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INFINITESIMAL ISOMETRIES ON DEVELOPABLE SURFACES AND ASYMPTOTIC THEORIES FOR THIN DEVELOPABLE SHELLS

PETER HORNUNG, MARTA LEWICKA AND MOHAMMAD REZA PAKZAD

Abstract. We perform a detailed analysis of first order Sobolev-regular infinitesimal isometries on developable surfaces without affine regions. We prove that given enough regularity of the surface, any first order infinitesimal isometry can be matched to an infinitesimal isometry of an arbitrarily high order. We discuss the implications of this result for the elasticity of thin developable shells.

1. Introduction

The derivation of asymptotic theories for thin elastic films has been a longstanding problem in the mathematical theory of elasticity [2]. Recently, various lower dimensional theories have been rigorously derived from the nonlinear 3 dimensional model, through Γ-convergence [5] methods. Consequently, what seemed to be competing and contradictory theories for elastica (rods, plates, shells, etc) are now revealed to be each valid in their own specific range of parameters such as material elastic constants, boundary conditions and force magnitudes [20, 7, 8, 27, 4]. In this line, Friesecke, James and Müller gave a detailed description of the so called hierarchy of plate theories in [8], corresponding to distinct energy scaling laws in terms of the plate thickness. Similar results have been established for elastic shells [21, 6, 22, 23, 24], however the description is still far from being complete.

In [25] Lewicka and Pakzad put forward a conjecture regarding existence of infinitely many small slope shell theories each valid for a corresponding range of energy scalings. This conjecture, based on formal asymptotic expansions, is in accordance with all therigorously obtained results for plates and shells. It predicts the form of the 2 dimensional limit energy functional, and identifies the space of admissible deformations as infinitesimal isometries of a given integer order $N > 0$ determined by the magnitude of the elastic energy. Hence, the influence of shell's geometry on its qualitative response to an external force, i.e. the shell's rigidity, is reflected in a hierarchy of functional spaces of isometries (and infinitesimal isometries) arising as constraints of the derived theories.

In certain cases, a given Nth order infinitesimal isometry can be modified by higher order corrections to yield an infinitesimal isometry of order $M > N$, a property to which we refer to by *matching property of infinitesimal isometries*. This feature, combined with certain density results for spaces of isometries, causes the theories corresponding to orders of infinitesimal isometries between N and M to collapse all into one and the same theory. Examples of such behavior are observed for plates [8], where any second order infinitesimal isometry can be matched to an exact isometry (and hence to an Mth-order isometry for all $M \in \mathbb{N}$), and for convex shells [24], where any first order infinitesimal isometry enjoys the same property. The effects of these observations on the elasticity of thin films are drastic. A plate possesses three types of small-slope theories: the linear theory, the von Kármán theory and the linearized Kirchhoff theory [8], whereas the only

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small slope theory for a convex shell is the linear theory [24]: a convex shell transitions directly from the linear regime to fully nonlinear bending if the applied forces are adequately increased. In other words, while the von Kármán theory describes buckling of thin plates, the equivalent variationally correct theory for elliptic shells is the purely nonlinear bending.

In this paper, we focus on developable surfaces (without flat regions). This class includes smooth cylindrical shells which are ubiquitous in nature and technology over a range of length scales. An example of a recently discovered structure is carbon nanotubes, i.e. molecular-scale tubes of graphitic carbon with outstanding rigidity properties [10]: they are among the stiffest materials in terms of the tensile strength and elastic modulus, but they easily buckle under compressive, torsional or bending stress [17]. The common approach in studying buckling phenomenon for cylindrical tubes has been to use the von Kármán-Donnel equations [26, 12]. However, as we establish here, the proper sublinear theory for this purpose is, again, the purely nonlinear bending theory. It seems likely that the von Kármán-Donnel equations could be rigorously derived and valid in another scaling limit, e.g. when the radius of the cylinder is very large as the thickness vanishes.

The key ingredient of our analysis is a study of $W^{2,2}$ first order isometries on developable surfaces, of which we address regularity, rigidity, density and matching properties. Our results depend on the regularity of the surface and a certain mild convexity property. More precisely, we establish that any $\mathcal{C}^{2N-1,1}$ regular first order infinitesimal isometry on a developable $\mathcal{C}^{2N,1}$ surface with a positive lower bound on the mean curvature, can be matched to an Nth-order infinitesimal isometry. Combined with a density result for $W^{2,2}$ first order isometries on such surfaces, we prove that the limit theories for the energy scalings of the order lower than $h^{2+2/N}$ collapse all into the linear theory. Our method is to inductively solve the linearized metric equation $\mathrm{sym}\nabla w = B$ on the surface with suitably chosen right hand sides, a process during which we lose regularity: only if the surface is $\mathcal{C}^\infty$ we can establish the total collapse of all small slope The importance of the solvability of $\mathrm{sym}\nabla w = B$ has been noted in [32], see also [9] and [1] in regards to relations of rigidity and elasticity. We remark here that a surface, e.g. a regular cylinder, may well be ill-inhibited according to the definition by Sanchez-Palencia in [32] and yet satisfy the adequate matching properties resulting in the aforementioned collapse.

Our analysis can be generalized to piecewise smooth surfaces which satisfy the convexity property as above, on each component. The question of existence of a developable or non-developable surface with no flat regions which shows a different elastic behavior (i.e. validity of an intermediate theory between the linear one and the purely nonlinear bending) remains open.

Finally, a word on the developability property of surfaces of vanishing Gaussian curvature, from which the term *developable* is derived, is to the point. For each point on such a surface there exists a straight segment passing through it and lying on the surface, in a characteristic direction. Developable surfaces are also locally identified with isometric images of domains in $\mathbb{R}^2$; this last property and, in particular, the developable structure of isometries of flat domains is heavily exploited in this paper. Such structure was established for $\mathcal{C}^2$ isometries in [11], for $\mathcal{C}^1$ isometries with total zero curvature in [30, Chapter II], [31, Chapter IX] and for $W^{2,2}$ isometries in [18, 29]. In [29] Pakzad proved that any $W^{2,2}$ isometry on a convex domain can be approximated in strong norm by smooth isometries, and in [28] the boundary regularity was discussed. Lately, Hornung systematically represented and generalized these results in [13, 14, 15] whose terminology we will adapt for the sake of simplicity and completeness. The above mentioned results have had other applications in nonlinear elasticity [3, 19, 16].

The paper is organized as follows. In section 2, we introduce and review preliminary facts about developable surfaces. In section 3, we study the linearized equation of isometric immersion of a developable surface and prove existence of a solution operator with suitable bounds in Theorem 3.1. We proceed to study the space of $W^{2,2}$ first order infinitesimal isometries and prove a compensated regularity and rigidity property of such mappings in Theorem 4.1 of section 4. We then use these results to prove in Theorem 5.2 that given enough regularity of the surface, any first order infinitesimal isometry can be matched to a higher order one. Combined with a straightforward density result in Theorem 5.3, we are finally able to derive the main application of this paper in elasticity theory, namely Theorem 6.2, which is the counterpart to Theorem 6.1 for deducing the Γ-limit of 3 dimensional nonlinear elasticity for thin developable shells.

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2. Developable surfaces

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $u \in \mathcal{C}^{2,1}(\Omega, \mathbb{R}^3)$ be an isometric immersion of Ω into $\mathbb{R}^3$:

$$\partial_i u(x) \cdot \partial_j u(x) = \delta_{ij} \quad \text{for all } x \in \Omega.$$

A classical result [11] asserts that, away from affine regions, such u must be developable: the domain Ω can be decomposed (up to a controlled remainder) into finitely many subdomains on which u is affine and finitely many subdomains on which u admits a line of curvature parametrization, see e.g. Theorem 4 in [14]. In light of this result, it is natural to restrict ourselves to the situation when Ω can be covered by a single line of curvature chart.

We now make the above more precise. Let $T > 0$, let $\Gamma \in \mathcal{C}^{1,1}([0,T], \mathbb{R}^2)$ be an arclength parametrized curve, and let $s^\pm \in \mathcal{C}^{0,1}([0,T])$ be positive functions (a rationale for assuming Lipschitz continuity here can be found in Lemma 2.2 in [16]). Following the notation in [15], we set:

$$N = (\Gamma')^\perp \in \mathcal{C}^{0,1}([0,T], \mathbb{R}^2) \quad \text{and} \quad \kappa = \Gamma'' \cdot N \in L^\infty(0,T)$$

the unit normal and the curvature of Γ. We introduce the bounded domain:

$$M_{s^\pm} = \{(s,t) : t \in (0,T),\ s \in (-s^-(t), s^+(t))\},$$

the mapping $\Phi : M_{s^\pm} \to \mathbb{R}^2$ given by:

$$\Phi(s,t) = \Gamma(t) + sN(t)$$

and the open line segments:

$$[\Gamma(t)] = \{\Gamma(t) + sN(t) : s \in (-s^-(t), s^+(t))\}.$$

From now on we assume that:

$$[\Gamma(t_1)] \cap [\Gamma(t_2)] = \emptyset \quad \text{for all unequal } t_1, t_2 \in [0,T]. \tag{2.1}$$

The next lemma is a consequence of Proposition 2 and Proposition 1 in [15].

Lemma 2.1. *Given $\kappa_n \in L^2(0,T)$, let $r = (\gamma', v, n)^T \in W^{1,2}((0,T), SO(3))$ satisfy the ODE:*

$$r' = \begin{pmatrix} 0 & \kappa & \kappa_n \\ -\kappa & 0 & 0 \\ -\kappa_n & 0 & 0 \end{pmatrix} r \tag{2.2}$$

with initial value $r(0) = Id$, where we set $\gamma(t) = \int_0^t \gamma'$. Define the mapping:

$$u : \Phi(M_{s^\pm}) \to \mathbb{R}^3, \qquad u(\Phi(s,t)) = \gamma(t) + sv(t) \quad \forall (s,t) \in M_{s^\pm}. \tag{2.3}$$

Then u is well defined and:

$$u \in W^{2,2}_{\rm loc}(\Phi(M_{s^\pm}), \mathbb{R}^3)$$

is an isometric immersion. Moreover, $\nabla u(\Phi) \in \mathcal{C}^0(\bar{M}_{s^\pm}, \mathbb{R}^{3\times 2})$ with:

$$\nabla u(\Phi(s,t)) = \gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t) \qquad \forall (s,t) \in \bar{M}_{s^\pm}. \tag{2.4}$$

and the functions:

$$a_{ij} = (\partial_1 u \times \partial_2 u) \cdot \partial_i \partial_j u$$

satisfy:

$$\partial_i \partial_j u = a_{ij}(\partial_1 u \times \partial_2 u) \qquad \forall i,j = 1,2, \tag{2.5}$$

$$a_{ij}(\Phi(s,t)) = \frac{\kappa_n(t)}{1 - s\kappa(t)} \Gamma_i'(t)\Gamma_j'(t) \quad \text{for almost every } (s,t) \in M_{s^\pm}. \tag{2.6}$$

If, in addition:

$$\int_0^T \left(\int_{s^-(t)}^{s^+(t)} \frac{\kappa_n^2(t)}{1 - s\kappa(t)} \mathrm{d}s \right) \mathrm{d}t < \infty \tag{2.7}$$

then $u \in W^{2,2}(\Phi(M_{s^\pm})), \mathbb{R}^3)$ with:

$$\int_{\Phi(M_{s^\pm})} |\nabla^2 u(x)|^2 \mathrm{d}x = \int_0^T \Big(\int_{s^-(t)}^{s^+(t)} \frac{\kappa_n^2(t)}{1 - s\kappa(t)} \ \mathrm{d}s \Big) \mathrm{d}t. \tag{2.8}$$

In order to keep the hypotheses short, we make the following definition:

Definition 2.2. *A surface $S \subset \mathbb{R}^3$ is said to be developable of class $\mathcal{C}^{k,1}$ if there are Γ, N, $s^\pm$, γ, v, n, κ, κ_n, Φ and u as in Lemma 2.1 such that*

$$\Phi(M_{s^\pm}) = \Omega, \tag{2.9}$$

$u \in \mathcal{C}^{k,1}(\overline{\Omega}, \mathbb{R}^3)$ and $S = u(\Omega)$.

Remark 2.3.
1. The curve γ is a line of curvature of surface S. Hence the mapping $(s,t) \mapsto \gamma(t) + sv(t)$ is a line of curvature parametrization of S and the condition (2.9) is the precise formulation of our assertion that Ω is covered by a single line of curvature chart.
2. The moving frame r is the Darboux frame on the surface S along γ. Therefore, (2.2) indicates that the geodesic curvature of γ coincides with the curvature κ of its preimage Γ, and that its geodesic torsion vanishes. This is naturally expected as u is an isometry. In the same vein, κ_n is the normal curvature of γ on S.
3. Of course, n is the unit normal vector to S and $-[a_{ij}]$ is the second fundamental form (expressed in u-coordinates). Equation (2.6) shows that it has rank one or zero, hence the Gauss curvature $\det[a_{ij}]$ is zero.
4. Condition (2.7) implies that $\kappa_n = 0$ almost everywhere on the set:
$$I_0 = \Big\{ t \in (0,T) : \kappa(t) \in \{1/s^-(t), 1/s^+(t)\} \Big\}.$$

Now, by Proposition 10 in [14] and in view of (2.1) we obtain:

$$\kappa(t) \in \left[-\frac{1}{s^-(t)}, \frac{1}{s^+(t)} \right] \text{ for almost every } t \in (0,T). \tag{2.10}$$

Moreover, the Lipschitz map Φ with:

$$\det \nabla \Phi(s,t) = -(1 - s\kappa(t)) \quad \text{for almost every } (s,t) \in M_{s^\pm}, \tag{2.11}$$

is a homeomorphism from $M_{s^\pm}$ onto Ω. We will frequently make the extra assumption that mean curvature of u be bounded away from zero:

$$|\mathrm{trace}\ [a_{ij}](x)| > 0 \text{ for all } x \in \overline{\Omega}. \tag{2.12}$$

Then (2.6) and (2.7) imply that there is $\delta > 0$ such that

$$\kappa(t) \in \left[\delta - \frac{1}{s^-(t)}, \frac{1}{s^+(t)} - \delta\right] \text{ for almost every } t \in (0,T). \tag{2.13}$$

By Proposition 10 (iii) in [14] and the bounds (2.13), Φ^{-1} is Lipschitz as well (this assertion is generally false if (2.13) is violated).

Lemma 2.4. *Assume that S is developable of class $\mathcal{C}^{k,1}$ for some $k \geq 2$ and that* (2.12) *holds. Then $\kappa, \kappa_n \in \mathcal{C}^{k-2,1}$ and Φ, Φ^{-1} are $\mathcal{C}^{k-1,1}$ up to the boundary of their respective domains $M_{s^\pm}$ and Ω.*

Proof. The hypothesis on S imply that $a_{ij} \in \mathcal{C}^{k-2,1}(\overline{\Omega})$. By continuity of a_{ij} and Φ and by (2.12), we may assume without loss of generality that Trace $[a_{ij}](\Phi) \geq c > 0$ on $M_{s^\pm}$. In view of (2.6) we have:

$$\mathrm{Trace}\ [a_{ij}](\Phi) = \frac{\kappa_n}{1 - s\kappa}. \tag{2.14}$$

Since Φ is bilipschitz, this implies that the right-hand side of (2.14) is Lipschitz. As $\kappa_n \geq c > 0$ it follows that κ, κ_n are Lipschitz. Thus Γ', N belong to $\mathcal{C}^{1,1}$, hence so does Φ. By (2.11), (2.13), the Jacobian of Φ is uniformly bounded away from zero on $M_{s^\pm}$, and so Φ^{-1} belongs to $\mathcal{C}^{1,1}$, too. If $k \geq 3$ then we return to (2.14), apply Lemma 2.5 and argue as before to conclude that κ, κ_n are in $\mathcal{C}^{1,1}$. The conclusion follows by iteration. ■

In the proof of Lemma 2.4 we used the following chain rule, which is a particular case of Theorem 2.2.2 from [33].

Lemma 2.5. *Let $U_1, U_2 \subset \mathbb{R}^n$ be two open, bounded sets and let $\Phi : U_1 \to U_2$ be a bilipschitz homeomorphism. Then $f \in W^{1,2}(U_2)$ if and only if $f \circ \Phi \in W^{1,2}(U_1)$. If this is the case, then the chain rule applies:*

$$\nabla(f \circ \Phi) = \big((\nabla f) \circ \Phi\big)\nabla\Phi \qquad a.e.\ in\ \ U_1. \tag{2.15}$$

3. Equation sym$\nabla w = B$ on developable surfaces

In this section we let S be a developable surface of class $\mathcal{C}^{2,1}$. By $\vec{n} : S \to \mathbb{S}^2$ we denote the unit normal to S which satisfies $\vec{n}(u(x)) = \partial_1 u(x) \times \partial_2 u(x)$ for all $x \in \Omega$, and $\vec{n}(\gamma(t)) = n(t)$ for all $t \in (0,T)$. By $\Pi = \nabla\vec{n}$ we denote the second fundamental form of S, defined as a symmetric bilinear form by

$$\Pi(p)(\tau, \eta) = \eta \cdot \partial_\tau \vec{n}, \quad \forall\ \tau, \eta \in T_pS,\ p \in S,$$

so that:

$$\Pi(u(x))(\partial_i u(x), \partial_j u(x)) = -a_{ij}(x) \text{ for all } x \in \Omega.$$

We continue to assume (2.12). Hence:

$$|\kappa_n(t)| > c > 0 \qquad \forall t \in [0,T]. \tag{3.1}$$

For a given symmetric bilinear form $B \in \mathcal{C}^{1,1}(S), \mathbb{R}^{2\times 2}$, we want to solve a first order PDE:

$$\mathrm{sym}\nabla w = B, \tag{3.2}$$

on S, where $w : S \to \mathbb{R}^3$ is a displacement field and the expression $\mathrm{sym}\nabla w$ in the left-hand side is the following bilinear form acting on the tangent space of S:

$$\mathrm{sym}\nabla w(p)(\tau,\eta) = \frac{1}{2}(\partial_\tau w(p)\cdot\eta + \partial_\eta w(p)\cdot\tau), \quad \forall\, \tau,\eta \in T_pS,\ p\in S.$$

1. We shall write $w\vec{n}$ to denote the scalar product $w\cdot\vec{n}$, and we decompose w as follows:

$$w = w_{tan} + (w\vec{n})\vec{n}.$$

Hence $w_{tan}(p) \in T_pS$ for all $p\in S$. We define the pulled back maps $w_3 = (w\vec{n})\circ u$ and $w' = (w_{tan}\circ u)^T\nabla u$, as well as the pulled back form:

$$B_{ij}(x) = B(u(x))(\partial_i u(x), \partial_j u(x)) \qquad \forall x\in\Omega \quad \forall i,j=1,2.$$

Using (2.5) and recalling that w_{tan} is tangent to S, we calculate:

$$\begin{aligned}\partial_j u(x)\cdot\nabla w(u(x))\partial_i u(x) &= \partial_i(w(u(x)))\cdot\partial_j u(x) = \partial_i\Big(w(u(x))\cdot\partial_j u(x)\Big) - w(u(x))\cdot\partial^2_{ij}u(x)\\ &= \partial_i w'_j - w_3 a_{ij}(x).\end{aligned}$$

Hence (3.2) can be written in terms of the pulled back quantities as the following matrix equality:

$$[B_{ij}] = \mathrm{sym}\nabla w' - w_3[a_{ij}], \tag{3.3}$$

where now sym $\nabla w'$ is understood in the usual way, with respect to the standard Euclidean coordinates in Ω.

2. Recalling that the condition for a matrix field $\tilde{B}$ to be of the form $\tilde{B} = \mathrm{sym}\nabla\tilde{w}$ for some vector field $\tilde{w}$ on Ω is equivalent to $\mathrm{curl}^T\mathrm{curl}\tilde{B} = 0$, the equation (3.3) becomes:

$$\begin{aligned}\mathrm{curl}^T\mathrm{curl}\ [B_{ij}] &= -\mathrm{curl}^T\mathrm{curl}\Big(w_3[a_{ij}]\Big)\\ &= -w_3\ \mathrm{curl}^T\mathrm{curl}\ [a_{ij}] - 2\nabla^\perp w_3\cdot\mathrm{curl}\ [a_{ij}] - \mathrm{cof}\ \nabla^2 w_3 : [a_{ij}].\end{aligned} \tag{3.4}$$

Notice now that:

$$\mathrm{curl}\ [a_{ij}] = -\mathrm{curl}\begin{bmatrix}\partial^2_{11}u\cdot n & \partial^2_{12}u\cdot n\\ \partial^2_{12}u\cdot n & \partial^2_{22}u\cdot n\end{bmatrix} = -\begin{bmatrix}\partial^2_{11}u\cdot\partial_2 n - \partial^2_{12}u\cdot\partial_1 n\\ \partial^2_{12}u\cdot\partial_2 n - \partial^2_{22}u\cdot\partial_1 n\end{bmatrix} = 0,$$

because $\partial_i(\vec{n}\circ u)\in T_pS$ and $\partial^2_{ij}u\cdot\partial_k u = 0$ by Lemma 2.1. Hence:

$$\theta = \mathrm{curl}^T\mathrm{curl}\ [B_{ij}] \tag{3.5}$$

belongs to L^∞. The problem (3.4) becomes:

$$\theta = -\mathrm{cof}\ \nabla^2 w_3 : [a_{ij}] \qquad \text{in } \Omega. \tag{3.6}$$

3. By Lemma 2.1 we have:

$$\big(\mathrm{cof}\ (\nabla^2 w_3) : [a_{ij}]\big)\,(\Phi(s,t)) = \frac{\kappa_n(t)}{1-s\kappa(t)}\partial^2_{ss}(w_3(\Phi(s,t))).$$

Consequently, problem (3.6) is equivalent to:

$$\partial^2_{ss}(w_3(\Phi(s,t))) = -\frac{1-s\kappa(t)}{\kappa_n(t)}\theta(\Phi(s,t)) \qquad \text{for all } (s,t)\in M_{s^\pm}. \tag{3.7}$$

The above calculations show that in order to solve (3.2), it is sufficient and necessary to solve the ODE (3.7) for w_3 and then recover w' from (3.3). Moreover, the solution (w', w_3) is unique after choosing the boundary conditions $w_3(\Phi(t,0))$ and $\partial_s(w_3(\Phi(t,0)))$, where uniqueness of w' is understood up to affine (linearized) rotations of the form $A(s,t)+b$, $A\in so(2)$, $b\in\mathbb{R}^2$.

Theorem 3.1. *Assume that S is developable of class $\mathcal{C}^{2,1}$ and satisfies (2.12), and let $\alpha \in (0,1)$. Then there exists a constant C such that the following is true. For every symmetric bilinear form $B \in \mathcal{C}^{1,1}(S, \mathbb{R}^{2\times 2})$ there exists a solution $w = w_{tan} + (w\vec{n})\vec{n}$ with $w_{tan} \in \mathcal{C}^{0,\alpha}$ and $(w\vec{n}) \in L^\infty$ of:*

$$\mathrm{sym}\nabla w = \mathrm{sym}\nabla w_{tan} + (w\vec{n})\Pi = B \tag{3.8}$$

satisfying the bounds:

$$\|w_{tan}\|_{\mathcal{C}^{0,\alpha}} + \|(w\vec{n})\|_\infty \leq C\|B\|_{\mathcal{C}^{1,1}}. \tag{3.9}$$

If, in addition, $S \in \mathcal{C}^{k+2,1}$ and $B \in \mathcal{C}^{k+1,1}$ for some $k \geq 1$, then

$$\|w_{tan}\|_{\mathcal{C}^{k,1}} + \|(w\vec{n})\|_{\mathcal{C}^{k-1,1}} \leq C\|B\|_{\mathcal{C}^{k+1,1}}. \tag{3.10}$$

Proof. **1.** Assume first the minimal regularity $u \in \mathcal{C}^{2,1}$ and $B \in \mathcal{C}^{1,1}$, so that $\theta = \mathrm{curl}^T\mathrm{curl}[B_{ij}] \in L^\infty(\Omega, \mathbb{R})$. Solving (3.7) by integrating twice in s from $w_3(\Phi(t,0)) = 0$ and $\partial_s w_3(\Phi(t,0)) = 0$, we obtain:

$$\|(w\vec{n})\|_{L^\infty} \leq C\|\theta\|_{L^\infty} \leq C\|B\|_{\mathcal{C}^{1,1}}. \tag{3.11}$$

Solving now (3.3) for w_{tan} so that:

$$\mathbb{P}_{so(2)} \int_\Omega \nabla(w_{tan} \circ u) = 0 \text{ and } \int_\Omega (w_{tan} \circ u) = 0,$$

we obtain by means of Korn's inequality, for any $p > 1$:

$$\|\nabla w_{tan}\|_{L^p} \leq C\|\mathrm{sym}\nabla(w_{tan} \circ u)\|_{L^p} \leq C(\|[B_{ij}]\|_{L^\infty} + \|(w\vec{n})\|_{L^\infty}) \leq C\|B\|_{\mathcal{C}^{1,1}},$$

where C may depend on p. Combining with the Poincaré inequality, we get:

$$\|w_{tan}\|_{W^{1,p}} \leq C\|B\|_{\mathcal{C}^{1,1}}. \tag{3.12}$$

By Sobolev embedding, (3.9) follows now from (3.11) and (3.12).

2. When $B \in \mathcal{C}^{k+1,1}$ and $u \in \mathcal{C}^{k+2,1}$, then $-\frac{1-s\kappa}{\kappa_n}\theta \in \mathcal{C}^{k-1,1}$ by Lemma 2.4, and so by (3.7):

$$\|(w\vec{n})\|_{\mathcal{C}^{k-1,1}} \leq C\|\theta\|_{\mathcal{C}^{k-1,1}} \leq C\|B\|_{\mathcal{C}^{k+1,1}}, \tag{3.13}$$

where C may depend on S. Recalling that $\nabla^2 w'$ can be expressed as the linear combination of partial derivatives of $\mathrm{sym}\nabla w'$, we have from (3.8), (3.13) that:

$$\|\nabla^2 w_{tan}\|_{\mathcal{C}^{k-2,1}} \leq C\|B\|_{\mathcal{C}^{k+1,1}},$$

which implies (3.10) in view of (3.9) and (3.13). ■

Proposition 3.2. *Assume that S is developable of class $\mathcal{C}^{k+2,1}$, satisfying (2.12). Also assume that $B = \mathrm{sym}((\nabla\phi)^T(\nabla\psi))$ where $\phi, \psi \in \mathcal{C}^{k+1,1}(S, \mathbb{R}^3)$. Then w as obtained in Theorem 3.1 satisfies:*

$$\|w_{tan}\|_{\mathcal{C}^{k,1}} + \|w\vec{n}\|_{\mathcal{C}^{k-1,1}} \leq C\|\psi\|_{\mathcal{C}^{k+1,1}}\|\phi\|_{\mathcal{C}^{k+1,1}}.$$

Proof. After straightforward calculations, we obtain:

$$B_{ij} = \frac{1}{2}\big(\partial_i(\phi \circ u) \cdot \partial_j(\psi \circ u) + \partial_j(\phi \circ u) \cdot \partial_i(\psi \circ u)\big).$$

Further calculations shows that in the expansion of θ, as defined in (3.5), the third derivatives of ψ and ϕ cancel out:

$$\begin{aligned}\theta &= \mathrm{curl}^T\mathrm{curl}\ [B_{ij}] = \partial^2_{11}B_{22} + \partial^2_{22}B_{11} - 2\partial^2_{12}B_{12}\\ &= \partial^2_{22}\big(\partial_1(\phi\circ u)\cdot\partial_1(\psi\circ u)\big) + \partial^2_{11}\big(\partial_2(\phi\circ u)\cdot\partial_2(\psi\circ u)\big)\\ &\qquad - \partial^2_{12}\big(\partial_1(\phi\circ u)\cdot\partial_2(\psi\circ u) + \partial_2(\phi\circ u)\cdot\partial_1(\psi\circ u)\big)\\ &= -\big(\partial^2_{11}(\phi\circ u)\cdot\partial^2_{22}(\psi\circ u) + \partial^2_{22}(\phi\circ u)\cdot\partial^2_{11}(\psi\circ u) - 2\partial^2_{12}(\phi\circ u)\cdot\partial^2_{12}(\psi\circ u)\big).\end{aligned}$$

As a consequence, if S is of class $\mathcal{C}^{k+2,1}$, the solution of (3.7) from Theorem 3.1 satisfies:

$$\|w\vec n\|_{\mathcal{C}^{k-1,1}} \leq C\|\theta\|_{\mathcal{C}^{k-1,1}} \leq C\|\phi\|_{\mathcal{C}^{k+1,1}}\|\psi\|_{\mathcal{C}^{k+1,1}}.$$

Reasoning as in the proof of Theorem 3.1, we obtain:

$$\|w_{tan}\|_{\mathcal{C}^{k,1}} + \|w\vec n\|_{\mathcal{C}^{k-1,1}} \leq C\|B\|_{\mathcal{C}^{k-1,1}} + C\|\phi\|_{\mathcal{C}^{k+1,1}}\|\psi\|_{\mathcal{C}^{k+1,1}} \leq C\|\phi\|_{\mathcal{C}^{k+1,1}}\|\psi\|_{\mathcal{C}^{k+1,1}},$$

proving the claim. ■

4. Spaces of $W^{2,2}$ infinitesimal isometries on developable surfaces

In this section, we establish some properties of $W^{2,2}$ first order infinitesimal isometries on developable surfaces S of $\mathcal{C}^{2,1}$ regularity. We give a classification of these displacements and prove that they are necessarily $\mathcal{C}^{1,1/2}$ regular. Define:

$$\mathcal{V} = \big\{V \in W^{2,2}(S,\mathbb{R}^3);\ \mathrm{sym}\nabla V = 0\big\}$$

Note that in view of Lemma 2.5, we may freely determine the regularity of any mapping on S, up to $\mathcal{C}^{2,1}$ regularity, by considering the regularity of its composition with the chart u. Here and in what follows we write $f \in W^{2,2}(S)$ precisely if $f\circ u \in W^{2,2}(\Omega)$.

The following is the main result of this section:

Theorem 4.1. *Let $V \in \mathcal{V}$ and assume that S is developable of class $\mathcal{C}^{2,1}$. Then $V \in \mathcal{C}^{1,1/2}(S,\mathbb{R}^3)$. More precisely, writing $V = V_{tan} + (V\vec n)\vec n$ we have:*

$$V_{tan} \in \mathcal{C}^{2,1/2}(S,\mathbb{R}^3) \quad \textit{and} \quad V\vec n \in \mathcal{C}^{1,1/2}(S).$$

Moreover, $V \in \mathcal{V}$ if and only if there exist $a, b \in W^{2,2}((0,T),\mathbb{R})$ such that

$$(V\vec n)(u(\Phi(s,t))) = a(t) + sb(t), \tag{4.1}$$

$$\mathrm{sym}\nabla V_{tan}(u(\Phi(s,t))) = \frac{a(t)+sb(t)}{1-s\kappa(t)}\kappa_n(t)\ (\Gamma'(t)\otimes\Gamma'(t)) \qquad \textit{for a.e. } (s,t)\in M_{s^\pm}, \tag{4.2}$$

and such that the following integrals are finite:

(4.3)

$$J_1(a,b) = \int_{M_{s^\pm}} \left(b'(t) + \frac{\kappa(a'(t)+sb'(t))}{1-s\kappa(t)}\right)^2 \frac{\mathrm{d}s\mathrm{d}t}{1-s\kappa(t)} < \infty,$$

(4.4)

$$J_2(a,b) = \int_{M_{s^\pm}} \left(a''(t) + sb''(t) - \kappa(t)(1-s\kappa(t))b(t) + \frac{s\kappa'(t)(a'(t)+sb'(t))}{1-s\kappa(t)}\right)^2 \frac{\mathrm{d}s\mathrm{d}t}{(1-s\kappa(t))^3} < \infty.$$

Proof. Since $u \in \mathcal{C}^{2,1}$, from (2.6) we conclude that κ_n is continuous up to the boundary and that κ is continuous on the open set where κ_n differs from zero.

1. Let $V \in \mathcal{V}$. From the observations following Lemma 2.1 and from (2.11) we deduce that Φ is bilipschitz on every set of the form:

$$M'_\delta = \{(s,t) \in M_{s^\pm} : s \in (\delta - s^-(t), s^+(t) - \delta)\}$$

with $\delta > 0$. Indeed, Φ^{-1} may fail to be globally Lipschitz on $\Phi(M_{s^\pm})$ unless (2.13) is satisfied. Set $V_3 = (V\vec{n}) \circ u$ and $f = V_3 \circ \Phi$. We have $V_3 \in W^{2,2}(\Omega)$. Since Φ is bilipschitz on M'_δ, Lemma 2.5 implies that $f \in W^{2,2}(M'_\delta)$ with:

$$\partial_s f(s,t) = \nabla V_3(x) N(t), \quad \partial_t f(s,t) = (1 - s\kappa(t))\nabla V_3(x)\Gamma'(t) \qquad \text{where } x = \Phi(t,s), \tag{4.5}$$

$$\begin{aligned} &\partial^2_{ss} f(s,t) = \big(\nabla^2 V_3(x) N(t)\big) N(t), \\ &\partial^2_{ts} f(s,t) = (1 - s\kappa(t))\big(\nabla^2 V_3(x)\vec{N}(t)\big)\Gamma'(t) - \kappa(t)\nabla V_3(x)\Gamma'(t), \\ &\partial^2_{tt} f(s,t) = (1 - s\kappa(t))^2\big(\nabla^2 V_3(x)\Gamma'(t)\big)\Gamma'(t) + \nabla V_3(x)\Big(\kappa(1 - s\kappa)N(t) - s\kappa'\Gamma'(t)\Big). \end{aligned} \tag{4.6}$$

Moreover, by (3.7):

$$\partial^2_{ss} f(s,t) = 0 \qquad \forall t \in [0,T] \text{ with } \kappa_n(t) \neq 0. \tag{4.7}$$

Indeed, $\theta = \mathrm{curl}^T \mathrm{curl}\ [B_{ij}] = 0$ in the present case where $\mathrm{sym}\nabla V = 0$.

Let $0 < \eta < \inf_{t\in[0,T]}\{s^-(t), s^+(t)\}$ so that $(-\eta,\eta) \times (0,T) \subset M_{s^\pm}$. Denote by f^* the precise representative of f and define:

$$a(t) = f^*(t,0), \qquad b(t) = \frac{1}{\eta}(f^*(t,\eta) - a(t)).$$

By (3.7) the definition of b does not depend on η, and:

$$f(s,t) = a(t) + sb(t) \qquad \forall (s,t) \in M_{s^\pm}. \tag{4.8}$$

2. Since $f \in W^{2,2}((0,T) \times (-\eta,\eta))$, for almost every pair $s_1, s_2 \in (-\eta,\eta)$, the traces $f(\cdot, s_1)$ and $f(\cdot, s_2)$ belong to $W^{2,2}(0,T)$. Hence $a, b \in W^{2,2}(0,T)$, which implies regularity of a and b by Sobolev embedding. In particular, $f \in \mathcal{C}^{1,1/2}(M_{s^\pm})$.

Since Φ is a $\mathcal{C}^{1,1}$ diffeomorphism, it also follows that $V\vec{n}$, $V_{tan} \in \mathcal{C}^{1,1/2}(S)$ which implies [22] that $V_{tan} \in \mathcal{C}^{2,1/2}$. Finally, (4.2) follows from Lemma 2.1.

3. We shall now prove that, given the structure (4.8), condition $V\vec{n} \in W^{2,2}(S)$ is equivalent to $a, b \in W^{2,2}(0,T)$ satisfying (4.3), (4.4). This will conclude the proof.

Inserting (4.5) into (4.6) we obtain, for all $t \in [0,T] \setminus I_0$:

$$b'(t) = \partial^2_{ts} f(s,t) = (1 - s\kappa)\Big(\nabla^2 V_3(x)\vec{N}(t)\Big)\Gamma'(t) - \frac{\kappa}{1 - s\kappa}(a'(t) + sb'(t)),$$

$$a''(t) + sb''(t) = \partial^2_{tt} f(s,t) = (1 - s\kappa)^2\big(\nabla^2 V_3(x)\Gamma'(t)\big)\Gamma'(t) + \kappa(1 - s\kappa)b(t) - \frac{s\kappa'}{1 - s\kappa}(a'(t) + sb'(t)).$$

Solving for $\nabla^2 V_3(x)$ we get:

$$\begin{aligned} &\big(\nabla^2 V_3(x)\Gamma'(t)\big)\Gamma'(t) = \frac{1}{(1 - s\kappa)^2}\Big(a''(t) + sb''(t) - \kappa(1 - s\kappa)b(t) + \frac{s\kappa'}{1 - s\kappa}(a'(t) + sb'(t))\Big), \\ &\big(\nabla^2 V_3(x)N'(t)\big)\Gamma'(t) = \frac{1}{1 - s\kappa}\Big(b'(t) + \frac{\kappa'}{1 - s\kappa}(a'(t) + sb'(t)\Big), \\ &\big(\nabla^2 V_3(x)N'(t)\big)N'(t) = 0. \end{aligned}$$

Now a change of variables shows that:

$$\int_\Omega |\nabla^2 V_3(x)|^2 \mathrm{d}x = \int_{M_{s\pm}} |(\nabla^2 V_3)(\Phi(s,t))|^2 (1-s\kappa)\mathrm{d}s\mathrm{d}t.$$

We see that $V_3 \in W^{2,2}(\Omega)$ if and only if (4.3) and (4.4) hold. ∎

We finish this section by pointing out a straightforward corollary of the above calculations:

Proposition 4.2. *Let $v \in W^{2,2}(S)$ satisfy*

$$v(u(\Phi(s,t))) = a(t) + sb(t) \quad \text{for a.e. } (s,t) \in M_{s\pm}.$$

Then $a, b \in W^{2,2}(0,T)$ and there exists a tangent vector field $V_{tan} \in W^{2,2}(S, \mathbb{R}^3)$ to S such that $V_{tan} + v\vec{n} \in \mathcal{V}$.

5. Matching and density of infinitesimal isometries

Definition 5.1. *A one parameter family $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}^{1,1}(\overline{S}, \mathbb{R}^3)$ is said to be a (generalized) Nth order infinitesimal isometry if the change of metric induced by u_ε is of order ε^{N+1}, that is:*

$$\|(\nabla u_\varepsilon)^T(\nabla u_\varepsilon) - \mathrm{Id}\|_{L^\infty(S)} = \mathcal{O}(\varepsilon^{N+1}) \text{ as } \varepsilon \to 0. \tag{5.1}$$

Here and in what follows we use the Landau symbols $\mathcal{O}(q)$ and $o(q)$. They denote, respectively, any quantity whose quotient with q is uniformly bounded or converges to 0 as $q \to 0$.

Note that if $V \in \mathcal{V} \cap C^{0,1}$, then $u_\varepsilon = \mathrm{id} + \varepsilon V$ is a (generalized) first order isometry.

Theorem 5.2. *Let S be a developable surface of class $\mathcal{C}^{2N,1}$, satisfying (2.12). Given $V \in \mathcal{V} \cap \mathcal{C}^{2N-1,1}(\bar{S})$, there exists a family $\{w_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}^{1,1}(S, \mathbb{R}^3)$, equibounded in $\mathcal{C}^{1,1}(S)$, such that for all small $\varepsilon > 0$ the family:*

$$u_\varepsilon = \mathrm{id} + \varepsilon V + \varepsilon^2 w_\varepsilon$$

is a (generalized) Nth order isometry of class $\mathcal{C}^{1,1}$.

Proof. **1.** The result is a consequence of the following claim. Let S be of class $\mathcal{C}^{k+2,1}$ and let u_ε be an $(i-1)$th order isometry of regularity $\mathcal{C}^{k+1,1}$ of the form:

$$u_\varepsilon = \mathrm{id} + \sum_{j=1}^{i-1} \varepsilon^j w_j, \qquad w_j \in \mathcal{C}^{k+1,1}.$$

Then there exists $w_i \in \mathcal{C}^{k-1,1}(S, \mathbb{R}^3)$ so that $\phi_\varepsilon = u_\varepsilon + \varepsilon^i w_i$ is an ith order infinitesimal isometry, and:

$$\|w_i\|_{\mathcal{C}^{k-1,1}} \leq C \sum_{j=1}^{i-1} \|w_j\|_{\mathcal{C}^{k+1,1}} \|w_{i-j}\|_{\mathcal{C}^{k+1,1}}. \tag{5.2}$$

Indeed, setting $w_1 = V \in \mathcal{C}^{2N-1,1}$ and applying the above result iteratively to find $w_j \in \mathcal{C}^{2N-2j+1,1}$, for $j = 2 \dots N$, we obtain the requested $w_\varepsilon = w_2 + \varepsilon w_3 + \cdots \varepsilon^{N-2} w_N \in \mathcal{C}^{1,1}$.

2. We now prove the claim. Set $w_0 = \mathrm{id}$. Calculating the change of metric induced by the deformation ϕ_ε we get:

$$|(\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon - \mathrm{Id}| = \left| \sum_{j=1}^{i} \varepsilon^j A_j \right| + \mathcal{O}(\varepsilon^{i+1}),$$

where the expression A_j indicating the change of metric of jth order induced by ϕ_ε, is given by:

$$A_j = \sum_{p=0}^{j} \mathrm{sym}\Big((\nabla w_p)^T \nabla w_{j-p}\Big).$$

Note that by the assumption $A_1 = \cdots = A_{i-1} = 0$. Consequently, in order for ϕ_ε to be an ith order isometry, we must have $A_i = 0$ or equivalently:

$$\mathrm{sym}\nabla w_i = -\frac{1}{2}\sum_{p=1}^{i-1} \mathrm{sym}\Big((\nabla w_p)^T \nabla w_{i-p}\Big).$$

Applying Theorems 3.1 and 3.2, we obtain that such w_k exists with the estimate:

$$\|w_{i,tan}\|_{\mathcal{C}^{k,1}} + \|w_{i,3}\|_{\mathcal{C}^{k-1,1}} \leq C\sum_{p=1}^{i-1} \|w_p\|_{\mathcal{C}^{k+1,1}}\|w_{i-p}\|_{\mathcal{C}^{k+1,1}},$$

provided that all $w_p \in \mathcal{C}^{k+1,1}$ and that S is of class $\mathcal{C}^{k+2,1}$. This completes the proof of the claim and of the theorem. ■

Theorem 5.3. *Assume that S is developable, of class $\mathcal{C}^{k+1,1}$ up to the boundary, and satisfying (2.12). Then, for every $V \in \mathcal{V}$ there exists a sequence $V_n \in \mathcal{V} \cap \mathcal{C}^{k,1}(\bar{S}, \mathbb{R}^3)$ such that:*

$$\lim_{n\to\infty} \|V_n - V\|_{W^{2,2}(S)} = 0.$$

Proof. Let $a, b \in W^{2,2}(0,T)$ be as in Proposition 4.1. Take $a_n, b_n \in \mathcal{C}^\infty([0,T])$ converging in $W^{2,2}$ to a, b respectively, and define:

$$v_n(s,t) = a_n(t) + s b_n(t).$$

By Propositions 4.1 and 4.2, there exist $V_n \in \mathcal{V}$ such that $(V_n\vec{n}) \circ u \circ \Phi = v_n$ and $\|V_n\vec{n} - V\vec{n}\|_{W^{2,2}(S)} \to 0$. Indeed, the last assertion is equivalent to proving that $J_i(a - a_n, b - b_n) \to 0$, $i = 1, 2$, which is established immediately after observing that $1 - s\kappa$ is bounded away from 0 by (2.13). Note that $(V_n)_{tan}$ can be chosen suitably such that also $\|(V_n)_{tan} - V_{tan}\|_{W^{2,2}(S)} \to 0$. In view of Lemma 2.4, $V_n \circ u \in \mathcal{C}^{k,1}(\overline{\Omega})$, that is $V_n \in \mathcal{C}^{k,1}$ up to the boundary of S. ■

6. The Γ-limit result

Consider a family $\{S^h\}_{h>0}$ of thin shells of thickness h around S:

$$S^h = \{z = p + t\vec{n}(p);\ p \in S,\ -h/2 < t < h/2\}, \qquad 0 < h < h_0,$$

where h_0 is small enough so that the projection map $\pi : S^{h_0} \to S, \pi(p + t\vec{n}(p) := p$ is well-defined. For a $W^{1,2}$ deformation $u^h : S^h \to \mathbb{R}^3$, we assume that its elastic energy (scaled per unit thickness) is given by the nonlinear functional:

$$E^h(u^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h).$$

The stored-energy density function $W : \mathbb{R}^{3\times3} \longrightarrow [0,\infty]$ is $\mathcal{C}^2$ in an open neighborhood of $SO(3)$, and it is assumed to satisfy the conditions of normalization, frame indifference and quadratic growth:

$$\begin{aligned} \forall F \in \mathbb{R}^{3\times3} \quad \forall R \in SO(3) \qquad & W(R) = 0, \quad W(RF) = W(F), \\ & W(F) \geq C\mathrm{dist}^2(F, SO(3)), \end{aligned}$$

with a uniform constant $C > 0$. The potential W induces the quadratic forms:

$$\mathcal{Q}_3(F) = D^2W(\mathrm{Id})(F,F), \qquad \mathcal{Q}_2(p, F_{tan}) = \min\{\mathcal{Q}_3(\tilde{F});\ (\tilde{F} - F)_{tan} = 0\}.$$

defined for $F \in \mathbb{R}^{3\times 3}$, and $p \in S$ respectively. Here and in what follows F_{tan} is the bilinear form induced by F on S through the formula:

$$F_{tan}(\tau, \eta) = \tau \cdot F(p)\eta \quad \forall p \in S,\ \tau, \eta \in T_pS.$$

Both forms $\mathcal{Q}_3$ and all $\mathcal{Q}_2(p,\cdot)$ depend only on the symmetric parts of their arguments, with respect to which they are positive definite [7].

In what follows we shall consider a sequence $e^h > 0$ such that:

$$0 < \lim_{h\to 0} e^h/h^\beta < +\infty, \qquad \text{for some } \ 2 < \beta < 4. \tag{6.1}$$

Also, let:

$$\beta_N = 2 + 2/N.$$

We assume that $N > 1$ (the case $N = 1$ is already covered in [22]). Recall the following result:

Theorem 6.1. [22]. *Let S be a surface embedded in $\mathbb{R}^3$, which is compact, connected, oriented, of class $\mathcal{C}^{1,1}$, and whose boundary ∂S is the union of finitely many Lipschitz curves. Let $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ be a sequence of deformations whose scaled energies $E^h(u^h)/e^h$ are uniformly bounded. Then there exist a sequence $Q^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that for the normalized rescaled deformations:*

$$y^h(p + t\vec{n}) = Q^h u^h(p + h/h_0 t\vec{n}) - c^h$$

defined on the common domain S^{h_0}, the following holds.

(i) *y^h converge in $W^{1,2}(S^{h_0})$ to π.*

(ii) *The scaled average displacements:*

$$V^h(p) = \frac{h}{\sqrt{e^h}} \int_{-h_0/2}^{h_0/2} y^h(p + t\vec{n}) - p \ \mathrm{d}t \tag{6.2}$$

converge (up to a subsequence) in $W^{1,2}(S)$ to some $V \in \mathcal{V}$.

(iii) *$\liminf_{h\to 0} 1/e^h E^h(u^h) \geq \mathcal{I}(V)$, where:*

$$\mathcal{I}(V) = \frac{1}{24}\int_S \mathcal{Q}_2\Big(p, \big(\nabla(A\vec{n}) - A\Pi\big)_{tan}\Big)\, dp. \tag{6.3}$$

Here, the matrix field $A \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ is such that:

$$\partial_\tau V(p) = A(p)\tau \quad \text{and} \quad A(p) \in so(3) \qquad \forall \text{a.e. } p \in S \quad \forall \tau \in T_pS.$$

In order to prove that the linear bending functional (6.3) restricted to $\mathcal{V}$ is the Γ-limit of the rescaled three dimensional nonlinear elasticity energy $(1/e^h)E^h$ we also need to establish the limsup counterpart of the Γ-convergence statement. This is the final contribution of this paper which we are formulating in the following theorem. For a full discussion of this topic in this context see [8] and [22].

Theorem 6.2. *Let $N > 1$ and assume that S is developable of class $\mathcal{C}^{2N,1}$ and satisfying (2.12). Assume that*

$$e^h = o(h^{\beta_N}). \tag{6.4}$$

Then for every $V \in \mathcal{V}$ there exists a sequence $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ such that:

(i) *the rescaled deformations $y^h(p + t\vec{n}) = u^h(p + th/h_0\vec{n})$ converge in $W^{1,2}(S^{h_0})$ to π.*

(ii) *the scaled average displacements V^h given in (6.2) converge in $W^{1,2}(S)$ to V.*

(iii) $\lim_{h\to 0} 1/e^h E^h(u^h) = \mathcal{I}(V)$.

Proof. We shall construct a recovery sequence for developable surfaces, based on Theorems 5.3 and 5.2. Indeed, by the density result and the continuity of the functional $\mathcal{I}$ with respect to the strong topology of $W^{2,2}(S)$, we can assume $V \in \mathcal{V} \cap \mathcal{C}^{2N-1,1}(\bar{S}, \mathbb{R}^3)$. In the general case the result will then follow through a diagonal argument.

1. Let $\varepsilon = \sqrt{e^h}/h$ so $\varepsilon \to 0$ as $h \to 0$, by assumption (6.1). Therefore, by Theorem 5.2 there exists a sequence $w_\varepsilon : \bar{S} \longrightarrow \mathbb{R}^3$, equibounded in $\mathcal{C}^{1,1}(\bar{S})$, such that for all small $h > 0$:

$$u_\varepsilon = \mathrm{id} + \varepsilon V + \varepsilon^2 w_\varepsilon \tag{6.5}$$

is a (generalized) Nth order infinitesimal isometry. Note that by (6.4) we have:

$$\frac{\varepsilon^{N+1}}{\sqrt{e^h}} = \frac{(\sqrt{e^h})^N}{h^{N+1}} = o(h^{N+1})/h^{N+1} \to 0,$$

hence $\varepsilon^{N+1} = o(\sqrt{e^h})$. We may thus replace $\mathcal{O}(\varepsilon^{N+1})$ with $o(\sqrt{e^h})$.

For every $p \in S$, let $\vec{n}_\varepsilon(p)$ denote the unit normal vector to $u_\varepsilon(S)$ at the point $u_\varepsilon(p)$. Clearly, $\vec{n}_\varepsilon \in \mathcal{C}^{0,1}(\bar{S}, \mathbb{R}^3)$, while by (6.5):

$$\vec{n}_\varepsilon = \frac{\partial_{\tau_1} u_\varepsilon \times \partial_{\tau_2} u_\varepsilon}{|\partial_{\tau_1} u_\varepsilon \times \partial_{\tau_2} u_\varepsilon|} = \vec{n} + \varepsilon A\vec{n} + \mathcal{O}(\varepsilon^2). \tag{6.6}$$

Here $\tau_1, \tau_2 \in T_pS$ are such that $\vec{n} = \tau_1 \times \tau_2$. Note that since $N > 1$ and u_ε is a (generalized) Nth order isometry, we have $|\partial_{\tau_i} u_\varepsilon|^2 = 1 + \mathcal{O}(\varepsilon^3)$ and $|\partial_{\tau_1} u_\varepsilon \cdot \partial_{\tau_2} u_\varepsilon| = \mathcal{O}(\varepsilon^3)$, which implies that:

$$|\partial_{\tau_1} u_\varepsilon \times \partial_{\tau_2} u_\varepsilon| = 1 + \mathcal{O}(\varepsilon^3).$$

Using now the Jacobi identity for vector product and the fact that $A \in so(3)$, we arrive at (6.6).

Here we introduce the recovery sequence u^h as required by the statement of the theorem. Note that the following suggestion for u^h is in accordance with the one used in [6] in the framework of the purely nonlinear bending theory for shells, corresponding to the scaling regime $\beta = 2$. Also, a comparison with the similar proof in [24] for convex shells, with which much of the following calculations overlap, is elucidating. Indeed, the main difference here is that instead of an exact isometry of the given shell, we make use of an Nth order isometry u_ε. Consider the sequence of deformations $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ defined by:

$$u^h(p + t\vec{n}) = u_\varepsilon(p) + t\vec{n}_\varepsilon(p) + \frac{t^2}{2}\varepsilon d^h(p), \tag{6.7}$$

where ε depends on h as above. The vector field $d^h \in W^{1,\infty}(S, \mathbb{R}^3)$ is taken so that:

$$\lim_{h\to 0} h^{1/2} \|d^h\|_{W^{1,\infty}(S)} = 0, \tag{6.8}$$

and:

$$\lim_{h\to 0} d^h = 2c\,(p, \mathrm{sym}(\nabla(A\vec{n}) - A\Pi)_{tan}) \quad \text{ in } L^\infty(S), \tag{6.9}$$

where $c(p, F_{tan})$ denotes the unique vector satisfying $\mathcal{Q}_2(p, F_{tan}) = \mathcal{Q}_3(F_{tan} + c \otimes \vec{n}(p) + \vec{n}(p) \otimes c)$ (see [22, Section 6]). We observe that, as $V \in \mathcal{C}^{1,1}(\bar{S}, \mathbb{R}^3)$ and c depends linearly on its second argument, the vector field:

$$\zeta(p) = c(p, \mathrm{sym}(\nabla(A\vec{n}) - A\Pi)_{tan}) \tag{6.10}$$

belongs to $L^\infty(S, \mathbb{R}^3)$. Properties (i) and (ii) now easily follow from the uniform bound on w_ε and the normalization (6.8).

2. To prove (iii) it is convenient to perform a change of variables in the energy $E^h(u^h)$, so to express it in terms of the scaled deformation y^h. By a straightforward calculation:

$$\frac{1}{e^h}E^h(u^h) = \frac{1}{e^h}\int_S \int_{-h_0/2}^{h_0/2} W(\nabla_h y^h(p+t\vec{n}))\det[\mathrm{Id} + th/h_0\Pi(p)]\ \mathrm{d}t\mathrm{d}p, \tag{6.11}$$

where $\nabla_h y^h(p+t\vec{n}) = \nabla u^h(p+th/h_0\vec{n})$. We also have:

$$\begin{aligned} \nabla_h y^h(p+t\vec{n})\vec{n}(p) &= \frac{h_0}{h}\partial_{\vec{n}} y^h(p+t\vec{n}) = \vec{n}_\varepsilon(p) + th/h_0\varepsilon d^h(p), \\ \nabla_h y^h(p+t\vec{n})\tau &= \nabla y^h(p+t\vec{n})\cdot(\mathrm{Id}+t\Pi(p))(\mathrm{Id}+th/h_0\Pi(p))^{-1}\tau \\ &= \Big(\nabla u_\varepsilon(p) + th/h_0\nabla\vec{n}_\varepsilon(p) + \frac{t^2}{2h_0^2}h^2\varepsilon\nabla d^h(p)\Big)(\mathrm{Id}+th/h_0\Pi(p))^{-1}\tau, \end{aligned} \tag{6.12}$$

for all $p \in S$ and $\tau \in T_pS$.

From (6.5), (6.6) and (6.8) it follows that $\|\nabla_h y^h - \mathrm{Id}\|_{L^\infty(S^{h_0})} \to 0$ as $h \to 0$. By polar decomposition theorem, $\nabla_h y^h$ is a product of a proper rotation and the well defined square root $\sqrt{(\nabla_h y^h)^T\nabla_h y^h}$. By frame indifference of W we deduce that:

$$W(\nabla_h y^h) = W\left(\sqrt{(\nabla_h y^h)^T\nabla_h y^h}\right) = W\left(\mathrm{Id} + \frac{1}{2}K^h + \mathcal{O}(|K^h|^2)\right),$$

where the last equality is obtained by the Taylor expansion, with:

$$K^h = (\nabla_h y^h)^T\nabla_h y^h - \mathrm{Id}.$$

As $\|K^h\|_{L^\infty(S^{h_0})}$ is infinitesimal as $h \to 0$, we can expand W around Id, using the formula:

$$W(\mathrm{Id}+K) = \frac{1}{2}D^2W(\mathrm{Id})(K,K) + \int_0^1 (1-s)[D^2W(\mathrm{Id}+sK) - D^2W(\mathrm{Id})](K,K)\mathrm{d}s,$$

and obtain, in view of using the assumption that W is $\mathcal{C}^2$ in a neighborhood of identity:

$$\frac{1}{e^h}W(\nabla_h y^h) = \frac{1}{2}\mathcal{Q}_3\left(\frac{1}{2\sqrt{e^h}}K^h + \frac{1}{\sqrt{e^h}}\mathcal{O}(|K^h|^2)\right) + \frac{1}{e^h}o(|K^h|^2). \tag{6.13}$$

Using (6.12) we now calculate K^h. We first consider the tangential minor of K^h, as usual conceived as a symmetric bilinear form on S:

$$\begin{aligned} K^h_{tan}(p+t\vec{n}) &= (\mathrm{Id}+th/h_0\Pi)^{-1}\Big[\mathrm{Id} + \mathcal{O}(\varepsilon^{N+1}) + 2th/h_0\,\mathrm{sym}((\nabla u_\varepsilon)^T\nabla\vec{n}_\varepsilon) \\ &\qquad + t^2h^2/h_0^2(\nabla\vec{n}_\varepsilon)^T\nabla\vec{n}_\varepsilon + o(\sqrt{e^h})\Big](\mathrm{Id}+th/h_0\Pi)^{-1} - \mathrm{Id} \\ &= (\mathrm{Id}+th/h_0\Pi)^{-1}\Big[2th/h_0\,\mathrm{sym}((\nabla u_\varepsilon)^T\nabla\vec{n}_\varepsilon) - 2th/h_0\Pi \\ &\qquad + t^2h^2/h_0^2(\nabla\vec{n}_\varepsilon)^T\nabla\vec{n}_\varepsilon - t^2h^2/h_0^2\Pi^2\Big](\mathrm{Id}+th/h_0\Pi)^{-1} + o(\sqrt{e^h}), \end{aligned}$$

where we used the fact that u_ε is a generalized Nth order infinitesimal isometry to see that $(\nabla u_\varepsilon)^T\nabla u_\varepsilon = \mathrm{Id} + \mathcal{O}(\varepsilon^{N+1}) = \mathrm{Id} + o(\sqrt{e^h})$, and the identity:

$$F_1^{-1}FF_1^{-1} - \mathrm{Id} = F_1^{-1}(F - F_1^2)F_1^{-1}.$$

By (6.5) and (6.6) we also deduce:

$$\begin{aligned} \mathrm{sym}((\nabla u_\varepsilon)^T\nabla\vec{n}_\varepsilon) &= \Pi + \varepsilon\,\mathrm{sym}(\nabla(A\vec{n}) - A\Pi) + \mathcal{O}(\varepsilon^2), \\ (\nabla\vec{n}_\varepsilon)^T\nabla\vec{n}_\varepsilon &= \Pi^2 + \mathcal{O}(\varepsilon). \end{aligned}$$

Combining these two identities with the expression of K^h_{tan} found above, we conclude that:

$$K^h_{tan}(p+t\vec{n}) = \sqrt{e^h}(\mathrm{Id} + th/h_0\Pi)^{-1}\Big[2t/h_0\,\mathrm{sym}(\nabla(A\vec{n}) - A\Pi)\Big](\mathrm{Id} + th/h_0\Pi)^{-1} + o(\sqrt{e^h}).$$

Now, as $|\vec{n}_\varepsilon| = 1$, the normal minor of K^h is calculated by means of (6.12) as:

$$\vec{n}^T K^h(p+t\vec{n})\vec{n} = |(\nabla_h y^h)\vec{n}|^2 - 1 = 2th/h_0\varepsilon d^h\cdot\vec{n}_\varepsilon + o(\sqrt{e^h}) = 2t/h_0\sqrt{e^h}d^h\cdot\vec{n} + o(\sqrt{e^h}).$$

The remaining coefficients of the symmetric matrix $K^h(p+t\vec{n})$ are, for $\tau\in T_xS$:

$$\begin{aligned}\tau^T K^h(p+t\vec{n})\vec{n} &= (\vec{n}_\varepsilon + th/h_0\varepsilon d^h)^T\Big(\nabla u_\varepsilon + th/h_0\nabla\vec{n}_\varepsilon + \frac{t^2}{2h_0^2}h^2\varepsilon\nabla d^h\Big)(\mathrm{Id} + th/h_0\Pi)^{-1}\tau\\ &= t/h_0\sqrt{e^h}(d^h)^T\nabla u_\varepsilon(\mathrm{Id} + th/h_0\Pi)^{-1}\tau + o(\sqrt{e^h}),\end{aligned}$$

where we have used that $\vec{n}_\varepsilon^T\nabla\vec{n}_\varepsilon = \vec{n}_\varepsilon^T\nabla u_\varepsilon = 0$.

3. From the previous computations we finally deduce, with some abuse of notation, that:

$$\lim_{h\to 0}\frac{1}{2\sqrt{e^h}}K^h = \frac{t}{h_0}K(p)_{tan} + \frac{t}{h_0}(\zeta\otimes\vec{n} + \vec{n}\otimes\zeta)\quad\text{ in } L^\infty(S^{h_0}),\tag{6.14}$$

where the vector field ζ is defined in (6.10) and the symmetric bilinear form $K_{tan}\in L^\infty(S)$ is:

$$K(p)_{tan} = \mathrm{sym}(\nabla(A\vec{n}) - A\Pi)_{tan}.\tag{6.15}$$

Using (6.11), (6.13), (6.14) and the dominated convergence theorem, we obtain:

$$\begin{aligned}\lim_{h\to 0}\frac{1}{e^h}E^h(u^h) &= \lim_{h\to 0}\frac{1}{e^h}\int_S\int_{-h_0/2}^{h_0/2} W(\nabla_h y^h)\det(\mathrm{Id} + th/h_0\Pi)\;\mathrm{d}t\mathrm{d}p\\ &= \frac{1}{2}\int_S\int_{-h_0/2}^{h_0/2}\mathcal{Q}_3\Big(\frac{t}{h_0}K(p)_{tan} + \frac{t}{h_0}(\zeta\otimes\vec{n} + \vec{n}\otimes\zeta)\Big)\;\mathrm{d}t\mathrm{d}p\\ &= \frac{1}{2}\int_S\int_{-h_0/2}^{h_0/2}\frac{t^2}{h_0^2}\mathcal{Q}_2\big(p, \mathrm{sym}(\nabla(A\vec{n}) - A\Pi)_{tan}\big)\;\mathrm{d}t\mathrm{d}p,\end{aligned}$$

the last equality following in view of (6.10) and (6.15). Property (iii) now follows, upon integration in t in the last integral above. ■

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