Long run input use-input price relations and the cost function Hessian

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Abstract

By definition, to compare alternative long run equilibria is to compare alternative points on the real input price frontier. It follows at once that one can never move between long run equilibria by changing just one input price; one must change at least two. And in some cases, indeed, such as the Wicksellian one, to change one price is *ipso facto* to change all the others in a determinate manner. Hence the Hessian of the cost function can – quite obviously – never represent the long run comparative statics of input price-input quantity relations with accuracy. More detailed investigation in fact shows the Hessian to be a hopeless guide to $[dl_i/dw_j]$, both qualitatively and quantitatively.

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The comparative statics analysis of input use and input price must, at a minimum, be able to include the comparison of alternative long run equilibria, since it would otherwise be incomplete. It might well be thought, moreover, that for policy purposes, it is indeed such long run comparisons that are of the greatest use. The purpose of this simple paper is to emphasize that the Hessian matrix of the cost function by no means provides the long run comparative statics required and, more positively, to show just how it is related to the relevant results.

Let w represent a vector of positive input prices and c(w) be a cost function relating to a single-product process. The output level does not appear in c(w) if we interpret the cost function in either of two ways. First, we may take c(w) to be a unit cost function for a constant-returns-to-scale production process. Alternatively, we may take it to be an 'indirect average cost function' (Silberberg, 1974) showing the minimum average cost at which output can be produced. In either case, the use of the ith input per unit of output, l_i, is given by $l_i = (\partial c/\partial w_i)$ and thus

$$dl = Cdw \qquad (1)$$

, where C is the Hessian matrix of c( ). It is of course a symmetric, negative semi-definite matrix. (On the indirect average cost function, see Silberberg, 1974) Here, we shall also assume that every off-diagonal element of C is positive, i.e., that all pairs of inputs are Hicksian substitutes; we make this (arbitrary) assumption merely in order to stress that nothing to be said below will presuppose the presence of Hicksian complementarity and the 'difficulties' to which it can give rise.

From (1), then,

$$(\partial l_i/\partial w_i) = c_{ii} < 0 \qquad (2)$$

and

$$(\partial l_i/\partial w_j) = c_{ij} = (\partial l_j/\partial w_i) > 0 \qquad (3)$$

Do the familiar results (2) and (3) constitute comparisons between alternative long run equilibria? Certainly not. By definition, any long run equilibrium involves that c(w) is equal to the product price, or – taking that price to be unity, so that the w are now real input prices – that

$$c(w) = 1 \qquad (4)$$

From (4), one can never change just one input price at a time when comparing long run equilibria. In (1), then, dw is constrained by

$$l^T dw = 0 \qquad (5)$$

and any study of the long run comparative statics of input use and input price must take account of both (1) and (5).

We shall begin slowly, by considering our old friend the two-input production process but later sections will of course generalize the argument.

Two inputs

In this basic case (1) and (5) simplify to

$$\begin{pmatrix} dl_1 \\ dl_2 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{pmatrix} dw_1 \\ dw_2 \end{pmatrix} \qquad (1^*)$$

and

$$l_1\, dw_1 + l_2\, dw_2 = 0 \qquad (5^*)$$

We also know, of course, that

$$\begin{aligned} w_1\, c_{11} + w_2\, c_{12} &\equiv 0 \\ w_1\, c_{12} + w_2\, c_{22} &\equiv 0 \end{aligned} \qquad (6)$$

because $c(w_1, w_2)$ is homogeneous of the first degree. Consider, for example, (dl_1/dw_1). From (1*) and (5*).

$$(dl_1/dw_1) = c_{11} - (l_1/l_2)\, c_{12}$$

and hence, from (6),

$$(dl_1/dw_1) = (c_{11}/w_2 l_2) \qquad (7)$$

, since $(w_1 l_1 + w_2 l_2) = 1$

Result (7) shows that

$$(dl_1/dw_1) < (\partial l_1/\partial w_1) < 0$$

and, indeed, that if the share of the second input is small, (dl_1/dw_1) will be <u>much</u> less than $(\partial l_1/\partial w_1)$.

Repeating the derivation of (7) for the other (dl_i/dw_j)

$$\begin{bmatrix} dl_1/dw_1 & dl_1/dw_2 \\ dl_2/dw_1 & dl_2/dw_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial l_1/\partial w_1}{w_2 l_2} & \dfrac{\partial l_1/\partial w_2}{w_1 l_1} \\ \dfrac{\partial l_2/\partial w_1}{w_2 l_2} & \dfrac{\partial l_2/\partial w_2}{w_1 l_1} \end{bmatrix} \qquad (8)$$

We see from (8) that the Hessian matrix C does give the correct sign pattern for the matrix $[dl_i/dw_j]$; beyond that, however, the former matrix is a very poor guide to the latter. Every element of C underestimates the absolute magnitude of the corresponding (dl_i/dw_j) – and may do so grossly. Moreover whilst C is necessarily symmetric, matrix $[dl_i/dw_j]$ will be so <u>only</u> in the marginal case $w_1 l_1 = w_2 l_2 = 0.5$. And then, of course,

$$[dl_i/dw_j] = 2\, [\partial l_i/\partial w_j]$$

so that the Hessian is a hopeless approximation to the true matrix of long run comparative statics effects.

(Note that $w_1 l_1 = w_2 l_2$ is equivalent to $(dw_1/w_1) = - (dw_2/w_2)$ and that there will be at most one such point on $c(w_1, w_2) = 1$. Note too that if either $l_1 w_1$ or $l_2 w_2$ becomes very small, one column of the Hessian becomes a very good – and the other column a very poor – approximation to the comparative statics matrix. Note finally that in the Cobb-Douglas case the ($w_j l_j$) terms become constants and that (generically) we <u>never</u> have (dl_1/dw_2) = (dl_2/dw_1); in the fluke case, c( ) = $\sqrt{w_1 w_2}$, however, they are <u>always</u> equal.)

Even in the basic, two-input case, then, the [$\partial l_i/\partial w_j$] matrix is a poor guide to the [dl_i/dw_j] matrix of long run comparative statics effects. We now ask how matters stand in the more general case.

<u>More inputs</u>

To have more than two inputs is not, of course, going to restore symmetry of [dl_i/dw_j]; that can be taken for granted. But we do need to ask whether (with Hicksian substitution throughout) we can at least continue to say that (dl_i/dw_i) < ($\partial l_i/\partial w_i$) < 0 and that ($dl_i/dw_j$) > ($\partial l_i/\partial w_j$) for $i \neq j$. It will suffice to consider the case of three inputs, since the interested reader can readily generalize further.

The relations

$$dl = Cdw \quad (1)$$

and

$$Cw \equiv 0 \quad (6)$$

now hold, C being the 3 x 3 Hessian of $c(w_1,w_2,w_3)$; relation (5) becomes

$$l_1 dw_1 + l_2 dw_2 + l_3 dw_3 = 0 \quad (5^{**})$$

The crucial difference between the present case and the above two-input case is that (5**) – unlike (5*) – no longer allows us to express the vector dw, in (1), in terms of a single (scalar) dw_j. Consequently, although the Hessian matrix C is fully defined at every point on the input price frontier $c(w_1, w_2, w_3) = 1$, the matrix [dl_i/dw_j] is not. The latter matrix depends on <u>how</u> vector w moves on the frontier, on <u>how</u> relative input prices change. (With only two inputs this was completely determined.)

Not all is lost, however. Consider for example, (dl_1/dw_1); we have

$$\left(\frac{dl_1}{dw_1}\right) = c_{11} + c_{12}\left(\frac{dw_2}{dw_1}\right) + c_{13}\left(\frac{dw_3}{dw_1}\right)$$

where
$$l_2\left(\frac{dw_2}{dw_1}\right) + l_3\left(\frac{dw_3}{dw_1}\right) = -l_1$$

If either of (dw_2/dw_1) and (dw_3/dw_1) is zero, or if they are both negative, then certainly

$$(dl_1/dw_1) < (\partial l_1/\partial w_1) < 0$$

However, if (dw_2/dw_1) and (dw_3/dw_1) are of opposite signs we can no longer be sure, a priori, that $(dl_1/dw_1) < (\partial l_1/\partial w_1)$; or even that $(dl_1/dw_1) < 0$! A more geometric way of seeing this is to note that $c(w_1,w_2,w_3) = 1$ and $l_1(w_1,w_2,w_3)$ = some constant, together define a <u>curve</u> on the input price frontier; by construction, along the curve $dl_1 = 0$ always. Starting from any point on that curve we may change w_1 a little and find corresponding changes in (w_2,w_3) that make dl_1 either positive, or zero or negative.

Intuition might suggest that if, say, w_2 and w_3 both change by the same percentage then some definite results should emerge for the (dl_i/dw_1) since we are almost back to the two-input case. And it can indeed be shown that, now,

$$(1 - w_1 l_1)\left(\frac{dl}{dw_1}\right) = \left(\frac{\partial l}{\partial w_1}\right) \qquad (9)$$

(9) is a nice enough result to be sure but, of course, the analogous result for (dl/dw_2) would require that w_1 and w_3 both change by the same percentage, which is completely inconsistent with the assumption underlying (9). Thus no general ‘nice’ relation between $[dl_i/dw_j]$ and C is going to be reachable by this route.

What if, say, $dw_3 = 0$? In this case, we find that $\left[\frac{dl}{dw_1} \quad \frac{dl}{dw_2} \right]$ has the sign pattern

$$\begin{bmatrix} - & + \\ + & - \\ (+/-) & (-/+) \end{bmatrix}$$

, where we note that (dl_3/dw_1) (dl_3/dw_2) is naturally negative. The complete sign pattern is again not predicted.

On a more constructive note, it is easy to show that at least one of

$$\left(\frac{dl_2}{dw_1} - c_{12} \right) \text{and} \left(\frac{dl_3}{dw_1} - c_{13} \right)$$

must be strictly positive (and similarly for the other two dw_j). Yet neither this result, nor the fact that $w^T [dl_i/dw_j] = 0$, can give one great cheer. Definite results have been sparse in this section and will not be made abundant by generalization to $n > 3$ inputs. In order to reach more specific conclusions, we shall need to supplement restriction (5) $[l^T dw = 0]$ with some further limitation on dw. Fortunately, this <u>can</u> arise quite naturally, as in the following section.

<u>The Wicksellian case</u>

Although the argument of this section can readily be extended to the general n-input case, it may be helpful to concentrate again on the case $n = 3$. Suppose now that the cost function $c(w_1, w_2, w_3)$ refers to a constant – returns, three period Wicksellian process of production. If wages are paid in advance, we have $w_j = w\,(1 + r)^{\,j}$, where w is the real (product) wage rate and r is the period rate of interest. On defining $\rho \equiv (1 + r)$, we see that in long run equilibrium

$$\begin{aligned} w_1\, c(1, \rho, \rho^2) &= 1 \\ w_2\, c(\rho^{-1}, 1, \rho) &= 1 \qquad (10) \\ w_3\, c(\rho^{-2}, \rho^{-1}, 1) &= 1 \end{aligned}$$

From the first and third elements of (10) we see that w_1 is monotonically decreasing and w_3 monotonically increasing in ρ; how w_2 varies with ρ will depend on the exact nature of the c( ) function. Alternatively, on noting that $w_2^2 \equiv w_1 w_3$, we may write

$$c\,(w_1, \sqrt{w_1 w_3}\,, w_3) = 1$$

and deduce immediately that w_1 is monotonically decreasing in w_3 but that how w_2 varies with w_3 will depend on c( ). Whether we think of ρ varying parametrically or, more directly, of w_3 varying parametrically, the upshot is the same. Although there is a complete, smooth input price surface $c(w_1, w_2\ w_3) = 1$, only the w vectors lying on a particular curve across that surface are potential long run equilibrium w. Hence only w vectors on that curve need ever be considered in calculating $[dl_i/dw_j]$ – and it can almost always be calculated exactly, because each w_j and therefore each l_i is a determinate function of w_3. The indeterminacy which plagued us in the previous section has been removed by our 'Wicksellian interpretation' of c ( ); note that this would continue to be true for an arbitrary number of dated labour inputs.

To keep our eye firmly fixed on (w_1, w_2, w_3), let us use the $w_2^2 \equiv w_1 w_3$ identity and treat w_3 (rather than ρ) as a parameter. Since

$$w_3\, dw_1 - 2\, w_2\, dw_2 + w_1\, dw_3 \equiv 0$$

and

$$l_1\, dw_1 + l_2\, dw_2 + l_3\, dw_3 = 0$$

we can always determine any two of the dw_j in terms of the third one. Using the relation $dl = Cdw$ we can thus determine all nine (dl_i/dw_j). What can we say about them?

Bearing in mind that all three first partial derivatives of c( ) are homogeneous of degree zero, we see that we can write

$$\begin{aligned} l_1 &= c_1 \left(1, \sqrt{w_3/w_1}, w_3/w_1\right) \\ l_2 &= c_2 \left(\sqrt{w_1/w_3}, 1, \sqrt{w_3/w_1}\right) \\ l_3 &= c_3 \left(w_1/w_3, \sqrt{w_1/w_3}, 1\right) \end{aligned} \tag{11}$$

If all the $c_{ij} > 0$ $(i \neq j)$, we see from the first and last elements of (11) that l_1 is increasing in w_3 and l_3 decreasing. (The movement of l_2 is less obvious.) Hence (dl_1/dw_1) and (dl_3/dw_3) are certainly both negative, while (dl_1/dw_3) and (dl_3/dw_1) are certainly both positive (although there is no reason to expect them to be equal).

It may be noted that

$$\left(\frac{dl_1}{dw_2}\right)\left(\frac{dl_3}{dw_2}\right) = \left(\frac{dl_1}{dw_1}\right)\left(\frac{dl_3}{dw_3}\right)\left(\frac{dw_1}{dw_3}\right)\left(\frac{dw_3}{dw_2}\right)^2$$

which is negative, being the product of three negative terms and one positive one; even when $(\partial l_1/\partial w_2)$ and $(\partial l_3/\partial w_2)$ are both positive, (dl_1/dw_2) and (dl_3/dw_2) must be of <u>opposite signs</u>. It may be noted also that if $dw_2 = 0$ at some point on the (w_1, w_2, w_3) curve then, at that point, $(dl_i/dw_i) < (\partial l_i/\partial w_i)$, for $i = 1,3$, and that

$$\begin{bmatrix} dl_1/dw_1 & dl_1/dw_3 \\ dl_2/dw_1 & dl_2/dw_3 \\ dl_3/dw_1 & dl_3/dw_3 \end{bmatrix} = 2\begin{bmatrix} \partial l_1/\partial w_1 & \partial l_1/\partial w_3 \\ \partial l_2/\partial w_1 & \partial l_2/\partial w_3 \\ \partial l_3/\partial w_1 & \partial l_3/\partial w_3 \end{bmatrix} + w_2 \begin{bmatrix} c_{12} \\ c_{22} \\ c_{23} \end{bmatrix} \left[w_1^{-1}, w_3^{-1}\right]$$

But, of course, we cannot evaluate the 'dl_i/dw_2' magnitudes at this particular point

We know that (dl_1/dw_1) and (dl_3/dw_3) are always negative in the Wicksellian model – but does (dl_2/dw_2) share this property? From (10) and (11)

$$w_2 c\left(\sqrt{w_1/w_3}, 1, \sqrt{w_3/w_1}\right) = 1$$

and

$$l_2 = c_2\left(\sqrt{w_1/w_3}, 1, \sqrt{w_3/w_1}\right)$$

so that each of w_2 and l_2 is a function of the single variable $\sqrt{w_3/w_1}$. As this variable increases, say, it exerts conflicting pressures on w_2 and, if $c_{12} > 0 < c_{23}$, on l_2. It is thus plausible to suppose that one or both of w_2 and l_2 may have a turning point with respect to (w_3/w_1). And if that is so for $(w_3/w_1) > 1$ (i.e., for $\rho > 1$) then the $l_2(w_2)$ relation will be non-monotonic for economically relevant values of (w_1, w_2, w_3).

Consider the cost function

$$c = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 + 2\lambda_{12}\sqrt{w_1 w_2} + 2\lambda_{13}\sqrt{w_1 w_3} + 2\lambda_{23}\sqrt{w_2 w_3}$$

for which

$$l_2 = \lambda_2 + \left(\frac{\lambda_{12}\sqrt{w_1} + \lambda_{23}\sqrt{w_3}}{\sqrt{w_2}}\right)$$

and thus

$$(l_2 - \lambda_2)^2 = 2\lambda_{12}\lambda_{23} + \lambda_{12}^2\sqrt{\frac{w_1}{w_3}} + \lambda_{23}^2\sqrt{\frac{w_3}{w_1}}$$

l_2 has a minimum value at $w_3 = (\lambda_{12}/\lambda_{23})^2 w_1$ and hence at a positive rate of interest if $\lambda_{12} > \lambda_{23}$. Also, on setting $c = 1$ and defining $x^4 \equiv (w_3/w_1)$ we find that

$$w_2[\lambda_1 x^{-2} + 2\lambda_{12} x^{-1} + (\lambda_2 + 2\lambda_{13}) + 2\lambda_{23} x + \lambda_3 x^2] = 1$$

Thus w_2 has a maximum value at

$$\lambda_{23} x^3 + \lambda_3 x^4 = \lambda_1 + \lambda_{12} x$$

and the solution will be greater than one when $\lambda_1 + \lambda_{12} > \lambda_3 + \lambda_{23}$.

It is thus certainly possible that, as w_3 increases, both a minimum l_2 and a maximum w_2 will occur at (different) positive rates of interest. Then a graph of the $l_2(w_2)$ relation will take the form of a (non-closed) loop, with initial and final downward sloping sections and an intermediate upward sloping section. (The direction of movement around the loop as w_3 and ρ increase will depend on which of the $l_2(w_3)$ and $w_2(w_3)$ turning points occurs first.) The $l_2(w_2)$ loop should not be called a 'demand curve', of course, since by construction both w_1 and w_3 are always changing along the curve. But it is the relationship obtaining between l_2 and w_2 when only long run equilibria are compared, i.e. it is relevant to long run comparative statics, whereas demand curves (with w_1, w_3 constant) are not.

It would naturally be possible to calculate the complete matrix $[dl_i/dw_j]$ for our quadratic-square-root cost function, noting for example that both (dl_1/dw_2) and (dl_3/dw_2) change sign along the above-mentioned loop, even when $(\partial l_1/\partial w_2)$ and $(\partial l_3/\partial w_2)$ are both positive throughout, and that both (dl_2/dw_1) and (dl_2/dw_3) do likewise. But the reader is probably tired by now and the general thrust of our argument will already be clear. Neither the symmetric nature, nor the sign pattern, nor again the absolute magnitudes displayed by the Hessian of the cost function give any useful guide to the corresponding properties of $[dl_i/dw_j]$, the matrix that does actually display the magnitudes relevant to the comparison of long run equilibria. And the underlying reason for this is very simple; input prices cannot be changed one at a time when such equilibria are being compared.

(It is simple to extend the Wicksellian case to n dated labour inputs. If $w_j = w\,(1+r)^j$, as before, then $w_i^{(n-1)} = w_1^{(n-i)}\,w_n^{(i-1)}$, for $2 \leq i \leq (n-1)$. It follows easily both that $(w_1, w_n; l_1, l_n)$ behave just like $(w_1, w_3; l_1, l_3)$ above and that every intermediate (w_i, l_i) for $2 \leq i \leq (n-1)$ is just as awkward a customer as (w_2, l_2) above; the reader who so desires can spend many a happy hour explicating all of this.)

Concluding remarks

By definition, to compare alternative long run equilibria is to compare alternative points on the real input price frontier. It follows at once that one can never move between long run equilibria by changing just one input price; one must change at least two. And in some cases, indeed, such as the Wicksellian one, to change one price can be *ipso facto* to change all the others in a determinate manner. Hence the Hessian of the cost function can – quite obviously – never represent the long run comparative statics of input price–input quantity relations with accuracy. More detailed investigation in fact shows the Hessian to be a hopeless guide to $[dl_i/dw_j]$, both qualitatively and quantitatively.

Reference

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