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Approaching criticality via the zero dissipation limit in the abelian avalanche model

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Abstract: The discrete height abelian sandpile model was introduced by Bak, Tang & Wiesenfeld and Dhar as an example for the concept of self-organized criticality. When the model is modified to allow grains to disappear on each toppling, it is called bulk-dissipative. We provide a detailed study of a continuous height version of the abelian sandpile model, called the abelian avalanche model, which allows an arbitrarily small amount of dissipation to take place on every toppling. We prove that for non-zero dissipation, the infinite volume limit of the stationary measure of the abelian avalanche model exists and can be obtained via a weighted spanning tree measure. We show that in the whole non-zero dissipation regime, the model is not critical, i.e., spatial covariances of local observables decay exponentially. We then study the zero dissipation limit and prove that the self-organized critical model is recovered, both for the stationary measure and for the dynamics. We obtain rigorous bounds on toppling probabilities and introduce an exponent describing their scaling at criticality. We rigorously establish the mean-field value of this exponent for $d > 4$.

Key-words: abelian sandpile model, abelian avalanche model, toppling probability exponent, burning algorithm, weighted spanning trees, Wilson's algorithm, zero dissipation limit, self-organized criticality.

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Contents

1 Introduction

The *abelian sandpile model*, introduced by Bak, Tang & Wiesenfeld [2], is a paradigm of the phenomenon of *self-organized criticality.* Its fundamental properties, including the abelianness, the set of recurrent configurations, and the group of toppling operators, was discovered by Dhar [5].

Just as for ordinary critical models in statistical mechanics, such as the Ising model or percolation, it is useful to add to the model a "magnetic field" (or a "mass") in order to tune it away from criticality, and in such a way that when the field becomes very small, new results can be obtained for the original critical model, for instance critical exponents, decay of covariances or percolation properties.

In the abelian sandpile model, a procedure to tune away from criticality is to introduce "bulk dissipation", i.e., to impose that in the bulk, upon each toppling mass is lost (whereas for the critical model this happens only at the boundary). Because to go back to criticality we want to tune this amount of lost mass to zero, we need a version of the abelian sandpile model with continuous heights. There are at least two reasons for considering such models. On the one hand, useful information can be gained about the critical model by letting the bulk dissipation go to zero; see for example [25]. On the other hand, with applications on random networks in mind, the underlying graph may have no natural boundary, but instead, dissipation can be present at all or some of the vertices.

A continuous model with deterministic additions, called the *abelian avalanche model*, has been introduced on finite graphs by Gabrielov [7]. We study a stochastic variant of this model, which has a dissipation parameter $\gamma \geq 0$, and we still call it the abelian avalanche model. For $\gamma = 0$ it is the natural continuous analogue of the (critical) abelian sandpile model. Moreover, this model generalizes the discrete dissipative models with integer dissipation studied by Maes, Redig & Saada in [23].

The main results of our paper are summarized as follows.

1. *Existence of infinite-volume limits for all $\gamma > 0$.* We prove the existence of the infinite-volume limit for the stationary measures and for the stochastic dynamics of the finite-volume abelian avalanche model.

2. *Non-criticality for every positive dissipation.* For every positive value of the dissipation, the infinite-volume model is not critical, i.e., covariances of local functions as well as avalanche sizes decay exponentially. This also extends the results in [23] for arbitrary small dissipation.

3. *Criticality for $\gamma \to 0$ and the toppling probability exponent.* We prove that for $\gamma \to 0$ the critical model is recovered, both for the stationary measures and for the dynamics. We prove lower bounds on the probability that vertex x topples in the dissipative model, and achieve a sharp lower bound when $d > 4$. In particular, we introduce the *toppling probability exponent* that describes the scaling of the above probability at criticality, and establish rigorously its mean-field value above the conjectured upper critical dimension, i.e. for $d > 4$.

2 Overview of the models and outline of the paper

2.1 Standard (discrete) abelian sandpile model

Let us start by briefly recalling the standard (i.e., discrete) abelian sandpile model. Let $\Lambda \subseteq \mathbb{Z}^d$ be a finite set. A *configuration* on Λ is a collection of particles occupying the sites in Λ, specified by a map $\eta : \Lambda \to \{0, 1, \dots\}$. If $\eta_x \geq 2d$ for $x \in \Lambda$, x is allowed to *topple*, that is to send one particle along each edge incident to x in $\mathbb{Z}^d$. Particles reaching $\mathbb{Z}^d \setminus \Lambda$ are *lost*, i.e. disappear. We say that η is *stable* if no site can topple, that is $\eta_x < 2d$ for all $x \in \Lambda$.

We define a Markov chain on the set of stable configurations as follows. At each time step, we add a particle to a stable configuration η at a randomly chosen site in Λ, then carry out all possible topplings (this succession of topplings is called an *avalanche* and its final result on the configuration is called stabilization) until a new stable configuration η' is reached. Going from η to η' is then a single transition in the Markov chain. It was shown by Dhar [5] that the resulting stable configuration does not depend on the order of topplings (hence the name "abelian"), and that the stationary distribution, denoted by ν_Λ, is unique and uniformly distributed on all recurrent states.

The abelian sandpile model is *critical*, in the sense that correlations in the stationary measure have *power law decay* (see [26] by Majumdar & Dhar): for all $d \geq 2$ there is a constant $c = c(d)$ such that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \{\nu_\Lambda(\eta_0 = 0, \eta_x = 0) - \nu_\Lambda(\eta_0 = 0)\nu_\Lambda(\eta_x = 0)\} = c|x|^{-2d}(1 + o(1)), \quad \text{as } |x| \to \infty,$$

where $|x|$ denotes the Euclidean norm of x, and where $o(1)$ goes to zero as $|x| \to \infty$.

2.2 Discrete dissipative sandpile model

The following modification of the abelian sandpile model, called the *discrete dissipative sandpile model* was introduced in [33] by Tsuchiya & Katori and studied rigorously by Maes, Redig & Saada in [23]. Let $\gamma \geq 1$ be *an integer*, and call a configuration γ-*stable* if $\eta_x < 2d + \gamma$ for all $x \in \Lambda$. A site $x \in \Lambda$ is allowed to γ-*topple* if $\eta_x \geq 2d + \gamma$, which means that it sends 1 particle along each edge incident to x in $\mathbb{Z}^d$ *and* γ particles are lost, or *dissipated*, thereby decreasing the number of particles at x by $2d+\gamma$. The presence of dissipation introduces exponential decay of avalanche sizes, hence the model becomes non-critical (for *large values of* γ it was shown in [23] that also covariances of local observables decay exponentially). The exponential decay of the massive Green's function enabled in [23] to define the dissipative sandpile model on $\mathbb{Z}^d$, and to extend Dhar's formalism of the sandpile group ([5]) to the infinite case.

2.3 Continuous dissipation

Mahieu & Ruelle [25] considered the limit $\gamma \downarrow 0$ in analytical expressions obtained for stationary expectation of special observables in the discrete dissipative sandpile model, these making sense for any real $\gamma > 0$. It is a natural question whether the limit $\gamma \downarrow 0$ can be made sense of in an appropriate continuous model. The problem of properly defining this model was circumvented in the physics literature, where only specific correlation functions involving the Green's function were considered, within which then the massive Green's function can be substituted. Our goal in this paper is to define and establish rigorously some of the basic properties of such a *continuous height* dissipative sandpile model and to use these to derive properties of the critical model.

Let us now briefly introduce this *abelian avalanche model*. It is a variant of a model introduced by Gabrielov [7], in which the system is driven deterministically; however, as we make use of it later, the main results of [7] apply to the stochastic case.

By a (continuous) *configuration* we mean a map $\eta : \Lambda \to [0, \infty)$, referred to as a collection of *heights*. Let $\gamma \geq 0$ be a *real* parameter. We say that η is γ*-stable* if $\eta_x \in [0, 2d + \gamma)$ for all $x \in \Lambda$. We say that x is *allowed to* γ*-topple* if $\eta_x \geq 2d + \gamma$. In this case a γ-toppling at x means that height 1 is sent along each edge incident to x in $\mathbb{Z}^d$, and height γ is lost, thereby decreasing the height at x by $2d + \gamma$. As in the discrete case it holds, by essentially the same proof, that any configuration has a unique γ-stabilization arrived at by carrying out all possible γ-topplings.

We define a pure jump Markov process on the set of γ-stable continuous configurations as follows. Let $\{\varphi(x)\}_{x\in\Lambda}$ be a collection of positive rates. For $x \in \Lambda$, height 1 is added at vertex x according to a Poisson process with rate $\varphi(x)$ (Poisson processes associated to different sites are mutually independent). After an addition of height 1 has occurred, the configuration is instantaneously γ-stabilized by carrying out all possible γ-topplings. By analogy with [7], we call this dynamics the abelian avalanche model.

2.4 Outline of the paper

Section 3 collects basic properties of the finite volume abelian avalanche model: In Section 3.1 we give the precise definitions of γ-toppling matrices, γ-stabilization and γ-toppling numbers. We discuss properties of the set $\mathcal{R}_\Lambda$ of "recurrent" configurations in Section 3.2. The natural invariant measure of this model is the normalized Lebesgue measure on $\mathcal{R}_\Lambda$, denoted by $m_\Lambda^{(\gamma)}$.

In Section 3.4 we consider rational $\gamma = k/n$. In that case the fractional part of the vector $(n\eta_x)_{x\in\Lambda}$ is invariant under the dynamics. When this is factored out, one obtains the discrete dissipative sandpile model, with an ergodic dynamics.

Due to the toppling rule, our continuous model retains some discreteness, for all values of γ. In fact, continuous heights will only be used to define the dynamics; the stationary measure has a natural description in terms of appropriate discrete height variables, that we define in Section 3.5. Those discretized heights allow us to adapt the burning bijection established by Majumdar & Dhar in [27], and to define a coding of those configurations with discretized heights in terms of weighted spanning trees. The coding by weighted spanning trees is extended to waves (an avalanche is decomposed in a succession of *waves*) in Section 3.6.

In Section 3.7 we discuss ergodic properties of the finite volume abelian avalanche model. We show that when γ is irrational, the stationary process started from the invariant measure $m_\Lambda^{(\gamma)}$ is ergodic in time. Another natural question is related to the following transformation: add unit height at a fixed vertex, and γ-stabilize. This is a measure preserving transformation on $(\mathcal{R}_\Lambda, m_\Lambda^{(\gamma)})$. We give a criterion for the transformation to be ergodic, and provide examples when the criterion can be verified.

The weak limit of the stationary measure $m^{(\gamma)} = \lim_{\Lambda\uparrow\mathbb{Z}^d} m_\Lambda^{(\gamma)}$ is constructed in Section 4, using the burning bijection and Wilson's algorithm [34]. We prove exponential decay of correlations of local functions, for all $\gamma > 0$, in Theorem 4.2. As $\gamma \downarrow 0$, the (discretized) stationary measure on $\mathbb{Z}^d$ converges to the critical sandpile measure; this is shown in Theorem 4.3. For certain special local events an estimate on the rate of convergence is given in Proposition 4.1.

The abelian avalanche model on $\mathbb{Z}^d$ is considered in Section 5. Theorem 5.1 proves the existence of a sandpile Markov process on $\mathbb{Z}^d$ with dissipation. In section 6, Theorem 6.1 gives estimates on toppling probabilities under $m^{(\gamma)}$, and Theorem 6.2 establishes the mean-field value of the toppling probability exponent in dimensions $d > 4$.

In Section 7 we prove that the abelian avalanche model on $\mathbb{Z}^d$ converges, as $\gamma \downarrow 0$, to the critical (non-dissipative) abelian sandpile process, when the latter is known to exist on $\mathbb{Z}^d$; see Theorem 7.1.

Our models will live on $\mathbb{Z}^d$, and for simplicity, we only consider the nearest neighbor case. Presumably one can extend our constructions to other infinite graphs on which each tree of the wired uniform spanning forest has one end; see [16, 18].

3 The abelian avalanche model in finite volume

3.1 Toppling matrices, stabilization, toppling numbers

For $x, y \in \mathbb{Z}^d$, we write $|x-y|$ for their Euclidean distance, and $\|x-y\|_1 := \sum_{i=1}^d |x_i - y_i|$. We denote $x \sim y$ if x and y are neighbors, that is $|x-y| = 1$. Each site x will carry a *continuous* height variable with value in the interval $[0, 2d+\gamma)$, where $\gamma \geq 0$ is a *real* parameter. On a γ-*toppling*, a site will give height 1 to each of its neighbors, and lose height $2d+\gamma$, that is, an amount γ of height is dissipated on each toppling (when it is clear in context, we sometimes write toppling instead of γ-toppling). This can be summarized using the γ-*toppling matrix* $\Delta^{(\gamma)}$ with elements

$$\Delta_{xy}^{(\gamma)} = \begin{cases} 2d+\gamma & \text{if } x = y; \\ -1 & \text{if } x \sim y; \\ 0 & \text{otherwise.} \end{cases}$$

For $\Lambda \subseteq \mathbb{Z}^d$, we write $\Delta_\Lambda^{(\gamma)} = (\Delta_{xy}^{(\gamma)})_{x,y\in\Lambda}$ for the γ-toppling matrix restricted to Λ.

We define the sets of *(continuous) height configurations* in $\mathbb{Z}^d$ and in Λ by

$$\mathcal{X} = [0,\infty)^{\mathbb{Z}^d} \qquad \text{and} \qquad \mathcal{X}_\Lambda = [0,\infty)^\Lambda.$$

A site x is called γ-*stable* in configuration η, if $\eta_x < \Delta^{(\gamma)}_{xx} = 2d+\gamma$; otherwise, x is γ-*unstable* in configuration η. The sets of γ-stable configurations are denoted by

$$\Omega^{(\gamma)} = [0, 2d+\gamma)^{\mathbb{Z}^d} \qquad \text{and} \qquad \Omega^{(\gamma)}_\Lambda = [0, 2d+\gamma)^\Lambda.$$

Sometimes we will need configurations that are only stable in Λ and are unrestricted in $\mathbb{Z}^d \setminus \Lambda$, so we will also use

$$\overline{\Omega}^{(\gamma)}_\Lambda = \{\eta \in \mathcal{X} : \eta_x < 2d+\gamma,\ x \in \Lambda\}.$$

A γ-*toppling* of site x in Λ is defined by the operator $T^{(\gamma)}_{\Lambda,x}$, via the formula

$$\left(T^{(\gamma)}_{\Lambda,x}\eta\right)_y = \eta_y - \Delta^{(\gamma)}_{xy}, \qquad y \in \Lambda, \eta \in \mathcal{X}_\Lambda. \tag{3.1}$$

We also define $T^{(\gamma)}_{\Lambda,x}\eta$, when $\eta \in \mathcal{X}$. In this case, the values of η_y for $y \in \mathbb{Z}^d \setminus \Lambda$ are left unchanged, and for $y \in \Lambda$ they change according to (3.1).

A γ-toppling is called *legal* if the toppled site $x \in \Lambda$ is γ-unstable before toppling, which ensures that each component of $T^{(\gamma)}_{\Lambda,x}\eta$ is still non-negative. A sequence of γ-topplings at sites $(x_1, \dots, x_n) \in \Lambda^n$ is called (Λ,γ)-*stabilizing* for $\eta \in \mathcal{X}_\Lambda$, if each toppling can be carried out and the final result is γ-stable; that is, if

(i) $T^{(\gamma)}_{\Lambda,x_k}$ is a legal γ-toppling of $T^{(\gamma)}_{\Lambda,x_{k-1}} \circ \cdots \circ T^{(\gamma)}_{\Lambda,x_1}\eta$, $1 \le k \le n$;

(ii) the final configuration is in $\Omega^{(\gamma)}_\Lambda$.

Note that for all finite $\Lambda \subseteq \mathbb{Z}^d$ and all $\eta \in \mathcal{X}_\Lambda$, a (Λ,γ)-stabilizing sequence exists. The number of times a site topples does not depend on the (Λ,γ)-stabilizing sequence, and hence there is a well-defined γ-*stabilization map* $\mathcal{S}^{(\gamma)}_\Lambda : \mathcal{X}_\Lambda \to \Omega^{(\gamma)}_\Lambda$, see e.g. [5], or [7, Appendix B] for a proof. The succession of topplings in a γ-stabilization is called a γ-avalanche.

The result of γ-stabilization is related to the original configuration as follows. Let $N^{(\gamma)}_\Lambda(\eta)$ be the vector consisting of the γ-*toppling numbers* associated to the (Λ,γ)-stabilization of η, i.e., $(N^{(\gamma)}_\Lambda(\eta))_x$ is the number of times $x \in \Lambda$ topples during γ-stabilization of η in Λ. Then by (3.1), the net effect of all topplings can be written as

$$(\mathcal{S}^{(\gamma)}_\Lambda\eta)_y = \eta_y - \sum_{x\in\Lambda} \Delta^{(\gamma)}_{yx}(N^{(\gamma)}_\Lambda(\eta))_x = \eta_y - (\Delta^{(\gamma)}_\Lambda N^{(\gamma)}_\Lambda(\eta))_y, \qquad y \in \Lambda. \tag{3.2}$$

Similarly, γ-stabilization in volume Λ can also be viewed as a map $\mathcal{S}^{(\gamma)}_\Lambda : \mathcal{X} \to \overline{\Omega}^{(\gamma)}_\Lambda$, where the coordinates outside Λ are left unchanged.

The *addition operators* are defined by adding height 1 at a site x, and then γ-stabilizing:

$$a^{(\gamma)}_{\Lambda,x} : \Omega^{(\gamma)}_\Lambda \to \Omega^{(\gamma)}_\Lambda; \qquad a^{(\gamma)}_{\Lambda,x}\eta = \mathcal{S}^{(\gamma)}_\Lambda(\eta + \delta_x), \tag{3.3}$$

where δ_x denotes the vector having entry equal to one at site x and zero elsewhere. The addition operators commute, that is, for all $x, y \in \Lambda$, $\eta \in \Omega^{(\gamma)}_\Lambda$,

$$a^{(\gamma)}_{\Lambda,x}(a^{(\gamma)}_{\Lambda,y}\eta) = a^{(\gamma)}_{\Lambda,y}(a^{(\gamma)}_{\Lambda,x}\eta). \tag{3.4}$$

This follows from the fact that γ-stabilization is well-defined: indeed both expressions in (3.4) are equal to $\mathcal{S}^{(\gamma)}_\Lambda(\eta + \delta_x + \delta_y)$.

We endow $\mathcal{X}$, $\mathcal{X}_\Lambda$, Ω, Ω_Λ with the product metric

$$\mathrm{dist}(\eta_1, \eta_2) = \sum_x 2^{-|x|} \min\{|(\eta_1)_x - (\eta_2)_x|, 1\}, \tag{3.5}$$

where the sum is over $\mathbb{Z}^d$ or over Λ.

Given a function $\varphi : \Lambda \to (0, \infty)$, we define a jump Markov process on γ-stable configurations. The action of the generator on Borel measurable functions $f : \Omega^{(\gamma)}_\Lambda \to \mathbb{R}$ is given by

$$\mathcal{L}_\Lambda f(\eta) = \mathcal{L}^{\gamma,\varphi}_\Lambda f(\eta) = \sum_{x\in\Lambda} \varphi(x) \left[f\left(a^{(\gamma)}_{\Lambda,x}\eta\right) - f(\eta) \right]. \tag{3.6}$$

The above process is described in words as follows: at each site $x \in \Lambda$ we have a Poisson process with intensity $\varphi(x)$, and at different sites these processes are independent. At the event times of this Poisson process we apply the addition operator $a^{(\gamma)}_{\Lambda,x}$ to the configuration.

Given a measure μ, we denote by $\mathbb{E}_\mu$ expectation with respect to μ.

3.2 Stationary measure, Dhar's formula

As in the discrete case [5], there is a subset $\mathcal{R}^{(\gamma)}_\Lambda \subseteq \Omega^{(\gamma)}_\Lambda$, called the set of "recurrent configurations", such that any invariant measure is concentrated on $\mathcal{R}^{(\gamma)}_\Lambda$. This is described via the notion of *γ-allowed configuration* defined below. A *γ-forbidden subconfiguration* (γ-FSC) is a pair (W, η_W) where $\varnothing \neq W \subseteq \mathbb{Z}^d$ is finite, $\eta_W \in \mathcal{X}_W$, such that for all $y \in W$, the number of neighbors of y in W is strictly larger than the height $(\eta_W)_y$:

$$(\eta_W)_y < \sum_{z: z\in W, z\neq y} (-\Delta^{(\gamma)}_{zy}). \tag{3.7}$$

Definition 3.1. *A configuration $\eta \in \Omega^{(\gamma)}_\Lambda$ (respectively $\eta \in \Omega^{(\gamma)}$) is called* γ-allowed *if there does not exist finite $W \subseteq \Lambda$ (respectively $W \subseteq \mathbb{Z}^d$) such that the pair consisting of W and the restriction η_W of η to W is a γ-FSC.*

Remark 3.1. *Since the off-diagonal elements of the γ-toppling matrix $\Delta^{(\gamma)}$ do not depend on γ, the right hand side of inequality* (3.7) *is independent of γ. Therefore we have the same forbidden subconfigurations for any value of γ. So from now on we use the words FSC rather than γ-FSC, and allowed rather than γ-allowed.*

Let

$\mathcal{R}_\Lambda^{(\gamma)} = \{\eta \in \Omega_\Lambda^{(\gamma)} : \eta \text{ is allowed}\}$,

$\mathcal{R}^{(\gamma)} = \{\eta \in \Omega^{(\gamma)} : \eta \text{ is allowed}\} := \{\eta \in \Omega^{(\gamma)} : \eta_V \in \mathcal{R}_V^{(\gamma)} \text{ for all finite } V \subseteq \mathbb{Z}^d\}$.

The results of [7, Sections 3,4] imply the following properties of allowed configurations. We write Vol for Lebesgue measure on $\mathbb{R}^\Lambda$.

Proposition 3.1. *(i) The addition operator $a_{\Lambda,x}^{(\gamma)}$ maps $\mathcal{R}_\Lambda^{(\gamma)}$ one-to-one and onto itself.*

(ii) $\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)}) = \det(\Delta_\Lambda^{(\gamma)})$.

(iii) Lebesgue measure on $\mathcal{R}_\Lambda^{(\gamma)}$ is invariant under $a_{\Lambda,x}^{(\gamma)}$, $x \in \Lambda$.

Hence, the probability measure $m_\Lambda^{(\gamma)}$ on $\mathcal{R}_\Lambda^{(\gamma)}$ defined by

$$m_\Lambda^{(\gamma)}(A) := \frac{\mathrm{Vol}(A)}{\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)})}$$

is stationary for the Markov process defined in (3.6) of Section 3.1. The set $\mathcal{R}_\Lambda^{(\gamma)}$ is a group under pointwise addition and γ-stabilization, isomorphic to $\mathbb{R}^\Lambda/\Delta_\Lambda^{(\gamma)}$, the quotient group obtained by identifying different elements of $\mathbb{R}^\Lambda$ that differ by an integer column multiple of the matrix $\Delta_\Lambda^{(\gamma)}$. For $\eta \in \Omega_\Lambda^{(\gamma)}$, we denote by $n_\Lambda^{(\gamma)}(x,y,\eta) = \left(N_\Lambda^{(\gamma)}(\eta+\delta_x)\right)_y$ the number of γ-topplings at y needed to γ-stabilize $\eta+\delta_x$. Then due to (3.2), we have the relation

$$\left(a_{\Lambda,x}^{(\gamma)}\eta\right)_z = \eta_z + \mathbf{1}_{\{x\}}(z) - \sum_{y\in\Lambda} \Delta_{zy}^{(\gamma)} n_\Lambda^{(\gamma)}(x,y,\eta). \tag{3.8}$$

Taking expectation with respect to $m_\Lambda^{(\gamma)}$ and using stationarity under the action of the addition operators (that is, Proposition 3.1*(iii)*) gives "Dhar's formula" [5]:

$$\mathbb{E}_{m_\Lambda^{(\gamma)}}(n_\Lambda^{(\gamma)}(x,y,\eta)) = (\Delta_\Lambda^{(\gamma)})^{-1}_{xy} =: G_\Lambda^{(\gamma)}(x,y), \qquad z \in \Lambda. \tag{3.9}$$

If $\gamma > 0$ or $d \geq 3$, the inverse $G^{(\gamma)}(x,y) := (\Delta^{(\gamma)})^{-1}_{xy}$ also exists and is equal to the limit $\lim_{\Lambda\uparrow\mathbb{Z}^d} G_\Lambda^{(\gamma)}(x,y)$. Note that $G^{(\gamma)}(x,y)$ equals the Green's function of a continuous time random walk that crosses an edge at rate 1, and is killed at rate γ. Similarly, $G_\Lambda^{(\gamma)}(x,y)$ equals the Green's function of a continuous time random walk that crosses an edge at rate 1, is killed upon exiting Λ and is killed (inside Λ) at rate γ.

Markov's inequality and (3.9) imply

$$m_\Lambda^{(\gamma)}\left(n_\Lambda^{(\gamma)}(x,y,\eta) \geq 1\right) \leq G_\Lambda^{(\gamma)}(x,y). \tag{3.10}$$

3.3 Green's function estimates

Here we provide estimates for the Green's function $G^{(\gamma)}(x,y)$. It is well-known that this Green's function decays exponentially in the distance to the origin. We prefer however to insert a proof for the sake of self-containedness, and to indicate the power of γ entering in the exponent. This will be important later on when we consider the limit $\gamma \downarrow 0$ at several places.

Lemma 3.1. *There exist $C > 0$ and $c > 0$ such that for all $\Lambda \subseteq \mathbb{Z}^d$, $0 < \gamma < 1$, and $x, y \in \mathbb{Z}^d$, $x \neq y$,*

$$G_\Lambda^{(\gamma)}(x,y) \leq G^{(\gamma)}(x,y)$$

$$G^{(\gamma)}(x,y) \leq \begin{cases} \dfrac{C\gamma^{d/4-1}}{|x-y|^{d/2}} e^{-c\sqrt{\gamma}|x-y|} & \text{if } |x-y| \geq \gamma^{-1/2}; \\ \dfrac{C}{|x-y|^{d-2}} & \text{if } |x-y| \leq \gamma^{-1/2},\ d \geq 3; \\ C + C\log(|x-y|^{-1}\gamma^{-1/2}) & \text{if } |x-y| \leq \gamma^{-1/2},\ d = 2. \end{cases} \tag{3.11}$$

Furthermore, there exist $C' > 0$, $c' > 0$ such that the reverse inequalities of (3.11) *hold with C' replacing C and c' replacing c.*

Proof. First note that $G_\Lambda^{(\gamma)}(x,y) \leq G^{(\gamma)}(x,y)$, since in defining $G_\Lambda^{(\gamma)}(x,y)$, the random walk is killed upon exiting Λ. Next, we have

$$G^{(\gamma)}(x,y) = \sum_{n=\|x-y\|_1}^{\infty} \left(\frac{2d}{2d+\gamma}\right)^n p_n(x,y), \tag{3.12}$$

where $p_n(x,y)$ denotes the n-step transition probability of simple (nearest neighbor) random walk $\{S_n\}$ on $\mathbb{Z}^d$, and where the sum over n starts at $n = \|x-y\|_1$ since the nearest neighbor random walk has to make at least that number of steps to reach y from x. Below we write $\mathbb{P}$ for the underlying probability measure.

We use the Gaussian upper and lower bounds:

$$p_n(x,y) \leq \frac{C_2}{n^{d/2}} e^{-C_1|x-y|^2/n}, \tag{3.13}$$

and for $\|x-y\|_1 \leq n$ with n of the same parity as $\|x-y\|_1$,

$$p_n(x,y) \geq \frac{C_2'}{n^{d/2}} e^{-C_1'|x-y|^2/n}. \tag{3.14}$$

These are well-known; see [9] for a much more general result on groups. For the reader's convenience, we supply the sketch of the proof in the case of $\mathbb{Z}^d$.

The upper bound follows from the large deviation bound $\mathbb{P}(|S_m| > |x|/2) \leq C_5 \exp(-C_1|x|^2/m)$, and the fact that $\mathbb{P}(S_m = y) \leq C_4 m^{-d/2}$, a consequence of the local limit theorem [21]. Taking $m = \lfloor n/2 \rfloor$ (where $\lfloor a \rfloor$ denotes the integer part of a real a), the two imply:

$$\begin{aligned} p_n(0,x) &\le \sum_{z:|z|\ge|x|/2} p_m(0,z)p_{n-m}(z,x) + \sum_{z:|z-x|\ge|x|/2} p_m(0,z)p_{n-m}(z,x) \\ &\le C_2 n^{-d/2} \exp(-C_1|x|^2/n). \end{aligned}$$

The Gaussian lower bound follows from a chaining argument: assuming $|x|^2 > n$, let $m = \lfloor |x|^2/n \rfloor$, and let $y_0 = 0, y_1, \dots, y_m = x$ be points such that $|y_{i-1} - y_i| \le 2n/|x|$. Consider the balls $B_i = B(y_i, n/|x|)$. Then for any $z \in B_{i-1}$, the central limit theorem implies $\mathbb{P}(S_{n/m} \in B_i | S_0 = z) \ge c > 0$. We get, using the local limit theorem,

$$p_n(0,x) \ge c^{m-1} C_2' (|x|^2/n^2)^{d/2} \ge C_2' n^{-d/2} \exp(-C_1'|x|^2/n).$$

Inserting the upper estimate (3.13) into (3.12), and using the notation $C_3 = 1/(2d)$, we have $2d/(2d+\gamma) \le \exp(-C_3\gamma)$, and we obtain

$$\begin{aligned} G^{(\gamma)}(0,x) &\le \sum_{n=1}^{\infty} \left(\frac{2d}{2d+\gamma} \right)^n \frac{C_2}{n^{d/2}} e^{-C_1|x|^2/n} \\ &\le \sum_{n=1}^{\infty} \frac{C_2}{n^{d/2}} \exp\left(-C_3\gamma n - C_1 \frac{|x|^2}{n} \right). \end{aligned} \tag{3.15}$$

In the case $|x| \ge \gamma^{-1/2}$, the bound in (3.11) follows from estimating separately the sums

$$\sum_{1 \le n \le |x|/\sqrt{\gamma}} \frac{C_2}{n^{d/2}} \exp(-C_1|x|^2/n) \quad \text{and} \quad \sum_{n > |x|/\sqrt{\gamma}} \frac{C_2}{n^{d/2}} \exp(-C_3\gamma n). \tag{3.16}$$

In the case $|x| < \gamma^{-1/2}$, $d = 3$, the bounds follow by using $2d/(2d+\gamma) \le 1$, and [20, Theorem 1.5.4]. In the case $d = 2$ we again estimate the sums (3.16).

The proof of the lower bound is similar, starting from the lower bound on $p_n(x,y)$. □

3.4 Rational γ

When γ is rational, the abelian avalanche model has a natural reduction to a discrete dissipative sandpile model that we now describe. The main results of the paper will not rely on this section. Let $\gamma = k/n$, with $k \ge 0$, $n \ge 1$ integers, and k and n relatively prime. Then the vector $(n\eta_x)_{x\in\Lambda}$ changes by integer amounts both during addition of unit height, and during γ-toppling. Hence the fractional part $(\{n\eta_x\})_{x\in\Lambda}$ remains invariant, and can be "factored out". We define the map $\varphi_\Lambda : \mathcal{X}_\Lambda \to \{0,1,2,\dots\}^\Lambda$, by $(\varphi_\Lambda(\eta))_x = \lfloor n\eta_x \rfloor$. The corresponding discrete γ-toppling matrix has elements

$$\Delta_{xy} = \begin{cases} 2dn + k & \text{if } x = y; \\ -n & \text{if } x \sim y; \\ 0 & \text{otherwise,} \end{cases}$$

and it has associated addition operators $a_{\Lambda,x}$, allowed configurations $\mathcal{R}_\Lambda$, and γ-toppling operators $T_{\Lambda,x}$. We write $\Delta_\Lambda = (\Delta_{xy})_{x,y\in\Lambda}$ for the γ-toppling matrix restricted to Λ.

Notice that (W, η_W) is a (k/n)-FSC if and only if $(W, \varphi_\Lambda(\eta_W))$ is an FSC with respect to Δ. This implies that $\varphi_\Lambda(\mathcal{R}_\Lambda^{(k/n)}) = \mathcal{R}_\Lambda$, and that the stationary measure for the discrete model coincides with $\varphi_\Lambda m_\Lambda^{(k/n)} := m_\Lambda^{(k/n)} \circ \varphi_\Lambda^{-1}$. The relation between toppling operators is $T_{\Lambda,x}\varphi_\Lambda = \varphi_\Lambda T_{\Lambda,x}^{(k/n)}$. Consequently, since adding unit height in the continuous model corresponds to adding n particles in the discrete model, with the notation $b_{\Lambda,x} = (a_{\Lambda,x})^n$, we have $b_{\Lambda,x}\varphi_\Lambda = \varphi_\Lambda a_{\Lambda,x}^{(k/n)}$.

The elements $\{b_{\Lambda,x}\}_{x\in\Lambda}$ generate the sandpile group for Δ_Λ. To see this, note that the order of the group, $\det(\Delta_\Lambda)$, is relatively prime to n. Hence, powers of $b_{\Lambda,x}$ yield all powers of $a_{\Lambda,x}$, and the claim follows. Therefore the reduced (discrete) sandpile model is ergodic.

3.5 Discretized heights, burning algorithm and spanning trees

The measure $m_\Lambda^{(\gamma)}$ can be described in terms of discrete height variables. We introduce the discretizing map

$$\psi_\Lambda : \Omega_\Lambda^{(\gamma)} \to \Omega_\Lambda^{\mathrm{discr}} := \{0, 1, \dots, 2d-1, 2d\}^\Lambda$$

defined by

$$\psi_\Lambda(\eta)_y = \begin{cases} m & \text{if } m \le \eta_y < m+1,\ m = 0, 1, \dots, 2d-1; \\ 2d & \text{if } 2d \le \eta_y < \gamma + 2d. \end{cases}$$

We define $\psi : \Omega^{(\gamma)} \to \Omega^{\mathrm{discr}} := \{0, 1, \dots, 2d-1, 2d\}^{\mathbb{Z}^d}$ analogously.

Notice that the height $2d$ is possible in the discretization of a stable configuration when $\gamma > 0$, whereas when $\gamma = 0$ the discretization of a stable configuration has possible heights up to $2d-1$ as in the standard discrete abelian sandpile model.

We define $\mathcal{R}_\Lambda^{\mathrm{discr}}$ to be the set of configurations $\xi \in \Omega_\Lambda^{\mathrm{discr}}$ which are allowed (cf. Definition 3.1). By Remark 3.1 and the fact that the right hand side of (3.7) is an integer, for any $\eta \in \Omega_\Lambda^{(\gamma)}$, we have

$$\eta \in \mathcal{R}_\Lambda^{(\gamma)} \qquad \text{if and only if} \qquad \psi_\Lambda(\eta) \in \mathcal{R}_\Lambda^{\mathrm{discr}}. \tag{3.17}$$

By a (γ, Λ)-*cell*, we will mean a subset of $\mathcal{R}_\Lambda^{(\gamma)}$ of the form $\mathcal{R}_\Lambda^{(\gamma)} \cap \psi_\Lambda^{-1}(\xi)$, for some $\xi \in \mathcal{R}_\Lambda^{\mathrm{discr}}$. It follows from the above discussion that $m_\Lambda^{(\gamma)}$ is uniform on each cell, hence $m_\Lambda^{(\gamma)}$ can be uniquely specified in terms of the measures of cells. Let $\nu_\Lambda^{(\gamma)} := \psi_\Lambda m_\Lambda^{(\gamma)}$. We proceed to give a description of $\nu_\Lambda^{(\gamma)}(\xi)$, $\xi \in \mathcal{R}_\Lambda^{\mathrm{discr}}$. Let

$$\mathcal{H}(\xi) = |\{y \in \Lambda : \xi_y = 2d\}|, \tag{3.18}$$

where $|A|$ denotes the cardinality of a set A. Then for $\gamma > 0$

$$\nu_\Lambda^{(\gamma)}(\xi) = \frac{\gamma^{\mathcal{H}(\xi)}}{\det(\Delta_\Lambda^{(\gamma)})}, \tag{3.19}$$

which follows from the fact that under ψ_Λ^{-1}, discrete heights $\xi_x \in \{0, \dots, 2d-1\}$ go to intervals of unit length, and heights $\xi_x = 2d$ to intervals of length γ, hence $\mathrm{Vol}(\psi_\Lambda^{-1}(\xi)) = \gamma^{\mathcal{H}(\xi)}$.

Remark 3.2. *When $\gamma = 0$, $\mathcal{H}(\xi) = 0$ for all ξ, and $\nu_\Lambda^{(0)}$ is uniform on allowed configurations such that all heights are $< 2d$.*

In order to study the infinite volume limit, we interpret $\nu_\Lambda^{(\gamma)}(\xi)$ in terms of *weighted spanning trees*. For this we adapt to our setting the *burning bijection* of Majumdar & Dhar that gives a one-to-one map between allowed configurations and spanning trees [27]. For more details and examples, see also [13, 24, 29].

Burning algorithm [5]. Fix $\eta \in \Omega_\Lambda^{(\gamma)}$, and let $\xi = \psi_\Lambda(\eta)$. Set

$$\begin{aligned} &U_0 = \Lambda, \\ &U_1 = \{y \in \Lambda : \xi_y < 2d\} = U_0 \setminus \{y \in U_0 = \Lambda : \xi_y \geq 2d\}. \end{aligned}$$

For $t = 1, 2, \dots,$ we recursively define

$$U_{t+1} = U_t \setminus \left\{ y \in U_t : \xi_y \geq \sum_{z: z \in U_t,\, z \neq y} (-\Delta_{zy}^{(\gamma)}) \right\}.$$

The sites y removed from U_t to obtain U_{t+1} are called "burnt" at time $t+1$, and we say that they have *burning time* $t(y) = t+1$. In particular at time 1 the sites in Λ with height $\geq 2d$ are burnt. By induction on t and (3.7), no site in $\Lambda \setminus U_t$ can be contained in any FSC. Hence we have $\cap_{t=0}^\infty U_t = \varnothing$ if and only if $\xi \in \mathcal{R}_\Lambda^{\mathrm{discr}}$ (if and only if $\eta \in \mathcal{R}_\Lambda^{(\gamma)}$ by (3.17)).

Now consider the graph $\widetilde{\mathbb{Z}^d}$ obtained by adding a new vertex ϖ to $\mathbb{Z}^d$ and connecting it to every vertex. Let us call the newly added edges *dissipative*, and the rest of the edges *ordinary*. Now we define a new graph $\widetilde{\Lambda}$, by identifying all vertices in $\mathbb{Z}^d \setminus \Lambda$ with ϖ (and removing loops). In $\widetilde{\Lambda}$, every $y \in \Lambda$ is connected to ϖ by exactly one dissipative edge. Boundary sites of Λ are connected to ϖ by one or more *ordinary* edges, in such a way that $2d$ ordinary edges emanate from each $y \in \Lambda$. We denote by $E(\widetilde{\Lambda})$ the set of edges of $\widetilde{\Lambda}$.

We define a *spanning tree* $\mathcal{T}_\Lambda(\xi)$ of $\widetilde{\Lambda}$. First, for each $y \in U_0 \setminus U_1$ (that is, when $\xi_y = 2d$), include the dissipative edge of y in the tree. We define by convention the burning time of ϖ to be 1, $t(\varpi) = 1$. We already said that sites $y \in U_0 \setminus U_1$ also have burning time 1: $t(y) = 1$; and for each $y \in U_1$, the burning time $t = t(y) \geq 2$ of y is the index t for which $y \in U_{t-1} \setminus U_t$. For $y \in U_{t-1} \setminus U_t$, i.e., $t(y) = t$, let

$$\begin{aligned} r(y) &= |\{\{z, y\} \in E(\widetilde{\Lambda}) : \{z, y\} \text{ ordinary and } t(z) = t-1\}|, \\ n(y) &= |\{\{z, y\} \in E(\widetilde{\Lambda}) : \{z, y\} \text{ ordinary and } t(z) < t\}|. \end{aligned}$$

In words, $n(y)$, resp. $r(y)$, is the number of neighbors of y excluding ϖ that are burnt before, resp. just before, y is burnt. From the construction, for all η such that $U_1, \ldots, U_{t-1}$ take some fixed values, we have the equivalence

$$\begin{aligned} r(y) = r,\ n(y) = n \qquad &\text{if and only if} \qquad 2d - n \le \eta_y < 2d - n + r \\ &\text{if and only if} \qquad 2d - n \le \xi_y < 2d - n + r. \end{aligned}$$

A one-to-one correspondence can be set up between the $2d$ directions of the ordinary edges and the values $\{0, 1, \ldots, 2d - 1\}$. This induces a one-to-one correspondence between the $r(y)$ ordinary edges and the values $\{2d - n, \ldots, 2d - n + r - 1\}$. Include in $\mathcal{T}_\Lambda(\xi)$ the ordinary edge $\{z, y\}$ corresponding to the value of ξ_y. Since each vertex in U_t is connected to a unique vertex in U_{t-1}, $\mathcal{T}_\Lambda(\xi)$ is a spanning tree. It follows by construction that the mapping

$$\mathcal{T}_\Lambda : \xi \mapsto \mathcal{T}_\Lambda(\xi)$$

is one-to-one and onto the set of spanning trees of $\widetilde{\Lambda}$.

Let $\mu_\Lambda^{(\gamma)}$ denote the distribution on $\{0,1\}^{E(\widetilde{\Lambda})}$ under which a spanning tree $\widetilde{t}$ has weight $\gamma^{\widetilde{N}(\widetilde{t})} / \det(\Delta_\Lambda^{(\gamma)})$, where $\widetilde{N}(\widetilde{t})$ denotes the number of dissipative edges in $\widetilde{t}$. By construction, for each ξ, $\widetilde{N}(\mathcal{T}_\Lambda(\xi)) = \mathcal{H}(\xi)$ (see (3.18)), and therefore, by (3.19),

$$\mu_\Lambda^{(\gamma)}(\mathcal{T}_\Lambda(\xi)) = \frac{\gamma^{\mathcal{H}(\xi)}}{\det(\Delta_\Lambda^{(\gamma)})} = \nu_\Lambda^{(\gamma)}(\xi) = m_\Lambda^{(\gamma)}(\psi_\Lambda^{-1}(\xi)).$$

3.6 Waves and spanning trees

We will need an extension of the results of Section 3.5 that allows us to represent waves in a γ-avalanche by spanning trees, in an analogous way to what happens for waves in an avalanche in the abelian sandpile model (see [11, 12, 17]). Let $\eta \in \Omega_\Lambda^{(\gamma)}$, and suppose we add unit height at a site, which we assume without loss of generality to be $0 \in \Lambda$. The γ-waves created by this addition are defined as follows. If $\eta_0 + 1 < 2d + \gamma$, there is no γ-avalanche and there are no γ-waves. Assuming $\eta_0 + 1 \ge 2d + \gamma$, γ-topple all sites that can be γ-toppled, not allowing 0 to topple more than once. Then all sites will topple at most once, and the set of sites that topple, call it $W_\Lambda^{(1)}(\eta)$, is the first γ-wave. If after the first γ-wave the height at 0 is still at least $2d + \gamma$, we start a second γ-wave, $W_\Lambda^{(2)}$ and so on. We set $W_\Lambda^{(i)} = \varnothing$, if the i-th wave does not exist. Note that after each γ-wave, the height at 0 has decreased by γ, and hence the number of γ-waves is, deterministically, bounded by $\lceil \gamma^{-1} \rceil$, where $\lceil a \rceil$ denotes the smallest integer larger than or equal to a real a.

We will represent the intermediate configurations between γ-waves as recurrent configurations on an auxiliary space, and show that they arise by applying the addition operator at 0 on this auxiliary space. Let for $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} \widehat{\Omega}_\Lambda^{(\gamma)} &= [0, 2d + \gamma + 1) \times [0, 2d + \gamma)^{\Lambda \setminus \{0\}}; \\ \widehat{\Delta}_{xy}^{(\gamma)} &= \begin{cases} 2d + \gamma + 1 & \text{if } x = y = 0; \\ \Delta_{xy}^{(\gamma)} & \text{otherwise,} \end{cases} \end{aligned} \tag{3.20}$$

and call $\widehat{\mathcal{R}}_\Lambda^{(\gamma)} \subseteq \widehat{\Omega}_\Lambda^{(\gamma)}$ the set of recurrent configurations for the toppling matrix $\widehat{\Delta}_\Lambda^{(\gamma)}$ (which is the above matrix restricted to Λ), and $\widehat{a}_x$ the corresponding addition operators (this amounts to a dissipation $\gamma+1$ at site 0 and γ elsewhere). By Remark 3.1 (which is valid also for non-homogeneous dissipation, as it is the case here), we have $\mathcal{R}_\Lambda^{(\gamma)} \subseteq \widehat{\mathcal{R}}_\Lambda^{(\gamma)}$. Let us show that there is a one-to-one mapping between $\widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)}$ and intermediate configurations between γ-waves at 0.

Let $\eta \in \Omega_\Lambda^{(\gamma)}$. Note that if $\eta + \delta_0 \in \mathcal{R}_\Lambda^{(\gamma)}$, then there are no γ-waves, and $\eta + \delta_0 = a_0\eta = \widehat{a}_0\eta$. If $(\eta+\delta_0)_0 = \eta_0+1 \geq 2d+\gamma$, then a γ-avalanche starts. Put $\eta^{(1)} := \eta+\delta_0 \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)}$. This is the intermediate configuration before the first γ-wave, and we have $\eta^{(1)} = \widehat{a}_0\eta$. Carrying out the first γ-wave in the original model is the same as applying $\widehat{a}_0$ to $\eta^{(1)}$ in the modified model. Let $\eta^{(2)} := \widehat{a}_0\eta^{(1)}$. If $(\eta^{(2)})_0 < 2d+\gamma$, then there is only one γ-wave, and $\eta^{(2)} = a_0\eta \in \mathcal{R}_\Lambda^{(\gamma)}$ is the configuration after the γ-avalanche. If $(\eta^{(2)})_0 \geq 2d+\gamma$, then $\eta^{(2)}$ is the intermediate configuration before the second γ-wave. Performing the second γ-wave in the original model amounts to applying $\widehat{a}_0$ to $\eta^{(2)}$ in the modified model. The result of the second γ-wave is $\eta^{(3)} := \widehat{a}_0\eta^{(2)}$. We continue inductively until we reach the smallest $K \geq 2$, such that $\eta^{(K)} \in \mathcal{R}_\Lambda^{(\gamma)}$. This happens precisely if there were $K-1$ γ-waves, and then $\eta^{(K)} = a_0\eta = \widehat{a}_0^K\eta$. Using invertibility of $\widehat{a}_0$ on $\widehat{\mathcal{R}}_\Lambda^{(\gamma)}$, it follows that the intermediate configurations $\eta^{(1)}, \ldots, \eta^{(K-1)}$ are all distinct, and also that distinct η's have distinct intermediate configurations.

We now show that any $\zeta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)}$ arises as an intermediate configuration. We first claim that there exist $n_x \geq 0$ such that $\prod_x \widehat{a}_x^{n_x}\zeta \in \mathcal{R}_\Lambda^{(\gamma)}$. To see this, let $\zeta' := \zeta + \sum_{x\in\Lambda} m_x\delta_x$, where $m_x \geq 0$, and $\zeta'_x \geq 2d+\gamma$, $x \in \Lambda$. The latter condition ensures that $\mathcal{S}_\Lambda^{(\gamma)}(\zeta') =: \zeta'' \in \mathcal{R}_\Lambda^{(\gamma)}$. Let m be the number of times 0 γ-topples during this stabilization of ζ'. Then in the modified dynamics, we have $\widehat{a}_0^m \prod_{x\in\Lambda} \widehat{a}_x^{m_x}\zeta = \zeta''$, as claimed. Now define $\zeta''' \in \mathcal{R}_\Lambda^{(\gamma)}$ by the equality $\zeta'' = \prod_{x\in\Lambda} a_x^{n_x}\zeta'''$. Let n be the number of times 0 γ-topples in computing $\prod_{x\in\Lambda} a_x^{n_x}\zeta'''$. Then we have, in the modified dynamics,

$$\prod_{x\in\Lambda} \widehat{a}_x^{n_x}\zeta = \zeta'' = \widehat{a}_0^n \prod_{x\in\Lambda} \widehat{a}_x^{n_x}\zeta'''.$$

The above implies $\zeta = \widehat{a}_0^n\zeta'''$. Since $\zeta''' \in \mathcal{R}_\Lambda^{(\gamma)}$, this proves that ζ is an intermediate configuration.

We now extend the spanning tree representation to $\widehat{\mathcal{R}}_\Lambda^{(\gamma)}$. Put

$$\widehat{\Omega}_\Lambda^{\mathrm{discr}} = \{0, 1, \ldots, 2d-1, 2d, *\} \times \{0, 1, \ldots, 2d-1, 2d\}^{\Lambda\setminus\{0\}},$$

and modify ψ_Λ by setting $\widehat{\psi}_\Lambda(\eta)_0 = *$, if $2d+\gamma \leq \eta_0 < 2d+\gamma+1$. We define the graph $\widehat{\Lambda}$ by adding an *extra* edge between 0 and ϖ in $\widetilde{\Lambda}$.

We use a particular burning rule when applying the burning bijection to $\widehat{\mathcal{R}}_\Lambda^{(\gamma)}$. We first send one unit of height along the extra edge from ϖ to 0. For configurations in $\mathcal{R}_\Lambda^{(\gamma)}$, nothing burns. For a configuration $\eta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)}$, a set of sites $W(\eta)$ burns. This is precisely the γ-wave in the intermediate configuration η. Following this, we send γ units of height along the dissipative edges from ϖ, continue burning, and then send 1 unit of height along ordinary edges from ϖ, and finish burning. Then under the burning bijection the spanning trees containing the extra edge are precisely the ones representing intermediate configurations, and the component of 0 in the forest obtained by removing the extra edge is the corresponding γ-wave.

3.7 Ergodicity of the finite-volume dynamics

In this section we show ergodicity of the Markov chain with generator (3.6) when γ is irrational and consider ergodicity of single addition operators. Because we work with continuous heights, even in finite volume the continuous-time Markov chain has infinite state space, and ergodicity is a non-trivial issue. The results of Sections 4–7 will not rely on this section, and therefore the reader can skip this somewhat independent section on first reading.

Proposition 3.2. *(a) For γ irrational, the continuous-time Markov chain with generator* (3.6) *is ergodic.*

(b) For $y \in \Lambda$, the addition operator $a^{(\gamma)}_{\Lambda,y}$ is an ergodic transformation on the measure space $(\mathcal{R}^{(\gamma)}_\Lambda, m^{(\gamma)}_\Lambda)$ if and only if $\{G^{(\gamma)}(x,y) : x \in \Lambda\} \cup \{1\}$ is rationally independent.

Proof. For *(a)* we have to show that $\mathcal{L}_\Lambda f = 0$, $f \in L^2(m^{(\gamma)}_\Lambda)$ implies f is constant, $m^{(\gamma)}_\Lambda$-a.s. The set $\mathcal{R}^{(\gamma)}_\Lambda$ is a group under pointwise addition and γ-stabilization, isomorphic to $\mathbb{R}^\Lambda/\Delta^{(\gamma)}_\Lambda$, see [7]. Analogously to the discrete case, the characters of this group are indexed by $m \in \mathbb{Z}^\Lambda$ via

$$\chi_m(\eta) = \exp\left(2\pi i \sum_{x,y\in\Lambda} m_x G^{(\gamma)}(x,y)\eta_y\right). \tag{3.21}$$

These characters form a complete orthogonal family in $L^2(m^{(\gamma)}_\Lambda)$. We have the identity

$$\chi_m(a^{(\gamma)}_{\Lambda,z}\eta) = \alpha_m(z)\chi_m(\eta), \tag{3.22}$$

where

$$\alpha_m(z) = \exp\left(2\pi i \sum_{x\in\Lambda} m_x G^{(\gamma)}(x,z)\right). \tag{3.23}$$

The generator applied to χ_m then gives

$$\mathcal{L}_\Lambda \chi_m = \left(\sum_{x\in\Lambda} \varphi(x)\,(\alpha_m(x) - 1)\right)\chi_m.$$

So we have to prove that if

$$\sum_{x\in\Lambda} \varphi(x)(\alpha_m(x) - 1) = 0, \tag{3.24}$$

then $m = 0$. Since for all $x \in \Lambda$, $\alpha_m(x)$ is a complex number of modulus one, and $\varphi(x) > 0$, (3.24) implies that $\alpha_m(x) = 1$. Hence for all $z \in \Lambda$

$$\sum_{x \in \Lambda} m_x G^{(\gamma)}(x, z) = k_z$$

where $k_z \in \mathbb{Z}$. In vector notation this reads $mG^{(\gamma)} = k$ and gives $m = k\Delta^{(\gamma)}$, which implies that for all $x \in \Lambda$, $k_x(2d + \gamma)$ is an integer. By the irrationality of γ this implies $k_x = 0$, and hence $m = k\Delta^{(\gamma)} = 0$.

To prove *(b)*, notice that the ergodicity of $a^{(\gamma)}_{\Lambda,y}$ is equivalent with the statement that $\chi_m \circ a^{(\gamma)}_{\Lambda,y} = \chi_m$ if and only if $m = 0$. By (3.22) this is the same as $\alpha_m(y) = 1$ if and only if $m = 0$. Now using formula (3.23) for α_m we have to prove that

$$\sum_{x \in \Lambda} m_x G^{(\gamma)}(x, y) = k$$

with $k \in \mathbb{Z}$ implies $m = 0$. This is exactly the condition of linear independence stated. ☐

The following Proposition gives a more explicit sufficient condition for ergodicity of an addition operator. For $x, y \in \Lambda$, let

$$P_\Lambda(x, y) = \begin{cases} \dfrac{1}{2d} & \text{if } x \sim y; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|\Lambda|}$ be the eigenvalues of P_Λ, and let $w_k(x)$, $1 \leq k \leq |\Lambda|$, be the corresponding eigenfunctions normalized to have ℓ_2-norm 1.

Proposition 3.3. *(a) Suppose that $0 \in \Lambda$, the eigenvalues λ_k are all distinct, and $w_k(0) \neq 0$ for $1 \leq k \leq |\Lambda|$. Suppose that γ is transcendental. Then $a^{(\gamma)}_{\Lambda,0}$ is ergodic.*

(b) Examples where the conditions in (a) are satisfied are given by: $\Lambda = [-a_1 + 1, b_1 - 1] \times \cdots \times [-a_d + 1, b_d - 1]$, where $p_i = a_i + b_i$ are distinct primes greater than 5, $i = 1, \ldots, d$.

Proof. (a) For each x, $\beta_x(\gamma) = G^{(\gamma)}(0, x) = (\Delta^{(\gamma)}_\Lambda)^{-1}_{0x}$ is a ratio of integer polynomials of γ. We show that under the assumptions of part *(a)*, the functions $\{\beta_x : x \in \Lambda\} \cup \{1\}$ are linearly independent over the rationals. This implies that if γ is transcendental, then no rational linear combination of the numbers $\{\beta_x(\gamma) : x \in \Lambda\}$ can take a rational value, and hence Proposition 3.2*(b)* can be applied.

We have the following spectral representation:

$$\begin{aligned} \beta_x(\gamma) = G^{(\gamma)}_\Lambda(0, x) &= \left[(2d + \gamma)I - 2dP_\Lambda\right]^{-1}(0, x) \\ &= \sum_{k=1}^{|\Lambda|} \frac{1}{2d + \gamma - 2d\lambda_k} w_k(0) w_k(x), \end{aligned} \tag{3.25}$$

(where I denotes the identity matrix). Suppose that $(\alpha(x))_{x\in\Lambda}$ is a system of rationals such that $\sum_{x\in\Lambda}\alpha(x)\beta_x(\gamma)\equiv c$, with c a rational constant. Since $\beta_x(\gamma)\to 0$ as $\gamma\to\infty$, we necessarily have $c=0$. Hence (3.25) implies

$$\sum_{k=1}^{|\Lambda|}\frac{c_k}{\gamma-2d(\lambda_k-1)}\equiv 0, \tag{3.26}$$

where $c_k = w_k(0)\sum_{x\in\Lambda}\alpha(x)w_k(x)$. Since the λ_k are all distinct, the sum in (3.26) can only vanish identically, if $c_k=0$ for $1\le k\le|\Lambda|$. Since $w_k(0)\neq 0$, this implies that $\sum_{x\in\Lambda}\alpha(x)w_k(x)=0$ for $1\le k\le|\Lambda|$. Since the collection of functions $\{k\mapsto w_k(x): x\in\Lambda\}$ forms a basis, it follows that $\alpha(x)=0$ for each $x\in\Lambda$. This proves the stated linear independence, and hence completes the proof of part *(a)*.

(b) For the rectangle Λ, the eigenfunctions and eigenvalues can be indexed by $\mathbf{k}=(k_1,\dots,k_d)\in\prod_{i=1}^d\{1,\dots,p_i-1\}=:\mathcal{K}$ and they are given by [21, Section 8.2]:

$$\lambda_{\mathbf{k}}=\frac{1}{d}\sum_{i=1}^d\cos\left(\frac{k_i\pi}{p_i}\right);$$
$$w_{\mathbf{k}}(x)=c_{\mathbf{k}}\prod_{i=1}^d\sin\left(\frac{(x_i+a_i)k_i\pi}{p_i}\right),$$

where $c_{\mathbf{k}}$ is a constant normalizing $w_{\mathbf{k}}$ to have ℓ_2-norm 1. Since p_i is prime, we have $w_{\mathbf{k}}(0)\neq 0$ for $\mathbf{k}\in\mathcal{K}$.

We conclude the proof by showing that when the p_i are as assumed, the eigenvalues are all distinct. Let $\eta_j=\exp(2\pi i k_j/2p_j)$, $\zeta_j=\exp(2\pi i l_j/2p_j)$, so that with $\mathbf{k}=(k_1,\dots,k_d)$ and $\mathbf{l}=(l_1,\dots,l_d)$ we have

$$2d\lambda_{\mathbf{k}}-2d\lambda_{\mathbf{l}}=\sum_{j=1}^d(\eta_j+\eta_j^{-1})-\sum_{j=1}^d(\zeta_j+\zeta_j^{-1}).$$

We prove by induction on d that $2d(\lambda_{\mathbf{k}}-\lambda_{\mathbf{l}})=0$ implies $\mathbf{k}=\mathbf{l}$.

When $d=1$, we have $\lambda_k=\cos(k\pi/p_1)$, $k=1,\dots,p_1-1$, which are all distinct, so the statement holds for $d=1$.

For the induction step, we will use some basic results from field theory, which can be found for example in [19, Chapter 8]. Assume $d\ge 2$, and that $2d(\lambda_{\mathbf{k}}-\lambda_{\mathbf{l}})=0$. For $m\ge 1$, write $\omega_m=\exp(2\pi i/m)$. Let $\mathbb{Q}(\omega_{m_1},\dots,\omega_{m_r})$ denote the field obtained by adjoining $\omega_{m_1},\dots,\omega_{m_r}$ to the rationals. The m-th cyclotomic polynomial has roots precisely the primitive m-th roots of unity, is irreducible over $\mathbb{Q}$, and has degree $\varphi(m)$, where $\varphi(m)$ is the number of residue classes mod m relatively prime to m. Hence, since the p_i are distinct, the degree of $K:=\mathbb{Q}(\omega_{p_1},\dots,\omega_{p_d})=\mathbb{Q}(\omega_{p_1\cdots p_d})$ over $\mathbb{Q}$ is $\varphi(p_1\cdots p_d)=(p_1-1)\cdots(p_d-1)$ [19, Theorem VII.1.3]. In particular, ω_{p_d}, and in fact all primitive p_d-th roots of unity, are not contained in $K':=\mathbb{Q}(\omega_{p_1},\dots,\omega_{p_{d-1}})$, and they have degree p_d-1 over K'. We distinguish three cases.

Case 1. The numbers k_d and l_d are both even. Then $\eta_d = \omega_{p_d}^a$, $\zeta_d = \omega_{p_d}^b$ with $a = k_d/2$, $b = l_d/2$, $1 \le a, b \le (p_d - 1)/2$. Let ω be a primitive p_d-th root of unity such that $\omega^{-1} \neq \omega_{p_d}^c$ for $c = a, b, -a, -b$. Since $p_d > 5$, this is possible. Since the degree of ω over K' is $p_d - 1$, the numbers $1, \omega, \omega^2, \dots, \omega^{p_d-2}$ are linearly independent over K'. Moreover, they contain $\eta_d, \eta_d^{-1}, \zeta_d, \zeta_d^{-1}$. Since $\sum_{j=1}^{d-1}(\eta_j + \eta_j^{-1} - \zeta_j - \zeta_j^{-1}) \in K'$ we need to have $\eta_d = \zeta_d$, and hence $k_d = l_d$. It follows that $\sum_{j=1}^{d-1}(\eta_j + \eta_j^{-1} - \zeta_j - \zeta_j^{-1}) = 0$, and the proof is completed using the induction hypothesis.

Case 2. The numbers k_d and l_d are both odd. Then $\eta_d = -\omega_{p_d}^a$, $\zeta_d = -\omega_{p_d}^b$ with $(p_d + 1)/2 \le a, b \le p_d - 1$. Let now ω be a primitive p_d-th root of unity such that $\omega^{-1} \neq \omega_{p_d}^c$ for $c = a, b, -a, -b$. The argument is then completed as in Case 1.

Case 3. The numbers k_d and l_d are of different parity. Without loss of generality, assume that k_d is even and l_d is odd, so that $\eta_d = \omega_{p_d}^a$, $1 \le a \le (p_d - 1)/2$, and $\zeta_d = -\omega_{p_d}^b$, $(p_d + 1)/2 \le b \le p_d - 1$. Again we can find ω such that $\omega^{-1} \neq \omega_{p_d}^c$ for $c = a, b, -a, -b$. Then linear independence implies that this case is impossible.

The above completes the argument that the eigenvalues are all distinct, and hence finishes the proof of part *(b)*. □

4 The infinite-volume limit of the stationary measure of the abelian avalanche model

4.1 Existence of the infinite-volume limit of $m_\Lambda^{(\gamma)}$

Recall the definition of $\widetilde{\Lambda}$ from Section 3.5. We can view $\widetilde{\Lambda}$ as a weighted network, where ordinary edges have weight 1, and dissipative edges have weight γ. Then $\mu_\Lambda^{(\gamma)}$ is the weighted spanning tree measure on this network, that is, the measure where the probability of a tree is proportional to the product of the weights of the edges it contains. Wilson's algorithm [34] can be used to sample from this distribution, analogously to what is done for the abelian sandpile model, see [1, 17]. This is described as follows. For a path σ, we denote by $LE(\sigma)$ its loop-erasure, that is the path obtained by removing loops from σ chronologically. Consider the network random walk on $\widetilde{\Lambda}$, that is, the reversible Markov chain that makes jumps with probabilities proportional to the weights. Let $\mathcal{F}_0 = \{\varpi\}$, and let $x_1, \dots, x_K$ be an enumeration of the vertices in Λ. If $\mathcal{F}_{j-1}$ has been defined, start a network random walk from x_j, and run it until it first hits $\mathcal{F}_{j-1}$. Let σ_j be the path obtained, and let $\mathcal{F}_j = \mathcal{F}_{j-1} \cup LE(\sigma_j)$. By Wilson's theorem, $\mathcal{F}_K$ is a tree with the stated distribution.

Let $\gamma \ge 0$. Using monotonicity arguments, in the spirit of [3, Section 5], we obtain that for any exhaustion $\Lambda_1 \subseteq \Lambda_2 \subseteq \dots$ of $\mathbb{Z}^d$, the weak limit

$$\lim_{n\to\infty} \mu_{\Lambda_n}^{(\gamma)} =: \mu^{(\gamma)} \tag{4.1}$$

exists. We will be interested in sampling from $\mu^{(\gamma)}$ when $\gamma > 0$. For this, Wilson's algorithm can be used. When $\gamma = 0$, $d \ge 3$, this is the statement of [3, Theorem 5.1].

The algorithm in this case uses simple random walk, and the "root" is at infinity. When $\gamma > 0$, the algorithm uses the network random walk on $\widetilde{\mathbb{Z}^d}$, stopped when it hits ϖ. The tree generated by the algorithm then gives a sample from $\mu^{(\gamma)}$. To see the convergence (4.1), couple the algorithms in $\widetilde{\Lambda}_n$ and $\widetilde{\mathbb{Z}^d}$, starting walks at a fixed finite number of sites $x_1, \dots, x_k$. The network random walks can be coupled until the first time the boundary of Λ_n is hit. The coupling shows that the joint distribution of $(LE(\sigma_1), \dots, LE(\sigma_k))$ in $\widetilde{\Lambda}_n$ converges as $n \to \infty$ to the joint distribution in $\widetilde{\mathbb{Z}^d}$. Since this information determines all finite dimensional distributions of the spanning tree, the claim (4.1) is proven.

Under $\mu^{(\gamma)}$, there is a unique path π_x in the tree from x to ϖ. Similarly, there is a unique path $\pi_{\Lambda,x}$ under $\mu_\Lambda^{(\gamma)}$. Let us write $\mathcal{P}_x$ for the set of self-avoiding paths from x to ϖ in $\widetilde{\mathbb{Z}^d}$, and $\mathcal{N}_x = \{y : |y - x| \le 1\}$. From the correspondence $\mathcal{T}_\Lambda$ we obtain (as in [1, 17])

Lemma 4.1. *Under the map $\mathcal{T}_\Lambda^{-1}$, the height ξ_x is a function of $\{\pi_{\Lambda,y}\}_{y \in \mathcal{N}_x}$ only.*

Lemma 4.2. *For any $\gamma > 0$, and any exhaustion $\Lambda_1 \subseteq \Lambda_2 \subseteq \dots$ of $\mathbb{Z}^d$, we have the unique weak limit*

$$\lim_{n \to \infty} \nu_{\Lambda_n}^{(\gamma)} =: \nu^{(\gamma)}.$$

Moreover, for all $x \in \mathbb{Z}^d$, there is a map $h_x : \prod_{y \in \mathcal{N}_x} \mathcal{P}_y \to \{0, 1, \dots, 2d\}$, such that $\{\xi_x\}_{x \in \mathbb{Z}^d}$ under $\nu^{(\gamma)}$ has the same law as $\{h_x(\pi_y : y \in \mathcal{N}_x)\}_{x \in \mathbb{Z}^d}$ under $\mu^{(\gamma)}$.

Proof. Coupling Wilson's algorithm in $\widetilde{\Lambda}_n$ and $\widetilde{\mathbb{Z}^d}$, we get that the joint distribution of $(\pi_{\Lambda_n,x_1}, \dots, \pi_{\Lambda_n,x_k})$ converges to the joint distribution of $(\pi_{x_1}, \dots, \pi_{x_k})$. This and Lemma 4.1 show that for any $x_1, \dots, x_k$, the joint distribution under $\nu_{\Lambda_n}^{(\gamma)}$ of $(\xi_{x_1}, \dots, \xi_{x_k})$ converges to a limit, which has the form stated. $\square$

For the next lemma, we make $m_\Lambda^{(\gamma)}$ into a measure on $\Omega^{(\gamma)}$ via the inclusion map $i : \Omega_\Lambda^{(\gamma)} \to \Omega^{(\gamma)}$, where

$$i(\eta)_x = \begin{cases} \eta_x & \text{if } x \in \Lambda; \\ 0 & \text{if } x \in \mathbb{Z}^d \setminus \Lambda. \end{cases}$$

We then obtain the existence of the thermodynamic limit of the stationary measures $m_\Lambda^{(\gamma)}$ as $\Lambda \uparrow \mathbb{Z}^d$.

Theorem 4.1. *For any $\gamma > 0$, and any exhaustion $\Lambda_1 \subseteq \Lambda_2 \subseteq \dots$ of $\mathbb{Z}^d$, the weak limit*

$$\lim_{n \to \infty} m_{\Lambda_n}^{(\gamma)} =: m^{(\gamma)}$$

exists, and we have $m^{(\gamma)}(\mathcal{R}^{(\gamma)}) = 1$.

Proof. Recall the fact that $m_\Lambda^{(\gamma)}$ is uniform on (γ,Λ)-cells. Therefore, under the measure $m_\Lambda^{(\gamma)}$ and conditioned on the value of $\xi = \psi_\Lambda(\eta) \in \Omega_\Lambda^{\text{discr}}$, the random variables $\{\eta_y - \xi_y\}_{y\in\Lambda}$ are (conditionally) independent, with distribution

$$\begin{aligned} \eta_y - \xi_y &\sim \text{Uniform}(0,1) \quad \text{for } y \text{ with } 0 \le \xi_y \le 2d-1; \\ \eta_y - \xi_y &\sim \text{Uniform}(0,\gamma) \quad \text{for } y \text{ with } \xi_y = 2d. \end{aligned} \tag{4.2}$$

Now let $V \subseteq \Lambda$. The above implies that if we condition on the values of $\{\xi_y\}_{y\in V}$ only, then $\{\eta_y - \xi_y\}_{y\in V}$ still has the conditional joint distribution in (4.2) (under $m_\Lambda^{(\gamma)}$). This and Lemma 4.2 imply that the joint law of $\{\eta_y\}_{y\in V}$ under $m_{\Lambda_n}^{(\gamma)}$ converges weakly as $n\to\infty$, in the space $\Omega_V^{(\gamma)}$.

This proves the weak convergence statement. Finally, the limit gives probability 1 to the event $\{\eta_V \in \mathcal{R}_V^{(\gamma)}\}$ for any finite V, because

$$m_W^{(\gamma)}\left(\{\eta : \eta_V \in \mathcal{R}_V^{(\gamma)}\}\right) = 1$$

for all $W \supset V$, since $m_W^{(\gamma)}$ is supported by $\mathcal{R}_W^{(\gamma)}$. □

Remark 4.1. *It follows from this proof that under the limiting measure $m^{(\gamma)}$ we still have $\{\eta_y - \xi_y\}_{y\in\mathbb{Z}^d}$ conditionally independent, given $\xi = \psi(\eta)$, with distribution* (4.2). *As we will see later on, when $\gamma = 0$, the distribution reduces to the first line of* (4.2).

4.2 Exponential decay of covariances for non-zero dissipation

We now prove that for dissipation $\gamma > 0$, we have exponential decay of correlations of local observables in the abelian avalanche model. In [22] this was shown for large enough dissipation for the discrete dissipative sandpile model.

By a *local function*, we mean a function $f : \Omega^{\text{discr}} \to \mathbb{R}$ (respectively $f : \Omega^{(\gamma)} \to \mathbb{R}$) such that f depends only on coordinates in a finite set $A \subseteq \mathbb{Z}^d$.

Theorem 4.2. *For all $\gamma > 0$, there exist $C = C^{(\gamma)}, c > 0$, such that for all bounded local functions $f, g : \Omega^{\text{discr}} \to \mathbb{R}$ with dependence sets A, B, we have*

$$|\mathbb{E}_{\nu^{(\gamma)}}[fg] - \mathbb{E}_{\nu^{(\gamma)}}[f]\mathbb{E}_{\nu^{(\gamma)}}[g]| \le C|A||B|\|f\|_\infty\|g\|_\infty \exp(-c\sqrt{\gamma}\,\text{dist}(A,B)). \tag{4.3}$$

The same statement holds for the measure $m^{(\gamma)}$.

Proof. The first statement deals with discretized heights. First consider the case when f depends only on ξ_x, and g only on ξ_y. Then by Lemma 4.2, f and g depend on the paths $\{\pi_z\}_{z\in\mathcal{N}_x}$, and $\{\pi_w\}_{w\in\mathcal{N}_y}$, respectively. Use Wilson's algorithm starting with the vertices in $\mathcal{N}_x$ and then using the vertices in $\mathcal{N}_y$. We couple the random walks appearing in the algorithm to a new set of random walks $\{S'_w\}_{w\in\mathcal{N}_y}$ as follows: $S'_w = S_w$ until the path hits $\{\varpi\} \cup (\cup_{z\in\mathcal{N}_x}\pi_z)$, and it moves independently afterwards. Let $\{\pi'_w\}_{w\in\mathcal{N}_y}$ be the paths created by Wilson's algorithm started from $\mathcal{N}_y$, using the S'_w's. Let g' be a copy of g, that is a function of $\{\pi'_w : w \in \mathcal{N}_y\}$ instead of

$\{\pi_w : w \in \mathcal{N}_y\}$. Then the left hand side of (4.3) equals $\mathbb{E}_{\nu^{(\gamma)}}[f(g-g')]$. This is bounded by $2\|f\|_\infty \|g\|_\infty$ times the probability that $g \neq g'$. The latter is bounded by the probability that one of the random walks used for g intersects one of the random walks used for f. This is bounded by

$$\sum_{z\in\mathcal{N}_x} \sum_{w\in\mathcal{N}_y} \sum_{u\in\mathbb{Z}^d} G^{\text{discr}}(z,u) G^{\text{discr}}(w,u), \tag{4.4}$$

where G^{discr} is the Green's function on $\mathbb{Z}^d$ for the discrete time walk that steps to each neighbor with probability $1/(2d+\gamma)$ and to ϖ with probability $\gamma/(2d+\gamma)$. Hence we obtain (4.3) from (3.11).

In the general case, we can repeat the argument with the vertices in $\mathcal{M}_f = \cup_{x\in A}\mathcal{N}_x$ and $\mathcal{M}_g = \cup_{y\in B}\mathcal{N}_y$. This leads to an estimate similar to (4.4), where now we sum over $z \in \mathcal{M}_f$ and $w \in \mathcal{M}_g$. This implies the claim.

Let us prove the statement for the continuous case, that is the abelian avalanche model. Due to the representation (4.2), if we set

$$\begin{aligned} f_0(\xi) &= \mathbb{E}_{m^{(\gamma)}}[f(\eta) | \psi(\eta)_{A\cup B} = \xi_{A\cup B}] \\ g_0(\xi) &= \mathbb{E}_{m^{(\gamma)}}[g(\eta) | \psi(\eta)_{A\cup B} = \xi_{A\cup B}] \end{aligned}$$

then we have

$$\begin{aligned} \mathbb{E}_{m^{(\gamma)}}[f(\eta)g(\eta)] &= \mathbb{E}_{\nu^{(\gamma)}}[f_0(\xi)g_0(\xi)], \\ \mathbb{E}_{m^{(\gamma)}}[f(\eta)] &= \mathbb{E}_{\nu^{(\gamma)}}[f_0(\xi)], \\ \mathbb{E}_{m^{(\gamma)}}[g(\eta)] &= \mathbb{E}_{\nu^{(\gamma)}}[g_0(\xi)]. \end{aligned}$$

Now we can apply the result for the model with discretized heights to f_0 and g_0. □

Remark 4.2. *Avalanches also satisfy exponential decay when $\gamma > 0$; see Theorem 6.1 item a) below.*

4.3 Convergence of the stationary measures when $\gamma \to 0$

For the abelian sandpile model, the measure $\nu = \nu^{(0)}$ was constructed in [1] and [17, Appendix], as the weak limit of $\nu_\Lambda^{(0)}$.

Theorem 4.3. *We have $\lim_{\gamma\to 0} \nu^{(\gamma)} = \nu^{(0)}$, and $\lim_{\gamma\to 0} m^{(\gamma)} = m^{(0)}$.*

Proof. For $\xi_\Lambda \in \Omega_\Lambda^{\text{discr}}$ chosen from $\nu_\Lambda^{(\gamma)}$, consider the random field of maximal heights $\{h_{\Lambda,x}\}_{x\in\Lambda} := \{I[\xi_{\Lambda,x} = 2d]\}_{x\in\Lambda}$ (where $I[\xi_{\Lambda,x} = 2d]$ denotes the indicator function of the event $\{\xi_{\Lambda,x} = 2d\}$), and the random set $H_\Lambda = \{x \in \Lambda : h_{\Lambda,x} = 1\}$. Due to the correspondence $\mathcal{T}_\Lambda$, $\mathbb{P}(x \in H_\Lambda)$ equals the probability that the dissipative edge containing x is included in $\mathcal{T}_\Lambda(\xi_\Lambda)$. Using Wilson's algorithm started at x and (3.19), we see that this probability vanishes in the limit $\gamma \to 0$, uniformly in Λ. Hence for any finite $V \subseteq \mathbb{Z}^d$, $\mathbb{P}(H_\Lambda \cap V = \varnothing)$ goes to 1 as $\gamma \to 0$, uniformly in Λ.

By (3.19), after removal of the sites with $\xi_{\Lambda,x} = 2d$, the joint distribution of the heights of the remaining sites is uniform on allowed configurations with heights in $\{0, 1, \ldots, 2d-1\}$. Therefore, given H_Λ, the conditional distribution of $\{\xi_{\Lambda,x}\}_{x\in\Lambda\setminus H_\Lambda}$ is given by $\nu^{(0)}_{\Lambda\setminus H_\Lambda}$, the measure for the abelian sandpile model in $\Lambda \setminus H_\Lambda$. Due to the convergence $\nu^{(0)}_W \to \nu^{(0)}$, as $W \uparrow \mathbb{Z}^d$, for large V and on the event $\{H_\Lambda \cap V = \varnothing\}$ the conditional distribution is close to $\nu^{(0)}$, uniformly in H_Λ and $\Lambda \supset V$. The above observations imply that as $\gamma \to 0$ and $\Lambda \to \mathbb{Z}^d$, $\nu^{(\gamma)}_\Lambda \to \nu^{(0)}$. This implies the first statement of the Proposition.

For the second statement, we again use the representation (4.2). Then the convergence of finite-dimensional distributions follows from the first part of the Proposition. Tightness holds trivially since all the measures under consideration have support contained in (the same) compact set. $\square$

Remark 4.3. *The process $\{h_x\}_{x\in\mathbb{Z}^d}$ is a determinantal process [10], that is there exists a kernel $\mathbf{K}^{(\gamma)}(x,y)$, such that for $n \geq 1$ and distinct $x_1, \ldots, x_n \in \mathbb{Z}^d$,*

$$\mathbb{P}(h_{x_1} = 1, \ldots, h_{x_n} = 1) = \det(\mathbf{K}^{(\gamma)}(x_i, x_j))_{i,j=1}^n.$$

This follows from the Transfer Current Theorem of Burton & Pemantle [4] applied to the collection of dissipative edges.

4.4 Speed of convergence for special events

The following proposition gives an estimate of the speed of convergence in Theorem 4.3 for the probability of having mimimal height, for which the method of Dhar & Majumdar (see [26, 25]) can be applied. Power law estimates for general cylinder events are proved in [15].

We call a set of points $\{x_1, \ldots, x_k\} \subseteq \mathbb{Z}^d$ isolated from each other if there does not exist $y \in \mathbb{Z}^d$ that is a neighbor of more than one site in the set. Denote

$$E_{x_1,\ldots,x_k} = \{\eta \in \mathcal{X} : \forall i \in \{1, \ldots, k\}, 0 \leq \eta_{x_i} < 1\}$$

Proposition 4.1. *For $d \geq 2$ and $k \in \mathbb{N}$ there exists $C(k,d) > 0$ such that for all sets $\{x_1, \ldots, x_k\}$ of isolated points we have*

$$|m^{(\gamma)}(E_{x_1,\ldots,x_k}) - m^{(0)}(E_{x_1,\ldots,x_k})| \leq \begin{cases} C(k,d)\gamma & \text{if } d \geq 3; \\ C(k,2)\gamma\log(1/\gamma) & \text{if } d = 2. \end{cases} \tag{4.5}$$

Proof. (a) Start with $d \geq 3$. First remark that, using the notation of section 3.5, for $\gamma \geq 0$, $m^{(\gamma)}(E_{x_1,\ldots,x_k}) = \nu^{(\gamma)}(\xi_{x_1} = 0, \ldots, \xi_{x_k} = 0)$. In terms of the discretized height-configuration, this probability can be computed using the method of [26]: the allowed discretized configurations with $(\xi_{x_1} = 0, \ldots, \xi_{x_k} = 0)$ are in one-to-one correspondence with allowed configurations on a modified graph. The latter is obtained by removing the dissipative edge at each x_j and all ordinary edges emanating from each x_j, except the one leading to $x_j + (1, 0, \ldots, 0)$. Then

$$\nu^{(\gamma)}(\xi_{x_1} = 0, \ldots, \xi_{x_k} = 0) = \det(I + K^{(\gamma)}B^{(\gamma)}),$$

where $B^{(\gamma)}$, $K^{(\gamma)}$ are finite matrices (and I denotes the identity matrix) with $B^{(\gamma)}$ not depending on the particular sites in $\{x_1, \dots x_k\}$, and elements of $K^{(\gamma)}$ have the form $G^{(\gamma)}(u, v)$. Moreover the off-diagonal elements of $B^{(\gamma)}$ do not depend on γ and for the diagonal elements we have $|B^{(\gamma)}(x, x) - B^{(0)}(x, x)| = \gamma$. We have the uniform estimate

$$0 \le G^{(0)}(x, y) - G^{(\gamma)}(x, y) \le C\gamma.$$

This can be seen for example from the Fourier representation (see [21, Propositions 4.2.3, 4.4.3], [30])

$$G^{(\gamma)}(x, y) = \int_{[0,2\pi)^d} \frac{e^{ik(x-y)}}{\Gamma^{(\gamma)}(k)} dk$$

with

$$\Gamma^{(\gamma)}(k) = 1 - (1 - \gamma) \left(\frac{1}{2d} \sum_{e \in \mathbb{Z}^d : |e|=1} \cos(k \cdot e) \right).$$

This gives

$$|\det(I + G^{(\gamma)} B^{(\gamma)}) - \det(I + G^{(0)} B^{(0)})| \le C(k)\gamma.$$

(b) In the case $d = 2$ we cannot directly work in infinite volume, as the Green's function is divergent. So we proceed by finite volume approximations: take Λ big enough so that it contains $x_1, \dots, x_k$ and all their neighbors:

$$\nu_\Lambda^{(\gamma)}(\xi_{x_1} = 0, \dots, \xi_{x_k} = 0) = \det(I + K_\Lambda^{(\gamma)} B^{(\gamma)}).$$

The matrix $B^{(0)}$ has zero column sums. Therefore, $A_\Lambda^{(\gamma)} B^{(0)} = 0$ for a matrix $A_\Lambda^{(\gamma)}$ with identical entries equal to $G_\Lambda^{(\gamma)}(0, 0)$. Hence, we can write

$$K_\Lambda^{(\gamma)} B^{(\gamma)} - K_\Lambda^{(0)} B^{(0)} = K_\Lambda^{(\gamma)} (B^{(\gamma)} - B^{(0)}) + \left((K_\Lambda^{(\gamma)} - A_\Lambda^{(\gamma)}) - (K_\Lambda^{(0)} - A_\Lambda^{(0)}) \right) B^{(0)}.$$

For a matrix $D^{(\gamma)}$, and a function $f : (0, 1] \to (0, \infty)$ we write $D^{(\gamma)} = O(f(\gamma))$ if for all entries (i, j), $|D_{ij}^{(\gamma)}| \le Cf(\gamma)$. Then if f is such that $f(\gamma) \to 0$ as $\gamma \to 0$, and if $D - D' = O(f(\gamma))$ we have $|\det(I + D) - \det(I + D')| \le C' f(\gamma)$.

Therefore, in order to prove the statement for $d = 2$, if suffices to see that

$$K_\Lambda^{(\gamma)} B^{(\gamma)} - K_\Lambda^{(0)} B^{(0)} = O(\gamma \log(1/\gamma)).$$

By Fourier representation of the potential kernel in $d = 2$

$$(K_\Lambda^{(\gamma)} - A_\Lambda^{(\gamma)}) - (K_\Lambda^{(0)} - A_\Lambda^{(0)}) = O(\gamma)$$

uniformly in Λ. Furthermore, uniformly in Λ,

$$K_\Lambda^{(\gamma)} (B^{(\gamma)} - B^{(0)}) = O(\gamma \log(1/\gamma)).$$

$\square$

5 The dynamics of the dissipative model

5.1 Legal and exhaustive sequences of topplings

Let $\varphi : \mathbb{Z}^d \to (0,\infty)$ be a bounded function, and let $(N_t^\varphi)_{t\geq 0} := \{N_{x,t}^\varphi\}_{x\in\mathbb{Z}^d,\, t\geq 0}$ be a collection of independent Poisson processes on a probability space $(X, \mathcal{F}, \mathbb{P})$, where $(N_{x,t}^\varphi)_{t\geq 0}$ has rate $\varphi(x)$. We want to define the dynamics $\{\eta_t\}_{t\geq 0}$ of the dissipative model for an initial configuration η_0 as the result of stabilizing $\eta_0 + N_t^\varphi$. Since this typically involves infinitely many topplings, it requires some care. The result of the next two lemmas will be that the procedure is well-defined.

First we consider γ-stabilization of infinite configurations. Fix $\eta \in \mathcal{X}$, and a sequence $x_1, x_2, \ldots$ of legal γ-topplings.

Definition 5.1. *A sequence $x_1, x_2, \ldots$ of legal γ-topplings of a configuration η is called* γ-exhaustive*, if for every $n \in \mathbb{N}$, and for every γ-unstable site x of $T_{x_n} \circ \ldots \circ T_{x_1}(\eta)$, there exists $m > n$ such that $x_m = x$. A sequence $x_1, x_2, \ldots$ of legal γ-topplings of a configuration η is called* γ-stabilizing *if the limit $\lim_{n\to\infty} T_{x_n} \circ \ldots \circ T_{x_1}(\eta)$ is γ-stable.*

The notion of an exhaustive sequence of topplings slightly generalizes what in the finite case corresponds to a stabilizing sequence of topplings. In particular in an exhaustive sequence sites are allowed to topple infinitely many times. Just as in the finite case, the number of times each site topples in a γ-exhaustive sequence (which may be infinite) is independent of the sequence (see [14, Lemma 4.1] for a proof). It can also be seen the same way that if $y_1, y_2, \ldots$ is another legal sequence of γ-topplings, then each site γ-topples at most as many times as in a γ-exhaustive sequence. Call

$$(N^{(\gamma)}(\eta))_x = \text{the number of times } x\ \gamma\text{-topples in a } \gamma\text{-exhaustive sequence.}$$

We say that $\eta \in \mathcal{X}$ is *γ-stabilizable*, if $(N^{(\gamma)}(\eta))_x < \infty$ for all $x \in \mathbb{Z}^d$. In this case, similarly to (3.2), the γ-stabilization is related to the original configuration by the formula:

$$\mathcal{S}^{(\gamma)}(\eta) = \eta - \Delta^{(\gamma)} N^{(\gamma)}(\eta).$$

Note that every γ-stabilizing sequence is γ-exhaustive and if the configuration is γ-stabilizable, every γ-exhaustive sequence is γ-stabilizing. Recall that $(N_\Lambda^{(\gamma)}(\eta))_x$ denotes the number of times $x \in \Lambda$ γ-topples during γ-stabilization in Λ. We will see in Lemma 5.2 that any configuration that does not grow too fast at infinity is γ-stabilizable.

Lemma 5.1. *Let $\gamma \geq 0$, $\eta \in \mathcal{X}$.*
(i) The vector $N_\Lambda^{(\gamma)}(\eta)$ is componentwise monotone increasing in Λ and η.
(ii) If η is γ-stabilizable, we have $(N^{(\gamma)}(\eta))_x = \sup_\Lambda (N_\Lambda^{(\gamma)}(\eta))_x$, $x \in \mathbb{Z}^d$.
(iii) If η is γ-stabilizable, we have $\mathcal{S}^{(\gamma)}(\eta) = \lim_\Lambda \mathcal{S}_\Lambda^{(\gamma)}(\eta) \in \Omega^{(\gamma)}$.
(iv) If $\zeta_1, \zeta_2 \in \mathcal{X}$, and $\zeta_1 + \zeta_2$ is γ-stabilizable, then we have

$$\mathcal{S}^{(\gamma)}(\zeta_1 + \zeta_2) = \mathcal{S}^{(\gamma)}(\mathcal{S}^{(\gamma)}(\zeta_1) + \zeta_2). \tag{5.1}$$

Proof. *(i)* Consider $\Lambda \subseteq \Lambda'$. First γ-stabilize η in Λ, and record the amount of height flowing out of Λ. Now γ-stabilize in Λ', with further γ-topplings, if necessary. To prove the other statement, consider $\eta \leq \eta'$. First γ-stabilize η. Adding height $\eta' - \eta$ does not affect the legality of the sequence of γ-topplings. Hence we can γ-stabilize η' by further γ-topplings if necessary.

(ii) Fix a sequence $\Lambda_1 \subseteq \Lambda_2 \subseteq \dots$ such that $\cup_k \Lambda_k = \mathbb{Z}^d$. First γ-stabilize η in Λ_1, then in Λ_2, and so on. We thus get a γ-exhaustive sequence, and each site x γ-topples $\sup_\Lambda (N_\Lambda^{(\gamma)} \eta)_x$ times.

(iii) Note that if η is γ-stabilizable, we have $N^{(\gamma)}\eta = \lim_\Lambda (N_\Lambda^{(\gamma)}\eta)$, and hence by (3.2)

$$\mathcal{S}^{(\gamma)}(\eta) = \eta - \Delta^{(\gamma)} N^{(\gamma)} \eta = \lim_\Lambda \left[\eta - \Delta^{(\gamma)} N_\Lambda^{(\gamma)} \eta \right] = \lim_\Lambda \mathcal{S}_\Lambda^{(\gamma)}(\eta).$$

(iv) This is essentially not more than the Abelian property. However, since there can be infinitely many γ-topplings in γ-stabilizing ζ_1, care needs to be taken. Note that $\zeta_1 \leq \zeta_1 + \zeta_2$ implies that any legal sequence for ζ_1 is legal for $\zeta_1 + \zeta_2$ as well. In particular, ζ_1 is also γ-stabilizable. Fix a γ-stabilizing sequence $y_1, y_2, \dots$ for ζ_1. We now construct a γ-exhaustive sequence for $\zeta_1 + \zeta_2$. Fix again $\Lambda_1 \subseteq \Lambda_2 \subseteq \dots$ We write

$$\begin{aligned} \partial_+ \Lambda &= \{x \in \Lambda^c : \exists\, y \in \Lambda \text{ such that } x \sim y\}; \\ \partial_- \Lambda &= \{y \in \Lambda : \exists\, x \in \Lambda^c \text{ such that } x \sim y\}; \\ \overline{\Lambda} &= \Lambda \cup \partial_+ \Lambda. \end{aligned}$$

Let m_1 be an index such that, for all $y \in \overline{\Lambda_1}$,

$$|\{1 \leq k \leq m_1 : y_k = y\}| = (N^{(\gamma)} \zeta_1)_y.$$

i.e., after index m_1 there will be no further γ-topplings in $\overline{\Lambda_1}$ in the γ-stabilization of ζ_1. Note that $y_1, \dots, y_{m_1}$ is legal for $\zeta_1 + \zeta_2$. Now stabilize $T_{y_{m_1}} \circ \dots T_{y_1}(\zeta_1 + \zeta_2)$ in $\Lambda_1 \setminus \partial_- \Lambda_1$, recording the amount of height flowing out of $\Lambda_1 \setminus \partial_- \Lambda_1$. This leads to potential further γ-topplings in $\Lambda_1 \setminus \partial_- \Lambda_1$ at $z_1, \dots, z_{n_1}$. By construction, $y_1, \dots, y_{m_1}, z_1, \dots, z_{n_1}$ is legal for $\zeta_1 + \zeta_2$, and $z_1, \dots, z_{n_1}$ is legal for $\mathcal{S}^{(\gamma)}(\zeta_1) + \zeta_2$. Moreover, the configuration in $\Lambda_1 \setminus \partial_- \Lambda_1$ after γ-topplings at $y_1, \dots, y_{m_1}, z_1, \dots, z_{n_1}$ coincides with the γ-stabilization of $\mathcal{S}^{(\gamma)}(\zeta_1) + \zeta_2$ in $\Lambda_1 \setminus \partial_- \Lambda_1$.

Now select an index $m_2 \geq m_1$, such that for all $y \in \overline{\Lambda_2}$,

$$|\{1 \leq k \leq m_2 : y_k = y\}| = (N^{(\gamma)} \zeta_1)_y.$$

Since the γ-topplings at $z_1, \dots, z_{n_1}$ do not change the configuration in Λ_1^c, the sequence

$$y_1, \dots, y_{m_1}, z_1, \dots, z_{n_1}, y_{m_1+1}, \dots, y_{m_2}$$

is legal for $\zeta_1 + \zeta_2$. Now we γ-stabilize in $\Lambda_2 \setminus \partial_- \Lambda_2$ via γ-topplings at $z_{n_1+1}, \dots, z_{n_2}$. Again by construction,

$$y_1, \dots, y_{m_1}, z_1, \dots, z_{n_1}, y_{m_1+1}, \dots, y_{m_2}, z_{n_1+1}, \dots, z_{n_2}$$

is legal for $\zeta_1 + \zeta_2$, and

$$z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_2}$$

is legal for $\mathcal{S}^{(\gamma)}(\zeta_1) + \zeta_2$. Continuing this argument, we obtain an interlacement of two sequences $y_1, y_2, \ldots$ and $z_1, z_2, \ldots$ γ-exhaustive for $\zeta_1 + \zeta_2$, and hence its final result is $\mathcal{S}^{(\gamma)}(\zeta_1 + \zeta_2)$. On the other hand, the configuration in $\Lambda_k \setminus \partial_- \Lambda_k$ after the γ-topplings at

$$y_1, \ldots, y_{m_1}, z_1, \ldots, z_{n_1}, \ldots y_{m_{k-1}+1}, \ldots, y_{m_k}, z_{n_{k-1}+1}, \ldots, z_{n_k}$$

coincides with the γ-stabilization of $\mathcal{S}^{(\gamma)}(\zeta_1) + \zeta_2$ in $\Lambda_k \setminus \partial_- \Lambda_k$. This yields the claim. $\square$

Lemma 5.2. *Let* $\gamma > 0$.
(i) If $\eta \in \mathcal{X}$ *satisfies* $\sum_{y \in \mathbb{Z}^d} G^{(\gamma)}(x, y)\eta_y < \infty$ *for all* $x \in \mathbb{Z}^d$*, then* η *is* γ*-stabilizable.*
(ii) With $\mathbb{P}$*-probability* 1*, for any* $\eta_0 \in \Omega^{(\gamma)}$ *and any* $t \geq 0$, $\eta_0 + N_t^\varphi$ *is* γ*-stabilizable.*

Proof. *(i)* By (3.2),

$$\Delta_\Lambda^{(\gamma)} N_\Lambda^{(\gamma)} \eta = \eta - \mathcal{S}_\Lambda^{(\gamma)}(\eta) \leq \eta.$$

Hence, by (3.11)

$$\sup_\Lambda (N_\Lambda^{(\gamma)} \eta)_x \leq \sup_\Lambda \sum_{y \in \Lambda} G_\Lambda^{(\gamma)}(x, y)\eta_y \leq \sum_{y \in \mathbb{Z}^d} G^{(\gamma)}(x, y)\eta_y < \infty. \tag{5.2}$$

(ii) Due to the boundedness of φ and estimate (3.11), the configuration $\eta_0 + N_t^\varphi$ satisfies the condition in part *(i)* for all $t \geq 0$, with probability 1. $\square$

By Lemmas 5.1 and 5.2, the process

$$\eta_t = \mathcal{S}^{(\gamma)}(\eta_0 + N_t^\varphi) \tag{5.3}$$

is well-defined for any initial configuration $\eta_0 \in \Omega^{(\gamma)}$. The computation in (5.2) also gives that the addition operators

$$a_x^{(\gamma)} \eta := \mathcal{S}^{(\gamma)}(\eta + \delta_x) = \lim_\Lambda \mathcal{S}_\Lambda^{(\gamma)}(\eta + \delta_x)$$

are defined for any $\eta \in \Omega^{(\gamma)}$, $\gamma > 0$.

5.2 Finiteness of avalanches

Recall that $n_\Lambda^{(\gamma)}(x, y, \eta_0)$ denotes the number of γ-topplings occurring at y in computing $\mathcal{S}_\Lambda^{(\gamma)}(\eta_0 + \delta_x)$. We also define $n^{(\gamma)}(x, y, \eta_0) := \sup_\Lambda n_\Lambda^{(\gamma)}(x, y, \eta_0)$. We call the sets

$$\begin{aligned} \mathrm{Av}_{\Lambda,x}^{(\gamma)}(\eta) &= \{y \in \Lambda : n_\Lambda^{(\gamma)}(x, y, \eta) \geq 1\} \\ \mathrm{Av}_x^{(\gamma)}(\eta) &= \{y \in \mathbb{Z}^d : n^{(\gamma)}(x, y, \eta) \geq 1\}. \end{aligned} \tag{5.4}$$

the γ-*avalanche clusters* started at x in Λ, resp. in $\mathbb{Z}^d$.

Lemma 5.3. *Let $\gamma > 0$.*
(i) We have $m^{(\gamma)}\left(|\mathrm{Av}_x^{(\gamma)}(\eta)| < \infty\right) = 1$.
(ii) The transformation $a_x^{(\gamma)}$ leaves $m^{(\gamma)}$ invariant.

Proof. Take η_0 distributed according to $m^{(\gamma)}$. Due to (3.9),

$$\begin{aligned}\mathbb{E}_{m^{(\gamma)}}[n^{(\gamma)}(x,y,\eta_0)] &= \lim_{\Lambda} \mathbb{E}_{m^{(\gamma)}}[n_{\Lambda}^{(\gamma)}(x,y,\eta_0)] \\ &= \lim_{\Lambda} \lim_{V} \mathbb{E}_{m_V^{(\gamma)}}[n_{\Lambda}^{(\gamma)}(x,y,\eta_0)] \\ &\leq \lim_{V} \mathbb{E}_{m_V^{(\gamma)}}[n_V^{(\gamma)}(x,y,\eta_0)] \\ &= G^{(\gamma)}(x,y).\end{aligned} \tag{5.5}$$

Inequality (5.5) and the estimates (3.10), (3.11) yield $\mathbb{E}_{m^{(\gamma)}}|\mathrm{Av}_x^{(\gamma)}(\eta)| < \infty$, for any x, implying *(i)*. As in [17, Section 4], we have that *(i)* implies *(ii)*. □

5.3 The stationary Markov process with positive dissipation and bounded addition rate

Theorem 5.1. *Let $\gamma > 0$.*
(i) The process $\{\eta_t\}_{t\geq 0}$ is Markovian.
(ii) With $\mathbb{P}$-probability 1, for any $\eta_0 \in \Omega^{(\gamma)}$ we have

$$\eta_t = \lim_{V} \prod_{x\in V} \left(a_x^{(\gamma)}\right)^{N_{x,t}^{\varphi}}(\eta_0).$$

(iii) If η_0 is distributed according to $m^{(\gamma)}$ then $\{\eta_t\}_{t\geq 0}$ is stationary.

Proof. *(i)* By (5.3), for $0 \leq t \leq t+s$, and $\eta_0 \in \Omega^{(\gamma)}$, η_t and η_{t+s} are well-defined, and $\eta_0 + N_t^{\varphi}$, $\eta_0 + N_{t+s}^{\varphi}$ are γ-stabilizable by Lemma 5.2 *(ii)*. By Lemma 5.1 *(iv)*, we have

$$\eta_{t+s} = \mathcal{S}^{(\gamma)}(\eta_0 + N_{t+s}^{\varphi}) = \mathcal{S}^{(\gamma)}(\eta_t + N_{t+s}^{\varphi} - N_t^{\varphi}), \qquad s \geq 0.$$

This implies the Markov property.

(ii) Condition on a realization of the Poisson processes such that, for all $y \in \mathbb{Z}^d$, $\sum_{x\in\mathbb{Z}^d} N_{x,t}^{\varphi} G^{(\gamma)}(x,y) < \infty$. Let $x_1, x_2, \ldots$ be an enumeration of $\mathbb{Z}^d$, and let

$$\begin{aligned}\eta_n &= \eta_0 + \sum_{i=1}^{n} N_{x_i,t}^{\varphi}\delta_{x_i}; \\ \eta &= \eta_0 + N_t^{\varphi}; \\ \zeta_n &= \mathcal{S}^{(\gamma)}(\eta_n) = \prod_{i=1}^{n} \left(a_{x_i}^{(\gamma)}\right)^{N_{x_i,t}^{\varphi}}(\eta_0) \\ \zeta &= \mathcal{S}^{(\gamma)}(\eta).\end{aligned}$$

Note that ζ is indeed well-defined by Lemma 5.2*(ii)*. Given $W \subseteq \mathbb{Z}^d$ finite, select Λ such that $(N_\Lambda^{(\gamma)}\eta)_y = (N^{(\gamma)}\eta)_y$ for all $y \in \overline{W}$. If $n \geq n_0(\Lambda)$, we have $\eta_n = \eta$ in Λ, and therefore

$$(N^{(\gamma)}(\eta))_y = (N_\Lambda^{(\gamma)}(\eta))_y = (N_\Lambda^{(\gamma)}(\eta_n))_y \leq (N^{(\gamma)}(\eta_n))_y \leq (N^{(\gamma)}(\eta))_y, \qquad y \in \overline{W},$$

where the first inequality is due to Lemma 5.1*(ii)*, and the second to Lemma 5.1*(i)*. This shows that $(\zeta_n)_y = \zeta_y$ for $y \in W$, when $n \geq n_0(\Lambda)$. Hence $\zeta_n \to \zeta$, proving *(ii)*.

(iii) Due to Lemma 5.3*(ii)*, and part *(ii)*, η_t is an almost sure limit of configurations distributed according to $m^{(\gamma)}$. This completes the proof. $\square$

6 Avalanche tails

6.1 Upper and lower bounds on avalanche tails

The next theorem gives upper and lower bounds on the probability that a γ-avalanche started at 0 contains a vertex x in the infinite-volume system. The upper bound is essentially Dhar's formula. In Theorem 6.2 we will improve the lower bound in part (a) to one that matches the upper bound up to a multiplicative constant when $d > 4$. Part (b), that follows by essentially the same proof as part (a), concerns the following modified model: let the dissipation at the origin be $0 \leq \gamma_0 < 1$, but keep the dissipation at all other vertices γ. We refer to this model by superscripts (γ_0, γ) in our notation.

Theorem 6.1. *Let $d \geq 1$ and $0 \leq \gamma < 1$.*

(a) We have
$$\gamma G^{(\gamma)}(0, x) \leq m^{(\gamma)}(x \in \mathrm{Av}_0^{(\gamma)}(\eta)) \leq G^{(\gamma)}(0, x). \tag{6.1}$$

(b) There are constants $0 < c = c(d) < C = C(d)$, such that
$$c\,\gamma_0\, G^{(\gamma)}(0, x) \leq m^{(\gamma_0,\gamma)}(x \in \mathrm{Av}_0^{(\gamma_0,\gamma)}(\eta)) \leq C\, G^{(\gamma)}(0, x). \tag{6.2}$$

Proof. The upper bound in *(a)* is immediate from Dhar's formula and Markov's inequality. We next prove the lower bound in *(a)*. We first work in finite volume Λ. Recall the representation of γ-waves from Section 3.6. For any $\zeta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)}$ we denote by $W(\zeta)$ the set of those sites that have to be toppled stabilizing $\zeta + \delta_0$ in order to obtain $\widehat{a}_0\zeta$, i.e., using the toppling matrix $\widehat{\Delta}^{(\gamma)}$ from (3.20). Let $\widehat{B}_x \subseteq \widehat{\mathcal{R}}_\Lambda^{(\gamma)}$ denote the set of those intermediate configurations whose wave contains x. Then $\widehat{m}_\Lambda^{(\gamma)}(\widehat{B}_x)$ is the probability that the weighted random spanning tree in $\widehat{\Lambda}$ contains the extra edge, and the component of 0 in the forest obtained by removing the extra edge contains x. By Wilson's algorithm, this is the same as the probability that a random walk started at x first reaches ϖ via the extra edge, which can be computed as:

$$\widehat{m}_\Lambda^{(\gamma)}(\widehat{B}_x) = \frac{G_\Lambda^{(\gamma)}(0, x)}{1 + G_\Lambda^{(\gamma)}(0, 0)}. \tag{6.3}$$

Indeed, $\widehat{m}_\Lambda^{(\gamma)}(\widehat{B}_x)$ is the probability that the random walk hits zero starting from x which is equal to $\frac{G_\Lambda^{(\gamma)}(0,x)}{G_\Lambda^{(\gamma)}(0,0)}$ times the probability that the random walk starting from zero reaches ϖ via the extra edge. The latter probability equals $\widehat{m}_\Lambda^{(\gamma)}(\widehat{B}_0)$ which is the probability that the weighted random spanning tree contains the extra edge which equals $\mathrm{Vol}(\widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)})/\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)})$. Using

$$\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)}) = \det(\widehat{\Delta}^{(\gamma)}) = \det(\widehat{\Delta}^{(\gamma)})(1 + G_\Lambda^\gamma(0,0))$$

we obtain $\widehat{m}_\Lambda^{(\gamma)}(\widehat{B}_0) = G_\Lambda^{(\gamma)}(0,0)/(1 + G_\Lambda^{(\gamma)}(0,0))$ and thus (6.3). Consider the set

$$C_x = \{\eta \in \widehat{B}_x : 2d + 1 \leq \eta_0 < 2d + 1 + \gamma\}.$$

By the burning algorithm, $\widehat{m}_\Lambda^{(\gamma)}(C_x) = \gamma \widehat{m}_\Lambda^{(\gamma)}(\widehat{B}_x)$.

Indeed we have $\eta \in \widehat{B}_x$ if and only if after adding height one at 0 we can topple 0 and then carry out a wave that topples x. The toppling of 0 will be possible if and only if

$$2d + \gamma \leq \eta_0 < 2d + \gamma + 1.$$

Once the toppling of 0 occurred, the height η_0 plays no role in the condition that x topples. If we condition on all the heights of η except η_0, then the conditional probability, given $\eta \in \widehat{B}_x$, that $\eta \in C_x$ is γ since the additional restriction is:

$$2d + 1 \leq \eta_0 < 2d + \gamma + 1.$$

Also note that every $\eta \in C_x$ is necessarily an intermediate configuration after the first γ-wave of its γ-avalanche, hence we have

$$D_x := \{\eta_1 \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} : \widehat{a}_0^{-1}\eta_1 \in \mathcal{R}_\Lambda^{(\gamma)},\, W(\eta_1) \ni x\} \supset C_x.$$

We have

$$\begin{aligned} \mathrm{Vol}(\eta \in \mathcal{R}_\Lambda^{(\gamma)} : \mathrm{Av}_{\Lambda,0}^{(\gamma)}(\eta) \ni x) &\geq \mathrm{Vol}(\eta \in \mathcal{R}_\Lambda^{(\gamma)} : W_\Lambda^{(1)}(\eta) \ni x) \\ &= \mathrm{Vol}(\eta \in \mathcal{R}_\Lambda^{(\gamma)} : W(\widehat{a}_0\eta) \ni x) \\ &= \mathrm{Vol}(D_x) \geq \mathrm{Vol}(C_x) = \gamma \mathrm{Vol}(\widehat{B}_x), \end{aligned}$$

where in the third step, we used invariance of Lebesgue measure under $\widehat{a}_0$. Indeed we have

$$\begin{aligned} \widehat{a}_0^{-1} D_x &= \{\widehat{a}_0^{-1}\eta_1 \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} : \widehat{a}_0^{-1}\eta_1 \in \mathcal{R}_\Lambda^{(\gamma)}, W(\eta_1) \ni x\} \\ &= \{\eta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} : \eta \in \mathcal{R}_\Lambda^{(\gamma)}, W(\widehat{a}_0\eta) \ni x\} \end{aligned}$$

This implies

$$\begin{aligned} m_\Lambda^{(\gamma)}(\mathrm{Av}_{\Lambda,0}^{(\gamma)}(\eta) \ni x) &\geq \gamma \widehat{m}_\Lambda^{(\gamma)}(\widehat{B}_x) \frac{\mathrm{Vol}(\widehat{\mathcal{R}}_\Lambda^{(\gamma)})}{\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)})} = \gamma \frac{G_\Lambda^{(\gamma)}(0,x)}{1 + G_\Lambda^{(\gamma)}(0,0)} \Big(1 + G_\Lambda^{(\gamma)}(0,0)\Big) \\ &= \gamma G_\Lambda^{(\gamma)}(0,x). \end{aligned}$$

To pass to the limit $\Lambda \to \mathbb{Z}^d$, approximate the event $E_x = \{\mathrm{Av}^{(\gamma)}_{\Lambda,0}(\eta) \ni x\}$ by $E_{x,V} = \{\mathrm{Av}^{(\gamma)}_{\Lambda,0}(\eta) \ni x,\, \mathrm{Av}^{(\gamma)}_{\Lambda,0}(\eta) \subseteq V\}$ with a large finite V. Since both $E_{x,V}$ and $F_V = \{\mathrm{Av}^{(\gamma)}_{\Lambda,0}(\eta) \not\subseteq V\}$ are local events, we have

$$
\begin{aligned}
&|m^{(\gamma)}_\Lambda(E_x) - m^{(\gamma)}(E_x)| \\
&\quad \le \limsup_V \Big[|m^{(\gamma)}_\Lambda(E_x) - m^{(\gamma)}_\Lambda(E_{x,V})| + |m^{(\gamma)}_\Lambda(E_{x,V}) - m^{(\gamma)}(E_{x,V})| \\
&\qquad + |m^{(\gamma)}(E_{x,V}) - m^{(\gamma)}(E_x)| \Big] \\
&\quad \le \limsup_\Lambda m^{(\gamma)}_\Lambda(F_V) + 0 + m^{(\gamma)}(F_V) \\
&\quad \le 2\varepsilon,
\end{aligned}
$$

if V is large enough.

Part *(b)* follows by essentially the same argument. Observe that only the dissipation at the origin played any significant role, and adjusting this to a different value only changes the Green's functions and the volume of recurrent configurations by at most a constant factor depending on d. □

Remark 6.1. *a) A natural length associated to the system with dissipation γ is the typical diameter of a γ-avalanche. By Lemma 3.1, $G^{(\gamma)}(0,x)$ decays, up to polynomial factors, as $e^{-|x|/L(\gamma)}$, with $L(\gamma) = C/\sqrt{\gamma}$. Thus Theorem 6.1 and also Theorem 6.2 below show that $1/\sqrt{\gamma}$ is the typical avalanche-diameter as $\gamma \to 0$, and supports the idea that in the system with dissipation γ, the "correlation length" scales as $1/\sqrt{\gamma}$, as $\gamma \to 0$ (compare with [33]).*

b) Inequalities (6.2) *generalize to inhomogeneous dissipation as follows. Let $\Gamma = (\gamma_x)_{x\in\mathbb{Z}^d}$, be a collection of non-negative real dissipations, and suppose that $\gamma_0 > 0$. Note that the Green's function $G^{(\Gamma)}(0,x)$ of the continuous-time random walk trapped at rate γ_x at each lattice site x is finite (even when $d = 2$). In particular, when $d \ge 3$ we can choose $\gamma_0 > 0$ and $\gamma_x = 0$ elsewhere in* (6.2). *Then the lower bound of* (6.2) *decays as $C\gamma_0/|x|^{d-2}$, i.e., this model has power-law decay of avalanches, and infinite expected avalanche area. It is an interesting problem to characterize the set of possible inhomogeneous dissipation functions Γ such that the expected area of the avalanche is infinite (i.e., the model is still "critical").*

6.2 The toppling probability exponent in $d > 4$

In Theorem 6.2 below we show that for $d > 4$ the factor γ on the left hand side of (6.1) can be replaced by a constant. This implies that the probability that a site x belongs to the avalanche initiated at the origin behaves asymptotically as $G^{(\gamma)}(0,x)$ for large x. By taking the limit $\gamma \to 0$ one obtains that the probability that x belongs to the avalanche initiated at 0 in the critical model behaves as $1/|x|^{d-2}$ for large x.

In analogy with the one-arm exponent (connectivity exponent) for percolation clusters [8], let us introduce the critical exponent θ by requiring that

$$m^{(0)}(x \in \mathrm{Av}_0^{(0)}(\eta)) \approx \frac{c}{|x|^{d-2+\theta}}, \quad \text{as } |x| \to \infty.$$

Here $\approx$ means logarithmic equivalence of the two sides (or possibly a stronger relation). Theorem 6.2 below shows that when $d > 4$, the exponent takes the mean-field value $\theta = 0$. In this context, Dhar's formula provides the mean-field bound $\theta \geq 0$. By analogy with other critical lattice systems, it is tempting to conjecture that below the critical dimension, i.e. when $d < 4$, we have $\theta > 0$, and that $\theta = 0$ with a logarithmic correction in dimension $d = 4$. One could heuristically argue for this conjecture as follows. When $d > 4$, avalanche clusters are tree-like [28], and repeated topplings are not significant. In this case the expected number of topplings gives the correct behavior of the probability that x topples. When $d < 4$, we can expect that repeated topplings are pronounced, which implies that if x topples at least once, it is likely to topple many times in the same avalanche. In this case the probability that x topples would be significantly smaller than the expected number of topplings, and hence $\theta > 0$.

Open question 6.1. *Is the above heuristic correct and $\theta > 0$ when $d = 2, 3$?*

We did not find any numerical work in the literature on the exponent θ in dimensions $d = 2, 3$.

Theorem 6.2. *When $d > 4$, there exists $c = c(d) > 0$ such that for all $\gamma \geq 0$ we have*

$$cG^{(\gamma)}(0,x) \leq m^{(\gamma)}(x \in \mathrm{Av}_0^{(\gamma)}(\eta)) \leq G^{(\gamma)}(0,x). \tag{6.4}$$

Proof. The upper bound follows from Dhar's formula, so we only need to prove the lower bound. The proof is divided in several steps.

Step 1: Reduction to a weighted spanning tree problem

Let Λ be a finite volume containing 0. For $\eta \in \mathcal{R}_\Lambda^{(\gamma)}$, let $W_\Lambda^{\mathrm{last}}(\eta)$ denote the *last wave* occurring when unit height is added at 0 to the configuration η.

We set $W_\Lambda^{\mathrm{last}}(\eta) = \varnothing$, if the avalanche is empty. We have

$$m_\Lambda^{(\gamma)}(x \in \mathrm{Av}_{0,\Lambda}^{(\gamma)}(\eta)) \geq m_\Lambda^{(\gamma)}(x \in W_\Lambda^{\mathrm{last}}).$$

Recall the correspondence between waves and intermediate configurations introduced in Section 3.6. We define a map $F : \mathcal{R}_\Lambda^{(\gamma)} \to \widehat{\mathcal{R}}_\Lambda^{(\gamma)}$ as follows. When the avalanche is empty, that is if $\eta + \delta_0 \in \mathcal{R}_\Lambda^{(\gamma)}$, we set $F(\eta) = \eta + \delta_0$. When the avalanche is non-empty, that is, if $\eta + \delta_0 \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)}$, we let $F(\eta)$ be the last intermediate configuration in the avalanche, i.e., the configuration corresponding to the last wave. The mapping F is one-to-one between $\mathcal{R}_\Lambda^{(\gamma)}$ and its image $F(\mathcal{R}_\Lambda^{(\gamma)})$, and preserves Lebesgue measure between these sets. Indeed, we can subdivide $\mathcal{R}_\Lambda^{(\gamma)}$ in finitely many disjoint pieces such that on each piece F is a composition of translations (individual topplings act as translations).

For an intermediate configuration $\zeta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)}$, let $W(\zeta)$ be the wave corresponding to it. Denote

$$e := (-1, 0, \ldots, 0) \in \mathbb{Z}^d. \tag{6.5}$$

We have that if $e \notin W(\zeta)$, then $(\hat{a}_0\zeta)_0 < 2d + \gamma$, and hence $W(\zeta)$ is a last wave (cf. also [11]). Hence we have

$$\begin{aligned} m_\Lambda^{(\gamma)}(x \in W_\Lambda^{\mathrm{last}}) &= \frac{\mathrm{Vol}\big(\{\eta \in \mathcal{R}_\Lambda^{(\gamma)} : x \in W^{\mathrm{last}}(\eta)\}\big)}{\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)})} \\ &= \frac{\mathrm{Vol}\big(\{\zeta \in F(\mathcal{R}_\Lambda^{(\gamma)}) : x \in W(\zeta)\}\big)}{\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)})} \\ &\geq \frac{\mathrm{Vol}\big(\{\zeta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)} : x \in W(\zeta),\, e \notin W(\zeta)\}\big)}{\mathrm{Vol}(\mathcal{R}_\Lambda^{(\gamma)})} \qquad (6.6) \\ &\geq \frac{\mathrm{Vol}\big(\{\zeta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)} \setminus \mathcal{R}_\Lambda^{(\gamma)} : x \in W(\zeta),\, e \notin W(\zeta)\}\big)}{\mathrm{Vol}(\widehat{\mathcal{R}}_\Lambda^{(\gamma)})} \\ &=: \widehat{m}_\Lambda^{(\gamma)}(E_x). \end{aligned}$$

The event E_x consists of all $\zeta \in \widehat{\mathcal{R}}_\Lambda^{(\gamma)}$ such that $2d + \gamma \leq \zeta_0 < 2d + \gamma + 1$, $x \in W(\zeta)$, and $e \notin W(\zeta)$. Observe that E_x is a union of (γ, Λ)-cells (here we use a natural extension of the notion of (γ, Λ)-cell to $\widehat{\mathcal{R}}_\Lambda^{(\gamma)}$). Therefore, letting $\widehat{\mu}_\Lambda^{(\gamma)}$ denote the weighted spanning tree measure for the graph $\widehat{\Lambda}$, we have

$$\widehat{m}_\Lambda^{(\gamma)}(E_x) = \widehat{\mu}_\Lambda^{(\gamma)}(x \text{ connects to } \varpi \text{ via the extra edge, and } e \text{ does not}).$$

We pass to the infinite volume limit (just as in the proof of Theorem 6.1) and get

$$m^{(\gamma)}(x \in \mathrm{Av}_0^{(\gamma)}(\eta)) \geq \widehat{\mu}^{(\gamma)}(x \text{ connects to } \varpi \text{ via the extra edge, and } e \text{ does not}). \tag{6.7}$$

Step 2: Reduction to a problem of two independent random walk paths

We now use Wilson's algorithm in $\widehat{\mathbb{Z}^d}$ to give a lower bound on the probability in the right hand side of (6.7). Roughly speaking, the required event will occur, if a network random walk started from x hits 0, then steps to ϖ, and an independent network random walk started at e avoids the loop-erasure of the first path until it hits ϖ. Since in $d > 4$, the probability that independent simple random walks do not intersect is bounded away from 0, the last event "should not matter", and the first two events will occur with a probability $cG_\Lambda^{(\gamma)}(x, 0)$.

In order to make the above precise, let S and S^- be independent simple random walks in $\mathbb{Z}^d$, with $S(0) = x$ and $S^-(0) = e$. Let T and T^- be independent $\mathrm{Geom}(\gamma/(2d + \gamma))$ random variables independent of the walks. Write

$$\begin{aligned} \tau_A &= \inf\{n \geq 0 : S \in A\}; \\ \tau_A^- &= \inf\{n \geq 0 : S^- \in A\} \end{aligned}$$

Write $\pi = \mathrm{LE}(S[0, T \wedge \tau_0])$.

$$\begin{aligned}
&\widehat{\mu}^{(\gamma)}(x \text{ connects to } \varpi \text{ via the extra edge, and } e \text{ does not}) \\
&\quad \geq \frac{1}{2d+\gamma+1} \mathbb{P}\left(\tau_0 < T \text{ and } S^-[0, T^-] \cap \pi = \varnothing\right).
\end{aligned}$$

Let S^+ be a simple random walk on $\mathbb{Z}^d$ with $S^+(0) = 0$, independent of S^-, and let T^+ be its $\mathrm{Geom}(\gamma/(2d+\gamma))$ killing time. Let

$$\begin{aligned}
\tau_A^+ &= \inf\{n \geq 0 : S^+ \in A\} \\
\sigma &= \sup\{0 \leq n < \tau_0 : S(n) = x\} \\
\sigma^+ &= \sup\{0 \leq n < \tau_x^+ : S^+(n) = 0\}.
\end{aligned}$$

Conditioned on the event $\tau_0 < T$, $\pi = \mathrm{LE}(S[0, T \wedge \tau_0]) = \mathrm{LE}(S[\sigma, \tau_0])$. Due to reversibility of simple random walk and [20, Lemma 7.2.1], the latter path has the same distribution as $\pi^+ := \mathrm{LE}(S^+[\sigma^+, \tau_x^+]) = \mathrm{LE}(S^+[0, \tau_x^+])$ conditioned on $\tau_x^+ < T^+$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\tau_0 < T \text{ and } S^-[0, T^-] \cap \pi = \varnothing\right) \\
&\quad = \mathbb{P}\left(\tau_x^+ < T^+ \text{ and } S^-[0, T^-] \cap \pi^+ = \varnothing\right) \\
&\quad \geq \mathbb{P}\left(\tau_x^+ < T^+ \text{ and } S^-[0, T^-] \cap S^+[0, \tau_x^+] = \varnothing\right).
\end{aligned} \tag{6.8}$$

Since $\mathbb{P}(\tau_x^+ < T^+) = \frac{G^{(\gamma)}(0,x)}{G^{(\gamma)}(0,0)}$, we are left to show that

$$\mathbb{P}\left(\tau_x^+ < T^+ \text{ and } S^-[0, T^-] \cap S^+[0, \tau_x^+] = \varnothing\right) \geq c\mathbb{P}(\tau_x^+ < T^+). \tag{6.9}$$

Notice that the event that two independent random walk paths do not meet has positive probability when $d > 4$. However we are asking that the first path S^+ hits x, and therefore, if x is large ($|x| > \gamma^{-1/2}$) the random walk survives anomalously long, and hence it is not clear whether it behaves as an "ordinary" random walk. In particular this will be reflected in the proof of (6.9), where a distinction between $|x| > \gamma^{-1/2}$ and $|x| \leq \gamma^{-1/2}$ has to be made.

For $\alpha \geq 0$ define the cones

$$\begin{aligned}
\mathcal{H}_\alpha^+ &= \{y \in \mathbb{Z}^d : y_1 \geq \alpha |y_j| \text{ for } j = 2, \ldots, d\} \\
\mathcal{H}_\alpha^- &= \{y \in \mathbb{Z}^d : y_1 < -\alpha |y_j| \text{ for } j = 2, \ldots, d\}.
\end{aligned}$$

We abbreviate $\mathcal{H}^+ = \mathcal{H}_1^+$ and $\mathcal{H}^- = \mathcal{H}_1^-$. We define the events

$$\begin{aligned}
H(R) &= \{S^+[0, \tau_{B(R)^c}^+] \cap S^-[0, \tau_{B(R)^c}^-] = \varnothing\} \\
G^+(r, R) &= \{S^+[\tau_{B(r)^c}^+, \tau_{B(R)^c}^+] \subseteq \mathcal{H}^+\} \\
G^-(r, R) &= \{S^-[\tau_{B(r)^c}^-, \tau_{B(R)^c}^-] \subseteq \mathcal{H}^-\}.
\end{aligned}$$

In showing (6.9), we assume, without loss of generality, that $x = (x_1, \ldots, x_d)$ satisfies

$$x_1 \geq |x_j|, \; j = 2, \ldots, d, \tag{6.10}$$

Step 3: proof of (6.9) **for "small"** x**:** $|x| \le \gamma^{-1/2}$**.**

Note that by transience of S^- we obtain (6.9) for all x such that $|x| \le k_0$ for any fixed $k_0 > 0$. So we will assume that $k_0 < |x| \le \gamma^{-1/2}$, and will choose k_0 suitably in course of the proof. Define the event:

$$A := H(|x|/2) \cap G^+(|x|/4, |x|/2) \cap G^-(|x|/4, 2|x|).$$

We show that $\mathbb{P}(A) \ge c_1 > 0$. From the invariance principle we derive that for all k large enough and $|x| > 8k$ we obtain

$$\mathbb{P}(S^+(\tau^+_{B(|x|/4)^c}) \in \mathcal{H}_2^+ \mid S^+(0) = (k, 0, \dots, 0)) \ge c$$
$$\mathbb{P}(S^-(\tau^-_{B(|x|/4)^c}) \in \mathcal{H}_2^- \mid S^-(0) = (-k, 0, \dots, 0)) \ge c.$$

Hence, using the invariance principle, there exists a constant $c' > 0$ such that for all x with large enough $|x|$, say $|x| > 8k$

$$\mathbb{P}(G^+(|x|/4, |x|/2) \mid S^+(0) = (k, 0, \dots, 0)) \ge c'$$
$$\mathbb{P}(G^-(|x|/4, 2|x|) \mid S^-(0) = (-k, 0, \dots, 0)) \ge c'.$$

When $d > 4$, the probability that two independent simple random walks starting at distance $2k$ apart intersect is $o(1)$ as $k \to \infty$ (see e.g. [20, Section 3.3]). Therefore we choose k large enough so that

$$\mathbb{P}(S^+[0,\infty) \cap S^-[0,\infty) \neq \varnothing \mid S^+(0) = (k, 0, \dots, 0),\, S^-(0) = (-k, 0, \dots, 0)) < c'/2,$$

and set $k_0 = 8k$. Requiring the occurrence of the event

$$\{S^+(\ell) = (\ell, 0, \dots, 0),\, \ell = 0, \dots, k\} \cap \{S^-(\ell) = (-\ell - 1, 0, \dots, 0),\, \ell = 0, \dots, k-1\}$$

we have $\mathbb{P}(A) \ge c_1$.

Let

$$B := \{S^-[\tau^-_{B(2|x|)^c}, \infty) \cap B(3|x|/2) = \varnothing\}.$$

By the strong Markov property, transience and the invariance principle, there exists $c_2 > 0$ such that $\mathbb{P}(A \cap B) \ge c_2$.

Let $\mathcal{D} = (B(3|x|/2) \setminus B(|x|/2)) \cap \mathcal{H}_0^+$. Let

$$C := \{S^+[\tau^+_{B(|x|/2)^c}, \infty) \text{ hits } x \text{ before exiting } \mathcal{D}\}.$$

For any $z \in \partial B(|x|/2)$ we have

$$\begin{aligned}
\mathbb{P}(C \mid S^+(0) = z) &\ge \mathbb{P}(\tau^+_{B(x,|x|/2\sqrt{d})} < \tau^+_{\mathcal{D}^c} \mid S^+(0) = z) \\
&\qquad \times \min_{w \in \partial B(x,|x|/2\sqrt{d})} \mathbb{P}(\tau^+_x < \tau^+_{B(x,|x|/\sqrt{d})^c} \mid S^+(0) = w) \\
&\ge c \frac{c'}{|x|^{d-2}},
\end{aligned}$$

where the second inequality follows e.g. from [20, Proposition 1.5.10]. By the strong Markov property, we get $\mathbb{P}(A \cap B \cap C) \geq c_3|x|^{d-2}$.

From the invariance principle we can deduce that for all x and for all $K \geq 1$ we have

$$\mathbb{P}(\tau_x^+ < \tau_{B(3|x|/2)^c}^+,\, \tau_{B(2|x|)^c}^+ \geq K|x|^2) \leq C_1|x|^{2-d}e^{-c_4K}.$$

Hence for a sufficiently large K_0 (whose value only depends on C_1 and c_4, and not on x, we have

$$\mathbb{P}(A \cap B \cap C \cap \{\tau_{B(3|x|/2)^c}^+ \leq K_0|x|^2\}) \geq (c_3/2)|x|^{2-d}.$$

Finally, note that due to $|x| \leq \gamma^{-1/2}$, we have $\mathbb{P}(T^+ \geq K_0|x|^2) \geq c_5 = c_5(K_0)$. We conclude the proof by observing that on the event

$$A \cap B \cap C \cap \{\tau_{B(3|x|/2)^c}^+ \leq K_0|x|^2\} \cap \{T^+ > K_0|x|^2\}$$

the required event in the left hand side of (6.9) occurs.

Step 4: Proof of (6.9) **for"large x": $|x| > \gamma^{-1/2}$.**

Let $\ell = \gamma^{-1/2}$. We need the following lemma.

Lemma 6.1. *There exist constants $0 < c_1 = c_1(d) < C_1 = C_1(d)$ such that for all $y \in B(\ell/8)$ and $z \in \partial B(\ell)$ we have*

$$\frac{c}{\ell^{d-1}} \leq \mathbb{P}^y(S(\tau_{B(\ell)^c}) = z,\, T > \tau_{B(\ell)^c}) \leq \frac{C}{\ell^{d-1}}.$$

Proof. Write $B = B(\ell)$. Were the event $T > \tau_{B^c}$ not present, the statement would be [20, Lemma 1.7.4]. With the event $T > \tau_{B^c}$, clearly the upper bound still holds. In order to deduce the lower bound, it is sufficient to show that uniformly in y and z, the random walk started at y and conditioned on exiting B at z has expected exit time at most $C'\ell^2$. Indeed, then

$$\begin{aligned}
\mathbb{P}^y(T > \tau_{B^c} \,|\, S(\tau_{B^c}) = z) &\geq \mathbb{P}^y(T > 2C'\ell^2,\, 2C'\ell^2 \geq \tau_{B^c} \,|\, S(\tau_{B^c}) = z) \\
&\geq \left(\frac{2d}{2d+\gamma}\right)^{2C'\ell^2} \frac{1}{2} \\
&\geq c'.
\end{aligned}$$

The conditional expected exit time can be written in the form

$$\sum_{x \in B} \frac{h(x)}{h(y)} G_B(y,x),$$

where $h(x) = \mathbb{P}^x(S(\tau_{B^c}) = z)$. Consider first the case $y = 0$. We can decompose the sum and bound it above as follows:

$$\begin{aligned}
&\frac{1}{h(0)}\sum_{k=0}^{\ell}\sum_{k\le|x|<k+1} h(x)G_B(0,x)\\
&\le \frac{1}{h(0)}\sum_{k=0}^{\ell}\sum_{k\le|x|<k+1}\frac{1}{c/k^{d-1}}\mathbb{P}^0(S(\tau_{B(k)^c})=x))h(x)G(0,x)\\
&\le \frac{1}{h(0)}\sum_{k=0}^{\ell}Ck^{d-1}\sum_{k\le|x|<k+1}\mathbb{P}^0(S(\tau_{B(k)^c})=x)h(x)k^{2-d}\\
&= \frac{1}{h(0)}\sum_{k=0}^{\ell}Ckh(0)\\
&= C'\ell^2.
\end{aligned}$$

For general $y \in B(\ell/8)$, we can repeat the above argument to bound the sum over $x : 0 \le |x-y| < \ell/2$ by $C'\ell^2$. Due to the Harnack principle, we can also bound the sum over $x : (3/8)\ell \le |x| \le \ell$ above by

$$\begin{aligned}
\frac{1}{h(y)}\sum_{k=(3/8)\ell}^{\ell}\sum_{k\le|x|<k+1}h(x)G_B(y,x) &\le \frac{C}{h(0)}\sum_{k=(3/8)\ell}^{\ell}\sum_{k\le|x|<k+1}h(x)G_B(0,x)\\
&\le C'\ell^2.
\end{aligned}$$

The two ranges of x's cover all of B, so the claim follows. □

Let us now fix $\delta > 0$ in such a way that for $y \in \partial B(\ell/16)\cap\mathcal{H}^+$ we have

$$\mathbb{P}^y(\tau_{B(\delta\ell)} < \infty) \le \frac{c_1}{2C_1},$$

where c_1 and C_1 are the constants in Lemma 6.1. This is possible, as long as ℓ is sufficiently large, since the probability scales as δ^{d-2}. Taking ℓ sufficiently large requires us to restrict to $\gamma \le$ some γ_0. We will comment on the case $\gamma > \gamma_0$ at the end.

Define the event

$$A := H(\delta\ell)\cap G^+(\delta\ell/2,\delta\ell)\cap G^-(\delta\ell/2,\delta\ell),$$

that satisfies $\mathbb{P}(A) \ge c_2$. For a sufficiently large K_0 we have $\mathbb{P}(\tau^-_{B(\delta\ell)^c} < \frac{1}{K_0}\ell^2) < c_2/4$ and $\mathbb{P}(\tau^+_{B(\delta\ell)^c} > K_0\ell^2) < c_2/4$. Hence with some $c_3 = c_3(K_0) > 0$ we have

$$\mathbb{P}(A\cap\{T^- < \tau^-_{B(\delta\ell)^c}\}\cap\{T^+ > \tau^-_{B(\delta\ell)^c}\}) \ge \frac{c_2}{2}c_3. \tag{6.11}$$

Denote by $\mathcal{D} = B(\delta\ell/2)\cup(\mathcal{H}^-\cap B(\delta\ell))$. Given $z \in \partial B(\delta\ell)\cap\mathcal{H}^+$, define the event

$$C_z := \{S^+(0) = z,\, \tau^+_x < \tau^+_{\mathcal{D}}\wedge T^+\}.$$

We show that

$$\inf_{z\in\partial B(\delta\ell)\cap\mathcal{H}^+} \mathbb{P}^z(C_z) \geq c\mathbb{P}^0(\tau_x^+ < T^+), \tag{6.12}$$

which implies the claim (6.9) in light of (6.11) and the strong Markov property of S^+.

First, it is clear from the invariance principle and $\gamma \asymp 1/\ell^2$ that the killed random walk started at $z \in \partial B(\delta\ell) \cap \mathcal{H}^+$ has probability bounded away from 0 to reach $\partial B(\ell/16) \cap \mathcal{H}^+$ without hitting $\mathcal{D}$. Hence (6.12) will be proved once we show

$$\inf_{y\in\partial B(\ell/16)\cap\mathcal{H}^+} \mathbb{P}^y(C_y) \geq c\mathbb{P}^0(\tau_x^+ < T^+). \tag{6.13}$$

Using Lemma 6.1 we write

$$\begin{aligned}\mathbb{P}^0(\tau_x^+ < T^+) &= \sum_{z\in\partial B(\ell)} \mathbb{P}^0(S^+(\tau^+_{B(\ell)^c}) = z,\, T^+ > \tau^+_{B(\ell)^c})\mathbb{P}^z(\tau_x^+ < T^+) \\ &\leq \frac{C_1}{\ell^{d-1}} \sum_{z\in\partial B(\ell)} \mathbb{P}^z(\tau_x^+ < T^+).\end{aligned} \tag{6.14}$$

On the other hand, for $y \in \partial B(\ell/16) \cap \mathcal{H}^+$ we have

$$\begin{aligned}&\mathbb{P}^y(\tau_x^+ < \tau_{\mathcal{D}}^+ \wedge T^+) \\ &\quad\geq \sum_{z\in\partial B(\ell)} \mathbb{P}^y(S(\tau^+_{B(\ell)^c}) = z,\, T^+ > \tau^+_{B(\ell)^c})\mathbb{P}^z(\tau_x^+ < T^+) \\ &\qquad - \mathbb{P}^y(\tau_{\mathcal{D}}^+ < \infty) \max_{y'\in\mathcal{D}} \sum_{z\in\partial B(\ell)} \mathbb{P}^{y'}(S(\tau^+_{B(\ell)^c}) = z,\, T^+ > \tau^+_{B(\ell)^c})\mathbb{P}^z(\tau_x^+ < T^+) \\ &\quad\geq \frac{c_1}{\ell^{d-1}} \sum_{z\in\partial B(\ell)} \mathbb{P}^z(\tau_x^+ < T^+) - \frac{c_1}{2C_1}\frac{C_1}{\ell^{d-1}} \sum_{z\in\partial B(\ell)} \mathbb{P}^z(\tau_x^+ < T^+) \\ &\quad= \frac{c_1}{2\ell^{d-1}} \sum_{z\in\partial B(\ell)} \mathbb{P}^z(\tau_x^+ < T^+).\end{aligned}$$

Together with (6.14) this yields the required statement (6.13), and completes the proof in the case $0 < \gamma \leq \gamma_0$.

When $\gamma_0 < \gamma < 1$, the statement of the theorem is implied by (6.1). When $\gamma \geq 1$, observe that every avalanche consists of only one wave, so $m^{(\gamma)}(x \in \mathrm{Av}_0^{(\gamma)}(\eta)) = G^{(\gamma)}(0, x)$. □

7 Zero dissipation limit of the stationary processes

We now consider the infinite volume dynamics in the case $\gamma = 0$, and show that it is the limit of the infinite volume dynamics with dissipation γ when $\gamma \downarrow 0$.

For the abelian sandpile model, this infinite volume dynamics was constructed in [17], in dimensions $d \geq 3$. We recall the main steps of this construction, and indicate how it applies to the abelian avalanche model.

Recall Remark 4.1, and note that when $\gamma = 0$, the dynamics preserves the fractional part of each coordinate.

Lemma 7.1. *(i) For $m^{(0)}$-a.e. η_0, the configuration $\eta_0 + \delta_x$ is 0-stabilizable for all $x \in \mathbb{Z}^d$.*

(ii) For all $x \in \mathbb{Z}^d$, $a_x^{(0)}$ leaves $m^{(0)}$ invariant.

(iii) Assume that φ satisfies $\sum_{x\in\mathbb{Z}^d} \varphi(x)G^{(0)}(x,0) < \infty$. Then $m^{(0)} \otimes \mathbb{P}$-a.s., the limit

$$\eta_t = \lim_V \prod_{x\in V} \left(a_x^{(0)}\right)^{N_{x,t}^{\varphi}} \eta_0$$

exists, and equals $\mathcal{S}^{(0)}(\eta_0 + N_t^{\varphi})$.

Proof. *(i)* This is because $\eta_0 + \delta_x$ is 0-stabilizable if and only if its image under ψ is stabilizable. It was shown in [17] that for $\nu^{(0)}$-a.e. configuration ξ, for all $x \in \mathbb{Z}^d$, $\xi + \delta_x$ is stabilizable. This implies *(i)*.

(ii) It was shown in [17] that the discrete addition operators leave $\nu^{(0)}$ invariant. Denoting the discrete stabilization operator by $\mathcal{S}^{\mathrm{discr}}$, we have

$$\psi(\mathcal{S}^{(0)}(\eta + \delta_x)) = \mathcal{S}^{\mathrm{discr}}(\psi(\eta) + \delta_x), \tag{7.1}$$

which implies *(ii)*.

(iii) Again, this was shown in the discrete case, and (7.1) implies it for the continuous case, that is the abelian avalanche model.

□

From now on, we denote $m := m^{(0)}$, $G(\cdot,\cdot) := G^{(0)}(\cdot,\cdot)$, $a_x := a_x^{(0)}$ and $\mathrm{Av}_x(\eta) := \mathrm{Av}_x^{(0)}(\eta)$.

Let φ be an addition rate such that

$$\sum_{x\in\mathbb{Z}^d} \varphi(x)G(x,y) < \infty \tag{7.2}$$

for all $y \in \mathbb{Z}^d$. Let $\eta_t^{(\gamma)}$ denote the stationary process obtained when starting from $\eta_0^{(\gamma)} := \eta^{(\gamma)}$ distributed according to $m^{(\gamma)}$, making additions according to independent Poisson processes with rate $\varphi(x)$ at $x \in \mathbb{Z}^d$, and stabilizing with dissipation γ. Similarly, let η_t denote the process starting from $\eta_0 = \eta$ distributed according to m, making additions according to independent Poisson processes with rate $\varphi(x)$ at $x \in \mathbb{Z}^d$, and stabilizing without dissipation, i.e., with $\gamma = 0$.

We write $D[0,1]$ for the space of càdlàg functions in $\mathcal{X}$ endowed with the Skorokhod topology.

Theorem 7.1. *When $d \geq 3$, and condition (7.2) is satisfied, the process $\eta_t^{(\gamma)}$ converges weakly in $D[0,1]$ to η_t.*

7.1 Convergence of the addition operator

We need the following two lemmas.

Lemma 7.2. *Suppose that* $0 \le \gamma' < \gamma$. *Then*
(i) $N^{(\gamma)}(\eta) \le N^{(\gamma')}(\eta)$;
(ii) $\mathrm{Av}_0^{(\gamma)}(\eta) \subseteq \mathrm{Av}_0(\eta)$.

Proof. (i) Consider a γ-exhaustive sequence $y_1, y_2, \dots$ of γ-legal γ-topplings for η. We show that the same sequence is γ'-legal for η. Since y_1 is γ-unstable in η, it is also γ'-unstable. After its γ-toppling let us add height $\gamma - \gamma' > 0$ at y_1. This has the same effect as if we performed a γ'-toppling, and the added extra height does not affect γ-legality of the sequence. Adding similarly after each γ-toppling shows that $y_1, y_2, \dots$ is γ'-legal, and *(i)* follows by the remarks preceding Lemma 5.1.

(ii) This follows immediately from *(i)*. □

Lemma 7.3. *Suppose that* $\gamma_n \downarrow 0$. *Then for* m*-a.e.* η,

$$\lim_{n\to\infty} a_0^{(\gamma_n)}(\eta) = a_0^{(0)}(\eta). \tag{7.3}$$

Proof. We have

$$m(|\mathrm{Av}_0(\eta)| < \infty) = 1; \tag{7.4}$$

see [17, Theorem 3.11]. Also, $m(\eta_x \notin \{0, 1, \dots, 2d-1\},\ x \in \mathbb{Z}^d) = 1$. On the intersection of the two events, there exists a random $\alpha > 0$ (depending on η), such that

$$\eta_x \notin [j, j+\alpha], \text{ for all } x \in \mathrm{Av}_0(\eta) \text{ and all } j = 0, 1, 2, \dots \tag{7.5}$$

Let $M = \max\{(N^{(0)}\eta)_x : x \in \mathrm{Av}_0(\eta)\}$. We claim that when $\gamma_n < \alpha(M+1)^{-1}$, then the γ_n-topplings satisfy $N^{(\gamma_n)}\eta = N^{(0)}\eta$. Observe that the first toppling $T_0^{(0)}$ is applied if and only if $\eta_0 \in (2d-1+\alpha, 2d)$. This means that $T_0^{(\gamma_n)}$ is also applied. After toppling, each $x \neq 0$ in the avalanche cluster still satisfies (7.5) (since the height changed by an integer amount). On the other hand, at $x = 0$, the condition is weakened to $\eta_0 \notin [j, j+\alpha-\gamma_n]$, which implies $\eta_0 \notin [j, j+\alpha M(M+1)^{-1}]$. Continuing inductively, we get that all topplings in the computation of $\mathcal{S}^{(0)}(\eta+\delta_0)$ occur under the γ_n-stabilization of $\eta+\delta_0$. Due to Lemma 7.2 *(i)*, this proves the claim. Together with $\Delta^{(\gamma_n)} \to \Delta^{(0)}$ this implies the statement. □

Proof of Theorem 7.1. Recall the definition (3.5) of the metric on $[0, 2d+\gamma]^{\mathbb{Z}^d}$, that is

$$\mathrm{dist}(\eta, \zeta) = \sum_{x\in\mathbb{Z}^d} 2^{-|x|} \min\{|\eta_x - \zeta_x|, 1\}.$$

As a first step, for a given $\delta > 0$, we will prove that for all $\varepsilon > 0$, there exists a coupling $\mathcal{M}^{(\gamma)}$ of $m^{(\gamma)}$ and m such that, in this coupling, with probability at least $1-\varepsilon$,

$$\mathrm{dist}(a_0^{(\gamma)}(\eta^{(\gamma)}), a_0^{(\gamma)}(\eta^{(0)})) \le \delta. \tag{7.6}$$

This will allow us to deal with the convergence of the processes for addition rates with finite support.

First, for given $\delta, \varepsilon > 0$, we choose V large enough such that if η, ζ agree on V, then $\mathrm{dist}(\eta, \zeta) < \delta$, and such that

$$m\left(\overline{\mathrm{Av}_0(\eta)} \not\subseteq V\right) \leq \varepsilon. \tag{7.7}$$

Such a choice of V is possible, since avalanches are finite with m-probability one by (7.4). By Lemma 7.2*(ii)*, we then have the same estimate (7.7) for $\mathrm{Av}_0^{(\gamma)}(\eta)$.

Since $m^{(\gamma)} \to m$ weakly, and restrictions to finite volumes $\Lambda \subseteq \mathbb{Z}^d$ of $m^{(\gamma)}$ and m are absolutely continuous with respect to Lebesgue measure, by [32, Proposition 1] we have the existence of $\gamma_0 > 0$ such that for all $\gamma < \gamma_0$ there exists a coupling $\mathcal{M}^{(\gamma)}$ of $m^{(\gamma)}$ and m such that

$$\mathcal{M}^{(\gamma)}(\eta_x^{(\gamma)} = \eta_x^{(0)},\ \forall x \in V) \geq 1 - \varepsilon. \tag{7.8}$$

In the coupling $\mathcal{M}^{(\gamma)}$ we then have, by (7.7)

$$\left(a_0^{(\gamma)}(\eta^{(\gamma)})\right)_y = \left(a_0^{(\gamma)}(\eta^{(0)})\right)_y$$

for all $y \in V$, with probability at least $1 - 3\varepsilon$. Therefore, the probability that the distance $\mathrm{dist}(a_0^{(\gamma)}(\eta^{(\gamma)}), a_0^{(\gamma)}(\eta^{(0)}))$ is larger that δ is smaller than 3ε.

So far, we can conclude that $a_0^{(\gamma)}(\eta^{(\gamma)}) \to a_0^{(\gamma)}(\eta^{(0)})$ weakly as $\gamma \to 0$. By Lemma 7.3, $a_0^{(\gamma)}(\eta) \to a_0(\eta)$ for m-a.e. η. Hence we have

$$a_0^{(\gamma)}(\eta^{(\gamma)}) \to a_0(\eta^{(0)}) \tag{7.9}$$

weakly as $\gamma \to 0$. Analogously, using finiteness of avalanches, we conclude that for any finite set $B \subseteq \mathbb{Z}^d$, and natural numbers $n_x, x \in B$, we have

$$\prod_{x \in B} (a_x^{(\gamma)})^{n_x}(\eta^{(\gamma)}) \to \prod_{x \in B} a_x^{n_x}(\eta^{(0)}) \tag{7.10}$$

weakly, as $\gamma \to 0$. Therefore, we have convergence of the processes $\eta_t^{(\gamma)} \to \eta_t^{(0)}$ for addition rates with finite support, i.e., such that $\varphi(x) = 0$ for $x \notin D$ with $D \subseteq \mathbb{Z}^d$ finite.

7.2 Convergence of the semigroups for general addition rates

The next step is to pass to general addition rates; we use the convergence of semigroups argument, as in the proof of [22, Proposition 4.1]. Let $S_t^{\varphi,\gamma}$ denote the semigroup of the process $\eta_t^{(\gamma)}$ with addition rate φ, and S_t^{φ} the semigroup of the process η_t (with zero dissipation) with addition rate φ. Both semigroups are well-defined as long as φ has finite support. For a local function f, with dependence set D_f, and for addition rates φ, φ' of finite support, we have, similarly to the estimate (51) in the proof of [22, Proposition 4.1]

$$\begin{aligned}\mathbb{E}_{m^{(\gamma)}}|S_t^{\varphi,\gamma}(f) - S_t^{\varphi',\gamma}(f)| &\le Ct \sum_{x\in\overline{D_f}} \sum_{y\in\mathbb{Z}^d} G^{(\gamma)}(x,y)|\varphi(y) - \varphi'(y)| \\ &\le Ct \sum_{x\in\overline{D_f}} \sum_{y\in\mathbb{Z}^d} G(x,y)|\varphi(y) - \varphi'(y)|;\end{aligned}$$

and

$$\mathbb{E}_m|S_t^{\varphi}(f) - S_t^{\varphi'}(f)| \le Ct \sum_{x\in\overline{D_f}} \sum_{y\in\mathbb{Z}^d} G(x,y)|\varphi(y) - \varphi'(y)|.$$

Note that this estimate in [22] is given in the context of the abelian sandpile model, that is a model with discrete heights and no dissipation. However, it is based only on the estimate (3.10) for the numbers of topplings, which is valid for the abelian avalanche model, and therefore it extends directly to the abelian avalanche model.

Hence, if for a sequence $\varphi^{(n)}$ of addition rates of finite support, for an addition rate φ (not necessarily of finite support) and for all $x \in \mathbb{Z}^d$

$$\sum_{y\in\mathbb{Z}^d} G(x,y)|\varphi(y) - \varphi^{(n)}(y)| \to 0$$

as $n \to \infty$, we conclude for all local f, $\gamma \ge 0$, $S_t^{\varphi^{(n)},\gamma}(f)$ is a Cauchy sequence in $L^1(m^{(\gamma)})$, and hence converges to $\Psi := S_t^{\varphi,\gamma}(f)$ for all $\gamma \ge 0$, as $n \to \infty$. This semigroup $S_t^{\varphi,\gamma}$ then defines a corresponding stationary Markov process $\eta_t^{\varphi,(\gamma)}$, and η_t^{φ} (for $\gamma = 0$). As the convergence of the semigroups implies the convergence of the finite dimensional distributions of the corresponding stationary processes, we conclude for all $\gamma > 0$,

$$\eta_t^{\varphi^{(n)},\gamma} \to \eta_t^{\varphi,\gamma} \tag{7.11}$$

and

$$\eta_t^{\varphi^{(n)}} \to \eta_t^{\varphi} \tag{7.12}$$

as $n \to \infty$ in the sense of convergence of finite dimensional distributions. Therefore, if φ satisfies (7.2), let $\varphi^{(n)}$ denote $\varphi^{(n)}(x) = \varphi(x)I(x \in [-n,n]^d)$, then we have for all $n \in \mathbb{N}$

$$\eta_t^{\varphi^{(n)},\gamma} \to \eta_t^{\varphi^{(n)}} \tag{7.13}$$

as $\gamma \to 0$. Combination of (7.11),(7.12),(7.13) together with a three epsilon argument then concludes the convergence of $\{\eta_t^{\varphi,\gamma} : t \ge 0\}$ to $\{\eta_t^{\varphi,0} : t \ge 0\}$ in the sense of convergence of finite dimensional distributions, as $\gamma \downarrow 0$.

7.3 Tightness

To finish the proof, we have to show that the processes $\{\eta_t^{\varphi,\gamma} : t \ge 0\}$ form a tight family if φ satisfies (7.2). By definition of the product distance between configurations, this reduces to showing that for all $x \in \mathbb{Z}^d$, and for all $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{0\le s\le t\le\delta} |\eta_s^{\varphi,\gamma}(x) - \eta_t^{\varphi,\gamma}(x)| \ge \varepsilon\right) \le C_\varepsilon \delta \tag{7.14}$$

where the constant C_ε only depends on ε. Let $L = \sum_{y\in\mathbb{Z}^d} \varphi(y)(a_y^{(\gamma)} - I)$ denote the generator of the process $\eta_t^{\varphi,\gamma}$, and $f_x(\eta) = \eta(x)$. We have that

$$\eta_t^{\varphi,\gamma}(x) - \eta_0^{\varphi,\gamma}(x) - \int_0^t \sum_{y\in\mathbb{Z}^d} \varphi(y)\left((a_y^{(\gamma)}\eta_s)(x) - \eta_s(x)\right) ds = M_t \tag{7.15}$$

is a martingale with quadratic variation

$$\begin{aligned} < M_t, M_t > &= \int_0^t \left(L(f_x^2) - 2f_x L(f_x)\right)(\eta_s) ds & (7.16)\\ &= \int_0^t \left(\sum_{y\in\mathbb{Z}^d} \varphi(y)\left((a_y^{(\gamma)}\eta_s)(x)\right)^2 - \eta_s(x)^2)\right. & (7.17)\\ &\qquad \left. + \eta_s(x) \sum_{y\in\mathbb{Z}^d} \varphi(y)\left((a_y^{(\gamma)}\eta_s)(x)\right) - \eta_s(x))\right) ds. \end{aligned}$$

Using that the heights are uniformly bounded by a constant, we estimate

$$|\left(L(f_x^2) - 2f_x L(f_x)\right)(\eta_s)| \le C \sum_{y\in\mathbb{Z}^d} \varphi(y) I\left((a_y^{(\gamma)}\eta_s)(x) \ne \eta_s(x)\right). \tag{7.18}$$

Now, since

$$\mathbb{P}\left((a_y^{(\gamma)}\eta_s)(x) \ne \eta_s(x)\right) \le \sum_{z\sim x} G^{(\gamma)}(y,z),$$

the stationary expectation of $< M_t, M_t >$ is bounded by

$$\mathbb{E}(< M_t, M_t >) \le Ct \sum_{y\in\mathbb{Z}^d} \sum_{z\sim x} \varphi(y) G(y,z) < C_1 t, \tag{7.19}$$

where (7.2) gives the final bound. Similarly,

$$\mathbb{E}\left|\int_s^t \sum_{y\in\mathbb{Z}^d} \varphi(y)\left((a_y^{(\gamma)}\eta_r)(x) - \eta_r(x)\right) dr\right| \le C_2(t-s). \tag{7.20}$$

Then use Markov's and Doob's inequality to conclude (7.14):

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\le s\le t\le\delta} |\eta_s^{\varphi,\gamma}(x) - \eta_t^{\varphi,\gamma}(x)| \ge \varepsilon\right)\\ \le\ & \mathbb{P}\left(\int_0^\delta |Lf_x(\eta_r)| dr \ge \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\sup_{0\le s\le t\le\delta} |M_t - M_s| \ge \frac{\varepsilon}{2}\right)\\ \le\ & \frac{C_1'\delta}{\varepsilon} + \frac{C_2'\delta}{\varepsilon^2}. \end{aligned}$$

□

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