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TRAVELING WAVES AND HOMOGENEOUS FRAGMENTATION

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We formulate the notion of the classical Fisher–Kolmogorov–Petrovskii–Piscounov (FKPP) reaction diffusion equation associated with a homogeneous conservative fragmentation process and study its traveling waves. Specifically, we establish existence, uniqueness and asymptotics. In the spirit of classical works such as McKean [*Comm. Pure Appl. Math*. **28** (1975) 323–331] and [*Comm. Pure Appl. Math.* **29** (1976) 553–554], Neveu [In *Seminar on Stochastic Processes* (1988) 223–242 Birkhäuser] and Chauvin [*Ann. Probab*. **19** (1991) 1195–1205], our analysis exposes the relation between traveling waves and certain additive and multiplicative martingales via laws of large numbers which have been previously studied in the context of Crump–Mode–Jagers (CMJ) processes by Nerman [*Z. Wahrsch. Verw. Gebiete* **57** (1981) 365–395] and in the context of fragmentation processes by Bertoin and Martinez [*Adv. in Appl. Probab.* **37** (2005) 553–570] and Harris, Knobloch and Kyprianou [*Ann. Inst. H. Poincaré Probab. Statist.* **46** (2010) 119–134]. The conclusions and methodology presented here appeal to a number of concepts coming from the theory of branching random walks and branching Brownian motion (cf. Harris [*Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999) 503–517] and Biggins and Kyprianou [*Electr. J. Probab.* **10** (2005) 609–631]) showing their mathematical robustness even within the context of fragmentation theory.

1. Introduction and main results.

1.1. *Homogeneous fragmentations and branching random walks*. Fragmentation is a natural phenomena that occurs in a wide range of contexts and at all scales. The stochastic models used to describe this type of process have attracted a lot of attention lately and form a fascinating class of mathematical objects in their own right, which are deeply connected to branching processes, continuum random trees and branching random walks. A good introduction to the study of fragmentation (and coalescence) is [6] which also contains many further references.

In the present work, we intend to explore and make use of the connection between random fragmentation processes, branching random walk (BRW) and

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branching Brownian motion (BBM). More precisely we define the notion of the fragmentation Fisher–Kolmogorov–Petrovskii–Piscounov (FKPP) equation and study the solutions of this equation.

Let us start by explaining the connection between fragmentation and BRW in the simple framework of *finite-activity* conservative fragmentations. In this context everything can be defined and constructed *by hand*. More general constructions will follow. Let $\nu(\cdot)$ be a finite measure on $\nabla_1 = \{s_1 \geq s_2 \geq \cdots \geq 0, \sum_i s_i = 1\}$ with total mass $\nu(\nabla_1) = \gamma$. The homogeneous mass-fragmentation process with dislocation measure ν is a ∇_1-valued Markov process $X := (X(t), t \geq 0)$, where $X(t) = (X_1(t), X_2(t), \ldots)$, which evolves as follows: $X(0) = (1, 0, \ldots)$; this initial fragment then waits an exponential time T_1 with parameter γ after which it splits according to the distribution $X(T_1) \sim \gamma^{-1}\nu(\cdot)$. Each of these pieces then starts to fragment independently of the others with the same law as the original object. That is, each fragment $X_i(T_1)$ waits an independent exponential-γ time after which it splits into $(X_i(T_1)s_1, X_i(T_1)s_2, \ldots)$ where $s = (s_1, s_2, \ldots) \sim \gamma^{-1}\nu(\cdot)$ and so on. When a fragment splits, we need to relabel all fragments since their relative ranks have changed.

This process can be seen as a continuous-time BRW. More precisely, if we let $Z(t) = (-\log X_1(t), -\log X_2(t), \ldots)$, then $Z(t)$ evolves according to the following dynamic. Suppose $Z(t) = (z_1, z_2, \ldots)$, then each individual in the population behaves independently, waits an independent and exponentially distributed length of time with parameter γ and then branches into offspring which are situated at distances $(-\log s_1, -\log s_2, \ldots)$ relative to their parent's position where, as before, $s \in \nabla_1$ has distribution $\gamma^{-1}\nu(\cdot)$. Figure 1 shows an example where $\nu(\cdot) = \delta_{\{1/3,1/3,1/3,0,\ldots\}} + \delta_{\{1/2,1/2,0,\ldots\}}$.

This is not, however, the whole story. In particular, it is possible to define homogenous fragmentation processes, also denoted X, such that the dislocation measure ν is infinite (but sigma-finite). In this case, fragmentation happens continuously in the sense that there is no first splitting event and along any branch of the fragmentation tree the branching points are dense in any finite interval of time.

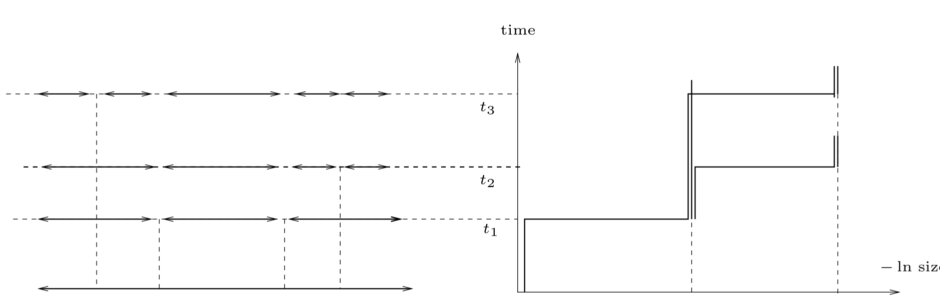

FIG. 1. *The initial fragment* $(0, 1)$ *splits into three equal parts* $(0, 1/3)$, $(1/3, 2/3)$, $(2/3, 1)$ *at time* t_1. *Then at* t_2, $(2/3, 1)$ *splits into two halves and at time* t_3 *the same thing happens to* $(0, 1/3)$.

Moreover, it is still the case that the Markov and fragmentation properties hold. In this more general setting the latter two properties are described more succinctly as follows. Given that $X(t) = (s_1, s_2, \dots) \in \nabla_1$, where $t \geq 0$, then for $u > 0$, $X(t+u)$ has the same law as the variable obtained by ranking in decreasing order the sequences $X^{(1)}(u), X^{(2)}(u), \dots$ where the latter are independent, random mass partitions with values in ∇_1 having the same distribution as $X(u)$ but scaled in size by the factors under $s_1, s_2, \dots$, respectively.

The construction of such processes, known as homogeneous fragmentation processes, requires some care and was essentially carried out by Bertoin [4, 6] (see also [1]). We defer a brief overview of the general construction to the next section. It is enough here to note that $\nu(ds)$ is the "rate" at which given fragments split into pieces whose relative sizes are given by $s = (s_1, s_2, \dots)$. This measure must verify the integrability condition $\int_{\nabla_1}(1 - s_1)\nu(ds) < \infty$. Some information about ν is captured by the function

$$\Phi(q) := \int_{\nabla_1} \left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right) \nu(ds), \qquad q > \underline{p}, \tag{1.1}$$

where

$$\underline{p} := \inf\left\{p \in \mathbb{R} : \int_{\nabla_1} \sum_{i=2}^{\infty} s_i^{p+1} \nu(ds) < \infty\right\} \leq 0.$$

From now on, we always assume that $\underline{p} < 0$. As we shall reveal in more detail later, the function $p \mapsto \Phi(p)$ turns out to be the Laplace exponent of a natural subordinator associated to the fragmentation. Hence, Φ is strictly increasing, concave and smooth such that $\Phi(0) = 0$. The equation

$$(p+1)\Phi'(p) = \Phi(p)$$

on $p > \underline{p}$ is known to have a unique solution in $(0, \infty)$ which we shall denote by $\bar{p}$; cf. [5]. Moreover, $(p+1)\Phi'(p) - \Phi(p) > 0$ when $p \in (\underline{p}, \bar{p})$. This implies that the function

$$c_p := \frac{\Phi(p)}{p+1} \tag{1.2}$$

reaches its unique maximum on $(\underline{p}, \infty)$ at $\bar{p}$ and this maximum is equal to $\Phi'(\bar{p})$.

1.2. *The fragmentation FKPP equation and traveling waves.* The main aim of this paper is to formulate the analogue of the Fisher–Kolmogorov–Petrovskii–Piscounov (FKPP) equation for fragmentation processes and thereby to analyze the existence, uniqueness and asymptotics of its traveling waves.

Consider a homogeneous fragmentation process Π with dislocation measure ν. The equation

$$-c\psi'(x) + \int_{\nabla_1} \left\{\prod_i \psi(x - \log s_i) - \psi(x)\right\} \nu(ds) = 0, \qquad x \in \mathbb{R}, \tag{1.3}$$

is the *fragmentation traveling wave* equation with wave speed $c \in \mathbb{R}$. Equation (1.3) is the analogue in the fragmentation context of the classical traveling wave equation associated with the FKPP equation. This is discussed in greater detail in Section 2 as well as existing results of the same flavor concerning BBM and BRWs. However, let us momentarily note the connection of (1.3) with a currently open problem and hence, the motivation for the current work.

A classical result, due to Bramson [12], establishes that the growth, as a function of time t, of the right-most particle in a one-dimensional, unit rate, binary splitting BBM is $\sqrt{2}t - 3 \cdot 2^{-3/2} \log t + O(1)$. The precise quantification of this result is done through a particular traveling wave solution of the FKPP equation. On account of the technical similarities between BBM and fragmentation processes, there are many reasons to believe that a similar result should also hold in the latter setting. Indeed, it is already known that the largest block at time $t > 0$, $X_1(t)$, satisfies a strong law of large numbers in the sense that

$$\lim_{t \uparrow \infty} \frac{-\log X_1(t)}{t} = c_{\bar{p}},$$

almost surely. Therefore, a natural conjecture, motivated by Bramson's result, is that there exists a deterministic function of time $(\gamma(t) : t \geq 0)$ such that

$$\mathbb{P}\big(-\log X_1(t) - \gamma(t) \geq x\big) \to \psi_{\bar{p}}(x),$$

where $\psi_{\bar{p}}(x)$ is a solution to (1.3) with $c = c_{\bar{p}}$ and moreover

$$\gamma(t) = c_{\bar{p}} t - \theta \log t + O(1)$$

for some constant θ.

We do not address the above conjecture in this paper. Instead, we shall study the existence, uniqueness and asymptotics of the solution of this *fragmentation traveling wave* equation. Below we introduce our main result in this direction. We first introduce two classes of functions.

DEFINITION 1. The class of functions $\psi \in C^1(\mathbb{R})$ such that $\psi(-\infty) = 0$ and $\psi(\infty) = 1$ and ψ is monotone increasing is denoted by $\mathcal{T}_1$. For each $p \in (\underline{p}, \bar{p}]$ we further define $\mathcal{T}_2(p) \subset \mathcal{T}_1$ as the set of $\psi_p \in \mathcal{T}_1$ such that $L_p(x) := e^{(p+1)x}(1 - \psi_p(x))$ is monotone increasing.

Let X be a homogenous mass fragmentation process as in the previous section and let $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ where $\mathcal{F}_t := \sigma\{X(u), u \leq t\}$ is its natural filtration. Our main result follows.

THEOREM 1. *Fix $p \in (\underline{p}, \bar{p}]$ and suppose that $\psi_p : \mathbb{R} \mapsto (0, 1]$ belongs to $\mathcal{T}_2(p)$. Then*

$$M(t, p, x) := \prod_i \psi_p\big(x - \log X(t) - c_p t\big), \qquad t \geq 0, \tag{1.4}$$

is a $\mathbb{F}$-martingale if and only if ψ_p solves (1.3) *with $c = c_p$. Furthermore, up to an additive translation, there is only one such function $\psi_p \in \mathcal{T}_2(p)$ which is given by*

$$\psi_p(x) = \mathbb{E}\big(\exp\{-e^{-(p+1)x}\Delta_p\}\big), \tag{1.5}$$

where $\Delta_p > 0$ a.s. is an $\mathcal{F}_\infty$-measurable martingale limit (*see Theorem* 2 *and Definition* 4 *for more details*).

Note that, $M(t, p, x)$ is a martingale, then it is necessarily uniformly integrable since it is bounded in [0, 1] and, therefore, converges in L^1 to its limit, which we denote by $M(\infty, p, x)$.

2. Discussion.

2.1. *Homogeneous fragmentations.* We start by stating the definition of and stating some results concerning homogeneous fragmentations.

The construction and manipulation of general homogeneous fragmentations are best carried out in the framework of partition valued fragmentations. Let $\mathcal{P} = \{\text{partitions of } \mathbb{N}\}$. An element π of $\mathcal{P}$ can be identified with an infinite collection of blocks (where a block is just a subset of $\mathbb{N}$ and can be the empty set), $\pi = (B_1, B_2, \dots)$ where $\bigcup_i B_i = \mathbb{N}$, $B_i \cap B_j = \varnothing$ when $i \neq j$ and the labeling corresponds to the order of the least element, that is, if w_i is the least element of B_i (with the convention $\min\varnothing = \infty$), then $i \leq j \Leftrightarrow w_i \leq w_j$. The reason for such a choice is that we can discretize the processes by looking at their restrictions to $[n] := \{1, \dots, n\}$ (if $\pi \in \mathcal{P}$, we denote by $\pi_{|[n]}$ the natural partition it induces on $[n]$). Roughly speaking, a homogeneous fragmentation is a $\mathcal{P}$-valued process $[\Pi(t), t \geq 0]$ such that blocks split independently of each other and with the same intensity. Given a subset $A = \{a_1, a_2, \dots\}$ of $\mathbb{N}$ and a partition $\pi = (B_1, B_2, \dots) \in \mathcal{P}$, we formally define the *splitting of B by π* to be the partition of B (i.e., a collection of disjoint subsets whose union is B) defined by the equivalence relation $a_i \sim a_j$ if and only if i and j are in the same block of π.

DEFINITION 2. Let $\Pi = (\Pi(t), t \geq 0)$ be a $\mathcal{P}$-valued Markov process with càdlàg[1] sample paths. Π is called a homogeneous fragmentation if its semi-group has the following, so-called, *fragmentation property*: For every $t, t' \geq 0$ the conditional distribution of $\Pi(t + t')$ given $\Pi(t) = \pi$ is that of the collection of blocks one obtains by splitting the blocks π_i, $i = 1, \dots,$ of π by an i.i.d. sequence $\Pi^{(i)}(t')$ of exchangeable random partitions whose distribution only depends on t'. We also impose the condition that $\Pi(0)$ is the trivial partition with a single block made up of the whole set $\mathbb{N}$ (see [6] for further discussion).

[1]Continuity is understood with respect to the following metric. The distance between two partitions, π and π', in $\mathcal{P}$, is defined to be $2^{-n(\pi,\pi')}$ where $n(\pi, \pi')$ is the largest integer such that $\pi_{|[n]} = \pi'_{|[n]}$.

Given $\pi = (\pi_1, \pi_2, \dots) \in \mathcal{P}$, we say that it has asymptotic frequencies if for each i

$$|\pi_i| = \lim_{n\to\infty} \frac{\#\pi_i \cap [n]}{n}$$

exists. We write $|\pi| = (|\pi_i|, i \in \mathbb{N})$ for the decreasing rearrangement of the frequencies of the blocks. It is known that if Π is a homogeneous fragmentation, then almost surely, for all $t \ge 0$, $\Pi(t)$ has asymptotic frequencies. The process $[|\Pi(t)|, t \ge 0]$ is called a mass fragmentation and coincides with the process described in the opening section. One can define directly mass fragmentations (i.e., Markov processes with state space ∇_1 such that each fragment splits independently with the same rate) but all such processes can be seen as the frequency process of an underlying integer partition fragmentation.

Given the "stationary and independent increments" flavor of the definition of a fragmentation process, it is no surprise that there is a Lévy–Khintchin type description of the law of these processes. Bertoin [6] shows that the distribution of a homogeneous fragmentation Π is completely characterized by a sigma-finite measure ν on $\nabla_1^{(-)} := \{s_1 \ge s_2 \ge \cdots \ge 0, \sum_i s_i \le 1\}$ (*the disclocation measure*) which satisfies

$$\int_{\nabla_1^{(-)}} (1 - s_1)\nu(ds) < \infty \tag{2.1}$$

and a parameter $\mathtt{c} \ge 0$ (*the erosion rate*).

The meaning of the dislocation measure ν is best understood through a Poissonian construction of the process Π which is given in [4]. Roughly speaking, for each label $i \in \mathbb{N}$ we have an independent Poisson point process on $\mathbb{R}_+ \times \nabla_1^{(-)}$ with intensity $dt \otimes \nu(ds)$. If (t_k, s_k) is an atom of the point process with label i, then at time t_k the block $B_i(t_k-)$ of $\Pi(t_k-)$ is split into fragments of relative size given by s_k (or more precisely, it is split according to Kingman's paintbox partition directed by s_k). To make the construction rigorous, one needs to show that the point processes can be used to construct a compatible family of Markov chains, $(\Pi^{(n)}(t), t \ge 0)$, each of which lives on $\mathcal{P}_n$, the space of partitions of $\{1, \dots, n\}$. Hence, $\nu(ds)$ is the rate at which blocks independently fragment into subfragments of relative size s.

The role of the erosion term $\mathtt{c}$ is easier to explain in terms of mass fragmentations. Indeed, if Π is a $(\nu, 0)$-fragmentation, then the process

$$e^{-\mathtt{c}t}|\Pi(t)| = (e^{-\mathtt{c}t}|\Pi_1(t)|, e^{-\mathtt{c}t}|\Pi_2(t)|, \dots), \qquad t \ge 0,$$

is a $(\nu, \mathtt{c})$-mass fragmentation. The erosion is essentially a deterministic phenomenon. The dislocation measure ν thus plays the same role as in our introductory example where ν was finite.

In the present work we will always suppose that the dislocation measure ν is *conservative*, that is, $\mathrm{supp}(\nu) \subseteq \nabla_1$. Moreover, we assume there is *no erosion*, namely $\mathtt{c} = 0$.

Recall the definition

$$\Phi(q) = \int_{\nabla_1} \left(1 - \sum_{i=1}^{\infty} s_i^{q+1}\right) \nu(ds), \qquad q > \underline{p},$$

and that the function $p \mapsto \Phi(p)$ is the Laplace exponent of a pure jump subordinator associated to the fragmentation. This subordinator is precisely the process $t \mapsto -\log|\Pi_1(t)|$. It is also not difficult to show that the associated Lévy measure is given by

$$m(dx) := e^{-x} \sum_{i=1}^{\infty} \nu(-\log s_i \in dx). \tag{2.2}$$

2.2. *Branching processes, product martingales and the FKPP equation.* Our main theorem follows in the spirit of earlier results which concern BBM or BRWs. We discuss these here and we explain why equation (1.3) is the analogue of the FKPP traveling wave equation for fragmentations.

The classical FKPP equation, in its simplest form, takes the form of the nonlinear parabolic differential equation

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} u(x,t) + u(x,t)^2 - u(x,t). \tag{2.3}$$

It is one of the simplest nonlinear partial differential equation which admits a traveling wave solution of the form $u(t,x) = \psi(x + ct)$ where $\psi : \mathbb{R} \mapsto [0,1]$ and c is the speed of the traveling wave. It is a classical result that a traveling wave solution to (2.3) exists with speed c if and only if $|c| \geq \sqrt{2}$, in which case it solves

$$\begin{aligned} &\tfrac{1}{2}\psi''(x) - c\psi'(x) + \psi^2(x) - \psi(x) = 0, \\ &\qquad\qquad \psi(-\infty) = 0, \qquad \psi(\infty) = 1. \end{aligned} \tag{2.4}$$

There are many different mathematical contexts in which the FKPP equation appears, not least of all within the original setting of gene concentrations; cf. [16, 23]. However, McKean [32], Ikeda, Nagasawa and Watanabe [19–21], Neveu [35] and Chauvin [13] all show that the FKPP equation has a natural relationship with one-dimensional dyadic BBM. This process consists of an individual particle initially positioned at zero that executes a Brownian motion until a unit-mean, independent and exponentially distributed time at which point it divides into two particles. Each of these particles behaves independently of one another and their parent and undergoes the same life-cycle as the initial ancestor but space–time shifted to its point of creation. The process thus propagates via the obvious iteration of this procedure.

Let $(V_1(t), V_2(t), \ldots, V_{N(t)})$ denote the positions of the particles at time t in the BBM. Neveu [35] (see also [13], Theorem 3.1) shows that $\psi : \mathbb{R} \mapsto [0,1]$ is a solution of (2.4) with speed $|c| \geq \sqrt{2}$ if and only if

$$M_t = \prod_{i=1}^{N(t)} \psi\big(x + V_i(t) - ct\big) \tag{2.5}$$

is a martingale for all $x \in \mathbb{R}$.

A natural extension of this model is to replace the BBM by a continuous-time BRW. This process can be described as follows: A particle initially positioned at the origin lives for a unit mean, exponentially distributed length of time and upon dying it scatters a random number of offsprings in space relative to its own position according to the point process whose atoms are $\{\zeta_i, i = 1, \dots\}$. Each of these particles then iterates this same procedure, independently from each other and from the past, starting from their new positions. As stated above, let $(V_1(t), V_2(t), \dots, V_{N(t)})$ denote the positions of the particles at time t.

This was, for instance, studied by Kyprianou in [25] where it was shown that in this context the analogue of (2.4) is

$$\begin{gathered} -c\psi'(x) + \left(E\left[\prod_i \psi(\zeta_i + x)\right] - \psi(x)\right) = 0, \\ \psi(-\infty) = 0, \qquad \psi(\infty) = 1 \end{gathered} \tag{2.6}$$

and that

$$M_t = \prod_{i=1}^{N(t)} \psi\big(x + V_i(t) - ct\big) \tag{2.7}$$

is a martingale for all $x \in \mathbb{R}$ if and only if ψ is a solution to (2.6) with speed c.

In the case of discrete time BRW, (2.6) reduces to the so-called smoothing transform and a similar result holds (see, e.g., [9, 10, 14, 24, 29]).

Given that homogeneous fragmentations can be seen as generalized BRWs, it is natural to formulate analogous results in this wider setting. To derive the analogue of (2.3), let us consider the following intuitive reasoning where, for simplicity, we assume that ν is a finite dislocation measure.

The classical technique for solving (2.3) with initial condition $u(x, 0) = g(x)$ consists of showing that

$$u(x, t) := \mathbb{E}\left(\prod_{i=1}^{N(t)} g\big(x + V_i(t)\big)\right)$$

is a solution. We take the same approach for fragmentations, where the role of the position $V_i(t)$ is played by $\{-\log|\Pi_i(t)| : i \geq 1\}$. It is easily seen from the fragmentation property in Definition 2 that

$$u(x, t + h) = \mathbb{E}\left(\prod_{i=1}^{N(t)} u\big(x - \log|\Pi_i(h)|, t\big)\right).$$

Recall also from the definition of a fragmentation process that in each infinitesimal period of time h, each block independently experiences a dislocation, fragmenting into smaller blocks of relative size $(s_1, s_2, \dots) \in \nabla_1$ with probability $\nu(ds)h + o(h)$ and more than one split dislocation occurs with probability $o(h)$. Roughly speaking, it follows that as $h \downarrow 0$,

$$\begin{aligned} &u(x, t+h) - u(x,t) \\ &\quad = \mathbb{E}\Bigl(\prod_i u\bigl(x - \log|\Pi_i(h)|, t\bigr)\Bigr) - u(x,t) \\ &\quad = \int_{\nabla_1} \Bigl\{\prod_i u(x - \log s_i, t) - u(x,t)\Bigr\} \nu(ds)h + o(h). \end{aligned}$$

This suggestively leads us to the parabolic integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) = \int_{\nabla_1} \Bigl\{\prod_i u(x - \log s_i, t) - u(x,t)\Bigr\} \nu(ds) \tag{2.8}$$

with initial condition $u(x, 0) = g(x)$, which we now formally identify as the analogue to the FKPP equation, even for the case that ν is an infinite measure.

A traveling wave $\psi : \mathbb{R} \mapsto [0, 1]$ of (2.8) with wave speed $c \in \mathbb{R}$, that is, $u(x,t) = \psi(x + ct)$, therefore solves

$$-c\psi'(x) + \int_{\nabla_1} \Bigl\{\prod_i \psi(x - \log s_i) - \psi(x)\Bigr\} \nu(ds) = 0, \qquad x \in \mathbb{R}. \tag{2.9}$$

2.3. *Additive martingales.* In the setting of both BBM and BRW, a key element to the analysis of (2.4) and (2.6) has been the study of certain "additive" martingales; cf. [10, 13, 17, 27, 32, 35]. In the current setting, these martingales are defined as follows:

DEFINITION 3. For all $p > \underline{p}$ we define the *additive martingale*

$$W(t, p) := e^{\Phi(p)t} \sum_i |\Pi_i(t)|^{p+1}, \qquad t \geq 0,$$

and the (critical) *derivative martingale*

$$\partial W(t, \bar{p}) := -\sum_i \bigl(t\Phi'(\bar{p}) + \log|\Pi_i(t)|\bigr) e^{\Phi(\bar{p})t} |\Pi_i(t)|^{\bar{p}+1}, \qquad t \geq 0.$$

The fact that the processes in the above definition are martingales is not difficult to establish; we refer the reader to Bertoin and Rouault [8] for further details. As we shall see, consistently with the case of BBM and BRW, it is the limits of these two martingales for certain parameter regimes in p which play a central role in the analysis of (2.9). Note that the additive martingale is positive and therefore, converges almost surely. The derivative martingale, so called because it is constructed from the derivative in p of the additive martingale, is a signed martingale and it is not a priori clear that it converges almost surely. Moreover, when limits for either of the two martingales exist, it is also not clear if they are nontrivial. The following theorem, lifted from Bertoin and Rouault [8], addresses precisely these questions.

THEOREM 2.

(i) *If* $p \in (\underline{p}, \bar{p})$ *then* $W(t, p)$ *converges almost surely and in mean to its limit,* $W(\infty, p)$, *which is almost surely strictly positive. If* $p \geq \bar{p}$ *then* $W(t, p)$ *converges almost surely to zero.*

(ii) *If* $p = \bar{p}$ *then* $\partial W(t, \bar{p})$ *converges almost surely to a nontrivial limit,* $\partial W(\infty, \bar{p})$, *which is almost surely strictly positive and has infinite mean.*

DEFINITION 4. Henceforth, we shall define $\Delta_p = W(\infty, p)$ if $p \in (\underline{p}, \bar{p})$ and $\Delta_{\bar{p}} = \partial W(\infty, \bar{p})$.

2.4. *Idea of the proof of Theorem* 1. Let us briefly discuss in informal terms the proof of Theorem 1. For convenience, we shall define the integro-differential operator

$$\mathcal{A}_p \psi(x) = -c_p \psi'(x) + \int_{\nabla_1} \left\{ \prod_i \psi(x - \log s_i) - \psi(x) \right\} \nu(ds)$$

whenever the right-hand side is well defined. Observe that the second term in the operator $\mathcal{A}_p$ is a jump operator driven by ν and is thus very close to the generator of the fragmentation itself. Indeed, $\mathcal{A}_p$ is the generator of the fragmentation process with dislocation measure ν and erosion coefficient c_p (albeit that the latter may take negative values).

Moreover, we shall say that $\psi_p \in \mathcal{T}_2(p)$ is a *multiplicative martingale function* if (1.4) is a martingale. The equivalence of the analytical property $\mathcal{A}_p \psi_p \equiv 0$ and the probabilistic property of ψ_p being a multiplicative martingale function emerges from a classical Feynman–Kac representation as soon as one can apply the appropriate stochastic calculus (which in the current setting is necessarily driven by the underlying Poisson random measure used to define the fragmentation process) in order to give the semi-martingale representation of $M(t, p, x)$.

To show the unique form of solutions in $\mathcal{T}_2(p)$, we start by studying the asymptotics of multiplicative martingale functions. We shall elaborate slightly here for the case $p \in (\underline{p}, \bar{p})$. To this end, consider $\psi_p \in \mathcal{T}_2(p)$ a multiplicative martingale function which makes (1.4) a martingale. As M is a uniformly integrable martingale $\psi_p(x) = M(0, p, x) = E(M(\infty, p, x))$. Our objective is, therefore, to understand in more detail the martingale limit $M(\infty, p, x)$. Taking logs of the multiplicative martingale we see that

$$-\log M(t, p, x) = -\sum_i \log \psi_p\bigl(x - \log|\Pi_i(t)| - c_p t\bigr).$$

Recall that $L_p(x) = e^{(p+1)x}(1 - \psi_p(x))$. Hence, if we replace $-\log x$ by $(1 - x)$ (assuming that the arguments of the log are all asymptotically close to 1 for large t) and multiply by $e^{(p+1)x}$, we see that

$$(2.10)\quad -e^{(p+1)x}\log M(t,p,x) \approx \sum_i |\Pi_i(t)|^{p+1} e^{\Phi(p)t} L_p\bigl(x - \log|\Pi_i(t)| - c_p t\bigr)$$

as $t \uparrow \infty$. We know, however, that the $\sum_i |\Pi_i(t)|^{p+1}e^{\Phi(p)t}$ is the additive martingale $W(t, p)$ which converges almost surely. On the other hand, for each fixed $i \in \mathbb{N}$ we know that $-\log(|\Pi_i(t)|) - c_p t$ is Bertoin's tagged fragment subordinator minus some drift so it is a Lévy process with no negative jumps. Accordingly, under mild assumptions, it respects the law of large numbers

$$-\log(|\Pi_i(t)|) - c_p t \sim \alpha_p t$$

as $t \uparrow \infty$ for some constant α_p which can be shown to be positive. Heuristically speaking, it follows that, if we can substitute $L(x + \alpha_p t)$ in place of $L_p(x - \log|\Pi_i(t)| - c_p t)$ in (2.10), then we can "factor out" the L_p terms to get

$$\frac{-e^{(p+1)x}\log M(t,p,x)}{W(t,p)} \approx L_p(x + \alpha_p t)$$

as $t \uparrow \infty$ and since both martingale limits $M(\infty, p, x)$ and $W(\infty, p)$ are nontrivial, we deduce that $\lim_{z\to\infty} L_p(z) = k_p$ for some constant $k_p \in (0,\infty)$. A direct consequence of this is that, irrespective of the multiplicative martingale function ψ_p that we started with, the L^1 limit $M(\infty, p, x)$ is always equal (up to an additive constant in x) to $\exp\{-e^{-(p+1)x}W(\infty, p)\}$. The stated uniqueness of ψ_p follows immediately.

Clearly this argument cannot work in the case $p = \bar{p}$ on account of the fact that Theorem 2 tells us that $W(\infty, p) = 0$ almost surely. Nonetheless, a similar, but more complex argument in which a comparison of the L^1 martingale limit $M(\infty, \bar{p}, x)$ is made with the derivative martingale limit $\partial W(\infty, \bar{p})$ is possible.

One particular technical difficulty in the above argument is that one needs to uniformly control the terms $L_p(x - \log|\Pi_i(t)| - c_p t)$ in order to "factor out" the common approximation $L_p(x + \alpha_p t)$. One could try to control the position of the left-most particle to do this (as in [17]), however, this turns out to be inconvenient and another technique, namely, projecting the martingales along the so-called stopping-lines, appears to work better. This is a classical idea in the context of BBM (see, e.g., Neveu [35] and Kyprianou [27]) and BRW (see, e.g., Biggins and Kyprianou [10, 11]).

Thanks to their uniform integrability, the martingales $M(t, p, x)$ and $W(t, p, x)$ may be seen as the projection of their limits on to $\mathcal{F}_t$, the information generated by the fragmentation tree when it is "cut" at fixed time t. However, we could also project these limits back on to filtrations generated by the fragmentation tree up to different "increasing" sequences of "cuts" or *stopping lines* as they turn out to be known. To give an example, a particular instance of an “increasing sequence” of stopping line is studied by Bertoin and Martinez [7] who *freeze* fragments as soon as their size is smaller than a certain threshold e^{-z}, $z \geq 0$. After some time all fragments are smaller than this threshold and the process stops, thereby “cutting” through the fragmentation tree. The collection of fragments one gets in the end, say $(\Pi_i(\ell^z) : i \in \mathbb{N})$, generates another filtration $\{\mathcal{G}_{\ell^z} : z \geq 0\}$. Projecting the martingale limits $M(\infty, p, x)$ and $W(\infty, p, x)$ back on to this filtration produces two new uniformly integrable martingales which have the same limits as before and which look the same as $M(t, p, x)$ and $W(t, p)$, respectively, except that the role of $(\Pi_i(t) : i \in \mathbb{N})$ is now played by $(\Pi_i(\ell^z) : i \in \mathbb{N})$. Considering the case $p = 0$, so that $c_p = 0$, one could now rework the heuristic argument given in the previous paragraphs for this sequence of stopping lines and take advantage of the uniform control that one now has over the fragment sizes on $(\Pi_i(\ell^z) : i \in \mathbb{N})$. A modification of this line of reasoning with a different choice of stopping line when $p \neq 0$ can and will be used as a key feature in the proof of our main theorem.

2.5. *Organization of the paper.* The rest of the paper is organized as follows. In Section 3 we show that the function ψ_p given by (1.5)

$$\psi_p(x) = \mathbb{E}\bigl(\exp\{-e^{-(p+1)x} \Delta_p\}\bigr)$$

indeed makes

$$M(t, p, x) = \prod_i \psi_p\bigl(x - \log|\Pi_i(t)| - c_p t\bigr), \qquad t \geq 0,$$

a martingale and that it belongs to $\mathcal{T}_2(p)$.

Section 4 introduces the notions and tools related to martingale convergence on stopping lines. More precisely, we generalize the notion of *frozen fragmentation* defined in [7] to other stopping lines which allow us to study the whole range of wave speeds. One particular feature which will emerge in this section will be the fact that, for certain parameter ranges, the stopping lines we consider will sweep out a Crump–Mode–Jagers (CMJ) process embedded within the fragmentation process.

A central result of the proof is given in Section 5 where we provide a law of large numbers for the empirical distribution of the sizes of the blocks on the stopping lines defined in Section 4. A key tool here is the connection between these stopped fragmentations and the aforementioned embedded CMJ processes which allows us to use classical results from Nerman [34].

Next, in Section 6, we show that $\psi_p(x) = \mathbb{E}(\exp(-e^{-(p+1)x} \Delta_p))$ are the only functions in $\mathcal{T}_2(p)$ for $(\underline{p}, \bar{p}]$ which make M a product martingale. We conclude by showing that on the one hand this function ψ_p solves (2.9) and on the other hand that, if $\psi_p \in \mathcal{T}_2(p)$ solves (2.9), then it makes M a martingale.

3. Existence of multiplicative martingale functions. Our main objective here is to establish the range of speeds c in (1.4) for which multiplicative martingale functions exist. In short, the existence of multiplicative martingale functions follows directly from the existence of a nontrivial limit of the martingales $W(\cdot, p)$ for $p \in (\underline{p}, \bar{p})$ and $\partial W(\cdot, \bar{p})$. We begin with a classification of possible speeds. We shall call wave speeds c *sub-critical* when $c \in (c_{\underline{p}}, c_{\bar{p}})$, *critical* when $c = c_{\bar{p}}$ and *super-critical* when $c > c_{\bar{p}}$.

THEOREM 3.
(i) *At least one multiplicative martingale function* [*which makes* (1.4) *a martingale*] *exists in* $\mathcal{T}_1$ (*the set of montone*, C^1 *functions with limit* 0 *in* $-\infty$ *and* 1 *in* $+\infty$) *for all wave speeds* $c \in (c_{\underline{p}}, c_{\bar{p}}]$.
(ii) *For all wave speeds* $c > c_{\bar{p}}$ *there is no multiplicative martingale function in* $\mathcal{T}_1$.

Before proceeding with the proof, we make the following observation.

REMARK 1. As we suppose that ν is conservative, we have that $\Phi(0) = 0$ and $\bar{p} > 0$, $\underline{p} \in [-1, 0]$. The map $p \mapsto \Phi(p)$ can have a vertical asymptote at $\underline{p}$ or a finite value with finite or infinite derivative.

The above theorem does not consider the case $c < c_{\underline{p}}$. Observe, however, that as soon as $\lim_{p \searrow \underline{p}} \Phi(p) = -\infty$, we have $c_{\underline{p}} = -\infty$, so this case is void. When $\Phi(\underline{p}+) > -\infty$, we have to look at the right-derivative of Φ at $\underline{p}$. If $\Phi'(\underline{p}+) < \infty$, then the fragmentation is necessarily finite activity (i.e., ν has finite mass) and dislocations are always finite (see [2]). Hence, we are dealing with a usual continuous-time BRW for which the results are available in the literature. We leave open the interesting case where $\Phi(\underline{p}+) > -\infty$ but $\Phi'(\underline{p}+) = \infty$.

The constants $\bar{p}$ and $\underline{p}$ also play an important role in [2]. More precisely, following Bertoin [5] it was shown in [2] that one could find fragments decaying like $t \mapsto e^{-\lambda t}$ provided that $\exists p \in (\underline{p}, \bar{p}]$ (the case $p = \underline{p}$ being particular) such that $\lambda = \Phi'(p)$. Hence, the set of admissible wave speeds $\{c_p : p \in (\underline{p}, \bar{p}]\}$ is closely related to the set of admissible speeds of fragmentation. However, the only p such that $c_p = \Phi'(p)$ is $\bar{p}$.

PROOF OF THEOREM 3. First, let us assume that $c < c_{\bar{p}}$. Since

$$W(t + s, p) = \sum_i e^{\Phi(p)t} |\Pi_i(t)|^{p+1} W^{(i)}(s, p),$$

where $W^{(i)}(\cdot, p)$ are i.i.d. copies of $W(\cdot)$ independent of $\Pi(t)$, it follows that taking Laplace transforms and then limits as $s \uparrow \infty$ and that if we take ψ_p as in (1.5)

$$\psi_p(x) = \mathbb{E}\big(\exp\{-e^{-(p+1)x} W(\infty, p)\}\big),$$

then

$$\psi_p(x) = \mathbb{E}\Big(\prod_i \exp\{-e^{-(p+1)x} e^{\Phi(p)t} |\Pi_i(t)|^{p+1} W^{(i)}(\infty, p)\}\Big) \tag{3.1}$$
$$= \mathbb{E}\prod_i \psi_p\big(x - \log|\Pi_i(t)| - c_p t\big).$$

Hence, we see that ψ_p is a multiplicative martingale function with wave speed c_p.

Note the fact that $\psi_p \in C^1(\mathbb{R})$ follows by dominated convergence. Note also that from the definition of ψ_p it follows automatically that $\psi_p(\infty) = 1$. Moreover, we know from Theorem 2 that $\mathbb{P}(W(\infty, p) > 0) = 1$ so $\psi_p(-\infty) = 0$ and $\psi_p \in \mathcal{T}_1$.

Next, assume that $c = c_{\bar{p}}$. The method in the previous part of the proof does not work because, according to the conclusion of Theorem 2(i), $W(\infty, \bar{p}) = 0$ almost surely. In that case, it is necessary to work instead with the derivative martingale. Using the conclusion of Theorem 2(ii) we note that by conditioning on the $\mathcal{F}_t$, with $\partial W^{(i)}(\infty, \bar{p})$ as i.i.d. copies of $\partial W(\infty, \bar{p})$, we have appealing to similar analysis to previously

$$\begin{aligned}\partial W(\infty, \bar{p}) &= -\sum_i \big(t\Phi'(\bar{p}) + \log|\Pi_i(t)|\big) e^{\Phi(\bar{p})t} |\Pi_i(t)|^{\bar{p}+1} W^{(i)}(\infty, \bar{p}) \\ &\quad + \sum_i e^{\Phi(\bar{p})t} |\Pi_i(t)|^{\bar{p}+1}\, \partial W^{(i)}(\infty, \bar{p}) \\ &= \sum_i e^{\Phi(\bar{p})t} |\Pi_i(t)|^{\bar{p}+1}\, \partial W^{(i)}(\infty, \bar{p}),\end{aligned}$$

where the second equality follows, since $W(\infty, \bar{p}) = 0$ and, as above, the Theorem 2 tells us that $\mathbb{P}(\partial W(\infty, \bar{p}) > 0) = 1$.

Now assume that $c > c_{\bar{p}}$. The following argument is based on ideas found in [17]. Suppose that $|\Pi_1(t)|^{\downarrow}$ is the largest fragment in the process $|\Pi|$. Then, as alluded to in the Introduction, we know that

$$\lim_{t\uparrow\infty} \frac{-\log|\Pi_1(t)|^{\downarrow}}{t} = \Phi'(\bar{p}) = c_{\bar{p}} \tag{3.2}$$

almost surely. See [2, 5].

Suppose that a multiplicative martingale function $\psi \in \mathcal{T}_1$ exists within this regime. Set $c > c_{\bar{p}}$. It follows by virtue of the fact that ψ is bounded in $(0, 1]$ that

$$\psi(x) = \mathbb{E}\prod_i \psi\big(x - \log|\Pi_i(t)| - ct\big) \le \mathbb{E}\psi\big(x - \log|\Pi_1(t)|^{\downarrow} - ct\big) \tag{3.3}$$

for all $t \ge 0$. From Remark 1 on the rate of decay of $|\Pi_1(t)|^{\downarrow}$, we can easily deduce that

$$-\log|\Pi_1(t)|^{\downarrow} - ct \to -\infty$$

almost surely as $t \uparrow \infty$ since $c_{\bar{p}} - c < 0$. Taking limits in (3.3) as $t \uparrow \infty$, we deduce by dominated convergence and the fact that $\psi(-\infty) = 0$, that $\psi(x) = 0$ for all $x \in \mathbb{R}$. This contradicts the assumption that ψ is in $\mathcal{T}_1$. $\square$

Recall that $\mathcal{T}_2(p) = \{\psi \in \mathcal{T}_1 : L_p(x) = e^{(p+1)x}(1 - \psi(x))$ is monotone increasing$\}$.

PROPOSITION 1. *Fix* $p \in (\underline{p}, \bar{p}]$. *Define* $\psi_p(x) = \mathbb{E}(\exp\{-e^{-(p+1)x}\Delta_p\})$, *then* $\psi_p \in \mathcal{T}_2(p)$.

PROOF. We have already established that $\psi_p \in \mathcal{T}_1$. Consider $\psi_p(x) = \mathbb{E}\exp\{-e^{-(p+1)x}\Delta_p\}$ for some nonnegative and nontrivial random variable Δ_p. Write $y = e^{-(p+1)x}$. From Feller [15], Chapter XIII.2, it is known that $[1 - \psi_p(-(1+p)^{-1}\log y)]/y$ is the Laplace transform of a positive measure and hence, is decreasing in y. In turn, this implies that $L_p(x)$ is an increasing function in x. $\square$

4. Stopping lines and probability tilting. The concept of *stopping line* was introduced by Bertoin [4] in the context of fragmentation processes, capturing in its definition the essence of earlier ideas on stopping lines for branching processes coming from the work of Neveu [35], Jagers [22], Chauvin [13] and Biggins and Kyprianou [10]. Roughly speaking, a stopping line plays the role of a stopping time for BRWs. The tools and techniques we now introduce also have an intrinsic interest in themselves and cast a new light on some earlier results by Bertoin.

4.1. *Stopping lines.* The following material is taken from [4]. Recall that for each integer $i \in \mathbb{N}$ and $s \in \mathbb{R}_+$, we denote by $B_i(s)$ the block of $\Pi(s)$ which contains i with the convention that $B_i(\infty) = \{i\}$ while $\Pi_i(t)$ is the ith block by order of least element. Then we write

$$\mathcal{G}_i(t) = \sigma\big(B_i(s), s \le t\big)$$

for the sigma-field generated by the history of that block up to time t. We will also use the notation $\mathcal{G}_t = \sigma\{\Pi(s), s \le t\}$ for the sigma-field generated by the whole (partition-valued) fragmentation. Hence, obviously, for all $t \ge 0$ we have $\mathcal{F}_t := \sigma\{|\Pi(u)|, u \le t\} \subset \mathcal{G}_t$ and for each $i \in \mathbb{N}$, $\mathcal{G}_i(t) \subset \mathcal{G}_t$.

DEFINITION 5. We call stopping line a family $\ell = (\ell(i), i \in \mathbb{N})$ of random variables with values in $[0, \infty]$ such that for each $i \in \mathbb{N}$:

(i) $\ell(i)$ is a $(\mathcal{G}_i(t))$-stopping time.
(ii) $\ell(i) = \ell(j)$ for every $j \in B_i(\ell(i))$.

For instance, first passage times such as $\ell(i) = \inf\{t \geq 0 : |B_i(t)| \leq a\}$ for a fixed level $a \in (0, 1)$ define a stopping line.

The key point is that it can be checked that the collection of blocks $\Pi(\ell) = \{B_i(\ell(i))\}_{i \in \mathbb{N}}$ is a partition of $\mathbb{N}$ which we denote by $\Pi(\ell) = (\Pi_1(\ell), \Pi_2(\ell), \ldots)$, where, as usual, the enumeration is by order of least element.

Observe that because $B_i(\ell(i)) = B_j(\ell(j))$ when $j \in B_i(\ell(i))$, the set $\{B_i(\ell(i))\}_{i \in \mathbb{N}}$ has repetitions, $(\Pi_1(\ell), \Pi_2(\ell), \ldots)$ is simply a way of enumerating each element only once by order of discovery. In the same way the $\ell(j)$'s can be enumerated as ℓ_i, $i = 1, \ldots,$ such that for each i, ℓ_i corresponds to the stopping time of $\Pi_i(\ell)$.

If ℓ is a stopping line, it is not hard to see that both $\ell + t := (\ell(i) + t, i \in \mathbb{N})$ and $\ell \wedge t := (\ell(i) \wedge t, i \in \mathbb{N})$ are also stopping lines. This allows us to define

$$\Pi \circ \theta_\ell(t) := \Pi(\ell + t)$$

and the sigma-field $\mathcal{G}_\ell = \bigvee_{i \in \mathbb{N}} \mathcal{G}_i(\ell(i))$.

The following lemma (see [4], Lemma 3.13) can be seen as the analogue of the strong Markov property for branching processes; it is also known as the Extended Fragmentation Property; cf. Bertoin [6], Lemma 3.14.

LEMMA 1. *Let $\ell = (\ell(i), i \in \mathbb{N})$ be a stopping line, then the conditional distribution of $\Pi \circ \theta_\ell : t \mapsto \Pi(\ell + t)$ given $\mathcal{G}_\ell$ is $\mathbb{P}_\pi$ where $\pi = \Pi(\ell)$.*

Heuristically, we are going along the rays from the root to the boundary, one at a time (each integer defines a ray, but observe that there are some rays which do not correspond to an integer). On each ray ξ we have a stopping time τ_ξ, that is, we look only at what happens along that ray $(\xi_t, t \geq 0)$ and based on that information, an alarm rings at a random time. When later we go along another ray ξ', if $\xi'(\tau_\xi) = \xi(\tau_\xi)$ (i.e., the two rays have not branched yet at τ_ξ), then $\tau_{\xi'} = \tau_\xi$.

Following Chauvin [13] and Kyprianou [26] we now introduce the notion of *almost sure dissecting* and *L^1-dissecting* stopping lines.

DEFINITION 6. Let $\ell = (\ell(i), i \in \mathbb{N})$ be a stopping line.

(i) We say that ℓ is a.s. dissecting if almost surely $\sup_i \{\ell_i\} < \infty$.
(ii) Let $A_\ell(t) = \{i : \ell_i > t\}$, then we say that ℓ is $p - L^1$-dissecting if

$$\lim_{t \to \infty} \mathbb{E}\biggl(\sum_{i \in A_\ell(t)} |\Pi_i(t)|^{p+1} e^{\Phi(p)t}\biggr) = 0.$$

4.2. *Spine and probability tilting.* In this section we will discuss changes of measures and subsequent path decompositions which were instigated by Lyons [30] for the BRW and further applied by Bertoin and Rouault [8] in the setting of fragmentation processes. The following lemma is a so-called many-to-one principle. It allows us to transform expectations of functionals along a stopping line into expectations of functions of a single particle, namely Bertoin's tagged fragment.

LEMMA 2 (Many-to-one principle). *Let ℓ be a stopping line. Then, for any measurable nonnegative f we have*

$$\mathbb{E}\biggl(\sum_i |\Pi_i(\ell)| f(|\Pi_i(\ell)|, \ell_i)\biggr) = \mathbb{E}\bigl(f(|\Pi_1(\ell)|, \ell_1)\mathbf{1}_{\{\ell_1<\infty\}}\bigr). \tag{4.1}$$

PROOF. To see this, observe that because $\Pi(\ell)$ is an exchangeable partition, the pair $(\Pi_1(\ell), \ell_1)$ is a size-biased pick from the sequence $((\Pi_i(\ell), \ell_i), i \in \mathbb{N})$ [i.e., given $((|\Pi_i(\ell)|, \ell_i), i \in \mathbb{N}) = ((x_i, l_i), i \in \mathbb{N})$ then $\mathbb{P}((|\Pi_1(\ell)|, \ell_1) = (x_i, l_i)) = x_i$]. The indicator function in the right-hand side comes from the possibility that there is some dust in $\Pi(\ell)$. $\square$

Observe that if ℓ is almost surely dissecting, as Π is conservative, then $\Pi(\ell)$ has no dust. The converse is not true.

The second tool we shall use is a probability tilting that was introduced by Lyons, Permantle and Peres [31] for Galton–Watson processes, Lyons [30] for BRWs and by Bertoin and Rouault [8] for fragmentation. First note that since $(-\log|\Pi_1(t)| : t \geq 0)$ is a subordinator with Laplace exponent Φ, it follows that

$$\mathcal{E}(t, p) := |\Pi_1(t)|^p e^{t\Phi(p)}$$

is a $(\mathbb{P}, \mathcal{G}_1(t))$-martingale. If we project this martingale on the filtration $\mathbb{F}$, we obtain the martingale $W(t, p)$. We can use these martingales to define the tilted probability measures

$$d\mathbb{P}^{(p)}_{|\mathcal{G}_t} = \mathcal{E}(t, p)\, d\mathbb{P}_{|\mathcal{G}_t} \tag{4.2}$$

and

$$d\mathbb{P}^{(p)}_{|\mathcal{F}_t} = W(t, p)\, d\mathbb{P}_{|\mathcal{F}_t}.$$

The effect of the latter change of measure is described in detail in [8], Proposition 5. More precisely, under $\mathbb{P}^{(p)}$ the process $\xi_t = -\log|\Pi_1(t)|$ is a subordinator with Laplace exponent

$$\Phi^{(p)}(q) = \Phi(p+q) - \Phi(p), \qquad q > \underline{p} - p,$$

and Lévy measure given by

$$m^{(p)}(dx) = e^{-px} m(dx), \tag{4.3}$$

where $m(dx)$ was given in (2.2). Under $\mathbb{P}^{(p)}$, the blocks with index not equal to unity split with the same dynamic as shown before. The block containing 1 splits according to the atoms of Poisson Point Process $\{(t, \pi(t)) : t \geq 0\}$ on $\mathbb{R}_+ \times \mathcal{P}$ with intensity $dr \otimes |\pi_1|^p \nu(d\pi)$.

Observe that because $\ell_1 = \ell(1)$ is a $\mathcal{G}_1(t)$-stopping time, it is measurable with respect to the filtration of the aforementioned Poisson Point Process and thus the above description is enough to determine the law of ℓ_1 under $\mathbb{P}^{(p)}$. Adopting the notation $\tau = \ell_1$ and writing ξ_t instead of $-\log|\Pi_1(t)|$, we have the following result:

LEMMA 3. *For all positive measurable g,*

$$\mathbb{E}\Big(\sum_i g(-\log|\Pi_i(\ell)|, \ell_i)|\Pi_i(\ell)|^{p+1}e^{\Phi(p)\ell_i}\Big) = \mathbb{E}^{(p)}\big(g(\xi_\tau, \tau)\mathbf{1}_{\{\tau<\infty\}}\big). \tag{4.4}$$

PROOF. From Lemma 2 with $f(x, \ell) = x^p g(x, \ell)e^{\Phi(p)\ell}$ we have

$$\begin{aligned}
&\mathbb{E}\Big(\sum_i g(-\log|\Pi_i(\ell)|, \ell_i)|\Pi_i(\ell)|^{p+1}e^{\Phi(p)\ell_i}\Big)\\
&\quad = \mathbb{E}\big(g(-\log|\Pi_1(\ell)|, \ell_1)|\Pi_1(\ell)|^p e^{\Phi(p)\ell_1}\mathbf{1}_{\{\ell_1<\infty\}}\big)\\
&\quad = \mathbb{E}\big(g(\xi_\tau, \tau)\mathcal{E}(\tau, p)\mathbf{1}_{\{\tau<\infty\}}\big)\\
&\quad = \mathbb{E}^{(p)}\big(g(\xi_\tau, \tau)\mathbf{1}_{\{\tau<\infty\}}\big)
\end{aligned}$$

and the result follows. □

As a first application of these tools we prove the analogue of Theorem 2 in [26] which gives a necessary and sufficient condition for a stopping line to be $p - L^1$ dissecting.

THEOREM 4. *A stopping-line ℓ is $p - L^1$-dissecting if and only if*

$$\mathbb{P}^{(p)}(\ell_1 < \infty) = 1.$$

PROOF. From Lemma 3,

$$\begin{aligned}
\mathbb{E}\Big(\sum_{i\in A_\ell(t)} |\Pi_i(t)|^{p+1}e^{\Phi(p)t}\Big) &= \mathbb{E}\Big(\sum_i \mathbf{1}_{\{\ell_i>t\}}|\Pi_i(t)|^{p+1}e^{\Phi(p)t}\Big)\\
&= \mathbb{P}^{(p)}(\ell_1 > t).
\end{aligned}$$

It follows that the limit of the left-hand side as $t \uparrow \infty$ is zero if and only if $P^{(p)}(\ell_1 < \infty) = 1$. □

4.3. *Martingales.* We define an ordering on stopping lines as follows: given $\ell^{(1)}$ and $\ell^{(2)}$ two stopping lines, we write $\ell^{(1)} \le \ell^{(2)}$ if almost surely, for all $i \in \mathbb{N}: \ell^{(1)}(i) \le \ell^{(2)}(i)$. So given a family $(\ell^z, z \ge 0)$ of stopping lines we say that ℓ^z is increasing if almost surely, for all $z \le z'$, $\ell^z \le \ell^{z'}$.

Given $(\ell^z, z \ge 0)$ an increasing family of stopping lines, we may define two filtrations $\mathcal{G}_{\ell^z}$ and $\mathcal{F}_{\ell^z}$ as follows:

$$\mathcal{G}_i(\ell^z) = \sigma\big(B_i(s) : s \le \ell^z(i)\big) \quad \text{and} \quad \mathcal{F}_i(\ell^z) = \sigma\big(|B_i(s)| : s \le \ell^z(i)\big)$$

and then define

$$\mathcal{G}_{\ell^z} := \bigvee_i \mathcal{G}_i(\ell^z) \quad \text{and} \quad \mathcal{F}_{\ell^z} := \bigvee_i \mathcal{F}_i(\ell^z).$$

Finally, we say that an increasing family of stopping lines ℓ^z is proper if $\lim_{z\to\infty} \mathcal{G}_{\ell^z} = \sigma\{\Pi(t), t \geq 0\}$ which is equivalent to $\ell_i^z \to \infty$ for all i almost surely.

The next lemma mirrors analogous results that were obtained for BBM by Chauvin [13]. Recall that $c_p = \Phi(p)/(p+1)$.

THEOREM 5. *Fix $p \in (\underline{p}, \bar{p}]$ and let ψ_p be a solution to (1.4), that is, a function that makes $(M(t, p, x), t \geq 0)$ a martingale and is bounded between 0 and 1. Then*:

(i) *Let $(\ell^z, z \geq 0)$ be a increasing family of a.s. dissecting lines. Then the stochastic process*

$$M(\ell^z, p, x) := \prod_i \psi_p\bigl(x - \log|\Pi_i(\ell^z)| - c_p \ell^z(i)\bigr), \qquad z \geq 0,$$

is a uniformly integrable martingale with respect to $\{\mathcal{F}_{\ell^z} : z \geq 0\}$ having limit equal to $M(\infty, p, x)$.

(ii) *Let $(\ell^z, z \geq 0)$ be a increasing family of $p - L^1$-dissecting lines, then the stochastic process*

$$W(\ell^z, p) := \sum_i |\Pi_i(\ell^z)|^{p+1} e^{\Phi(p)\ell^z(i)}, \qquad z \geq 0,$$

is a unit mean martingale with respect to $\{\mathcal{F}_{\ell^z} : z \geq 0\}$. Furthermore, when $p \in (\underline{p}, \bar{p})$

$$\lim_{z\uparrow\infty} W(\ell^z, p) = W(\infty, p)$$

in L^1 where $W(\infty, p)$ is the martingale limit described in Theorem 2(i).

REMARK 2. A straightforward analogue result cannot hold for ∂W as it is a signed martingale on the stopping lines we consider.

PROOF. We start with (i). With the help of Lemma 1, we have for all $x \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E} \prod_i \psi_p\bigl(x - \log|\Pi_i(t)| - c_p t\bigr) \\
&\quad = \mathbb{E} \prod_{i : \ell_i^z \geq t} \psi_p\bigl(x - \log|\Pi_i(t)| - c_p t\bigr) \\
&\qquad \times \prod_{i : \ell_i^z < t} \prod_{j : \Pi_j(t) \subset \Pi_i(\ell^z)} \psi_p\bigl(x - \log|\Pi_i(\ell^z)| - c_p \ell_i^z
\end{aligned}$$

$$- \log\big(|\Pi_j(t)|/|\Pi_i(\ell^z)|\big) - c_p(t - \ell_i^z)\big)$$

$$= \mathbb{E} \prod_{i\,:\,\ell_i^z \geq t} \psi_p\big(x - \log|\Pi_i(t)| - c_p t\big)$$

$$\times \prod_{i\,:\,\ell_i^z < t} M^{(i)}\big(t - \ell_i^z, p, x - \log|\Pi_i(\ell^z)| - c_p \ell_i^z\big)$$

$$= \mathbb{E} \prod_{i\,:\,\ell_i^z \geq t} \psi_p\big(x - \log|\Pi_i(t)| - c_p t\big) \prod_{i\,:\,\ell_i^z < t} \psi_p\big(x - \log|\Pi_i(\ell^z)| - c_p \ell_i^z\big),$$

where in the second equalities, given $\mathcal{F}_{\ell^z}$, the quantities $M^{(i)}(\cdot, p, \cdot)$ are independent copies of $M(\cdot, p, \cdot)$. As ℓ^z is almost surely dissecting, we know that as $t \to \infty$ the set $\{i : \ell_i^z \geq t\}$ becomes empty almost surely. Since ψ_p is positive and bounded by unity, we may apply dominated convergence to deduce as $t \uparrow \infty$ that for all $x \in \mathbb{R}$,

$$\psi_p(x) = \mathbb{E} \prod_i \psi_p\big(x - \log|\Pi_i(\ell^z)| - c_p \ell_i^z\big). \tag{4.5}$$

Since $z \geq 0$ is arbitrarily valued, the last equality in combination with the Extended Fragmentation Property is sufficient to deduce the required martingale property. Indeed, suppose that $z' > z \geq 0$; we have that

$$M(\ell^{z'}, p, x)$$

$$= \prod_i \psi_p\big(x - \log|\Pi_i(\ell^{z'})| - c_p \ell_i^{z'}\big)$$

$$= \prod_i \prod_{j\,:\,\Pi_j(\ell^{z'}) \subset \Pi_i(\ell^z)} \psi_p\big(x - \log|\Pi_j(\ell^{z'})| - c_p \ell_j^{z'}\big)$$

$$= \prod_i \prod_{j\,:\,\Pi_j(\ell^{z'}) \subset \Pi_i(\ell^z)} \psi_p\big((x - \log|\Pi_i(\ell^z)| - c_p \ell_i^z\big)$$

$$- \log\big(|\Pi_j(\ell^{z'})|/|\Pi_i(\ell^z)|\big) - c_p(\ell_j^{z'} - \ell_i^z)\big).$$

Using (4.5) and Lemma 1 we see that for all $\mathcal{F}_{\ell^z}$-measurable x',

$$\psi_p(x') = \mathbb{E}\Big(\prod_{j\,:\,\Pi_j(\ell^{z'}) \subset \Pi_i(\ell^z)} \psi_p\big(x' - \log\big(|\Pi_j(\ell^{z'})|/|\Pi_i(\ell^z)|\big) - c_p(\ell_j^{z'} - \ell_i^z)\big) \Big| \mathcal{F}_{\ell^z} \Big)$$

so the martingale property follows. Uniform integrability follows on account of the fact that $0 \leq \psi_p \leq 1$ and since ℓ^z is a.s. dissecting Lemma 1 (and more precisely the independence of the subtrees which start at ℓ^z) gives us that

$$\mathbb{E}(M(\infty, p, x) | \mathcal{F}_{\ell^z}) = M(\ell^z, p, x).$$

We now prove (ii). Let ℓ be a $p-L^1$ dissecting stopping line. By Lemma 3 and the Monotone Convergence Theorem we have that

$$\begin{aligned}\mathbb{E}(W(\ell,p)) &= \lim_{t\to\infty}\mathbb{E}\Big[\sum_i \mathbf{1}_{\{\ell_i\le t\}}|\Pi_i(\ell_i)|^{p+1}e^{\Phi(p)\ell_i}\Big]\\ &= \lim_{t\to\infty}\mathbb{E}^{(p)}[\mathbf{1}_{\tau\le t}]\\ &= 1,\end{aligned}$$

where we have used the many-to-one principle, the probability tilting (4.2) and Theorem 4.

To prove the martingale property, fix $0\le z\le z'$ and observe that

$$\begin{aligned}W(\ell_z,p) &= \sum_i \sum_{j:\Pi_j(\ell^{z'})\subset\Pi_i(\ell^z)} |\Pi_j(\ell^{z'})|^{p+1}e^{\Phi(p)\ell_j^{z'}}\\ &= \sum_i |\Pi_i(\ell^z)|^{p+1}e^{\Phi(p)\ell_i^z}\sum_{j:\Pi_j(\ell^{z'})\subset\Pi_i(\ell^z)}\left(\frac{|\Pi_j(\ell^{z'})|}{|\Pi_i(\ell^z)|}\right)^{p+1}e^{\Phi(p)(\ell_j^{z'}-\ell_i^z)}.\end{aligned}$$

We apply the strong Markov property in Lemma 1 to obtain that

$$\mathbb{E}\left(\sum_{j:\Pi_j(\ell^{z'})\subset\Pi_i(\ell^z)}\left(\frac{|\Pi_j(\ell^{z'})|}{|\Pi_i(\ell^z)|}\right)^{p+1}e^{\Phi(p)(\ell_j^{z'}-\ell_i^z)}\Big|\mathcal{F}_{\ell^z}\right)=1.$$

The L^1 convergence of $W(\ell^z,p)$ when $p\in(\underline{p},\bar{p})$ is a consequence of the fact that Lemma 1 applied to ℓ^z again gives us

$$\mathbb{E}(W(\infty,p)|\mathcal{F}_{\ell^z})=W(\ell^z,p)$$

together with Theorem 2(i). $\square$

4.4. *First-passage stopping lines*. In this paragraph, we introduce the families of stopping lines we will be using and we show that they satisfy the desired properties.

Fix $p\in(\underline{p},\bar{p}]$ and for each $z\ge 0$ let $\ell^{(p,z)}$ be the stopping line defined as follows: For each $i\in\mathbb{N}$

$$\ell^{(p,z)}(i)=\inf\{t\ge 0:-\ln|B_i(t)|>z+c_pt\},$$

where $c_p=\Phi(p)/(p+1)$. In other words, $\ell^{(p,z)}(i)$ is the first time when $-\ln|B_i(t)|$ crosses the line $x=c_pt+z$ (see Figure 2). Recall that $p\mapsto c_p$ is increasing on $(\underline{p},\bar{p}]$ with $c_0=0$.

PROPOSITION 2. *For any fixed $p\in(\underline{p},\bar{p}]$, the family of stopping lines $(\ell^{(p,z)},z\ge 0)$ is a.s. dissecting and $p-L^1$ dissecting as well as proper.*

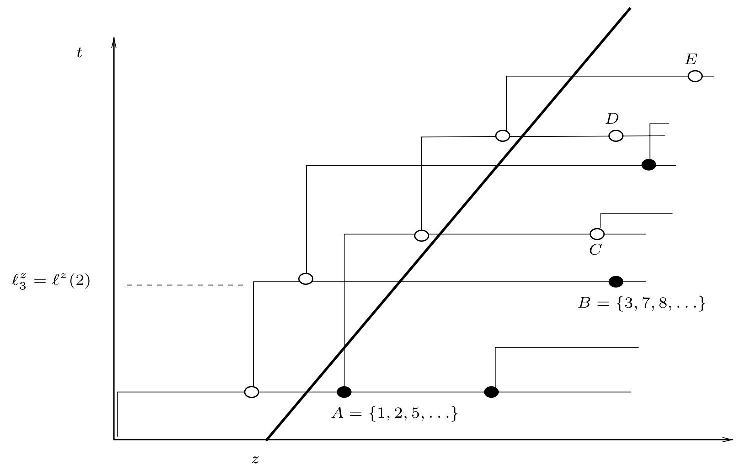

FIG. 2. *Each dot represents a fragment (block) at its birth. The time of birth is its vertical position while its* $-\log$ *size is given by its position on the x axis. Each horizontal line thus corresponds to a single splitting event in which multiple fragments are created. The blocks of* $\Pi(\ell^{(p,z)})$ *are the dots which are the first in their line of descent to be on the right-hand side of the bold line* $x = z + c_p t$. *The collection of this dots is* the coming generation. *As z increases, it reaches fragment A which then splits giving birth to $C, D, E, \dots$ which will then replace A in the coming generation. For simplification this picture corresponds to the case of a finite activity dislocation kernel* ν.

PROOF. We know from Theorem 2 that when $p = \bar{p}$ the martingale

$$W(t, \bar{p}) := e^{\Phi(\bar{p})t} \sum_i |\Pi_i(t)|^{\bar{p}+1}, \qquad t \geq 0,$$

converges almost surely to 0. Hence, $e^{\Phi(\bar{p})t}(|\Pi_1(t)|^{\downarrow})^{\bar{p}+1} = \exp(-(\bar{p}+1) \times (-\log(|\Pi_1(t)|^{\downarrow}) - c_{\bar{p}}t))$ also tends to 0 a.s. and we conclude that

$$\mathbb{P}\bigl(-\log(|\Pi_1(t)|^{\downarrow}) - c_{\bar{p}}t \to +\infty\bigr) = 1.$$

As $c_p < c_{\bar{p}}$ when $p < \bar{p}$, this entails immediately that

$$\forall p \in (\underline{p}, \bar{p}] \qquad \mathbb{P}\bigl(-\log(|\Pi_1(t)|^{\downarrow}) - c_p t \to +\infty\bigr) = 1$$

and hence, that the stopping lines $(\ell^{(p,z)} : z \geq 0)$ are a.s. dissecting.

To prove that for any $z \geq 0$, $\ell^{(z,p)}$ is L^1-dissecting we use Theorem 4. Recall that under $\mathbb{P}^{(p)}$ the process $(\xi_t = -\log|\Pi_1(t)|, t \geq 0)$ is a subordinator with Laplace exponent $\Phi^{(p)}$.

$$\mathbb{P}^{(p)}\bigl(\ell_1^{(p,z)} < \infty\bigr) = \mathbb{P}^{(p)}(\inf\{t : \xi_t - c_p t > z\} < \infty)$$

and this is equal to one if and only if the mean of the Lévy process $(\xi_t - c_p, t \geq 0)$ is positive under $\mathbb{P}^{(p)}$. That is to say, if and only if $\Phi^{(p)\prime}(0) - c_p = \Phi'(p) - c_p \geq 0$ which is equivalent to $p \leq \bar{p}$. $\square$

Consider the stopping line $\ell^{(p,0)}$. Two fundamental differences in the way this stopping line dissects the fragmentation process occurs in the regimes $p \leq 0$ and $p > 0$. When $p \leq 0$, it is easily seen that for all $i \in \mathbb{N}$, we have that $\ell^{(p,0)}(i) = 0$ so that $\Pi(\ell^{(p,0)})$ is the trivial partition with all integers in one block. On the other hand, when $p > 0$, we claim that almost surely, for all $i \in \mathbb{N}$ we have $\ell^{(p,0)}(i) > 0$. By exchangeability it is enough to prove it for $\ell^{(p,0)}(1)$. Observing that $-\log|\Pi_1(t)| - c_p t$ is a spectrally positive Lévy process with negative drift, standard theory (cf. [3], Chapter VII) tells us it has the property that $(0, \infty)$ is irregular for 0 which, in turn, implies the claim. Hence, when $p > 0$, the partition $\Pi(\ell^{(p,0)})$ is a nontrivial collection of nonsingleton blocks.

4.5. *Embedded CMJ process.* A CMJ process is a branching process in which a typical individual reproduces at ages according to a random point process on $[0, \infty)$ and may or may not live forever. The coming generation at time t of a CMJ process consists of the collection of individuals born after time t whose parent was born before time t.

In this section we show that, for appropriate values of p, the collection of blocks $\Pi(\ell^{(p,z)})$ is also the coming generation at time z of certain CMJ process which is path-wise embedded into the fragmentation process. Our observation builds on ideas which go back to Neveu [35] and Biggins and Kyprianou [10].

In the following, we let $p \in (0, \bar{p}]$ be fixed and we consider the collection of distances

$$d_i = -\ln\bigl|\Pi_i\bigl(\ell^{(p,0)}\bigr)\bigr| - c_p \ell_i^{(p,0)},$$

that is, the point process of distances to the line $\ell^{(p,0)}$ of the individuals in the first generation. Note specifically that the latter point process cannot be defined when $p \leq 0$ on account of the fact that for all i we have that $\ell^{(p,0)}(i) = 0$. Define $D^{(p)}(\cdot) := \sum_i \delta_{d_i}(\cdot)$ to be the point process of the d_i's and let $\mu^{(p)} = E[D^{(p)}]$ be its intensity measure. The following proposition shows that the associated intensity measure has several convenient properties.

PROPOSITION 3. *Fix $p \in (0, \bar{p}]$. Then $\mu^{(p)}$ is a nonlattice measure with the following properties*:

(i) *Its Malthusian parameter is equal to $p + 1$, that is,*

$$\int_0^\infty e^{-(p+1)t} \mu^{(p)}(dt) = 1.$$

(ii) *For all $\varepsilon > 0$ such that $|\Phi(p - \varepsilon)| < \infty$,*

$$\int_0^\infty e^{-(p+1-\varepsilon)t} \mu^{(p)}(dt) < \infty.$$

PROOF. We first introduce some more notation: we define the *martingale weights*

$$y_i^{(p)}\big(\ell^{(p,z)}\big) := \big|\Pi_i\big(\ell^{(p,z)}\big)\big|^{(p+1)} e^{\Phi(p)\ell_i^{(p,z)}}. \tag{4.6}$$

Note that for any Borel set $A \in [0,\infty)$ we have, with the help of the many-to-one principle in Lemma 3, that

$$\begin{aligned}\int_A e^{-(p+1)t}\mu^{(p)}(dt) &= E\Big(\sum_i e^{-(p+1)d_i}\mathbf{1}_{\{d_i\in A\}}\Big)\\ &= E\Big(\sum_i y_i^{(p)}\big(\ell^{(p,0)}\big)\mathbf{1}_{\{-\ln|\Pi_i(\ell^{(p,0)})|-c_p\ell_i^{(p,0)}\in A\}}\Big)\\ &= \mathbb{P}^{(p)}(Y_{\tau_0}\in A),\end{aligned}$$

where $Y_t = \xi_t - c_p t$ and $\tau_0 = \inf\{t : Y_t > 0\}$. It is well known that, since Y is spectrally positive, the law of Y_{τ_0} is diffuse and hence, nonlattice. Note also that

$$\int_0^\infty e^{-(p+1)t}\mu^{(p)}(dt) = \mathbb{E}\Big(\sum y_i\big(\ell^{(p,0)}\big)\Big) = \mathbb{E}(W(p,0)) = 1,$$

which establishes the proof of part (i).

For the proof of part (ii), our objective is to compute

$$\int_0^\infty e^{-(p+1-\varepsilon)t}\mu^{(p)}(dt) = \mathbb{E}^{(p)}(e^{\varepsilon Y_{\tau_0}}). \tag{4.7}$$

Noting that $(0,\infty)$ is irregular for 0 for Y (cf. [3], Chapter VII), and moreover, that $\mathbb{E}^{(p)}(Y_1) = \Phi^{(p)\prime}(0) - c_p = \Phi'(p) - c_p \ge 0$, it follows that the ascending ladder height of Y is a compound Poisson process. The jump measure of the latter, say $m_H^{(p)}(dx)$, is, therefore, proportional to $\mathbb{P}^{(p)}(Y_{\tau_0}\in dx)$ and from [28], Corollary 7.9, it can also be written in the form

$$m_H^{(p)}(dx) = m^{(p)}(x,\infty)\,dx - \eta\Big(\int_x^\infty e^{-\eta(y-x)}m^{(p)}(y,\infty)\,dy\Big)\,dx,$$

where η is the largest root in $[0,\infty)$ of the equation $c_p\theta - \Phi(\theta+p) + \Phi(p) = 0$.

In order to verify (4.7) it therefore suffices to prove that

$$\int_0^\infty e^{\varepsilon x} m_H^{(p)}(dx) < \infty$$

whenever $|\Phi(p-\varepsilon)| < \infty$. To this end, note that with the help of Fubini's theorem and the fact that $m^{(p)}(dx) = e^{-px}m(dx)$, we have

$$\begin{aligned}&\int_0^\infty e^{\varepsilon x} m_H^{(p)}(dx)\\ &\qquad = \int_0^\infty e^{\varepsilon x}\int_x^\infty e^{-zp}m(dz)\,dx\end{aligned}$$

$$
\begin{aligned}
&- \eta \int_0^\infty e^{\varepsilon x} \left(\int_x^\infty e^{-\eta(y-x)} \int_y^\infty e^{-pz} m(dz)\, dy \right) dx \\
&= \int_0^\infty (1 - e^{-\varepsilon x}) e^{-(p-\varepsilon)z} m(dz) \\
&\quad - \frac{\eta}{\eta + \varepsilon} \int_0^\infty (e^{\varepsilon y} - e^{-\eta y}) \left(\int_y^\infty e^{-pz} m(dz) \right) dy \\
&= \frac{1}{\varepsilon} [\Phi(p) - \Phi(p - \varepsilon)] - \frac{\eta}{\varepsilon(\eta + \varepsilon)} \int_0^\infty (1 - e^{-\varepsilon z}) e^{-(p-\varepsilon)z} m(dz) \\
&\quad + \frac{1}{\eta + \varepsilon} \int_0^\infty (1 - e^{-\eta z}) e^{-pz} m(dz) \\
&= \frac{1}{\eta + \varepsilon} [\Phi(p) - \Phi(p - \varepsilon)] + \frac{1}{\eta + \varepsilon} [-\Phi(p)] \\
&= \frac{1}{\eta + \varepsilon} [\Phi(p + \eta) - \Phi(p - \varepsilon)],
\end{aligned}
$$

which is indeed finite when $|\Phi(p - \varepsilon)| < \infty$. $\square$

We conclude this section by showing an important relationship between first passage stopping lines and the coming generation of an embedded CMJ process. To this end, define for each $z \geq 0$

$$
\mathcal{L}(z) := \{ |\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)}, i \in \mathbb{N} \}.
$$

THEOREM 6. *For each $p \in (0, \bar{p}]$, the process $z \mapsto \mathcal{L}(z)$ is the process of the birth times of the individuals in the coming generation at time z of a CMJ process whose individuals live and reproduce according to the point process $D^{(p)}$.*

PROOF. Note that Proposition 3(i) implies that $\mu^{(p)}(0, \varepsilon) < \infty$ for all $\varepsilon > 0$ and hence, there is an almost sure first atom in the point process $D^{(p)}$. The theorem is trivially true for each $z \leq \inf\{d_i, i \in \mathbb{N}\}$. The process $\Pi(\ell^{(p,z)})$ is constant on $[0, \inf\{d_i\})$ and has a jump at time $z_1 = \inf\{d_i\}$ where a single block splits. Thanks to the fragmentation property, it gives birth to a collection of blocks in $\Pi(\ell^{(p,z_1)})$ whose positions to the right of their parent is again an instance of the point process $D^{(p)}$. This shows that as the line $\ell^{(p,z)}$ sweeps to the right, the coming generation process $\mathcal{L}(z)$ describes a CMJ process which is our claim. $\square$

5. Laws of large numbers. Before proceeding to the proof of asymptotics and uniqueness of multiplicative-martingale functions in the class $\mathcal{T}_2(p)$, we need to establish some further technical results which will play an important role. The following result is of a similar flavor to the types of laws of large numbers found in Bertoin and Martinez [7] and Harris, Knobloch and Kyprianou [18].

The next theorem gives us a strong law of large numbers for fragments in $\Pi(\ell^{(p,z)})$ as $z \uparrow \infty$ with respect to the weights (4.6) when $p \in (\underline{p}, 0]$. Recall that the Lévy measure m was defined in (2.2) and the definition of the martingale weights (4.6) gives

$$y_i^{(p)}(\ell^{(p,z)}) = |\Pi_i(\ell^{(p,z)})|^{(p+1)} e^{\Phi(p)\ell_i^{(p,z)}}.$$

THEOREM 7. *Fix $p \in (\underline{p}, 0]$. Let $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$. Suppose $f(x) \leq C e^{\varepsilon x}$ for some $C > 0$ and $\varepsilon > 0$ satisfying $|\Phi(p - \varepsilon)| < \infty$. Then*

$$\begin{aligned}\lim_{z\uparrow\infty} \sum_i y_i^{(p)}(\ell^{(p,z)}) f(-\log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)} - z)\\ = Q^{(p)}(f) W(\infty, p)\end{aligned} \tag{5.1}$$

in probability where

$$Q^{(p)}(f) = \frac{1}{\Phi'(p) - c_p} \int_{(0,\infty)} \left(\int_0^y f(t)\, dt \right) e^{-py} m(dy). \tag{5.2}$$

If f is uniformly bounded, then the above convergence may be upgraded to almost sure convergence.

PROOF. For convenience we shall define

$$W(\ell^{(p,z)}, p, f) := \sum_i y_i^{(p)}(\ell^{(p,z)}) f(-\log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)} - z).$$

Note that from the many-to-one principle in Lemma 3 that

$$\mathbb{E}(W(\ell^{(p,z)}, p, f)) = \mathbb{E}^{(p)}(f(Y_{\tau_z} - z)), \tag{5.3}$$

where $\tau_z = \inf\{t > 0 : Y_t > z\}$ and for $t \geq 0$, $Y_t = \xi_t - c_p t$. The process Y is in fact a subordinator on account of the fact that when $p \in (\underline{p}, 0]$, $c_p \leq 0$. Moreover, it has finite mean with $\mathbb{E}^{(p)}(Y_t) = (\Phi'(p) - c_p)t$ for $t \geq 0$. Note also that the assumption that $f(0) = 0$ implies that the expectation on the right-hand side of (5.3) does not include the possible contribution that comes from the event that Y creeps over z.

Let us first prove that if $f(x) \leq C e^{\varepsilon x}$ for some $C > 0$ and $\varepsilon > 0$ satisfying $|\Phi(p - \varepsilon)| < \infty$ (and, in particular, for uniformly bounded f), we have that

$$\lim_{z\uparrow\infty} \mathbb{E}(W(\ell^{(p,z)}, p, f)) = Q^{(p)}(f). \tag{5.4}$$

A classical result from the theory of subordinators (cf. [3], Chapter 3) tells us that for $y > 0$

$$\mathbb{P}^{(p)}(Y_{\tau_z} - z \in dy) = \int_{[0,z)} U^{(p)}(dx) m^{(p)}(z - x + dy),$$

where $U^{(p)}$ is the potential measure associated with Y under $\mathbb{P}^{(p)}$, meaning that $U^{(p)}(dx) = \int_0^\infty \mathbb{P}^{(p)}(Y_t \in dx)\,dt$ and we recall that $m^{(p)}(dx) = e^{-px}m(dx)$ is the Lévy measure of ξ under $\mathbb{P}^{(p)}$. Hence, it follows that

$$\mathbb{E}\big(W\big(\ell^{(p,z)}, p, f\big)\big) = U^{(p)} * g(z),$$

where

$$g(u) = \int_{(u,\infty)} f(y-u)m^{(p)}(dy).$$

It can also be shown that $V(dx) := U^{(p)}(dx) + \delta_0(dx)$ is a classical renewal measure of a renewal process with mean inter-arrival time given by $\mathbb{E}^{(p)}(Y_1)$ (see [28], Lemma 5.2). The latter result also indicates that the associated inter-arrival time of V has distribution $\int_0^\infty e^{-t}\mathbb{P}^{(p)}(Y_t \in dx)\,dt$ and hence, an easy computation shows that the mean inter-arrival time is equal to $\mathbb{E}^{(p)}(Y_1) = \Phi'(p) - c_p$. Applying the Key Renewal Theorem to $V * g(z)$, we deduce that, whenever g is directly Riemann integrable,

$$\begin{aligned}\lim_{z\uparrow\infty} \mathbb{E}\big(W\big(\ell^{(p,z)}, p, f\big)\big) &= \frac{1}{\mathbb{E}^{(p)}(Y_1)} \int_0^\infty g(t)\,dt \\ &= \frac{1}{\mathbb{E}^{(p)}(Y_1)} \int_{(0,\infty)} \biggl(\int_0^y f(t)\,dt\biggr) m^{(p)}(dy).\end{aligned}$$

Note that g has no discontinuities on account of the fact that, for each $u > 0$, $g(u+) - g(u-) = f(0)m^{(p)}(\{u\})$ which equals zero thanks to the assumption that $f(0) = 0$. Moreover, thanks to the assumption that $f(x) \le C^{\varepsilon x}$ and $|\Phi(p-\varepsilon)| < \infty$, we have that

$$\begin{aligned}\int_0^\infty g(t)\,dt &\le \frac{C}{\varepsilon} \int_{(0,\infty)} (e^{\varepsilon y} - 1)m^{(p)}(dy) \\ &= \frac{C}{\varepsilon} \int_{(0,\infty)} (1 - e^{-\varepsilon y})e^{-(p-\varepsilon)y} m(dy) \\ &= \frac{C}{\varepsilon}\big(\Phi(p) - \Phi(p-\varepsilon)\big) \\ &< \infty,\end{aligned}$$

which shows that g is directly Riemann integrable. We have thus established (5.4).

Next we turn to establishing the limit (5.2) in the almost sure sense when f is uniformly bounded. Harris, Knobloch and Kyprianou [18] show that, when $p = 0$, the required strong law of large numbers holds for all bounded measurable f in the sense of almost sure convergence. Although we are interested in conservative fragmentation processes in this paper, the proof of (5.2) for the case that $p \in (\underline{p}, 0)$ is mathematically similar to the dissipative case that was handled when $p = 0$ in [18]. In the notation of [18], the role of the quantity $X_{j,\eta}^{1+p^*}$ is now played by the martingale weights $y_j(\ell^{(p,z)})$. In that case, using (5.3) in place of the limit (9) in [18], all of the proofs go through verbatim, or with obvious minor modification, with the exception of their Lemma 5 which incurs a moment condition. In fact, this moment condition turns out to be unnecessary as we shall now demonstrate. The aforementioned lemma requires us to show that, in the notation of the current setting,

$$\sup_{z\geq 0}\mathbb{E}(W(\ell^z, p)^q) < \infty \qquad \text{for some } q > 1. \tag{5.5}$$

Thanks to Jensen's inequality, the process $(W(\ell^z, p)^q, z \geq 0)$ is a submartingale. Hence, recalling that the almost sure limit of $W(\ell^z, p)$ is $W(\infty, p)$, as soon as it can be shown that $\mathbb{E}(W(\infty, p)^q) < \infty$ for some $q > 1$, then (5.5) is satisfied. Note however, the latter has been clearly established in the proof of Theorem 2 of [5] despite the fact that the aforementioned theorem itself does not state this fact. This completes the proof of the almost sure convergence (5.2) for uniformly bounded f.

We shall now obtain the required weak law of large numbers for $W(\ell^{(p,z)}, p, f)$. To this end, let us suppose that $\{f_k : k \geq 1\}$ is an increasing sequence of bounded positive functions such that, in the pointwise sense, $f_k \uparrow f$. It follows from the aforementioned strong law of large numbers for each f_k that

$$\liminf_{z\uparrow\infty} W\big(\ell^{(p,z)}, p, f\big) \geq \liminf_{z\uparrow\infty} W\big(\ell^{(p,z)}, p, f_k\big) = Q^{(p)}(f_k)W(\infty, p)$$

for all k and hence, by monotone convergence,

$$\liminf_{z\uparrow\infty} W\big(\ell^{(p,z)}, p, f\big) \geq Q^{(p)}(f)W(\infty, p)$$

almost surely. Next note by Fatou's Lemma,

$$\begin{aligned}
0 &\leq \mathbb{E}\Big(\liminf_{z\uparrow\infty} W\big(\ell^{(p,z)}, p, f\big) - Q^{(p)}(f)W(\infty, p)\Big)\\
&= \mathbb{E}\Big(\liminf_{z\uparrow\infty} W\big(\ell^{(p,z)}, p, f\big)\Big) - Q^{(p)}(f)\\
&\leq \liminf_{z\uparrow\infty} \mathbb{E}\big(W\big(\ell^{(p,z)}, p, f\big)\big) - Q^{(p)}(f),\\
&= 0,
\end{aligned}$$

where the final equality follows by (5.4) and hence, we are led to the conclusion that

$$\liminf_{z\uparrow\infty} W\big(\ell^{(p,z)}, p, f\big) = Q^{(p)}(f)W(\infty, p) \tag{5.6}$$

almost surely.

Next define for $z \geq 0$

$$\Theta_z = W\big(\ell^{(p,z)}, p, f\big) - \inf_{u\geq z} W\big(\ell^{(p,u)}, p, f\big) \geq 0.$$

Note by (5.4), monotone convergence and (5.6) that

$$\begin{aligned}\lim_{z\uparrow\infty}\mathbb{E}(\Theta_z) &= \lim_{z\uparrow\infty}\mathbb{E}\big(W\big(\ell^{(p,z)},p,f\big)\big) - \lim_{z\uparrow\infty}\mathbb{E}\Big(\inf_{u\geq z} W\big(\ell^{(p,u)},p,f\big)\Big) \\ &= Q^{(p)}(f) - \mathbb{E}\Big(\liminf_{z\uparrow\infty} W\big(\ell^{(p,z)},p,f\big)\Big) \\ &= 0.\end{aligned}$$

It now follows as a simple consequence from the Markov inequality that Θ_z converges in probability to zero. Using the latter, together with the almost sure convergence in (5.6), it follows that

$$\begin{aligned}&W\big(\ell^{(p,z)},p,f\big) - Q^{(p)}(f)W(\infty,p) \\ &\qquad = \Theta_z + \inf_{u\geq z} W\big(\ell^{(p,u)},p,f\big) - Q^{(p)}(f)W(\infty,p)\end{aligned}$$

converges in probability, to zero as $z\uparrow\infty$. $\square$

REMARK 3. In the above proof, when dealing with the almost sure convergence (5.2) for uniformly bounded f, the replacement argument we offer for Lemma 5 of Harris, Knobloch and Kyprianou [18] applies equally well to the case that the fragmentation process is dissipative. This requires however, some bookkeeping based around the computations in Bertoin [5], which assumes conservativeness, in order to verify that the L^q estimates are still valid. The consequence of this observation is that the moment condition (A3) in [18] is unnecessary. In fact, the aforementioned condition (A3) is equivalent to the condition that the dislocation measure has finite total mass in that context.

The next result gives us a strong law with respect to the weights (4.6) for the regime $p\in(0,\bar{p}]$. For this we recall also that the measure $D^{(p)}$ was defined for $p\in(0,\bar{p}]$ in Proposition 3 by

$$D^{(p)}(dx) = \sum_i \delta_{\{-(p+1)^{-1}\log y_i(\ell^{(p,0)})\in dx\}}$$

and its intensity was denoted by $\mu^{(p)}$. It is also worth recalling that the Lévy measure associated with the tagged fragment $\xi_\cdot := -\log|\Pi_1(\cdot)|$ is denoted by $m(dx)$ for $x>0$. Moreover, under the measure $\mathbb{P}^{(p)}$ where $p>\underline{p}$, the aforementioned Lévy measure takes the form $e^{-px}m(dx)$ for $x>0$.

THEOREM 8. *Fix $p\in(0,\bar{p}]$. Let $f:[0,\infty)\to[0,\infty)$ such that $f(x)\leq Ce^{\varepsilon x}$ for some $C>0$ and $\varepsilon>0$ satisfying $|\Phi(p-\varepsilon)|<\infty$. Then, almost surely,*

$$\lim_{z\uparrow\infty}\frac{\sum_i y_i^{(p)}(\ell^{(p,z)})f(-\log|\Pi_i(\ell^{(p,z)})| - c_p\ell_i^{(p,z)} - z)}{W(\ell^{(p,z)},p)} = Q^{(p)}(f), \tag{5.7}$$

where

$$Q^{(p)}(f) = \frac{\int_0^\infty \int_u^\infty e^{-(p+1)y} f(y-u)\mu^{(p)}(dy)\,du}{\int_{(0,\infty)} ye^{-(p+1)y}\mu^{(p)}(dy)}. \tag{5.8}$$

PROOF. First recall from Theorem 6 that when $p \in (0, \bar{p}]$, the sequence of stopping lines $(\ell^{(p,z)} : z \geq 0)$ sweeps out the coming generation of an embedded CMJ process with Malthusian parameter $p+1$ and whose associated birth process is described by the point process $D^{(p)}(\cdot)$. For the aforementioned CMJ process we denote by $\{\sigma_i : i \geq 0\}$ the birth times of individuals, where the enumeration is in order of birth times starting with the initial ancestor, counted as 0, having birth time $\sigma_0 = 0$. Define

$$\phi_0^f(u) = \mathbf{1}_{\{u>0\}} e^{(p+1)u} \int_{(u,\infty)} e^{-(p+1)y} f(y-u) D^{(p)}(dy).$$

Then, following Jagers' classical theory of counting with characteristics (cf. Jagers [22]), our CMJ processes have count at time $z \geq 0$ given by

$$\eta_z^f := \sum_{i : \sigma_i \leq z} \phi_i^f(z - \sigma_i),$$

where, for each i, ϕ_i^f has the same definition as ϕ_0^f except that the counting measure $D^{(p)}$ is replaced by the counting measure of the point process describing the age of the ith individual at the moment it reproduces. In this respect, the characteristics $\{\phi_i : i\}$ are i.i.d. Writing $\{\sigma_j^i : j \geq 1\}$ for the ages at which the ith individual reproduces and $\mathcal{C}_z$ for the index set of individuals which form the coming generation at time $z \geq 0$, we have

$$\begin{aligned} e^{-(p+1)z}\eta_z^f &= \sum_{i : \sigma_i \leq z} e^{-(p+1)\sigma_i} \sum_{j : \sigma_j^i > z - \sigma_i} e^{-(p+1)\sigma_j^i} f(\sigma_j^i + \sigma_i - z) \\ &= \sum_{k : k \in \mathcal{C}_z} e^{-(1+p)\sigma_k} f(\sigma_k - z) \\ &= \sum_i y_i^{(p)}(\ell^{(p,z)}) f(-\log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)} - z), \end{aligned}$$

where the final equality follows from Theorem 6. In the particular case that f is identically equal to unity, we denote η_z^f by η_z^1 and we see that $e^{-(p+1)z}\eta_z^1 = W(\ell^{(p,z)}, p)$.

Recall that $\mu^{(p)}$ was defined in Proposition 3 as the intensity of the counting measure $D^{(p)}$. The strong law of large numbers (5.7) now follows from the classical strong law of large numbers for CMJ processes given in [34], Theorem 6.3, which says that

$$\lim_{z\uparrow\infty} \frac{\eta_z^f}{\eta_z^1} = \frac{\int_0^\infty \int_u^\infty e^{-(p+1)y} f(y-u)\mu^{(p)}(dy)\,du}{\int_{(0,\infty)} ye^{-(p+1)y}\mu^{(p)}(dy)} \tag{5.9}$$

almost surely provided that the following two conditions hold for some $\beta < p+1$. First,

$$\int_{(0,\infty)} e^{-\beta z}\mu^{(p)}(dz) < \infty$$

and second,

$$\mathbb{E}\left(\sup_{u\geq 0} e^{(p+1-\beta)u}\int_{(u,\infty)} e^{-(p+1)y} f(y-u) D^{(p)}(dy)\right) < \infty.$$

The first condition holds thanks to Proposition 3(ii) for all β sufficiently close to $p+1$. For the second condition, we note that when $f(x) \leq Ce^{\varepsilon x}$, we may estimate for all β sufficiently close to $p+1$,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{u\geq 0} e^{(p+1-\beta)u}\int_{(u,\infty)} e^{-(p+1)y} f(y-u) D^{(p)}(dy)\right)\\
&\quad\leq C\mathbb{E}\left(\sup_{u\geq 0} e^{(p+1-\beta-\varepsilon)u}\int_{(u,\infty)} e^{-(p+1-\varepsilon)y} D^{(p)}(dy)\right)\\
&\quad\leq C\mathbb{E}\left(\int_{(0,\infty)} e^{-(p+1-\varepsilon)y} D^{(p)}(dy)\right)\\
&\quad= C\int_{(0,\infty)} e^{-(p+1-\varepsilon)y}\mu^{(p)}(dy),
\end{aligned}$$

which is finite, again thanks to Proposition 3, providing $|\Phi(p-\varepsilon)| < \infty$. Note in particular that these conditions also ensure that the right-hand side of (5.9) is positive but neither zero nor infinity in value. The proof of part (i) of the theorem is thus complete as soon as we note that (5.9) is the desired limit. $\square$

6. Exact asymptotics and uniqueness.

In this section we establish the asymptotics of multiplicative martingale functions in the class $\mathcal{T}_2(p)$ which will quickly lead to the property of uniqueness within the same class.

For any product martingale function ψ_p, with speed c_p, where $p \in (\underline{p}, \bar{p}]$, which belongs to the class $\mathcal{T}_2(p)$, recall that we have defined

$$L_p(x) = e^{(p+1)x}\bigl(1-\psi_p(x)\bigr).$$

We start with the following first result.

THEOREM 9. *Suppose that $p \in (\underline{p}, \bar{p}]$ and that $\psi_p \in \mathcal{T}_2(p)$ is a product martingale function which makes $(M(t, \underline{p}, x), t \geq 0)$ in (1.4) a martingale. Then for all $\beta \geq 0$*

$$\lim_{x\uparrow\infty} \frac{L_p(x+\beta)}{L_p(x)} = 1.$$

That is to say, L_p is additively slowly varying.

PROOF. The proof is an adaptation of arguments found in [24]. First note that the monotonicity of L_p implies that for all $\beta \geq 0$

$$\limsup_{x\uparrow\infty} \frac{L_p(x+\beta)}{L_p(x)} \geq \liminf_{x\uparrow\infty} \frac{L_p(x+\beta)}{L_p(x)} \geq 1.$$

In turn, this implies that for each $\beta \geq 0$ there exists an increasing subsequence $\{x_k(\beta) : k \geq 1\}$ tending to infinity along which we have the following limit:

$$\lim_{k\uparrow\infty} \frac{L_p(x_k(\beta)+\beta)}{L_p(x_k(\beta))} = \limsup_{x\uparrow\infty} \frac{L_p(x+\beta)}{L_p(x)} \geq 1.$$

Suppose now that there exists a $\beta_0 > 0$ and $\eta > 1$ such that

$$\lim_{k\uparrow\infty} \frac{L_p(x_k(\beta_0)+\beta_0)}{L_p(x_k(\beta_0))} > \eta.$$

The monotonicity of L_p implies that for all $\beta \geq \beta_0$

$$\liminf_{k\uparrow\infty} \frac{L_p(x_k(\beta_0)+\beta)}{L_p(x_k(\beta_0))} = \lim_{k\uparrow\infty} \frac{L_p(x_k(\beta_0)+\beta_0)}{L_p(x_k(\beta_0))} > \eta.$$

The crux of the proof will be to show that this leads to a contradiction. To this end, recall the identity (4.5) in the proof of Theorem 5. Starting from this expression and with the help of a telescopic sum we have that for all $\beta \in \mathbb{R}$

$$\begin{aligned} 1 - \psi_p(x) = \mathbb{E}\sum_i &[1 - \psi_p(x - \log|\Pi_i(\ell^{(p,z)})| - c_p\ell_i^{(p,z)})] \\ &\times \prod_{j<i} \psi_p(x - \log|\Pi_j(\ell^{(p,z)})| - c_p\ell_j^{(p,z)}). \end{aligned}$$

Recalling the definition of L_p, it follows easily that

$$\begin{aligned} 1 = \mathbb{E}\Bigg[\sum_i &\frac{L_p(x_k(\beta_0) - \log|\Pi_i(\ell^{(p,z)})| - c_p\ell_i^{(p,z)})}{L_p(x_k(\beta_0))} |\Pi_i(\ell^{(p,z)})|^{p+1} \\ &\times e^{\Phi(p)\ell_i^{(p,z)}} \prod_{j<i} \psi_p(x_k(\beta_0) - \log|\Pi_j(\ell^{(p,z)})| - c_p\ell_j^{(p,z)})\Bigg]. \end{aligned} \tag{6.1}$$

Now pick $z \geq \beta_0$. Next, recalling that $\psi_p(\infty) = 1$ and that $-\log|\Pi_i(\ell_i^{(p,z)})| - c_p\ell_i^{(p,z)} \geq z \geq \beta_0$, we can take limits in (6.1) as $k \uparrow \infty$, applying Fatou's Lemma twice, to reach the inequality

$$1 \geq \eta\mathbb{E}\sum_i |\Pi_i(\ell^{(p,z)})|^{p+1} e^{\Phi(p)\ell_i^{(p,z)}}.$$

However, Theorem 5 implies that the expectation above is equal to unity and we reach a contradiction. We are forced to conclude that $\limsup_{x\uparrow\infty} L_p(x+\beta)/L_p(x) = 1$ and the required additive slow variation follows. $\square$

The following lemma is a key ingredient which will help extract exact asymptotics.

LEMMA 4. *Fix* $p \in (\underline{p}, \bar{p}]$. *Suppose that* $g : \mathbb{R} \to (0, \infty)$ *is a monotone increasing function and additive slowly varying at* $+\infty$, *that is to say, it satisfies the property that for all* $\beta \geq 0$

$$\lim_{x \uparrow \infty} \frac{g(x+\beta)}{g(x)} = 1.$$

Then

$$\lim_{z \uparrow \infty} \frac{\sum_i y_i^{(p)}(\ell^{(p,z)}) g(x - \log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)})}{g(x+z) W(\ell^{(p,z)}, p)} = 1,$$

where the limit is understood almost surely when $p \in (0, \bar{p}]$ *and in probability when* $p \in (\underline{p}, 0]$.

PROOF. The proof will follow closely ideas in Biggins and Kyprianou [10], Theorem 8.6. First define

$$\mathcal{I}_c = \{ i : -\log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)} > z + c \}.$$

Then, using the fact that g is increasing and $-\log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)} \geq z$, we have

$$\begin{aligned} 1 &\leq \frac{\sum_i y_i^{(p)}(\ell^{(p,z)}) g(x - \log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)})}{g(x+z) W(\ell^{(p,z)}, p)} \\ &\leq \frac{g(x+z+c)}{g(x+z)} + \frac{\sum_{i \in \mathcal{I}_c} y_i^{(p)}(\ell^{(p,z)}) g(x - \log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)})}{g(x+z) W(\ell^{(p,z)}, p)}. \end{aligned} \tag{6.2}$$

As g is additively slowly varying, we may appeal to the classical representation of slowly varying functions (cf. Feller [15], VIII.9) to deduce that for all $\varepsilon_1, \varepsilon_2 > 0$ there exists a $z_0 > 0$ such that for all $u > 0$

$$\sup_{z > z_0} \frac{g(z+u)}{g(z)} \leq (1 + \varepsilon_1) e^{\varepsilon_2 u}.$$

This allows for the upper estimate on the second term on the right-hand side of (6.2) for all z sufficiently large

$$\begin{aligned} &\frac{\sum_{i \in \mathcal{I}_c} y_i^{(p)}(\ell^{(p,z)}) g(x - \log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)})}{g(x+z) W(\ell^{(p,z)}, p)} \\ &\qquad \leq \frac{\sum_i y_i^{(p)}(\ell^{(p,z)}) f_c(-\log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)} - z)}{W(\ell^{(p,z)}, p)}, \end{aligned} \tag{6.3}$$

where

$$f_c(x) = \mathbf{1}_{\{x>c\}}(1+\varepsilon_1)e^{\varepsilon_2 x}.$$

Now note from Theorem 8 that the right-hand side of (6.3) converges almost surely to $Q^{(p)}(f_c)$ when $p \in (0, \bar{p}]$, where the definition of $Q^{(p)}(f_c)$ is given by (5.8). When $p \in (\underline{p}, 0]$, the convergence occurs in probability and $Q^{(p)}(f_c)$ is defined by (5.2). In either case, thanks to the appropriate integrability of the function $e^{\varepsilon_2 x}$ for sufficiently small ε_2, we have that $Q^{(p)}(f_c) \downarrow 0$ as $c \uparrow \infty$. Moreover, in both cases, using (6.3) and (6.2) we have

$$\begin{aligned}
0 &\le \frac{\sum_i y_i^{(p)}(\ell^{(p,z)})g(x - \log|\Pi_i(\ell^{(p,z)})| - c_p\ell_i^{(p,z)} - z)}{g(x+z)W(\ell^{(p,z)}, p)} - 1 \\
&\le \left|\frac{g(x+z+c)}{g(x+z)} - 1\right| + Q^{(p)}(f_c) \\
&\quad + \left|\frac{\sum_i y_i^{(p)} f_c(-\log|\Pi_i(\ell^{(p,z)})| - c_p\ell_i^{(p,z)} - z)}{W(\ell^{(p,z)}, p)} - Q^{(p)}(f_c)\right|.
\end{aligned}$$

When $p \in (0, \bar{p}]$, thanks to the preceding remarks, as $z \uparrow \infty$, the almost sure limit of the right-hand side above can be made arbitrarily small by choosing c sufficiently large. When $p \in (\underline{p}, 0]$, again thanks to the preceding remarks, we see that for each $\varepsilon > 0$, we may choose c sufficiently large such that

$$\lim_{z\uparrow\infty} \mathbb{P}\left(\left|\frac{\sum_i y_i^{(p)}(\ell^{(p,z)})g(x - \log|\Pi_i(\ell^{(p,z)})| - c_p\ell_i^{(p,z)} - z)}{g(x+z)W(\ell^{(p,z)}, p)} - 1\right| > \varepsilon\right) = 0,$$

thus establishing the required convergence in probability. $\square$

We are now ready to establish the asymptotics of multiplicative martingale functions from which uniqueness will follow. It has already been shown in Theorem 9 that $L_p = e^{(p+1)x}(1 - \psi_p(x))$ is additively slowly varying [with $\psi_p, \in \mathcal{T}_2(p)$ a product martingale function with speed c_p].

THEOREM 10. *Suppose that ψ_p is any product martingale function in $\mathcal{T}_2(p)$ with speed c_p [i.e., which makes $M(t, p, x)$ in (1.4) a martingale] such that $p \in (\underline{p}, \bar{p}]$. Then there exist some constants $k_p \in (0, \infty)$ such that when $p \in (\underline{p}, \bar{p})$, we have that*

$$L_p(x) \to k_p \qquad \text{as } x \uparrow \infty$$

and when $p = \bar{p}$ we have

$$\frac{L_{\bar{p}}(x)}{x} \to k_{\bar{p}} \qquad \text{as } x \uparrow \infty.$$

PROOF. Suppose that $p \in (\underline{p}, \bar{p}]$. It is not difficult to show that for any given $\varepsilon > 0$ we may take z sufficiently close to 1 such that

$$-\frac{\log z}{1-z} \in \left[1, (1-\varepsilon)^{-1}\right]. \tag{6.4}$$

Thanks to Theorem 5 and, in particular, the fact that $M(\ell^{(p,z)}, p, x)$ is a uniformly integrable martingale with limit $M(\infty, p, x)$,

$$-\log M(\infty, p, x) = -\lim_{z\uparrow\infty} \sum_i \log \psi_p\left(x - \log\left|\Pi_i\left(\ell^{(p,z)}\right)\right| - c_p \ell_i^{(p,z)}\right).$$

Since $-\log|\Pi_i(\ell^{(p,z)})| - c_p \ell_i^{(p,z)} \geq z$ and $\psi_p(\infty) = 1$, we may apply the estimate in (6.4) and deduce that

$$\begin{aligned} &-e^{(p+1)x} \log M(\infty, p, x) \\ &\quad = \lim_{z\uparrow\infty} \sum_i y_i^{(p)}\left(\ell^{(p,z)}\right) L_p\left(x - \log\left|\Pi_i\left(\ell^{(p,z)}\right)\right| - c_p \ell_i^{(p,z)}\right). \end{aligned} \tag{6.5}$$

Next, we consider the restriction that $p \in (\underline{p}, \bar{p})$. Recalling from Theorem 5 that $W(\ell^{(p,z)}, p)$ converges almost surely in L^1 to $W(\infty, p)$ and from Theorem 9 that L_p is additive slowly varying, we may apply the conclusion of Lemma 4 to (6.6) and deduce that

$$\frac{-e^{(p+1)x} \log M(\infty, p, x)}{W(\infty, p)} = \lim_{z\uparrow\infty} L_p(x+z). \tag{6.6}$$

The right-hand side above is purely deterministic and the left-hand side is bounded and strictly positive which leads us to the conclusion that there exists a constant $k_p \in (0, \infty)$ such that $L_p(x) \sim k_p$ as $x \uparrow \infty$.

Now suppose that $p = \bar{p}$. From Bertoin and Rouault [8] (see also the method in Kyprianou [27]) it is known that

$$\partial W(t, \bar{p}, x) := \sum_{i \in \mathcal{I}(t,x)} \left(x - \log|\Pi_i(t)| - c_{\bar{p}} t\right) y_i^{(\bar{p})}(t),$$

where

$$\mathcal{I}(t, x) = \left\{ i : \inf_{s \leq t} \{x - \log|\Pi_i(t)| - c_{\bar{p}} t\} > 0 \right\}$$

is a uniformly integrable positive martingale with mean x; we denote its limit by $\partial W(\infty, \bar{p}, x)$. Moreover, thanks to (3.2), it is also true that there exists a part of the probability space, say γ_x, satisfying $\lim_{x\uparrow\infty} \mathbb{P}(\gamma_x) = 1$, such that $\partial W(\infty, \bar{p}, x) = \partial W(\infty, \bar{p})$ on γ_x.

Again, thanks to Lemma 1, we may project the limit $\partial W(\infty, \bar{p}, x)$ back on to the filtration $\mathcal{F}_{\ell(p,z)}$ to obtain

$$\begin{aligned}\partial W\big(\ell^{(\bar{p},z)}, \bar{p}, x\big) &:= \sum_{i\in\mathcal{I}(\ell^{(\bar{p},z)},x)} y_i^{(\bar{p})}\big(\ell^{(\bar{p},z)}\big)\big(x - \log\big|\Pi_i\big(\ell^{(\bar{p},z)}\big)\big| - c_{\bar{p}}\ell^{(\bar{p},z)}\big)\\ &= \mathbb{E}\big(\partial W(\infty, \bar{p}, x)\big|\mathcal{F}_{\ell(\bar{p},z)}\big)\end{aligned}$$

is a positive uniformly integrable martingale with almost sure and L^1 limit $\partial W(\infty, \bar{p}, x)$ where

$$\mathcal{I}\big(\ell^{(\bar{p},z)}, x\big) = \Big\{i : \inf_{s\le\ell^{(\bar{p},z)}}\{x - \log|\Pi_i(s)| - c_{\bar{p}}s\} > 0\Big\}.$$

Note also that this implies that for each $x > 0$, on γ_x we have

$$\lim_{z\uparrow\infty}\sum_i y_i^{(\bar{p})}\big(\ell^{(\bar{p},z)}\big)\big(x - \log\big|\Pi_i\big(\ell^{(\bar{p},z)}\big)\big| - c_{\bar{p}}\ell^{(\bar{p},z)}\big) = \partial W(\infty, \bar{p}) \tag{6.7}$$

almost surely. As we may take x arbitrarily large, the above almost sure convergence occurs on the whole of the probability space.

Next, turning to Lemma 4, we note that both $g(x) = L_{\bar{p}}(x)$ and $g(x) = x$ are suitable functions to use within this context. We, therefore, have for $x > 0$

$$\lim_{z\uparrow\infty}\frac{x+z}{L_{\bar{p}}(x+z)}\frac{\sum_i y_i^{(\bar{p})}(\ell^{(\bar{p},z)})L_{\bar{p}}(x - \log|\Pi_i(\ell^{(\bar{p},z)})| - c_{\bar{p}}\ell_i^{(\bar{p},z)})}{\sum_i y_i^{(\bar{p})}(\ell^{(\bar{p},z)})(x - \log|\Pi_i(\ell^{(\bar{p},z)})| - c_{\bar{p}}\ell^{(\bar{p},z)})} = 1$$

almost surely. Thanks to (6.6) and (6.7), it follows that

$$\lim_{z\uparrow\infty}\frac{x+z}{L_{\bar{p}}(x+z)} = \frac{-e^{(\bar{p}+1)x}\log M(\infty, \bar{p}, x)}{\partial W(\infty, \bar{p})} \tag{6.8}$$

almost surely. In particular, as the left-hand side above is deterministic and the right-hand side is a random variable in $(0, \infty)$, it follows that the limit must be equal to some constant in $\in (0, \infty)$ which we identify as $1/k_{\bar{p}}$. The proof is complete. $\square$

THEOREM 11. *For $p \in (\underline{p}, \bar{p}]$, there is a unique multiplicative martingale function ψ_p which is a solution to (1.4) with speed c_p in $\mathcal{T}_2$ (up to additive translation in its argument). In particular, when $p \in (\underline{p}, \bar{p})$, the shape of the multiplicative martingale function is given by*

$$\psi_p(x) = \mathbb{E}\big(e^{-e^{-(p+1)x}W(\infty,p)}\big)$$

and the shape of the critical multiplicative martingale function is given by

$$\psi_{\bar{p}}(x) = \mathbb{E}\big(e^{-e^{-(\bar{p}+1)x}\partial W(\infty,\bar{p})}\big).$$

PROOF. First, suppose that $p \in (\underline{p}, \bar{p})$ and take any traveling wave ψ_p at wave speed c_p. Thanks to the uniform integrability of the associated multiplicative martingale, as well as (6.6), we have that

$$\psi_p(x) = \mathbb{E}(M(\infty, p, x)) = \mathbb{E}\big(e^{-k_p e^{-(p+1)x} W(\infty, p)}\big). \tag{6.9}$$

Note that from (2.9), if $\psi_p(x)$ is a traveling wave, then so is $\psi_p(x+k)$ for any $k \in \mathbb{R}$. We, therefore, deduce from (6.9) that traveling waves at wave speed c_p and $p \in (\underline{p}, \bar{p})$ are unique up to an additive translation in the argument. Moreover, without loss of generality, the shape of the traveling wave may be taken to be of the form given on the right-hand side of (6.9) but with $k_p = 1$. Exactly the same reasoning applies in the case $p = \bar{p}$ except that we appeal to the distributional identity (6.8) instead of (6.6). □

7. Proof of Theorem 1. Given the conclusion of Theorem 11, it remains only to prove the first part of Theorem 1. To this end, first suppose that $\psi_p \in \mathcal{T}_2(p)$ and that

$$\mathcal{A}_p \psi_p \equiv 0 \tag{7.1}$$

(and hence, implicitly we understand that ψ_p is in the domain of $\mathcal{A}_p$). Define $u(x,t) := \mathbb{E}(M(t,p,x))$ for $x \in \mathbb{R}$ and $t \geq 0$ where $M(t,p,x)$ is given by (1.4). Also, for convenience, write

$$\mathcal{L}\psi_p(x) := \int_{\nabla_1} \Big\{ \prod_i \psi_p(x - \log s_i) - \psi_p(x) \Big\} \nu(ds).$$

The change of variable formula gives us

$$u(x,t) - u(x,0) = \mathbb{E}\Big[\int_0^t \frac{\partial}{\partial t} M(s,p,x)\, ds + \sum_{s \leq t} \Delta M(s,p,x) \Big].$$

Henceforth, we will use the notation

$$z_i(s) = x - \log|\Pi_i(s)| - cs.$$

Recall the Poisson point process construction of the fragmentation X described in the Introduction. Write $N(\cdot)$ for the Poisson random measure on $\mathbb{R}_+ \times \mathbb{N} \times \nabla_1$ with measure intensity $dt \otimes \# \otimes \nu(ds)$ which describes the evolution of the fragmentation process. Using classical stochastic analysis of semi-martingales and the Poissonian construction of fragmentation processes, we deduce that

$$\begin{aligned} u(x,t) &- u(x,0) \\ &= \mathbb{E} \int_0^t \frac{\partial}{\partial t} M(\tau, p, x)\, d\tau \\ &\quad + \mathbb{E} \int_0^t \int_{\mathbb{N}} \int_{\nabla_1} \Big\{ \prod_{i \neq k} \psi_p(z_i(\tau)) \prod_{j \geq 1} \psi_p\big(z_k(\tau-) - \log s_j\big) \\ &\qquad\qquad - \prod_i \psi_p(z_i(\tau-)) \Big\} N(d\tau, dk, ds) \end{aligned} \tag{7.2}$$

$$= \mathbb{E}\int_0^t d\tau \cdot \prod_i \psi_p(z_i(\tau)) \cdot \sum_k \frac{1}{\psi_p(z_k(\tau))}(-c\psi_p'(z_k(\tau)))$$

$$+ \mathbb{E}\int_0^t d\tau \cdot \prod_i \psi_p(z_i(\tau-)) \cdot \sum_k \frac{1}{\psi_p(z_k(\tau-))}\mathcal{L}\psi_p(z_k(\tau-))$$

$$= \mathbb{E}\int_0^t d\tau \cdot \prod_i \psi_p(z_i(\tau)) \cdot \sum_k \frac{1}{\psi_p(z_k(\tau))}\mathcal{A}_p\psi_p(z_k(\tau))$$

and it is obvious that changing $\tau-$ into τ does not affect the value of the integral in the final two steps above.

Assumption (7.1) now implies that for all $x \in \mathbb{R}$ and $t \geq 0$

$$u(x,t) = E\biggl(\prod_i \psi_p(z_i(t))\biggr) = u(x,0) = \psi_p(x).$$

It is now a simple application of the fragmentation property to deduce that

$$\mathbb{E}\biggl(\prod_i \psi_p(z_i(t+s))\Big|\mathcal{F}_t\biggr)$$
$$= \prod_i \mathbb{E}\biggl(\prod_{j\geq 1} \psi_p\bigl(z_i(t) - \log\bigl|\Pi_j^{(i)}(s)\bigr| - cs\bigr)\Big|\mathcal{F}_t\biggr)$$
$$= \prod_i \psi_p(z_i(t)),$$

where $|\Pi^{(i)}|$ is the fragmentation process initiated by the i-fragment at time t of the original fragmentation process. Hence, $\prod_i \psi_p(z_i(t))$ is a $\mathbb{F}$-martingale.

For the converse direction, we know from Theorem 11 that if $\psi_p \in \mathcal{T}_2(p)$ makes $(M(t,p,x), t \geq 0)$ a martingale then, without loss of generality, we may take

$$\psi_p(x) = \mathbb{E}\bigl(\exp\bigl(-e^{-(p+1)x}\Delta_p\bigr)\bigr), \tag{7.3}$$

where $\Delta_p = W(\infty, p)$ if $p \in (\underline{p}, \bar{p})$ and $\Delta_p = \partial W(\infty, p)$ if $p = \bar{p}$. For the rest of the proof, ψ_p will be given by the above expression. Note that since $L_p(x) = e^{(p+1)x}(1-\psi_p(x))$ is monotone increasing (see Proposition 1), we have that

$$0 \leq L_p'(y) = (p+1)L_p(y) - e^{(p+1)y}\psi_p'(y)$$

so that

$$\psi_p'(y) \leq (p+1)\bigl(1-\psi_p(y)\bigr). \tag{7.4}$$

This estimate and the fact that it implies the uniform boundedness of ψ_p', will be used at several points in the forthcoming text.

We start with a lemma which shows that $\mathcal{A}_p\psi_p$ is well defined and that it is continuous.

LEMMA 5.

$$[\mathcal{L}\psi_p(x)| < \infty \qquad \forall x.$$

Furthermore, $x \mapsto \mathcal{A}_p\psi_p(x)$ *is continuous.*

PROOF. We will use the following fact. Given a_n and b_n two sequences in [0, 1], we have that

$$\left|\prod a_n - \prod b_n\right| \le \sum |a_n - b_n|. \tag{7.5}$$

As an immediate application we have that

$$\begin{aligned}
|\mathcal{L}\psi_p(x)| &= \left|\int_{\nabla_1}\left\{\prod_{i=1}^{\infty}\psi_p(x-\log s_i) - \psi_p(x)\right\}\nu(ds)\right| \\
&\le \int_{\nabla_1}\left\{\left|\prod_{i}^{\infty}\psi_p(x-\log s_i) - \psi_p(x)\right|\right\}\nu(ds) \\
&\le \int_{\nabla_1}\{|\psi_p(x-\log s_1) - \psi_p(x)|\}\nu(ds) \\
&\quad + \int_{\nabla_1}\left\{\sum_{i=2}^{\infty}|\psi_p(x-\log s_i) - 1|\right\}\nu(ds).
\end{aligned} \tag{7.6}$$

We bound the two terms separately. For the first term, chose ε small enough so that $(1-\varepsilon \le x \le 1) \Rightarrow \log x \le 2(1-x)$ to obtain

$$\begin{aligned}
&\int_{\nabla_1}\bigl(\psi_p(x-\log s_1) - \psi_p(x)\bigr)\nu(ds) \\
&\quad \le \int_{\{1-s_1\ge\varepsilon\}}\bigl(\psi_p(x-\log s_1) - \psi_p(x)\bigr)\nu(ds) \\
&\qquad + \int_{\nabla_1/\{1-s_1<\varepsilon\}}\bigl(\psi_p(x-\log s_1) - \psi_p(x)\bigr)\nu(ds) \\
&\quad \le \bigl(1-\psi_p(x)\bigr)\nu(\{1-s_1\ge\varepsilon\}) + (p+1)\int_{\nabla_1/\{1-s_1<\varepsilon\}}(-\log s_1)\nu(ds) \\
&\quad \le C\bigl(1-\psi_p(x)\bigr) + (p+1)\int_{\nabla_1/\{1-s_1<\varepsilon\}}2(1-s_1)\nu(ds) \\
&\quad \le C\bigl(1-\psi_p(x)\bigr) + C',
\end{aligned} \tag{7.7}$$

where C and C' are finite constants and we have used the Mean Value Theorem and (7.4) in the second inequality and $\int_{\nabla_1}(1-s_1)\nu(ds) < \infty$ in the final inequality. This shows that $\int_{\nabla_1}(\psi_p(x-\log s_1) - \psi_p(x))\nu(ds) < \infty$.

For the second term in (7.6), we first observe that, thanks to Theorem 9, for each $\varepsilon > 0$ we can bound

$$|1 - \psi_p(x)| \le ce^{-(p+1-\varepsilon)x}, \tag{7.8}$$

where c is a constant. Hence,

$$\begin{aligned}\int_{\nabla_1}\left\{\sum_{i=2}^{\infty}|\psi_p(x-\log s_i)-1|\right\}\nu(ds) &\le c\int_{\nabla_1}\left\{\sum_{i=2}^{\infty}e^{-(p+1-\varepsilon)(x-\log s_i)}\right\}\nu(ds)\\ &\le ce^{-(p+1-\varepsilon)x}\int_{\nabla_1}\left\{\sum_{i=2}^{\infty}s_i^{p+1-\varepsilon}\right\}\nu(ds)\end{aligned}$$

as $p > \underline{p}$ we can chose ε small enough so that $p - \varepsilon > \underline{p}$ which then implies that

$$\int_{\nabla_1}\left\{\sum_{i=2}^{\infty}s_i^{p+1-\varepsilon}\right\}\nu(ds) < \infty.$$

Hence, putting the two bounds together we see that, for each $x \in \mathbb{R}$, it holds that $|\mathcal{L}\psi_p(x)| < \infty$ and hence ,ψ_p belongs to the domain of $\mathcal{A}_p$.

Let us now show that $\mathcal{A}_p\psi_p$ is continuous. As ψ_p is $C^1(\mathbb{R})$, it is enough to show that $\mathcal{L}\psi_p$ is continuous. We start by writing

$$\begin{aligned}&|\mathcal{L}\psi_p(x+\varepsilon)-\mathcal{L}\psi_p(x)|\\ &\quad\le\int_{\nabla_1}\left\{\left|\prod_i^{\infty}\psi_p(x+\varepsilon-\log s_i)\right.\right.\\ &\qquad\qquad\left.\left.-\psi_p(x+\varepsilon)-\prod_i^{\infty}\psi_p(x-\log s_i)+\psi_p(x)\right|\right\}\nu(ds).\end{aligned}$$

Next, we decompose the integrand as a sum

$$\begin{aligned}&\prod_i^{\infty}\psi_p(x+\varepsilon-\log s_i)-\psi_p(x+\varepsilon)-\prod_i^{\infty}\psi_p(x-\log s_i)+\psi_p(x)\\ &\quad=\big(\psi_p(x+\varepsilon-\log s_1)-\psi_p(x+\varepsilon)\big)\prod_{i\ge2}^{\infty}\psi_p(x+\varepsilon-\log s_i)\\ &\qquad+\psi_p(x+\varepsilon)\left(\prod_{i\ge2}^{\infty}\psi_p(x+\varepsilon-\log s_i)-1\right)\\ &\qquad-\big(\psi_p(x-\log s_1)-\psi_p(x)\big)\prod_{i\ge2}^{\infty}\psi_p(x-\log s_i)\end{aligned}\tag{7.9}$$

$$
\begin{aligned}
&\quad - \psi_p(x)\left(\prod_{i\geq 2}^{\infty} \psi_p(x - \log s_i) - 1\right) \\
&= \big(\psi_p(x+\varepsilon - \log s_1) - \psi_p(x - \log s_1) - \psi_p(x+\varepsilon) + \psi_p(x)\big) \\
&\quad \times \left(\prod_{i\geq 2}^{\infty} \psi_p(x+\varepsilon - \log s_i)\right) \\
&\quad + \psi_p(x - \log s_1)\left(\prod_{i\geq 2}^{\infty} \psi_p(x+\varepsilon - \log s_i) - \prod_{i\geq 2}^{\infty} \psi_p(x - \log s_i)\right) \\
&\quad + \big(\psi_p(x+\varepsilon) - \psi_p(x)\big)\left(\prod_{i\geq 2}^{\infty} \psi_p(x+\varepsilon - \log s_i) - 1\right).
\end{aligned}
$$

The proof will be complete once we will have shown that the integral with respect to $\nu(ds)$ of each term on the right-hand side of (7.9) goes to 0 as $\varepsilon \to 0$.

First term. The first term is

$$
\begin{aligned}
&\big(\psi_p(x+\varepsilon - \log s_1) \\
&\quad - \psi_p(x - \log s_1) - \psi_p(x+\varepsilon) + \psi_p(x)\big)\left(\prod_{i\geq 2}^{\infty} \psi_p(x+\varepsilon - \log s_i)\right).
\end{aligned}
$$

The term $\prod_{i\geq 2}^{\infty} \psi_p(x+\varepsilon - \log s_i)$ is uniformly bounded between 0 and 1. On the other hand,

$$
\begin{aligned}
&\big(\psi_p(x+\varepsilon - \log s_1) - \psi_p(x - \log s_1) - \psi_p(x+\varepsilon) + \psi_p(x)\big) \\
&\quad = \big(\psi_p(x+\varepsilon - \log s_1) - \psi_p(x+\varepsilon)\big) - \big(\psi_p(x - \log s_1) - \psi_p(x)\big) \\
&\quad \leq (-\log s_1)\psi_p'\big(x + \eta_1(\varepsilon, s_1)\big) + (\log s_1)\psi_p'\big(x + \eta_2(\varepsilon, s_1)\big),
\end{aligned}
$$

where $\eta_1(\varepsilon, s_1) \in [\varepsilon, \varepsilon - \log s_1]$ and $\eta_2(\varepsilon, s_1) \in [0, -\log s_1]$. Observe that for each s_1, because ψ_p is $C^\infty(\mathbb{R})$ [since (6.9) holds], we have that $|\eta_1(\varepsilon, s_1) - \eta_2(\varepsilon, s_1)| \to 0$ as $\varepsilon \to 0$. (The choice of η_1 and η_2 might not be unique but by adopting the convention that we always chose the lowest possible such value, the above argument becomes tight.)

Fix $\delta \in (0, 1)$ and decompose

$$
\begin{aligned}
&\int_{\nabla_1} (-\log s_1)\big[\psi_p'\big(x + \eta_1(\varepsilon, s_1)\big) - \psi_p'\big(x + \eta_2(\varepsilon, s_1)\big)\big]\nu(ds) \\
&\quad = \int_{\{1-s_1>\delta\}} (-\log s_1)\big[\psi_p'\big(x + \eta_1(\varepsilon, s_1)\big) - \psi_p'\big(x + \eta_2(\varepsilon, s_1)\big)\big]\nu(ds) \\
&\quad\quad + \int_{1-s_1\leq\delta} (-\log s_1)\big[\psi_p'\big(x + \eta_1(\varepsilon, s_1)\big) - \psi_p'\big(x + \eta_2(\varepsilon, s_1)\big)\big]\nu(ds)
\end{aligned}
$$

$$\leq C(\delta) + C'(\delta) \int_{\{1-s_1 \leq \delta\}} \bigl[\psi_p'\bigl(x + \eta_1(\varepsilon, s_1)\bigr) \\ - \psi_p'\bigl(x + \eta_2(\varepsilon, s_1)\bigr)\bigr](1 - s_1)\nu(ds),$$

where the first integral is bounded by a constant $C(\delta)$ which is arbitrarily small according to the choice of δ since ψ_p' is uniformly bounded by (7.4). As $(1 - s_1)\nu(ds)$ is a finite measure, we can use the dominated convergence theorem and we see that

$$\lim_{\varepsilon \to 0} C'(\delta) \int_{s_1 \geq 1-\delta} \bigl[\psi_p'\bigl(x + \eta_1(\varepsilon, s_1)\bigr) - \psi_p'\bigl(x + \eta_2(\varepsilon, s_1)\bigr)\bigr](1 - s_1)\nu(ds) = 0$$

which proves that the first term converges to 0.

Second term. Again using (7.5) to bound the difference of the two products in the second term we see, with the help of (7.4) and the monotonicity of ψ_p, that

$$\begin{aligned} &\left|\psi_p(x - \log s_1)\left(\prod_{i\geq 2}^{\infty} \psi_p(x + \varepsilon - \log s_i) - \prod_{i\geq 2}^{\infty} \psi_p(x - \log s_i)\right)\right| \\ &\quad \leq \sum_{i\geq 2} |\psi_p(x + \varepsilon - \log s_i) - \psi_p(x - \log s_i)| \\ &\quad \leq \varepsilon \sum_{i\geq 2} \max\{\psi_p'(y) : y \in [x - \log s_i, x - \log s_i + \varepsilon]\} \\ &\quad \leq \varepsilon(p + 1) \sum_{i\geq 2} \bigl(1 - \psi(x - \log s_i)\bigr) \end{aligned}$$

and we can use (7.8) to see that

$$\int_{\nabla_1} \sum_{i\geq 2} (1 - \psi_p(x - \log s_i))\nu(ds) < \infty.$$

We conclude that the integral of the second term converges to 0 as $\varepsilon \to 0$.

Third term. Let us now consider the third term

$$\bigl(\psi_p(x + \varepsilon) - \psi_p(x)\bigr)\left(\prod_{i\geq 2}^{\infty} \psi_p(x + \varepsilon - \log s_i) - 1\right).$$

We have already shown [for the second term of (7.6)] that

$$\int_{\nabla_1} \left(\prod_{i\geq 2}^{\infty} \psi_p(x + \varepsilon - \log s_i) - 1\right)\nu(ds) < \infty$$

and $(\psi_p(x + \varepsilon) - \psi_p(x)) \to 0$ as $\varepsilon \to 0$, so the integral of this term also converges to 0. $\square$

We now return to the proof of Theorem 7 and show that $\mathcal{A}_p\psi_p \equiv 0$ where ψ_p is as above. Suppose for contradiction that there exists some x such that $\mathcal{A}_p\psi_p(x) > 0$ [a similar argument will work to refute the case $\mathcal{A}_p\psi_p(x) < 0$]. We introduce the process $t \mapsto F(t)$ [which is a functional of the fragmentation process $t \mapsto \Pi(t)$]

$$F(t) := \prod_i \psi_p(z_i(t)) \cdot \sum_k \frac{1}{\psi_p(z_k(t))} \mathcal{A}_p\psi_p(z_k(t)).$$

Observe that since M is a martingale, we have as in (7.2) that for all $t \geq 0, 0 = \mathbb{E}(M(t, p, x)) - \mathbb{E}(M(0, p, x)) = \mathbb{E} \int_0^t F(s)\, ds$.

We claim that almost surely $\lim_{t\to 0} F(t) = \mathcal{A}_p\psi_p(x)$. Indeed, we start by observing that as $M(x, t) = \prod_i \psi_p(z_i(t))$ is a uniformly bounded martingale, then as $t \to 0$

$$\prod_i \psi_p(z_i(t)) \to M(0, p, x) = \psi_p(x).$$

So we only need to show that

$$\sum_k \frac{1}{\psi_p(z_k(t))} \mathcal{A}_p\psi_p(z_k(t)) \to \frac{\mathcal{A}_p\psi_p(x)}{\psi_p(x)}.$$

However, since

$$\frac{1}{\psi_p(z_1(t))} \mathcal{A}_p\psi_p(z_1(t)) \to \frac{\mathcal{A}_p\psi_p(x)}{\psi_p(x)},$$

because $\mathcal{A}_p\psi_p$ is continuous (Lemma 5) and $z_1(t) \to x$, it is enough to show that

$$\sum_{k\geq 2} \mathcal{A}_p\psi_p(z_k(t)) \to 0$$

[where we have used that for all t and k, $1 < 1/\psi_p(z_k(t)) < 1/\psi_p(x)$].

Note that

$$\sum_{k\geq 2} \mathcal{A}_p\psi_p(z_k(t)) = -c_p \sum_{k\geq 2} \psi_p'(z_k(t)) + \sum_{k\geq 2} \mathcal{L}\psi_p(z_k(t)).$$

We now turn to the sum $\sum_{k\geq 2} \mathcal{L}\psi_p(z_k(t))$. Using the bounds in (7.7) and (7.4) and the same arguments as in Lemma 5, we see that

$$\begin{aligned}
|\mathcal{L}\psi_p(x)| &\leq \int_{\nabla_1} \{|\psi_p(x - \log s_1) - \psi_p(x)|\}\nu(ds) \\
&\quad + \int_{\nabla_1} \left\{ \sum_{i=2}^{\infty} |\psi_p(x - \log s_i) - 1| \right\} \nu(ds) \\
&\leq C\big(1 - \psi_p(x)\big) + C'\psi_p'(x) + C'' e^{-(p+1-\varepsilon)x} \\
&\leq C\big(1 - \psi_p(x)\big) + C'' e^{-(p+1-\varepsilon)x},
\end{aligned}$$

where C, C' and C'' are (uniform in x) constants which may change value from line to line.

We thus have, again appealing to (7.4),

$$\left|\sum_{k\geq 2} \mathcal{A}_p \psi_p(z_k(t))\right| \leq C \sum_{k\geq 2}\bigl(1-\psi_p(z_k(t))\bigr) + C'' e^{-(p+1-\varepsilon)z_k(t)}.$$

Using that for any $\varepsilon > 0$, we have $1-\psi_p(x) \leq Ce^{-(p+1-\varepsilon)x}$ for some constant C (which, again, may change from line to line) we see that

$$\begin{aligned}\left|\sum_{k\geq 2} \mathcal{A}_p \psi_p(z_k(t))\right| &\leq C \sum_{k\geq 2} |\Pi_k(t)|^{p+1-\varepsilon} e^{-(p+1)(x-c_p t)}\\ &\leq C \sum_{k\geq 2} |\Pi_k(t)|^{p+1-\varepsilon}\end{aligned}$$

and for ε small enough so that $p-\varepsilon > \underline{p}$, $\sum_{k\geq 2} |\Pi_k(t)|^{p+1-\varepsilon} \to 0$ almost surely when $t \to 0$ on account of the fact that $W(t, p-\varepsilon) \to 1$ almost surely as $t \to 0$. since $p > \underline{p}$. The claim that $\lim_{t\to 0} F(t) = \mathcal{A}_p\psi_p(x)$ now follows.

The almost sure right-continuity at 0 of F implies that the stopping time $\tau = \inf\{t : F(t) < \mathcal{A}_p\psi_p(x)/2\}$ is almost surely strictly positive. Because M is a uniformly integrable martingale we must have $\mathbb{E}(M(\tau, p, x)) = \mathbb{E}(M(0, p, x))$ but

$$\begin{aligned}\mathbb{E}(M(\tau, p, x)) - \mathbb{E}(M(0, p, x)) &= \mathbb{E}\int_0^\tau F(s)\,ds\\ &\geq \mathcal{A}_p\psi_p(x)/2\mathbb{E}(\tau)\\ &> 0,\end{aligned}$$

so we have a contradiction to the assumption that there exists some x such that $\mathcal{A}_p\psi_p(x) > 0$. This completes the proof of Theorem 7.

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