# P-matrix recognition is co-NP-complete

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18 October 2007



This is a summary of the proof by G.E. Coxson [1] that P-matrix recognition is co-NP-complete. The result follows by a reduction from the MAX CUT problem using results of S. Poljak and J. Rohn [5].

## 1 Considered problems

Our main interest is the complexity of deciding whether an input matrix is a P-matrix. A *P-matrix* is a square matrix $M \in \mathbb{R}^{n\times n}$ such that all its principal minors are positive. Such matrices were first studied by Fiedler and Pták [2].

### P-MATRIX

**Instance:** A square matrix $M \in \mathbb{Q}^{n\times n}$.
**Question:** Are all the principal minors of $M$ positive?

To start with, we use a well-known combinatorial problem.

### SIMPLE MAX CUT

**Instance:** A graph $G = (V, E)$, a positive integer $K$.
**Question:** Is there a partition of the vertex set $V$ into sets $V_1$ and $V_2$ such that the number of edges with one end in $V_1$ and the other end in $V_2$ is at least $K$?

Garey, Johnson and Stockmeyer [4] showed that the SIMPLE MAX CUT problem is NP-complete.

The reduction from SIMPLE MAX CUT to P-MATRIX uses two intermediate steps. The first of them is the computation of the $r$-norm of a matrix.

For an arbitrary matrix $A \in \mathbb{R}^{n\times n}$, let

$$r(A) = \max\left\{z^{\mathsf{T}}Ay : z, y \in \{-1,1\}^n\right\}.$$

**Remark.** The function $r$ is a matrix norm.

*Proof.* For an arbitrary square matrix $A$, we have $r(A) \geq 0$ because $z^{\mathsf{T}}Ay = -(-z)^{\mathsf{T}}Ay$. Moreover if $r(A) = 0$, then $z^{\mathsf{T}}Ay = 0$ for all choices of $z, y \in \{-1,1\}^n$, hence $A = 0$. If $k \in \mathbb{R}$, then $z^{\mathsf{T}}(kA)y = k \cdot z^{\mathsf{T}}Ay$, so $r(kA) = |k| \cdot r(A)$.

Let $A, B \in \mathbb{R}^{n\times n}$. Then

$$\begin{aligned} r(A+B) = \max\{z^{\mathsf{T}}(A+B)y : y, z \in \{-1,1\}^n\} &= \max\{z^{\mathsf{T}}Ay + z^{\mathsf{T}}By : y, z \in \{-1,1\}^n\} \\ \leq \max\{z^{\mathsf{T}}Ay : y, z \in \{-1,1\}^n\} &+ \max\{z^{\mathsf{T}}By : y, z \in \{-1,1\}^n\} \\ &= r(A) + r(B). \end{aligned}$$

Thus $r$ is also subadditive. □

The decision problem corresponding to $r$-norm computation is defined as follows.

## MATRIX R-NORM

**Instance:** A matrix $A \in \mathbb{Q}^{n\times n}$ and a rational number $K$.
**Question:** Is $r(A) \geq K$?

For the last of the decision problems considered here, we need the notion of matrix interval. If $A_-$ and $A_+$ are $n \times n$ real matrices such that $A_- \leq A_+$ (that is, for each $r$ and $s$ we have $(A_-)_{r,s} \leq (A_+)_{r,s}$), then the *matrix interval*[*] $[A_-, A_+]$ is the set of all matrices $A$ satisfying $A_- \leq A \leq A_+$.

A matrix interval is *singular* if it contains a singular matrix; otherwise it is *non-singular*.

The decision problem we consider consists in testing whether a given matrix interval is singular. We will see that this is a computationally hard problem even when the difference $A_+ - A_-$ has rank 1.

## RK1-MATRIX-INTERVAL SINGULARITY

**Instance:** A non-singular matrix $A \in \mathbb{Q}^{n\times n}$ and a non-negative matrix $\Delta \in \mathbb{Q}^{n\times n}$ of rank 1.
**Question:** Is the matrix interval $[A - \Delta, A + \Delta]$ singular?

The rest of this exposition contains three polynomial reductions of these problems, ultimately proving that P-MATRIX is co-NP-complete.

[*] This object is usually called an *interval matrix*. Since it is actually an *interval* and not a *matrix*, I beg the reader to pardon my decision to call it an uncommon but appropriate name.

## 2 Reduction from SIMPLE MAX CUT to MATRIX R-NORM

Let $G = (V, E)$ be an undirected graph with $n = |V|$ and let $\ell = 2|E| + 1$. If $A(G)$ is the adjacency matrix of $G$, define $A = \ell \cdot I_n - A(G)$. Thus

$$A_{u,v} = \begin{cases} \ell & \text{if } u = v, \\ -1 & \text{if } uv \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for $y, z \in \{-1, 1\}^n$ we have $z^\mathsf{T} A y \leq y^\mathsf{T} A y$ because of the choice of $\ell$. Hence $r(A) = y^\mathsf{T} A y$ for some $y \in \{-1, 1\}^n$.

Let $S \subseteq V$ be defined by $S = \{u : y_u = 1\}$ and let $m'$ be the number of edges of $G$ with one end in $S$ and the other end in $V \setminus S$. In this way, $m'$ is the size of the cut defined by $S$ and $V \setminus S$.

Then

$$y^\mathsf{T} A y = n\ell + 4m' - 2|E|$$

and therefore there is a cut in $G$ of size at least $K$ if and only if $r(A) \geq n\ell - 2|E| + 4K$.

The described reduction (by Poljak and Rohn [5]) establishes the hardness of computing the $r$-norm.

**Theorem 1.** *MATRIX R-NORM is NP-complete, even if input is restricted to non-singular matrices.*

*Proof.* It follows from the reduction above that MATRIX R-NORM is NP-hard. Observe that by the choice of $\ell$ the matrix $A$ in the reduction is strictly diagonally dominant and thus non-singular.

A non-deterministic Turing machine can guess the values of $y, z \in \{-1, 1\}^n$ and check in polynomial time that $z^\mathsf{T} A y \geq K$, so the problem is in the class NP. □

## 3 Reduction from MATRIX R-NORM to RK1-MATRIX-INTERVAL SINGULARITY

For a matrix $A \in \mathbb{R}^{n \times n}$ define

$$\rho_0(A) = \max\{|\lambda| : \lambda \text{ is a real eigenvalue of } A\}$$

and set $\rho_0(A) = 0$ if $A$ has no real eigenvalue.

Further for a vector $y \in \mathbb{R}^n$ define $D(y)$ to be the diagonal $n \times n$ matrix with diagonal vector $y$.

The following fact was proved by Rohn [6].

**Lemma 2.** *Let $A$ be a real non-singular $n \times n$ matrix and let $\Delta$ be a real non-negative $n \times n$ matrix. Then the matrix interval $[A - \Delta, A + \Delta]$ is singular if and only if $\rho_0(A^{-1} D(y) \Delta D(z)) \geq 1$ for some $y, z \in \{-1, 1\}^n$.*

*Proof.* For $y, z \in \{-1, 1\}^n$ let $\Delta_{y,z}$ denote the matrix $D(y)\Delta D(z)$.

First suppose that $A^{-1}\Delta_{y,z}$ has a real eigenvalue $\lambda$ such that $|\lambda| \geq 1$ and $A^{-1}\Delta_{y,z}x = \lambda x$ for some $y, z \in \{-1, 1\}^n$ and a non-zero vector $x$. Then

$$\left(1 - \tfrac{1}{\lambda}A^{-1}\Delta_{y,z}\right)x = 0,$$
$$\left(A - \tfrac{1}{\lambda}\Delta_{y,z}\right)x = 0.$$

Hence $A - (1/\lambda)\Delta_{y,z}$ is a singular matrix in the interval $[A - \Delta, A + \Delta]$ because

$$\left|\tfrac{1}{\lambda}\Delta_{y,z}\right| = \left|\tfrac{1}{\lambda}D(y)\Delta D(z)\right| \leq \Delta.$$

Therefore the interval $[A - \Delta, A + \Delta]$ is singular.

To prove the converse, suppose that $B$ is a singular matrix, $B \in [A - \Delta, A + \Delta]$. Let $x$ be a non-zero vector for which $Bx = 0$.

For $i = 1, 2, \ldots, n$ set

$$t_i = \frac{(Ax)_i}{(\Delta|x|)_i}.$$

We claim that $t \in [0, 1]^n$. Indeed, $|Ax| = |(A - B)x| \leq \Delta|x|$ because $Bx = 0$ and $B \in [A - \Delta, A + \Delta]$.

Moreover, set $z = \operatorname{sgn} x$. Then $D(z)x = |x|$ and

$$(A - \Delta_{t,z})x = Ax - D(t)\Delta D(z)x = Ax - D(t)\Delta|x| = 0$$

by the definition of $t$. Thus the matrix $A - \Delta_{t,z}$ is a singular matrix in the interval $[A - \Delta, A + \Delta]$.

Define $\psi(s) = \det(A - \Delta_{s,z})$. The function $\psi$ is affine in each of the variables $s_1, \ldots, s_n$. Since $\psi(t) = \det(A - \Delta_{t,z}) = 0$, either there exists $y \in \{-1, 1\}^n$ such that $\det(A - \Delta_{y,z}) = 0$, or there exist $y, y' \in \{-1, 1\}^n$ such that $\det(A - \Delta_{y,z}) \cdot \det(A - \Delta_{y',z}) < 0$.

In the latter case, without loss of generality we may assume that $\det A \cdot \det(A - \Delta_{y,z}) < 0$. The function $\phi$ defined by $\phi(\alpha) = \det(A - \alpha\Delta_{y,z})$ is continuous and $\phi(0)\phi(1) < 0$, so $\phi$ has a root in $(0, 1)$.

In either case, there exist $y \in \{-1, 1\}^n$ and $\alpha \in (0, 1]$ such that $\det(A - \alpha\Delta_{y,z}) = 0$. Then

$$\det\left(\tfrac{1}{\alpha}A - \Delta_{y,z}\right) = 0,$$
$$\det\left(\tfrac{1}{\alpha}I - A^{-1}\Delta_{y,z}\right) = 0,$$

hence $\frac{1}{\alpha}$ is a real eigenvalue of the matrix $A^{-1}D(y)\Delta D(z)$ and $\frac{1}{\alpha} \geq 1$, as we were supposed to prove. □

This lemma provides a useful connection between singularity of matrix intervals and a parameter $\rho_0$ dependent on the two matrices $A$, $\Delta$ that define the interval. Next we establish a connection between $\rho_0$ and the $r$-norm of matrices.

From now on let $\mathbb{1}$ be the all-one vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$ and let $J = \mathbb{1} \cdot \mathbb{1}^\mathsf{T}$ be the all-one $n \times n$ matrix.

**Lemma 3.** *Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix, let $\alpha$ be a positive real number and let $\Delta = \alpha J$. Then*

$$\max \{\rho_0(AD(y)\Delta D(z)) : y, z \in \{-1, 1\}^n\} = \alpha \cdot r(A).$$

*Proof.* First observe that $D(y)\Delta D(z) = \alpha \cdot D(y)\mathbb{1} \cdot \mathbb{1}^\mathsf{T} D(z) = \alpha \cdot yz^\mathsf{T}$ for arbitrary $y, z \in \{-1, 1\}^n$. If $\lambda$ is a non-zero real eigenvalue of $\alpha \cdot Ayz^\mathsf{T}$ and $x$ is a non-zero vector such that

$$\alpha \cdot Ayz^\mathsf{T} x = \lambda x \neq 0,$$

then $z^\mathsf{T} x \neq 0$ and

$$\alpha \cdot z^\mathsf{T} Ayz^\mathsf{T} x = \lambda \cdot z^\mathsf{T} x,$$
$$\alpha \cdot z^\mathsf{T} Ay = \lambda.$$

Thus $\rho_0(AD(y)\Delta D(z)) = \alpha \cdot |z^\mathsf{T} Ay|$. Hence

$$\max \{\rho_0(AD(y)\Delta D(z)) : y, z \in \{-1, 1\}^n\} = \alpha \cdot \max \left\{ |z^\mathsf{T} Ay| : y, z \in \{-1, 1\}^n \right\} = \alpha \cdot r(A).$$

□

Now everything is set for Poljak and Rohn's reduction [5].

**Theorem 4.** *Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix, let $K$ be a positive real number and let $\Delta = (1/K) \cdot J$. Then $r(A) \geq K$ if and only if the matrix interval $[A^{-1} - \Delta, A^{-1} + \Delta]$ is singular.*

*Proof.* By Lemma 2, the matrix interval $[A^{-1} - \Delta, A^{-1} + \Delta]$ is singular if and only if $\rho_0(AD(y)\Delta D(z)) \geq 1$ for some $y, z \in \{-1, 1\}^n$. By Lemma 3, $\rho_0(AD(y)\Delta D(z)) \geq 1$ for some $y, z \in \{-1, 1\}^n$ if and only if $r(A) \geq K$. □

**Corollary 5.** *RK1-MATRIX-INTERVAL SINGULARITY is NP-hard.* □

**Remark.** Poljak and Rohn [5] show that RK1-MATRIX-INTERVAL SINGULARITY belongs to the class NP by proving the existence of a singular matrix in every singular matrix interval, with a polynomial bound on the size of all entries of that matrix.

# 4 Reduction from RK1-MATRIX-INTERVAL SINGULARITY to P-MATRIX

The described reduction is by Coxson [1].

Let $A, \Delta \in \mathbb{R}^{n \times n}$. Consider the matrix interval $[A, A+\Delta]$. Let $\Delta^{i,j}$ be the matrix whose element in the $i$th row and $j$th column is $\Delta_{i,j}$ and which has zeros elsewhere. Then each matrix $M$ in the interval $[A, A+\Delta]$ can be uniquely expressed as

$$M = A + \sum_{i,j=1}^{n} p_{i,j} \Delta^{i,j}, \tag{1}$$

where $p_{i,j} \in [0,1]$ for all values of $i, j$.

Each matrix $\Delta^{i,j}$ is a rank-1 matrix (even if $\Delta$ has higher rank), and so $\Delta^{i,j} = r_{i,j} s_{i,j}^{\mathsf{T}}$ for some vectors $r_{i,j}, s_{i,j} \in \mathbb{R}^n$. We can actually take $r_{i,j}$ to be $\Delta_{i,j}$ in its $i$th entry and zero elsewhere, and $s_{i,j}$ to be 1 in its $j$th entry and zero elsewhere.

Now let $R$ be the matrix whose columns are all the $n^2$ vectors $r_{i,j}$ and let $S$ be the matrix whose columns are all the $n^2$ vectors $s_{i,j}$. Thus $\Delta = RS^{\mathsf{T}}$. Moreover, if $p \in \mathbb{R}^{n^2}$ is the vector formed by the numbers $p_{i,j}$, we can write (1) as

$$M = A + RD(p)S^{\mathsf{T}}.$$

Suppose that $A$ is non-singular. Then the matrix interval $[A, A+\Delta]$ is non-singular if and only if

$$\det(A + RD(p)S^{\mathsf{T}}) = \det(A)\det(I_n + A^{-1}RD(p)S^{\mathsf{T}}) \neq 0 \tag{2}$$

for each vector $p \in [0,1]^{n^2}$.

Supposing that the matrix $A$ is non-singular, inequality (2) holds if and only if

$$\det(I_n + A^{-1}RD(p)S^{\mathsf{T}}) \neq 0. \tag{3}$$

In this way we have proved that for a non-singular matrix $A$, singularity of the matrix interval $[A, A+\Delta]$ is equivalent to the existence of a vector $p \in [0,1]^{n^2}$ that does not satisfy inequality (3). Since the expression in (3) is a multi-affine function of $p$, we can actually derive another condition.

**Lemma 6.** *Let* $\psi(p) = \det(I_n + A^{-1}RD(p)S^{\mathsf{T}})$. *Then inequality* (3) *holds for each* $p \in [0,1]^{n^2}$ *if and only if* $\psi(p) > 0$ *for each* $p \in \{0,1\}^{n^2}$.

*Proof.* First observe that $\psi(p) = \det(I_n + A^{-1}RD(p)S^{\mathsf{T}})$ is a multi-affine function of $p$, that is, for each $i$ we have $\psi(p) = c_1 + c_2 p_i$, where $c_1$, $c_2$ depend on $i$ and $p_j$ for $j \neq i$.

We claim that any multi-affine function $\phi : [0,1]^k \to \mathbb{R}$ is non-zero on the whole domain if and only if its values on the vertices $\{0,1\}^k$ have all the same sign. Assuming this claim holds, we notice that $\psi(0) = \det I_n = 1 > 0$, so $\psi$ is non-zero on $[0,1]^{n^2}$ if and only if it is positive on $\{0,1\}^k$.

To prove the claim, first suppose that $\phi$ is non-zero on $[0,1]^k$ but there are two vertices $u, v \in \{0,1\}^k$ such that $\phi(u) < 0$ and $\phi(v) > 0$. Following the path along the edges of $\{0,1\}$, we will find two vertices $u', v' \in \{0,1\}$ that differ in exactly one coordinate and such that $\phi(u') < 0$ and $\phi(v') > 0$. Without loss of generality we may assume that $u'_1 = 0$ and $v'_1 = 1$, while $u'_i = v'_i$ for $i \geq 2$. Let $x \in [0,1]^k$ be defined by $x_1 = \phi(u')/(\phi(u') - \phi(v'))$ and $x_i = u'_i$ for $i \geq 2$. Then $\phi(x) = 0$, a contradiction.

Conversely, if $\phi$ is positive (negative) on all the vertices, it is easy to prove by induction on face dimension that $\phi$ is positive (negative) in every internal point of each face. $\square$

Lemma 6 together with the discussion that precedes it imply the following characterisation.

**Lemma 7.** *Let $A$ be a non-singular matrix and let $R, S$ be defined as above. Then the matrix interval $[A, A + \Delta]$ is singular if and only if*

$$\det(I_n + A^{-1}RD(p)S^\mathsf{T}) \leq 0$$

*for some $p \in \{0, 1\}^{n^2}$.* □

In order to get $D(p)$ from the middle of the product to the beginning, we use the following lemma, whose proof we present in the Appendix.

**Lemma 8.** *Let $F \in \mathbb{R}^{k \times n}$ and $G \in \mathbb{R}^{n \times k}$. Then $\det(I_k + FG) = \det(I_n + GF)$.* □

This fact can be exploited to prove the following equivalence.

**Theorem 9.** *Let $A$ be a non-singular matrix and let $R, S$ be defined as in Lemma 7. Then the matrix interval $[A, A + \Delta]$ is singular if and only if the matrix $M = I_{n^2} + S^\mathsf{T}A^{-1}R$ is not a P-matrix.*

*Proof.* Because of Lemma 8,

$$\psi(p) = \det(I_{n^2} + A^{-1}RD(p)S^\mathsf{T}) = \det(I_{n^2} + D(p)S^\mathsf{T}A^{-1}R).$$

If $p \in \{0, 1\}^{n^2}$ and $p \neq 0$, the expression $\det(I_{n^2} + D(p)S^\mathsf{T}A^{-1}R)$ is equal to the principal minor of the matrix $M$ obtained by selecting exactly those rows and columns that correspond to the 1-entries of the vector $p$. Thus $\psi(p)$ is non-positive for some $p \in \{0, 1\}^{n^2}$ if and only if the matrix $M$ is not a P-matrix.

The proof is now completed by applying Lemma 7. □

**Corollary 10.** *The problem P-MATRIX is co-NP-complete.*

*Proof.* NP-hardness follows from Corollary 5 and Theorem 9.

The problem belongs to co-NP because after guessing the rows and columns, the corresponding principal minor, which certifies the negative answer, can be computed in polynomial time. □

## Appendix: Proof of Lemma 8

One of the basic facts about determinants is that adding a multiple of a row to another row does not change the determinant. The following lemma (Theorem 3 in Section 2.5 of Gantmacher's book [3]) is a block version of this fact. Even though it holds for matrices with an arbitrary number of blocks, we state it just for $2 \times 2$ blocks. This variant is sufficient for the proof of Lemma 8.

**Lemma 11.** *Let $A \in \mathbb{R}^{m \times n}$ be a matrix with block structure*

$$A = \begin{matrix} m_1\{ \\ m_2\{ \end{matrix} \begin{pmatrix} \overbrace{A_{1,1}}^{n_1} & \overbrace{A_{1,2}}^{n_2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

*and let $X \in \mathbb{R}^{m_1 \times m_2}$, $Y \in \mathbb{R}^{n_1 \times n_2}$. Then*

$$\det A = \det \begin{pmatrix} A_{1,1} + XA_{2,1} & A_{1,2} + XA_{2,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \det \begin{pmatrix} A_{1,1} & A_{1,2} + A_{1,1}Y \\ A_{2,1} & A_{2,2} + A_{2,1}Y \end{pmatrix}.$$

*Proof.* Since

$$\begin{pmatrix} A_{1,1} + XA_{2,1} & A_{1,2} + XA_{2,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} I_{m_1} & X \\ 0 & I_{m_2} \end{pmatrix} A,$$

we have

$$\det \begin{pmatrix} A_{1,1} + XA_{2,1} & A_{1,2} + XA_{2,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \det \begin{pmatrix} I_{m_1} & X \\ 0 & I_{m_2} \end{pmatrix} \cdot \det A = \det A.$$

Similarly

$$\det \begin{pmatrix} A_{1,1} & A_{1,2} + A_{1,1}Y \\ A_{2,1} & A_{2,2} + A_{2,1}Y \end{pmatrix} = \det A \cdot \det \begin{pmatrix} I_{n_1} & Y \\ 0 & I_{n_2} \end{pmatrix} = \det A.$$

□

Finally comes the proof of Lemma 8.

*Proof of Lemma 8.* Applying Lemma 11 twice, we get

$$\det(I_k + FG) = \det \begin{pmatrix} I_k + FG & 0 \\ G & I_n \end{pmatrix} \overset{(*)}{=} \det \begin{pmatrix} I_k & -F \\ G & I_n \end{pmatrix}$$
$$\overset{(\dagger)}{=} \det \begin{pmatrix} I_k & 0 \\ G & I_n + GF \end{pmatrix} = \det(I_n + GF).$$

Here $(*)$ follows by applying Lemma 11 to rows with $X = F$ and $(\dagger)$ follows by applying it to columns with $Y = F$. □

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