EFFECTIVE ALGEBRAIC INTEGRATION IN BOUNDED GENUS

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ABSTRACT. We introduce and study birational invariants for foliations on projective surfaces built from the adjoint linear series of positive powers of the canonical bundle of the foliation. We apply the results in order to investigate the effective algebraic integration of foliations on the projective plane. In particular, we describe the Zariski closure of the set $\Sigma_{d,g}$ of foliations on $\mathbb{P}^2$ of degree d admitting rational first integrals with fibers having geometric genus bounded by g.

CONTENTS

1. INTRODUCTION

1.1. **Effective algebraic integration.** It seems fair to say that the simplest class of algebraic ordinary differential equations consists of the class of equations having all its solutions algebraic. In general, given an explicit differential equation, it is a difficult to problem to decide whether or not it belongs to this distinguished class. Perhaps the first positive result on the subject is Schwarz's list of parameters for which Gauss' hypergeometric equation belongs to this class [35].

Motivated by this remarkable result, a lot of activity on the study of algebraic solutions of linear differential equations took place in the XIXth century leading to a fairly good understanding of the problem for homogeneous linear differential equations. Among the works dealing with this question one can find contributions by Fuchs, Gordan, Jordan, Halphen, and Klein just to name a few. At that time, the community seemed to believe that it would be possible to decide whether or not all solutions of a given linear differential

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equations are algebraic, see for instance the concluding remarks[1] of [20, Section 3, Chapter V].

By the end of XIXth century mathematicians like Painlevé, Autonne, and Poincaré [32, 33] started to study the next case, that is, polynomial differential equations of first order and of first degree. In modern language, they studied foliations on the projective plane with special emphasis on the existence of methods/algorithms to decide whether or not all leaves are algebraic. We will call this general line of enquiring effective algebraic integration. The results obtained at the time relied on strong assumptions on the nature of the singularities of the foliations and were not considered definitive as one can learn from the Introduction[2] of [33]. For a modern account of some of these classical results see [17] and [30, Chapter 7].

The results of the XIXth century on effective integration of linear differential equations were revisited in the course of the XXth century. It was then made clear that a full solution for the problem was not available, but instead it was reduced to a similar problem for rank one linear differential equations over curves. More precisely, in order to be able to decide whether or not a homogeneous linear differential equations $P(x, y, y', y'', y''', \ldots, y^{(n)}) = 0$ has all its solutions algebraic it suffices to be able to solve the following problem: given an element u belonging to an algebraic extension of the field $\mathbb{C}(x)$, decide if u is the logarithmic derivative of an element v also belonging to an algebraic extension of $\mathbb{C}(x)$. Some authors expressed doubts on the possibility of solving this problem. For instance, in [19, page 51] one can find the view of Hardy[3] on the subject.

Despite the scepticism of Hardy and others (cf. [34]), in the late 1960's Risch (loc. cit.) showed that this problem, in its turn, can be reduced to the following one: given an explicit divisor on an explicit algebraic curve C, decide whether or not such divisor is of finite order in the Jacobian of C. Risch proved that this problem can be solved by restricting the data modulo two distinct primes and using the resulting bounds in positive characteristic to devise an explicit bound in characteristic zero. For a detailed account on the case of second order homogeneous differential equations see [1]. More about the history of effective algebraic integration of linear differential equations can be found in [37, page 124], [18, Chapter III], and references therein.

The corresponding problem for (non-linear) differential equations of the first order and of the first degree is still wide open and received considerably less attention. After being dormant for a good while, the interest towards it has been revived by experts in foliation theory who considered the problem of bounding the degree of algebraic leaves of foliations on $\mathbb{P}^2$, see for instance [9, 7, 6, 16] and references therein. The influence of arithmetic on the subject was rediscovered by Lins Neto [23] who determined algebraic families (pencils) of foliations on the projective plane with fixed number and analytical type of singularities and with algebraic leaves of arbitrarily large degree.

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[1] "Thus is the problem, which we formulated at the beginning of this paragraph [present all linear homogenous differential equations of the second order with rational coefficients: $y'' + py' + qy = 0$ which possess altogether algebraic solutions], fully solved."

[2] "Je me suis occupé de nouveau de la même question dans ces derniers temps, dan l' espoir que je parviendrais à généraliser les résultats obtenus. Cet espoir a été déçu. J'ai obtenu cependant quelques résultats partiels, que je prends la liberté de publier, estimant qu'on pourra s'en servir plus tard pour obtenir, par un nouvel effort, une solution plus satisfaisante du problème."

[3] "But no method has been devised as yet by which we can always determine in a finite number of steps whether a given elliptic integral is pseudo-elliptic, and integrate it if it is, and there is reason to suppose that no such method can be given."

1.2. **Degenerations of planar foliations admitting a rational first integrals.** This work investigates the problem of effective algebraic integration for foliations on projective surfaces. In order to focus the discussion and clarify the framework in which we are going to carry it, we introduce the following conjecture.

Conjecture 1.1. *The Zariski closure in $\mathbb{P}H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(d-1))$ of the set of foliations of degree d on $\mathbb{P}^2$ which admit a rational integral consists of transversely projective foliations.*

This conjecture is inspired by a remark made by Painlevé[4] ([28, pp. 216–217]) in his Stockholm's lectures. Knowledge of a transverse projective structure for a given foliation, in view of their recent description [12, 24], would allow to reduce the problem to either the determination of periods of differential forms – when, after passing to a ramified covering, the foliation is defined by a closed rational 1-form – or to the algebraic integrability of Riccati equations.

The main results of this paper provide evidence in favor of this conjecture and are obtained using birational techniques. More precisely, we use basic results on adjoint linear series, the birational classification of foliated surfaces according to their Kodaira dimension [26, 5, 27], and a variant of it which we now proceed to explain.

1.3. **Adjoint dimension of foliations.** The works of the Italian school of algebraic geometry in the beginning of the XXth century showed how much of the geometry of a smooth projective surface X can be determined by the order of growth of the function

$$n \mapsto h^0(X, K_X^{\otimes n}).$$

Whenever this function grows slower than a quadratic polynomial, one has a rather precise description of the surface (the so called Enriques-Kodaira classification). A similar classification is also available in dimension three thanks to the works of the modern school of birational geometry, and there is also a similar picture in arbitrary dimensions conditional on the so-called Abundance Conjecture.

In the case of foliations on surfaces, McQuillan, Brunella and Mendes obtained a very precise classification – analogue to the Enrique-Kodaira classification – in terms of the Kodaira dimension of the foliation. As in the case of surfaces, the Kodaira dimension of a foliation $\mathcal{F}$, $\mathrm{kod}(\mathcal{F})$, measures the growth of the function $h^0(X, K_{\mathcal{F}}^{\otimes n})$ where $K_{\mathcal{F}}$ is the bundle of holomorphic 1-forms along the leaves of the foliation.

As the terminology suggests the canonical bundle together with its dual are the most obvious naturally determined line-bundles on a variety. Combined with the fact that integers $h^0(X, K_X^{\otimes n})$ ($n > 0$) are birational invariants for smooth projective varieties, its study is rather natural if one wants to understand varieties birationally. For foliations of arbitrary dimension/codimension, besides the canonical bundle, one also has another naturally attached line-bundle: the determinant of the conormal bundle. If $\mathcal{F}$ is a foliation on a projective surface X with canonical singularities then it turns out that for arbitrary $n, m \geq 0$ the integers $h^0(X, K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m})$ are birational invariants. Most of the results obtained in this paper steam from this simple observation. We define the adjoint dimension of a foliation according to the order of growth of the function $h^0(X, K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m})$, see Section 3.

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[4]"J'ajoute qu'on ne peut espérer résoudre d'un coup qui consiste à limiter n. L'énoncé vers lequel il faut tendre doit avoir la forme suivante: *"On sait reconnaître si l'intégrale d'une équation $F(y\prime, y, x) = 0$ donnée est algébrique ou ramener l'équation aux quadratures."* Dans ce dernier cas, la question reviendrait à reconnaître si une certaine intégrale abélienne (de première ou de troisième espèce) n'a que deux ou une périodes."

Building on the classification of foliations on surfaces according to their Kodaira dimension, in Section 6 we present a classification in function of the adjoint dimension. The results we obtain are summarized in Table 1. Thee outcome of the classification provides a framework well-suited to deal with families of foliations (Section 7) mainly due to the fact that it is more flexible with respect to type of singularities which are allowed (Section 4). The classification in terms of the adjoint dimension also reflects distinct cases of the problem of effective algebraic integration (Section 8).

adj	kod	Description
$-\infty$	$-\infty$	Rational fibration
	0	Finite quotient of Riccati foliation generated by global vector field
	1	Riccati foliation
0	0	Finite quotient of linear foliation on a torus
1	0	Finite quotient of $E \times C \to C$, $g(C) \geq 2$
	1	Finite quotient of $E \times C \to E$, $g(C) \geq 2$
	1	Turbulent foliation
	1	Non-isotrivial elliptic fibration
2	$-\infty$	Irreducible quotient of $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$
	1	Finite quotient of $C_1 \times C_2 \to C_1$, $g(C_i) \geq 2$
	2	General type

TABLE 1. Classification of foliations according to their adjoint/Kodaira dimensions.

1.4. **Plan of the paper and statement of main results.** The bulk of the paper starts by reviewing classification of foliations with respect to their Kodaira dimension in Section 2. Then we introduce new birational invariants for foliations on surfaces, notably the effective threshold and the adjoint dimension, in Section 3. Section 4 is devoted to the study of a variation of the concept of canonical singularities, the so-called ε-canonical singularities. We prove in Corollary 4.10 that, for $\varepsilon > 0$, this concept is stable for small perturbations of the singularity of the foliation. This fact will be particularly important in the study of families of foliations carried out in Section 7.

Section 5 is devoted to the proof of boundness of non-isotrivial fibrations of bounded genus in families, see Theorem 5.7. In the particular case of $\mathbb{P}^2$, the result reads as follows.

Theorem A. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. Assume that $\mathcal{F}$ is birationally equivalent to a non-isotrivial fibration of genus $g \geq 1$. Then the degree of the general leaf of $\mathcal{F}$ is bounded by*

$$\left(7\Big(42(2g-2)\Big)!\right)^2 \deg(\mathcal{F}).$$

Theorem A refines the main result of [29] where it was established the existence of a bound for the degree of the general leaf depending on its genus and on the first $k > 0$ for which the linear system $|K_{\mathcal{F}}^{\otimes k}|$ defines a rational map with two dimensional image. The existence of universal k working for every non-isotrivial fibration of genus g was not known then - and is still not known at present time - hence the existence of a bound depending only on the degree of the foliation and on the genus was unclear. In comparison to [29] the proof of the result above has two new ingredients. The first is a bound on multiplicities of irreducible components of fibers of relatively minimal non-isotrivial fibrations of genus

$g \geq 2$ (Proposition 5.6). The second new ingredient is the use of standard results on adjoint linear series (recalled in Section 5.1) in order to obtain effective $(n, m) \in \mathbb{N}^2$ such that the rational map defined by $|K_{\mathcal{F}}^{\otimes n} \otimes K_X^{\otimes m}|$ has two dimensional image. By imposing further assumptions on the nature of the singularities of a foliation on $\mathbb{P}^2$ we obtain significantly better bounds (sub-linear on g), refining a classical result of Poincaré, cf. Theorem 5.9.

In Section 6 we carry out the classification of foliations on surfaces according to the adjoint dimension, see Table 1. The proof strongly relies on the classification of foliations according to the Kodaira dimension, but it does need to dwell with its subtlest point: the classification of non–abundant foliations. A nice corollary of the classification is a cohomological characterization of rational fibrations, which is a weak analogue of Castelnuovo's Criterion for the rationality of surfaces, cf. [2, Thm. V.1].

Theorem B. *Let $\mathcal{F}$ be a foliation with at worst canonical singularities on a smooth projective surface X. The foliation $\mathcal{F}$ is a rational fibration if, and only if, $h^0(X, K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m}) = 0$ for every $n \geq 1$ and every $m > 0$.*

Section 7 investigates families of foliations. There it is shown that the set of effective thresholds in a family does not accumulate at zero (Theorem 7.5). More important, it prepares the ground for the proof of the most compelling evidence we have so far in favor of Conjecture 1.1.

Theorem C. *The Zariski closure in $\mathbb{P}(H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(d-1)))$ of the set of degree d foliations admitting a rational first integral with general fiber of genus $\leq g$ is formed by transversely projective foliations.*

Its proof is presented in Section 8 and relies on Theorem A, on the birational classification of foliations, and on basic properties of families of foliations.

2. KODAIRA DIMENSION OF FOLIATIONS

We start things off by reviewing the birational classification of foliations on surfaces following [26] and [5]. No new results are presented in this section. We have only included proofs of a few key properties of the Zariski decomposition of the canonical bundle of a foliation which will be used in the sequel.

2.1. **Singularities of foliations.**

Definition 2.1. *Let $\mathcal{F}$ be a foliation on X and let $\pi : Y \to X$ be a birational morphism. Denote by $\mathcal{G}$ the pull-back of $\mathcal{F}$ under π. If E is an exceptional divisor of π then the discrepancy of $\mathcal{F}$ along E is*

$$a(\mathcal{F}, E) = \mathrm{ord}_E(K_{\mathcal{G}} - \pi^* K_{\mathcal{F}}).$$

Definition 2.2. *Let $\mathcal{F}$ be a foliation on X. A point $x \in X$ is canonical for $\mathcal{F}$ if and only if $a(\mathcal{F}, E) \geq 0$ for every divisor E over x. A point $x \in X$ is log canonical for $\mathcal{F}$ if and only $a(\mathcal{F}, E) \geq -1$ for every divisor E over x.*

Example 2.3. *Consider the pencil of foliations on $X = \mathbb{P}^2$ defined by the vector fields $sx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}$ where $(s : t) \in \mathbb{P}^1$. If $s \cdot t \cdot (s - t) \neq 0$ then $\mathcal{F}_{(s:t)}$ is a foliation with trivial canonical bundle and three singularities at the points $(0 : 0 : 1), (0 : 1 : 0),$ and $(1 : 0 : 0)$. For $(s : t) \notin \mathbb{P}^1(\mathbb{Q})$ the three singularities are canonical. For $(s : t) \in \mathbb{P}^1(\mathbb{Q}) - \{(0 : 1), (1 : 0), (1 : 1)\}$, two of the singularities are log canonical but not canonical, while the third singularity is canonical. Finally, when $s \cdot t \cdot (s - t) = 0$, the vector field will have one of the coordinate axis as a line of singularities. The corresponding foliation will have canonical bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$ and only one singularity which is log canonical but not canonical.*

Any foliation on a projective surface is birationally equivalent a foliation having at worst canonical singularities thanks to the following which is essentially due to Seidenberg.

Theorem 2.4. *Let $\mathcal{F}$ be a foliation on a smooth projective surface X. Then there exists a finite composition of blow-ups $\pi : Y \to X$ such that all the singularities of $\pi^*\mathcal{F}$ are canonical.*

2.2. **Kodaira dimension.**

Definition 2.5. *Let $\mathcal{F}$ be a foliation with at worst canonical singularities on a smooth projective surface X. The Kodaira dimension of $\mathcal{F}$, $\mathrm{kod}(\mathcal{F})$, is by definition*

$$\mathrm{kod}(\mathcal{F}) := \mathrm{kod}(K_\mathcal{F}) = \max_{m \in \mathbb{N}}\{\phi_m(X)\},$$

where $\phi_m \colon X \dashrightarrow \mathbb{P}(H^0(X, K_\mathcal{F}^{\otimes m})^*)$ and we adopt the convention that $\dim \phi_m(X) = -\infty$ when $h^0(X, K_\mathcal{F}^{\otimes m}) = 0$. (and it is not possible to define the associated map).

The numerical Kodaira dimension of $\mathcal{F}$, $\nu(\mathcal{F})$, is defined to be the numerical dimension of $K_\mathcal{F}$, that is:

- *$\nu(\mathcal{F}) = -\infty$ if $K_\mathcal{F}$ is not pseudo-effective, while*
- *if $K_\mathcal{F}$ is pseudoeffective with Zariski decompositon $K_\mathcal{F} = P + N$ then $\nu(\mathcal{F}) = 0$ if P is numerically zero, $\nu(\mathcal{F}) = 1$ if $P \neq 0$ but $P^2 = 0$, and $\nu(\mathcal{F}) = 2$ if $P^2 > 0$.*

The classification of foliation with negative numerical Kodaira dimension stated in the next result is due to Miyaoka.

Theorem 2.6. *Let $\mathcal{F}$ be a foliation on a projective surface X. If $K_\mathcal{F}$ is not pseudo-effective then $\mathcal{F}$ is birationally equivalent to a $\mathbb{P}^1$-bundle over a curve.*

2.3. **Relatively minimal models.**

Definition 2.7. *Let $\mathcal{F}$ be a foliation with canonical singularities on a smooth projective surface X. An irreducible curve $C \subset X$ is called $\mathcal{F}$-exceptional if $K_X \cdot C = -1$ (i.e. $C\mathbb{P}^1$ and $C^2 = -1$) and the contraction of C gives rise to a foliation with canonical singularities.*

Definition 2.8. *Let $\mathcal{F}$ be a foliation with canonical singularities on a smooth projective surface X. A relatively minimal model for $\mathcal{F}$ is the datum of a foliation $\mathcal{G}$ with canonical singularities and without $\mathcal{G}$-exceptional curves on a smooth projective surface Y which is birationally equivalent to $\mathcal{F}$. We say that $\mathcal{G}$ is a minimal model if for any birational map $\pi : Z \dashrightarrow Y$ and any foliation $\mathcal{H}$ on Z with canonical singularities such that $\pi_*\mathcal{H} = \mathcal{G}$, π is a birational morphism.*

The definitions above and the next result are essentially due to Brunella [3]. The only minor difference is that in the original definition of $\mathcal{F}$-exceptional curve Brunella only

considered reduced singularities instead of canonical singularities. Nonetheless, his proof works also in this slightly more general situation.

Theorem 2.9. *Let $\mathcal{F}$ be a foliation with at worst canonical singularities on a smooth surface X. There exists a birational morphism $\pi : X \to Y$ such that $\pi_*\mathcal{F}$ is a relatively minimal model for $\mathcal{F}$. Moreover, $\pi_*\mathcal{F}$ is a minimal model for $\mathcal{F}$ unless $\mathcal{F}$ is birationally equivalent to a rational fibration, a Riccati foliation, or Brunella's special foliation $\mathcal{H}$.*

The reader will find the explicit construction of the foliation $\mathcal{H}$ from the theorem in the paper just cited.

Remark 2.10. *The above theorem highlights the main difference between the birational classification of projective surfaces and that of foliations on surfaces: while surfaces of non-negative Kodaira dimension always have a unique minimal model, there are foliations of Kodaira dimension zero and one which do not have unique minimal models.*

2.4. **Zariski decomposition and nef models.** If $\mathcal{L}$ is a pseudo-effective line bundle on a smooth projective surface then $\mathcal{L}$ is numerically equivalent to $P_{\mathcal{L}} + N_{\mathcal{L}}$ where $P_{\mathcal{L}}$ is a nef $\mathbb{Q}$-divisor and $N_{\mathcal{L}}$ is a contractible effective $\mathbb{Q}$-divisor satisfying $P_{\mathcal{L}} \cdot N_{\mathcal{L}} = 0$. This is the so-called Zariski decomposition of $\mathcal{L}$. We will denote by $\mathrm{i}(\mathcal{F})$ the index of $K_{\mathcal{F}}$, i.e., the minimum of the set $\{n \in \mathbb{N} \mid nN \text{ has integral coefficients}\}$.

Theorem 2.11. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth projective surface X. If $K_{\mathcal{F}}$ is pseudo-effective and $P + N$ is its Zariski decomposition then the support of N is a disjoint union of Hirzebruch-Jung strings.*

A Hirzebruch-Jung string is a chain of smooth rational curves of self-intersection smaller ≤ -2. At one end of the chain, the handle of the Hirzebruch-Jung string, the foliation has only one singularity. Every other curve in the chain contains two singularities of the foliation. There is only one singularity of $\mathcal{F}$ on the Hirzebruch-Jung string which does not coincide with a singularity of its support. There exists a unique leaf of $\mathcal{F}$ not contained in the Hirzebruch-Jung string that passes through this singularity. Such curve is called the tail of the Hirzebruch-Jung string.
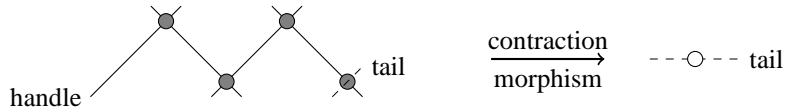
Definition 2.12. *Let $\mathcal{F}$ be a relatively minimal foliation with pseudo-effective $K_{\mathcal{F}}$ on a smooth projective surface X. The order of a maximal Hirzebruch-Jung string contained in the support of N is the determinant of the negative of the intersection matrix of its support.*

The following proposition shows that the order and the index are closely related.

Proposition 2.13. *Notation as in the definition above. The following assertions hold true.*

(1) *The order of a maximal Hirzebruch-Jung string J contained in the support of N coincides with the smallest $o \in \mathbb{N}$ such that the coefficients of N corresponding to curves in J belong to $\frac{1}{o}\mathbb{N}$.*

(2) *The contraction of a Hirzebruch-Jung string of order o is locally isomorphic to the quotient of a smooth foliation on $(\mathbb{C}^2, 0)$ by the cyclic group generated by an automorphism of the form $(x, y) \mapsto (\xi_o \cdot x, \xi_o^a \cdot y)$ where ξ_o is a primitive root of unity of order o and a is a natural number relatively prime to o.*

Proof. The statement is local so we may very well assume that the support of N is connected. Let us write $N = \sum_{i=1}^{k} a_i E_i$ where E_i are the irreducible components of N. We denote by E_1 the handle of the Hirzebruch-Jung string while the other curves are numbered following the order in which they appear in the chain.

Let $A = (E_i \cdot E_j)_{i,j}$ be the intersection matrix of the Hirzebruch-Jung string and let $o = \det(-A)$ be the order of the Hirzebruch-Jung string. To determine the coefficients $a_1, \ldots, a_k$ we have to solve the linear system $(-A) \cdot (a_1, a_2, \ldots, a_k)^T = (1, 0, \ldots, 0)^T$. Therefore the coefficients a_i certainly lie in $\frac{1}{o}\mathbb{N}$. To see that o is the minimal number with such property it suffices to notice that $a_k = 1/o$, cf. [26, proof of Proposition III.1.4]. This proves item (1). Item (2) is [26, Reinterpretation III.2.bis.3.a] $\qquad\qquad\square$

In the Lemma below, we collect some properties of tails of Hirzebruch-Jung strings for later use.

Lemma 2.14. *Let $\mathcal{F}$ be a relatively minimal foliation with pseudo-effective canonical bundle on a smooth projective surface X. Let T be an irreducible invariant curve not contained in the support of N and let $o_1, \ldots, o_k$ be the orders of Hirzebruch-Jung strings intersecting T. Then the following assertions hold true.*

(1) *The intersection of the positive part of the Zariski decomposition of $K_{\mathcal{F}}$ with T is given by the formula*

$$P \cdot T = K_{\mathcal{F}} \cdot T - \sum_{i=1}^{k} \frac{1}{o_i}.$$

(2) *If $\mathcal{F}$ admits a holomorphic first integral $f : U \to \mathbb{C}$ defined on a $\mathcal{F}$-invariant neighborhood of T which vanishes along T then the vanishing order along T is a multiple of the least common multiple of $o_1, \ldots, o_k$.*

Proof. Item (1) is [26, Remark III.1.3.a]. To verify item (2) let us work locally on a neighborhood V of a Hirzebruch-Jung string intersecting T. Let $\pi : V \to W$ be the contraction of the Hirzebruch-Jung string we are considering and o be its order. Perhaps after restricting V to a smaller neighborhood we can assume that W is the quotient of a neighborhood $\tilde{V}$ of the origin in $\mathbb{C}^2$ by a cyclic group generated by $\varphi(x, y) = (\xi_o \cdot x, \xi_o^a \cdot y)$ according to Proposition 2.13. We can also assume that the pull-back $\mathcal{G}$ of $\pi_*(\mathcal{F}_{|V})$ to $\tilde{V}$ is the foliation defined by the level sets of the coordinate function y. The pull-back of $\pi_*(f_{|V})$ to $\tilde{V}$ is a holomorphic function g constant along the leaves of $\mathcal{G}$. The φ invariance of g implies that $g(x, y) = h(y^o)$ for some one variable holomorphic function h. Item (2) follows. $\qquad\square$

Definition 2.15. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth surface X with pseudo-effective canonical divisor. The nef model of $\mathcal{F}$ is the foliation obtained by contracting the negative part of the Zariski decomposition of $K_{\mathcal{F}}$.*

2.5. **Canonical models.**

Definition 2.16. *A foliation $\mathcal{F}$ on a normal projective surface X is called a canonical model if $K_{\mathcal{F}}$ is nef and $K_{\mathcal{F}} \cdot C = 0$ implies $C^2 \geq 0$ for every irreducible curve $C \subset X$.*

Theorem 2.17. *Let $\mathcal{F}$ be relatively minimal foliation with pseudo-effective $K_{\mathcal{F}}$ on a smooth surface X. Then there exists a morphism $\pi : X \to Y$ from X to a normal projective surface Y such that $\mathcal{G} = \pi_*\mathcal{F}$ is a canonical model. The singular points of Y and the corresponding exceptional fibers of π are of one of the following forms.*

(1) *The singular point is a cyclic quotient singularity and the exceptional divisor over it is a chain of rational curves of self-intersection at most -2*
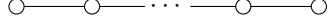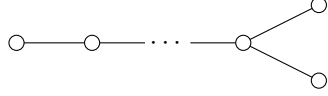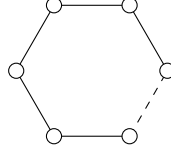

The foliation around the singular is the quotient of a smooth foliation; or the quotient of a canonical foliation singularity on a (germ of) smooth surface;

(2) *The singular point is dihedral quotient singularity and the exceptional divisor over it has the following dual graph:*

The foliation around the singularity is again the quotient of a smooth foliation or of a canonical singularity on a (germ of) smooth surface.

(3) *The singular point is an elliptic Gorenstein singularity and the exceptional divisor is a cycle of smooth rational curves each of self-intersection at most -2; or a unique nodal rational curve of negative self-intersection*

The foliation around the singular point is isomorphic to a cusp of a Hilbert modular foliation (cf. [26, Theorem IV.2.2]). The corresponding germ of foliation is a transversely affine and transversely hyperbolic on the complement of the singular point. Moreover, the canonical bundle of the foliation on the canonical model is never $\mathbb{Q}$-Cartier.

When compared with the theory for projective surfaces, item (3) of the above Theorem is quite surprising. The fact that the canonical bundle is never $\mathbb{Q}$-Cartier is a clear obstruction to the base point freeness of $|K_{\mathcal{F}}^{\otimes n}|$ and for the finite generation of the canonical algebra of the foliation. It turns out that this is the only obstruction, cf. [26, Corollary IV.2.3].

2.6. **Kodaira dimension zero.**

Theorem 2.18. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth projective surface X with $\nu(\mathcal{F}) = 0$. Let $\pi : X \to Z$ be the contraction of the negative part of $K_{\mathcal{F}}$, i.e. $\pi_* \mathcal{F}$ is a nef model for $\mathcal{F}$. Then there exists a smooth projective surface Y and a quasi-étale cyclic covering $p : Y \to Z$ of degree $\mathrm{i}(\mathcal{F})$ such that $p^* \pi_* \mathcal{F}$ is a foliation with trivial canonical bundle. In particular, $\mathrm{kod}(\mathcal{F}) = 0$.*

The resulting surface Y belongs to the following list:
(1) Product of a hyperbolic curve and an elliptic curve;
(2) Abelian surfaces;
(3) Projective bundle over an elliptic curve;
(4) Rational surface.

Consequently the klt surface Z has Kodaira dimension 1, 0, or $-\infty$ according to whether Y fits in case (1), (2), or (3)/(4). One can also determine the possibilities for the index of $\mathcal{F}$. This is done in [29]. There it is shown that

$$\mathrm{i}(\mathcal{F}) \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$$

when $\mathcal{F}$ has Kodaira dimension zero.

2.7. **Kodaira dimension one.** The classification of foliations of Kodaira dimension one is essentially due to Mendes, see [27, Theorem 3.3.1]

Theorem 2.19. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth projective surface X. Assume that $\mathrm{kod}(\mathcal{F}) = 1$ and let $f : X \to C$ be the Iitaka fibration of $K_{\mathcal{F}}$. If $\mathcal{F}$ coincides with the foliation defined by f then f is non-isotrivial elliptic fibration. Otherwise $\mathcal{F}$ is completely transverse to a general fiber F of f and we have the following possibilities:*

(1) *The genus of F is zero and $\mathcal{F}$ is a Riccati foliation; or*
(2) *The genus of F is one and $\mathcal{F}$ is a turbulent foliation; or*
(3) *The genus of F is at least two and $\mathcal{F}$ is an isotrivial fibration of genus at least two.*

2.8. **Non-abundant foliations.** The most striking difference between the birational classification of projective surfaces and the classification of rank one foliations in dimension two is the existence of foliations having canonical bundle with numerical dimension one and negative Kodaira dimension. This phenomenon is restricted to a rather special class of foliations as pointed out by the next result.

Theorem 2.20. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth projective surface X. If the numerical dimension of $\mathcal{F}$ does not coincide with the Kodaira dimension of $\mathcal{F}$ then*

(1) $\nu(\mathcal{F}) = 1$,
(2) $\mathrm{kod}(\mathcal{F}) = -\infty$,
(3) X *is the minimal desingularization of the Bayle-Borel compactification of an irreducible quotitent of $\mathbb{H} \times \mathbb{H}$, and*
(4) $\mathcal{F}$ *is induced by one of the two natural fibrations on $\mathbb{H} \times \mathbb{H}$.*

Arguably this result constitutes the hardest part of the classification of foliations. The known proofs of this result rely heavily on Brunella's plurisubharmonic variation of the Poincaré metric and where obtained by Brunella and McQuillan in a collaborative effort.

In Section 6 we will carry out a classification of foliations in terms of another birational invariants. It relies heavily on the classification of foliations on surfaces according to their Kodaira dimension but it does not need its full power. In particular, all that we need to know about non-abundant foliations in contained in the following Lemma.

Lemma 2.21. *Let $\mathcal{F}$ be a relatively minimal foliation with $\nu(\mathcal{F}) = 1$ and $\mathrm{kod}(\mathcal{F}) = -\infty$. Then $h^1(X, \mathcal{O}_X) = 0$ and $P \cdot N_{\mathcal{F}}^* = P \cdot K_X > 0$ where P is the positive part of the Zariski decomposition of $K_{\mathcal{F}}$.*

Proof. If $h^1(X, \mathcal{O}_X) = h^0(X, \Omega_X^1) \neq 0$ then the restriction of a holomorphic 1-form to the leaves of $\mathcal{F}$ either vanishes identically or gives rise to a non-zero section of $K_{\mathcal{F}}$. Thus if $\mathrm{kod}(\mathcal{F}) = -\infty$ we obtain that $\mathcal{F}$ factors through the Albanese map of X and is a fibration. Hence $\mathrm{kod}(\mathcal{F}) \geq 0$ contrary to our assumptions. Thus $h^1(X, \mathcal{O}_X) = 0$.

Since $h^1(X, \mathcal{O}_X) = 0$ we obtain that $\chi(\mathcal{O}_X) \geq 1$. Let $\mathcal{L} = \mathcal{O}_X(mP)$ where m is a sufficiently divisible positive integer. By Riemann-Roch,

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + 1/2(m^2 P^2 - mP \cdot K_X)$$

If $P \cdot K_X < 0$ then $\chi(\mathcal{L}) > 0$. Thus $h^0(X, \mathcal{L}) + h^2(X, \mathcal{L}) > 0$. But if m is sufficiently large then $K_X \otimes \mathcal{L}^*$ is not pseudoeffective and consequently $h^2(X, \mathcal{L}) = h^0(X, K_X \otimes \mathcal{L}^*) = 0$. It follows that $h^0(X, K_{\mathcal{F}}^{\otimes m}) = h^0(X, \mathcal{L}) > 0$, contradicting $\mathrm{kod}(\mathcal{F}) = -\infty$. □

3. EFFECTIVE THRESHOLD AND ADJOINT DIMENSION

In this section we define the effective threshold and the adjoint dimension of a foliation on a smooth projective surface and prove their birational invariance.

3.1. Effective threshold.

Definition 3.1. *Let $\mathcal{F}$ be a foliation with canonical singularities on a smooth projective surface X. If the canonical bundle of $\mathcal{F}$ is pseudo-effective then we define the effective threshold of $\mathcal{F}$, $\mathrm{eff}(\mathcal{F})$, as the largest $\varepsilon \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $K_\mathcal{F} + \varepsilon N_\mathcal{F}^*$ is pseudo-effective. If $K_\mathcal{F}$ is not pseudo-effective, then we set $\mathrm{eff}(\mathcal{F}) = -\infty$.*

Example 3.2. *Let $\mathcal{F}$ be a very general foliation on $\mathbb{P}^2$ of degree d. It is well known that $\mathcal{F}$ has reduced, and in particular canonical, singularities. Recall that the degree of $\mathcal{F}$ is defined as the number of tangencies between $\mathcal{F}$ and a general line. In this case $K_\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}(d-1)$ and $N_\mathcal{F}^* = \mathcal{O}_{\mathbb{P}^2}(-d-2)$. If $d = 0$ then $K_\mathcal{F}$ is not pseudoeffective. If instead $d \geq 1$ then $K_\mathcal{F}$ is pseudo-effective and*

$$\mathrm{eff}(\mathcal{F}) = \frac{d-1}{d+2}.$$

The reader should notice that $\mathrm{eff}(\mathcal{F}) < 1$ for every foliation on $\mathbb{P}^2$.

This is by no means a coincidence since $K_X = K_\mathcal{F} + N_\mathcal{F}^*$ and foliations on a surface X of negative Kodaira dimension will always have $\mathrm{eff}(\mathcal{F}) < 1$ as K_X is not pseudo-effective. If instead X has non-negative Kodaira dimension then K_X is pseudo-effective and consequently $\mathrm{eff}(\mathcal{F}) \geq 1$ for every foliation on X.

Similarly, one sees that $\mathrm{eff}(\mathcal{F}) = \infty$ if and only if both $K_\mathcal{F}$ and $N_\mathcal{F}^*$ are pseudo-effective. Foliations with pseudo-effective conormal bundle have recently been classified by Touzet, [36]. They fit in one of the following descriptions:

(1) after a finite étale cover $\mathcal{F}$ is defined by a closed holomorphic 1-form; or
(2) there exists a morphism from X to a quotient of a polydisc $\mathbb{D}^m$ by an irreducible lattice and $\mathcal{F}$ is the pull-back of one of the m tautological foliations on the polydisk. In particular $\mathcal{F}$ is transversely hyperbolic.

Notice that the dimension of the ambient manifold is not necessarily equal to the dimension of the polydisk.

Remark 3.3. *Using the identity $K_X = K_\mathcal{F} + N_\mathcal{F}^*$ we can write*

$$K_\mathcal{F} + \varepsilon N_\mathcal{F}^* = (1-\varepsilon)(K_\mathcal{F} + \frac{\varepsilon}{1-\varepsilon}K_X),$$

when $\varepsilon \neq 1$.

When $\mathrm{eff}(\mathcal{F})$ is small, we will often work with $K_\mathcal{F} + \varepsilon K_X$ as that is more convenient.

3.2. Adjoint dimension.

Definition 3.4. *Let $\mathcal{F}$ be a foliation with canonical singularities on a projective surface X. Consider the pluricanonical maps*

$$\varphi_{m,n} : X \dashrightarrow \mathbb{P}H^0(X, K_\mathcal{F}^{\otimes m} \otimes N_\mathcal{F}^{* \otimes n})^*$$

for $m \geq 1$, $n \geq 1$. The adjoint dimension of $\mathcal{F}$, denoted $\mathrm{adj}(\mathcal{F})$, is the maximal dimension of the image of these maps. If $h^0(X, K_\mathcal{F}^{\otimes m} \otimes N_\mathcal{F}^{* \otimes n}) = 0$ for every $m, n \geq 1$ then we set $\mathrm{adj}(\mathcal{F}) = -\infty$.

Definition 3.5. *Let $\mathcal{F}$ be a foliation with canonical singularities on a projective surface X. The numerical adjoint dimension of $\mathcal{F}$, $\mathrm{adj}_{\mathrm{num}}(\mathcal{F})$, is equal to $-\infty$ if $\mathrm{eff}(\mathcal{F}) \le 0$ and equal to the maximal numerical dimension of $K_{\mathcal{F}} + \varepsilon N_{\mathcal{F}}^*$ for $\varepsilon \in (0, \mathrm{eff}(\mathcal{F}))$ otherwise.*

Of course $\mathrm{adj}(\mathcal{F}) \le \mathrm{adj}_{\mathrm{num}}(\mathcal{F})$.

3.3. **Birational invariance.** The significance of the concepts of effective threshold and of (numerical) adjoint dimension for the purpose of the birational classification of foliations on surfaces is assured by the next proposition.

Proposition 3.6. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two birationally equivalent foliations. If $\mathcal{F}$ and $\mathcal{G}$ have at worst canonical singularities then $\mathrm{eff}(\mathcal{F}) = \mathrm{eff}(\mathcal{G})$, $\mathrm{adj}(\mathcal{F}) = \mathrm{adj}(\mathcal{G})$ and $\mathrm{adj}_{\mathrm{num}}(\mathcal{F}) = \mathrm{adj}_{\mathrm{num}}(\mathcal{G})$. Furthermore, $h^0(X, K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m}) = h^0(Y, K_{\mathcal{G}}^{\otimes n} \otimes N_{\mathcal{G}}^{*\otimes m})$ for every $n, m \ge 0$.*

Proof. The proof is standard. Since we can choose a foliation $(Z, \mathcal{H})$ on a smooth projective surface Z dominating both $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, there is no loss of generality in assuming the existence of a birational morphism $\pi : (X, \mathcal{F}) \to (Y, \mathcal{G})$. Indeed, we can even assume (and will) that π is the blow-up of a point $p \in Y$. Let E be the exceptional divisor.

We will first prove that $\mathrm{eff}(\mathcal{F}) = \mathrm{eff}(\mathcal{G})$. First notice that $K_{\mathcal{G}} + \varepsilon N_{\mathcal{G}}^* = \pi_*(K_{\mathcal{F}} + \varepsilon N_{\mathcal{F}}^*)$. Therefore if $K_{\mathcal{F}} + \varepsilon N_{\mathcal{F}}^*$ is pseudo-effective then the same holds true for $K_{\mathcal{G}} + \varepsilon N_{\mathcal{G}}^*$. This shows that $\mathrm{eff}(\mathcal{G}) \ge \mathrm{eff}(\mathcal{F})$. To prove the converse inequality, we will need to use that $\mathcal{G}$ has canonical singularities. Since π is the blow-up of a point by assumption, we have that $K_{\mathcal{F}} - \pi^* K_{\mathcal{G}} = aE$ for some $a \in \{0, 1\}$. Since $K_X - \pi^* K_Y = E$ we also have that $N_{\mathcal{F}}^* - \pi^* N_{\mathcal{G}}^* = (1-a)E$, and consequently $K_{\mathcal{F}} + \varepsilon N_{\mathcal{F}}^* = \pi^*(K_{\mathcal{G}} + \varepsilon N_{\mathcal{G}}^*) + (a + \varepsilon(1-a))E$. Therefore, if $K_{\mathcal{G}} + \varepsilon N_{\mathcal{G}}^*$ is pseudo-effective then the same holds true for $K_{\mathcal{F}} + \varepsilon N_{\mathcal{F}}^*$. We conclude that $\mathrm{eff}(\mathcal{G}) \le \mathrm{eff}(\mathcal{F})$ and the equality between the effective thresholds follow. The same argument also shows the equality $\mathrm{adj}_{\mathrm{num}}(\mathcal{F}) = \mathrm{adj}_{\mathrm{num}}(\mathcal{G})$.

To conclude the proof of the proposition it suffices to verify that $h^0(X, K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m}) = h^0(Y, K_{\mathcal{G}}^{\otimes n} \otimes N_{\mathcal{G}}^{*\otimes m})$ for every $n, m \ge 0$. Once these equalities are proved, the equality $\mathrm{adj}(\mathcal{F}) = \mathrm{adj}(\mathcal{G})$ follows. Let us fix $n, m \ge 0$. From the isomorphism $K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m} = \pi^*(K_{\mathcal{G}}^{\otimes n} \otimes N_{\mathcal{G}}^{*\otimes m}) \otimes \mathcal{O}_X((na + m(1-a))E)$ we deduce the short exact sequence

$$0 \to \pi^*(K_{\mathcal{G}}^{\otimes n} \otimes N_{\mathcal{G}}^{*\otimes m}) \to K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m} \to \mathcal{O}_E((na + m(1-a))E) \to 0 \,.$$

Since $h^0(E, \mathcal{O}_E((na + (1-a))E) = 0$, we obtain the sought identity. $\qquad\square$

3.4. **Convention.** For an arbitrary foliation $\mathcal{F}$ on a smooth projective surface X we define the adjoint dimension, the numerical adjoint dimension and the effective threshold as the corresponding quantity for any foliation $\mathcal{G}$ with canonical singularities birationally equivalent to $\mathcal{F}$.

4. SINGULARITIES

4.1. **Adjoint discrepancy and ε-canonical singularities.**

Definition 4.1. *Let $\mathcal{F}$ be a foliation on X and let $\pi : Y \to X$ be a birational morphism. Denote by $\mathcal{G}$ the pull-back of $\mathcal{F}$ under π. If E is an exceptional divisor of π then the adjoint discrepancy of $\mathcal{F}$ along E is the function*

$$a(\mathcal{F}, E) : [0, \infty) \longrightarrow \mathbb{R}$$
$$t \longmapsto \mathrm{ord}_E(K_{\mathcal{G}} + t N_{\mathcal{G}}^* - (\pi^* K_{\mathcal{F}} + t \pi^* N_{\mathcal{F}}^*)) \,.$$

Definition 4.2. *Let $\mathcal{F}$ be a foliation on X and $\varepsilon \geq 0$ a real number. A point $x \in X$ is ε–canonical if and only if the adjoint discrepancy of $\mathcal{F}$ along any divisor E over x satisfies $a(\mathcal{F}, E)(t) \geq 0$ for every $t \geq \varepsilon$. The foliation $\mathcal{F}$ is said to have ε–canonical singularities if every point $x \in X$ is ε–canonical. The smallest ε for which $x \in X$ is ε–canonical will be called the canonical threshold of $\mathcal{F}$ at x.*

Proposition 4.3. *Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two foliations on smooth projective surfaces. Assume that $\mathcal{F}$ and $\mathcal{G}$ are birationally equivalent. If both $\mathcal{F}$ and $\mathcal{G}$ have ε-canonical singularities, then for any pair of integers n, m satisfying $m/n \geq \varepsilon$ we have that*

$$h^0(X, K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m}) = h^0(Y, K_{\mathcal{G}}^{\otimes n} \otimes N_{\mathcal{G}}^{*\otimes m}) .$$

In particular, if $\operatorname{eff}(\mathcal{F}) \geq \varepsilon$ then $\operatorname{eff}(\mathcal{F}) = \operatorname{eff}(\mathcal{G})$.

Proof. The proof is completely analogue to the proof of Proposition 3.6. $\qquad\square$

Remark 4.4. *We point out that ε'-canonical singularities are ε-canonical for every $\varepsilon \geq \varepsilon'$. In particular, canonical singularities are ε-canonical singularities for every $\varepsilon \geq 0$. Also note that the canonical threshold of a log canonical singularity is at most $1/2$, i.e. log canonical singularities are ε-canonical for every $\varepsilon \geq 1/2$. This is a straightforward consequence of the simple fact that for every divisor E exceptional over X extracted on a smooth birational surface $\pi \colon Y \to X$ then $\operatorname{ord}_E(K_Y - \pi^* K_X) \in \mathbb{Z}_{>0}$.*

Notation 4.5. *If $p, q \geq 1$ are relatively prime integers then we will write*

$$\frac{p}{q} = [u_0, u_1, \ldots, u_n] = u_0 + \cfrac{1}{u_1 + \cfrac{1}{\cdots + \cfrac{1}{u_n}}}$$

for the continued fraction presentation of their quotient.

Definition 4.6. *Let $p, q \geq 1$ be relatively prime positive integers and consider the germ of foliation on $X = (\mathbb{C}^2, 0)$ defined by $v = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$. Let $\pi : Y \to X$ be the minimal reduction of singularities of $\mathcal{F}$, let $\mathcal{G}$ be the transformed foliation $\pi^*\mathcal{F}$, and let E be the irreducible component of the exceptional divisor which is not $\mathcal{G}$ invariant. We will denote the order of $K_Y - \pi^* K_X$ along E by $\varphi(p, q)$.*

Lemma 4.7. *Notations as in Definition 4.6. If we write $p/q = [u_0, u_1, \ldots, u_n]$ then the following assertions hold true.*

  (1) *π is the composition of exactly $\sum_{i=0}^n u_i$ blow-ups; and*
  (2) *the order of $K_Y - \pi^* K_X$ along E satisfies $\varphi(p, q) \geq \sum_{i=0}^n u_i$.*

Proof. The key observation is that the reduction of singularities of v follows step-by-step Euclid's algorithm for the computation of $\gcd(p, q)$.

Assume that $p \geq q$ and write p/q as a continued fraction $[u_0, u_1, \ldots, u_n]$. The proof will by induction on the number $N = \sum_{i=1}^n u_i$.

If $p = q = 1$ then clearly $N = 1$ and the result is obvious in this case. Assume $p > q$ and consider the blow-up $s \colon Z \to X$ of the origin with exceptional divisor E_0. Over the exceptional divisor we will find two singularities with eigenvalues $(p-q, q)$ and $(p, q-p)$. Since we are assuming that $p > q$ then the pair $(p, q-p)$ corresponds to a canonical singularity while the pair $(p-q, q)$ corresponds to a non-canonical singularity. Observe that

$$\frac{p - q}{q} = [u_0 - 1, u_1, \ldots, u_n]$$

Assuming that the result is true for $N-1$ then the firs part of the statement follows.

To verify item (2), notice that $K_Z = s^* K_X + E_0$. If $\pi \colon Y \to Z$ is the minimal desingularization of $r^* \mathcal{F}$ then by induction hypothesis $\mathrm{ord}_E(K_Y - r^* K_Z) \geq N-1$. Since $\pi = s \circ r$, we can write

$$\mathrm{ord}_E(K_Y - \pi^* K_X) = \mathrm{ord}_E(K_Y - r^*(K_Z - E_0))$$
$$\geq \mathrm{ord}_E(K_Y - r^* K_Z) + \mathrm{ord}_E(r^* E_0) \geq N.$$

Then the Lemma follows by induction. $\qquad \square$

Remark 4.8. *The inequality in part (2) of the Lemma becomes an equality only for singularities with eigenvalues of the form $(1, q)$. If p and q are both strictly greater than one, at some intermediate step we will be forced to blow-up at the intersection of two exceptional divisors and one will get a greater order at the end. For instance, if $p/q = [u_0, u_1]$ then order of $K_Y - \pi^* K_X$ along the last exceptional divisor is $\varphi(p, q) = (u_1 + 1)u_0 - 1$.*

As a consequence of the above description we are able to characterize ε-canonical singularities for small values of $\varepsilon > 0$.

Proposition 4.9. *Let $\mathcal{F}$ be a germ of foliation on $(\mathbb{C}^2, 0)$. If the canonical threshold of $\mathcal{F}$ at 0 is strictly less than $1/4$ then 0 is a log-canonical singularity.*

Proof. Let v be a generator of $T_{\mathcal{F}}$. Assume first that the linear part of v is zero. If $\pi : Y \to (\mathbb{C}^2, 0)$ is the blow-up of the origin, $\mathcal{G} = \pi^* \mathcal{F}$ and E is the exceptional divisor then $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}} - aE$, where $a \geq 1$. On the other hand $N_{\mathcal{G}}^* = \pi^* N_{\mathcal{F}}^* + (a+1)E$. Therefore, if $\varepsilon < 1/2$ then the origin is not ε-canonical.

Assume now that the linear part of v is non-zero but nilpotent. We will use the description of the resolution process of this kind of singularities presented in [5, Chapter 1, proof of Theorem 1]. If we blow-up the origin then we obtain only one singularity over the exceptional divisor which is invariant by the transformed foliation. This new singularity can have zero linear part or non-zero but nilpotent linear part. Let us analyze the two possibilities. Start with the case where the linear part is zero and let $\pi : Y \to (\mathbb{C}^2, 0)$ be the composition of the two obvious blow-ups. As before we will set $\mathcal{G} = \pi * \mathcal{F}$ and will let E_1, E_2 be the two irreducible components of the exceptional divisor of π with E_2 corresponding to the last blow-up. Notice that $K_{\mathcal{G}} = K_{\mathcal{F}} - aE_2$ for some $a \geq 1$ and $N_{\mathcal{G}}^* = \pi^* N_{\mathcal{F}}^* + E_1 + (a+2)E_2$. Hence if $\varepsilon < 1/3$ then 0 is not an ε-canonical singularity. Let us now deal with the second possibility. If the blow-up of a nilpotent singularity with non-zero linear part is still a singularity with these two properties then one further blow-up gives rise to a singularity with trivial linear part. Let now $\pi : Y \to (\mathbb{C}^2, 0)$ be the composition of the three obvious blow-ups, and let E_1, E_2, E_3 be the irreducible components of the exceptional divisor numbered according to the order of appearance. If we set $\mathcal{G} = \pi^* \mathcal{F}$ then $K_{\mathcal{G}} = \pi^* K_{\mathcal{F}} - aE_3$ for some $a \geq 1$ and $N_{\mathcal{G}}^* = \pi^* N_{\mathcal{F}}^* + E_1 + 2E_2 + (a+3)E_3$. Thus if $\varepsilon < 1/4$ then 0 is not a ε-canonical singularity.

Therefore if $\varepsilon < 1/4$ then the linear part of v is non-nilpotent and we can apply [26, Fact I.1.8] to conclude that 0 is a log-canonical singularity of $\mathcal{F}$. $\qquad \square$

Corollary 4.10. *Let $\mathcal{F}$ be a germ of foliation on $(\mathbb{C}^2, 0)$ defined by a germ of vector field v. If $0 < \varepsilon < 1/4$ then 0 is a ε-canonical singularity of $\mathcal{F}$ if and only if the linear part of v is non-nilpotent and one of the following holds:*

   (1) *the singularity of v is canonical; or*
   (2) *the singularity of v is not canonical, v is analytically conjugated to $px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$ with p, q relatively prime positive integers, and $\varphi(p, q) \geq \frac{1-\varepsilon}{\varepsilon}$.*

Proof. Proposition 4.9 implies that the linear part of v is non-nilpotent. If 0 is not a canonical singularity then by [26, Fact I.1.9] we know that v is analytically conjugated to $px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$ for suitable relatively prime positive integers p, q. If $\pi : Y \to X = (\mathbb{C}^2, 0)$ is the minimal reduction of singularities of $\mathcal{F}$, E denotes the last exceptional divisor and $\mathcal{G} = \pi^*\mathcal{F}$ then $K_{\mathcal{G}} = \pi^*K_{\mathcal{F}} - E$. Therefore the adjoint discrepancy of $\mathcal{F}$ along E is (cf. Remark 3.3)

$$a(\mathcal{F}, E)(t) = (1 - t)\operatorname{ord}_E(K_{\mathcal{G}} + \frac{t}{1-t}K_Y - \pi^*(K_{\mathcal{F}} + \frac{t}{1-t}K_X)) =$$

$$= (1 - t)(-1 + \frac{t}{1-t}\varphi(p, q)).$$

Since the adjoint discrepancy is clearly non-negative along all the other divisors in the minimal resolution it follows that 0 is an ε-canonical singularity if and only if $\varphi(p, q) \geq \frac{1-\varepsilon}{\varepsilon}$. $\square$

4.2. **Example: log canonical foliations on the projective plane.** For a foliation $\mathcal{F}$ on the projective plane with log-canonical singularities one can easily verify the following assertions.

- (1) If $d = \deg(\mathcal{F}) \geq 4$ then $\operatorname{eff}(\mathcal{F}) = \frac{d-1}{d+2}$.
- (2) If $d = \deg(\mathcal{F}) = 3$ then $\operatorname{eff}(\mathcal{F}) = 2/5$ unless $\mathcal{F}$ has radial singularities.
- (3) If $d = \deg(\mathcal{F}) = 2$ then $\operatorname{eff}(\mathcal{F}) = 1/4$ unless $\mathcal{F}$ has radial singularities or dicritical singularities of type $(1, 2)$.

One could try to pursue a case-by-case analysis in order to provide an explicit lower bound for the positive effective thresholds of foliations of degree two and three with log-canonical singularities. We will show later in Section 7 that the positive effective thresholds of foliations varying in an algebraic family do not accumulate at zero. Unfortunately, our proof is not effective and, a priori, the bound might depend on the family.

5. Non-isotrivial fibrations

5.1. **Seshadri constants.** Our original motivation to introduce and study the adjoint dimension of foliations lies on our poor understanding of the linear systems $|K_{\mathcal{F}}^{\otimes n}|$. When $\mathcal{F}$ is a foliation of general type we are not aware of lower bounds on n such that $|K_{\mathcal{F}}^{\otimes n}|$ is not empty. For the linear systems $|K_{\mathcal{F}}^{\otimes n} \otimes K_X^{\otimes m}|$ the situation is considerably better. We can apply the current knowledge on adjoint linear systems to obtain effective bounds on n, m such that $|K_{\mathcal{F}}^{\otimes m} \otimes K_X^{\otimes n}|$ defines a rational map with two dimensional image.

To be more precise we recall the definition of Seshadri constants and a pair of fundamental results about them.

Definition 5.1. *Let $\mathcal{L}$ be a nef line-bundle on a projective manifold X and $x \in X$ be a closed point. The Seshadri constant $\varepsilon(X, \mathcal{L}; x) = \varepsilon(\mathcal{L}; x)$ is the non-negative real number*

$$\varepsilon(\mathcal{L}; x) = \max\{\varepsilon \geq 0 \,|\, \mu^*\mathcal{L} - \varepsilon \cdot E \text{ is nef}\},$$

where μ is the blow-up of X at x.

Knowledge of lower bounds of Seshadri constants allows to produce plenty of sections of adjoint linear systems through the use of Kawamata-Viehweg vanishing Theorem.

Proposition 5.2. *Let X be a projective manifold of dimension n and $\mathcal{L}$ be a big and nef line-bundle on X. If $\varepsilon(\mathcal{L}; x) > n + s$ then $K_X + \mathcal{L}$ separates s-jets at x. In particular, if $\varepsilon(\mathcal{L}; x) > n + 1$ then the image of $|K_X \otimes \mathcal{L}|$ has dimension n.*

Proof. This is contained [22, Proposition 5.1.19]. □

Combined with the homogeneity of Seshadri constants, i.e. $\varepsilon(\mathcal{L}^{\otimes m}; x) = m\varepsilon(\mathcal{L}, x)$ for any integer $m > 0$, this result reduces the problem of finding sections of adjoint linear systems to the one of providing a lower bound for Seshadri constants. In this direction we recall the main result of [15].

Theorem 5.3. *Let $\mathcal{L}$ be a nef and big line-bundle on an irreducible projective variety X of dimension n. Then $\varepsilon(\mathcal{L}; x) \geq 1/n$ for all $x \in X$ outside a countable union of proper closed subvarieties.*

5.2. **Producing sections.** We can, rather straightforwardly, apply the results just introduced to produce sections of the linear systems $K_X + nK_{\mathcal{F}}|$ for suitable n.

Proposition 5.4. *Let $\mathcal{F}$ be a foliation with canonical singularities on a smooth projective surface. If $\mathrm{kod}(\mathcal{F}) = 2$ then the linear system $|K_X + 7\,\mathrm{i}(\mathcal{F})K_{\mathcal{F}}|$ defines a rational map with two dimensional image.*

Proof. Suppose first that $\mathrm{kod}(\mathcal{F}) = 2$. Then $\mathrm{i}(\mathcal{F})K_{\mathcal{F}} = \mathrm{i}(\mathcal{F})P + \mathrm{i}(\mathcal{F})N$ is a sum of a nef and big divisor with an effective divisor. Theorem 5.3 implies that the Seshadri constant of $\mathrm{i}(\mathcal{F})P$ is at least $1/2$. Therefore we can apply Proposition 5.2 to guarantee that $|K_X + 7\,\mathrm{i}(\mathcal{F})P|$ defines a rational map with two dimensional image. Then the same holds true for $|K_X + 7\,\mathrm{i}(\mathcal{F})K_{\mathcal{F}}|$, as $7\,\mathrm{i}(\mathcal{F})N$ is an effective Cartier divisor. □

The proposition above is certainly not optimal. There are are many refinements of the results of Section 5.1 in the literature that lead to better constants. See for instance [14] and references therein. The real question underlying the whole issue here is whether or not one can provide universal bounds which do not depend on the index of the foliation. The reader will find a more precise formulation of this question in Problem 6.8.

5.3. **Bound for the index of hyperbolic fibrations.** In order to use the results above to provide explicit bounds for the degree of leaves of non-isotrivial hyperbolic fibrations we need to obtain bounds for the index of the foliation.

Lemma 5.5. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth projective surface X. Assume $\mathcal{F}$ is defined by a fibration $f : X \to C$ and that the general fiber of f has genus at least two. If T is an irreducible curve invariant by $\mathcal{F}$ which intersects the support of the negative part of $K_{\mathcal{F}}$ and it is not contained in it (i.e. T is a tail) then one of the following holds:*

(1) *$P \cdot T = 0$ and T intersects exactly two connected components of the support of N, both of them of order 2; or*
(2) *$P \cdot T \geq \frac{1}{42}$.*

Proof. It follows from Lemma 2.14 that

$$(5.1) \qquad P \cdot T = K_{\mathcal{F}} \cdot T - \sum_{i=1}^{k} \frac{1}{o_i} = -\chi(T) + s + k - \sum_{i=1}^{k} \frac{1}{o_i}$$

where s is the number of singularities of $\mathcal{F}$ on T which do are not contained in the support of N, [5, Chapter 2, Prop. 3].

Assume $P \cdot T = 0$. If $s = 0$ then we have the following possibilities for k and $o = (o_1, \ldots, o_k)$: $k = 3$ and $o = (3, 3, 3)$; or $k = 3$ and $o = (2, 3, 6)$; or $k = 4$ and $o = (2, 2, 2, 2)$. In all cases the whole fiber F containg T is the union of k Hirzebruch-Jung strings joined by a single common tail T and $\chi(F) = \chi_{orb}(\tilde{T}) = 0$. Since $\chi(F) < 0$

by assumption, we get that $P \cdot T > 0$ contradicting our assumption. The only remaining possibility is $s = 1$, $k = 2$ and $o = (2, 2)$. Item (1) follows.

If $P \cdot T > 0$ then it is an elementary and well known fact that the lower bound for (5.1) is equal to $1/42$ and is attained by $s = 0$, $k = 3$, and $o = (2, 3, 7)$. $\qquad \square$

Proposition 5.6. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth projective surface X. Assume $\mathcal{F}$ is defined by a fibration $f : X \to C$ and that the general fiber of f has genus $g \geq 2$. Then*

$$\mathrm{i}(\mathcal{F}) \leq (42(2g - 2))! \,.$$

Proof. Let $F = \sum m_i C_i$ be a fiber of f and let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$. If C_i is a tail then, according to Lemma 5.5, either the Hirzebruch-Jung strings intersecting it have order two or $P \cdot C_i \geq 1/42$. In the later case we get that $m_i \leq 42(2g - 2)$ since $P \cdot F = K_{\mathcal{F}} \cdot F = -\chi(F) = 2g - 2$. Moreover, Lemma 2.14 item (b) implies that the least common multiple of the orders of the Hirzebruch-Jung strings intersecting C_i divides $m_i \leq 42(2g - 2)$. The Lemma follows. $\qquad \square$

5.4. Boundness of fibers of non-isotrivial fibrations of a given genus.
Theorem A of the Introduction will follow rather easily from the more general result below.

Theorem 5.7. *Let $\mathcal{F}$ be a foliation with canonical singularities on a projective surface X. Suppose that $\mathcal{F}$ is a fibration with general fiber F of genus g. If $\mathrm{kod}(\mathcal{F}) = 2$ (i.e. the fibration is a non isotrivial hyperbolic fibration) then for every big and nef divisor H we have*

$$F \cdot H \leq M(K_X + 7\,\mathrm{i}(\mathcal{F})K_{\mathcal{F}}) \cdot H,$$

where $M = M(g)$ satisfies the following inequality

$$M \leq 2(7\,\mathrm{i}(\mathcal{F}) + 1)(2g - 2) \leq \big(7\big(42(2g - 2)\big)! + 1\big)(4g - 4)\,.$$

Proof. Let $\mathcal{L} = K_X \otimes K_{\mathcal{F}}^{\otimes 7\,\mathrm{i}(\mathcal{F})}$ and F be a general leaf of $\mathcal{F}$. If $m \geq 1$ is an integer then $\mathcal{L}_{|F}^{\otimes m} = K_F^{\otimes m(7\,\mathrm{i}(\mathcal{F})+1)}$. On the one hand, Riemann-Roch Theorem implies that

$$h^0(F, \mathcal{L}_{|F}^{\otimes m}) = m(7\,\mathrm{i}(\mathcal{F}) + 1)(2g - 2) - g + 1.$$

On the other hand, since according to Theorem 5.3 the linear system $|\mathcal{L}|$ defines a rational map with two dimensional image, $h^0(X, \mathcal{L}^{\otimes m}) \geq \binom{m+2}{2}$. If we take $M = 2(7\,\mathrm{i}(\mathcal{F}) + 1)(2g(F) - 2)$ then $h^0(X, \mathcal{L}^{\otimes M}) - h^0(F, \mathcal{L}_{|F}^{\otimes M}) \geq \binom{M+2}{2} - M(7\,\mathrm{i}(\mathcal{F}) + 1)(2g - 2) + g - 1 = g$. In particular, there exists a non-zero section σ of $\mathcal{L}^{\otimes M}$ vanishing on F.

If H is an arbitrary big and nef divisor on X then the intersection of F with H is bounded by the intersection of the divisor cut out by σ with H. But the later intersection number is nothing but $M(K_X + 7\,\mathrm{i}(\mathcal{F})K_{\mathcal{F}}) \cdot H$. Proposition 5.6 then concludes the proof. $\qquad \square$

5.5. Proof of Theorem A.
Let $\mathcal{F}$ be a foliation of $\mathbb{P}^2$. Notice that its canonical bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(\deg(\mathcal{F}) - 1)$. Let $\pi : X \to \mathbb{P}^2$ a birational morphism such that all the singularities of $\mathcal{G} = \pi^* \mathcal{F}$ are canonical. If we take $H = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$ then the degree of an algebraic leaf L of $\mathcal{F}$ is given by

$$\deg(L) = H \cdot \pi^* L = H \cdot \hat{L}\,,$$

where $\hat{L}$ is the strict transform of L. We can thus apply Theorem 5.7 to deduce that

$$\begin{aligned}
\deg(L) &\leq \big(7\big(42(2g-2)\big)! + 1\big)(4g-4)(K_X + 7\,\mathrm{i}(\mathcal{F})K_{\mathcal{F}}) \cdot H \\
&\leq \big(7\big(42(2g-2)\big)! + 1\big)\big(-3 + 7\big(42(2g-2)\big)!(\deg(\mathcal{F}) - 1)\big) \\
&\leq \Big(7\big(42(2g-2)\big)!\Big)^2 \deg(\mathcal{F})\,.
\end{aligned}$$

This concludes the proof of Theorem A. $\qquad\square$

5.6. **Log canonical foliations on $\mathbb{P}^2$ of high degree.** The bounds appearing in Theorem 5.7 are ridiculously large and far from optimal. Proposition 5.8 below combined with the results presented in Section 7 (notably Theorem 7.5) indicate that the dependence of M on g in Theorem 5.7 should be at worst linear on g. The results of [25] also indicate the existence of such linear bounds which are not universal but depend on the family of foliations in question.

Proposition 5.8. *Let $\mathcal{F}$ be a foliation with canonical singularities on a projective surface X. Assume that $\mathcal{F}$ is a fibration with general fiber F of geometric genus $g \geq 2$ and that $H^0(X, K_{\mathcal{F}}^{\otimes a} \otimes N_{\mathcal{F}}^{*\,\otimes b})$ admits three algebraically independent sections for some $a > 0$ and $b \geq 0$. Then for every nef divisor H we have*

$$F \cdot H \leq 2a(2g-2)(aK_{\mathcal{F}} + bN_{\mathcal{F}}^*) \cdot H\,.$$

Proof. Let $\mathcal{L} = K_{\mathcal{F}}^{\otimes a} \otimes N_{\mathcal{F}}^{*\,\otimes b}$ and F be a general leaf of $\mathcal{F}$. If $m \geq 1$ is an integer then $\mathcal{L}_{|F}^{\otimes m} = K_F^{\otimes am}$. On the one hand, by Riemann-Roch Theorem

$$h^0(F, \mathcal{L}_{|F}^{\otimes m}) = ma(2g-2) - g + 1\,.$$

On the other hand, our assumption on $H^0(X, \mathcal{L})$ implies that $h^0(X, \mathcal{L}^{\otimes m}) \geq \binom{m+2}{2}$. If we take $m = 2a(2g-2)$ then

$$\begin{aligned}
h^0(X, \mathcal{L}^{\otimes m}) - h^0(F, \mathcal{L}_{|F}^{\otimes m}) &\geq \binom{2a(2g-2)+2}{2} - 2a^2(2g-2)^2 + (g-1) \\
&= 6a(g-1) + g > 0\,.
\end{aligned}$$

In particular, there exists a non-zero section σ of $\mathcal{L}^{\otimes 2a(2g-2)}$ vanishing on F.

If H is an arbitrary nef divisor on X then the intersection of F with H is bounded by the intersection of the divisor cut out by σ with H. But this intersection number is $2a(2g-2)(aK_{\mathcal{F}} + bN_{\mathcal{F}}^*) \cdot H$. $\qquad\square$

In the case of foliations of the projective plane with log canonical singularities and of degree greater or equal to 5, we can actually obtain bounds that are better than linear using a simple variation of the argument used to prove Proposition 5.8.

Theorem 5.9. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ of degree $d \geq 5$. Assume that $\mathcal{F}$ has log canonical singularities and admits a rational first integral with general fiber of geometric genus $g \geq 2$. If F is a general leaf of $\mathcal{F}$ then*

$$\deg(F) \leq \left\lceil \frac{4(2g-2)}{(d-4)^2} \right\rceil (d-4)\,.$$

Proof. Since the singularities of $\mathcal{F}$ are ε-canonical for $\varepsilon = 1/2$ (see Remark 4.4) we have that the dimension of the vector spaces $H^0(\mathbb{P}^2, K_{\mathcal{F}}^{\otimes 2m} \otimes N_{\mathcal{F}}^{*\,\otimes m})$, $m > 0$ is unaltered after replacing $\mathcal{F}$ by a model with at worst canonical singularities.

Let F be a general fiber of the rational first integral of $\mathcal{F}$ and consider the real valued function

$$f(m) = \binom{m(d-4)+2}{2} - 2m(2g-2) - g + 1.$$

Its values on positive integers correspond to the difference $h^0(\mathbb{P}^2, K_{\mathcal{F}}^{\otimes 2m} \otimes N_{\mathcal{F}}^{*\otimes m}) - h^0(\tilde{F}, K_{\tilde{F}}^{\otimes 2m})$, where $\tilde{F}$ is the normalization of F. Since $f(4(2g-2)/(d-4)^2) = (dg + 8g - 12)/(d-4)$ which is clearly positive and moreover the derivative of f satisfies $f'(4(2g-2)/(d-4)^2) = (3/2)d + 4g - 10 > 0$, it follows that if m is the smallest integer greater than $4(2g-2)/(d-4)^2$ then there exists a section of $K_{\mathcal{F}}^{\otimes 2m} \otimes N_{\mathcal{F}}^{*\otimes m} \simeq \mathcal{O}_{\mathbb{P}^2}(m(d-4))$ vanishing identically on F. The Theorem follows. $\qquad\square$

As already mentioned in the Introduction, this Theorem 5.9 refines a classical result of Poincaré, see [32, pages 169 and 176] and [30, Chapter 7, Corollary 14].

6. CLASSIFICATION VIA ADJOINT DIMENSION

In this section we apply the results recalled in Section 2 to obtain a classification of foliations on surfaces according to their adjoint dimension.

6.1. K_X-**negative extremal rays.** Recall that for a smooth projective surface X the K_X-negative extremal rays are spanned by numerical classes of rational curves of self-intersection either $-1, 0$ or 1. The first case corresponds to the exceptional divisor of the blow-up of a smooth point, the second to a smooth fiber of a $\mathbb{P}^1$-bundle, while the last one is just the class of a line in $\mathbb{P}^2$.

Lemma 6.1. *Let $\mathcal{F}$ be relatively minimal foliation with pseudo-effective $K_{\mathcal{F}}$ on a smooth projective surface X, and let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$. Assume there exists a K_X–negative extremal curve $C \subset X$ and $P \cdot C = 0$. Then the Kodaira dimension of $\mathcal{F}$ is either 0 or 1. Moreover, if $\mathrm{kod}(\mathcal{F}) = 1$, then the image of C in the canonical model $\pi : X \to Z$ of $\mathcal{F}$ is proportional to $\pi_* K_{\mathcal{F}}$.*

Proof. If C is an extremal ray with $C^2 \geq 1$ then Hodge index theorem implies that P is numerically zero. Theorem 2.18 implies $\mathrm{kod}(\mathcal{F}) = 0$.

If instead $C^2 = 0$ then P is numerically proportional to a non-negative multiple of C and we deduce that either $\nu(\mathcal{F}) = 0$ or $\nu(\mathcal{F}) = 1$. The case $\nu(\mathcal{F}) = 0$ follows as before. If $\nu(\mathcal{F}) = 1$ and since P is numerically proportional to an effective divisor, we can apply Theorem 2.20 and Lemma 2.21 to deduce that $\mathrm{kod}(\mathcal{F}) = 1$.

From now on assume that $C^2 = -1$ and let $\pi : X \to Y$ be the contraction of $\mathcal{F}$ into its canonical model. If C is not contracted by π then write $\pi^* \pi_* C = C + \sum a_i E_i$ where $a_i > 0$ and the E_i are π-exceptional divisors. Thus $\pi_* P \cdot \pi_* C = P \cdot \pi^* \pi_* C = P \cdot C$ since P is the pull-back of a nef divisor from Y and hence π-exceptional curves intersect P trivially. As we are assuming $P \cdot C = 0$ we deduce from Hodge index Theorem that either P is numerically trivial, or that $\pi_* C^2 = 0$ and $\pi_* P$ is numerically equivalent to a positive multiple of $\pi_* C$. Hence $\nu(\mathcal{F}) \in \{0, 1\}$. As before, we obtain that in both cases $\nu(\mathcal{F}) = \mathrm{kod}(\mathcal{F})$.

Suppose now that C is contracted by π. In this case C is $\mathcal{F}$-invariant according to Theorem 2.17. Since $C^2 = -1$ and $\mathcal{F}$ is relatively minimal we have that $Z(\mathcal{F}, C) \geq 3$. Notice that $K_{\mathcal{F}} \cdot C = -2 + Z(\mathcal{F}, C)$ and, as we are assuming $P \cdot C = 0$, according to Lemma 2.14 we also have that $K_{\mathcal{F}} \cdot C = \sum_{i=1}^k 1/o_i$ where o_i are the orders of the Hirzebruch-Jung strings intersecting C. Then we must have $k = 2$ and $o_1 = o_2 = 2$; or $k = 3$ and $(o_1, o_2, o_3) \in \{(2, 3, 6), (3, 3, 3)\}$; or $k = 4$ and $(o_1, o_2, o_3, o_4) = (2, 2, 2, 2)$.

If we contract the Hirzebruch-Jung strings intersecting C, we obtain that the direct image of C has self-intersection ≥ 0, cf. [26, Remark III.2.2]. Thus C cannot be contracted by π contrary to our assumption. □

6.2. **Kodaira dimension zero.**

Lemma 6.2. *Let $\mathcal{F}$ be a relatively minimal foliation with pseudo-effective $K_{\mathcal{F}}$ on a smooth projective surface X. If $\pi : X \to Z$ is the contraction of the negative part of $K_{\mathcal{F}}$ (i.e. $\pi_* \mathcal{F}$ is a nef model of $\mathcal{F}$) and we write $K_X + \Delta = \pi^* K_Z$ then $i(\mathcal{F})N - \Delta$ is effective.*

Proof. If $E_1, \ldots, E_k$ are the exceptional divisors of π then Δ is defined by the relations

$$\Delta \cdot E_i = -K_X \cdot E_i = 2 + E_i^2 \, .$$

Notice that $2 + E_i^2 \leq 0$ for every i, while $2 + E_i^2 \geq (E_1 + \cdots + E_k) \cdot E_i$ for every i and the latter inequality is strict when E_i is either a handle or a tail in a Hirzebruch-Jung string. Therefore by [21, Corollary 4.2] the coefficients of Δ lie in $[0, 1)$. Since N is effective the lemma follows. □

Proposition 6.3. *Let $\mathcal{F}$ be a relatively minimal foliation of Kodaira dimension zero on a smooth projective surface X. If $\pi : X \to Z$ is the contraction of the negative part of the Zariski decomposition of $K_{\mathcal{F}}$ and (X, Δ) is the pair satisfying $K_X + \Delta = \pi^* K_Z$ then the adjoint dimension and the numerical adjoint dimension of $\mathcal{F}$ coincide with the Kodaira dimension of (X, Δ). Moreover, when $\mathrm{adj}(\mathcal{F}) \geq 0$ then $\mathrm{eff}(\mathcal{F}) \geq \frac{1}{i(\mathcal{F})+1} \geq \frac{1}{13}$.*

Proof. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$. Since we are assuming that $\mathcal{F}$ has Kodaira dimension zero we have that $P = 0$. Let $\pi : X \to Z$ be the contraction of the support of N and notice that we can write

$$K_{\mathcal{F}} + \varepsilon K_X = \varepsilon \pi^* K_Z + (N - \varepsilon \Delta).$$

Assume that ε is rational and satisfies $\varepsilon < 1/i(\mathcal{F})$. Lemma 6.2 implies that $(N - \varepsilon\Delta)$ is effective. Hence for any k sufficiently divisible, $h^0(X, k(\varepsilon\pi^* K_Z + (N - \varepsilon\Delta))) \geq h^0(X, k\pi^* K_Z) = h^0(Z, kK_Z)$. Since every irreducible component E of the support of $(N - \varepsilon\Delta)$ is π-exceptional we also have the opposite inequality. This shows that the Kodaira dimension of Z is equal to the adjoint dimension of $\mathcal{F}$.

To verify that the adjoint dimension and the numerical adjoint dimension of $\mathcal{F}$ coincide first observe that every irreducible component E of of $N - \varepsilon\Delta$ satisfies $\pi^* K_Z \cdot E = 0$. Therefore the numerical dimension of $K_{\mathcal{F}} + \varepsilon K_X$ coincides with the numerical dimension of K_Z. As the numerical dimension of K_Z and the Kodaira dimension of (X, Δ) coincide, the Proposition follows. □

6.3. **Kodaira dimension one.**

Proposition 6.4. *Let $\mathcal{F}$ be a relatively minimal foliation of Kodaira dimension one on a smooth projective surface X. Let g be the genus of a general fiber of the Iitaka's fibration of $\mathcal{F}$. If $g = 0$ then $\mathrm{adj}(\mathcal{F}) = \mathrm{adj}_{\mathrm{num}}(\mathcal{F}) = -\infty$. Otherwise*

$$\mathrm{adj}(\mathcal{F}) = \mathrm{adj}_{\mathrm{num}}(\mathcal{F}) = \min\{g, 2\} \text{ and } \mathrm{eff}(\mathcal{F}) \geq \frac{1}{4\,i(\mathcal{F}) + 1}.$$

Proof. Let $f : X \to B$ be the Iitaka's fibration of $\mathcal{F}$. Assume first that $g = 0$. Then for a general fiber F of f we have that $K_{\mathcal{F}} \cdot F = 0$ and $K_X \cdot F = -2$. Hence $K_{\mathcal{F}} + \varepsilon K_X$ is not pseudoeffective for every $\varepsilon > 0$. It follows that $\mathrm{adj}(\mathcal{F}) = \mathrm{adj}_{\mathrm{num}}(\mathcal{F}) = -\infty$.

Assume now that $g \geq 1$. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$ and let $\pi : X \to Z$ be the contraction of the negative part of $K_{\mathcal{F}}$. Denote by $\mathcal{G}$ the direct image of

$\mathcal{F}$. We claim that $4\,\mathrm{i}(\mathcal{F})K_{\mathcal{G}}+K_Z$ is nef. Suppose not, and let D be an effective divisor such that $(4\,\mathrm{i}(\mathcal{F})K_{\mathcal{G}}+K_Z)\cdot D < 0$. By the Cone Theorem we can numerically decompose D as a sum $\sum a_i C_i + R$ where R is a pseudo-effective divisor and satisfies $K_X \cdot R \geq 0$; C_i are K_Z–negative extremal rays satisfying $0 < -K_Z \cdot C_i \leq 4$ and $a_i \in \mathbb{R}_{>0}$. Therefore, there exists a K_Z–negative extremal ray C such that $(4\,\mathrm{i}(\mathcal{F})K_{\mathcal{G}}+K_Z)\cdot C < 0$. If $K_{\mathcal{G}}\cdot C = 0$ then Lemma 6.1 implies that C is numerically proportional to $K_{\mathcal{G}}$. Consequently C is proportional to a general fiber of $f \circ \pi^{-1}$ and must intersect K_Z non-negatively. Thus $K_{\mathcal{G}}\cdot C > 0$. Since $\mathrm{i}(F)K_{\mathcal{G}}$ is Cartier we deduce that $4\,\mathrm{i}(\mathcal{F})K_{\mathcal{G}}\cdot C \geq 4$. It follows that also in this case $(4\,\mathrm{i}(\mathcal{F})K_{\mathcal{G}}+K_Z)\cdot C \geq 0$. We conclude that $4\,\mathrm{i}(\mathcal{F})K_{\mathcal{G}}+K_Z$ is nef. Consequently we obtain that

$$(6.1) \qquad K_{\mathcal{F}} + \frac{1}{4\,\mathrm{i}(\mathcal{F})}K_X = \pi^*\left(K_{\mathcal{G}} + \frac{1}{4\,\mathrm{i}(\mathcal{F})}K_Z\right) + \left(N - \frac{1}{4\,\mathrm{i}(\mathcal{F})}\Delta\right)$$

where Δ is defined by $K_X + \Delta = \pi^* K_Z$. Since the singularities of Z are klt, it follows that $N - \frac{1}{4\,\mathrm{i}(\mathcal{F})}\Delta$ is effective and that $K_{\mathcal{F}} + \frac{1}{4\,\mathrm{i}(\mathcal{F})}K_X$ is pseudo-effective. Thus $\mathrm{eff}(\mathcal{F}) \geq \frac{1}{4\,\mathrm{i}(\mathcal{F})+1}$.

It remains to determine the adjoint dimension of $\mathcal{F}$. For that, notice that (6.1) is the Zariski decomposition of $K_{\mathcal{F}} + \frac{1}{4\,\mathrm{i}(\mathcal{F})}K_X$. When $g = 1$, since K_X is trivial when restricted to the general fiber of f it follows that the positive part $\pi^*\left(K_{\mathcal{G}} + \frac{1}{4\,\mathrm{i}(\mathcal{F})}K_Z\right)$ is numerically proportional a general fiber and also that there exists an a effective $\mathbb{Q}$-divisor D on B such that $\pi^*\left(K_{\mathcal{G}} + \frac{1}{4\,\mathrm{i}(\mathcal{F})}K_Z\right) = f^*B$. Hence $\mathrm{adj}_{\mathrm{num}}(\mathcal{F}) = \mathrm{adj}(\mathcal{F}) = 1$.

To prove the claim for $g \geq 2$ it suffices to verify that $\pi^*(K_{\mathcal{G}} + \varepsilon K_Z)^2 > 0$ for ε sufficiently small. If this were not the case then $K_{\mathcal{G}} \cdot K_Z = 0$ and $K_Z \cdot K_Z = 0$. Hodge index theorem would imply that $\pi^* K_Z$ is proportional to a general fiber f. But this is not possible since $\pi^* K_Z \cdot F = 2g - 2 > 0$ for any fiber F of f. $\qquad \square$

6.4. **Kodaira dimension two and non-abundant foliations.**

Lemma 6.5. *Let $\mathcal{F}$ be a relatively minimal foliation with canonical singularities which is not a fibration by rational curves. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$. If $\mathrm{kod}(\mathcal{F}) \notin \{0, 1\}$ then $P + \frac{1}{3\,\mathrm{i}(\mathcal{F})}K_X$ is nef.*

Proof. Aiming at a contradiction, let C be a curve such that $(P + 1/3\,\mathrm{i}(\mathcal{F})K_X)\cdot C < 0$. As in the proof of Proposition 6.3 we can assume that C is a K_X-negative extremal curve and therefore $K_X \cdot C \in \{-3, -2, -1\}$. By Lemma 6.1, $P \cdot C > 0$. Hence

$$-K_X \cdot C_i > 3\,\mathrm{i}(\mathcal{F})(P \cdot C_i) \geq 3$$

gives the sought contradiction. $\qquad \square$

Proposition 6.6. *Let $\mathcal{F}$ be a relatively minimal foliation with canonical singularities and pseudo-effective canonical bundle. If $\mathrm{kod}(\mathcal{F}) \notin \{0, 1\}$ then $\mathrm{adj}_{\mathrm{num}}(\mathcal{F}) = \mathrm{adj}(\mathcal{F}) = 2$.*

Proof. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$. Since $\mathrm{kod}(\mathcal{F}) \neq 0$ we have that $\nu(\mathcal{F}) \geq 1$. Lemma 6.5 implies that $P + \varepsilon K_X$ is nef for ε sufficiently small. If $\mathcal{F}$ is not of adjoint general type then $(P + \varepsilon K_X)^2$ must vanish identically. It follows $P^2 = P \cdot K_X = K_X{}^2 = 0$. Lemma 2.21 implies that $\mathrm{kod}(\mathcal{F}) \geq 0$. Since this is excluded by assumption, the result follows. $\qquad \square$

6.5. **Characterization of rational fibrations (Proof of Theorem B).** One immediate consequence of the classification of foliations according to their adjoint dimension is the characterization of rational fibrations stated in the Introduction as Theorem B.

Theorem 6.7. *Let $\mathcal{F}$ be a foliation with canonical singularities on a smooth projective surface X. Then $\mathcal{F}$ is a rational fibration if and only if $h^0(X, K_{\mathcal{F}}^{\otimes m} \otimes N_{\mathcal{F}}^{*\otimes n}) = 0$ for every $m > 0$ and every $n \geq 0$.*

Proof. If $\mathrm{adj}(\mathcal{F}) \geq 0$ then $h^0(X, K_{\mathcal{F}}^{\otimes m} \otimes N_{\mathcal{F}}^{*\otimes n}) \neq 0$ for some $m, n > 0$ by definition. If instead $\mathrm{adj}(\mathcal{F}) = -\infty$ and $\mathcal{F}$ is not a fibration by rational curves then $\mathcal{F}$ is either a finite quotient of a Riccati foliation of Kodaira dimension zero or $\mathcal{F}$ is a Riccati foliation of Kodaira dimension one. In both cases $h^0(X, K_{\mathcal{F}}^{\otimes m}) \neq 0$ for some $m > 0$. $\qquad\square$

For foliations on smooth surfaces of Kodaira dimension 0 or 1, $h^0(X, K_{\mathcal{F}}^{\otimes n}) > 0$ for some n between 1 and 12, see [31] and [11, Section 4]. It is a simple matter to obtain effective non-vanishing of $h^0(X, K_{\mathcal{F}}^{\otimes n} \otimes N_{\mathcal{F}}^{*\otimes m})$ for foliations $\mathcal{F}$ of adjoint general type as functions of their index $\mathrm{i}(\mathcal{F})$. This is what we did in the proof of 5.4 when $\kappa(\mathcal{F}) = 2$. The real question here is if one can do that that regardless of the index of the the foliation.

Problem 6.8. *Find universal bounds on $(n, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ in order to ensure the non-vanishing of $h^0(X, K_{\mathcal{F}}^{\otimes m} \otimes N_{\mathcal{F}}^{*\otimes n})$ for foliations of adjoint general type.*

For bounded families of foliations, the results of Section 7 imply the existence of bounds depending on the family.

7. VARIATION IN MODULI

7.1. **Families of foliations.** We start by spelling out the definition of family of foliated surfaces.

Definition 7.1. *Let $\pi : \mathscr{X} \to T$ be a family of smooth projective surfaces, i.e. $\mathscr{X}$ and T are irreducible complex manifolds and π is a proper submersion with projective surfaces as fibers. A family of foliations parametrized by T is a foliation $\mathscr{F}$ of dimension one on $\mathscr{X}$ which is everywhere tangent to the fibers of π. If $\mathscr{X}, T, \pi$ and $\mathscr{F}$ are all algebraic then we say that $\mathscr{F}$ is an algebraic family of foliations.*

Notice that in the definition above we do not impose any condition on the nature of singularities of $\mathscr{F}$, contrary to what is done in [4]. Also when the dimension of T is at least two it may happen that some fibers of π are contained in the singular set of $\mathscr{F}$.

It is useful to think of an algebraic family of foliations parametrized by T as a foliation defined over the function field $\mathbb{C}(T)$. Algebraic properties of a very general member $\mathscr{F}_t$ of the family – like existence of invariant algebraic curves, rational first integrals, transversely projective structures – are displayed already when one considers the foliation as defined over $\mathbb{C}(T)$. Also the Kodaira dimension (resp. the adjoint dimension) of the foliation defined over $\mathbb{C}(T)$ coincides with the Kodaira dimension (adjoint dimension) of a very general member of the family.

7.2. **Partial reduction of singularities for families.** One of the sources of difficulties of applying birational techniques to understand the behavior of the plurigenera in families of foliations comes from the fact that canonical singularities are not stable in the Zariski topology, i.e. the set of foliations with at worst canonical singularities can fail to be Zariski open as the family of foliations on $\mathbb{C}^2$ parametrized by $\mathbb{C}$ and defined by $xdy - tydx$ shows. In this family the singularity at the origin is canonical if and only if $t \notin \mathbb{Q}_+$. Thus

a very general foliation in the family has canonical singularities, but the set of foliations with non-canonical singularities is Zariski dense. This unpleasant situation can be avoided if instead one considers ε-canonical singularities for $\varepsilon > 0$.

Lemma 7.2. *Let $\mathscr{F}$ be an algebraic family of foliations parametrized by an algebraic variety T. If $0 < \varepsilon < 1/4$ then the subset of T corresponding to foliations with isolated and ε-canonical singularities is a Zariski open subset of T.*

Proof. This is a simple consequence Corollary 4.10. If a singularity is not ε-canonical, $0 < \varepsilon < 1/4$, then either its linear part is nilpotent or the singularity is formally equivalent to one of finitely many singularities of the form $px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$ with p, q relatively prime positive integers satisfying $\varphi(p, q) < \frac{\varepsilon}{1-\varepsilon}$ (see Definition 4.6 for the meaning of φ). Since both conditions are clearly closed the lemma follows. $\square$

Proposition 7.3. *Given an algebraic family of foliation $\mathscr{F}$ parametrized by an algebraic variety T and a real number $\varepsilon > 0$, there exists a Zariski open subset $U \subset T$ and a family of foliations $\mathscr{G}$ on $\mathscr{Y} \to U$ obtained from $\mathscr{F}_{|U}$ by a finite composition of blow-ups over (multi)-sections such that for every closed point $t \in U$, the foliation $\mathscr{G}_t$ has at worst ε-canonical singularities.*

Proof. First consider $\mathscr{F}$ as foliation defined over $\mathbb{C}(T)$ and apply Seidenberg's Theorem to obtain a foliation over $\mathbb{C}(T)$ with reduced singularities. Then restrict to a Zariski open subset of T in order to guarantee that we still have a family of foliations in the sense of Definition 7.1 and apply Lemma 7.2 to conclude. $\square$

7.3. **Families of foliations of negative adjoint dimension.** Foliations of negative adjoint dimension also behave better in families compared to foliations of negative Kodaira dimension.

Lemma 7.4. *Let $(\pi : \mathscr{X} \to T, \mathscr{F})$ be an algebraic family of foliations. If for a very general closed point $t_0 \in T$ the foliation $\mathscr{F}_{t_0}$ is reduced and has negative adjoint dimension then there exists a Zariski open subset $U \subset T$ such that for every closed point $t \in U$ the foliation $\mathscr{F}_t$ has negative adjoint dimension.*

Proof. Assume first that for a very general point $t \in T$ the foliation $\mathscr{F}_t$ has Kodaira dimension one. Since the adjoint dimension is negative, $\mathscr{F}_t$ must be a Riccati foliation. It follows from [11, Proposition 4.3] that for some $n \leq 42$ the linear system $|K_{\mathscr{F}_t}^{\otimes n}|$ is non-empty and defines the reference rational fibration. Moreover, the general fiber of the reference fibration intersects $K_{\mathscr{F}_t}$ trivially. By semi-continuity the same holds true over a Zariski open subset U of T. Consequently we can apply [5, Proposition 4.1] to deduce that for every $t \in U$ the foliation $\mathscr{F}_t$ is a Riccati foliation and as such has negative adjoint dimension.

Assume now that for a very general point $t \in T$ the foliation $\mathscr{F}_t$ has Kodaira dimension zero. Interpret $\mathscr{F}$ as a foliation defined over $\overline{\mathbb{C}(T)}$ and apply Theorem 2.18. We deduce that after restricting T to a Zariski open subset U and base changing the family $\mathscr{F}$ through an étale covering $V \to U$ we obtain that the resulting family $\mathscr{X}' \to U$ is birationally equivalent to a finite quotient of a smooth family of foliations $\mathscr{G}$ on $\mathscr{Z} \to V$ defined by global holomorphic vector fields. Since we are assuming that for a very general $t \in T$ the foliation has negative adjoint dimension it follows that the very general fiber of $\mathscr{Z} \to V$ is a surface of negative Kodaira dimension and the corresponding foliation is a Riccati foliation. It follows that for every $t \in U$, $\mathscr{F}_t$ has negative adjoint dimension.

Finally, if for a very general $t \in T$ the foliation $\mathscr{F}_t$ is a rational fibration then for every $t \in T$ the foliation admits a rational first integral, and by semi-continuity of the genus of curves, for every $t \in T$ the foliation $\mathscr{F}_t$ is birationally equivalent to a rational fibration. □

7.4. **Boundness of the effective threshold in families.** We have now all the ingredients to prove the result mentioned at the end of Section 4.2.

Theorem 7.5. *Let* $(\pi : \mathscr{X} \to T, \mathscr{F})$ *be an algebraic family of foliations. Then there exists* $\delta > 0$ *such that, for every* $t \in T$, *the following holds true:* $\mathrm{adj}(\mathscr{F}_t) = -\infty$ *or* $\mathrm{eff}(\mathscr{F}_t) \geq \delta$. *In other words, if* $\mathrm{eff}(\mathscr{F}_t) < \delta$ *then* $\mathrm{adj}(\mathscr{F}_t) = -\infty$.

Proof. Proposition 7.3 guarantees that there is no loss of generality in assuming that $\mathscr{F}_t$ has canonical singularities for a very general $t \in T$.

If $\mathrm{adj}(\mathscr{F}_t) \geq 0$ for a very general $t \in T$ then there exists $m, n > 0$ such that $h^0(\mathscr{X}_t, K_{\mathscr{F}_t}^{\otimes m} \otimes N_{\mathscr{F}_t}^{*\otimes n}) > 0$ for a very general $t \in T$. Choose $\varepsilon > 0$ small enough and apply Proposition 7.3 to obtain a Zariski open $U \subset T$ such that $\mathscr{F}_t$ has at worst ε-canonical singularities for every $t \in U$. By semi-continuity it follows that $\mathrm{eff}(\mathscr{F}_t) \geq \frac{n}{m}$ for every $t \in U$.

If instead $\mathrm{adj}(\mathscr{F}_t) = -\infty$ for a very general $t \in T$ then Lemma 7.4 implies that the same holds true for every t in a Zariski open subset of T.

In any case, we have just proved that the result is true for the restriction of $\mathscr{F}$ to a Zariski open subset of T. The Theorem follows by Noetherian induction. □

8. FOLIATIONS WITH RATIONAL FIRST INTEGRALS

This section is devoted to the proof of the following result.

Theorem 8.1. *Let* $(\pi : \mathscr{X} \to T, \mathscr{F})$ *be an algebraic family of foliations and* $g \geq 0$ *be an integer. Let* $\Sigma_g \subset T$ *be the Zariski closure of the set of parameters corresponding to foliations birationally equivalent to a fibration of geometric genus at most* g. *Then for every* $t \in \Sigma_g$ *the foliation* $\mathscr{F}_t$ *is transversely projective.*

If one considers the universal family of degree d foliations on $\mathbb{P}^2$ then one promptly realizes that Theorem C is nothing but a particular case of this more general statement.

8.1. **Example.** Before dealing with the proof of Theorem 8.1 let us analyze the Zariski closure of the set of foliations admitting a rational first integral in a family derived from Gauss hypergeometric equation.

Whenever $c \notin \mathbb{Z}$, Gauss hypergeometric equation

$$z(1 - z)w'' + (c - (a + b + 1)z)w' - abw = 0,$$

admits as general solution in a neighboorhud of the origin the function

$$\varphi(z) = C_1 F(a, b, c; z) + C_2 z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z),$$

where C_1, C_2 are arbitrary constants to be determined by boundary conditions and

$$F(a, b, c; z) = 1 + \sum \frac{(a)_n (b)_n}{(c)_n} z^n, \quad (p)_n := p(p+1)(p+2) \cdots (p + n - 1).$$

The change of variable $y(z) = -d \log w(z)$ associates a Riccati equation/foliation to any second order differential equation. In this new coordinate the family of foliations induced by Gauss hypergeometric equation can be written as

$$\omega = z(1 - z)dy - z(1 - z)y^2 + (c - (a + b + 1)z)y + abdz.$$

If $\varphi(z)$ is an arbitrary solution of Gauss hypergeometric equation then $y = -d \log \varphi(z)$ is a solution of the corresponding Riccati equation. If we choose $c \in \mathbb{Q} - \mathbb{Z}$, $a \in \mathbb{Z}_{<0}$, and $b = c - 1 + \beta$ where $\beta \in \mathbb{Z}_{<0}$ then it is clear from the explicit form of the solutions that all leaves of the foliation corresponding to this choice of parameters are algebraic. It follows that the set of foliations in this family admitting a rational integral is Zariski dense. Since there are parameters for which the foliation is not transversely affine it follows that one cannot replace transversely projective by transversely affine in the conjecture proposed at the Introduction. Indeed, one can show that for the choice of parameters made above the foliations are birationally equivalent to fibrations by rational curves. We conclude that one cannot hope to replace transversely projective by transversely affine in the statement of Theorem 8.1.

8.2. **Non-isotrivial fibrations.** We now start the proof of Theorem 8.1. We first treat the case of foliations birationally equivalent to non-isotrivial fibrations.

Proposition 8.2. *Let $g \geq 1$ be a natural number and let $(\pi : \mathscr{X} \to T, \mathscr{F})$ be an algebraic family of foliations. The Zariski closure in T of the set of parameters corresponding to foliations birationally equivalent to non-isotrivial fibrations of genus at most g consists of foliations admitting rational first integrals.*

Proof. According to [13, Proposition 2.1] it suffices to prove that the fibers of the non-isotrivial fibrations in the family belong to a bounded family of curves.

For $g = 1$ the boundness is clear since the fibers of non-isotrivial elliptic fibration $\mathscr{F}_t$ are contained in zero sets of sections of $K_{\mathscr{F}_t}^{\otimes 12}$, see for instance [11, Proposition 4.2]. The boundness of fibers of non-isotrivial fibrations of genus $g \geq 2$ is guaranteed by Theorem A. $\qquad\square$

8.3. **Isotrivial fibrations of adjoint general type.** For isotrivial fibrations of adjoint general type the situation is better when compared to non-isotrivial fibrations as there is no need to bound the genus in order to obtain boundness of the leaves.

Proposition 8.3. *Let $(\pi : \mathscr{X} \to T, \mathscr{F})$ be an algebraic family of foliations. The Zariski closure in T of the set of parameters corresponding to foliations of adjoint general type birationally equivalent to isotrivial fibrations consists of foliations admitting rational first integrals.*

Proof. If $\mathcal{F}$ is an isotrivial fibration of adjoint general type on a projective surface X then $\mathcal{F}$ has Kodaira dimension one and the Iitaka fibration of $K_{\mathcal{F}}$ is an isotrivial fibration of genus $g \geq 2$. According to [11, Proposition 4.10] there are at least two linearly independent sections σ_1, σ_2 of $K_{\mathcal{F}}^{\otimes k}$ for some $k \leq 42$. Consider the rational map $f = (\sigma_1 : \sigma_2) : X \dashrightarrow \mathbb{P}^1$ defined by them. The foliation $\mathcal{G}$ defined f coincides with the foliation defined by the Iitaka fibration of $K_{\mathcal{F}}$. Its normal bundle is of the form $N_{\mathcal{G}} = f^* T_{\mathbb{P}^1} \otimes \mathcal{O}_X(-\Delta) = K_{\mathcal{F}}^{\otimes 2k} \otimes \mathcal{O}_X(-\Delta)$ where Δ is an effective divisor. Since the leaves of $\mathcal{F}$ are contained in fibers of the Iitaka fibration of $K_{\mathcal{G}}$, we repeat the argument to obtain the existence of a $k' \leq 42$ such that the leaves of $\mathcal{F}$ are contained in zero set of sections of $K_X^{\otimes k'} \otimes K_{\mathcal{F}}^{\otimes 2k'k} \otimes \mathcal{O}_X(-k'\Delta)$. This suffices to prove the boundness of the leaves of foliations in a family having adjoint general type and birationally equivalent to isotrivial fibrations. $\square$

8.4. **First integrals and transverse structures.** A foliation on projective surface X is called a transversely affine if for any rational 1-form ω_0 defining $\mathcal{F}$, there exists a rational 1-form ω_1 such that

$$d\omega_0 = \omega_0 \wedge \omega_1 \quad \text{and} \quad d\omega_1 = 0 \,.$$

Similarly, a foliation $\mathcal{F}$ on X is called transversely projective if for any rational 1-form ω_0 defining $\mathcal{F}$ there exists rational 1-forms ω_1 and ω_2 such that

$$
\begin{aligned}
d\omega_0 &= \omega_0 \wedge \omega_1 \\
d\omega_1 &= 2\omega_0 \wedge \omega_2 \\
d\omega_2 &= \omega_1 \wedge \omega_2 \,.
\end{aligned}
$$

For a thorough discussion about transversely affine and transversely projective foliations of codimension one on projective manifolds the reader should consult [12] and [24] respectively.

Proposition 8.4. *Let $\mathcal{F}$ be a foliation on a projective surface X. If $\mathrm{adj}(\mathcal{F}) < 2$ then $\mathcal{F}$ is a transversely projective foliation. Moreover, if $\mathrm{adj}(\mathcal{F}) \in \{0,1\}$ then $\mathcal{F}$ is a transversely affine foliation.*

Proof. This is a straight-forward consequence of the classification. If $\mathcal{F}$ has adjoint dimension zero then it is birationally equivalent to a finite quotient of a foliation defined by a closed rational 1-form. Since the property of being transversely is invariant by dominant rational maps, $\mathcal{F}$ is transversely affine. If $\mathcal{F}$ has adjoint dimension one then $\mathcal{F}$ is either a fibration (and therefore clearly transversely affine) or $\mathcal{F}$ is a turbulent foliation which is well-known to be transversely affine (see for instance [30, Proposition 22]). Finally if $\mathcal{F}$ has negative adjoint dimension then it is either a fibration, a Riccati foliation, or a finite quotient of a Riccati foliation. In any case we have that $\mathcal{F}$ is a transversely projective foliation. $\square$

Proposition 8.5. *Let $(\pi : \mathscr{X} \to T, \mathscr{F})$ be an algebraic family of foliations. If for a very general closed point $t_0 \in T$ the foliation $\mathscr{F}_{t_0}$ is a transversely projective foliation then for every closed point $t \in T$ the foliation $\mathscr{F}_t$ is a transversely projective foliation. Similarly, if for a very general closed point $t_0 \in T$ the foliation $\mathscr{F}_{t_0}$ is a transversely affine foliation then for every closed point $t \in T$ the foliation $\mathscr{F}_t$ is a transversely affine foliation.*

Proof. We can interpret the family of foliation as a single foliation defined over the function field $\mathbb{C}(T)$. By assumption, this foliation is transversely projective. Hence there exists a triplet $(\omega_0, \omega_1, \omega_2)$ of rational differential 1-forms with coefficients in $\overline{\mathbb{C}(T)}$, the algebraic closure of $\mathbb{C}(T)$, satisfying the equations

$$
\begin{aligned}
d\omega_0 &= \omega_0 \wedge \omega_1 \\
d\omega_1 &= 2\omega_0 \wedge \omega_2 \\
d\omega_2 &= \omega_1 \wedge \omega_2 \,.
\end{aligned}
$$

and such that ω_0 is a 1-form differential form defined over $\mathbb{C}(T)$ which defines $\mathscr{F}$. According to [8, Lemma 3.2] we can assume that ω_1, ω_2 are also defined over $\mathbb{C}(T)$ (no need to pass to the algebraic closure). Therefore, over $\mathbb{C}$, we have the equations

$$
\begin{aligned}
d\omega_0 \wedge d\pi &= \omega_0 \wedge \omega_1 \wedge d\pi \\
d\omega_1 \wedge d\pi &= 2\omega_0 \wedge \omega_2 \wedge d\pi \\
d\omega_2 \wedge d\pi &= \omega_1 \wedge \omega_2 \wedge d\pi \,.
\end{aligned}
$$

If $t \in T$ is such that $\pi^{-1}(t)$ is not contained in the polar set of $(\omega_i)_\infty$ for $i = 0, 1, 2$ nor in the zero set of ω_0 then the restriction of the triple $(\omega_0, \omega_1, \omega_2)$ to the fiber over t defines a (singular) projective structure for the foliation $\mathscr{F}_t$ on $X_t = \pi^{-1}(t)$.

Let us fix $t_0 \in T$ such that $X_0 = \pi^{-1}(t_0)$ is contained in the polar set of ω_i ($i = 0, 1, 2$) or in the zero set of ω_0 and let $f \in \pi^* \mathcal{O}_{T,t_0}$ be a rational function on X_0 corresponding to a

generator of the maximal ideal of $\mathcal{O}_{T,t_0}$. Notice that we can replace the triplet $(\omega_0, \omega_1, \omega_2)$ by $(f^k\omega_0, \omega_1, f^{-k}\omega_2)$. Thus, there is no loss of generality in assuming that $\pi^{-1}(t_0)$ is not contained in $(\omega_0)_\infty \cup (\omega_0)_0$.

For $i = 0, 1, 2$, let a_i be the order of ω_i along X_0 and set $\alpha_i = \mathrm{Res}_{X_0} f^{-a_i}\omega_i \wedge \frac{df}{f}$. As mentioned above we will assume that $a_0 = 0$ and, therefore, α_0 is just the restriction of ω_0 to the fiber X_0.

If a_1 is negative then, comparing the orders along X_0 of $d\omega_0 \wedge df$ and of $\omega_0 \wedge \omega_1 \wedge df$, we deduce that $\alpha_0 \wedge \alpha_1 = 0$ and we can write $\alpha_0 = g\alpha_1$ for some rational function $g \in \mathbb{C}(X_0)$. Let $G \in \mathbb{C}(\mathcal{X})$ be a rational function on $\mathcal{X}$ extending g. According to formula (14) of [10] we can replace the triplet $(\omega_0, \omega_1, \omega_2)$ by the triplet

$$\left(\omega_0, \ \omega_1 - f^{-a_1}G\omega_0, \ \omega_2 + f^{-a_1}G\omega_1 + f^{-2a_1}G^2\omega_0 - f^{-a_1}dG \right).$$

This increases a_1. After a finite number of changes we may assume that $a_0 = 0$ and $a_1 \geq 0$.

Finally, if a_2 is negative and $a_1 > 0$ then α_0 is closed and it is clear that $\mathscr{F}_{t_0}$ is transversely projective. If instead $a_2 < 0$ and $a_0 = a_1 = 0$ then comparing the orders along X_0 of $d\omega_1 \wedge df$ and $\omega_0 \wedge \omega_2 \wedge df$ we deduce that $\alpha_0 \wedge \alpha_2 = 0$. Thus we can write $\alpha_2 = h\alpha_0$ for a suitable rational function $h \in \mathbb{C}(X_0)$. From the equation $d\omega_2 \wedge df = \omega_1 \wedge \omega_2 \wedge df$ we deduce that $d\alpha_2 = \alpha_1 \wedge \alpha_2$. Combining these two identities we obtain

$$d(h\alpha_0) = \alpha_1 \wedge (h\alpha_0) \implies d\alpha_0 = (\alpha_1 - \frac{dh}{h}) \wedge \alpha_0.$$

Finally, comparing this identity with $d\alpha_0 = \alpha_0 \wedge \alpha_1$ (first equation) we obtain that $d\alpha_0 = -(1/2)\frac{dh}{h} \wedge \alpha_0$. Thus $\mathscr{F}_{t_0}$ is transversely projective also in this case. $\qquad\square$

8.5. **Proof of Theorem 8.1 (and of Theorem C).** Let $(\pi : \mathcal{X} \to T, \mathscr{F})$ be an algebraic family of foliations and $g \geq 0$ be an integer. We want to prove that the Zariski closure of $\Sigma_g \subset T$ (subset parametrizing foliations with rational first integral of genus at most g) corresponds to transversely projective foliations.

If a very general member of the family, say $\mathcal{F}_t$, is not of adjoint general type then Proposition 8.4 implies that $\mathcal{F}_t$ is transversely projective. We can apply Proposition 8.5 to conclude that every foliation in the family is also transversely projective.

If instead a very general member is of adjoint general type then we will argue as in the proof of Theorem 7.5 to obtain a non-empty Zariski open subset of T such that every foliation parametrized by this subset is of adjoint general type.

Proposition 7.3 allow us to assume the existence of a non-empty Zariski open subset $U_0 \subset T$ that for a very general (i.e. outside a countable union of Zariski closed subsets) $t \in U_0$, the foliation $\mathscr{F}_t$ has canonical singularities. Since $\mathbb{C}$ is uncountable we also know that there exists $n, m > 0$ and an open subset $U_1 \subset T$ such that for every $t \in U_1$, the linear $|K_{\mathscr{F}_t}^{\otimes m} \otimes N_{\mathscr{F}_t}^{*\otimes n}|$ defines a rational map with two dimensional image. Notice that there may exist foliations in $U_0 \cap U_1$ which are not of adjoint general type because of the presence of non-canonical singularities. To remedy this we take $\varepsilon > 0$ sufficiently small in order to obtain from Lemma 7.2 a non-empty Zariski open $U_2 \subset T$ such that $\mathscr{F}_t$ has ε-canonical singularities. Every foliation parametrized by non-empty Zariski open $U = U_0 \cap U_1 \cap U_2$ is of adjoint general type.

Propositions 8.2 and 8.3 imply that the Zariski closure in T of $\Sigma_g \cap U$ corresponds to foliations with rational first integrals. The Theorem follows by Noetherian induction. $\quad\square$

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