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Modelling Nonlinear Hydroelastic Waves

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This paper uses the special Cosserat theory of hyperelastic shells satisfying Kirchoff's hypothesis and irrotational flow theory to model the interaction between a heavy thin elastic sheet and an infinite ocean beneath it. From a general discussion of three-dimensional motions, involving an Eulerian description of the flow and a Lagrangian description of the elastic sheet, a special case of two-dimensional travelling waves with two wave-speed parameters, one for the sheet and another for the fluid, is developed in terms of Eulerian coordinates only.

Keywords: Hydroelastic waves, Cosserat theory of shells, Kirchoff's hypothesis, Willmore functional

1. Nonlinear Hydroelastic Waves

Our goal is to derive nonlinear equations that govern the motion of an ideal incompressible liquid under an elastic sheet. For example, we would like to describe the propagation of waves of finite amplitude on the surface of an ocean under ice, regarding the ice sheet as an elastic shell. It is assumed throughout that there are no frictional forces between the sheet and the fluid beneath.

Fluid motion

We restrict attention to flows of infinite depth that at time t occupy a domain \mathcal{O}_t in \mathbb{R}^3 the boundary of which is a surface Σ_t that lies in a horizontal slab of finite height. Let points of \mathbb{R}^3 be identified by coordinates $\mathbf{x} = (x^1, x^2, x^3)$ and let Δ and ∇ denote the usual Laplacian and gradient operators with respect to these variables. We assume that the flow is incompressible, irrotational, satisfies the usual kinematic condition at Σ_t and vanishes at infinity. In particular, its velocity at a point \mathbf{x} at time t is given by $\nabla\varphi(\mathbf{x}, t)$ where the velocity potential φ satisfies

$$\begin{aligned}\Delta\varphi(\mathbf{x}, t) &= 0 \quad \text{in } \mathcal{O}_t, \\ n^t(\mathbf{x}, t) + \nabla\varphi(\mathbf{x}, t) \cdot \mathbf{n} &= 0 \quad \text{on } \Sigma_t, \\ \varphi(\mathbf{x}, t) &\rightarrow 0 \quad \text{as } x^3 \rightarrow -\infty.\end{aligned}\tag{1.1}$$

Here $\mathbf{n} \in \mathbb{R}^3$ is the unit normal vector to Σ_t outward from \mathcal{O}_t and

$$(n^t, \mathbf{n}) = \pm|\nabla F|^{-1}(\partial_t F, \nabla F)\tag{1.2}$$

when locally $\Sigma = \{(\mathbf{x}, t) : F(\mathbf{x}, t) = 0\}$ and $\Sigma_t = \{\mathbf{x} : F(\mathbf{x}, t) = 0\}$. So the kinematic condition is equivalent to $\partial_t F(\mathbf{x}, t) + \nabla F(\mathbf{x}, t) \cdot \nabla \varphi(\mathbf{x}, t) = 0$ on Σ_t . The fluid pressure is

$$p = -\partial_t \varphi - \frac{1}{2} |\nabla \varphi|^2 - gx^3 + C(t) \text{ in } \mathcal{O}_t.$$

After changing coordinates the function $C(t)$ can be taken to be zero.

Elastic shell dynamics

There are many approaches to elastic shell dynamics (see Antman 2005, Ciarlet 2005 and Naghdi 1972 for discussions and references). One possibility (see Friesecke *et al.* 2002, Le Dret & Raoul 1995, 1996 and Neff 2004) is to reduce a three-dimensional theory, using physically reasonable assumptions on the kinematics and careful asymptotic analysis, to a two-dimensional model. The conclusions of that approach will be used here to specify realistic stored elastic energy functions when we regard the sheet on the ocean surface as a thin elastic shell that can bend, twist and stretch, but cannot shear or change its thickness. To be precise, we adopt the special Cosserat theory of shells, with one director satisfying Kirchoff's hypothesis, see Antman 2005, Ch. 17.8, to describe the surface sheet.

We assume that there is an *unloaded rest state* in which the thin sheet occupies the plane $x^3 = 0$ and has constant density (mass per unit area) $\rho_0 > 0$. We regard this as the reference plane and the coordinates $\mathbf{X} = (X^1, X^2)$ of its material points are the Lagrangian coordinates of points of the sheet after deformation. In Cosserat theory (Antman 2005 gives a thorough mathematical account) the elastic state of a shell, considered as an oriented surface in \mathbb{R}^3 , is described by the position of its material points in Lagrangian coordinates together with a family of vector fields \mathbf{d}^k , which are called directors, on the surface.

To proceed, we recall some basic geometric notation. Suppose a smooth oriented surface in \mathbb{R}^3 is given in local coordinates (Ω, \mathbf{r}) by $\mathbf{x} = \mathbf{r}(\mathbf{X})$, $\mathbf{X} \in \Omega \subset \mathbb{R}^2$, where $\mathbf{r} \in C^3(\Omega)$ and the partial derivatives of \mathbf{r} are denoted by

$$\partial_{X^i} \mathbf{r} := \mathbf{r}_{,i} := \mathbf{r}_{,i}.$$

When $\mathbf{r}_{,1}$ and $\mathbf{r}_{,2}$ are linearly independent at every point of Ω , they form a basis for the tangent space at $\mathbf{r}(\mathbf{X})$ and the normal vector $\mathbf{n}(\mathbf{X})$ at $\mathbf{r}(\mathbf{X})$ is a C^2 vector field given by

$$\mathbf{n} = \frac{\mathbf{r}_{,1} \times \mathbf{r}_{,2}}{|\mathbf{r}_{,1} \times \mathbf{r}_{,2}|}.$$

Let the corresponding coefficients a_{ij} , b_{ij} of the first and second fundamental forms be

$$a_{ij} = \mathbf{r}_{,i} \cdot \mathbf{r}_{,j}, \quad b_{ij} = -\mathbf{r}_{,i} \cdot \mathbf{n}_{,j} = \mathbf{r}_{,ij} \cdot \mathbf{n}. \quad (1.3)$$

Then the surface area element is

$$d\Sigma = \sqrt{a(\mathbf{X})} d\mathbf{X} \quad \text{where} \quad a = |\det(a_{ij})| := |a_{11}a_{22} - a_{12}^2|.$$

Let A , B denote the matrices (a_{ij}) , (b_{ij}) and let $A^{-1} = (a^{ij})$, $B^{-1} = (b^{ij})$. Then the principal curvatures are the eigenvalues k_1 and k_2 of the Weingarten matrix

$W = A^{-1}B$ (Priestley 2010, Ch. 8), and the mean curvature H and Gauss curvature K are, with the usual summation convention,

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\text{Tr}(A^{-1}B) = \frac{1}{2}a^{ij}b_{ij}, \quad K = k_1k_2 = 2H^2 - \frac{1}{2}\text{Tr}((A^{-1}B)^2),$$

where $\text{Tr}((A^{-1}B)^2) = k_1^2 + k_2^2 = a^{ip}b_{pk}a^{kq}b_{qi}$.

In general Cosserat theory, there may be many directors, $\mathbf{d}^1, \dots, \mathbf{d}^k$, and the elastic state of the shell is given by $(\mathbf{r}(\mathbf{X}), \mathbf{d}^1(\mathbf{X}), \dots, \mathbf{d}^k(\mathbf{X})) \in \mathbb{R}^3 \times \mathbb{R}^{3k}$. However, in the special Cosserat theory adopted here there is only one director which, by Kirchoff's hypothesis, is the unit vector normal to the surface. Hence the time-dependent state of the elastic shell is given in local coordinates by a mapping

$$(\mathbf{X}, t) \rightarrow (\mathbf{r}(\mathbf{X}, t), \mathbf{n}(\mathbf{X}, t)) \in \mathbb{R}^3 \times S^2$$

in which $\mathbf{r}(\mathbf{X}, t) \in \mathbb{R}^3$ is the position at time t of a material point with Lagrangian coordinates \mathbf{X} , and \mathbf{n} , the unit normal to the surface at $r(\mathbf{X}, t)$, is the only director.

Since we assume that the shell has zero thickness, we take $\tilde{\mathbf{p}} = \mathbf{r}$ and $\varphi = 0$ in Ch.17, eqns. (1.11), (1.12) and (2.1c), and hence $\mathbf{h} = \mathbf{l} = \mathbf{0}$ and $I = 0$ in eqns. (2.13b,c) and (2.5c-e), in Antman 2005. Under these assumptions the impulse-momentum law, which follows from the principle of virtual power, can be written in weak form as (take $\mathbf{h} = \mathbf{l} = \mathbf{0}$ in Antman 2005, Ch. 17, (8.44)–(8.46) where, since ρ_0 is its mass per unit surface area, we have taken $2h\rho = \rho_0$)

$$\int_0^\infty \int_{\mathbb{R}^2} \left(\rho_0 \mathbf{r}_{,t} \cdot \delta \mathbf{r}_{,t} - \mathbf{N}^\alpha \cdot \delta \mathbf{r}_{,\alpha} - \mathbf{M}^\alpha \cdot \boldsymbol{\omega}_{,\alpha} + \mathbf{f} \cdot \delta \mathbf{r} + (\mathbf{r}_{,\alpha} \times \mathbf{N}^\alpha) \cdot \boldsymbol{\omega} \right) d\mathbf{X} dt = 0,$$

for any $\delta \mathbf{r} \in C_0^\infty(\mathbb{R}^2 \times (0, \infty))$, a smooth, compactly-supported, vector-valued test function, and $\boldsymbol{\omega}$ given by

$$\boldsymbol{\omega} = \mathbf{n} \times \frac{\delta \mathbf{r}_{,1} \times \mathbf{r}_{,2} + \mathbf{r}_{,1} \times \delta \mathbf{r}_{,2}}{|\mathbf{r}_{,1} \times \mathbf{r}_{,2}|}.$$

Here \mathbf{f} is a given external force, and \mathbf{N}^α and \mathbf{M}^α are functions of $\mathbf{r}_{,i}$, $\mathbf{n}_{,j}$ and \mathbf{n} which will be determined by constitutive hypotheses on the elasticity of the shell. An integration by parts leads to

$$\int_0^\infty \int_{\mathbb{R}^2} \left((-\rho_0 \mathbf{r}_{,tt} + \mathbf{N}_{,\alpha}^\alpha + \mathbf{f}) \cdot \delta \mathbf{r} + (\mathbf{M}_{,\alpha}^\alpha + (\mathbf{r}_{,\alpha} \times \mathbf{N}^\alpha)) \cdot \boldsymbol{\omega} \right) d\mathbf{X} dt = 0. \quad (1.4)$$

In Lemma 1.1 below we derive this law from a geometric variational principle.

Hyperelasticity

We consider a special Cosserat shell with director \mathbf{n} , which in addition is *hyperelastic* (Antman 2005 Ch. 17, eqn. (8.34)). In other words, there exists an elastic energy density function $W(\mathbf{r}_{,1}, \mathbf{r}_{,2}, \mathbf{n}_{,1}, \mathbf{n}_{,2})$ with

$$\mathbf{N}^\alpha = \frac{\partial W}{\partial \mathbf{r}_{,\alpha}}, \quad \mathbf{M}^\alpha = \mathbf{n} \times \frac{\partial W}{\partial \mathbf{n}_{,\alpha}}, \quad \alpha = 1, 2,$$

where, to ensure that the stored energy is frame-indifferent (see Antman 2005, Ch. 17, eqns. (8.7) & (8.34)) we suppose that

$$W \text{ is a function of } \mathbf{r}_{,\alpha} \cdot \mathbf{r}_{,\beta} \text{ and } \mathbf{r}_{,\alpha} \cdot \mathbf{n}_{,\beta}, \quad \alpha, \beta \in \{1, 2\}, \quad \mathbf{r} \in \mathbb{R}^3, \mathbf{n} \in S^2. \quad (1.5)$$

In other words, W is a function of the coefficients of the first and second fundamental forms (1.3) of the sheet in local coordinates (Ω, \mathbf{r}) . In addition we note that the kinetic energy density of the sheet is

$$Ke = \frac{\rho_0}{2} |\mathbf{r}_{,t}|^2.$$

At this point we introduce notation for variations of functions due to variations of the surface Σ in Section 1. Suppose G is a function of \mathbf{r} and its derivatives with respect to \mathbf{X} and t . For an arbitrary $\mathbf{r} \in C^3(\Omega \times (0, \infty))$ and $\delta\mathbf{r} \in C_0^3(\Omega \times (0, \infty))$ consider the one-parameter family Σ^τ of perturbed surfaces given by

$$\mathbf{r}^\tau(\mathbf{X}, t) := \mathbf{r}(\mathbf{X}, t) + \tau \delta\mathbf{r}(\mathbf{X}, t).$$

Let $g(\mathbf{X}, t; \tau)$ denote the value of G at $\mathbf{r}^\tau(\mathbf{X}, t)$ and let

$$\delta G(\mathbf{X}, t) := \left. \frac{\partial}{\partial \tau} g(\mathbf{X}, t; \tau) \right|_{\tau=0}.$$

The corresponding variations of the energy densities are: $\delta Ke = \rho_0 \mathbf{r}_{,t} \cdot \delta \mathbf{r}_{,t}$ and

$$\delta W = \frac{\partial W}{\partial \mathbf{r}_{,\alpha}} \cdot \delta \mathbf{r}_{,\alpha} + \frac{\partial W}{\partial \mathbf{n}_{,\alpha}} \cdot \delta \mathbf{n}_{,\alpha} = \mathbf{N}^\alpha \cdot \delta \mathbf{r}_{,\alpha} + \frac{\partial W}{\partial \mathbf{n}_{,\alpha}} \cdot \delta \mathbf{n}_{,\alpha}. \quad (1.6)$$

The next lemma gives a formula for the energy variation of a hyperelastic shell that leads to the impulse momentum law (1.4).

Lemma 1.1. *When W is given by (1.5), for $\delta\mathbf{r} \in C_0^\infty(\Omega \times (0, \infty))$,*

$$\begin{aligned} \int_0^\infty \int_\Omega (\delta Ke - \delta W + \mathbf{f} \cdot \delta \mathbf{r}) d\mathbf{X} dt &= \int_0^\infty \int_\Omega (\mathbf{N}_{,\alpha}^\alpha + \mathbf{f}) \cdot \delta \mathbf{r} d\mathbf{X} dt \\ &+ \int_0^\infty \int_\Omega (\mathbf{M}_{,\alpha}^\alpha + (\mathbf{r}_{,\alpha} \times \mathbf{N}^\alpha)) \cdot \boldsymbol{\omega} d\mathbf{X} dt + \int_0^\infty \int_\Omega \rho_0 \mathbf{r}_{,t} \cdot \delta \mathbf{r}_{,t} d\mathbf{X} dt. \end{aligned}$$

In particular, (1.4) can be written

$$\int_0^\infty \int_\Omega (\delta(Ke - W) + \mathbf{f} \cdot \delta \mathbf{r}) d\mathbf{X} dt = 0. \quad (1.7)$$

Proof. Since $(\mathbf{n} \times \mathbf{m}) \cdot (\mathbf{n} \times \delta \mathbf{n}) = \mathbf{m} \cdot \delta \mathbf{n}$ for any \mathbf{m} , because $|\mathbf{n}| = 1$ and $\mathbf{n} \perp \delta \mathbf{n}$, and since it is easily checked that $\boldsymbol{\omega} = \mathbf{n} \times \delta \mathbf{n}$, (1.6) gives

$$\begin{aligned} - \int_0^\infty \int_\Omega \delta W d\mathbf{X} dt &= \int_0^\infty \int_\Omega \left(\mathbf{N}_{,\alpha}^\alpha \cdot \delta \mathbf{r} + \left(\frac{\partial W}{\partial \mathbf{n}_{,\alpha}} \right)_{,\alpha} \cdot \delta \mathbf{n} \right) d\mathbf{X} dt \\ &= \int_0^\infty \int_\Omega \left\{ \mathbf{N}_{,\alpha}^\alpha \cdot \delta \mathbf{r} + \mathbf{n} \times \left(\frac{\partial W}{\partial \mathbf{n}_{,\alpha}} \right)_{,\alpha} \cdot \boldsymbol{\omega} \right. \\ &\quad \left. + \left(\mathbf{n}_{,\alpha} \times \frac{\partial W}{\partial \mathbf{n}_{,\alpha}} \right) \cdot \boldsymbol{\omega} + \left(\mathbf{r}_{,\alpha} \times \mathbf{N}^\alpha \right) \cdot \boldsymbol{\omega} \right\} d\mathbf{X} dt, \end{aligned}$$

since $\mathbf{n}_{,\alpha} \times \frac{\partial W}{\partial \mathbf{n}_{,\alpha}} + \mathbf{r}_{,\alpha} \times \mathbf{N}^\alpha = \mathbf{n}_{,\alpha} \times \frac{\partial W}{\partial \mathbf{n}_{,\alpha}} + \mathbf{r}_{,\alpha} \times \frac{\partial W}{\partial \mathbf{r}_{,\alpha}} = 0$ because (1.5) holds,

$$= \int_0^\infty \int_\Omega \left(\mathbf{N}_{,\alpha}^\alpha \cdot \delta \mathbf{r} + \mathbf{M}_{,\alpha}^\alpha \cdot \boldsymbol{\omega} + \left(\mathbf{r}_{,\alpha} \times \mathbf{N}^\alpha \right) \cdot \boldsymbol{\omega} \right) d\mathbf{X} dt.$$

The result follows. \square

The function $Ke - W$ is the density of an action integral and (1.7) is the minimum action principle for this hyperelastic material.

Governing equations for a shell

We now use (1.7) to derive the governing equation for the motion of an elastic shell. Note that $\{\mathbf{r}_{,1}, \mathbf{r}_{,2}, \mathbf{n}\}$ spans a natural coordinate system at points of a surface defined by $\mathbf{x} = \mathbf{r}(\mathbf{X})$, $\mathbf{r} \in C_0^3(\Omega)$. We consider perturbations of the form

$$\delta \mathbf{r} = \phi^n \mathbf{n} + \phi^1 \mathbf{r}_{,1} + \phi^2 \mathbf{r}_{,2},$$

where the normal variation ϕ^n and tangent variations ϕ^α , $\alpha = 1, 2$, are arbitrary functions. The following lemma gives formulae for normal and tangential variations of geometrical characteristics of the surface.

Lemma 1.2. (i) If $\delta \mathbf{r}(\mathbf{X}) = \phi^n(\mathbf{X}) \mathbf{n}(\mathbf{X})$ where $\phi^n \in C_0^3(\Omega)$ then

$$\delta a_{ij} = -2\phi^n b_{ij}, \quad \delta \sqrt{a} = -2\phi^n H \sqrt{a}, \quad \delta b_{ij} = \nabla_i \nabla_j \phi^n - \phi^n a^{kp} b_{pi} b_{kj}, \quad (1.8)$$

where $\nabla_i \nabla_j \phi^n = \phi_{,ij}^n - \Gamma_{ij}^k \phi_{,k}^n$ is a covariant derivative of the co-vector field $(\phi_{,j}^n)$ in which the Christoffel symbols are $\Gamma_{ij}^k = \frac{1}{2} a^{hk} (a_{jh,i} + a_{ih,j} - a_{ij,h})$.

(ii) If $\delta \mathbf{r}(\mathbf{X}) = \phi^\alpha(\mathbf{X}) \mathbf{r}_{,\alpha}(\mathbf{X})$ where $\phi^\alpha \in C_0^3(\Omega)$, $\alpha = 1, 2$, then

$$\begin{aligned} \delta a_{ij} &= a_{j\alpha} \phi_{,i}^\alpha + a_{i\alpha} \phi_{,j}^\alpha + \phi^\alpha a_{ij,\alpha}, \quad \delta \sqrt{a} = \partial_\alpha (\sqrt{a} \phi^\alpha), \\ \delta b_{ij} &= \phi_{,i}^\alpha b_{j\alpha} + \phi_{,j}^\alpha b_{i\alpha} + \phi^\alpha b_{ij,\alpha}. \end{aligned}$$

Proof. Relations (1.8) are well known in differential geometry, see Willmore 1982, §5.8. The formulae for tangential variations follow by similar arguments. \square

It follows from Lemma 1.2 and the identity $\delta W(a_{ij}, b_{ij}) = \frac{\partial W}{\partial b_{ij}} \delta b_{ij} + \frac{\partial W}{\partial a_{ij}} \delta a_{ij}$ that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^2} \delta W d\mathbf{X} dt &= \int_0^\infty \int_{\mathbb{R}^2} \frac{\partial W}{\partial b_{ij}} \left\{ (\nabla_i \nabla_j \phi^n - \phi^n a^{kp} b_{pi} b_{kj}) + \phi_{,i}^\alpha b_{j\alpha} + \phi_{,j}^\alpha b_{i\alpha} + \phi^\alpha b_{ij,\alpha} \right\} d\mathbf{X} dt \\ &\quad + \int_0^\infty \int_{\mathbb{R}^2} \frac{\partial W}{\partial a_{ij}} \left(-2 \phi^n b_{ij} + a_{j\alpha} \phi_{,i}^\alpha + a_{i\alpha} \phi_{,j}^\alpha + \phi^\alpha a_{ij,\alpha} \right) d\mathbf{X} dt. \end{aligned}$$

Then integration by parts gives

$$\int_0^\infty \int_{\mathbb{R}^2} \delta W d\mathbf{X} dt = \int_0^\infty \int_{\mathbb{R}^2} \left(\phi^n \Psi_n + \phi^\alpha \Psi_\alpha \right) d\mathbf{X} dt \quad (1.9)$$

where

$$\Psi_n = (\nabla_i \nabla_j)^* \left(\frac{\partial W}{\partial b_{ij}} \right) - \frac{\partial W}{\partial b_{ij}} a^{kp} b_{pi} b_{kj} - 2 \frac{\partial W}{\partial a_{ij}} b_{ij}, \quad (1.10a)$$

the asterix denotes the $L^2(dx)$ adjoint and

$$\Psi_\alpha = -\partial_i \left(b_{j\alpha} \frac{\partial W}{\partial b_{ij}} + a_{j\alpha} \frac{\partial W}{\partial a_{ij}} \right) - \partial_j \left(b_{i\alpha} \frac{\partial W}{\partial b_{ij}} + a_{i\alpha} \frac{\partial W}{\partial a_{ij}} \right) + \partial_\alpha W \quad (1.10b)$$

Since $\phi^n = \delta \mathbf{r} \cdot \mathbf{n}$ and $\phi^\alpha = a^{\alpha j} \mathbf{r}_{,j} \cdot \delta \mathbf{r}$, (1.9) can be written

$$\int_0^\infty \int_{\mathbb{R}^2} \delta W d\mathbf{X} dt = \int_0^\infty \int_{\mathbb{R}^2} \left(\Psi_n \mathbf{n} + a^{j\alpha} \Psi_\alpha \mathbf{r}_{,j} \right) \cdot \delta \mathbf{r} d\mathbf{X} dt, \quad (1.11)$$

and since

$$\int_0^\infty \int_{\mathbb{R}^2} \delta K e d\mathbf{X} dt = \rho_0 \int_0^\infty \int_{\mathbb{R}^2} \partial_t \mathbf{r} \cdot \partial_t \delta \mathbf{r} d\mathbf{X} dt = -\rho_0 \int_0^\infty \int_{\mathbb{R}^2} \partial_t^2 \mathbf{r} \cdot \delta \mathbf{r} d\mathbf{X} dt,$$

(1.11) and (1.7) yield the system governing the motion of the shell

$$\rho_0 \partial_t^2 \mathbf{r} + \Psi_n \mathbf{n} + a^{j\alpha} \Psi_\alpha \mathbf{r}_{,j} - \mathbf{f} = \mathbf{0}, \quad (1.12)$$

where Ψ_n and Ψ_α are defined by (1.10). The system (1.12) consists of three nonlinear partial differential equations for the vector-valued function $\mathbf{r}(\mathbf{X}, t)$. Henceforth we consider physically reasonable hypotheses under which (1.12) can be simplified.

Notice that the elastic energy is not a geometric invariant, in the sense that it depends not only on the shape of the surface Σ_t , but also on a specific parametrization $\mathbf{r}(\cdot, t)$ associated with the reference frame. It is common to decompose the elastic energy density W into two parts, the *membrane* energy W_m and the *bending* energy W_b . The membrane energy is associated with stretching and depends only on the first derivatives of \mathbf{r} . However the energy due to bending depends both on

the first derivatives of \mathbf{r} and on the curvatures of Σ . When combined with the requirements of frame indifference, this leads to the following special class of stored energies in terms of the fundamental forms of \mathbf{r} :

$$W = W_b(a_{ij}, b_{ij}) + W_m(a_{ij})$$

Remark. A possible justification of this splitting is the observation that its terms arise at different orders in an asymptotic expansion, in terms of a small thickness parameter, of a three-dimensional elastic-plate energy function. A function W of the form $W(a_{ij}, b_{ij})$ is a geometric invariant if, for any diffeomorphism $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}(\mathbf{X})$ and $\tilde{\mathbf{r}}(\tilde{\mathbf{X}}) := \mathbf{r}(\mathbf{X}(\tilde{\mathbf{X}}))$,

$$W(a_{ij}, b_{ij})\sqrt{a}d\mathbf{X} = W(\tilde{a}_{ij}, \tilde{b}_{ij})\sqrt{\tilde{a}}d\tilde{\mathbf{X}},$$

where $\tilde{a}_{ij}, \tilde{b}_{ij}$ are the coefficients of the first and second fundamental form in local coordinates $\tilde{\mathbf{X}}$. There is only one first order geometric invariant, \sqrt{a} , and the only second order geometric invariants are H and K , the mean and Gauss curvatures, respectively. An elastic energy of the form

$$W = W_b(H, K)\sqrt{a} + c_m\sqrt{a},$$

in which the membrane energy is simply the surface area, is geometrically invariant. The first term is consistent with the expression for bending energy of a shell that emerges from rigorous asymptotic analysis (see Friesecke *et al.* 2002) under hypotheses that the three-dimensional elastic energy density is an isotropic function $W_{3D}(\nabla\mathbf{v})$ of the gradient of the displacement vector \mathbf{v} , displacements satisfy the appropriate boundary conditions on $\partial\Omega$ (for example, periodic when Ω is a rectangular), the plate has a smooth middle surface and its thickness h is a small positive parameter. In Friesecke *et al.* (2002) it was shown that the functional $h^{-3}E(\cdot, h)$, where $E(\cdot, h)$ is the total elastic energy of a plate of thickness h , Γ -converges to

$$W_b(\Sigma) = \frac{1}{24} \int_{\Sigma} \left(2\mu|b|^2 + \frac{\lambda\mu}{\mu + \lambda/2}|2H|^2 \right) d\Sigma,$$

$|b|^2 = k_1^2 + k_2^2$ and the constants λ, μ are related to W at $\mathbf{v} = \text{Id}$ by

$$\frac{\partial^2 W_{3D}}{\partial F^2}(\text{Id})(F, F) = 2\mu|e|^2 + \lambda(\text{Tr } e)^2, \quad e = \frac{1}{2}(F + F^T).$$

If $\Omega = \mathbb{R}^2$ and Σ converges asymptotically to a horizontal plane, or if $\mathbf{r}(\mathbf{X})$ is periodic on \mathbb{R}^2 , then, by the Gauss-Bonnet Theorem, the integral of the Gauss curvature K over Σ depends only on the topological type of Σ . Since $|b|^2 = 4H^2 - 2K$ we may take the total bending energy to be

$$W_b = C_b \int_{\Sigma} |H|^2 d\Sigma, \quad \text{where } C_b = \frac{\mu}{3} \left(\frac{\mu + \lambda}{\mu + \lambda/2} \right).$$

Since $d\Sigma = \sqrt{a}d\mathbf{X}$ we conclude that the asymptotic theory in the case of pure bending yields $W = C_b H^2 \sqrt{a}$. In other words the bending energy, up to multiplicative constant, coincides with the Willmore functional (see Willmore 1982). \square

For geometrically invariant energy functions, the total elastic energy depends only on the shape of Σ_t , the tangent variations of the total energy are zero and

$$\int_0^\infty \int_{\mathbb{R}^2} \delta W d\mathbf{X} dt = \int_0^\infty \int_{\mathbb{R}^2} \Psi_n \mathbf{n} \cdot \delta \mathbf{r} d\mathbf{X} dt.$$

The formula for variation of mean curvature (Willmore 1982, five lines below (5.55) where a factor 1/2 is missing on the right), is

$$\delta H = \frac{1}{2} \left(\Delta \phi^n + (4H^2 - 2K)\phi^n \right)$$

where

$$\Delta \phi := a^{ij} \nabla_i \nabla_j \phi = \frac{1}{\sqrt{a}} \partial_{X^i} \left(a^{ij} \sqrt{a} \partial_{X^j} \phi \right).$$

Hence the function Ψ_n has a straightforward representation in the important case when $W = W_b(H)\sqrt{a}$ and $W_b \in C^3$:

$$\frac{1}{\sqrt{a}} \Psi_n = \frac{1}{2} \Delta W_b'(H) + 2H(HW_b'(H) - W_b(H)) - KW_b'(H)$$

and (1.12) has the form

$$\varrho \partial_t^2 \mathbf{r} + \left\{ \frac{1}{2} \Delta W_b'(H) + 2H(HW_b'(H) - W_b(H)) - KW_b'(H) \right\} \mathbf{n} - \frac{1}{\sqrt{a}} \mathbf{f} = 0,$$

where $\varrho = \rho_0/\sqrt{a}$ is the density of the deformed elastic sheet at the point \mathbf{r} .

2. Equations for Hydroelastic Waves

The equations governing the elastic shell on the water surface follow from system (1.1) for the fluid dynamics and system (1.12) for the shell dynamics, if we assume that the liquid surface coincides with the elastic shell (no cavities) and the force term in the equation for the shell's dynamics is due to water pressure and gravity:

$$\mathbf{f} = \sqrt{a} p(\mathbf{r}, t) \mathbf{n} - g \rho_0 \mathbf{j}, \quad \text{where } \mathbf{j} = (0, 0, 1).$$

This leads to the free boundary problem of finding a mapping $\mathbf{r} : \mathbb{R}^2 \times [0, \infty) \rightarrow \mathbb{R}^3$ and a velocity potential $\varphi(\cdot, t) : \mathcal{O}_t \rightarrow \mathbb{R}$, where \mathcal{O}_t is bounded by $\Sigma_t = \{\mathbf{r}(\mathbf{X}, t) : \mathbf{X} \in \mathbb{R}^2\}$, satisfying the equations

$$\rho_0 \partial_t^2 \mathbf{r} + \Psi_n \mathbf{n} + a^{j\alpha} \Psi_\alpha \mathbf{r}_{,j} = \sqrt{a} p(\mathbf{r}(\mathbf{X}, t), t) \mathbf{n} - g \rho_0 \mathbf{j} \quad \text{in } \mathbb{R}^2 \times [0, \infty), \quad (2.1a)$$

$$n^t(\mathbf{x}, t) + \nabla \varphi(\mathbf{x}, t) \cdot \mathbf{n} = 0 \quad \text{on } \Sigma_t \text{ for } t \in (0, \infty), \quad (2.1b)$$

$$\Delta \varphi(\mathbf{x}, t) = 0, \quad p = -\partial_t \varphi - \frac{1}{2} |\nabla \varphi|^2 - g x^3 \quad \text{in } \mathcal{O}_t \text{ for } t \in (0, \infty), \quad (2.1c)$$

where (n^t, \mathbf{n}) is defined by (1.2) and the functions Ψ_n, Ψ_α are defined in terms of \mathbf{r} by (1.10). Equations (2.1) may be supplemented by initial conditions

$$\mathbf{r}(\mathbf{X}, 0) = \mathbf{r}_0(\mathbf{X}) \quad \text{for } \mathbf{X} \in \mathbb{R}^2, \quad \varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x}) \quad \text{in } \mathcal{O}_0,$$

and by hypotheses determined by the behavior of the flow at infinity.

There is an essential difference between this problem with general W and free boundary problems such as the familiar water-wave problem that arise in hydrodynamics. A solution of (2.1) involves not only a free boundary Σ_t but also a particular parametrization $\mathbf{x} = \mathbf{r}(\mathbf{X}, t)$ of it. In contrast, capillary, capillary-gravity and gravity water waves problems are geometrically invariant in that their solutions are independent of the parametrization of the free surface. When the elastic energy density function is $W_b(H)\sqrt{a}$ (pure bending), equation (2.1a) becomes

$$\varrho \partial_t^2 \mathbf{r} + \left\{ \frac{1}{2} \Delta W'_b(H) + 2H(HW'_b(H) - W_b(H)) - KW'_b(H) \right\} \mathbf{n} = p(\mathbf{r}, t) \mathbf{n} - g \varrho \mathbf{j},$$

where $\varrho = \rho_0/\sqrt{a}$ is the actual density (mass per unit area) of the deformed shell. If we neglect the inertial forces by taking $\varrho = 0$ and assume that Σ_t is the graph $x^3 = \eta(x^1, x^2, t)$ of a smooth function η , the system becomes geometrically invariant with coefficients of first and second fundamental forms

$$\alpha_{ij} = \delta_{ij} + (\partial_{x^i} \eta)(\partial_{x^j} \eta), \quad \beta_{ij} = (\partial_{x^i} \partial_{x^j} \eta) / \sqrt{\alpha} \quad \text{where } \alpha = \det(\alpha_{ij}) = 1 + |\nabla \eta|^2,$$

δ_{ij} is the Kronecker δ , and the governing equations are

$$\frac{1}{2} \Delta W'_b(H) + 2H(HW'_b(H) - W_b(H)) - KW'_b(H) = p \quad \text{for } x^3 = \eta(x^1, x^2), \quad (2.2a)$$

where $H = \frac{1}{2\sqrt{\alpha}} (\alpha^{11} \partial_{x^1}^2 \eta + 2\alpha^{12} \partial_{x^1} \partial_{x^2} \eta + \alpha^{22} \partial_{x^2}^2 \eta)$, and

$$\partial_t \eta + \partial_{x^1} \eta \partial_{x^1} \varphi + \partial_{x^2} \eta \partial_{x^2} \varphi - \partial_{x^3} \varphi = 0 \quad \text{for } x^3 = \eta, \quad (2.2b)$$

$$\Delta \varphi(\mathbf{x}, t) = 0, \quad p = -\partial_t \varphi - \frac{1}{2} |\nabla \varphi|^2 - g x^3 \quad \text{for } x^3 < \eta, \quad (2.2c)$$

where $\Delta h = \frac{1}{\sqrt{\alpha}} (\partial_{x^i} (\sqrt{\alpha} \alpha^{ij} \partial_{x^j} h))$, and

$$\alpha^{11} = \frac{1}{\alpha} (1 + (\partial_{x^2} \eta)^2), \quad \alpha^{22} = \frac{1}{\alpha} (1 + (\partial_{x^1} \eta)^2), \quad \alpha^{12} = \frac{-1}{\alpha} (\partial_{x^1} \eta)(\partial_{x^2} \eta).$$

Travelling waves: two speeds of propagation.

Returning to (2.1) we consider waves of permanent form travelling with speed c on a steady irrotational flow which is at rest at infinite depth beneath an elastic sheet. To be precise, suppose the position at time t of the point with Lagrangian coordinates \mathbf{X} is

$$\mathbf{r}(\mathbf{X}, t) := \tilde{\mathbf{r}}(\mathbf{X} - \underline{c}t(1, 0)) + ct(1, 0, 0) = (\mathbf{X}, 0) + \hat{\mathbf{r}}(\mathbf{X} - \underline{c}t(1, 0)) + dt(1, 0, 0),$$

where $\underline{c}, c \in \mathbb{R}$, $d := c - \underline{c}$ and $\hat{\mathbf{r}}(\tilde{\mathbf{X}}) := \tilde{\mathbf{r}}(\tilde{\mathbf{X}}) - (\tilde{\mathbf{X}}, 0)$, $\tilde{\mathbf{X}} \in \mathbb{R}^2$. This describes a surface of fixed shape moving with constant speed c and the point with Lagrangian coordinate \mathbf{X} moves relative to a frame moving with speed d . We call c *the wave speed*, d *the drift velocity of the surface sheet* and \underline{c} *the mean velocity of the surface sheet relative to the wave*. Thus we have two velocities of wave propagation, one for the fluid and another for the elastic shell. If the fluid velocity field is supposed

stationary with respect to a frame moving with speed c , the fluid pressure p and the flow potential φ are functions of the variable $\tilde{\mathbf{x}} = (x^1 - ct, x^2, x^3)$. After this change of variables and after introducing the modified potential $\Phi(\tilde{\mathbf{x}}) = \varphi(\tilde{\mathbf{x}}) - c\tilde{x}^1$ and pressure $P = p - c^2/2$, (2.1) gives the stationary equations and boundary conditions for functions $\tilde{\mathbf{r}}(\tilde{\mathbf{X}})$ and $\Phi(\tilde{\mathbf{x}})$ in a fixed domain $\tilde{\mathcal{O}}$ bounded by $\tilde{\Sigma} = \{\tilde{\mathbf{x}} = \tilde{\mathbf{r}}(\tilde{\mathbf{X}})\}$. If, to simplify notation, we henceforth write \mathbf{r} , \mathbf{X} and \mathbf{x} instead of $\tilde{\mathbf{r}}$, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{x}}$, the governing equations for steady travelling hydroelastic waves in a moving frame become

$$\rho_0 \underline{c}^2 \partial_{X^1}^2 \mathbf{r} + \Psi_n \mathbf{n} + a^{j\alpha} \Psi_\alpha \mathbf{r}_{,j} = \sqrt{a} p(\mathbf{r}(\mathbf{X})) \mathbf{n} - g \rho_0 \mathbf{j} \quad \text{in } \mathbb{R}^2, \quad (2.3a)$$

$$\nabla \Phi(\mathbf{x}, t) \cdot \mathbf{n} = 0 \quad \text{on } \Sigma, \quad (2.3b)$$

$$\Delta \Phi = 0, \quad P = -\frac{1}{2} |\nabla \Phi|^2 - g x^3 \quad \text{in } \mathcal{O}. \quad (2.3c)$$

In the geometrically invariant case, when $W = W_b(H) \sqrt{a}$ and $\rho_0 = 0$, (2.3) becomes

$$\frac{1}{2} \Delta W_b'(H) + 2H(HW_b'(H) - W_b(H)) - KW_b'(H) = p \quad \text{for } x^3 = \eta,$$

$$\partial_{x^1} \eta \partial_{x^1} \Phi + \partial_{x^2} \eta \partial_{x^2} \Phi - \partial_{x^3} \Phi = 0 \quad \text{for } x^3 = \eta,$$

$$\Delta \Phi(\mathbf{x}, t) = 0, \quad P = -\frac{1}{2} |\nabla \Phi|^2 - g x^3 \quad \text{for } x^3 < \eta,$$

where H and Δ are defined by (2.2). These equations should be supplemented with the conditions at infinity

$$\nabla \Phi(\mathbf{x}) \rightarrow -(c, 0, 0) \quad \text{as } x^3 \rightarrow -\infty.$$

Steady 2D-flows

As remarked earlier, an important difference between hydroelastic problems and many other free boundary problems in mathematical physics is that here we are looking, not only for the shape of the free surface, but also for a special parametrization $\mathbf{x} = \mathbf{r}(\mathbf{X}, t)$ of it relative to a prescribed reference frame. In particular, equations (2.1) and (2.3) involve both a Lagrangian variable \mathbf{X} and an Eulerian variable \mathbf{x} . Because of the first term in (2.3a), we can not in general eliminate the Lagrangian part and write the equation in Eulerian coordinates, even when the stored energy density is geometrically invariant and the elastic energy depends only on the shape of the sheet. A remarkable property of two-dimensional travelling-wave flows is that the governing equations admit a scalar integral and the stretches can be expressed algebraically as a function of the wave elevation. Consequently the whole system can be described in terms of one type of coordinates. There follows a brief survey of these results.

In (2.3) assume that $\mathbf{r}(\mathbf{X}) = (r_1(X_1), X_2, r_3(X_1))$ and that the velocity potential of the flow beneath Σ is independent of x^2 . So the steady waves have surface profiles Σ given by their cross sections $\mathbb{E} = \{(r_1(X), r_3(X)) : X \in \mathbb{R}\}$. With $a := r_1'^2 + r_3'^2$, the only non-zero coefficients of the first and second fundamental forms of Σ are $a_{11} = a$, $b_{11} = (r_1' r_3'' - r_1'' r_3') / \sqrt{a}$ and $a_{22} = 1$. Following Antman, set

$$\nu = \sqrt{a} = \sqrt{a_{11}}, \quad \mu = b_{11} / \sqrt{a_{11}} = b_{11} / \nu$$

and note that $\sigma := 2H = \mu/\nu$, is the curvature of the curve \mathbb{E} , where H is the mean curvature of the surface Σ .

We assume that the material is hyperelastic with frame-indifferent stored energy density $W(a_{11}, b_{11})$ in the form $W = \mathcal{E}(\nu, \mu)$ and let

$$\mathcal{N} = \partial_\nu \mathcal{E}(\nu, \mu), \quad \mathcal{M} = \partial_\mu \mathcal{E}(\nu, \mu).$$

In this notation,

$$a_{11} = \nu^2; \quad a_{22} = 1; \quad b_{11} = \mu\nu; \quad a^{11} = \nu^{-2}; \quad a^{22} = 1; \quad a_{ij} = a^{ij} = b_{ij} = 0 \text{ otherwise};$$

$$\frac{\partial W}{\partial a_{11}} = \frac{1}{2} \left(\frac{\mathcal{N}}{\nu} - \frac{\mu}{\nu^2} \mathcal{M} \right); \quad \frac{\partial W}{\partial b_{11}} = \frac{\mathcal{M}}{\nu}; \quad (\nabla_1 \nabla_1)^* \phi = \phi_{,11} + \left(\frac{\nu'}{\nu} \phi \right)_{,1},$$

and (1.10) becomes

$$\Psi_n = \left(\frac{\mathcal{M}_{,1}}{\nu} \right)_{,1} - \mu \mathcal{N}, \quad \Psi_1 = -\mu \mathcal{M}_{,1} - \nu \mathcal{N}_{,1}, \quad \Psi_2 = 0.$$

Let the so-called pseudo elastic stress and moment functions, $N(\nu, \mu)$ and $M(\nu, \mu)$, be defined by

$$M = \mathcal{M} = \partial_\mu E(\nu, \mu), \quad N(\nu, \mu) = \mathcal{N} - \rho_0 \underline{c}^2 \nu = \partial_\nu E(\nu, \mu), \quad (2.4)$$

where the *pseudo elastic energy* is $E = \mathcal{E}(\nu, \mu) - \rho_0 \underline{c}^2 \nu^2 / 2$. Let $\mathbf{r} = (r_1, r_3) : \mathbb{R} \rightarrow \mathbb{R}^2$, let $\mathbf{t} = \mathbf{r}' / \nu$ be the unit tangent vector to the curve \mathbb{E} and note that $\mathbf{r}'' \cdot \mathbf{n} = \nu \mu$ and $\mathbf{r}'' \cdot \mathbf{t} = \nu'$. Then, relative to the moving frame, the wave propagation in the sheet is governed by a two-dimension version of the steady system (2.3a), which in this notation is a system of ordinary differential equations for the functions $\mu(X)$ and $\nu(X)$:

$$N' - \mu R - g\rho_0(\mathbf{j} \cdot \mathbf{t}) = 0, \quad R' + \mu N + \nu P(\mathbf{r}) - g\rho_0(\mathbf{j} \cdot \mathbf{n}) = 0, \quad (2.5)$$

where $'$ denotes the derivative with respect to X , $M' + \nu R = 0$ and $\mathbf{j} = (0, 1)$ (N and R are tangential and normal stresses and M is a moment). Remarkably, the first of equations (2.5) has a first integral independent of the pressure P ,

$$\nu \partial_\nu E(\nu, \mu) + \mu \partial_\mu E(\nu, \mu) - E(\nu, \mu) - g\rho_0 \eta = \gamma, \quad (2.6)$$

where $\eta(X) = \mathbf{r}(X) \cdot \mathbf{j}$ is the wave elevation and γ is a constant. So we can simplify (2.5) by substituting for R and using the identity (2.6). To study the resulting system it is convenient and natural (see Baldi & Toland 2010, Toland 2007, 2008) to let $\phi = 1/\nu$ and put

$$F(\phi, \sigma) := \phi E(1/\phi, \sigma/\phi). \quad (2.7)$$

Notice that $\phi = 1/\sqrt{a} = \varrho/\rho_0$ is the ratio of the deformed elastic material density ϱ at point $\mathbf{r}(X)$ and the elastic material density ρ_0 in the unloaded state. With this density ratio ϕ and the curvature σ of the elastic sheet as new unknowns we have

$$N = F(\phi, \sigma) - \phi \partial_\phi F(\phi, \sigma) - \sigma \partial_\sigma F(\phi, \sigma), \quad M = \partial_\sigma F(\phi, \sigma),$$

$$\nu \partial_\nu E(\nu, \mu) + \mu \partial_\mu E(\nu, \mu) - E(\nu, \mu) = -\partial_\phi F(\phi, \sigma), \quad (2.8)$$

and the governing equations (2.5) for the elastic sheet can be rewritten as

$$\phi \left(\phi (\partial_\sigma F)' \right)' - \sigma \left(F - \phi \partial_\phi F - \sigma \partial_\sigma F \right) + g \rho_0 \phi (\mathbf{j} \cdot \mathbf{n}) - P(\mathbf{r}) = 0 \quad (2.9a)$$

$$\partial_\phi F + \gamma + g \rho_0 \eta = 0, \quad (2.9b)$$

where $\eta = \mathbf{r} \cdot \mathbf{j}$ and P is the pressure in the flow at the free surface and γ is a constant. For hydroelastic waves, equations (2.9) should be supplemented with the hydrodynamical equations that determine P . Before doing so we make the following important observation.

Equations (2.9) are written in Lagrangian coordinates and $'$ denotes the derivative with respect to the Lagrangian variable X . However, from the definition of ϕ , we have $ds = dX/\phi(X)$ where s is arc length along \mathbb{E} . Using this relation we can rewrite the governing equations (2.9) in Eulerian coordinates. To illustrate this observation, we focus on the natural case when the density W of the stored elastic energy is a sum of bending and membrane energies. Moreover, in accordance with the asymptotic theory, we assume that the bending energy is a geometric invariant, which leads to the representation

$$W = W_b(H)\sqrt{a} + W_m(a_{11})\sqrt{a}, \text{ equivalently, } \mathcal{E}(\nu, \mu) = \nu(W_b(\mu/2\nu) + W_m(\nu^2)),$$

because $H = \sigma/2$ is the mean curvature of Σ and $a_{11} = a = \nu^2$. The changes of variables (2.4) and (2.7) yield that

$$F(\phi, \sigma) = F_b(\sigma) + F_m(\phi) - \frac{\rho_0 \underline{c}^2}{2\phi}, \quad \text{where } F_b(\sigma) = W_b(\sigma/2), \quad F_m(\phi) = W_m(1/\phi^2).$$

Now assume that the elastic sheet is represented in Eulerian coordinate as a graph, $x^3 = \eta(x^1)$, and note that along the elastic sheet

$$\phi F' \equiv \partial_s F \equiv \frac{1}{\sqrt{1 + \partial_{x^1} \eta^2}} \partial_{x^1} F, \quad \sigma = \frac{\partial_{x^1}^2 \eta}{(1 + \partial_{x^1} \eta^2)^{3/2}}.$$

Therefore, after the change of variables $X \rightarrow x^1$, we arrive at a system of equations and boundary conditions (in physical variables) for the wave profile $\eta(x^1)$, the normalized density $\phi(x^1)$, and the stream function $\psi(x^1, x^3)$, which determine the propagation of two-dimensional hydroelastic travelling waves. The system consists of the elasticity equations

$$\frac{1}{\sqrt{1 + \partial_{x^1} \eta^2}} \partial_{x^1} \left\{ \frac{1}{\sqrt{1 + \partial_{x^1} \eta^2}} \partial_{x^1} \left(\partial_\sigma F_b(\sigma) \right) \right\} - \sigma (F_b(\sigma) - \sigma \partial_\sigma F_b(\sigma)) \quad (2.10a)$$

$$- \sigma (F_m(\phi) - \phi \partial_\phi F_m(\phi)) + \rho_0 \underline{c}^2 \frac{\sigma}{\phi} = P(x^1, \eta(x^1)) - \frac{g \rho_0 \phi}{\sqrt{1 + \partial_{x^1} \eta^2}}, \quad x^1 \in \mathbb{R},$$

$$\partial_\phi F_m(\phi) + \frac{\rho_0 \underline{c}^2}{2\phi^2} + \gamma + g \rho_0 \eta(x^1) = 0, \quad x^1 \in \mathbb{R}, \quad (2.10b)$$

and the equations from hydrodynamics,

$$P = -\frac{|\nabla\psi|^2}{2} - gx^3 \quad \text{for } x^3 \leq \eta(x^1), \quad (2.10c)$$

$$\Delta\psi = 0 \quad \text{for } x^3 < \eta(x^1), \quad (2.10d)$$

$$\psi(x^1, \eta(x^1)) = \text{constant for } x^1 \in \mathbb{R}, \quad (2.10e)$$

$$\nabla\psi(x^1, x^3) \rightarrow (0, -c) \quad \text{as } x^3 \rightarrow -\infty. \quad (2.10f)$$

Here ψ and the velocity potential Φ satisfy the Cauchy-Riemann equations

$$\partial_{x^3}\psi = \partial_{x^1}\Phi, \quad \partial_{x^1}\psi = -\partial_{x^3}\Phi.$$

System (2.10) is coupled via the pressure function P which appears in both the elastic equation (2.10a) and the hydrodynamic equation (2.10c). In the case of pure bending, $F_m = 0$ and we can calculate ϕ (which for physical reasons should be non-negative) from equation (2.10b),

$$\phi(x^1) = |\underline{c}| \sqrt{\frac{-\rho_0}{2(\gamma + g\rho_0\eta(x^1))}},$$

and (2.10a) can be replaced by

$$\begin{aligned} & \frac{1}{\sqrt{1 + \partial_{x^1}\eta^2}} \partial_{x^1} \left\{ \frac{1}{\sqrt{1 + \partial_{x^1}\eta^2}} \partial_{x^1} \left(\partial_\sigma F_b(\sigma) \right) \right\} - \sigma (F_b(\sigma) - \sigma \partial_\sigma F_b(\sigma)) \\ & + \sqrt{2\rho_0} |\underline{c}| \sigma \sqrt{-(\gamma + g\rho_0\eta)} = P(x^1, \eta) - \frac{g\rho_0^{3/2} |\underline{c}|}{\sqrt{-2(\gamma + g\rho_0\eta)} \sqrt{1 + \partial_{x^1}\eta^2}}, \quad x^1 \in \mathbb{R}. \end{aligned}$$

Finally we note from the conclusions of Friesecke *et al.* (2002) that in many applications it often suffices to take $F(\sigma) = \text{const. } \sigma^2$.

3. Summary

The main outcome of this paper is a mathematical model (2.10) in physical coordinates for the propagation of two-dimensional nonlinear hydroelastic travelling waves. It was derived from a general treatment of three-dimensional flows beneath an elastic sheet that involved both Lagrangian and Eulerian coordinates. We finish by mentioning some particular features of these systems.

Singular waves. Singularities due to the self-intersection of a three-dimensional elastic sheet are, of course, physically impossible, but the governing equations themselves do not guarantee the absence of self-intersections. It follows from work of Li & Yau 1982 that if the surfaces Σ_t in three dimensional space tends asymptotically to a plane at infinity and if

$$\int_{\Sigma_t} H^2 d\Sigma < 8\pi,$$

then Σ_t does not intersect itself. The left side of this inequality, which is the Willmore functional (its critical points are called Willmore surfaces), can be regarded as the simplest example of the bending energy. The expectation is that a Willmore

surface can develop singularities if the value of the Willmore functional exceeds 8π . Hence if this happens in our problem we might expect the formations of singularities in the elastic sheet similar to the formation of singularities in an Euler elastica.

Jumps. Consider the two-dimensional case when the membrane energy is not zero. The natural assumption is that the function $F_m(\phi)$ is convex and tends to $+\infty$ as $\phi \rightarrow 0$ or to $+\infty$. Even then, the left side of equation (2.10b) is not a monotone function of ϕ for sufficiently large values of the mean velocity \underline{c} . In this case (2.10b), for given γ and η , can have more than one solution ϕ . This means that system (2.10) may have a solution with discontinuous ϕ . Plotnikov & Toland 2010 found the following results on the structure of such solutions.

If the function F admits the representation $F_b(\sigma) + F_m(\phi)$ in which F_b is convex and F_b, F_m satisfy certain growth condition, then the problem (2.10) admits periodic solutions which minimize the energy per period. For such solutions the functions ϕ may have jumps, but the functions $N(\sigma, \phi)$ and $M(\sigma, \phi)$ in (2.8) are continuous everywhere. The continuity of M is equivalent to continuity of σ , but continuity of N yields a Hugoniot type condition relating the values of ϕ on both sides of the jump where it was shown that certain stability conditions hold. \square

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