*Citation for published version:*
Chen, C, Davenport, JH, Moreno Maza, M, Xia, B & Xiao, R 2011, Computing with semi-algebraic sets represented by triangular decomposition. in ISSAC '11 Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation. Proceedings of the International Symposium on Symbolic and Algebraic Computation, ISSAC, Association for Computing Machinery, New York, pp. 75-82, 36th International Symposium on Symbolic and Algebraic Computation, ISSAC 2011, June 8, 2011 - June 11, 2011, San Jose, CA, USA United States, 1/01/11. https://doi.org/10.1145/1993886.1993903

*DOI:*
10.1145/1993886.1993903

*Publication date:*
2011

*Document Version*
Peer reviewed version

# Computing with Semi-Algebraic Sets Represented by Triangular Decomposition


Changbo Chen
University of Western Ontario
cchen252@csd.uwo.ca

James H. Davenport
University of Bath
J.H.Davenport@bath.ac.uk

Marc Moreno Maza
University of Western Ontario
moreno@csd.uwo.ca

Bican Xia
Peking University
xbc@math.pku.edu.cn

Rong Xiao
University of Western Ontario
rong@csd.uwo.ca



## ABSTRACT

This article is a continuation of our earlier work [3], which introduced triangular decompositions of semi-algebraic systems and algorithms for computing them. Our new contributions include theoretical results based on which we obtain practical improvements for these decomposition algorithms.

We exhibit new results on the theory of *border polynomials* of parametric semi-algebraic systems: in particular a geometric characterization of its "true boundary" (Definition 2). In order to optimize these algorithms, we also propose a technique, that we call *relaxation*, which can simplify the decomposition process and reduce the number of redundant components in the output. Moreover, we present procedures for basic set-theoretical operations on semi-algebraic sets represented by triangular decomposition. Experimentation confirms the effectiveness of our techniques.


## 1. INTRODUCTION

Triangular decompositions of semi-algebraic systems were introduced in [3]. The key notions and notations of this paper are reviewed in the next section.

That paper presents also an algorithm for generating those decompositions. This algorithm can either be *eager*, computing the entire decomposition, or *lazy*, only computing the decomposition corresponding to the highest (complex) dimensional components, and deferring lower-dimensional components. While a complete decomposition is known to have a worst-case complexity which is doubly-exponential in the number of variables [8], under plausible assumptions the lazy variant has a singly-exponential complexity. Nevertheless, it is still desirable to improve the practical efficiency of both types of decomposition.

The notion of a *border polynomial* [15] is at the core of our work. A strongly related notion, discriminant variety, was introduced in [9] and the link between them was investigated in [14]. Other similar but more restrictive notions like "generalised discriminant" and "generalised resultant" were introduced in [10]. For a squarefree regular chain $T$, regarded as a real parametric system in its free variables $\mathbf{u}$, the border polynomial $BP(T)$ encodes the locus of the $\mathbf{u}$-values at which $T$ has lower rank or at which $T$ is no longer a squarefree regular chain. (See §2 for the notions related to triangular decomposition and regular chains.) Consequently, for each connected component $C$ of the complement of the real hypersurface defined by $BP(T)$ the number of real solutions of the regular chain $T$ is constant at any point of $C$. However, $BP(T)$ is not an invariant of the variety $\overline{W(T)}$, which is a bottleneck in designing better algorithms based on the notion of a border polynomial. We overcome this difficulty in two ways.

Firstly, in §3, we prove that among all regular chains $T'$ satisfying $\mathrm{sat}(T') = \mathrm{sat}(T)$ there is one and only one (characterized in Theorem 1) for which $BP(T')$ is minimal w.r.t. inclusion. Secondly, in §4, we introduce the concept of an *effective boundary* of a squarefree semi-algebraic system, see Definition 2. This allows us to identify a subset of $BP(T)$ which is an invariant of $\overline{W(T)}$, that is, unchanged when replacing $T$ by $T'$ as long as $\overline{W(T)} = \overline{W(T')}$ holds. In many ways, our notion of effective boundary is similar to the "better projection" ideas in the classical [7, and many others] approach to cylindrical algebraic decomposition.

In §5, we introduce the technique of *relaxation* which we shall motivate by an example. Consider the semi-algebraic system $sys = [f = 0, x - b > 0]$, where $f = ax^3 + bx - a$ for the variable ordering $a < b < x$. The LazyRealTriangularize algorithm of [3] will compute the border polynomial set $B = \{a, b_1, b_2\}$ and the fingerprint polynomial set (FPS) $F = \{a, b_1, b_2, b, p_1, p_2, p_3\}$. where $b_1 = ab^3 + b^2 - a$, $b_2 = 27a^3 + 4b^3$, $p_1 = 2b^3 + 1$, $p_2 = b^3 - 4$ and $p_3 = b - 1$. Thus the LazyRealTriangularize$(sys)$ will produce 1 regular semi-algebraic system $S_1 = [Q_1, \{f = 0, x - b > 0\}]$, and 7 unevaluated recursive calls, where

$$Q_1 = (b < 0 \wedge p_1 \neq 0 \wedge b_1 \neq 0 \wedge a \neq 0 \wedge b_2 \neq 0)$$
$$\bigvee (p_1 > 0 \wedge b_1 > 0 \wedge a < 0 \wedge p_3 > 0 \wedge p_2 \neq 0 \wedge b_2 \neq 0)$$
$$\bigvee (b > 0 \wedge p_1 > 0 \wedge b_1 \neq 0 \wedge a < 0 \wedge p_3 < 0 \wedge p_2 < 0 \wedge b_2 \neq 0)$$
$$\bigvee (b > 0 \wedge p_1 > 0 \wedge b_1 < 0 \wedge a > 0 \wedge p_3 < 0 \wedge p_2 < 0 \wedge b_2 > 0)$$

and the 7 calls are made for each $p \in F$ with the form $\mathsf{LazyRealTriangularize}([p = 0, f = 0, x - b > 0])$. The key observation is that some of these recursive calls can simply be avoided if some of the strict inequalities in $Q_1$ can be relaxed, that is, replaced by non-strict inequalities. The results of §5, and in particular Theorem 5 provide criteria for this purpose. Returning to our example, when relaxation techniques are used $\mathsf{LazyRealTriangularize}(sys)$ will produce 1 regular semi-algebraic system $S_2 = [Q_2, \{f = 0, x-b > 0\}]$, and 3 un-evaluated recursive calls, where

$$Q_2 = (b \leq 0 \wedge b_1 \neq 0 \wedge a \neq 0 \wedge b_2 \neq 0)$$
$$\bigvee (p_1 \geq 0 \wedge b_1 > 0 \wedge a < 0 \wedge p_3 \geq 0 \wedge b_2 \neq 0)$$
$$\bigvee (b \geq 0 \wedge p_1 \geq 0 \wedge b_1 \neq 0 \wedge a < 0 \wedge p_3 \leq 0 \wedge p_2 \leq 0 \wedge b_2 \neq 0)$$
$$\bigvee (b \geq 0 \wedge p_1 \geq 0 \wedge b_1 < 0 \wedge a > 0 \wedge p_3 \leq 0 \wedge p_2 \leq 0 \wedge b_2 > 0)$$

Moreover, it turns that the the 3 un-evaluated recursive calls are of the form $\mathsf{LazyRealTriangularize}([p = 0, f = 0, x - b > 0])$, for $p \in B$. Continuing with that example, one can check that the full triangular decomposition of *sys* produces 16 and 9 regular semi-algebraic systems, without and with relaxation techniques, respectively. Therefore, relaxation techniques can help simplify the output of our algorithms.

Nevertheless, even with relaxation techniques, our algorithms can produce redundant components, that is, a regular semi-algebraic system $S$ for which there exists another regular semi-algebraic system $S'$ in the same decomposition and such that $Z_{\mathbb{R}}(S) \subseteq Z_{\mathbb{R}}(S')$ holds. This is actually the case for our example where 1 out of the 9 regular semi-algebraic systems is redundant.

To perform inclusion test on the zero sets of regular semi-algebraic systems, we have developed algorithms for set-theoretical operations on semi-algebraic sets represented by triangular decomposition, see §7. Those algorithms rely on a new algorithm, presented in §6, for computing triangular decomposition of semi-algebraic systems in an incremental manner, which is a natural adaption of the idea presented in [11] for computing triangular decomposition of algebraic systems incrementally.

The experimentation illustrates the effectiveness of the different techniques presented in this paper. In particular, we observe that with relaxation, the decomposition algorithm will produce output with less redundancy without paying a lot, and accelerate on some hard systems; the incremental algorithm for computing triangular decomposition of semi-algebraic systems often outperforms the one in [3]. Moreover, we observe that our techniques for removing redundant components can usually process in a "reasonable" amount time the output of the systems that $\mathsf{RealTriangularize}$ can decompose.

## 2. TRIANGULAR DECOMPOSITION

We summarize below the notions and notations of [3], including triangular decompositions of semi-algebraic systems.

**Zero sets and topology**. In this paper, we use "$Z$" to denote the zero set of a polynomial system, involving equations and inequations, in $\mathbb{C}^n$ and "$Z_{\mathbb{R}}$" to denote the zero set of a semi-algebraic system in $\mathbb{R}^n$. If a semi-algebraic set $S$ is finite, we denote by $\#(S)$ the number of distinct points in it. In $\mathbb{R}^n$, we use the Euclidean topology; in $\mathbb{C}^n$, we use the Zariski topology. Given a semi-algebraic set $S$, we denote by $\partial S$ the boundary of $S$, by $\overline{S}$ the closure of $S$.

**Notations on polynomials.** Throughout this paper, all polynomials are in $\mathbb{Q}[\mathbf{x}]$, with ordered variables $\mathbf{x} = x_1 < \cdots < x_n$. We order monomials of $\mathbb{Q}[\mathbf{x}]$ by the lexicographical ordering induced by $x_1 < \cdots < x_n$. Then, we require that the leading coefficient of every polynomial in a regular chain or in a border polynomial set (defined hereafter) is equal to 1. Let $F \subset \mathbb{Q}[\mathbf{x}]$. We denote by $V(F)$ the set of common zeros of $F$ in $\mathbb{C}^n$. Let $p$ be a polynomial in $\mathbb{Q}[\mathbf{x}] \setminus \mathbb{Q}$. Then denote by $\mathrm{mvar}(p)$, $\mathrm{init}(p)$, and $\mathrm{mdeg}(p)$ respectively the greatest variable appearing in $p$ (called the *main variable* of $p$), the leading coefficient of $p$ w.r.t. $\mathrm{mvar}(p)$ (called the *initial* of $p$), and the degree of $p$ w.r.t. $\mathrm{mvar}(p)$ (called the *main degree* of $p$). Let $v \in \mathbf{x}$. Denote by $\mathrm{lc}(p, v)$, $\deg(p, v)$, $\mathrm{der}(p, v)$, $\mathrm{discrim}(p, v)$ respectively the leading coefficient, the degree, the derivative and the discriminant of $p$ w.r.t. $v$.

**Triangular set.** Let $T \subset \mathbb{Q}[\mathbf{x}]$ be a *triangular set*, that is, a set of non-constant polynomials with pairwise distinct main variables. Denote by $\mathrm{mvar}(T)$ the set of main variables of the polynomials in $T$. A variable $v$ in $\mathbf{x}$ is called *algebraic* w.r.t. $T$ if $v \in \mathrm{mvar}(T)$, otherwise it is said *free* w.r.t. $T$. If no confusion is possible, we shall always denote by $\mathbf{u} = u_1, \ldots, u_d$ and $\mathbf{y} = y_1, \ldots, y_m$ $(m + d = n)$ respectively the free and the main variables of $T$. When $T$ is regarded as a *parametric system*, the free variables in $T$ are its parameters.

Let $h_T$ be the product of the initials of the polynomials in $T$. We denote by $\mathrm{sat}(T)$ the *saturated ideal* of $T$: if $T$ is the empty triangular set, then $\mathrm{sat}(T)$ is defined as the trivial ideal $\langle 0 \rangle$, otherwise it is the colon ideal $\langle T \rangle : h_T^{\infty}$. The *quasi-component* $W(T)$ of $T$ is defined as $V(T) \setminus V(h_T)$. Denote by $\overline{W(T)}$ the Zariski closure of $W(T)$, which is equal to $V(\mathrm{sat}(T))$. Denote by $W_{\mathbb{R}}(T)$ the set $Z_{\mathbb{R}}(T) \setminus Z_{\mathbb{R}}(h_T)$.

**Iterated resultant**. Let $p, q \in \mathbb{Q}[\mathbf{x}] \setminus \mathbb{Q}$. Let $v = \mathrm{mvar}(q)$. Denote by $\mathrm{res}(p, q, v)$ the resultant of $p, q$ w.r.t. $v$. Let $T \subset \mathbb{Q}[\mathbf{x}]$ be a triangular set. We define $\mathrm{res}(p, T)$ inductively: if $T$ is empty, then $\mathrm{res}(p, T) = p$; otherwise let $v$ be the largest variable occurring in $T$, then $\mathrm{res}(p, T) = \mathrm{res}(\mathrm{res}(p, T_v, v), T_{<v})$, where $T_v$ and $T_{<v}$ denote respectively the polynomials of $T$ with main variables equal to and less than $v$.

**Regular chain**. A triangular set $T \subset \mathbb{Q}[\mathbf{x}]$ is called a *regular chain* if: either $T$ is empty; or (letting $t$ be the polynomial in $T$ with maximum main variable), $T \setminus \{t\}$ is a regular chain, and the initial of $t$ is regular w.r.t. $\mathrm{sat}(T \setminus \{t\})$. Let $H \subset \mathbb{Q}[\mathbf{x}]$. The pair $[T, H]$ is a *regular system* if each polynomial in $H$ is regular modulo $\mathrm{sat}(T)$. A regular chain $T$ or a regular system $[T, H]$, is *squarefree* if for all $t \in T$, $\mathrm{der}(t)$ is regular w.r.t. $\operatorname{sat}(T)$. Given $u \in \mathbb{R}^d$, we say that a squarefree regular system $[T, H]$ *specializes well* at $u$ if $h_T(u) \neq 0$ and $[T(u), H(u)]$ is a squarefree regular system. A regular chain is called *$d$-dimensional* if it has $d$ free variables.

**Semi-algebraic system**. Consider four finite polynomial sets $F = \{f_1, \ldots, f_s\}$, $N = \{n_1, \ldots, n_k\}$, $P = \{p_1, \ldots, p_e\}$, and $H = \{h_1, \ldots, h_\ell\}$ of $\mathbb{Q}[\mathbf{x}]$. Let $N_{\geq}$ denote the set of non-negative inequalities $\{n_1 \geq 0, \ldots, n_k \geq 0\}$. Let $P_{>}$ denote the set of positive inequalities $\{p_1 > 0, \ldots, p_e > 0\}$. Let $H_{\neq}$ denote the set of inequations $\{h_1 \neq 0, \ldots, h_\ell \neq 0\}$. We denote by $\mathfrak{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$ the *semi-algebraic system* (SAS) defined as the conjunction of the constraints $f_1 = \cdots f_s = 0$, $N_{\geq}$, $P_{>}$, $H_{\neq}$. When $N_{\geq}$, $H_{\neq}$ are empty, $\mathfrak{S}$ is called a *basic semi-algebraic system* and denoted by $[F, P_{>}]$.

**Regular semi-algebraic system**. We call a basic SAS $[T, P_{>}]$ in $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$ a squarefree semi-algebraic system, SFSAS for short, if $[T, P]$ forms a squarefree regular system. Let $[T, P_{>}]$ be an SFSAS. Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{u}]$. We say that $R := [\mathcal{Q}, T, P_{>}]$ is a *regular semi-algebraic system* if

$(i)$ $\mathcal{Q}$ defines a non-empty open semi-algebraic set $S$ in $\mathbb{R}^d$;
$(ii)$ $[T, P]$ specializes well at every point of $S$,
$(iii)$ at each $u \in S$, the specialized system $[T(u), P(u)_{>}]$ has at least one real zero.

**Border polynomial [15, 16, 3]**. We review briefly the notion of *border polynomial* of a regular chain, a regular system, or an SFSAS. Let $R$ be either a squarefree regular chain $T$, or a squarefree regular system $[T, P]$, or an SFSAS $[T, P_{>}]$ in $\mathbb{Q}[\mathbf{x}]$. We denote by $B_{sep}(T)$, $B_{ini}(T)$, $B_{ineqs}([T, P])$ the set of irreducible factors of: $\prod_{t \in T} \operatorname{res}(\operatorname{discrim}(t, \operatorname{mvar}(t)), T)$, $\prod_{t \in T} \operatorname{res}(\operatorname{init}(t), T)$, and $\prod_{f \in P} \operatorname{res}(f, T)$, respectively. Denote by $\operatorname{BP}(R)$ the set $B_{sep}(T) \cup B_{ini}(T) \cup B_{ineqs}([T, P])$. Then $\operatorname{BP}(R)$ (resp. the polynomial $\prod_{f \in \operatorname{BP}(R)} f$) is called the *border polynomial set* (resp. *border polynomial*) of $R$.

Lemma 1 (Lemma 2 in [3]). *Let $R = [T, P_{>}]$ be an SFSAS of $\mathbb{Q}[\mathbf{x}]$. Let $u_1, u_2$ be two parameter values in a same connected component of $Z_{\mathbb{R}}(\prod_{f \in \operatorname{BP}(R)} f \neq 0)$ in $\mathbb{R}^d$. Then $\#Z_{\mathbb{R}}(R(u_1)) = \#Z_{\mathbb{R}}(R(u_2))$.*

**Fingerprint polynomial set.** $R = [B_{\neq}, T, P_{>}]$ is called a *pre-regular semi-algebraic system*, if for each $p \in \operatorname{BP}([T, P_{>}])$, $p$ is a factor of some polynomial in $B$. Suppose $R$ is a pre-regular semi-algebraic system. A polynomial set $D \subset \mathbb{Q}[\mathbf{u}]$ is called a *fingerprint polynomial set* (FPS) of $R$ if:

$(i)$ $Z_{\mathbb{R}}(D_{\neq}) \subseteq Z_{\mathbb{R}}(B_{\neq})$ holds,
$(ii)$ for all $\alpha, \beta \in Z_{\mathbb{R}}(D_{\neq})$ with $\alpha \neq \beta$, if the signs of $p(\alpha)$ and $p(\beta)$ are the same for all $p \in D$, then $R(\alpha)$ has real solutions if and only if $R(\beta)$ does.

**Open CAD operator [12, 2, 3].** Let $\mathbf{u} = u_1 < \cdots < u_d$ be ordered variables. For a polynomial $p \in \mathbb{Q}[\mathbf{u}]$, denote by $\operatorname{factor}(p)$ the set of the non-constant irreducible factors of $p$; for $A \subset \mathbb{Q}[\mathbf{u}]$, define $\operatorname{factor}(A) = \cup_{p \in A} \operatorname{factor}(p)$. For a squarefree polynomial $p$, the *open projection operator* (oproj) w.r.t. a variable $v \in \mathbf{u}$ is defined as below:

$$\operatorname{oproj}(p, v) := \operatorname{factor}(\operatorname{discrim}(p, v) \operatorname{lc}(p, v)).$$

If $p$ is not squarefree, then we define $\operatorname{oproj}(p, v) := \operatorname{oproj}(p^*, v)$, where $p^*$ is the squarefree part of $p$; then for a polynomial set $A$, we define $\operatorname{oproj}(A, v) := \operatorname{oproj}(\Pi_{f \in A} f, v)$.

Given $A \subset \mathbb{Q}[\mathbf{u}]$ and $x \in \{u_1, \ldots, u_d\}$, denote by $\operatorname{der}(A, x)$ the *derivative closure* of $A$ w.r.t. $x$. The *open augmented projected factors* of $A$, denoted by $\operatorname{oaf}(A)$, is defined as follows. Let $k$ be the smallest positive integer such that $A \subset \mathbb{Q}[u_1, \ldots, u_k]$ holds. Let $C = \operatorname{factor}(\operatorname{der}(A, u_k))$; we have:

1. if $k = 1$, then $\operatorname{oaf}(A) := C$;
2. if $k > 1$, then $\operatorname{oaf}(A) := C \cup \operatorname{oaf}(\operatorname{oproj}(C, u_k))$.

## 3. BORDER POLYNOMIAL

The relation "having the same saturated ideal" is an equivalence relation among regular chains of $\mathbb{Q}[\mathbf{x}]$. We show in this section that, for each equivalence class, there exists a unique representative whose border polynomial set is contained in the border polynomial set of any other representative.

To this end, we rely on the concept of *canonical regular chain*. In the field of triangular decompositions, several authors have used this term to refer to different notions. To be precise, we make use of the one defined in [13].

Definition 1 (canonical regular chain). *Let $T$ be a regular chain of $\mathbb{Q}[\mathbf{x}]$. If each polynomial $t$ of $T$ satisfies:*

1. *the initial of $t$ involves only the free variables of $T$,*
2. *for any polynomial $f \in T$ with $mvar(f) < mvar(t)$, we have $\deg(t, mvar(f)) < mdeg(f)$,*
3. *$t$ is primitive over $\mathbb{Q}$, w.r.t. its main variable,*

*then we say that $T$ is* canonical.

Remark 1. *Let $T = \{t_1, \cdots, t_m\}$ be a regular chain; let $d_k = mdeg(t_k)$, for $k = 1 \cdots m$. One constructs a canonical regular chain $T^* = \{t_1^*, t_2^*, \ldots, t_m^*\}$ such that $sat(T) = sat(T^*)$ in the following way:*

1. *set $t_1^*$ to be the primitive part of $t_1$ w.r.t. $y_1$;*
2. *for $k = 2, \ldots, m$, let $r_k$ be the iterated resultant $res(init(t_k), \{t_1, \ldots, t_{k-1}\})$. Suppose $r_k = a_k \operatorname{init}(t_k) + \sum_{i=1}^{k-1} c_i t_i$. Compute $t$ as the pseudo-reminder of $a_k t_k + (\sum_{i=1}^{k-1} c_i t_i) y_k^{d_k}$ by $\{t_1^*, \ldots, t_{k-1}^*\}$. Set $t_k^*$ to be the primitive part of $t$ w.r.t. $y_k$.*

A canonical regular chain has the minimal border polynomial set among the family of regular chains having the same saturated ideal, which is stated in the following theorem.

Theorem 1. *Given a squarefree regular chains $T$ of $\mathbb{Q}[\mathbf{x}]$, there exists a unique canonical regular chain $T^*$ such that $sat(T) = sat(T^*)$. Moreover, we have $\operatorname{BP}(T^*) \subseteq \operatorname{BP}(T)$.*

The proof of the above theorem relies on some basic properties of border polynomial set recalled below.

Given a constructible set $\mathcal{C}$ defined by a parametric polynomial system, *the minimal discriminant variety* (MDV) [9] of $\mathcal{C}$, denoted by $\mathrm{mdv}(\mathcal{C})$, is an intrinsic geometric object attached to $\mathcal{C}$ and the parameters. The following results relate the border polynomial of a regular chains $T$ and the discriminant variety of the algebraic variety $V(T)$.

Lemma 2 ([14]). *Let $T$ be a squarefree regular chain of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Then we have $mdv(V(T)) = V(\prod_{f \in \mathrm{BP}(T)} f)$.*

Lemma 3 ([14, Lemma 17]). *Let $T$ be a squarefree regular chain of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Then we have $mdv(\overline{W(T)}) \subseteq mdv(V(T))$ and $mdv(V(T)) \setminus mdv(\overline{W(T)}) \subseteq V(\prod_{f \in B_{ini}(T)} f)$.*

Lemma 4. *Let $T_1$ and $T_2$ be squarefree regular chains of $\mathbb{Q}[\mathbf{x}]$ such that $sat(T_1) = sat(T_2)$. If $B_{ini}(T_1) \subseteq B_{ini}(T_2)$, then we have $\mathrm{BP}(T_1) \subseteq \mathrm{BP}(T_2)$.*

Proof. Firstly, we have $V(\prod_{f \in B_{ini}(T_i)} f) \subseteq \mathrm{mdv}(V(T_i))$ by Lemma 2. Then with Lemma 3, we have $\mathrm{mdv}(V(T_i)) = V(\prod_{f \in B_{ini}(T_i)} f) \cup \mathrm{mdv}(\overline{W(T_i)})$. Since $\mathrm{sat}(T_1) = \mathrm{sat}(T_2)$, we have $\overline{W(T_1)} = \overline{W(T_2)}$. Therefore we have $\mathrm{mdv}(V(T_2)) \subseteq \mathrm{mdv}(V(T_1))$ by the assumption $B_{ini}(T_1) \subseteq B_{ini}(T_2)$, which implies the lemma. $\square$

Next we prove Theorem 1.

Proof. By Remark 1, we can always construct a canonical regular chain $T^*$ such that $\mathrm{sat}(T) = \mathrm{sat}(T^*)$. Moreover, for each $t \in T$, we have $\mathrm{init}(t^*)$ divides $\mathrm{res}(\mathrm{init}(t), T)$. Therefore, $B_{ini}(T^*) \subseteq B_{ini}(T)$ holds, which implies $\mathrm{BP}(T^*) \subseteq \mathrm{BP}(T)$ by Lemma 4.

Suppose $T^\diamond$ is any given canonical regular chain such that $\mathrm{sat}(T^\diamond) = \mathrm{sat}(T)$ holds. It is sufficient to show that $T^* = T^\diamond$ holds to complete the proof.

Note that $T^\diamond$, $T^*$ and $T$ have the same set of free and algebraic variables, denoted respectively by $\mathbf{u}$ and $\mathbf{y}$. Given $\mathcal{I}$ an ideal in $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$, denote by $\mathcal{I}^{ext}$ the extension of $\mathcal{I}$ in $\mathbb{Q}(\mathbf{u})[\mathbf{y}]$. Since $\mathfrak{p}^{ext} = \langle 1 \rangle$ holds for any prime ideal $\mathfrak{p}$ in $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$ with $\mathbf{u}$ algebraically dependent, we have $\langle T^* \rangle^{ext} = \langle T^\diamond \rangle^{ext} = \mathrm{sat}(T)^{ext}$ holds. Therefore, the polynomials in $T^*$ (or $T^\diamond$) form a Gröbner basis of $\mathrm{sat}(T)^{ext}$ (w.r.t. the lexicographical ordering on $\mathbf{y}$) since their leading power products are pairwise coprime. Dividing each polynomial in $T^*$ (or $T^\diamond$) by its initial, we obtain the unique reduced Gröbner basis of $\mathrm{sat}(T)^{ext}$. This implies $T^* = T^\diamond$. $\square$

## 4. EFFECTIVE BOUNDARY AND FPS

In this subsection, we will focus on an SFSAS $\mathfrak{S} = [T, P_>]$ in $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$ where $\mathbf{u} = u_1, \ldots, u_d$ are the free variables of $T$.

Definition 2 (Effective boundary). *Let $\mathbf{h}$ be a $(d-1)$-dimensional hypersurface in the parameter space $\mathbb{R}^d$ of $\mathfrak{S}$. We call $\mathbf{h}$ an* effective boundary *of $\mathfrak{S}$ if for every hypersurface $\mathcal{H} \not\supseteq \mathbf{h}$ in $\mathbb{R}^d$, there exists a point $u^*$ in $\mathbf{h} \setminus \mathcal{H}$ satisfying: for any open ball $O(u^*)$ of $u^*$, there exist two points $\alpha_1$, $\alpha_2 \in O(u^*) \setminus \mathbf{h}$, s.t. $\#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_1)) \neq \#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_2))$. Denote by $\mathcal{E}(\mathfrak{S})$ the union of all effective boundaries of $\mathfrak{S}$.*

Recall that the hypersurface defined by the border polynomial of an SFSAS partitions the parametric space into regions, where the number of real solutions is locally invariant. One might imagine that the effective boundaries are strongly related to the border polynomial set. Indeed, we have the following Lemma stating the relation.

Lemma 5. *We have $\mathcal{E}(\mathfrak{S}) \subseteq Z_{\mathbb{R}}(\prod_{f \in \mathrm{BP}(\mathfrak{S})} f = 0)$.*

Proof. Let $\mathbf{h} \in \mathcal{E}(\mathfrak{S})$ such that $\mathbf{h} \not\subseteq Z_{\mathbb{R}}(\prod_{f \in \mathrm{BP}(\mathfrak{S})} f = 0)$ holds. Then for each $u \in \mathbf{h} \setminus Z_{\mathbb{R}}(\prod_{f \in \mathrm{BP}(\mathfrak{S})} f = 0)$, we can choose an open ball $O(u)$ of $u$ contained in a connected component of the set $Z_{\mathbb{R}}(\prod_{f \in B} f \neq 0)$. By Lemma 1, for any two points $\alpha_1, \alpha_2 \in O(u)$, $\#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_1)) = \#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_2))$ holds. That is a contradiction to the assumption of $\mathbf{h}$ being an effective boundary. $\square$

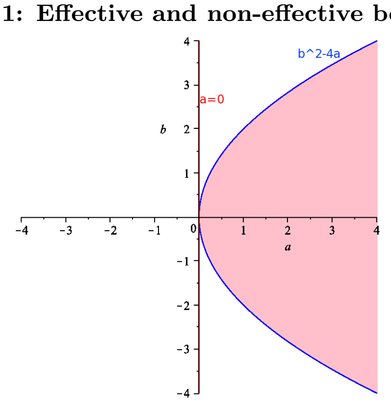
Lemma 5 implies that the set of effective boundaries represented by irreducible polynomials of $\mathbb{Q}[\mathbf{u}]$ is finite and can be given by polynomials from the border polynomial set.

Definition 3. *A polynomial $p$ in $\mathrm{BP}(\mathfrak{S})$ is called an* effective border polynomial factor *if $Z_{\mathbb{R}}(p = 0)$ is an effective boundary of $\mathfrak{S}$. We denote by $ebf(\mathfrak{S})$ the set of effective border polynomial factors.*

The example below shows that some of the polynomials in a border polynomial may not be effective. Roughly speaking, the factors in $B_{ini}$ are not effective. This property is formally stated in a soon coming extended version of this article.

Example 1. *Consider an SFSAS $R = [\{ax^2+bx+1\}, \{\ \}]$. Its border polynomial set is $\{a, b^2-4a\}$. One can verify from Figure 1 that $Z_{\mathbb{R}}(b^2-4a = 0)$ is an effective boundary of $R$, while $Z_{\mathbb{R}}(a = 0)$ is not. Indeed, all $a, b$-values in the blank (resp, filled) area specialize $R$ to have 2 (resp. 0) real solutions.*

**Figure 1: Effective and non-effective boundary**




Since $\mathcal{E}(\mathfrak{S})$ can be described by border polynomial factors, we derive the following theorem, which can be viewed as a "computable-version" of Definition 2.

Theorem 2. *A polynomial $p$ in $\mathrm{BP}(\mathfrak{S})$ is an effective border polynomial factor if and only if there exist two connected components $C_1, C_2$ of $Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f \neq 0)$ satisfying*

(1) *$\partial C_1 \cap \partial C_2 \cap Z_{\mathbb{R}}(p=0)$ is of dimension $d-1$,*
(2) *for all point $\alpha_1 \in C_1$ and for all point $\alpha_2 \in C_2$ we have $\#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_1)) \neq \#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_2))$.*

Proof. "$\Rightarrow$". Suppose $p$ is an effective border polynomial factor. By definition, there exists a point $u^* \in Z_{\mathbb{R}}(p = 0) \setminus Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})\setminus\{p\}} f = 0)$ and a sufficiently small open ball $O(u^*)$ centered at $u^*$ satisfying the properties below:

($i$) $O(u^*) \setminus Z_{\mathbb{R}}(p=0) \subseteq Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f \neq 0)$;
($ii$) $O(u^*) \cap Z_{\mathbb{R}}(\prod_{f\in \mathrm{ebf}(\mathfrak{S})} f = 0) \subseteq Z_{\mathbb{R}}(p=0)$;
($iii$) there exist two points $\alpha_1, \alpha_2$ in $O(u^*)\setminus Z_{\mathbb{R}}(p=0)$, such that we have $\#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_1)) \neq \#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_2))$;
($iv$) $O(u^*) \setminus Z_{\mathbb{R}}(p=0) = O(u^*) \cap C_1 \cup O(u^*) \cap C_2$, where $C_i$ is the connected component of $Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f \neq 0)$ containing $\alpha_i$, for $i = 1, 2$.

Property ($iv$) can be achieved by imposing that $Z_{\mathbb{R}}(p = 0)$ is not singular at $u^*$. Property ($iii$) and Lemma 1 imply that $O(u^*) \setminus Z_{\mathbb{R}}(p = 0)$ is not a connected set. Since $O(u^*)$ is an open ball, we deduce that $O(u^*) \cap Z_{\mathbb{R}}(p = 0)$ must be $d-1$ dimensional. Then Property ($iv$) implies $O(u^*) \cap Z_{\mathbb{R}}(p = 0) = \overline{O(u^*) \cap C_1} \cap \overline{O(u^*) \cap C_2}$. and $\overline{O(u^*) \cap C_i} = \overline{O(u^*)} \cap \overline{C_i}$ for $i = 1, 2$. Therefore, we have: $O(u^*) \cap Z_{\mathbb{R}}(p = 0) \subseteq (\partial C_1 \cap \partial C_2) \cap Z_{\mathbb{R}}(p = 0)$. Hence $(\partial C_1 \cap \partial C_2) \cap Z_{\mathbb{R}}(p = 0)$ is also of dimension $d - 1$.

"$\Leftarrow$". Suppose there exist two connected components $C_1, C_2$ of $Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f \neq 0)$ satisfying the above (1) and (2) in the theorem statement. Let $\mathcal{H}$ be a hypersurface with $\mathcal{H} \not\supseteq Z_{\mathbb{R}}(p=0)$. Since the dimension of $(\mathcal{H}\cup Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})\setminus\{p\}} f = 0)) \cap Z_{\mathbb{R}}(p = 0)$ cannot be $d-1$, the set $S$ defined by

$$(\partial C_1 \cap \partial C_2 \cap Z_{\mathbb{R}}(p=0)) \setminus \left(\mathcal{H} \cup Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})\setminus\{p\}} f = 0)\right)$$

is not empty. Let $u^*$ be a point of $S$. Any open ball $O(u^*)$ centered at $u^*$ contains at least one point $\alpha_1$ (resp. $\alpha_2$) from $C_1$ (resp. $C_2$). From (2) we deduce $\#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_1)) \neq \#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_2))$. That is, $Z_{\mathbb{R}}(p = 0)$ is an effective boundary according to Definition 2. $\square$

The above theorem suggests some practical ways to compute the effective border polynomial factors, using the adjacency information and sample points of the connected components of $Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} \neq 0)$.

Corollary 1. *If two points $\alpha_1, \alpha_2 \in Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f \neq 0)$ are in the same connected component of the complement of $\mathcal{E}(\mathfrak{S})$, then $\#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_1)) = \#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_2))$ holds.*

Proof. Let $C$ be a connected component of the complement of $\mathcal{E}(\mathfrak{S})$. Observe that $C \setminus Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f = 0)$ is the union of a finite set $\mathcal{O}$ of connected components of $Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f \neq 0)$. Indeed, $\mathcal{E}(\mathfrak{S}) \subseteq Z_{\mathbb{R}}(\prod_{f\in \mathrm{BP}(\mathfrak{S})} f = 0)$ holds by Lemma 5. If $\mathcal{O}$ contains only one element, the conclusion is trivially true.

Assume from now that $\mathcal{O}$ contains more than one elements. We can number the elements of $\mathcal{O}$ such that for any two elements with consecutive numbers, say $C_i, C_{i+1}$, the dimension of $\partial C_i \cap \partial C_{i+1}$ is $d-1$. Proceeding by contradiction, assume that the conclusion of the corollary is false. Thus, there exist two consecutive elements of $\mathcal{O}$, say $C_i, C_{i+1}$, and two points $\alpha_i \in C_i$, $\alpha_{i+1} \in C_{i+1}$, such that $\#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_i)) \neq \#Z_{\mathbb{R}}(\mathfrak{S}(\alpha_{i+1}))$ holds. Since $C$ lies in the complement of $\mathcal{E}(\mathfrak{S})$, there exists a non-effective border polynomial factor $p$ such that $\partial C_i \cap \partial C_{i+1} \subseteq Z_{\mathbb{R}}(p = 0)$ holds. However, this also implies that $p$ is an effective border polynomial factor by Theorem 2, which is a contradiction. $\square$

Given a pre-regular system $R = [B_{\neq}, T, P_>]$, we can rely on $\mathrm{ebf}([T, P_>])$ to compute an FPS of $R$ rather than $B$ (which is often much larger than $\mathrm{ebf}([T, P_>])$).

Theorem 3. *Given a pre-regular system $R = [B_{\neq}, T, P_>]$, let $D = oaf(ebf([T, P_>]))$. Then $D \cup B$ is an FPS of $R$.*

Proof. By Theorem 3 in [3] on the property of the oaf operator, each realizable strict sign conditions on $D$ defines a connected components of $Z_{\mathbb{R}}(\prod_{f\in D} f \neq 0)$. Therefore, for any two points $\alpha_1, \alpha_2 \in Z_{\mathbb{R}}(\prod_{f\in B} f \neq 0)$ satisfying the same realizable sign condition of $D$, we have $\#Z_{\mathbb{R}}(R(\alpha_1)) = \#Z_{\mathbb{R}}(R(\alpha_2))$ by Corollary 1. Hence $D \cup B$ is an FPS of $R$ by definition. $\square$

Theorem 4. *Given two SFSASes $R_1 = [T_1, P_>]$ and $R_2 = [T_2, P_>]$ with $sat(T_1) = sat(T_2)$, then $\mathcal{E}(R_1) = \mathcal{E}(R_2)$ holds.*

Proof. Let $B = \mathrm{BP}(R_1) \cup \mathrm{BP}(R_2)$ and let $\mathbf{h}$ be an effective boundary of $R_1$ defined by a polynomial $p$ in $\mathrm{BP}(R_1)$. Let $\mathcal{H} \not\supseteq \mathbf{h}$ be any hypersurface and denote by $S$ the set $\mathbf{h} \setminus \left(Z_{\mathbb{R}}(\prod_{f\in B\setminus\{p\}} f = 0) \cup \mathcal{H}\right)$. Observe that $S$ is not empty.

We can find a point $u^* \in S$ satisfying: for any open ball $O(u^*)$ centered at $u^*$, there exist two points $\alpha_1, \alpha_2 \in O(u^*)\setminus \mathbf{h}$, such that $\#Z_{\mathbb{R}}(R_1(\alpha_1)) \neq \#Z_{\mathbb{R}}(R_1(\alpha_2))$ holds.

Since $\mathrm{sat}(T_1) = \mathrm{sat}(T_2)$ holds, for all $u \in Z_{\mathbb{R}}(\prod_{f\in B} f \neq 0)$, we have $Z_{\mathbb{R}}(R_1(u)) = Z_{\mathbb{R}}(R_2(u))$. When an open ball $O$ at $u^*$ is sufficiently small, $O \cap Z_{\mathbb{R}}(\prod_{f\in B\setminus p} f = 0) = \emptyset$ holds. Therefore, $Z_{\mathbb{R}}(R_1(u)) = Z_{\mathbb{R}}(R_2(u))$ holds for any $u \in O \setminus \mathbf{h}$.

From the above arguments and Definition 2, we deduce that $\mathbf{h}$ is also an effective boundary of $R_2$. This shows $\mathcal{E}(R_1) \subseteq \mathcal{E}(R_2)$. Similarly $\mathcal{E}(R_1) \supseteq \mathcal{E}(R_2)$ can be proved. $\square$

Let $R = [T, P_>]$, $R_i = [T_i, P_>]$ $(i = 1, 2)$ be three SFSASes with $\mathrm{sat}(T) = \mathrm{sat}(T_1) \cap \mathrm{sat}(T_2)$. One can prove that $\mathcal{E}(R) \subseteq \mathcal{E}(R_1) \cup \mathcal{E}(R_2)$ holds. Moreover, one can prove that $\mathrm{ebf}(R_1) \cap \mathrm{ebf}(R_2) = \emptyset$ implies $\mathcal{E}(R) = \mathcal{E}(R_1) \cup \mathcal{E}(R_2)$. These results and their proofs will appear in an extended version of this article.

**Table 1** The timing and number of components in the output of different algorithms

| sys | RTD$\|_{re}-$relax | | | | RTD$\|_{re}+$relax | | | | RTD$\|_{inc}-$relax | | | | RTD$\|_{inc}+$relax | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | RTD | | RR | | RTD | | RR | | RTD | | RR | | RTD | | RR | |
| 8-3-config-Li | 418.6 | 203 | 1727 | 45 | 410.6 | 203 | 1688 | 45 | 30.5 | 47 | 129.5 | 47 | 30.4 | 47 | 129.1 | 47 |
| dgp6 | 65.17 | 20 | 17.44 | 15 | 64.37 | 20 | 17.59 | 15 | 47.73 | 19 | 22.38 | 17 | 47.43 | 19 | 22.24 | 17 |
| Leykin-1 | 4.9 | 28 | 20.1 | 18 | 4.9 | 28 | 20.8 | 18 | 6.5 | 19 | 13.9 | 19 | 6.5 | 19 | 14.0 | 19 |
| L | 14.9 | 69 | 94.3 | 20 | 14.9 | 69 | 96.9 | 20 | 2.6 | 19 | 11.7 | 19 | 2.6 | 19 | 11.7 | 19 |
| Mehta0 | 1294 | 21 | NA | NA | 713.6 | 15 | NA | NA | 1558 | 20 | NA | NA | 998.9 | 15 | NA | NA |
| EdgeSquare | 247.7 | 116 | NA | NA | 725.3 | 91 | NA | NA | 116.8 | 43 | NA | NA | 629.4 | 33 | NA | NA |
| Enneper | 6.1 | 18 | 12.4 | 13 | 5.4 | 13 | 11.0 | 12 | 4.9 | 17 | 12.7 | 12 | 4.9 | 12 | 10.1 | 11 |
| IBVP | 14.1 | 8 | NA | NA | 16.8 | 4 | NA | NA | 2.5 | 8 | NA | NA | 7.6 | 4 | NA | NA |
| MPV89 | 2.7 | 6 | 84.1 | 6 | 2.4 | 5 | 53.1 | 5 | 2.1 | 7 | 73.4 | 6 | 2.1 | 6 | 53.4 | 5 |
| SEIT | NA | NA | NA | NA | 1411 | 1 | 0.00 | 1 | NA | NA | NA | NA | NA | NA | NA | NA |
| Solotareff-4b | 3223 | 3 | 229.0 | 3 | 3222 | 3 | 228.4 | 3 | 3424 | 3 | 230.0 | 3 | 3424 | 3 | 228.4 | 3 |
| Xia | 223.7 | 12 | NA | NA | 224.8 | 10 | NA | NA | 21.4 | 9 | NA | NA | 20.5 | 8 | NA | NA |
| Lanconelli | 1.1 | 7 | 2.4 | 6 | 1.1 | 7 | 2.4 | 6 | 1.0 | 7 | 2.2 | 6 | 1.0 | 7 | 2.2 | 6 |
| MacLane | 17.4 | 79 | 240.5 | 28 | 17.3 | 79 | 239.5 | 28 | 5.8 | 27 | 35.8 | 27 | 5.8 | 27 | 35.6 | 27 |
| MontesS12 | 197.8 | 163 | 346.5 | 62 | 197.4 | 163 | 344.7 | 62 | 49.9 | 85 | 413.9 | 61 | 49.7 | 85 | 433.8 | 61 |
| MontesS14 | 3.4 | 23 | 14.1 | 13 | 3.4 | 23 | 14.1 | 13 | 2.8 | 15 | 11.0 | 13 | 2.9 | 15 | 11.1 | 13 |
| Pappus | 750.5 | 409 | NA | NA | 748.2 | 409 | NA | NA | 29.1 | 119 | 1127.6 | 119 | 29.0 | 119 | 1125 | 119 |
| Wang168 | 7.0 | 16 | 8.4 | 10 | 7.1 | 16 | 8.4 | 10 | 3.4 | 11 | 5.6 | 10 | 3.5 | 11 | 5.6 | 10 |
| xia-issac07-1 | 2.7 | 13 | NA | NA | 4.4 | 11 | NA | NA | 2.2 | 12 | NA | NA | 4.2 | 10 | NA | NA |

# 5. RELAXATION TECHNIQUES

Given a pre-regular semi-algebraic system $R = [B_{\neq}, T, P_{>}]$ as input, the algorithm GenerateRegularSas in [3] generates an FPS $\mathbf{F} \supseteq B$ of $R$ and a regular semi-algebraic system $[\mathcal{Q}, T, P_{>}]$ such that $Z_{\mathbb{R}}(\mathbf{F}_{\neq}) \cap Z_{\mathbb{R}}(R) = Z_{\mathbb{R}}([\mathcal{Q}, T, P_{>}])$. Denote by $B^*$ the polynomial set $\mathrm{oaf}(B)$, which is proved to be an FPS of $R$ by Theorem 4 in [3]. The notations $R$, $B$, $T$, $P$, $\mathbf{F}$, $B^*$ will be fixed in this section.

Note that if $\mathbf{F} = B$, then we have $Z_{\mathbb{R}}(R) = Z_{\mathbb{R}}([\mathcal{Q}, T, P_{>}]$; otherwise, for each $b \in \mathbf{F} \setminus B$, we have to compute recursively a triangular decomposition of $[T \cup \{b\}, \{\ \}, P_{>}, B_{\neq}]$ to obtain a complete triangular decomposition of $R$. There are two directions to reduce the number of such recursive calls, which will help to produce output with less redundancy:

- $(i)$ minimize the number of polynomials in $\mathbf{F}$, where the effective boundary theory in Section 4 can help;
- $(ii)$ relax some polynomials in $\mathbf{F} \setminus B$ such that there is no need to make recursive calls for those polynomials, which we will discuss in this section.

The following notions of *sign condition* and *relaxation* appear in [1] in a more general setting. We adapt them to our study of regular semi-algebraic systems. Throughout this subsection, we consider a finite set $F \subset \mathbb{Q}[\mathbf{x}]$ of coprime polynomials.

Definition 4. *We call any semi-algebraic system of the form*

$$\bigwedge_{f \in F} f \ \sigma_f \ 0, \qquad (1)$$

*where $\sigma_f$ is one of $>, <, \geq, \leq$, a* sign condition *on $F$, or an $F$*-sign condition. *An $F$-sign condition is called* strict *if every $\sigma_f$ involved belongs to $\{>, <\}$. An $F$-sign condition $C$ is called* realizable *if $C$ has at least one real solution.*

Definition 5 (Relaxation of sign condition). *For an $F$-sign condition $C$ given as in (1) and a subset $E$ of $F$, the* (partial) relaxation *of $C$* w.r.t. *$E$, denoted by $\widetilde{C}^E$, is defined by*

$$\bigwedge_{p \in F} p \ \widetilde{\sigma_p} \ 0 \quad \text{where} \quad \widetilde{\sigma_p} = \begin{cases} \leq, & \text{if } p \in E \text{ and } \sigma_p \text{ is } <, \\ \geq, & \text{if } p \in E \text{ and } \sigma_p \text{ is } >, \\ \sigma_p, & \text{otherwise.} \end{cases}$$

*Let $Q = \vee_{i=1}^{e} C_i$ be a quantifier free formula, where each $C_i$ is an $F$-sign condition. The* relaxation *of $Q$ w.r.t. $E$, denoted by $\widetilde{Q}^E$, is defined as $\vee_{i=1}^{e} \widetilde{C_i}^E$. If $E$ contains only one polynomial $h$, then we also denote the relaxation by $\widetilde{Q}^h$.*

Let us fix the following notations as well in the rest of this section. Let $D \subseteq \mathbb{Q}[\mathbf{u}]$ such that $B \subseteq D \subseteq \mathbf{F}$. Let $Q_i$ $(i = 0, 1)$ be a quantifier free formula in disjunctive form such that each conjunction clause $C$ of it is in the following form: $C = \wedge_{f \in \mathbf{F}} f \ \sigma_f \ 0$, where $\sigma_f \in \{>, <\}$ if $f \in D$ and $\sigma_f \in \{\geq, \leq\}$ if $f \in \mathbf{F} \setminus D$. Moreover, assume that for any $u$ such that $D(u) \neq 0$, $R(u)$ has (resp. has no) real solutions if and only if $Q_1(u)$ (resp. $Q_0(u)$) is true. Let $h$ be a polynomial in $D \setminus B$. Denote by $D^h$ the set $D \setminus \{h\}$. Denote by $\partial_i$ $(i = 0, 1)$ the boundary of the set $Z_{\mathbb{R}}(Q_i)$. Denote by $G_i$ $(i = 0, 1)$ the set $Z_{\mathbb{R}}(\widetilde{Q_i}^h) \cap \overline{Z_{\mathbb{R}}(Q_i)}$. Let $S_i$ $(i = 0, 1)$ be the semi-algebraic set such that $Z_{\mathbb{R}}(\widetilde{Q_i}^h) = G_i \uplus S_i$, where the symbol $\uplus$ denotes disjoint union.

The following Theorem states an criterion for relaxation.

Theorem 5. *The following two statements are equivalent:*

- $(i)$ $Z_{\mathbb{R}}(\widetilde{Q_1}^h) \cap Z_{\mathbb{R}}(\widetilde{Q_0}^h) = \emptyset$,
- $(ii)$ *for any $u \in Z_{\mathbb{R}}(D^h_{\neq})$, $R(u)$ has real solutions if and only if $\widetilde{Q_1}^h(u)$ is true; $R(u)$ has no real solutions if and only if $\widetilde{Q_0}^h(u)$ is true.*

Before providing the proof, we supply several lemmas on the properties of the objects we defined.

Lemma 6. *$Z_{\mathbb{R}}(Q_0)$ and $Z_{\mathbb{R}}(Q_1)$ are both open sets.*

Proof. On one hand, $Z_{\mathbb{R}}(D_{\neq}) = Z_{\mathbb{R}}(Q_0) \uplus Z_{\mathbb{R}}(Q_1)$. On the other hand, there exists a finite set of connected open sets, $\mathcal{O} = \{C_1, \ldots, C_e\}$, such that $Z_{\mathbb{R}}(D_{\neq}) = \cup_{i=1}^{e} C_i$ holds. By Lemma 1, for each $C_i \in \mathcal{O}$, either $C_i \subseteq Z_{\mathbb{R}}(Q_0)$ or $C_i \subseteq Z_{\mathbb{R}}(Q_1)$ holds. Therefore, both $Z_{\mathbb{R}}(Q_0)$ and $Z_{\mathbb{R}}(Q_1)$ are a union of finitely many elements of $\mathcal{O}$ and thus are open. $\square$

Lemma 7. *For any $u \in \overline{Z_{\mathbb{R}}(Q_0)} \cap Z_{\mathbb{R}}(B_{\neq})$, $R(u)$ has no real solutions; for any $u \in \overline{Z_{\mathbb{R}}(Q_1)} \cap Z_{\mathbb{R}}(B_{\neq})$, $R(u)$ has real solutions.*

Proof. Suppose $u$ is in $\overline{Z_{\mathbb{R}}(Q_0)} \cap Z_{\mathbb{R}}(B_{\neq})$. There exists a connected component $C$ of $Z_{\mathbb{R}}(Q_0)$ and a connected component $C'$ of $Z_{\mathbb{R}}(B_{\neq})$ such that $u \in \overline{C} \cap Z_{\mathbb{R}}(B_{\neq}) \subseteq C'$ holds. Since $C \subseteq Z_{\mathbb{R}}(Q_0) \subseteq Z_{\mathbb{R}}(B_{\neq})$, we have $C \subseteq C'$. Since the number of real solutions of $R$ is constant above $C'$ (by Lemma 1) and $R$ has no real solutions above $C$, we conclude that $R(u)$ has no real solutions. The other part of the lemma can be proved similarly. $\square$

Note that $G_i = Z_{\mathbb{R}}(\widetilde{Q_i}^h) \cap \overline{Z_{\mathbb{R}}(Q_i)} \subseteq Z_{\mathbb{R}}(B_{\neq}) \cap \overline{Z_{\mathbb{R}}(Q_i)}$ $(i = 0, 1)$ holds. We have the following proposition as a direct consequence of Lemma 7.

Proposition 1. *For any $u \in G_0$, $R(u)$ has no real solutions; for any $u \in G_1$, $R(u)$ has real solutions.*

Lemma 8. *The following relations hold: $(i)$ $\partial_0 \cup \partial_1 = Z_{\mathbb{R}}(\prod_{f \in D} f)$; $(ii)$ $\partial_0 \cap \partial_1 \subseteq Z_{\mathbb{R}}(\prod_{f \in B} f)$.*

Proof. By Lemma 6, both $Z_{\mathbb{R}}(Q_0)$ and $Z_{\mathbb{R}}(Q_1)$ are open sets. We have $\partial_0 \cup \partial_1 = \partial(Z_{\mathbb{R}}(Q_0) \cup Z_{\mathbb{R}}(Q_1))$, since $Z_{\mathbb{R}}(Q_0) \cap Z_{\mathbb{R}}(Q_1) = \emptyset$ holds. Therefore, we have

$$\begin{aligned} \partial_0 \cup \partial_1 &= \overline{Z_{\mathbb{R}}(Q_0) \cup Z_{\mathbb{R}}(Q_1)} \setminus (Z_{\mathbb{R}}(Q_0) \cup Z_{\mathbb{R}}(Q_1)) \\ &= \overline{Z_{\mathbb{R}}(D_{\neq})} \setminus (Z_{\mathbb{R}}(D_{\neq})) \\ &= Z_{\mathbb{R}}(\textstyle\prod_{f \in D} f). \end{aligned}$$

By Lemma 7, $\overline{Z_{\mathbb{R}}(Q_0)} \cap Z_{\mathbb{R}}(B_{\neq})$ and $\overline{Z_{\mathbb{R}}(Q_1)} \cap Z_{\mathbb{R}}(B_{\neq})$ has no intersection. Therefore $\overline{Z_{\mathbb{R}}(Q_0)} \cap \overline{Z_{\mathbb{R}}(Q_1)} \cap Z_{\mathbb{R}}(B_{\neq}) = \emptyset$. Then the conclusion follows by $\partial_i \subseteq \overline{Z_{\mathbb{R}}(Q_i)}$ $(i = 0, 1)$. $\square$

Lemma 9. *The following relations hold: $(a)$ for $(i = 0, 1)$, $\overline{Z_{\mathbb{R}}(Q_i)} \cap Z_{\mathbb{R}}(D^h_{\neq}) \subseteq Z_{\mathbb{R}}(\widetilde{Q_i}^h)$; $(b)$ $Z_{\mathbb{R}}(\widetilde{Q_0}^h) \cup Z_{\mathbb{R}}(\widetilde{Q_1}^h) = Z_{\mathbb{R}}(D^h_{\neq})$.*

Proof. Since $Z_{\mathbb{R}}(\widetilde{Q_i}^D)$ is a closed set, we have $\overline{Z_{\mathbb{R}}(Q_i)} \subseteq Z_{\mathbb{R}}(\widetilde{Q_i}^D)$. Therefore, we have

$$Z_{\mathbb{R}}(D^h_{\neq}) \cap \overline{Z_{\mathbb{R}}(Q_i)} \subseteq Z_{\mathbb{R}}(D^h_{\neq}) \cap Z_{\mathbb{R}}(\widetilde{Q_i}^D) = Z_{\mathbb{R}}(\widetilde{Q_i}^h).$$

By $(a)$, we have $Z_{\mathbb{R}}(D^h_{\neq}) \cap (\cup_{i=0,1} \overline{Z_{\mathbb{R}}(Q_i)}) \subseteq \cup_{i=0,1} Z_{\mathbb{R}}(\widetilde{Q_i}^h)$, which implies that $Z_{\mathbb{R}}(D^h_{\neq}) \subseteq Z_{\mathbb{R}}(\widetilde{Q_0}^h) \cup Z_{\mathbb{R}}(\widetilde{Q_1}^h)$. And $Z_{\mathbb{R}}(\widetilde{Q_0}^h) \cup Z_{\mathbb{R}}(\widetilde{Q_1}^h) \subseteq Z_{\mathbb{R}}(D^h_{\neq})$ holds since all polynomials in $D^h$ remain strict after relaxing $h$. $\square$

Proposition 2. *For $i = 0, 1$, we have $S_i \subseteq Z_{\mathbb{R}}(h = 0) \cap Z_{\mathbb{R}}(\widetilde{Q_i}^h)$ holds.*

Proof. Recall that $G_i = Z_{\mathbb{R}}(\widetilde{Q_i}^h) \cap \overline{Z_{\mathbb{R}}(Q_i)}$, $Z_{\mathbb{R}}(\widetilde{Q_i}^h) = G_i \uplus S_i$. Therefore, we have $Z_{\mathbb{R}}(Q_i) \subseteq G_i$ and $S_i = Z_{\mathbb{R}}(\widetilde{Q_i}^h) \setminus G_i \subseteq Z_{\mathbb{R}}(\widetilde{Q_i}^h) \setminus Z_{\mathbb{R}}(Q_i) \subseteq Z_{\mathbb{R}}(h = 0)$ hold. Hence, we deduce that $S_i \subseteq Z_{\mathbb{R}}(h = 0) \cap Z_{\mathbb{R}}(\widetilde{Q_i}^h)$ holds. $\square$

Lemma 10. *Both $S_1 \subseteq G_0$ and $S_0 \subseteq G_1$ hold.*

Proof. By Lemma 8, we have $\partial_0 \cup \partial_1 = Z_{\mathbb{R}}(\prod_{f \in D} f = 0)$. Since $h \in D$, we have $Z_{\mathbb{R}}(h = 0) \subseteq \partial_0 \cup \partial_1$, which implies $Z_{\mathbb{R}}(h = 0)$ can be rewriten as

$$Z_{\mathbb{R}}(h = 0) \cap ((\partial_0 \setminus \partial_1) \uplus (\partial_1 \setminus \partial_0) \uplus (\partial_0 \cap \partial_1)).$$

By Lemma 8, we have $\partial_0 \cap \partial_1 \subseteq Z_{\mathbb{R}}(\prod_{f \in B} f = 0)$, which implies that $Z_{\mathbb{R}}(D^h_{\neq}) \cap \partial_0 \cap \partial_1 = \emptyset$. Let $S_h$ be $Z_{\mathbb{R}}(h = 0) \cap Z_{\mathbb{R}}(D^h_{\neq})$. Then $S_h$ can be rewriten as $(Z_{\mathbb{R}}(h = 0) \cap Z_{\mathbb{R}}(D^h_{\neq}) \cap (\partial_0 \setminus \partial_1)) \uplus (Z_{\mathbb{R}}(h = 0) \cap Z_{\mathbb{R}}(D^h_{\neq}) \cap (\partial_1 \setminus \partial_0))$.

Intersecting both sides of relation $(a)$ of Lemma 9 with $\overline{Z_{\mathbb{R}}(Q_i)}$, we obtain $Z_{\mathbb{R}}(D^h_{\neq}) \cap \overline{Z_{\mathbb{R}}(Q_i)} \subseteq G_i$, which implies that $Z_{\mathbb{R}}(D^h_{\neq}) \cap \partial_i \subseteq G_i$. Therefore, $S_h \subseteq G_0 \cup G_1$ holds.

Since $Z_{\mathbb{R}}(\widetilde{Q_i}^h) \subseteq Z_{\mathbb{R}}(D^h_{\neq})$, we have $Z_{\mathbb{R}}(h = 0) \cap Z_{\mathbb{R}}(\widetilde{Q_i}^h) \subseteq S_h$. By Proposition 2, we have $S_i \subseteq Z_{\mathbb{R}}(h = 0) \cap Z_{\mathbb{R}}(\widetilde{Q_i}^h)$. Therefore, $S_i \subseteq S_h$ holds.

We then deduce the conclusion by combining the facts $S_i \cap G_i = \emptyset$, $S_i \subseteq S_h$, and $S_h \subseteq G_0 \cup G_1$. $\square$

Corollary 2. *We have $Z_{\mathbb{R}}(\widetilde{Q_1}^h) \cap Z_{\mathbb{R}}(\widetilde{Q_0}^h) = S_0 \uplus S_1$.*

Proof. We can rewrite $Z_{\mathbb{R}}(\widetilde{Q_1}^h) \cap Z_{\mathbb{R}}(\widetilde{Q_0}^h)$ as the disjoint union $(S_1 \cap G_0) \uplus (S_0 \cap G_1) \uplus (S_1 \cap S_0) \uplus (G_0 \cap G_1)$. By Proposition 1, $G_0 \cap G_1 = \emptyset$. Together with Lemma 10, we have $Z_{\mathbb{R}}(\widetilde{Q_1}^h) \cap Z_{\mathbb{R}}(\widetilde{Q_0}^h) = S_0 \uplus S_1$. $\square$

Next, we complete the proof for Theorem 5.

Proof. By Lemma 9, $Z_{\mathbb{R}}(\widetilde{Q_0}^h) \cup Z_{\mathbb{R}}(\widetilde{Q_1}^h) = Z_{\mathbb{R}}(D^h_{\neq})$.

$(i) \Rightarrow (ii)$. By Corollary 2, we have $S_0 = S_1 = \emptyset$ and $Z_{\mathbb{R}}(\widetilde{Q_i}^h) = G_i$ $(i = 0, 1)$. Then the conclusion follows from Proposition 1.

$(ii) \Rightarrow (i)$. We prove by contradiction. Assume $(i)$ does not hold. There exists $u \in Z_{\mathbb{R}}(D^h_{\neq})$, such that both $\widetilde{Q_0}^h(u)$ and $\widetilde{Q_1}^h(u)$ are true. This is a contradiction to $(ii)$. $\square$

We have the following remarks on relaxation once $(i)$ of Theorem 5 is checked to be true.

- One can verify that, $\widetilde{Q_i}^h$ $(i = 0, 1)$ and $D^h$ have the same configuration as that we assumed on $Q_i$ $(i = 0, 1)$ and $D$. So $Z_{\mathbb{R}}(\widetilde{Q_1}^h)$ is still **open** by Lemma 6.
- If $[Q_1, T, P_>]$ is a regular semi-algebraic system, then so is $[\widetilde{Q_1}^h, T, P_>]$.

# 6. INCREMENTAL DECOMPOSITION

In this section, we present algorithms to compute a full triangular decomposition of a semi-algebraic system in an incremental manner, which serves as a counterpart of the recursive algorithm in our previous paper [3]. Given a semi-algebraic system $\mathfrak{S} := [F, N_{\geq}, P_>, H_{\neq}]$, the incremental decomposition is realized by passing the empty regular chain $\varnothing$ and $\mathfrak{S}$ to Algorithm 1, whose incrementality is mainly due to its subroutine Triangularize, which computes a Lazard triangular decomposition by solving equations one by one [11].

**External algorithms.** We recall the specifications of the algorithms BorderPolynomialSet and GenerateRegularSas, (see [3]), Triangularize, Intersect, RegularOnly (see [11]). BorderPolynomialSet computes the border polynomial set of a regular system, whereas GenerateRegularSas decomposes the zero set of a pre-regular semi-algebraic system as a union of zero sets of regular semi-algebraic systems. Let $p$ be a polynomial, $F$ be a polynomial list, and $T$ be a regular chain. The algorithm $\mathsf{Triangularize}(F, T, \mathrm{mode} = \mathrm{Lazard})$ computes regular chains $T_i$, $i = 1, \ldots, e$, such that $V(F) \cap W(T) \subseteq \cup_{i=1}^{e} W(T_i) \subseteq V(F) \cap \overline{W(T)}$. The algorithm $\mathsf{Intersect}(p, T)$ is equivalent to $\mathsf{Triangularize}(\{p\}, T, \mathrm{mode} = \mathrm{Lazard})$. The algorithm $\mathsf{RegularOnly}(T, F)$ computes regular chains $T_i$, $i = 1, \ldots, e$, s.t. $W(T) \setminus V(\prod_{h \in F} h) = \cup_{i=1}^{e} W(T_i) \setminus V(\prod_{h \in F} h)$ and every polynomial in $F$ is regular modulo $\mathrm{sat}(T_i)$.

The proof of the termination and correctness of the algorithms rely on standard arguments used in the proof of algorithm PCTD in paper [4]. Limited to space, we will not expand the proof here.

**Algorithm 1:** $\mathsf{RealTriangularize}(T, F, N_{\geq}, P_>, H_{\neq})$

**Input**: a regular chain $T$ and a semi-algebraic system $\mathfrak{S} = [F, N_{\geq}, P_>, H_{\neq}]$
**Output**: a set of regular semi-algebraic systems $R_i$, $i = 1 \cdots e$, such that $W_{\mathbb{R}}(T) \cap Z_{\mathbb{R}}(\mathfrak{S}) = \cup_{i=1}^{e} Z_{\mathbb{R}}(R_i)$.

```
𝔗 := Triangularize(F, T, mode = Lazard);
for C ∈ 𝔗 do
    output RealTriangularize(C, N≥, P>, H≠ ∪ init(T)≠);
```

# 7. SET THEORETICAL OPERATIONS

In paper [3], we proved that every semi-algebraic set can be represented by the union of zero sets of finitely many regular semi-algebraic systems. It is natural to ask how to perform set theoretical operations, such as union, intersection, complement and difference of semi-algebraic sets based on such a representation.

Note that each (regular) semi-algebraic system can also be seen as a quantifier free formula. So one can implement the set operations naively based on the algorithm RealTriangularize and logic operations. However, an obvious drawback of such an implementation is that it totally neglects the structure of a regular semi-algebraic system.

Indeed, if the structure of the computed object can be exploited, it is possible to obtain more efficient algorithms. One good example of this is the Difference algorithm, which computes the difference of zero sets of two regular systems, presented in [6]. This algorithm exploits the structure of a regular chain and outperforms the naive implementation by several orders of magnitude.

Apart from the algebraic computations, the idea behind the Difference algorithm of paper [6] is to compute the difference $(A_1 \cap A_2) \setminus (B_1 \cap B_2)$ in the following way:

$$(A_1 \cap B_1) \cap (A_2 \setminus B_2) \bigcup (A_1 \setminus B_1) \cap A_2.$$

Observe that if $A_1 \cap B_1 = \emptyset$, then the difference is $(A_1 \cap A_2)$. Moreover, computing $\cap_{i=1}^{s} A_i \setminus \cap_{i=1}^{t} B_i$ $(s, t \geq 2)$ can be reduced to the above base case.

In this section, we present algorithms (Algorithm 4 and 5) which take advantage of the algorithm Difference (also an algorithm Intersection derived from it) and the idea presented above for computing the intersection and difference of semi-algebraic sets represented by regular semi-algebraic systems.

**Algorithm 2:** $\mathsf{RealTriangularize}(T, N_{\geq}, P_>, H_{\neq})$

**Input**: a regular chain $T$ and a semi-algebraic system $\mathfrak{S} = [\emptyset, N_{\geq}, P_>, H_{\neq}]$
**Output**: a set of regular semi-algebraic systems $R_i$, $i = 1, \ldots, e$, such that $W_{\mathbb{R}}(T) \cap Z_{\mathbb{R}}(\mathfrak{S}) = \cup_{i=1}^{e} Z_{\mathbb{R}}(R_i)$.

```
H := init(T) ∪ H;
𝔗 := {[T, ∅]}; 𝔗' := ∅;
for p ∈ N do
    for [T', N'] ∈ 𝔗 do
        𝔗' := 𝔗' ∪ {[C, N'] | C ∈ Intersect(p, T')};
        𝔗' := 𝔗' ∪ {[T', N' ∪ {p}]}
    𝔗 := 𝔗'; 𝔗' := ∅;
𝔗 := {[T', N' ∪ P, H] | [T', N'] ∈ 𝔗};
while 𝔗 ≠ ∅ do
    let [T', P', H] ∈ 𝔗; 𝔗 := 𝔗 \ {[T', P', H]};
    for C ∈ RegularOnly(T', P' ∪ H) do
        BP := BorderPolynomialSet(C, P' ∪ H);
        (DP, R) = GenerateRegularSas(BP, C, P');
        if R ≠ ∅ then output R;
        for f ∈ DP \ (P' ∪ H) do
            𝔗 := 𝔗 ∪ {[D, P', H] | D ∈ Intersect(f, C)};
```

**Algorithm 3:** $\mathsf{RealTriangularize}(T, \mathcal{Q})$

**Input**: $T$, a regular chain; $\mathcal{Q}$, a quantifier free formula
**Output**: a set of regular semi-algebraic systems $R_i$, $i = 1, \ldots, e$, such that $W_{\mathbb{R}}(T) \cap Z_{\mathbb{R}}(\mathcal{Q}) = \cup_{i=1}^{e} Z_{\mathbb{R}}(R_i)$.

```
for each conjunctive formula F ∧ N≥ ∧ P> ∧ H≠ do
    output RealTriangularize(T, F, N≥, P>, H≠);
```

# 8. EXPERIMENTATION

In this section, we report on the experimental results of the techniques presented in this paper. The systems were tested

**Algorithm 4:** $\mathsf{DifferenceRsas}(R, R')$

**Input**: two regular semi-algebraic systems $R = [\mathcal{Q}, T, P_>]$ and $R' = [\mathcal{Q}', T', P'_>]$
**Output**: a set of regular semi-algebraic systems $R_i$, $i = 1, \ldots, e$, such that $Z_{\mathbb{R}}(R) \setminus Z_{\mathbb{R}}(R') = \cup_{i=1}^{e} Z_{\mathbb{R}}(R_i)$.
**begin**
$\mathcal{Q} := \mathcal{Q} \wedge P_>$;
$\mathcal{Q}' := \mathcal{Q}' \wedge P'_>$;
$\mathfrak{T} := \mathsf{Difference}(T, T')$;
$\mathfrak{T}' := \mathsf{Intersection}(T, T')$;
**if** $\mathfrak{T}' = \emptyset$ **then** return $R$;
**for** $[T^*, H^*] \in \mathfrak{T}'$ **do**
$\mathcal{Q}^* = \mathcal{Q} \setminus \mathcal{Q}' \wedge H^*_{\neq}$;
output $\mathsf{RealTriangularize}(T^*, \mathcal{Q}^*)$
**for** $[T^*, H^*] \in \mathfrak{T}$ **do**
$\mathcal{Q}^* = \mathcal{Q} \wedge H^*_{\neq}$;
output $\mathsf{RealTriangularize}(T^*, \mathcal{Q}^*)$

**Algorithm 5:** $\mathsf{IntersectionRsas}(R, R')$

**Input**: two regular semi-algebraic systems $R = [\mathcal{Q}, T, P_>]$ and $R' = [\mathcal{Q}', T', P'_>]$
**Output**: a set of regular semi-algebraic systems $R_i$, $i = 1, \ldots, e$, such that $Z_{\mathbb{R}}(R) \cap Z_{\mathbb{R}}(R') = \cup_{i=1}^{e} Z_{\mathbb{R}}(R_i)$.
$\mathcal{Q}^* := \mathcal{Q} \wedge P_> \wedge \mathcal{Q}' \wedge P'_>$;
**for** $[T^*, H^*] \in \mathsf{Intersection}(T, T')$ **do**
output $\mathsf{RealTriangularize}(T^*, \mathcal{Q}^* \wedge H^*_{\neq})$

on a machine with Intel Core 2 Quad CPU (2.40GHz) and 3.0Gb total memory. The time-out is set as 3600 seconds. The memory usage is limited to 60% of total memory. $NA$ means the computation does not finish in the resource (time or memory) limit.

In Table 1, RTD denotes RealTriangularize. The subscripts $re$ and $inc$ denote respectively the recursive and incremental implementation of RealTriangularize. The suffixes $+relax$ and $-relax$ denote respectively applying and not applying relaxation techniques. The name RR, short name for RemoveRedundantComponents, is an algorithm, implemented based on the algorithm DifferenceRsas, to remove the redundant components in the output of RTD. For each algorithm, the left column records the time (in seconds) while the right one records the number of components in the output.

Table 1 illustrates the effectiveness of the techniques presented in this paper. For system 8-3-config-Li, $RTD|_{inc}$ greatly outperforms $RTD|_{re}$. Moreover, $RR$ helps reduce the number of the output components of $RTD|_{re}$ from 203 to 45. For system Metha0, with the relaxation technique, both timing and the number of components in the output are reduced. For system SEIT, with the help of relaxation, $RTD|_{re}$ can now solve it within half an hour.

To conclude, the algorithms of [3] can, in practice, be often substantially improved by better analysis of the border polynomials, by relaxation (where allowed) and by the incremental approach. The experimentation shows that the latter can sometimes result in a speed-up by more than 10.

**Acknowledgments.** The authors would like to thank the reviewers for their valuable remarks, the support from MAPLESOFT, MITACS and NSERC of Canada, and the EXACTA project supported by ANR (ANR-09-BLAN-0371-01) and NSFC (60911130369 and 91018012).

# 9. ADDITIONAL AUTHORS

# 10. REFERENCES

[1] S. Basu, R. Pollack, and M-F. Roy. *Algorithms in real algebraic geometry.* Springer-Verlag, 2006.

[2] C. W. Brown. Improved projection for cylindrical algebraic decomposition. *J. Symb. Comput.*, 32(5):447–465, 2001.

[3] C. Chen, J.H. Davenport, J. May, M. Moreno Maza, B. Xia, and R. Xiao. Triangular decomposition of semi-algebraic systems. In *Proc. of ISSAC 2010*, pages 187–194, 2010.

[4] C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, and W. Pan. Comprehensive triangular decomposition. In *Proc. of CASC'07*, volume 4770 of *Lecture Notes in Computer Science*, pages 73–101, 2007.

[5] C. Chen and M. Moreno Maza. Algorithms for Computing Triangular Decompositions of Polynomial Systems In *Proc. of ISSAC 2011*, ACM Press, 2011.

[6] C. Chen, M. Moreno Maza, W. Pan, and Y. Xie. On the verification of polynomial system solvers. *Frontiers of Computer Science in China*, 2(1):55–66, 2008.

[7] G.E. Collins. Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition. In *Proc. of 2nd. GI Conference Automata Theory & Formal Languages*, pages 134–183, 1975.

[8] J.H. Davenport and J. Heintz. Real Quantifier Elimination is Doubly Exponential. *J. Symbolic Comp.*, 5:29–35, 1988.

[9] D. Lazard and F. Rouillier. Solving parametric polynomial systems. *J. Symb. Comput.*, 42(6):636–667, 2007.

[10] S. McCallum and C. W. Brown. On delineability of varieties in cad-based quantifier elimination with two equational constraints. In *Proc. of ISSAC 2009*, pages 71–78, 2010.

[11] M. Moreno Maza. On triangular decompositions of algebraic varieties. Technical Report TR 4/99, NAG Ltd, Oxford, UK, 1999. Presented at MEGA-2000, Bath, England.

[12] A. Strzeboński. Solving systems of strict polynomial inequalities. *J. Symb. Comput.*, 29(3):471–480, 2000.

[13] D. M. Wang. *Elimination Methods.* Springer, New York, 2000.

[14] R. Xiao. *Parametric Polynomial System Solving.* PhD thesis, Peking University, Beijing, 2009.

[15] L. Yang, X. Hou, and B. Xia. A complete algorithm for automated discovering of a class of inequality-type theorems. *Science in China, Series* **F**, 44(6):33–49, 2001.

[16] L. Yang and B. Xia. Real solution classifications of a class of parametric semi-algebraic systems. In *Proc. of the A3L'05*, pages 281–289, 2005.

[17] N. Phisanbut, R.J. Bradford, and J.H. Davenport. Geometry of Branch Cuts. *Communications in Computer Algebra*, 44:132–135, 2010.