any smooth projective curve C there is a map $g\colon C \to X$ such that $H^1\big(g^*T_X(-P-Q)\big) = 0$ for any two distinct points $P,\ Q \in C$.*

Proof. Suppose C has genus g. We choose a map $g_1\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $g_1^*g_0^*T_X$ is sufficiently ample: it is enough to require that $g_1^*g_0^*T_X \cong \bigoplus \mathcal{O}(a_i)$ with each $a_i > 2g$, which can be achieved by taking g_1 to have sufficiently high degree. Take any surjection $g_2\colon C \to \mathbb{P}^1$ and let F be a fibre of $g_1g_2\colon C \to \mathbb{P}^1$. If we put $g = g_0g_1g_2\colon C \to X$, we have

$$g^*T_X(-P-Q) = \bigoplus \mathcal{O}_C(a_iF - P - Q).$$

This is the direct sum of line bundles of degree $> 2g-2$ and therefore $H^1\big(g^*T_X(-P-Q)\big) = 0$. $\square$

Proposition 1.6 *If C is any smooth projective curve and X is any smooth SRC projective variety of dimension ≥ 3, then C can be embedded in X.*

Proof. Choose $g\colon C \to X$ as in Lemma 1.5. According to [Ko2, II.1.8.2] (with $B = \emptyset$), a general deformation of g is an embedding because

$$\dim H^1\big(g^*T_X(-P-Q)\big) \leq \dim X - 3 = 0.$$

$\square$

Over the complex numbers we can show more.

Lemma 1.7 *Let X be any smooth quasi-projective variety over $\mathbb{C}$, and suppose $x \in X$. Then there exists a subset $X_1(x) \subset X$, the complement of a countable union of Zariski-closed sets, such that if $y \in X_1(x)$ and if the image of $f\colon \mathbb{P}^1 \to X$ passes through both x and y then f is very free.*

Proof. This follows from [AK, Proposition 13], exactly as [AK, Remark 10] follows from [AK, Proposition 10]. We consider one of the countably many irreducible components R of $\mathrm{Hom}_x(\mathbb{P}^1, X) = \{f\colon \mathbb{P}^1 \to X \mid f(0) = x\}$ and the evaluation morphism $u_R\colon \mathbb{P}^1 \times R \to X$ given by $u_R(P, f) = f(P)$. The morphisms that are not very free form a closed subscheme $R' \subseteq R$: but $u_R|_{\mathbb{P}^1\times R'}$ is not dominant because of [AK, Proposition 13(2)], so its image lies in a proper closed subset $X_R \subset X$. So any f that is not very free has image contained in some X_R, and we take $X_1 = X \setminus \bigcup_R X_R$. $\square$

This yields a characterisation of RC varieties of dimension ≥ 3 in terms of maps from curves.

Theorem 1.8 *If X is a smooth projective variety of dimension ≥ 3 over $\mathbb{C}$, then X is rationally connected if and only if the following holds: for every smooth projective curve C and zero-dimensional subscheme $Z \subset C$, and every embedding $f_Z\colon Z \hookrightarrow X$, there is an embedding $f_C\colon C \hookrightarrow X$ such that $f_C|_Z = f_Z$.*

Proof. One direction is trivial: if every f_Z extends then taking $C = \mathbb{P}^1$ and $Z = \{0, 1\}$ we recover the definition of RC.

Conversely, suppose that X is RC of dimension at least 3 and suppose first that $Z = \{P_1, \ldots, P_n\}$ is reduced, and write $x_i = f(P_i)$. If $Z = \emptyset$ then the statement reduces to Proposition 1.6. Otherwise, we may choose $x_0 \in X_1(x_1)$ as in Lemma 1.7. By [Ko1, (4.1.2.4)] we can find a map $f_0\colon \mathbb{P}^1 \to X$ such that $x_0, \ldots, x_n$ are all in the image of f_0. See also [Ko1, (5.2)]. If Z is not reduced, we can still arrange for $f_C|_Z = f_Z$ because [Ko1, (4.1.2.4)] allows us to specify the Taylor expansion of f_0 as far as we like.

The map f_0 is very free by Lemma 1.7. Exactly as in Lemma 1.5 we may compose f_0 with suitable maps $f_2\colon C \to \mathbb{P}^1$ and $f_1\colon \mathbb{P}^1 \to \mathbb{P}^1$ so as to get a map $f\colon C \to X$ such that $f|_Z = f_Z$ and $\dim H^1\big(C, f^*T_X \otimes I_Z(-P-Q)\big) = 0$ for every $P,\ Q \in C$. To do this we choose f_2 first to be a surjection such that $f_2|_Z$ is an isomorphism. Then we choose f_1, of sufficiently large degree, so that $f|_Z = f_Z$: to do so we need only choose a polynomial with prescribed values at each point of $f_2(Z)$ and injective on $f_2(Z)$, which is trivial to do as the degree of f_1 may be as large as we please. Although f need not be an embedding, we may take f_C to be a general deformation of f preserving $f|_Z$, and this is an embedding by [Ko2, II.1.8.2]. $\square$

Remark 1.9 *The condition in Theorem 1.8 that C be smooth is not strictly necessary. It is enough for C to be a reduced curve whose singularities have embedding dimension $\leq \dim X$.*

Indeed, let $\nu\colon \widetilde{C} \to C$ be the normalisation. Consider the subscheme $Z' = Z \cup \operatorname{Sing} C \subset C$ and let $\widetilde{Z}$ be the subscheme of $\widetilde{C}$ given by $I_{\widetilde{Z}} = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_{\widetilde{C}}, I_{Z'}) \cdot \mathcal{O}_{\widetilde{C}}$. If $f_{Z'}$ is an embedding of Z' in X, extending f_Z, then the argument above allows us to extend $f_{\widetilde{Z}} = f_{Z'}\nu\colon \widetilde{Z} \to X$ to $f_{\widetilde{C}}\colon \widetilde{C} \to X$ in such a way that $f_{\widetilde{C}}$ is an embedding away from $\widetilde{Z}$. The image of $f_{\widetilde{C}}$ is then isomorphic to C.

2 Toric varieties

In this section we look at the particular case in which X is a smooth projective toric 3-fold over $\mathbb{C}$. As toric varieties are rational, they are in particular SRC, so by Proposition 1.6 a smooth projective toric 3-fold contains all curves. However, in the toric case it is possible to give a more direct proof, and one that shows rather more concretely how to construct an embedding of a given curve in a given toric variety X.

2.1 Maps to toric varieties

We need a good description of maps to a smooth projective toric variety. Several descriptions available of maps to toric varieties exist, due to Cox [Co], Kajiwara [Ka] and others. The version that we use here appeared in [Sa1, Section 2][1] but the proof, which is largely due to Tadao Oda, is very short, so we give it here. We refer to [Od] for general background on toric varieties.

Let Δ be a finite (but not necessarily complete) smooth fan for a free $\mathbb{Z}$-module N of rank r. Denote the corresponding toric variety by X, and write M for the dual lattice $\operatorname{Hom}(N, \mathbb{Z})$, with pairing $\langle\ ,\ \rangle\colon M \times N \to \mathbb{Z}$. The torus is then $\mathbb{T} = \operatorname{Spec}(\mathbb{C}[M])$, where $\mathbb{C}[M] = \bigoplus_{m \in M} \mathbb{C}\mathbf{e}(m)$ is the semigroup ring of M over $\mathbb{C}$. Here, as in [Od, 1.2], the character $\mathbf{e}(m)\colon \mathbb{T} \to \mathbb{C}^*$ may be thought of after identifying M with $\mathbb{Z}^n$ as $(u_1, \ldots, u_n) \mapsto u_1^{m_1} \ldots u_n^{m_n}$ for suitable coordinates u_i on $\mathbb{T} \cong (\mathbb{C}^*)^n$.

[1] Warning: some other unrelated parts of [Sa1] are incorrect.

As usual, $\Delta(d)$ denotes the set of d-dimensional cones in Δ. For each $\rho \in \Delta(1)$, we denote by $V(\rho)$ the corresponding irreducible Weil divisor on X and by n_ρ the generator of the semigroup $N \cap \rho$.

Theorem 2.1 *Let Y be a normal algebraic variety over $\mathbb{C}$. A morphism $f\colon Y \to X$ such that $f(Y) \cap \mathbb{T} \neq \emptyset$ corresponds to a collection of effective reduced Weil divisors $D(\rho)$ on Y indexed by $\rho \in \Delta(1)$ and a group homomorphism $\varepsilon\colon M \to \mathcal{O}_Y(Y_0)^\times$ to the multiplicative group of invertible regular functions on $Y_0 = Y \setminus \bigcup_{\rho\in\Delta(1)} D(\rho)$, such that*

$$D(\rho_1) \cap D(\rho_2) \cap \cdots \cap D(\rho_s) = \emptyset \qquad \textit{if } \rho_1 + \rho_2 + \cdots + \rho_s \not\in \Delta \tag{1}$$

and

$$\operatorname{div}(\varepsilon(m)) = \sum_{\rho\in\Delta(1)} \langle m, n_\rho\rangle D(\rho) \qquad \textit{for all} \quad m \in M. \tag{2}$$

Proof. Suppose $f\colon Y \to X$ is a morphism with $f(Y) \cap \mathbb{T} \neq \emptyset$. For each $\rho \in \Delta(1)$, we take $D(\rho)$ to be the pull-back Weil divisor $f^{-1}(V(\rho))$, which is well-defined since Y is normal, X is smooth and $f(Y) \not\subset V(\rho)$.

If $\rho_1+\cdots+\rho_s \not\in \Delta$, then $V(\rho_1)\cap\cdots\cap V(\rho_s) = \emptyset$ so $D(\rho_1)\cap\cdots\cap D(\rho_s) = \emptyset$. In this case $Y_0 = f^{-1}(\mathbb{T})$ is nonempty by assumption, and $f|_{Y_0}$ induces

$$f|_{Y_0}^*\colon \mathbb{C}[M] \to \mathcal{O}_Y(Y_0)^\times.$$

The composite $\varepsilon := f|_{Y_0}^* \circ \mathbf{e}$ satisfies (2), since

$$\operatorname{div}(\mathbf{e}(m)) = \sum_{\rho\in\Delta(1)} \langle m, n_\rho\rangle V(\rho) \qquad \text{for all} \quad m \in M.$$

Conversely, suppose $\{D(\rho)\}_{\rho\in\Delta(1)}$ and ε satisfy (1) and (2). For $\sigma \in \Delta$, put $\hat\sigma = \{\rho \in \Delta(1) \mid \rho \not\prec \sigma\}$. Then the corresponding open piece U_σ of X satisfies

$$\begin{aligned} U_\sigma &= X \setminus \bigcup_{\rho\in\hat\sigma} V(\rho) \\ &= \bigcap_{\rho\in\hat\sigma} (X \setminus V(\rho)) \\ &\cong \operatorname{Spec}(\mathbb{C}[M \cap \sigma^\vee]). \end{aligned}$$

Put $Y_\sigma = f^{-1}(U_\sigma) = Y \setminus \bigcup_{\rho\in\hat\sigma} D(\rho)$. Then $Y = \bigcup_{\sigma\in\Delta} Y_\sigma$ since the U_σ cover X (or one can check this directly).

For each $\sigma \in \Delta$, $M \cap \sigma^\vee$ is the semigroup consisting of $m \in M$ such that $\mathbf{e}(m)$ is regular on U_σ. Thus $\varepsilon(M \cap \sigma^\vee)$ consists of regular functions on Y_σ, and defines a morphism $f_\sigma\colon Y_\sigma \to U_\sigma$. These morphisms glue together to give a morphism $f\colon Y \to X$. $\square$

In choosing the collection of divisors $\{D(\rho)\}$ we determine how the toric divisors are to intersect the image $f(Y)$. Not all choices are possible: if $\{D_\rho\}$ is chosen arbitrarily then possibly no map corresponding to that collection exists. Although the $D(\rho)$ themselves are not required to be Cartier divisors, the left-hand side of (2) is Cartier, so one condition for such an f to exist is that the right-hand side of (2) is Cartier.

The condition that X be smooth is stronger than we need. See [Ka] for related results for singular toric varieties.

2.2 Embedding a curve

Now we apply Theorem 2.1 to the case where $Y = C$ is a smooth projective curve and X is projective of dimension 3.

Let L be an effective (hence ample) divisor on C. Let $\Delta(1) = \{\rho_1, \ldots, \rho_r\}$, so $r - 3 = \operatorname{rk} \operatorname{Pic} X > 0$. We write n_j and V_j (rather than n_{ρ_j} and $V(\rho_j)$) for the generator and the divisor corresponding to $\rho_j \in \Delta(1)$.

Let $\{m_1, m_2, m_3\}$ be a $\mathbb{Z}$-basis for M and put $a_{ij} = \langle m_i, n_j \rangle$. The system of linear equations $\sum_{j=1}^{r} a_{ij}\xi_j = 0$ has rank at most 3, so we can find nontrivial integer solutions. In the projective case we can do better.

Lemma 2.2 *If X is projective, then $\sum_{j=1}^{r} a_{ij}\xi_j = 0$ has integer solutions with $\xi_j > 0$ for all j.*

Proof. Let H be a very ample divisor on X. We have $\sum_j a_{ij}V_j = 0$ in $\operatorname{Pic} X$, since it is the divisor of $\mathbf{e}(m_i)$. But H^2V_j is the degree of the surface V_j in the projective embedding of X under $|H|$ and is therefore positive, so it is enough to take $\xi_j = H^2V_j$. $\square$

On C we take the line bundles $\mathcal{D}_j = \mathcal{O}_C(\xi_j H)$, with ξ_j as in Lemma 2.2. We may assume that $\xi_j > 2g(C)$ for all j, so that any nonzero linear combination of the $\mathcal{D}_j$ with nonnegative integer coefficients is very ample.

We want to specify a map $f\colon C \to X$ by means of data as in Theorem 2.1. Thus we must give elements D_j of the linear system $|\mathcal{D}_j|$.

Lemma 2.3 *If the D_j are general in $|\mathcal{D}_j|$ then they are reduced divisors and $\bigcap_j D_j = \emptyset$. In particular they satisfy* (1) *from Theorem 2.1.*

Proof. This follows from the very ampleness of the linear systems $|\mathcal{D}_j|$. $\square$

To specify a map $f\colon C \to X$ we now need only choose ε according to Theorem 2.1. This amounts to choosing suitable trivialisations of each of the three bundles $\mathcal{O}_C(\sum a_{ij}\mathcal{D}_j)$, i.e. non-vanishing sections of $\mathcal{O}_C(\sum a_{ij}\mathcal{D}_j)$ with order $-a_{ij}$ along D_j. Such trivialisations are unique up to multiplication by non-zero scalars. This means that the map $f = f_{\mathbb{D},\mathbf{t}}$ is determined by choices of $\mathbb{D} = (D_1, \ldots, D_r) \in |\mathcal{D}_1| \times \cdots \times |\mathcal{D}_r|$ together with a choice of an element

$\mathbf{t} \in (\mathbb{C}^*)^3 = \mathbb{T} \subset \operatorname{Aut} X$. In other words, choosing the D_j determines f up to composition with an element of $\mathbb{T}$ acting as an automorphism of X.

We note that the action of $\mathbb{T}$ has no effect on the question of whether or not the map is an embedding, and accordingly we suppress $\mathbf{t}$ in the notation.

Later we shall see that $f_{\mathbb{D}}$ will turn out to be an embedding for all sufficiently general $\mathbb{D} \in |\mathcal{D}_1| \times \cdots \times |\mathcal{D}_r|$. The next lemma shows that in order to determine whether the general $f_{\mathbb{D}}$ is an immersion, it is enough to check it over the standard affine pieces of X.

Lemma 2.4 *Suppose that, for every $\tau \in \Delta(3)$, there is a nonempty open subset $A_\tau \subset \prod_j |\mathcal{D}_j|$ such that*

$$f_{\mathbb{D}} \colon C_\tau = f_{\mathbb{D}}^{-1}(U_\tau) \to U_\tau$$

is a closed immersion if $\mathbb{D} \in A_\tau$. Then $f_{\mathbb{D}} \colon C \to X$ is a closed immersion for general $\mathbb{D} \in \prod_j |\mathcal{D}_j|$.

Proof. It is enough to take $\mathbb{D} \in \bigcap_{\tau \in \Delta(3)} A_\tau$. □

Theorem 2.5 *If X is a projective smooth toric 3-fold, C is a smooth projective curve and $\mathcal{D}_j$ are as above, the map $f_{\mathbb{D},\mathbf{t}} \colon C \to X$ is an embedding for almost all $\mathbb{D} \in \prod_j |\mathcal{D}_j|$.*

Proof. In view of Lemma 2.4 it remains to check that the set A_τ for which $f_{\mathbb{D}}$ is an embedding above U_τ is indeed nonempty.

After renumbering, we have $\tau = \rho_1 + \rho_2 + \rho_3$ and we consider the semigroup $M \cap \tau^\vee$. It is generated by l_1, l_2, $l_3 \in M$ with the property that $\langle l_i, n_i \rangle > 0$ and $\langle l_i, n_k \rangle = 0$ if $1 \le k \le 3$ and $k \ne i$. The function $p_i = \varepsilon_{\mathbb{D},\mathbf{t}}(l_i) = f_{\mathbb{D},\mathbf{t}}|_{C_\tau} \circ \mathbf{e}(l_i) f_{\mathbb{D},\mathbf{t}}$ is the ith coordinate function: it takes the value 0 on D_i and is nonzero on D_k for $1 \le k \le 3$, $k \ne i$.

We first pick D_j for $j > 3$ once and for all, only requiring them to be general in the sense of Lemma 2.3. Now choose D_3 so that D_3 is also reduced and disjoint from the other D_j chosen so far. This is enough to determine p_3 up to the torus action, since $\operatorname{div}(p_3) = \langle l_3, n_3 \rangle D_3 + \sum_{j>3} \langle l_3, n_j \rangle D_j$ is independent of D_1 and D_2. Similarly a choice of D_1 or of D_2 determines p_1 or p_2 up to the torus action, independently of the choice.

After making such a choice of D_3, we claim that for general $D_2 \in |\mathcal{D}_2|$ the map $(p_2, p_3) \colon C_\tau \to \mathbb{A}^2$ is generically injective. We shall check this by exhibiting a choice of D_2 which makes this map injective near D_3. Observe that for any pair P, $Q \in D_3$ (so $p_3(P) = p_3(Q) = 0$) we can find $D_2 \in |\mathcal{D}_2|$ such that $P \in D_2$ but $Q \notin D_2$ (although such a choice of D_2 will not be general in the sense of Lemma 2.3), because $\mathcal{D}_2$ is sufficiently ample. For this choice of D_2, we have $0 = p_2(P) \ne p_2(Q)$, so $p_2(P) \ne p_2(Q)$ for general D_2 and hence for general D_2 the values of p_2 on the points of D_3 are all different from one another. In particular (p_2, p_3) corresponding to a general D_2 is injective at any point of D_3 and is therefore injective generically.

By exactly the same argument, a general choice of $D_1 \in |\mathcal{D}_1|$ separates points not separated by the other choices. If P' and Q' are (possibly infinitely near) points such that $p_2(P') = p_2(Q')$ and $p_3(P') = p_3(Q')$, then $p_1(P') \neq p_1(Q')$ if $P' \in D_1$ and $Q' \notin D_1$. Such D_1 exist if $\mathcal{D}_1$ is sufficiently ample. So for general D_1 we also have $p_1(P') \neq p_1(Q')$, as required. $\square$

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