Citation for published version:
Crosby, PG 2012, 'On the structure of right 3-Engel subgroups', *Journal of Algebra*, vol. 365, no. 1, pp. 205-220. https://doi.org/10.1016/j.jalgebra.2011.12.031

DOI:
10.1016/j.jalgebra.2011.12.031

Publication date:
2012

Document Version
Peer reviewed version

On the structure of right 3-Engel subgroups

Peter G. Crosby
Department of Mathematical Sciences,
University of Bath,
Bath BA2 7AY, UK
peter.g.crosby@bath.edu

March 20, 2012

Abstract

We state and prove two sharp results on the structure of normal subgroups consisting of right 3-Engel elements. First we prove that if H is a 3-torsion-free such subgroup of a group G and $x \in G$, then $[H,_4 \langle x\rangle^G] = \{1\}$. When H is additionally $\{2,5\}$-torsion-free, we prove that $[H,_8 G] = \{1\}$.

1 Introduction

Recall that an element a of a group G is a right 3-Engel element if it satisfies the commutator identity $[a, g, g, g] = 1$ for every $g \in G$. Let $R_3(G)$ be the set of right 3-Engel elements of a group G. $R_3(G)$ need not be a subgroup of G. However, in certain classes of groups it does form a subgroup, for example when G is a group such that $\gamma_5(G)$ has no element of order 2 [1]. We are interested in the structure of subgroups contained in $R_3(G)$. We call such a subgroup a right 3-Engel subgroup.

When $G = R_3(G)$, G is a 3-Engel group. Kappe and Kappe [6] have shown that every 3-Engel group is Fitting, with Fitting degree at most 2. That is, the normal closure of any element is nilpotent, with nilpotency class at most 2. M. Newell has shown that in any group G the normal closure of an element of $R_3(G)$ is nilpotent of class at most 3 [7]. We prove the following related theorem.

Theorem 1. *Let H be a 3-torsion-free normal right 3-Engel subgroup of a group G. Let $x \in G$. Then $[H,_4 \langle x\rangle^G] = \{1\}$.*

It is necessary to exclude the prime 3 here. To see this, let $r \in \mathbb{N}$ be arbitrary. Let $F = \langle x_1, x_2, \dots, x_r\rangle$ be the relatively free group of rank r and nilpotency class 2 and let $M = F/F^3$. Consider $G = C_3 \operatorname{wr} M = \prod_{g\in M} \langle a^g\rangle \rtimes M$. Letting $H = \prod_{g\in M} \langle a^g\rangle$, $h, k \in H$ and $g \in M$ we have that

$$[h, kg, kg, kg] = [h, g, g, g] = h^{(-1+g)^3} = h^{-1+g^3} = 1.$$

Hence H is a normal right 3-Engel subgroup of G. Also,

$$[a^1, [x_1, x_2], [x_1, x_3], \ldots, [x_1, x_r]] = a^{(-1+[x_1,x_2])\cdots(-1+[x_1,x_r])},$$

which is non-trivial as, for example, the $\langle a^1 \rangle$ part is. As $[x_1, x_i] \in \langle x_1 \rangle^G$, for each i, we have $[H,_r \langle x_1 \rangle^G] \neq \{1\}$ for any positive integer r.

We use Theorem 1 to prove the following theorem.

Theorem 2. *Let H be a $\{2,3,5\}$-torsion-free normal right 3-Engel subgroup of a group G. Then $[H,_8 G] = \{1\}$.*

The primes 2 and 5 are exceptional for 3-Engel groups and hence are exceptional for right 3-Engel subgroups too and need to be excluded. Let H be a normal upper central right n-Engel subgroup of a group G. In [3] it was shown that there exist positive integers $c = c(n)$, $e = e(n)$ and $f = f(n)$, only depending on n, such that $H^e \leq Z_c(G)$ and $[H,_c G]^f = \{1\}$. Theorem 2 shows that we can take $c(3) = 8$ and that the best possible values for $e(3)$ and $f(3)$ have prime divisors 2, 3 and 5. In fact $c(3) = 8$ is best possible and we will outline the construction for this. The detailed calculations can be found in [2] (Section 6.5 and Appendix A).

Remark 1. From the proofs of these two theorems one sees that in fact one only needs $[H,_4 \langle x \rangle^G]$ to be 3-torsion-free or $[H,_8 G]$ to be $\{2, 3, 5\}$-torsion-free respectively, instead of H.

2 Proof of Theorem 1

Let H be a normal right 3-Engel subgroup of a group G and let $h \in H$, $x, a, b, c \in G$. It suffices to show that when H is 3-torsion-free, commutators with an h entry and four x entries are trivial. For this we will show that each of

$$[h, x, x, b, x, x], [h, x, a, x, b, x, x], [h, x, a, x, x, c, x], [h, x, x, b, x, c, x], [h, x, a, x, b, x, c, x]$$

is trivial modulo commutators of higher multiweight with at least two h entries. It will then follow, by replacing h by an arbitrary commutator with first entry h, and replacing a, b and c by arbitrary products and expanding, that an arbitrary commutator of multiweight (1,4,1,...,1) in $h, x, a_1, \ldots, a_n$, for arbitrary $a_1, \ldots, a_n \in G$, can be written as a product of commutators of multiweight at least (2,4,1,...,1). Replacing a_1 by h shows that commutators of multiweight (2,4,1,...,1) in $h, x, a_2, \ldots, a_n$ can be written as a product of commutators of multiweight at least (3,4,1,...,1). Similarly, replacing both a_1 and a_2 by h shows that commutators of multiweight (3,4,1,...,1) in $h, x, a_3, \ldots, a_n$ can be written as a product of commutators of multiweight at least (4,4,1,...,1). Thus commutators with an h entry and four x entries are trivial modulo commutators with at least four h and four x entries. By the result of Newell [7], any commutator with four h entries is trivial and hence this will prove the theorem.

Thus it suffices to prove that $[h, x, a, x, b, x, c, x] = 1$ modulo commutators of multiweight at least (2,4,1,1,1) in h, x, a, b, c. We may therefore assume that commutators with at least two h entries and four x entries are trivial.

We start with some identities for normal right 3-Engel subgroups, which will be useful throughout. We consider commutators with an entry set $\{h, x, y\}$, where $y \in G$. We prove the identities modulo commutators of certain multiweights in h, x, y. For the first two identities we do not require that H is 3-torsion-free.

Lemma 1. *Modulo commutators of multiweight at least (2,1,1) in* h, x, y,
(i) $[h, x, x, y][h, x, y, x][h, y, x, x][h, x, y, y][h, y, x, y][h, y, y, x][h, y, x, y, x]$
$[h, x, x, y, y]^{-1} = 1$.
(ii) $[h, y, x, y, x]^{-1}[h, y, y, x, x]^{-1}[h, x, y, x, y][h, x, x, y, y] = 1$.

Proof. We work in the endomorphism ring of $\langle h\rangle^G$, which we may assume is abelian as we are working modulo commutators of multiweight at least (2,1,1) in h, x, y. For $g \in G$ we define $R_g : \langle h\rangle^G \mapsto \langle h\rangle^G$ by $R_g(k) = [k, g]$. For $h_1, h_2 \in \langle h\rangle^G$, $[h_1h_2, g] = [h_1, g][h_2, g]$ and thus R_g is an endomorphism. We also have

$$R_{xy} = R_x + R_y + R_xR_y = R_x + (1 + R_x)R_y = R_x(1 + R_y) + R_y.$$

Thus, since $R_x^3 = R_y^3 = 0$,

$$\begin{aligned}(1 - R_x + R_x^2)R_{xy} &= R_x - R_x^2 + R_y,\\ R_{xy}(1 - R_y + R_y^2) &= R_x + R_y - R_y^2.\end{aligned}$$

Hence,

$$\begin{aligned}0 &= (1 - R_x + R_x^2)R_{xy}^3(1 - R_y + R_y^2)\\ &= (R_x - R_x^2 + R_y)(R_x + R_y + R_xR_y)(R_x + R_y - R_y^2)\\ &= (R_x + R_y)^3 - R_x^2R_y(R_x + R_y) + (R_x + R_y)R_xR_y(R_x + R_y) - (R_x + R_y)R_xR_y^2\\ &= (R_x + R_y)^3 + R_yR_xR_yR_x - R_x^2R_y^2.\end{aligned}$$

Expanding and applying to h gives (i). Swapping the roles of x and y in (i) and multipling by the inverse of (i) gives (ii). $\square$

From now on we exclude the prime 3. We use Lemma 1(i) to give us some more identities, this time modulo commutators of multiweight at least (2,3,1) in h, x, y.

Lemma 2. *Suppose that H is 3-torsion-free. Then, modulo commutators of multiweight at least (2,3,1) in* h, x, y,
(i) $[h, x, x, y, x, x] = 1$.
(ii) $[h, x, x, y, x][h, x, y, x, x] = 1$.
(iii) $[h, x, y, x, y, x] = 1$.

Proof. From Lemma 1(i) we may assume that

$$\begin{aligned}&[h,x,x,y][h,x,y,x][h,y,x,x][h,x,y,y][h,y,x,y][h,y,y,x]\\ &[h,y,x,y,x][h,x,x,y,y]^{-1}uv=1,\end{aligned}\tag{1}$$

where u is a product of commutators of multiweight at least (2,1,1) in h,x,y with one entry of x and v is a product of commutators of multiweight at least (2,2,1) in h,x,y. Let $n\in\mathbb{N}$. Now,

$$[h,x^n]=[h,x]^n[h,x,x]^{\binom{n}{2}}.$$

Hence replacing x by x^n in (1) gives

$$\begin{aligned}1=&[h,x,x,y]^{n^2}[h,x,y,x]^{n^2}[h,x,x,y,x]^{n\binom{n}{2}}[h,x,y,x,x]^{n\binom{n}{2}}[h,x,x,y,x,x]^{\binom{n}{2}^2}\\ &[h,y,x,x]^{n^2}[h,x,y,y]^{n}[h,x,x,y,y]^{\binom{n}{2}}[h,y,x,y]^{n}[h,y,x,x,y]^{\binom{n}{2}}[h,y,y,x]^{n}\\ &[h,y,y,x,x]^{\binom{n}{2}}[h,y,x,y,x]^{n^2}[h,y,x,x,y,x]^{n\binom{n}{2}}[h,y,x,y,x,x]^{n\binom{n}{2}}\\ &[h,y,x,x,y,x,x]^{\binom{n}{2}^2}[h,x,x,y,y]^{-n^2}u^nv^{n^2}w^{\binom{n}{2}},\end{aligned}\tag{2}$$

where w is a product of commutators of multiweight at least (2,2,1) in h,x,y. Note that by the well known Hall-Petrescu identity,

$$[h,x,y,y]^n[h,y,x,y]^n[h,y,y,x]^nu^n=([h,x,y,y][h,y,x,y][h,y,y,x]u)^na^{\binom{n}{2}},$$

where a is a product of commutators of multiweight at least (2,2,1) in h,x,y. Multiplying (2) by (1) raised to the power of $-n$ and using the above gives

$$\begin{aligned}1=&[h,x,x,y]^{2\binom{n}{2}}[h,x,y,x]^{2\binom{n}{2}}[h,y,x,x]^{2\binom{n}{2}}[h,x,x,y,x]^{n\binom{n}{2}}[h,x,y,x,x]^{n\binom{n}{2}}\\ &[h,x,x,y,y]^{-\binom{n}{2}}[h,y,x,x,y]^{\binom{n}{2}}[h,y,y,x,x]^{\binom{n}{2}}[h,y,x,y,x]^{2\binom{n}{2}}[h,y,x,x,y,x]^{n\binom{n}{2}}\\ &[h,y,x,y,x,x]^{n\binom{n}{2}}[h,x,x,y,x,x]^{\binom{n}{2}^2}[h,y,x,x,y,x,x]^{\binom{n}{2}^2}v^{2\binom{n}{2}}t^{\binom{n}{2}},\end{aligned}$$

where t is a product of commutators of multiweight at least (2,2,1) in h,x,y. Setting $n=2$ gives

$$\begin{aligned}1=&[h,x,x,y]^2[h,x,y,x]^2[h,y,x,x]^2[h,x,x,y,x]^2[h,x,y,x,x]^2[h,x,x,y,y]^{-1}\\ &[h,y,x,x,y][h,y,y,x,x][h,y,x,y,x]^2[h,y,x,x,y,x]^2[h,y,x,y,x,x]^2\\ &[h,x,x,y,x,x][h,y,x,x,y,x,x]v^2t.\end{aligned}\tag{3}$$

Multiplying this to the power of $-\binom{n}{2}$ by the previous equation gives

$$\begin{aligned}1=&[h,x,x,y,x]^{(n-2)\binom{n}{2}}[h,x,y,x,x]^{(n-2)\binom{n}{2}}[h,y,x,x,y,x]^{(n-2)\binom{n}{2}}\\ &[h,y,x,y,x,x]^{(n-2)\binom{n}{2}}[h,x,x,y,x,x]^{\binom{n}{2}^2-\binom{n}{2}}[h,y,x,x,y,x,x]^{\binom{n}{2}^2-\binom{n}{2}}.\end{aligned}$$

Setting $n=3$ gives

$$\begin{aligned}1=&[h,x,x,y,x]^3[h,x,y,x,x]^3[h,y,x,x,y,x]^3\\ &[h,y,x,y,x,x]^3[h,x,x,y,x,x]^6[h,y,x,x,y,x,x]^6.\end{aligned}\tag{4}$$

Replacing h by $[h, y, y]$ and commuting by x gives $[h, y, y, x, x, y, x, x] = 1$, as H is 3-torsion-free. Replacing h by $[h, y, y]$ in (4) gives $[h, y, y, x, x, y, x][h, y, y, x, y, x, x] = 1$. Replacing h by $[h, y]$ in (4) gives

$$1 = [h, y, x, x, y, x]^3[h, y, x, y, x, x]^3[h, y, x, x, y, x, x]^6.$$

Hence (4) becomes

$$1 = [h, x, x, y, x]^3[h, x, y, x, x]^3[h, x, x, y, x, x]^6.$$

Commuting by x gives (i) and (ii). Replacing y by y^2 in (ii) gives

$$1 = [h, x, x, y, y, x][h, x, y, y, x, x].$$

Commuting (1) by x and replacing h by $[h, x]$ gives, using (i) and the above,

$$1 = [h, x, y, x, y, x][h, x, y, x, y, x, x].$$

Commuting by x gives $1 = [h, x, y, x, y, x, x]$ and hence we have (iii). □

We now work again modulo commutators of multiweight at least (2,1,1) in h, x, y and find three more identities.

Lemma 3. *Suppose that H is 3-torsion-free. Then, modulo commutators of multiweight at least (2,1,1) in h, x, y,*
(i) $[h, y, x, x, y][h, y, x, y, x][h, y, y, x, x] = 1$.
(ii) $[h, x, x, y]^2[h, x, y, x]^2[h, y, x, x]^2[h, x, x, y, y]^{-1}[h, y, x, y, x] = 1$.
(iii) $[h, y, x, x, y][h, x, y, y, x]^{-1} = 1$.

Proof. Replacing h by $[h, y]$ in Lemma 1(i) and using Lemma 2(ii) gives

$$1 = [h, y, x, x, y][h, y, x, y, x][h, y, y, x, x][h, y, x, x, y, y]^{-1}[h, y, y, x, y, x].$$

Replacing h by $[h, y]$ in Lemma 1(ii) and using Lemma 2(iii) gives

$$1 = [h, y, y, x, y, x]^{-1}[h, y, x, x, y, y]$$

and hence (i) follows. For (ii) we use (3) from the proof of Lemma 2. Recall that v and t were products of commutators of multiweight at least (2,2,1) in h, x, y. Hence, also using Lemma 2(i) and (ii), (3) becomes

$$\begin{aligned}1 =& [h, x, x, y]^2[h, x, y, x]^2[h, y, x, x]^2 \\ & [h, x, x, y, y]^{-1}[h, y, x, x, y][h, y, y, x, x][h, y, x, y, x]^2.\end{aligned}$$

Multiplying by the inverse of (i) gives (ii).

Commuting Lemma 1(i) by x and using Lemma 2(ii) gives

$$1 = [h, x, y, y, x][h, y, x, y, x][h, y, y, x, x][h, x, x, y, y, x]^{-1}[h, y, x, y, x, x].$$

Multiplying by Lemma 1(ii) commuted by x, and using Lemma 2(iii), gives

$$1 = [h, x, y, y, x][h, y, x, y, x][h, y, y, x, x].$$

Together with (i) this gives (iii). □

From now on H is always assumed to be 3-torsion-free. Recall that we are also assuming that commutators with at least two h entries and four x entries are trivial. We use the identities above to show that $[h, x, a, x, b, x, c, x] = 1$. The commutators that arise in the following calculations from the fact that the identities only hold modulo certain multiweights will contain at least two h entries and four x entries and hence cancel.

First note that by commuting Lemma 2(ii) by x we have $1 = [h, x, x, y, x, x]$. Hence, by replacing y by an arbitrary product of elements from G, left normed commutators with first entry h and two pairs of consecutive x entries are trivial. Let $k = [h, x, x]$. Note that $[k, x] = [k, \ldots, x, x] = 1$. Using these we now show that $[k, b, x, c, x] = 1$.

Lemma 3(iii) gives $[k, b, x, c, c, x] = 1$ and Lemma 3(i) gives $[k, c, x, c, x] = 1$ and $[k, b, x, c, x, c] = 1$. We now use these to show that $[k, b, x, c, x, b] = 1$. First,

$$\begin{aligned}1 =&[k, b, x, bc, x, bc]\\ =&[k, b, x, c, x, b][k, b, x, c, x, b, c][k, b, x, b, c, x, c][k, b, x, b, c, x, b][k, b, x, b, c, x, b, c]\\ =&[k, b, x, c, x, b][k, b, x, c, x, b, c][k, b, b, x, c, x, c]^{-1}[k, b, x, b, c, x, b][k, b, x, b, c, x, b, c]\\ =&[k, b, x, c, x, b][k, b, x, c, x, b, c][k, b, x, b, c, x, b][k, b, x, b, c, x, b, c],\end{aligned} \tag{5}$$

where the third equality follows from Lemma 1(i). Thus $f = [f, c]^{-1}$, where $f = [k, b, x, c, x, b][k, b, x, b, c, x, b]$. Since f is a right 3-Engel element, it follows that $f = 1$. Hence,

$$\begin{aligned}1 =&[k, b, x, c, x, b][k, b, x, b, c, x, b]\\ =&[k, b, x, c, x, b][k, b, b, x, c, x, b]^{-1}, \text{ by Lemma 1(i)},\\ =&[k, b, x, c, x, b][k, b, x, c, x, b, b], \text{ by Lemma 2(ii)}.\end{aligned}$$

Commuting by b gives $[k, b, x, c, x, b, b] = 1$ and hence we have $[k, b, x, c, x, b] = 1$. Note that replacing b by cb gives $[k, c, b, x, c, x, b] = 1$.

We now use this to prove that $[k, b, x, c, x] = 1$. We expand $[k, bc, bc, x][k, bc, x, bc] = 1$, which holds by Lemma 1(i). This gives

$$\begin{aligned}1 =&[k, b, c, x][k, c, b, x][k, b, x, c][k, c, x, b][k, b, b, c, x][k, c, b, c, x][k, b, c, b, x][k, b, c, c, x]\\ &[k, b, x, b, c][k, c, x, b, c][k, b, c, x, b][k, b, c, x, c][k, b, c, b, c, x][k, b, c, x, b, c]\\ =&[k, b, c, x][k, c, b, x][k, b, x, c][k, c, x, b][k, c, b, b, x]^{-1}[k, c, c, b, x]^{-1}[k, b, x, b, c]\\ &[k, c, x, b, c][k, b, c, x, b][k, b, c, x, c][k, c, c, b, b, x][k, b, c, x, b, c], \text{ by Lemma 1(i)}.\end{aligned}$$

Replacing c by xc gives, using the identities found in showing that $[k, b, x, c, x, b] = 1$,

$$\begin{aligned}1 =&[k, b, x, c, x][k, c, x, b, x]^{-1}[k, c, x, c, b, x]^{-1}[k, c, x, b, x][k, c, x, c, b, b, x][k, b, c, x, b, x]\\ =&[k, b, x, c, x][k, c, c, x, b, x][k, c, c, x, b, b, x]^{-1}[k, b, c, x, b, x], \text{ by Lemma 1(i)},\\ =&[k, b, x, c, x][k, c, c, x, b, x][k, b, c, x, b, x].\end{aligned} \tag{6}$$

Replacing b by b^{-1} and using Lemma 2(ii) we get

$$1 = [k,b,x,c,x]^{-1}[k,b,b,x,c,x][k,c,c,x,b,x]^{-1}[k,b,c,x,b,x].$$

Multiplying by (6) gives

$$1 = [k,b,b,x,c,x][k,b,c,x,b,x]^2. \tag{7}$$

Instead replacing c by bc in (6) we have

$$\begin{aligned}1 =&[k,b,x,b,c,x][k,b,c,x,b,x][k,b,c,c,x,b,x][k,c,b,c,x,b,x][k,b,b,c,x,b,x]\\ &[k,b,c,b,c,x,b,x][k,b,b,c,x,b,x]\\ =&[k,b,x,b,c,x][k,b,c,x,b,x], \text{ by Lemma 1(i) and Lemma 2(ii)}\\ =&[k,b,b,x,c,x]^{-1}[k,b,c,x,b,x].\end{aligned}$$

Multiplying by (7) gives $1 = [k,b,c,x,b,x]^3$ and as H is 3-torsion-free we have $1 = [k,b,c,x,b,x] = [k,b,b,x,c,x]$. Thus (6) gives the desired result of $1 = [k,b,x,c,x]$.

It follows by Lemma 2(ii) that

$$[h,x,x,b,x,c,x] = [h,x,b,x,x,c,x] = [h,x,b,x,c,x,x] = 1.$$

Letting $l = [h,x,a,x]$ it follows that $[l,x,\ldots,x] = [l,\ldots,x,x] = 1$. The above calculations, with l in place of k, then show that $[l,b,x,c,x] = 1$, which completes the proof of Theorem 1.

3 Theorem 1 is best possible

We now show that Theorem 1 cannot be improved by giving an example satisfying the conditions of Theorem 1, but with $[H,_3 \langle x\rangle^G] \neq \{1\}$.

Let $G = \langle h,x,y\rangle$ be the relatively free group on these 3 generators, such that any commutator with at least two entries of h, two entries of y or four entries of x is trivial and $[h,x,x,x] = [h,y,x,x,x] = [h,x,x,y][h,x,y,x][h,y,x,x] = 1$. Define $H = \langle h\rangle^G$. It is important that there are no relations between left normed commutators in H with first entry h and entry set $\{h,x,y\}$, other than those which follow directly from those stated above in the definition of G. This follows from a lemma of M. Hall (Lemma 11.2.1 in [5]), which gives a bijection between the set of basic commutators (with an arbitrary ordering) of a particular weight with exactly one h entry and the set of left normed commutators in H with first entry h and entry set $\{h,x,y\}$ of the same weight. These sets have the same spans modulo higher multiweights, and hence are linearly independent modulo higher multiweights, as the first is by a theorem of M. Hall [4].

One can then check from the relations above that H is 3-torsion-free. We claim that H is a right 3-Engel subgroup of G in which $[h,x,y,x,x] \neq 1$. The latter claim is clear, since $[h,x,y,x,x]$ and $[h,x,x,y,x]$ only ever occur together and with the same power from the relations imposed on G. To show that H is a right 3-Engel subgroup of G, one needs only check that

$$1 = [h, x^a y^b [x, y]^c, x^a y^b [x, y]^c, x^a y^b [x, y]^c],\ \forall a, b, c \in \mathbb{Z}.$$

By expanding the right hand side one sees that this holds in G. This calculation is straightforward and can be found in [2] (Section 6.3).

4 Proof of Theorem 2

Let H be a $\{2, 3, 5\}$-torsion-free normal right 3-Engel subgroup of a group G. Let $h \in H$ and $g_1, \ldots, g_8 \in G$. It suffices to show that $[h, g_1, \ldots, g_8] = 1$. Suppose that $[h, g_1, \ldots, g_8] \in H'$. Independently setting $g_i = 1$ for $i \in \{1, \ldots, 8\}$, we see that $[h, g_1, \ldots, g_8]$ can be written as a product of commutators with multiweight at least $(2, 1, 1, \ldots, 1)$ in $(h, g_1, \ldots, g_8)$. Using $[h, g_1, \ldots, g_8] \in H'$ again, each of these commutators can be written as a product of commutators with multiweight at least $(3, 1, 1, \ldots, 1)$ in $(h, g_1, \ldots, g_8)$ and each of these as a product of commutators with multiweight at least $(4, 1, 1, \ldots, 1)$ in $(h, g_1, \ldots, g_8)$. However, these commutators are trivial, by the result of Newell [7]. Hence we would have $[h, g_1, \ldots, g_8] = 1$. Thus we may assume that H is abelian.

We will use the linearised 3-Engel identity, which holds for all right 3-Engel elements h in a group G. If $g_1, g_2, g_3 \in G$, then expanding $1 = [h, g_1g_2g_3, g_1g_2g_3, g_1g_2g_3]$ gives, modulo higher multiweights,

$$1 = \prod_{\sigma \in S_3} [h, g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}]. \tag{8}$$

We start by considering $\langle h, x, y \rangle$, where $x, y \in G$. Here we only need assume that H is a $\{2, 3\}$-torsion-free normal right 3-Engel subgroup of G. We will exclude the prime 5 later. For a set X and positive integer i we will use the notation $[h,_i X]$ to denote the set of left normed commutators with first entry h and i entries from X.

Lemma 4. $[H,_6 \langle x, y \rangle] = \{1\}$.

Proof. By Theorem 1, $[H,_7 \langle x, y \rangle] = \{1\}$. Let $g \in [h,_6 \{x, y\}]$, where g has three x and three y entries. It suffices to see that $g = 1$. Without loss of generality we may assume that the second entry is an x. Expanding $1 = [h, [x, y], [x, y], [x, y]]$ gives

$$\begin{aligned} 1 =& [h, x, y, x, y, x, y][h, x, y, x, y, y, x]^{-1}[h, x, y, y, x, x, y]^{-1}[h, x, y, y, x, y, x] \\ & [h, y, x, x, y, x, y]^{-1}[h, y, x, x, y, y, x][h, y, x, y, x, x, y][h, y, x, y, x, y, x]^{-1}. \end{aligned}$$

By Lemma 2(iii) and repeated use of Lemma 3(iii) this becomes

$$1 = [h, x, y, x, y, y, x]^{-3}[h, x, y, y, x, y, x]^3.$$

Hence by Lemma 2(ii), $1 = [h,x,y,y,x,y,x]^6$ and as H is $\{2,3\}$-torsion-free, $1 = [h,x,y,y,x,y,x]$. By Lemma 3(iii) and Lemma 2(ii) we have

$$[h,x,y,y,x,x,y] = [h,x,y,x,y,y,x] = [h,x,y,y,x,y,x]^{-1} = 1.$$

By Lemma 2(ii), Lemma 3(iii) and Lemma 1(ii) respectively,

$$\begin{aligned}[h,x,x,y,x,y,y] =&[h,x,x,y,y,x,y]^{-1}\\ =&[h,x,y,x,x,y,y]^{-1}\\ =&[h,x,y,x,y,x,y][h,x,y,y,x,y,x]^{-1}\\ =&1.\end{aligned}$$

Thus g is trivial. □

We now introduce another element $z \in G$ and consider $\langle h,x,y,z\rangle$. We show that commutators in $[h,_6 \{x,y,z\}]$ with a triple entry are trivial, as long as $[H,_7 \langle x,y,z\rangle]$ is. By Lemma 4 and Theorem 1, we need only consider commutators of multiweight (1,3,2,1) in h,x,y,z. We will then use this to show that indeed $[H,_7 \langle x,y,z\rangle] = \{1\}$.

We first introduce some notation to make certain calculations easier to read. We write $(a_1;a_2;\ldots;a_n)$ for an arbitrary commutator starting with h and containing entries $a_1,\ldots,a_n$ in that order, with possibly other entries too. Within a calculation the remainder of the commutator is assumed to stay the same.

Lemma 5. *Suppose that $[H,_7 \langle x,y,z\rangle] = \{1\}$. Then any commutator of multiweight (1,3,2,1) in h,x,y,z is trivial.*

Proof. Consider such a commutator with first entry h. Suppose that the second entry is $a \neq x$. By Theorem 1, $(ax;ax;ax;ax) = 1$ and so, modulo commutators of multiweight (1,3,2,1) in h,x,y,z with first entry h and second entry x, $1 = (a;x;x;x)uv$, where u consists of commutators with one x entry and v consists of commutators with two x entries. Replacing x by x^{-1} gives $1 = (a;x;x;x)^{-1}u^{-1}v$. Hence, $1 = (a;x;x;x)^2u^2$. Replacing x by x^2 now gives $1 = (a;x;x;x)^{16}u^4$. Hence, $1 = (a;x;x;x)^{12}$ and so, as H is {2,3}-torsion-free, we may assume that the second entry is an x.

Next consider the y and z entries. Replacing these by yz we have that $(yz;yz;yz) = 1$, by Lemma 4. Hence

$$1 = (y;y;z)(y;z;y)(z;y;y)(y;z;z)(z;y;z)(z;z;y).$$

Replacing y by y^{-1} and multiplying gives

$$1 = (y;y;z)^2(y;z;y)^2(z;y;y)^2 = (y;y;z)(y;z;y)(z;y;y).$$

So we may assume that the z entry is not before the first y entry.

Next we show that we may also assume that the third entry is x. Suppose not and that the last two x entries aren't consecutive, else by Lemma 2(ii) we could move one third. If the last two x's are within 3 entries, then we can move them together by Lemma 3(ii). Hence it remains to consider the commutators $[h,x,y,x,y,z,x]$ and $[h,x,y,x,z,y,x]$. But, by Lemma 3(ii),

$$[h,x,y,x,y,z,x]=[h,x,x,y,y,z,x]^{-1}[h,x,y,y,x,z,x]^{-1}$$

and by Lemma 3(iii) with y replaced by yz,

$$[h,x,y,x,z,y,x]=[h,x,y,x,y,z,x]^{-1}[h,x,y,y,x,x,z][h,x,y,z,x,x,y].$$

Hence we are left with the following commutators.

$$\begin{aligned}
&g_1=[h,x,x,y,x,y,z],\ g_2=[h,x,x,y,x,z,y],\ g_3=[h,x,x,y,y,x,z],\\
&g_4=[h,x,x,y,y,z,x],\ g_5=[h,x,x,y,z,x,y],\ g_6=[h,x,x,y,z,y,x].
\end{aligned}$$

We find relations involving these to show that they are trivial.

First notice that by Lemma 3(ii), $g_1g_3=1$ (α). Next note that the linearised 3-Engel identity (8) gives $g_2g_4g_5g_6=1$ (β). By Lemma 3(ii),

$$\begin{aligned}
1=&[h,x,x,y,y,[x,z]][h,x,x,y,[x,z],y][h,x,x,[x,z],y,y]\\
=&g_3g_4^{-1}g_2g_5^{-1}[h,x,x,z,x,y,y]^{-1}\\
=&g_3g_4^{-1}g_2g_5^{-1}[h,x,x,y,x,z,y][h,x,x,y,x,y,z]\\
=&g_1g_2^2g_3g_4^{-1}g_5^{-1}\ (\gamma).
\end{aligned}$$

Also,

$$\begin{aligned}
1=&[h,x,x,y,y,x,z][h,x,x,y,y,z,x][h,x,z,y,y,x,x][h,z,x,y,y,x,x]\\
=&g_3g_4[h,x,x,z,y,y,x]^{-1}[h,z,x,x,y,y,x]^{-1}\\
=&g_3g_4[h,x,z,x,y,y,x]\\
=&g_3g_4[h,x,z,y,x,x,y]\\
=&g_3g_4[h,x,x,z,y,x,y]^{-1}\\
=&g_3^2g_4g_5\ (\delta)
\end{aligned}$$

and

$$\begin{aligned}
1=&[h,x,y,x,x,[y,z]][h,x,y,x,[y,z],x][h,x,y,[y,z],x,x]\\
=&[h,x,y,x,x,y,z][h,x,y,x,x,z,y]^{-1}[h,x,y,x,y,z,x][h,x,y,x,z,y,x]^{-1}\\
&[h,x,y,y,z,x,x][h,x,y,z,y,x,x]^{-1}\\
=&[h,x,x,y,x,y,z]^{-1}[h,x,x,y,x,z,y][h,x,y,x,y,z,x]^2[h,x,y,y,x,x,z]^{-1}\\
&[h,x,y,z,x,x,y]^{-1}[h,x,x,y,y,z,x]^{-1}[h,x,x,y,z,y,x]\\
=&g_1^{-1}g_2[h,x,y,y,x,z,x]^{-2}[h,x,x,y,y,z,x]^{-2}[h,x,x,y,y,x,z]\\
&[h,x,x,y,z,x,y]g_4^{-1}g_6\\
=&g_1^{-1}g_2[h,x,y,y,x,x,z]^2[h,x,y,y,z,x,x]^2g_4^{-2}g_3g_5g_4^{-1}g_6\\
=&g_1^{-1}g_2g_3^{-1}g_4^{-5}g_5g_6\ (\epsilon).
\end{aligned}$$

Now, $(\alpha)(\beta)^{-1}(\epsilon)$ implies $1 = g_4^{-6}$ and so $g_4 = 1$. By the same argument, reversing the order of the entries except h in all commutators, we get $1 = [h, x, z, y, y, x, x]$ and so $1 = [h, x, x, z, y, y, x]^{-1} = g_6$. Then $(\alpha)^{-1}(\beta)(\gamma)$ gives $1 = g_2^3$ and hence $g_2 = 1$. Then (β) gives $g_5 = 1$. (δ) then gives $1 = g_3^2$ and thus $g_3 = 1$. Finally (α) gives $g_1 = 1$, which completes the proof. □

Lemma 6. $[H,_7 \langle x, y, z\rangle] = \{1\}$.

Proof. We may assume that $[H,_8 \langle x, y, z\rangle] = \{1\}$. Let $h \in H$ and consider an arbitrary commutator in $[h,_7 \{x, y, z\}]$. By Lemma 5 we may assume that there is no triple entry within six consecutive entries. Thus we need only consider commutators of multiweight (1,3,2,2) in h, x, y, z with second and eighth entries both x.

Suppose that the third entry is y. Then by Lemma 5 we have

$$1 = [h, xy, xy, \ldots, xy, \ldots],$$

where the third xy is in place of the second x entry. Expanding this we have

$$1 = [h, x, x, \ldots, y, \ldots][h, x, y, \ldots, x, \ldots].$$

Hence we may assume that the third entry is also x. Assuming without loss of generality that the fourth entry is a y, this leaves the following commutators.

$$g_1 = [h, x, x, y, y, z, z, x],\ g_2 = [h, x, x, y, z, y, z, x],\ g_3 = [h, x, x, y, z, z, y, x].$$

By Lemma 3(ii) on the last three entries, $g_1 = 1$. Then using Lemma 3(ii) on the fifth to seventh entries, $g_2 g_3 = 1$. Finally,

$$1 = [h, x, x, y, z, z, [y, x]][h, x, x, y, z, [y, x], z][h, x, x, y, [y, x], z, z] = g_3.$$

Thus $g_2 = g_3^{-1} = 1$ and the lemma is proved. □

In particular we have shown that commutators of multiweight (1,3,2,1) or higher in h, x, y, z are trivial. We now show that commutators in $[H,_7 G]$ with a triple entry are trivial as long as commutators of higher multiweight are.

Lemma 7. *Any commutator containing an element of H, a triple entry and four other entries is trivial if commutators of higher multiweight are.*

Proof. Consider commutators with first entry $h \in H$, a triple entry of $x \in G$ and other entries $a, b, c, d \in G$. Suppose that higher multiweight commutators are trivial. As in the proof of Lemma 5, we may assume that the second entry is an x. Also, by Lemma 3(ii) and Lemma 5,

$$\begin{aligned} 1 =& [h, x, a, x, x, b][h, x, a, x, b, x][h, x, a, b, x, x] \\ =& [h, x, x, a, x, b]^{-1}[h, x, a, x, b, x][h, x, x, a, b, x]^{-1}. \end{aligned}$$

Replacing a and b by arbitrary products we see that we may assume that the third entry is also x. So it suffices to see that the following commutators are trivial.

$$g_1 = [h, x, x, a, x, b, c, d],\ g_2 = [h, x, x, a, b, x, c, d]$$
$$g_3 = [h, x, x, a, b, c, x, d],\ g_4 = [h, x, x, a, b, c, d, x].$$

Note that, by Lemma 5, g_1, g_2 and g_3 are alternating in the entries a, b and c. Further,

$$1 = [h, x, x, ad, x, [b, c], ad] = [h, x, x, ad, x, b, c, ad]^2,$$
$$1 = [h, x, x, [a, b], x, cd, cd] = [h, x, x, a, b, x, cd, cd]^2,$$
$$1 = [h, x, x, ad, [b, c], x, ad] = [h, x, x, ad, b, c, x, ad]^2.$$

Hence, as H is 2-torsion-free, g_1, g_2 and g_3 are alternating in the d entries as well. Now, again by Lemma 5,

$$1 =[h, x, x, a, x, [b, c], d][h, x, x, a, x, d, [b, c]] = g_1^4.$$

and hence $g_1 = 1$. Also,

$$1 =[h, x, x, [a, b], x, c, d][h, x, x, c, x, [a, b], d] = g_2^2,$$
$$1 =[h, x, x, a, [b, c], x, d][h, x, x, [b, c], a, x, d] = g_3^4.$$

Thus $g_1 = g_2 = g_3 = 1$.

It remains to see that $g_4 = 1$. Now, by (8),

$$\begin{aligned}1 =&[h, x, x, a, b, c, d, x][h, x, x, a, b, c, x, d][h, x, x, a, b, d, c, x]\\ &[h, x, x, a, b, d, x, c][h, x, x, a, b, x, c, d][h, x, x, a, b, x, d, c]\\ =&[h, x, x, a, b, c, d, x][h, x, x, a, b, d, c, x].\end{aligned}$$

Thus g_4 is alternating in the c and d entries. Also,

$$\begin{aligned}1 =&[h, x, x, a, b, c, [d, x]][h, x, x, a, b, [d, x], c][h, x, x, a, c, b, [d, x]]\\ &[h, x, x, a, c, [d, x], b][h, x, x, a, [d, x], b, c][h, x, x, a, [d, x], c, b]\\ =&[h, x, x, a, b, c, d, x][h, x, x, a, c, b, d, x].\end{aligned}$$

Thus g_4 is also alternating in the b entry. Hence, by Lemma 5,

$$1 =[h, x, x, a, b, [c, d], x][h, x, x, a, [c, d], b, x] = g_4^4.$$

Thus, as H is 2-torsion-free, $g_4 = 1$, which completes the proof. $\square$

We now use this to show that the same is true in $[H,_8 G]$. Since we are aiming to show that under certain conditions $[H,_8 G] = \{1\}$, and we are assuming that H is upper central, we assume from now on that $[H,_9 G] = \{1\}$.

Corollary 1. *Any commutator in $[H,_8 G]$ with a triple entry is trivial.*

Proof. Consider commutators with first entry $h \in H$, a triple entry of $x \in G$ and other entries $a, b, c, d, e \in G$. As in the proof of Lemma 7, we may assume that the second and third entry are both x. By Lemma 7 we are left with the commutator

$$g = [h, x, x, a, b, c, d, e, x].$$

First, by (8),

$$\begin{aligned}1 =&[h, x, x, a, b, c, d, e, x][h, x, x, a, b, c, d, x, e][h, x, x, a, b, c, e, d, x]\\ &[h, x, x, a, b, c, e, x, d][h, x, x, a, b, c, x, d, e][h, x, x, a, b, c, x, e, d]\\ =&[h, x, x, a, b, c, d, e, x][h, x, x, a, b, c, e, d, x].\end{aligned}$$

Thus g is alternating in the d and e entries. Also,

$$\begin{aligned}1 =&[h, x, x, a, b, c, d, [e, x]][h, x, x, a, b, c, [e, x], d][h, x, x, a, b, d, c, [e, x]]\\ &[h, x, x, a, b, d, [e, x], c][h, x, x, a, b, [e, x], c, d][h, x, x, a, b, [e, x], d, c]\\ =&[h, x, x, a, b, c, d, e, x][h, x, x, a, b, d, c, e, x]\end{aligned}$$

and thus g is also alternating in the c entry. Hence, again by (8), we have

$$\begin{aligned}1 =&[h, x, x, a, b, c, [d, e], x][h, x, x, a, b, c, x, [d, e]][h, x, x, a, b, [d, e], c, x]\\ &[h, x, x, a, b, [d, e], x, c][h, x, x, a, b, x, c, [d, e]][h, x, x, a, b, x, [d, e], c]\\ =&[h, x, x, a, b, c, [d, e], x][h, x, x, a, b, [d, e], c, x] = g^4.\end{aligned}$$

Hence $g = 1$. □

The next step is to show that commutators in $[H,_8 G]$ with a double entry are trivial, from which Theorem 2 will easily follow. We impose that H is 5-torsion-free at this point.

Lemma 8. *Any commutator in $[H,_8 G]$ with a double entry is trivial.*

Proof. Consider commutators with first entry $h \in H$, double entry x and other entries a, b, c, d, e and f. By Corollary 1 we may assume that the second entry is x, by use of $1 = (a; x; x)(x; a; x)(x; x; a)$. First we show that it suffices to see that

$$g_1 =[h, x, x, a, b, c, d, e, f],\ g_2 = [h, x, a, x, b, c, d, e, f],\ g_3 = [h, x, a, b, x, c, d, e, f],$$

are trivial.

By Corollary 1,

$$(a; b; x)(a; x; b)(b; a; x)(b; x; a)(x; a; b)(x; b; a) = 1.$$

Hence, modulo commutators with second entry x and other x entry further to the left, the entries between the two x entries are alternating. So, modulo these commutators,

$$\begin{aligned}1 =&[\ldots, [a, b], c, x, \ldots][\ldots, [a, b], x, c, \ldots][\ldots, c, [a, b], x, \ldots]\\ &[\ldots, c, x, [a, b], \ldots][\ldots, x, [a, b], c, \ldots][\ldots, x, c, [a, b], \ldots]\\ =&[\ldots, a, b, c, x, \ldots]^2[\ldots, c, a, b, x, \ldots]^2\\ =&[\ldots, a, b, c, x, \ldots]^4.\end{aligned}$$

Hence, as H is 2-torsion-free, it does indeed suffice to see that $g_1 = g_2 = g_3 = 1$.

Next suppose that g_1 and g_2 are trivial. Then g_3 is alternating in the entries a and b. Hence, by Lemma 3(ii),

$$\begin{aligned}
1 =& [h, x, x, [a, b], c, d, e, f]^2[h, x, [a, b], x, c, d, e, f]^2[h, [a, b], x, x, c, d, e, f]^2 \\
=& g_3^4[h, a, b, x, x, c, d, e, f]^2[h, b, a, x, x, c, d, e, f]^{-2} \\
=& g_3^4[h, x, b, a, x, c, d, e, f]^{-2}[h, x, a, b, x, c, d, e, f]^2, \text{ by Corollary 1,} \\
=& g_3^8.
\end{aligned}$$

Thus it remains to show that $g_1 = g_2 = 1$. To show this we can instead, by Lemma 3(ii), show that g_1 and $\hat{g}_2 = [h, a, x, x, b, c, d, e, f]$ are trivial. For this we will show that g_1 is alternating in the entries a, b, c, d, e, f and $\hat{g}_2$ in the entries b, c, d, e, f. By the proof that we need only check that g_1 and $\hat{g}_2$ are trivial, it suffices for this to show that the elements $[h, x, x, a, y, y, b, c, d]$, $[h, x, x, y, a, y, b, c, d]$, $[h, a, x, x, b, y, y, c, d]$ and $[h, a, x, x, y, b, y, c, d]$ are trivial.

Consider the following commutators in $[H,_8 G]$, where $k \in H$.

$$\begin{aligned}
&h_1 = [k, x, x, a, y, y, \ldots],\ h_2 = [k, x, x, y, a, y, \ldots],\ h_3 = [k, x, a, x, y, y, \ldots], \\
&h_4 = [k, x, a, y, x, y, \ldots],\ h_5 = [k, x, y, x, a, y, \ldots],\ h_6 = [k, x, y, a, x, y, \ldots].
\end{aligned}$$

The four commutators listed earlier are all of the same form as h_1 or h_2. We first show that $h_1 = 1$. First, by (8), $1 = h_1h_2h_3h_4h_5h_6$ (α). We also have

$$\begin{aligned}
1 =& [k, x, x, [y, a, y], \ldots][k, x, [y, a, y], x, \ldots][k, [y, a, y], x, x, \ldots] \\
=& [k, x, x, y, a, y, \ldots]^3[k, x, y, a, y, x, \ldots]^3[k, y, a, y, x, x, \ldots]^3 \\
=& h_2^3h_1^{-3}h_6^{-3}h_3^3 \\
=& h_1^{-1}h_2h_3h_6^{-1}\ (\beta),
\end{aligned}$$

$$\begin{aligned}
1 =& [k, x, x, [a, y], y, \ldots][k, x, [a, y], x, y, \ldots][k, [a, y], x, x, y, \ldots] \\
=& h_1h_2^{-1}h_4h_6^{-1}[k, a, y, x, x, y, \ldots][k, y, a, x, x, y, \ldots]^{-1} \\
=& h_1h_2^{-1}h_3h_4^2h_5^{-1}h_6^{-2}\ (\gamma),
\end{aligned}$$

$$\begin{aligned}
1 =& [k, x, [x, a], y, y, \ldots][k, x, y, [x, a], y, \ldots][k, x, y, y, [x, a], \ldots] \\
=& h_1h_3^{-1}h_5h_6^{-1}[k, x, y, y, x, a, \ldots][k, x, y, y, a, x, \ldots]^{-1} \\
=& h_1h_2h_3^{-1}h_4^{-1}h_5^2h_6^{-2}\ (\delta),
\end{aligned}$$

$$\begin{aligned}
1 =& [k, x, x, y, [a, y], \ldots][k, x, [a, y], y, x, \ldots][k, [a, y], x, y, x, \ldots], \text{ by Lemma 7,} \\
=& h_2[k, x, x, y, y, a, \ldots]^{-1}[k, x, a, y, y, x, \ldots][k, x, y, a, y, x, \ldots]^{-1} \\
& [k, a, y, x, y, x, \ldots][k, y, a, x, y, x, \ldots]^{-1} \\
=& h_1^2h_2h_3^{-1}h_4^{-3}h_6\ (\epsilon).
\end{aligned}$$

Now, $(\alpha)^5(\beta)^{-5}(\gamma)^7(\delta)(\epsilon)^6$ gives $h_1^{30} = 1$ and hence, as H is $\{2,3,5\}$-torsion-free, $h_1 = 1$. Then $(\alpha)(\beta)^{-1}(\gamma)(\epsilon)$ gives $h_6 = 1$. (β) is then $h_2h_3 = 1$, $(\alpha)(\beta)^{-1}$ is $h_4h_5 = 1$ and $(\alpha)(\gamma)$ is $h_3^2h_4^3 = 1$.

Now consider commutators in $[H,_8 G]$ with first entry h and double entries of x, y and z within six entries. We will show that these are trivial. First we may assume, by Corollary 1, that of these six entries, the first is x, fifth is y and sixth is z. Since $h_1 = h_6 = 1$ we are left with

$$k_1 = [k,x,x,y,z,y,z,\ldots],\ k_2 = [k,x,y,x,z,y,z,\ldots],\ k_3 = [k,x,z,x,y,y,z,\ldots].$$

By (8) we get $k_1k_2k_3 = 1$. Also, $h_2h_3 = 1$ gives $k_1k_3 = 1$. Hence $k_2 = 1$. Further, $h_3^2h_4^3 = 1$ implies that $k_3^2 = 1$ and hence that $k_3 = 1$. Thus $k_1 = 1$ as well. So these commutators are trivial. In particular, any commutator in $[H,_8 G]$ with first entry h and double entries of x and y within six other entries will be alternating in the other two entries of the six.

By Lemma 7,

$$\begin{aligned}
1 =&[h,x,y,y,x,[a,b],c,d][h,x,y,y,[a,b],x,c,d][h,[a,b],y,y,x,x,c,d]\\
=&[h,x,y,y,x,a,b,c,d]^2[h,x,y,y,a,b,x,c,d]^2[h,a,b,y,y,x,x,c,d]^2\\
=&[h,x,y,y,x,a,b,c,d]^2[h,x,y,y,x,b,a,c,d]^{-2}[h,a,y,y,x,b,x,c,d]^{-2}\\
&[h,a,y,b,y,x,x,c,d]^{-2}, \text{ since } h_1 = 1,\\
=&[h,x,y,y,x,a,b,c,d]^4[h,a,y,y,x,b,x,c,d]^{-2}[h,a,y,b,y,x,x,c,d]^{-2}.
\end{aligned}$$

Since $h_2h_3 = 1$, we get $[h,x,y,y,x,a,b,c,d]^4 = [h,x,y,y,x,a,b,c,d] = 1$. Hence, as $h_6 = 1$, $[h,x,a,y,x,y,b,c,d] = 1$. From $h_3^2h_4^3 = 1$ and $h_2h_3 = 1$ it follows that $h_2^{-2}h_4^3 = 1$. Hence $[h,x,x,y,a,y,b,c,d] = 1$ and similarly $[h,a,x,x,y,b,y,c,d] = 1$.

This proves that g_1 and $\hat{g}_2$ are alternating in the entries b,c,d,e,f. Thus, for g_1 and $\hat{g}_2$,

$$\begin{aligned}
1 =&[h,\ldots,[b,c],[d,e],f][h,\ldots,[b,c],f,[d,e]][h,\ldots,[d,e],[b,c],f]\\
&[h,\ldots,[d,e],f,[b,c]][h,\ldots,f,[b,c],[d,e]][h,\ldots,f,[d,e],[b,c]]\\
=&[h,\ldots,b,c,d,e,f]^{24}
\end{aligned}$$

Hence $g_1 = \hat{g}_2 = 1$, which completes the proof. □

We now have that commutators in $[H,_8 G]$ with a double entry are trivial. This shows that commutators in $[H,_8 G]$ are alternating in all entries, except for the entry from H. For $g_1,\ldots,g_8 \in G$ we then have

$$\begin{aligned}
1 =&[h,g_1,[g_2,g_3],[g_4,g_5],g_6,g_7,g_8][h,g_1,[g_4,g_5],[g_2,g_3],g_6,g_7,g_8]\\
&[h,[g_2,g_3],g_1,[g_4,g_5],g_6,g_7,g_8][h,[g_2,g_3],[g_4,g_5],g_1,g_6,g_7,g_8]\\
&[h,[g_4,g_5],g_1,[g_2,g_3],g_6,g_7,g_8][h,[g_4,g_5],[g_2,g_3],g_1,g_6,g_7,g_8]\\
=&[h,g_1,g_2,g_3,g_4,g_5,g_6,g_7,g_8]^{24}.
\end{aligned}$$

As H is {2,3}-torsion-free, this completes the proof of Theorem 2.

5 Theorem 2 is best possible

We now show that Theorem 2 is best possible, by finding a group G and subgroup H satisfying the conditions of Theorem 2, but with $[H,_7 G] \neq \{1\}$.

Let $G = \langle h, x, y, a, b, c \rangle$ be the free group on six generators. Let $H = \langle h \rangle^G$. Quotient G out by H' to make H abelian. Further, quotient out by any commutator containing a triple entry of x or y, or a double entry of either a, b or c. Note that, as in Section 3, there are no relations between left normed commutators in H with first entry h and entry set $\{h, x, y, a, b, c\}$, other than those that follow directly from what is quotiented out by above. We now place some more structure on $[H,_7 G]$.

Note that currently $[H,_7 G]$ is generated by left normed commutators with first entry h, double entries of both x and y and single entries of a, b and c, and that these commutators are independent. We quotient out by products of commutators to make all commutators in $[H,_7 G]$ alternating in entries of a, b and c and such that we can swap both x entries with both y entries. Further, we quotient out by every product of the form

$$[h, g_1, g_2, \ldots, g_7][h, g_2, g_3, \ldots, g_7, g_1]^{-1},$$

where $g_1, \ldots, g_7 \in \{x, y, a, b, c\}$, so that the entries after h in commutators in $[H,_7 G]$ can be cycled around. We also quotient out by products of the form

$$[h, g_1, g_2, \ldots, g_7][h, g_7, g_6, \ldots, g_1],$$

so that the entries after h in commutators in $[H,_7 G]$ can be reflected, which inverses the commutator.

Using the alternating, cycling, reflecting and swapping properties we can seperate the generators of $[H,_7 G]$ into equivalence classes, where two commutators are equivalent if one is equal to the other or the inverse of the other. For each equivalence class we can pick as a representative the smallest commutator in the class with respect to the lexicographical order where $x < y < a < b < c$. Note that these representatives are independent. We now place these in sets as follows. Let

$$\begin{aligned}
A_{-4} &= \{[h, x, x, a, y, b, y, c]\} \\
A_{-3} &= \{[h, x, x, y, y, a, b, c]\} \\
A_{-2} &= \{[h, x, y, x, a, y, b, c], [h, x, y, a, y, x, b, c]\} \\
A_0 &= \{[h, x, y, a, x, y, b, c]\} \\
A_1 &= \{[h, x, x, y, a, y, b, c], [h, x, x, y, a, b, y, c], [h, x, y, x, y, a, b, c], [h, x, y, a, x, b, y, c]\} \\
A_2 &= \{[h, x, x, y, a, b, c, y], [h, x, x, a, y, y, b, c]\} \\
A_3 &= \{[h, x, y, a, y, b, x, c]\}.
\end{aligned}$$

For each $x_i \in A_i$ we quotient G out by $[h, x, x, y, a, y, b, c]^{-i} x_i$. Thus we have that $x_i = [h, x, x, y, a, y, b, c]^i$. Since the elements above were previously independent we have that $[h, x, x, y, a, y, b, c]$ is still a torsion-free element of $[H,_7 G]$.

Using the alternating, cycling, reflecting and swapping properties above, every commutator in $[H,_7 G]$ may be written as a product of elements from the A_i's. Thus $[H,_7 G]$ is generated by $[h, x, x, y, a, y, b, c]$. Let

$$N = \langle [h, g, g, g] : g \in G \rangle^G.$$

Then H/N is a right 3-Engel subgroup of G/N and it remains to check that no non-zero power of $[h, x, x, y, a, y, b, c]$ is in N. The calculations for this, along with some techniques to shorten the calculations, can be found in [2] (Section 6.5 and Appendix A).

References

[1] A. Abdollahi and H. Khosravi, When right n-Engel elements of a group form a subgroup? (http://arxiv.org/abs/0906.2439v1).

[2] P. G. Crosby, Right Engel Subgroups, Ph.D. Thesis, University Of Bath (2011)

[3] P. G. Crosby and G. Traustason, On right n-Engel subgroups, *J. Algebra* **324** (2010), 875-883.

[4] M. Hall Jr., A basis for free Lie rings and higher commutators in free groups, *Proc. Amer. Math. Soc.* **1** (1950), 575-581.

[5] M. Hall Jr., The theory of groups, Chelsea Publishing Co. (1976).

[6] L. C. Kappe and W. P. Kappe, On three-Engel groups, *Bull. Austral. Math. Soc.* **7** (1972), 391-405.

[7] M. L. Newell, On right-Engel elements of length three, *Proceedings of the Royal Irish Academy*, Vol. 96A **1** (1996), 17-24.