Norm-Resolvent Estimates and Perforated Domains

Frank Rösler

A Thesis presented for the degree of
Doctor of Philosophy

Department of Mathematical Sciences
Durham University
United Kingdom

April 2018

Norm-Resolvent Estimates and Perforated Domains

Submitted for the degree of Doctor of Philosophy

April 2018

Frank Rösler

Abstract In this thesis we are concerned with norm-resolvent estimates for unbounded linear operators. The text is structured into four parts. The first two parts contain mathematical preliminaries, reviews of previous work and an introduction into the two results which constitute parts three and four.

In the third part we are concerned with the non-normal Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$, where $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$ and $\operatorname{Re} V(x) \geq c|x|^2 - b$ for some $c, b > 0$. The spectrum of this operator is discrete and its real part is bounded below by $-b$. In general, the ε-pseudospectrum of H will have an unbounded component for any $\varepsilon > 0$ and thus will not approximate the spectrum in a global sense [KSTV15].

By exploiting the fact that the semigroup e^{-tH} is immediately compact, we show a complementary result, namely that for every $\delta > 0$, $R > 0$ there exists an $\varepsilon > 0$ such that the ε-pseudospectrum

$$\sigma_\varepsilon(H) \subset \{z : \operatorname{Re} z \geq R\} \cup \bigcup_{\lambda \in \sigma(H)} \{z : |z - \lambda| < \delta\}.$$

In particular, the unbounded component of the pseudospectrum escapes towards $+\infty$ as ε decreases. Additionally, we give two examples of non-selfadjoint Schrödinger operators outside of our class and study their pseudospectra in more detail.

In Part IV, we prove norm-resolvent convergence for the operator $-\Delta$ in the perforated domain $\Omega \setminus \bigcup_{i \in 2\varepsilon\mathbb{Z}^d} B_{a_\varepsilon}(i)$, $a_\varepsilon \ll \varepsilon$, to the limit operator $-\Delta + \mu_\iota$ on $L^2(\Omega)$, where $\mu_\iota \in \mathbb{C}$ is a constant depending on the choice of boundary conditions on the holes (we consider Dirichlet, Neumann and Robin boundary conditions).

This is an improvement of previous results [CM97], [Kai85], which show *strong* resolvent convergence. In particular, our result implies Hausdorff convergence of the spectrum of the resolvent for the perforated domain problem.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Acknowledgements

I would like to thank my supervisors P. Dondl, P. Dorey and my collaborator K. Cherednichenko for guidance and inspiration during the course of this thesis.

Furthermore, I am grateful to P. Siegl, D. Krejčiřík and F. Schwenninger for inspiring conversations during conferences and research visits.

Finally, my special thanks goes to the examiners of this thesis, D. Bourne and I. Kamotski for their constructive and interesting feedback and thorough scrutiny.

Contents

I. Mathematical Preliminaries

I.1. Spectral Theory of Unbounded Operators

In this section we will review fundamental definitions and theorems about unbounded operators on Hilbert spaces. We will mostly follow [Wer08],[Kat95],[RS80].

I.1.1. Closed and Closable Operators

Let us first recall the definition of a closed operator. We will restrict ourselves to the case of Hilbert spaces which will be sufficient for our purposes. In this section, $\mathcal{H}$ will denote a complex Hilbert space and $\langle \cdot, \cdot \rangle$, $\|\cdot\|$ its scalar product and norm. All operators in the following are assumed to be linear and we do not distinguish in notation between the norm on $\mathcal{H}$ and the operator norm in $\mathcal{L}(\mathcal{H})$ defined as $\|B\| := \sup_{\|x\|_{\mathcal{H}}=1} \|Bx\|_{\mathcal{H}}$.

Definition I.1.1. Let $D \subset \mathcal{H}$ be a linear subspace and $A : D \to \mathcal{H}$ a linear operator. A is called *closed* if

> If a sequence $(x_n) \subset D$ converges to $x \in \mathcal{H}$ and the sequence (Ax_n) converges to $y \in \mathcal{H}$, then $x \in D$ and $Ax = y$.

An operator A is closed if and only if its graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$. The closed graph theorem from functional analysis states that every closed operator with $D = \mathcal{H}$ is bounded. The domain of an operator A is denoted $\operatorname{dom}(A)$.

Definition I.1.2. An operator A is called *closable*, if there exists a closed extension of A. The smallest closed extension is called the *closure* of A and is denoted $\overline{A}$.

A convenient tool for determining the closure of an operator A is given by

Lemma I.1.3 ([RS80, Kapitel VIII])**.** *Let A be closable. Then*

$$\overline{\Gamma(A)} = \Gamma(\overline{A}),$$

where $\Gamma(A)$ denotes the graph of A.

I.1.2. Selfadjoint Operators

Definition I.1.4. An operator $A : \mathrm{dom}(A) \to \mathcal{H}$ is called

(i) *symmetric*, if
$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \text{for all } x, y \in \mathrm{dom}(A)$$

(ii) *densely defined*, if $\mathrm{dom}(A) \subset \mathcal{H}$ is dense.

Definition I.1.5. Let $A : \mathrm{dom}(A) \to \mathcal{H}$ be a densely defined operator and let

$$\mathrm{dom}(A^*) := \{x \in \mathcal{H} : \exists z \in \mathcal{H} \text{ such that } \langle Ay, x\rangle = \langle y, z\rangle \ \forall y \in \mathrm{dom}(A)\}$$

For such $x \in \mathcal{H}$ we define an operator A^* by $A^*x := z$. This operator is called the *adjoint* of A.

The Riesz-Fréchet theorem implies that $x \in \mathrm{dom}(A^*)$ if and only if $|\langle Ay, x\rangle| \leq C\|y\|$ for all $y \in D(A)$.

Note that the definition of A^* only makes sense if $\mathrm{dom}(A)$ is dense in $\mathcal{H}$, since otherwise the condition $\langle Ay, x\rangle = \langle y, z\rangle \ \forall y \in \mathrm{dom}(A)$ does not uniquely determine z.

Lemma I.1.6 ([Wer08])**.** *Let A be densely defined and symmetric. Then A is closable.*

Definition I.1.7. An operator $A : \mathrm{dom}(A) \to \mathcal{H}$ is called

(i) *selfadjoint*, if $A = A^*$.

(ii) *essentially selfadjoint*, if A is symmetric and $\overline{A}$ is selfadjoint.

In particular, for a selfadjoint operator, one necessarily has $\mathrm{dom}(A) = \mathrm{dom}(A^*)$. The following classical theorem is known as the fundamental criterion for selfadjointness.

Theorem I.1.8 ([Wer08])**.** *Let $A : \mathrm{dom}(A) \to \mathcal{H}$ be densely defined and symmetric. Then the following are equivalent.*

(a) A is selfadjoint.

(b) A is closed and $\ker(A^* \pm i) = \{0\}$

(c) $\mathrm{Ran}(A \pm i) = \mathcal{H}$,

where $\mathrm{Ran}(A \pm i)$ *denotes the range of $A \pm i$, i.e.* $\mathrm{Ran}(A \pm i) = \{y \in \mathcal{H} : y = Ax \pm ix$ *for some* $x \in \mathrm{dom}(A)\}$.

Corollary I.1.9. *Let* $A : \mathrm{dom}(A) \to \mathcal{H}$ *be symmetric. Then the following are equivalent:*

(a) *A is essentially selfadjoint;*

(b) $\ker(A^* \pm i) = \{0\}$*;*

(c) $\mathrm{Ran}(A \pm i)$ *is dense in* $\mathcal{H}$*.*

I.1.3. Basic Spectral Theory

Definition I.1.10. Let $A : \mathrm{dom}(A) \to \mathcal{H}$ be a closed operator. The *resolvent set* of A is defined by

$$\rho(A) := \{\lambda \in \mathbb{C} : (A - \lambda) : \mathrm{dom}(A) \to \mathcal{H} \text{ is bijective}\}.$$

Note that for $\lambda \in \rho(A)$ the open mapping theorem implies that

$$(A - \lambda)^{-1} : \mathcal{H} \to \mathcal{H}$$

is bounded. The map $(A - \lambda)^{-1}$ is called the *resolvent of A at λ*. A modification of the argument for bounded operators shows the following:

Theorem I.1.11 ([Wer08])**.** *Let* $A : \mathrm{dom}(A) \to \mathcal{H}$ *be closed and densely defined. Then*

(i) $\rho(A)$ *is open;*

(ii) *The resolvent mapping* $\lambda \mapsto (A - \lambda)^{-1}$ *is analytic and for* $\lambda, \lambda_0 \in \rho(A)$ *with* $|\lambda - \lambda_0| < \|(\lambda_0 - A)^{-1}\|^{-1}$ *one has the series expansion*

$$(\lambda - A)^{-1} = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k (\lambda_0 - A)^{-k-1}, \tag{I.1}$$

which converges in operator norm.

(iii) *For every pair* $\lambda, \mu \in \rho(A)$ *the* resolvent identity

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1} \tag{I.2}$$

holds.

Let us now define the spectrum of a closed operator.

Definition I.1.12. Let A be as in Definition I.1.10.

(i) The *spectrum* of A is defined to be the closed set

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

(ii) A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of A if there exists a $x \in \operatorname{dom}(A)$ such that $Ax = \lambda x$. The set of eigenvalues of A is also called the *point spectrum* of A and denoted $\sigma_p(A)$.

(iii) The *spectral radius* of A is defined as $r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\} \in \mathbb{R} \cup \{\infty\}$.

Clearly, one has $\sigma_p(A) \subset \sigma(A)$, but the converse inclusion is not necessarily true.

Lemma I.1.13 ([Wer08])**.** *For any bounded operator $T : \mathcal{H} \to \mathcal{H}$ one has*

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

The question arises, whether there is a connection between the spectrum of a closed operator A and the spectrum of its resolvent $(\lambda_0 - A)^{-1}$. Naively, one would expect that if $\mu \in \sigma(A)$ then $\frac{1}{\lambda_0 - \mu} \in \sigma((\lambda_0 - A)^{-1})$. Under mild assumptions, this is in fact the case, as the next theorem shows.

Theorem I.1.14. *Let $A : \mathcal{H} \supset \operatorname{dom}(A) \to \mathcal{H}$ be a closed operator with nonempty resolvent set. Then*

$$\sigma((\lambda_0 - A)^{-1}) \setminus \{0\} = \left\{ \frac{1}{\lambda_0 - \mu} : \mu \in \sigma(A) \right\} \qquad \text{for each } \lambda_0 \in \rho(A). \tag{I.3}$$

Proof. Let $0 \neq \mu \in \mathbb{C}$ and $\lambda_0 \in \rho(A)$. We have

$$\begin{aligned}\big(\mu - (\lambda_0 - A)^{-1}\big)x &= \mu\big(\lambda_0 - \tfrac{1}{\mu} - A\big)(\lambda_0 - A)^{-1}x \qquad \text{for all } x \in \mathcal{H} & \text{(I.4)}\\ &= \mu(\lambda_0 - A)^{-1}\big(\lambda_0 - \tfrac{1}{\mu} - A\big)x \qquad \text{for all } x \in \operatorname{dom}(A). & \text{(I.5)}\end{aligned}$$

Now (I.5) shows that $\big(\mu - (\lambda_0 - A)^{-1}\big)x = 0$, if and only if $\big(\lambda_0 - \frac{1}{\mu} - A\big)x = 0$, since $(\lambda_0 - A)^{-1}$ is bijective (note that $\big(\mu - (\lambda_0 - A)^{-1}\big)x = 0$ implies that $x \in \operatorname{dom}(A)$). Hence $\ker\big(\mu - (\lambda_0 - A)^{-1}\big) = \ker\big(\lambda_0 - \frac{1}{\mu} - A\big)$. Moreover, (I.4) immediately yields that $\operatorname{Ran}\big(\mu - (\lambda_0 - A)^{-1}\big) = \operatorname{Ran}\big(\lambda_0 - \frac{1}{\mu} - A\big)$ (again by bijectivity of $(\lambda_0 - A)^{-1}$).

Hence, $\mu \in \sigma((\lambda_0 - A)^{-1})$ if and only if $\lambda_0 - \frac{1}{\mu} \in \sigma(A)$. □

Corollary I.1.15. *Let there be a $\lambda_0 \in \rho(A)$ such that $(\lambda_0 - A)^{-1}$ is a compact operator. Then $(\lambda - A)^{-1}$ is compact for every $\lambda \in \rho(A)$ and $\sigma(A)$ consists of isolated eigenvalues of finite multiplicity.*

Proof. Let $\lambda \in \rho(A)$. By (I.2) we have

$$(\lambda - A)^{-1} = (\lambda_0 - \lambda)(\lambda - A)^{-1}(\lambda_0 - A)^{-1} + (\lambda_0 - A)^{-1}.$$

Both operators on the right-hand side are compact, hence so is $(\lambda - A)^{-1}$. The remaining assertions follow immediately from the spectral theory of compact operators and the proof of Theorem I.1.14. □

Corollary I.1.16. *For every $\lambda \in \rho(A)$ one has*

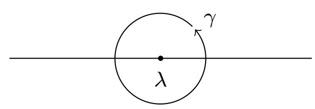
$$\|(\lambda - A)^{-1}\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \tag{I.6}$$

Proof. Just note that, as for any bounded operator, one has $r((\lambda - A)^{-1}) \leq \|(\lambda - A)^{-1}\|$ □

For any continuous Banach space valued function $u : [0, T] \to \mathcal{H}$ one can define the Riemann integral $\int_a^b u(t)\,dt$ (for $a, b \in [0, T]$) in the usual way. Fundamental properties of the integral such as linearity, the standard estimate $\left\|\int_a^b u(t)\,dt\right\| \leq \int_a^b \|u(t)\|\,dt$ and the fundamental theorem of calculus can be shown just like in the scalar case. Moreover, the definition of improper integrals $\int_a^\infty u(t)\,dt := \lim_{b\to\infty} \int_a^b u(t)\,dt$ carries over from the scalar case verbatim. This definition also enables us to define complex line integrals along piecewise smooth paths and Cauchy's integral formula carries over to vector valued analytic functions. In particular, integrals of meromorphic functions do not depend on the specific path chosen, as long as the number of singularities inside the curve remains unchanged.

Definition I.1.17. Let $A : \mathcal{H} \supset \operatorname{dom}(A) \to \mathcal{H}$ be a closed operator and $\lambda \in \sigma(A)$ be an *isolated point*. Then the *Riesz projection* $P_\lambda : \mathcal{H} \to \mathcal{H}$ associated with λ is defined by

$$P_\lambda := \frac{1}{2\pi i} \oint_\gamma (z - A)^{-1}\,dz,$$



where $\gamma \subset \mathbb{C}$ is any small circle such that $\operatorname{int}(\gamma) \cap \sigma(A) = \{\lambda\}$.

Theorem I.1.18 ([GGK90]). *Let $A : \mathcal{H} \supset \mathrm{dom}(A) \to \mathcal{H}$ be a closed operator and $\lambda \in \sigma(A)$ be an isolated point. The Riesz projections P_λ satisfy the following*

(i) $P_\lambda^2 = P_\lambda$;

(ii) $\mathrm{Ran}(P_\lambda) \subset \mathrm{dom}(A)$ *and* $A|_{\mathrm{Ran}(P_\lambda)}$ *is bounded;*

(iii) $\sigma(A|_{\mathrm{Ran}(P_\lambda)}) = \{\lambda\}$

In particular, if $\mathrm{Ran}(P_\lambda)$ is finite-dimensional, then $A|_{\mathrm{Ran}(P_\lambda)}$ is given by a matrix and we can conclude from (iii) that λ is an eigenvalue of $A|_{\mathrm{Ran}(P_\lambda)}$ and hence of A.

I.1.4. The Spectral Theorem

In this section we will take a closer look at selfadjoint operators and their spectral properties. A first simple observation is the following.

Proposition I.1.19 ([Wer08]). *Let $A : \mathrm{dom}(A) \to \mathcal{H}$ be selfadjoint. Then $\sigma(A) \subset \mathbb{R}$.*

Proof. Let $z = \lambda + i\mu$ with $\mu \neq 0$. Define the operator $S := \frac{T}{\mu} - \frac{\lambda}{\mu}$ on $\mathrm{dom}\, T$. Then S is selfadjoint. Note that since $\|\cdot\|$ is induced by a scalar product, we have

$$\|(z-T)x\|^2 = \mu^2\|(S-i)x\|^2 = \mu^2\|Sx\|^2 + \mu^2\|x\|^2 \geq \mu^2\|x\|^2.$$

Hence $(z-T)$ is injective. But by Theorem I.1.8 we have $\mathrm{Ran}(S-i) = \mathrm{Ran}(z-T) = \mathcal{H}$, so $z - T$ is surjective. □

We conclude this section by quoting the spectral theorem for unbounded selfadjoint operators. A proof can be found in [RS80, Ch. VIII].

Theorem I.1.20 (Spectral Theorem - Functional calculus form). *Let A be a selfadjoint operator on $\mathcal{H}$. Then there exists a unique map Φ from the bounded Borel functions on $\mathbb{R}$ into $\mathcal{L}(\mathcal{H})$ such that*

(i) $\Phi(fg) = \Phi(f)\Phi(g)$ *and* $\Phi(\bar{f}) = \Phi(f)^*$.

(ii) $\|\Phi(f)\|_{\mathcal{L}(\mathcal{H})} \leq \|f\|_\infty$.

(iii) If $f_n(x) \to f(x)$ pointwise and if $\|f\|_\infty$ is bounded, then $\Phi(f_n) \to \Phi(f)$ strongly.

(iv) If $Ax = \lambda x$ then $\Phi(f)x = f(\lambda)x$.

As an intuitive notation one usually writes $\Phi(f) = f(A)$.

Corollary I.1.21. *If A is selfadjoint and $\lambda \in \rho(A)$, then one has equality in* (I.6).

Proof. Let $f(t) = \frac{1}{\lambda - t}$. This is a bounded Borel function on $\mathbb{R}$. Now use (ii) in Theorem I.1.20. □

I.1.5. The Numerical Range

Let $A : \mathcal{H} \supset \mathrm{dom}(A) \to \mathcal{H}$ be a closed operator. In this section we briefly study the so-called *numerical range* of A which can give a rough, but easily computable estimate for the location of the spectrum.

Definition I.1.22. The *numerical range* of A is the set

$$\Theta(A) := \{\langle Ax, x\rangle : x \in \mathrm{dom}(A),\ \|x\| = 1\}.$$

It can be shown that $\Theta(A)$ is always a convex set [Dav80, Ch. 6].

Proposition I.1.23. *Let $S := \mathbb{C} \setminus \overline{\Theta(A)}$ be connected and $S \cap \rho(A) \neq \emptyset$. Then one has $S \subset \rho(A)$ and*

$$\|(\lambda - A)^{-1}\| \le \frac{1}{\mathrm{dist}\left(\lambda, \overline{\Theta(A)}\right)} \qquad \text{for all } \lambda \in S.$$

Proof. By assumption we have $S \cap \rho(A) \neq \emptyset$. Note first that for any $\lambda \in \rho(A) \cap S$ we have

$$\|(\lambda - A)x\| \ge |\langle(\lambda - A)x, x\rangle| \ge \mathrm{dist}(\lambda, \overline{\Theta(A)})\|x\| \qquad \text{for all } x \in \mathrm{dom}(A)$$

Since $(\lambda - A)$ is invertible, we obtain

$$\|(\lambda - A)^{-1}\| \le \mathrm{dist}(\lambda, \overline{\Theta(A)})^{-1}$$

We will now show that $S \cap \rho(A)$ is both open and closed in S. Since S is connected, this will imply $S \cap \rho(A) = S$ and conclude the proof.

Since $\rho(A)$ is open in $\mathbb{C}$, it is clear that $\rho(A) \cap S$ is relatively open in S. To show closedness, let (λ_n) be a sequence in $\rho(A) \cap S$ converging to $\lambda \in S$. Then we have for all $x \in \mathrm{dom}(A)$

$$\limsup_{n\to\infty} \|(\lambda_n - A)^{-1}\| \le \mathrm{dist}(\lambda, \overline{\Theta(A)})^{-1}$$

for all $n \in \mathbb{N}$. Applying Corollary I.1.16, we obtain

$$\begin{aligned}\frac{1}{\operatorname{dist}(\lambda, \sigma(A))} &\leq \limsup_{n\to\infty} \frac{1}{\operatorname{dist}(\lambda_n, \sigma(A))} \\ &\leq \limsup_{n\to\infty} \|(\lambda_n - A)^{-1}\| \\ &\leq \frac{1}{\operatorname{dist}(\lambda, \overline{\Theta(A)})}.\end{aligned}$$

Hence

$$\operatorname{dist}(\lambda, \sigma(A)) \geq \operatorname{dist}(\lambda, \overline{\Theta(A)}) > 0$$

and consequently, $\lambda \in \rho(A)$ which proves that $\rho(A) \cap S = S$. □

The numerical range will become important later in the context of one-parameter semigroups which we will discuss next.

I.2. One-Parameter Semigroups

I.2.1. General Facts about Semigroups and Generators

In this section we review the theory for the treatment of *abstract Cauchy problems* of the form

$$\begin{cases} \frac{du}{dt} &= Au \\ u(0) &= x_0 \end{cases} \tag{I.7}$$

where A is a closed operator and $u : [0, \infty) \to \mathcal{H}$ is an unknown vector-valued function. Formally, eq. (I.7) is solved by $u(t) = e^{tA}x_0$. We will now develop a mathematically rigorous construction of a bounded linear operator $e^{tA} : \mathcal{H} \to \mathcal{H}$ in order to solve problem (I.7). Our discussion follows [Wer08, EN00, Dav80, Kat95].

Definition I.2.1. A *strongly continuous semigroup* (or C_0 semigroup) is a family $T(t) : \mathcal{H} \to \mathcal{H}$ of bounded linear operators on a Hilbert space $\mathcal{H}$ such that

(i) $T(0) = \mathrm{id}$

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$

(iii) $\lim_{t\to 0} T(t)x = x$ for all $x \in \mathcal{H}$.

Lemma I.2.2 ([Wer08]). *If $\mathcal{T} = (T(t))_{t\geq 0}$ is a C_0 semigroup on a Hilbert space $\mathcal{H}$ then there exist $M > 0$, $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t} \qquad \forall t \geq 0. \tag{I.8}$$

The number

$$\omega_0 := \omega_0(\mathcal{T}) := \inf\{\omega : \exists M > 0 \text{ s.t. (I.8) holds}\} \tag{I.9}$$

is called the *growth bound* for $\mathcal{T}$. If (I.8) holds with $M = 1$ and $\omega = 0$, $\mathcal{T}$ is called a *contraction* semigroup.

Definition I.2.3. Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on a Hilbert space $\mathcal{H}$. The *infinitesimal generator* (or simply *generator*) of $(T(t))_{t\geq 0}$ is defined to be the operator

$$Ax := \lim_{h\to 0} \frac{T(h)x - x}{h}$$

on the domain $\operatorname{dom}(A) = \Big\{x \in \mathcal{H} : \lim_{h\to 0} \frac{T(h)x-x}{h} \text{ exists}\Big\}$.

It can be shown that the generator of a C_0 semigroup is always closed, densely defined and determines the semigroup uniquely. A commonly used notation for the semigroup $(T(t))_{t\geq 0}$ generated by an operator A is $T(t) =: e^{tA}$. We will frequently adopt this notation in Parts III and IV.

Theorem I.2.4 ([Wer08]). *Let A be the generator of a C_0 semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$ and let $x_0 \in \operatorname{dom}(A)$. Then the function $u : [0,T] \to \mathcal{H}$; $u(t) = T(t)x_0$ is continuously differentiable, maps into* $\operatorname{dom}(A)$ *and solves the abstract Cauchy problem* (I.7). *Furthermore, u is the only solution with these properties and it depends continuously on the initial condition x_0.*

Lemma I.2.5 ([EN00]). *For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ the following hold.*

(i) For all $x \in \mathcal{H}$, $\tau > 0$ one has $\displaystyle\int_0^\tau T(t)x\,dt \in \operatorname{dom}(A)$,

(ii) If $x \in \operatorname{dom}(A)$, then $T(t)x \in \operatorname{dom}(A)$ and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \qquad \textit{for all } t \geq 0,$$

(iii) For every $t \geq 0$ one has

$$\begin{aligned} T(t)x - x &= A\int_0^t T(s)x\,ds && \textit{for all } x \in \mathcal{H} \\ &= \int_0^t T(s)Ax\,ds && \textit{for all } x \in \operatorname{dom}(A) \end{aligned}$$

The following proposition is the first step towards the important Hille-Yosida characterisation theorem for generators of strongly continuous semigroups.

Proposition I.2.6. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on $\mathcal{H}$ and let M, ω be chosen such that $\|T(t)\| \leq Me^{\omega t}$ (cf. Lemma I.2.2). Let A denote the generator of $(T(t))_{t\geq 0}$. If $\lambda \in \mathbb{C}$ is such that $\int_0^\infty e^{-\lambda t}T(t)x\,dt$ exists for all $x \in \mathcal{H}$, then $\lambda \in \rho(A)$ and*

$$(\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)x\,dt. \tag{I.10}$$

Proof. Denote $Ux := \int_0^\infty e^{-\lambda t}T(t)x\,dt$. By rescaling we may assume $\lambda = 0$. Then we have for $h > 0$ and $x \in \mathcal{H}$

$$\begin{aligned} \frac{T(h) - \mathrm{id}}{h}Ux &= \frac{T(h) - \mathrm{id}}{h}\int_0^\infty T(t)x\,dt \\ &= h^{-1}\int_0^\infty T(s+h)x\,ds - h^{-1}\int_0^\infty T(s)x\,ds \\ &= h^{-1}\int_h^\infty T(s)x\,ds - h^{-1}\int_0^\infty T(s)x\,ds \\ &= -h^{-1}\int_0^h T(s)x\,ds. \end{aligned}$$

Since the limit for $h \to 0$ of the right-hand side exists and is equal to $T(0)x = x$, we conclude that $\operatorname{Ran}(U) \subset \operatorname{dom}(A)$ and $AU = -\mathrm{id}_{\mathcal{H}}$. To show $UA = -\mathrm{id}_{\operatorname{dom}(A)}$, let $x \in \operatorname{dom}(A)$ and note that by Lemma I.2.5 we have

$$A\int_0^t T(s)x\,ds = \int_0^t T(s)Ax\,ds.$$

By assumption, the limit for $t \to \infty$ of the right-hand side in the above equation exists and is equal to UAx. Hence, the limit $\lim_{t\to\infty} A\int_0^t T(s)x\,ds$ exists as well. From closedness of A we conclude that $Ux \in \operatorname{dom}(A)$ and $\lim_{t\to\infty} A\int_0^t T(s)x\,ds = AUx$.

Since $AU = -\mathrm{id}_{\mathcal{H}}$, this implies

$$\begin{aligned} & \lim_{t\to\infty} \int_0^t T(s)Ax\,ds = -x \\ \Leftrightarrow \qquad & UAx = -x, \end{aligned}$$

which concludes the proof. ☐

We will often use the shorthand notation

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\,dt \tag{I.11}$$

to mean that (I.10) be satisfied for all $x \in \mathcal{H}$. Notice that $\int_0^\infty e^{-\lambda t} T(t)dt$ does not necessarily converge in operator norm.

Corollary I.2.7. *Let $(T(t))_{t\geq 0}$ and A be as in Proposition I.2.6. Then*

(i) Let $\operatorname{Re}\lambda > \omega$. Then $\lambda \in \rho(A)$ and (I.11) holds.

(ii) One has $\|(\lambda - A)^{-1}\| \leq \frac{M}{\operatorname{Re}\lambda - \omega}$ for all $\operatorname{Re}\lambda > \omega$.

I.2.2. The Hille-Yosida Theorem

From the discussion in the previous subsection we can infer several necessary conditions that a linear operator A needs to satisfy in order to be the generator of a strongly continuous semigroup:

1. A is closed and densely defined;
2. there exists $\omega \in \mathbb{R}$ such that $\rho(A) \supset \{z \in \mathbb{C} : \operatorname{Re} z > \omega\}$;
3. for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ there exists $M > 0$ such that $\|(\lambda - A)^{-1}\| \leq \frac{M}{\operatorname{Re}\lambda - \omega}$.

These facts suggest that generation properties of C_0 semigroups are intimately connected to the resolvent of A. The question immediately arises to what extent the above conditions are *sufficient* for A to be a generator. This question is resolved by the famous Hille-Yosida theorem which we will prove next. We will consider separately the case of contraction semigroups (i.e. semigroups with $\|T(t)\| \leq 1$ for all $t \geq 0$) and the general case.

Theorem I.2.8 (Hille-Yosida)**.** *Let A be any linear operator on a Hilbert space $\mathcal{H}$. Then the following are equivalent.*

(i) A generates a strongly continuous semigroup of contractions.

(ii) A is closed, densely defined and for every $\lambda > 0$ one has $\lambda \in \rho(A)$ and

$$\|\lambda(\lambda - A)^{-1}\| \leq 1. \tag{I.12}$$

This theorem has been proved independently by E. Hille and K. Yosida in 1948 using different methods of proof. We will give Yosida's proof here.

Proof. The implication $(i) \Rightarrow (ii)$ has been shown in the previous section. It remains to prove $(ii) \Rightarrow (i)$. To this end, define the *Yosida Approximation*

$$A_n := nA(n-A)^{-1} = n^2(n-A)^{-1} - n\,\mathrm{id}$$

which is a sequence of bounded, commuting operators. Consider the sequence T_n of associated semigroups defined by

$$T_n := e^{tA_n} := \sum_{k=1}^{\infty} \frac{(tA_n)^k}{k!}.$$

Claim: One has $A_n x \to Ax$ for all $x \in \mathrm{dom}(A)$.

Proof of claim: Let $y \in \mathrm{dom}(A)$ and note that $n(n-A)^{-1}y = (n-A)^{-1}Ay + y$. The first summand converges to 0 as $n \to \infty$ since by assumption $\|(n-A)^{-1}\| \leq \frac{1}{n}$ and hence $n(n-A)^{-1}y \to y$. Since $\|n(n-A)^{-1}\|$ is uniformly bounded, this implies $n(n-A)^{-1}x \to x$ for all $x \in \mathcal{H}$. Now compute

$$A_n y = An(n-A)^{-1}y = n(n-A)^{-1}Ay \to Ay$$

by the above.

To conclude the proof, we will show the following three properties of (T_n) from which the assertion of the theorem follows.

(a) The limit $T(t)x := \lim_{n\to\infty} T_n(t)x$ exists for each $x \in \mathcal{H}$.

(b) $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{H}$.

(c) This semigroup has generator A.

Proof of (a): Note that $\|T_n(t)\|$ are uniformly bounded in n and t, since

$$\|T_n(t)\| \leq e^{-nt}e^{\|n^2(n-A)^{-1}\|t} \leq e^{-nt}e^{nt} = 1.$$

Hence it suffices to prove (a) for $x \in \mathrm{dom}(A)$. To this end, let $0 \leq s \leq t$ and $m, n \in \mathbb{N}$ and compute

$$\begin{aligned} T_n(t)x - T_m(t)x &= \int_0^t \frac{d}{ds}\big(T_m(t-s)T_n(s)x\big)\, ds \\ &= \int_0^t T_m(t-s)T_n(s)(A_n x - A_m x)\, ds \end{aligned}$$

$$\Rightarrow \qquad \|T_n(t)x - T_m(t)x\| \leq t\|A_n x - A_m x\|$$

By pointwise convergence of A_n, we infer that $(T_n(t)x - T_m(t)x)$ is a Cauchy sequence and converges uniformly in t on bounded intervals.

Proof of (b): By passing to the limit in the semigroup law $T_n(s+t) = T_n(s)T_n(t)$, we see that $(T(t))_{t\geq 0}$ satisfies condition (ii) of Definition I.2.1. Moreover, one has $\|T(t)x\| = \lim_{n\to\infty} \|T_n(t)x\| \leq 1$ for all $x \in \mathcal{H}$, so $(T(t))_{t\geq 0}$ is a contraction semigroup. Finally, the strong continuity property (iii) of Definition I.2.1 follows because for every $x \in \mathcal{H}$, the map $t \mapsto T(t)x$ is (locally) the uniform limit of a sequence of continuous functions $T_n(t)x$.

Proof of (c): Let B denote the generator of $(T(t))_{t\geq 0}$ and fix $x \in \mathrm{dom}(A)$ and note that the functions $\xi_n : t \mapsto T_n(t)x$ converge uniformly on compact intervals to $\xi : t \mapsto T(t)x$. Moreover, the sequence of derivatives $\xi_n'(t) = T_n(t)A_n x$ converge uniformly on compact intervals to $\eta : t \mapsto T(t)Ax$. By a standard theorem from Analysis these two facts imply that ξ is differentiable and $\xi'(0) = \eta(0)$. Hence every $x \in \mathrm{dom}(A)$ is in $\mathrm{dom}(B)$ and $Ax = Bx$ for all $x \in \mathrm{dom}(A)$. Now let $\lambda > 0$. Then

- $(\lambda - A)^{-1}$ is a bijection between $\mathrm{dom}(A)$ and $\mathcal{H}$ by assumption and
- $(\lambda - B)^{-1}$ is a bijection between $\mathrm{dom}(B)$ and $\mathcal{H}$ by Corollary I.2.7.

But we have $\lambda - A = \lambda - B$ on $\mathrm{dom}(A)$. This is only possible if $\mathrm{dom}(A) = \mathrm{dom}(B)$ and $A = B$. $\square$

Next we will state the Hille-Yosida theorem in the general case first proved by Feller, Miyadera and Phillips in 1952.

Theorem I.2.9 (Feller-Miyadera-Phillips)**.** *Let A be any linear operator on a Hilbert space $\mathcal{H}$ and let $\omega \in \mathbb{R}$ and $M > 0$ be constants. Then the following are equivalent.*

(i) A generates a strongly continuous semigroup satisfying

$$\|T(t)\| \leq Me^{\omega t} \qquad \text{for } t \geq 0.$$

(ii) A is closed, densely defined and for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and

$$\left\|(\lambda-\omega)^n(\lambda-A)^{-n}\right\| \le M \qquad \text{for } n \in \mathbb{N}.$$

Proof. We only give the general idea of the proof. The central idea is to introduce a new norm

$$|||x||| := \sup_{\mu>\omega} \sup_{n\in\mathbb{N}_0} \left\|\mu^n(\mu-A)^{-n}x\right\|$$

on $\mathcal{H}$ which can be shown to be equivalent to the previous norm $\|\cdot\|_{\mathcal{H}}$. With respect to this new norm, the operator A can be seen to satisfy the assumptions of Theorem I.2.8 and hence generates a contraction semigroup w.r.t. $|||\cdot|||$. Rewriting everything in terms of $\|\cdot\|_{\mathcal{H}}$ yields the assertion. □

I.2.3. Accretive and Sectorial Operators

As a first step towards the spectral theory for semigroups of operators, let us briefly study accretive and sectorial operators which will turn out to be generators for special classes of semigroups. Let us fix the following convenient notation. By a *sector* in the complex plane we mean a set of the form

$$\Sigma_\theta := \{z \in \mathbb{C} : |\arg(z)| \le \theta\} \tag{I.13}$$

for some $\theta \in (0, \pi)$.

Definition I.2.10. A linear operator $A : \mathcal{H} \supset \operatorname{dom}(A) \to \mathcal{H}$ is said to be

(i) *accretive* if $\Theta(A)$ is a subset of the right half-plane, that is, if $\operatorname{Re}\langle Ax, x\rangle \ge 0$ for all $x \in \operatorname{dom}(A)$. It is called *dissipative,* if $-A$ is accretive.

(ii) *maximally accretive*, or *m-accretive*, if A is accretive and $\{z \in \mathbb{C} : \operatorname{Re}(z) < 0\} \subset \rho(A)$ with

$$\|(\lambda-A)^{-1}\| \le \frac{1}{|\operatorname{Re}\lambda|} \qquad \text{for } \operatorname{Re}\lambda < 0.$$

(iii) *sectorial*, if $\Theta(A) \subset \Sigma_\theta + \gamma$ for some $\theta \in (0, \frac{\pi}{2})$ and $\gamma \in \mathbb{C}$. The numbers γ and θ are called the *vertex* and *semi-angle* of A, respectively.

(iv) *m-sectorial*, if A is sectorial and $A + z$ is m-accretive for some $z \in \mathbb{C}$.

Note that the vertex and semi-angle of a sectorial operator are not uniquely defined. The key statement of this section is the Lumer-Phillips theorem which gives a convenient characterisation for generators of *contraction* semigroups.

Remark I.2.11. The reader should be cautious and note that there are different notions of sectoriality used in the literature. The notion we use in this text is sectoriality in the sense of Kato (cf. [Kat95]). The authors of [EN00, Haa06] use a less restrictive definition which is implied by sectoriality in Kato's sense.

Our distinction between accretive and dissipative operators is convenient because in practice one often encounters operators A such that $-A$ generates a contraction semigroup.

Lemma I.2.12. *A is dissipative if and only if*

$$\|(\lambda - A)x\| \geq \lambda\|x\| \tag{I.14}$$

for all $\lambda > 0$ and $x \in \operatorname{dom}(A)$.

Proof. If A is dissipative, then

$$\|(\lambda - A)x\|\|x\| \geq |\langle(\lambda - A)x, x\rangle| \geq \lambda\|x\|^2 - \underbrace{\operatorname{Re}\langle Ax, x\rangle}_{\leq 0} \geq \lambda\|x\|^2.$$

Conversely, assume $\|(\lambda - A)x\| \geq \lambda\|x\|$ $\forall\lambda > 0$, $x \in \operatorname{dom}(A)$. Then we have for $x \in \operatorname{dom}(A)$

$$\begin{aligned}
\lambda\|x\| \leq \|(\lambda - A)x\| &= \left\langle (\lambda - A)x, \frac{(\lambda - A)x}{\|(\lambda - A)x\|} \right\rangle \\
&= \|(\lambda - A)x\|^{-1}\left(\lambda^2\|x\|^2 + \|Ax\|^2 - 2\lambda\operatorname{Re}\langle x, Ax\rangle\right) \\
\Leftrightarrow \quad \lambda\|x\|\big(\underbrace{\|(\lambda - A)x\| - \lambda\|x\|}_{\geq 0}\big) &= \|Ax\|^2 - 2\lambda\operatorname{Re}\langle x, Ax\rangle \\
\Rightarrow \quad \operatorname{Re}\langle Ax, x\rangle &\leq \frac{\|Ax\|^2}{2\lambda}
\end{aligned}$$

The result follows by letting $\lambda \to \infty$. $\square$

Theorem I.2.13 (Lumer-Phillips)**.** *Let A be a densely defined linear operator on $\mathcal{H}$. Then A generates a contraction semigroup if and only if A is dissipative and there exists $\lambda_0 > 0$ such that $\operatorname{Ran}(\lambda_0 - A) = \mathcal{H}$.*

Proof. If A generates a contraction semigroup, Theorem I.2.8 shows that $(0,\infty) \subset \rho(A)$ and $\|\lambda(\lambda - A)^{-1}\| \leq 1$ which immediately yields (I.14).

To show the converse, let A be dissipative and note that (I.14) implies that $\lambda_0 - A$ is injective. Since by assumption, $\lambda_0 - A$ is surjective as well, we have $\lambda_0 \in \rho(A)$. Hence $(\lambda_0 - A)^{-1}$ is bounded and A closed. Since A is dissipative, eq. (I.14) shows that $\|\lambda(\lambda - A)^{-1}\| \leq 1$ for all $\lambda \in \rho(A) \cap (0,\infty)$. It remains to show that actually $(0,\infty) \subset \rho(A)$. Then by Theorem I.2.8, A will generate a contraction semigroup. We will show that $\emptyset \neq \rho(A) \cap (0,\infty)$ is both open and closed in $(0,\infty)$ which will yield the result. First, it is clear by definition that $\rho(A) \cap (0,\infty)$ is open in $(0,\infty)$. To see closedness, let $(\lambda_n) \subset \rho(A) \cap (0,\infty)$ be a sequence with $\lambda_n \to \lambda > 0$. By (I.14) and (I.6) we have

$$\operatorname{dist}(\lambda_n, \sigma(A)) \geq \frac{1}{\|(\lambda_n - A)^{-1}\|} \geq \lambda_n.$$

Passing to the limit, this yields $\operatorname{dist}(\lambda, \sigma(A)) \geq \lambda > 0$ and concludes the proof. □

Corollary I.2.14. *If A is m-accretive, then $-A$ generates a strongly continuous contraction semigroup.*

I.2.4. Compact and Analytic Semigroups

Next we will discuss special subclasses of semigroups. As we will see in the next section, these classes exhibit interesting spectral behaviour.

Norm continuous semigroups

Definition I.2.15. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called

(i) *norm continuous* if the map $t \mapsto T(t)$ is continuous from $[0,\infty) \to \mathcal{L}(\mathcal{H})$;

(ii) *eventually norm continuous* if there exists $t_0 > 0$ such that the map $t \mapsto T(t)$ is continuous from $(t_0,\infty) \to \mathcal{L}(\mathcal{H})$;

(iii) *immediately norm continuous* if one can choose $t_0 = 0$ in (ii);

(iv) *eventually differentiable* if there exists $t_0 > 0$ such that the maps $t \mapsto T(t)x$ are differentiable on (t_0,∞) for every $x \in \mathcal{H}$;

(v) *immediately differentiable* if one can choose $t_0 = 0$ in (iv)

Lemma I.2.16. *If $(T(t))_{t\geq 0}$ is norm continuous, the generator A is bounded.*

Proof. Let $(T(t))_{t\geq 0}$ be a norm continuous semigroup. By assumption, there exists $\tau > 0$ such that

$$\left\| \frac{1}{\tau} \int_0^\tau T(t)\, dt - \mathrm{id} \right\| \leq \frac{1}{\tau} \int_0^\tau \|T(t) - \mathrm{id}\|\, dt < 1.$$

By the Neumann series, $\frac{1}{\tau}\int_0^\tau T(t)\, dt$ is surjective. But $\operatorname{Ran}\left(\frac{1}{\tau}\int_0^\tau T(t)\, dt\right) \subset \operatorname{dom}(A)$, by Lemma I.2.5 (i). Hence $\operatorname{dom}(A) = \mathcal{H}$ and A is bounded by the closed graph theorem. $\square$

Note the difference between a norm continuous semigroup and an immediately norm continuous semigroup. While the former always has a bounded generator, as we have just seen, there is no reason why this should be true for the latter. Indeed, we will see examples of immediately norm continuous semigroups with unbounded generators in Part III.

A first observation about the spectral properties of eventually norm continuous semigroups which we will need later on is the following.

Lemma I.2.17. *Let A be the generator of an eventually norm continuous semigroup $(T(t))_{t\geq 0}$. Then for every $b \in \mathbb{R}$ the set*

$$\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq b\}$$

is bounded.

Proof. Fix $a > \omega_0$ (cf. (I.9)). Proposition I.2.6 yields the formula

$$(\lambda - A)^{-n-1}x = \frac{1}{n!}\int_0^\infty e^{-\lambda t}\, t^n\, T(t)x\, dt$$

for $x \in \mathcal{H}$, $\operatorname{Re}\lambda > \omega_0$ and $n \in \mathbb{N}$. Indeed, this follows from (I.10) using the formula $(\lambda - A)^{-n-1} = \frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}(\lambda - A)^{-1}$ which easily follows from the resolvent identity (I.2) by induction. We need to show that choosing $r > 0$ large enough we will obtain a uniform bound on $\|(a + ir - A)^{-1}\|$.

To this end, let $\varepsilon > 0$, $x \in \mathcal{H}$ and choose $t_1 > 0$ such that $(T(t))_{t\geq 0}$ is norm continuous on $[t_1, \infty)$. Furthermore, let $t_2 > t_1$ to be determined later and choose $\omega \in (\omega_0, a)$ such that (I.2.2) holds. Then for every $n \in \mathbb{N}$ we have

$$\|(a + ir - A)^{-n-1}x\| = \left\| \frac{1}{n!}\int_0^\infty e^{-(a+ir)t}\, t^n\, T(t)x\, dt \right\|$$

$$
\begin{aligned}
&\leq \frac{1}{n!}\int_0^{t_1} e^{-at}t^n\|T(t)x\|\,dt + \frac{1}{n!}\left\|\int_{t_1}^{t_2} e^{-irt}e^{-at}\,t^n\,T(t)\,dt\right\|\,\|x\| \\
&\qquad + \frac{1}{n!}\int_{t_2}^{\infty} e^{-at}\,t^n\;\|T(t)x\|\;dt \\
&\leq \frac{t_1^n}{n!}M\int_0^{t_1} e^{-at}e^{\omega t}\,dt\|x\| \;+ \frac{1}{n!}\left\|\int_{t_1}^{t_2} e^{-irt}e^{-at}\,t^n\,T(t)\,dt\right\|\,\|x\| \\
&\qquad + \frac{M}{n!}\int_{t_2}^{\infty} t^n e^{-at}e^{\omega t}\,dt\|x\|
\end{aligned}
$$

Next, choose n large enough such that $\frac{t_1^n}{n!}M\int_0^{t_1} e^{-at}e^{\omega t}\,dt < \frac{\varepsilon^{n+1}}{3}$ and t_2 large enough such that $\frac{M}{n!}\int_{t_2}^{\infty} t^n e^{-at}e^{\omega t}\,dt < \frac{\varepsilon^{n+1}}{3}$. These choices leave us with

$$
\|(a+ir-A)^{-n-1}x\| \leq \frac{2}{3}\varepsilon^{n+1}\|x\| \;+\; \frac{1}{n!}\left\|\int_{t_1}^{t_2} e^{-irt}e^{-at}\,t^n\,T(t)\,dt\right\|\,\|x\|
$$

Finally, choose $r_0 > 0$ such that $\left\|\frac{1}{n!}\int_{t_2}^{\infty} e^{irt}t^n e^{-at}T(t)\,dt\right\| < \frac{\varepsilon^{n+1}}{3}$ whenever $|r| > r_0$. This is possible by the Riemann-Lebesgue-Lemma applied to the norm continuous function $t \mapsto t^n e^{-at}T(t)$ (note that by norm continuity this function is measurable). We have shown that for n large enough

$$
\|(a+ir-A)^{-n-1}x\| \leq \varepsilon^{n+1}\|x\| \qquad \text{for } |r| > r_0.
$$

To conclude the proof, let $b \in \mathbb{R}$ be an arbitrary constant and define $\varepsilon := \frac{1}{2|b-a|}$. Then by the above, there exist $r_0 > 0$ and $n \in \mathbb{N}$ such that

$$
\begin{aligned}
\operatorname{dist}(a+ir,\sigma(A)) \geq \|(a+ir-A)^{-1}\|^{-1} &\geq \|(a+ir-A)^{-n}\|^{-1/n} \\
&\geq \frac{1}{\varepsilon} \\
&= 2|b-a|
\end{aligned}
$$

for $|r| > r_0$, where we have used Corollary I.1.16 in the first line. Hence,

$$
\begin{aligned}
\operatorname{dist}(b+ir,\sigma(A)) &\geq \operatorname{dist}(a+ir,\sigma(A)) - |b-a| \\
&\geq |b-a|
\end{aligned}
$$

for $|r| > r_0$ which immediately yields the assertion. □

Compact semigroups. An important subclass of eventually norm continuous semigroups are semigroups which are *compact* operators for some $t > 0$. In fact, we have the following

Lemma I.2.18. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on $\mathcal{H}$ and assume that there exists $t_0 > 0$ such that $T(t_0)$ is a compact operator. Then $T(t)$ is compact for all $t > t_0$ and the map $t \mapsto T(t)$ is norm continuous on $[t_0, \infty)$.*

Proof. The first assertion follows immediately from the semigroup law (cf. Definition I.2.1 (ii)). To prove norm continuity, note that for $t > t_0$

$$T(t+h) - T(t) = (T(h) - \mathrm{id})T(t_0).$$

Thus, if (x_n) is any bounded sequence, the sequence $(T(t_0)x_n)$ has a convergent subsequence $T(t_0)x_{n_k} \to y$. To conclude, let $h_n \searrow 0$, and compute

$$\begin{aligned}(T(t+h_{n_k}) - T(t))x_{n_k} &= (T(h_{n_k}) - \mathrm{id})T(t_0)x_{n_k} \\ &\to (T(0) - \mathrm{id})y \\ &= 0.\end{aligned}$$

Applying the above argument to every subsequence yields the assertion. □

Definition I.2.19. A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called

(i) *eventually compact* if there exists $t_0 > 0$ such that $T(t_0)$ is compact;

(ii) *immediately compact* if $T(t)$ is compact for all $t > 0$.

Eventually compact semigroups are a convenient tool because compactness is often easier to verify directly than norm continuity. This point is emphasised by the following example.

Example 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open subset with smooth boundary and let $A = \Delta$ on $\mathcal{H} = L^2(\Omega)$ with $\mathrm{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ be the Dirichlet Laplacian. Then the Lumer-Phillips theorem shows that A generates a strongly continuous contraction semigroup. This semigroup is given by

$$(e^{t\Delta}f)(x) = \int_\Omega K(t,x,y)f(y)\,dy \qquad \text{for } f \in L^2(\Omega),$$

with an integral kernel $K(t,x,y) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} + \varphi(t,x,y)$, where φ is a smooth, bounded function depending on Ω. Clearly, we have $\int_{\Omega\times\Omega} |K(t,x,y)|^2\,dxdy < \infty$ for $t > 0$, that is, $e^{t\Delta}$ is Hilbert-Schmidt and thus compact. We conclude that the semigroup $(e^{t\Delta})_{t\geq 0}$ is immediately norm continuous.

Analytic semigroups. Finally, we will discuss analytic semigroups which are even more tame than eventually compact semigroups. As a necessary evil, the restrictions on the associated generators are more severe. The idea behind the definition of analytic operator semigroups is to use Cauchy's integral formula to define

$$e^{zA} := \frac{1}{2\pi i} \int_\gamma e^{\mu z} (\mu - A)^{-1} \, d\mu$$

for $z \in \mathbb{C}$ and a suitable path γ enclosing z. This definition is justified if the integral on the right-hand side converges. In order to investigate the above idea, let us make the following

Hypothesis I.2.20. Let A be a closed, densely defined linear operator such that

(i) there exists $\delta > 0$ such that the sector $\Sigma_{\frac{\pi}{2}+\delta}$ is contained in the resolvent set of A,

(ii) for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon > 0$ such that for all $z \in \overline{\Sigma}_{\frac{\pi}{2}+\delta-\varepsilon}$ one has $\|(z - A)^{-1}\| \leq \frac{M_\varepsilon}{|z|}$.

For A satisfying Hypothesis I.2.20, let $\delta > 0$ be as in (i), $\delta' \in (0, \delta)$ and fix $z \in \Sigma_{\delta'}$. Furthermore, set $\varepsilon := \frac{\delta - \delta'}{2}$. We first choose an explicit path $\gamma_z \subset \mathbb{C}$ as the concatenation of the following

$$\begin{aligned} \gamma_z^1 &= \left\{ -\rho e^{-i(\frac{\pi}{2}+\delta-\varepsilon)} : -\infty < \rho < -r \right\} \\ \gamma_z^2 &= \left\{ r e^{i\alpha} : -(\tfrac{\pi}{2} + \delta - \varepsilon) < \alpha < \tfrac{\pi}{2} + \delta - \varepsilon \right\} \\ \gamma_z^3 &= \left\{ \rho e^{i(\frac{\pi}{2}+\delta-\varepsilon)} : r < \rho < \infty \right\} \end{aligned} \tag{I.15}$$

where $r = \frac{1}{|z|}$ (cf. Figure I.1). Elementary geometric considerations lead to the estimates

$$\|e^{\mu z}(\mu - A)^{-1}\| \leq e^{-|\mu z| \sin(\varepsilon)} \frac{M_\varepsilon}{|\mu|} \qquad \text{for } z \in \Sigma_{\delta'} \text{ and } \mu \in \gamma_z^1 \cup \gamma_z^3 \tag{I.16}$$

$$\|e^{\mu z}(\mu - A)^{-1}\| \leq e M_\varepsilon |z| \qquad \text{for } z \in \Sigma_{\delta'} \text{ and } \mu \in \gamma_z^2. \tag{I.17}$$

We conclude that

$$\begin{aligned} \left\| \int_{\gamma_z} e^{\mu z} (\mu - A)^{-1} \, d\mu \right\| &\leq \sum_{k=1}^{3} \int_{\gamma_z^k} \|e^{\mu z}(\mu - A)^{-1}\| \, d\mu \\ &\leq 2M_\varepsilon \int_{|z|^{-1}}^{\infty} \frac{1}{s} e^{-s|z|\sin(\varepsilon)} \, ds \, + \, e M_\varepsilon |z| \frac{2\pi}{|z|} \end{aligned}$$

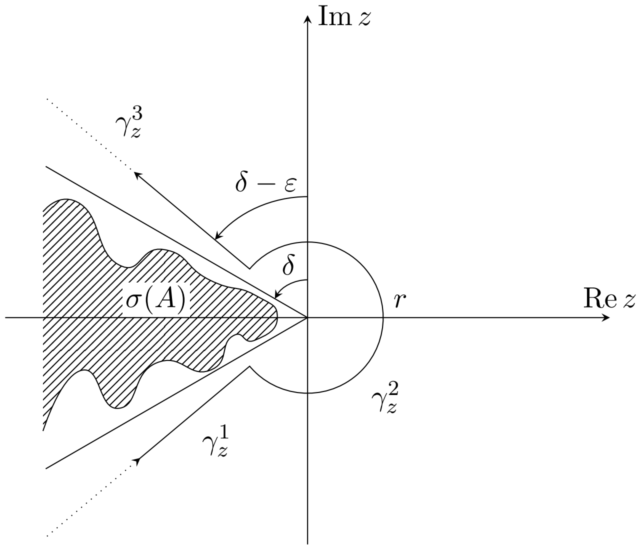


Figure I.1.: Sketch of the path of integration composed of $\gamma_z^1, \gamma_z^2, \gamma_z^3$ (originally from [EN00]).

$$= 2M_\varepsilon \int_1^\infty \frac{e^{-s\sin(\varepsilon)}}{s}\, ds + 2\pi e M_\varepsilon$$

The right-hand side is just a finite constant independent of z which shows that the integral along γ_z converges absolutely and uniformly for $z \in \Sigma_{\delta'}$. Furthermore, since the integrand is an analytic function (cf. Theorem I.1.11), the integral does not depend on the specific path chosen. The above considerations also imply that the integral $\int_{\gamma_z} e^{\mu z}(\mu - A)^{-1}\, d\mu$ defines an analytic function for $z \in \Sigma_\delta$. We recapitulate our results in the following

Theorem and definition I.2.21. *Let A satisfy Hypothesis I.2.20 and let $\delta > 0$ be as in (i), $\delta' \in (0, \delta)$. For $z \in \Sigma_{\delta'}$, the formula*

$$T(z) := \frac{1}{2\pi i} \int_\gamma e^{\mu z}(\mu - A)^{-1}\, d\mu \tag{I.18}$$

specifies a well-defined analytic family of uniformly bounded operators for any piecewise smooth path $\gamma : \mathbb{R} \to \rho(A)$ such that asymptotically $\gamma(-\infty) = \infty e^{-(\frac{\pi}{2}+\delta')i}$ and $\gamma(\infty) = \infty e^{(\frac{\pi}{2}+\delta')i}$.

The above observation is the starting point for the theory of analytic semigroups. Note that up to now we have merely defined an analytic family of bounded operators without any additional structure. To make progress, let us make the following

Definition I.2.22. A family of bounded operators $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ is called an *analytic semigroup* of angle $\delta \in (0, \frac{\pi}{2}]$, if

(i) $T(0) = \mathrm{id}$ and $T(z + w) = T(z)T(w)$ for all $z, w \in \Sigma_\delta$;

(ii) the map $z \mapsto T(z)$ is analytic in Σ_δ;

(iii) $\lim_{\substack{z\to 0\\ z\in\Sigma_{\delta'}}} T(z)x = x$ for all $x \in \mathcal{H}$ and $\delta' \in (0, \delta)$.

Theorem I.2.23. *Let A satisfy Hypothesis I.2.20. Then* (I.18) *defines an analytic semigroup.*

Proof. Let δ be as in I.2.20. Condition (ii) of Definition I.2.22 has already been proven above. To verify (i), let $z, w \in \Sigma_\delta$ and choose $\delta' \in (0, \delta)$ such that $z, w \in \Sigma_{\delta'}$. Next choose γ as in (I.15) and let $\tilde{\gamma} := \gamma + c$, where $c \in \mathbb{C}$ is such that $\gamma \cap \tilde{\gamma} = \emptyset$. Now compute

$$
\begin{aligned}
T(z)T(w) &= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\tilde{\gamma}} e^{\mu z}e^{\lambda w}(\mu - A)^{-1}(\lambda - A)^{-1}\,d\lambda d\mu\\
&= \frac{1}{(2\pi i)^2}\int_\gamma\int_{\tilde{\gamma}} e^{\mu z}e^{\lambda w}(\lambda - \mu)^{-1}\left[(\mu - A)^{-1} - (\lambda - A)^{-1}\right]\,d\lambda d\mu\\
&= \frac{1}{2\pi i}\int_\gamma e^{\mu z}(\mu - A)^{-1}\left(\frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{e^{\lambda w}}{\lambda - \mu}\,d\lambda\right)d\mu\\
&\qquad\qquad - \frac{1}{2\pi i}\int_{\tilde{\gamma}} e^{\lambda w}(\lambda - A)^{-1}\left(\frac{1}{2\pi i}\int_\gamma \frac{e^{\mu z}}{\lambda - \mu}\,d\mu\right)d\lambda,
\end{aligned}
$$

where we have used Fubini's theorem and the resolvent identity (I.2). Now, Cauchy's integral theorem implies that

$$
\frac{1}{2\pi i}\int_\gamma \frac{e^{\mu z}}{\lambda - \mu}\,d\mu = 0,
$$

since all $\lambda \in \tilde{\gamma}$ lie outside γ. On the other hand, again by Cauchy's integral formula,

$$
\frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{e^{\lambda w}}{\lambda - \mu}\,d\lambda = e^{\mu w}.
$$

Plugging these identities back into our expression for $T(z)T(w)$ we obtain

$$
\begin{aligned}
T(z)T(w) &= \int_{\tilde{\gamma}} e^{\mu(z+w)}(\mu - A)^{-1}\,d\mu\\
&= T(z + w).
\end{aligned}
$$

It remains to verify (iii) of Definition I.2.22. Since the definition of $T(z)$ is independent of the path γ, let us assume that $\gamma = \gamma_1$ in the following (cf. (I.15)). Since by Cauchy's integral theorem, $\frac{1}{2\pi i}\int_{\gamma_1} \frac{e^{\mu z}}{\mu} d\mu = 1$ for $z \in \Sigma_{\delta'}$, we can compute for $x \in \operatorname{dom}(A)$

$$\begin{aligned} T(z)x - x &= \frac{1}{2\pi i}\int_{\gamma_1} e^{\mu z}\left((\mu - A)^{-1} - \frac{1}{\mu}\right) x\, d\mu \\ &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{e^{\mu z}}{\mu}(\mu - A)^{-1} A x\, d\mu \end{aligned}$$

for all $z \in \Sigma_{\delta'}$, where we have used the identity $(\mu - A)^{-1}Ax = \mu(\mu - A)^{-1}x - x$ which holds for all $x \in \operatorname{dom}(A)$. By (I.16), we have the estimate

$$\left\| \frac{e^{\mu z}}{\mu}(\mu - A)^{-1}Ax \right\| \le \frac{M_\varepsilon}{|\mu|^2}\left(1 + e^{|z|}\right)\|Ax\| \tag{I.19}$$

for all $\mu \in \gamma$ and $z \in \Sigma_{\delta'}$. This yields an integrable majorant uniformly in z near 0. Applying Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{\substack{z\to 0\\ z\in\Sigma_{\delta'}}} T(z)x - x = \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{\mu}(\mu - A)^{-1}Ax\, d\mu = 0,$$

where the second equality follows by closing the path γ_1 on the right by circles of increasing diameter and using Cauchy's integral theorem. The integrals over the circles tend to zero with increasing diameter by estimate (I.19).

This settles condition (iii) for all $x \in \operatorname{dom}(A)$ and the corresponding statement for all $x \in \mathcal{H}$ follows by the uniform boundedness (i). □

This theorem finally justifies the

Definition I.2.24. If A satisfies Hypothesis I.2.20 and the semigroup $(T(z))_{z\in\Sigma_\delta}$ is defined by (I.18), then we call A the *generator* of $(T(z))_{z\in\Sigma_\delta}$.

It can be shown that if A generates the analytic semigroup $(T(z))_{z\in\Sigma_\delta}$ in the sense of Definition I.2.24, then A is also the generator of the strongly continuous semigroup $(T(z))_{z\ge 0}$ in the sense of Definition I.2.3 (cf. [EN00, Ch. II.4]).

Recalling Definition I.2.10, we immediately conclude the following

Proposition I.2.25. *Let A be a sectorial operator with vertex γ such that* $\operatorname{Re}\gamma \ge 0$. *Then $-A$ generates an analytic semigroup.*

Proof. It follows immediately from Proposition I.1.23 that $-A$ satisfies the conditions in Hypothesis I.2.20. □

I.2.5. Spectral Theory for Semigroups and Generators

We have already seen in Corollary I.2.7 that being a generator imposes certain restrictions on the spectrum and resolvent of an operator A. In this section we will investigate this point further and ask to what extend the special classes of semigroups discussed in the previous section impose stronger restrictions on the spectrum of the generator.

Spectral bound. As a first step to execute the above plan, let us study how growth and decay properties of the semigroup affect the location of its generator's spectrum. The reader is encouraged to recall the definition of the growth bound, eq. (I.9).

Definition I.2.26. Let $A : \mathcal{H} \supset \mathrm{dom}(A) \to \mathcal{H}$ be a closed operator. Then

$$s(A) := \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$$

is called the *spectral bound* of A.

In order to prove the next proposition we need the following elementary fact from analysis which we quote without proof.

Lemma I.2.27. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be bounded on compact intervals and subadditive, i.e. $\varphi(s + t) \le \varphi(s) + \varphi(t)$ for all $s, t \ge 0$. Then*

$$\inf_{t>0} \frac{\varphi(t)}{t} = \lim_{t\to 0} \frac{\varphi(t)}{t}$$

exists. □

Proposition I.2.28. *Let A be the generator of a strongly continuous semigroup with growth bound $\omega_0 := \omega_0((T(t))_{t\ge 0})$. Then one has*

$$\begin{aligned} -\infty \le s(A) \le \omega_0 &= \inf_{t>0} \frac{\log \|T(t)\|}{t} = \lim_{t\to 0} \frac{\log \|T(t)\|}{t} \\ &= \frac{\log r(T(t_0))}{t_0} < \infty \end{aligned}$$

for any $t_0 > 0$, where $r(T(t))$ denotes the spectral radius (cf. Definition I.1.12 (iii). In particular, one has

$$r(T(t)) = e^{\omega_0 t} \qquad \text{for all } t \ge 0. \tag{I.20}$$

Proof. Define $\varphi(t) := \log\|T(t)\|$. Then φ is bounded on compact intervals because of (I.8) and it is subadditive because $\varphi(t+s) = \log\|T(t+s)\| = \log\|T(t)T(s)\| \leq \log(\|T(t)\|\|T(s)\|) = \log\|T(t)\| + \log\|T(s)\| = \varphi(t)+\varphi(s)$. Hence we can apply Lemma I.2.27 and infer that

$$v := \inf_{t>0} \frac{\log\|T(t)\|}{t} = \lim_{t\to 0} \frac{\log\|T(t)\|}{t}.$$

exists. It follows that $e^{vt} \leq e^{\log\|T(t)\|} = \|T(t)\|$ for all $t \geq 0$, hence $v \leq \omega_0$, by the definition of ω_0. Now let $w > v$. Then by the definition of v there exists $t_0 > 0$ such that

$$\frac{\log\|T(t)\|}{t} \leq w \qquad \text{for all } t \geq t_0,$$

hence $\|T(t)\| \leq e^{tw}$ for $t \geq t_0$. This implies that there exists $M > 0$ such that for *all* $t \geq 0$

$$\|T(t)\| \leq Me^{wt},$$

i.e. $w \geq \omega_0$. Overall we have proved that $v \leq \omega_0$ and $w > \omega_0$ for every $w > v$ and hence $v = \omega_0$.

To prove (I.20), we use Lemma I.1.13 to compute

$$\begin{aligned} r(T(t)) &= \lim_{n\to\infty} \|T(t)^n\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} e^{t\cdot\frac{\log\|T(nt)\|}{nt}} \\ &= e^{t\cdot\lim_{n\to\infty}\frac{\log\|T(nt)\|}{nt}} \\ &= e^{t\omega_0}. \end{aligned}$$

The inequalities $-\infty \leq s(A) \leq \omega_0 < \infty$ follow immediately from Corollary I.2.7. $\square$

Spectral Mapping Theorems. A question which is immediate in the spectral theory of semigroups and their generators is whether there exist any relations between the spectrum of an operator A and its semigroup $(T(t))_{t\geq 0}$. Naively one would expect a relation of the form

$$\sigma(T(t)) = \left\{ e^{\lambda t} : \lambda \in \sigma(A) \right\}$$

similar to the situation in Theorem I.1.14. However, in the case of semigroups the situation is more complicated and one cannot expect a spectral mapping theorem of the above form without assuming any additional structure. In the most general situation the best one can achieve is the following spectral inclusion which is an immediate consequence of Lemma I.2.5.

Theorem I.2.29 (Spectral inclusion theorem)**.** *Let A be the generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$. Then for all $t \geq 0$*

$$\sigma(T(t)) \supset \left\{ e^{\lambda t} : \lambda \in \sigma(A) \right\}, \tag{I.21}$$

$$\sigma_p(T(t)) \supset \left\{ e^{\lambda t} : \lambda \in \sigma_p(A) \right\} \tag{I.22}$$

Proof. To prove (I.21), let $\lambda \in \mathbb{C}$ and denote by $S(t) := e^{-\lambda t} T(t)$ the rescaled semigroup whose generator is $A - \lambda$ as can be seen by differentiating at $t = 0$. Lemma I.2.5 (iii) applied to $(S(t))_{t\geq 0}$ yields

$$e^{-\lambda t} T(t)x - x = (A - \lambda) \int_0^t e^{-\lambda s} T(s)x \, ds \qquad \text{for } x \in \mathcal{H} \tag{I.23}$$

$$= \int_0^t e^{-\lambda s} T(s)(A - \lambda)x \, ds \qquad \text{for } x \in \mathrm{dom}(A). \tag{I.24}$$

Multiplying these identities with $e^{\lambda t}$ shows that $e^{\lambda t} - T(t)$ is not bijective if $\lambda - A$ is not bijective.

To see (I.22), let $\lambda_0 \in \sigma_p(A)$ and let $x_0 \in \mathrm{dom}(A)$ be a corresponding eigenvector. From (I.24) we conclude

$$\begin{aligned} T(t)x_0 - e^{\lambda_0 t} x_0 &= \int_0^t e^{\lambda_0 (t-s)} T(s)(A - \lambda_0)x_0 \, ds \\ &= 0. \end{aligned}$$

Hence, x_0 is an eigenvector of $T(t)$ with eigenvalue $e^{\lambda_0 t}$ for all $t \geq 0$. □

In the following we will limit ourselves to proving a spectral mapping theorem for the *point spectrum* σ_p which is enough for our purposes, i.e. we will show the converse inclusion in eq. (I.22). In fact, the converse inclusion in (I.21) also holds true under certain conditions, e.g. when the semigroup is eventually norm continuous. The interested reader may indulge in [EN00, Ch. IV.3].

In order to prove our spectral mapping theorem, we have to take a quick excursion into the theory of periodic semigroups.

Definition I.2.30. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$ is called *periodic* if there exists $t_0 > 0$ such that $T(t_0) = \mathrm{id}_{\mathcal{H}}$. In such a case, we call

$$\tau := \inf\{t > 0 : T(t) = \mathrm{id}\}$$

the *period* of $(T(t))_{t\geq 0}$.

Lemma I.2.31. *Let $(T(t))_{t\geq 0}$ be a periodic strongly continuous semigroup with period $\tau > 0$ and generator A. Then*

$$\sigma(A) \subset \frac{2\pi i}{\tau}\mathbb{Z} \qquad \textit{and} \tag{I.25}$$

$$(\lambda - A)^{-1} = \frac{1}{1 - e^{-\lambda\tau}} \int_0^\tau e^{-\lambda s} T(s)\, ds \qquad \textit{for } \lambda \notin \frac{2\pi i}{\tau}\mathbb{Z}. \tag{I.26}$$

Proof. Let $\lambda \in \mathbb{C} \setminus \frac{2\pi i}{\tau}\mathbb{Z}$ and consider eqs. (I.23), (I.24) with $t = \tau$

$$\begin{aligned}(e^{-\lambda\tau} - 1)x &= (A - \lambda) \int_0^\tau e^{-\lambda s} T(s) x\, ds && \text{for } x \in \mathcal{H} \\ &= \int_0^\tau e^{-\lambda s} T(s)(A - \lambda) x\, ds && \text{for } x \in \mathrm{dom}(A).\end{aligned}$$

Since $(e^{-\lambda\tau} - 1)$ is nonzero by assumption, the first equation shows that $\lambda - A$ is surjective while the second shows that $\lambda - A$ is injective. Hence $\lambda \notin \sigma(A)$. □

The formula (I.26) for the resolvent of A shows that near a point $\frac{2\pi i k}{\tau}$, $(\lambda - A)^{-1}$ has at worst a simple pole. This fact can be exploited to prove the following

Lemma I.2.32. *Let A be as in Lemma I.2.31. Then $\sigma(A)$ is nonempty and we have*

$$\sigma(A) = \sigma_p(A).$$

Proof. Denote $\mu_k := \frac{2\pi i k}{\tau}$ with $k \in \mathbb{Z}$ and let $x \in \mathcal{H}$. Applying $(\lambda - A)$ to eq. (I.26) yields

$$x = \frac{1}{1 - e^{-\lambda\tau}} (\lambda - A) \int_0^\tau e^{-\lambda s} T(s) x\, ds$$

(note that $\int_0^\tau e^{-\lambda s} T(s)\, ds$ maps into $\mathrm{dom}(A)$ by Lemma I.2.5). Multiplying this equation by $(\lambda - \mu_k)$ and letting $\lambda \to \mu_k$ we get

$$\begin{aligned}0 &= (\mu_k - A) \frac{1}{\tau} \int_0^\tau e^{-\mu_k s} T(s) x\, ds \\ &=: (\mu_k - A) P_k x,\end{aligned}$$

that is, we have $\operatorname{Ran} P_k \subset \ker(\mu_k - A)$ for every $k \in \mathbb{Z}$.

It remains to show that $P_k \neq 0$ if $\mu_k \in \sigma(A)$. To this end, let us first note that we have in fact

$$\begin{aligned}\left\|\frac{1}{\tau}\int_0^\tau e^{-\lambda s}T(s)\,ds - P_k\right\| &= \left\|\int_0^\tau (e^{-\lambda s} - e^{\mu_k s})T(s)\,ds\right\| \\ &\leq \|e^{-\lambda \cdot} - e^{\mu_k \cdot}\|_{L^\infty([0,\tau])}\int_0^\tau \|T(s)\|\,ds \\ &\to 0 \qquad \text{as } \lambda \to \mu_k,\end{aligned}$$

i.e. we have $\frac{1}{\tau}\int_0^\tau e^{-\lambda s}T(s)\,ds \to P_k$ in the *operator norm* topology. Now, let $\mu_k \in \sigma(A)$ and go back to eq. (I.26) which immediately yields

$$\begin{aligned}\operatorname{dist}(\lambda, \sigma(A))\|(\lambda - A)^{-1}\| &= \frac{\operatorname{dist}(\lambda, \sigma(A))}{|1 - e^{-\lambda\tau}|}\left\|\int_0^\tau e^{-\lambda s}T(s)\,ds\right\| \\ &\leq \frac{|\mu_k - \lambda|}{|1 - e^{-\lambda\tau}|}\left\|\int_0^\tau e^{-\lambda s}T(s)\,ds\right\|.\end{aligned}$$

Now, if $P_k = 0$, the right-hand side of this equation converges to 0 as $\lambda \to \mu_k$. By Corollary I.6, this is only possible if $\mu_k \notin \sigma(A)$. □

Theorem I.2.33 (Spectral mapping theorem for the point spectrum)**.** *Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathcal{H}$. Then one has*

$$\sigma_p(T(t)) \setminus \{0\} = \left\{e^{\lambda t} : \lambda \in \sigma_p(A)\right\} \qquad \textit{for all } t \geq 0.$$

Proof. By Theorem I.2.29 it only remains to prove the inclusion "$\subset$". Let $t_0 > 0$ and $\lambda \in \sigma_p(T(t_0)) \setminus \{0\}$. First note that by considering the rescaled semigroup $S(t) = e^{-t\log\lambda}T(tt_0)$ with generator $B := t_0 A - \log\lambda$ we can assume w.l.o.g. that $t_0 = \lambda = 1$. Indeed, for this rescaled semigroup, 1 is an eigenvalue of $S(1)$.

Using these assumptions, consider the subspace

$$V := \{x \in \mathcal{H} : T(1)x = x\}$$

which is invariant under $T(t)$ for every $t \geq 0$ and nonempty by assumption. This allows us to define the family of restrictions $\left(T(t)|_V\right)_{t\geq 0}$ which can easily be seen to be a strongly continuous one-parameter semigroup with generator $A|_V$. Moreover, this semigroup is periodic by definition of V with some period $\tau \in \left\{n^{-1} : n \in \mathbb{N}\right\}$. By Lemmas I.2.31, I.2.32, we have $\emptyset \neq \sigma_p(A|_V) \subset \frac{2\pi i}{\tau}\mathbb{Z}$, that is, we can find $k \in \mathbb{Z}$ such that $\frac{2\pi i k}{\tau} \in \sigma_p(A|_V) \subset \sigma_p(A)$. Accordingly,

$$e^{2\pi i \frac{k}{\tau}} = 1,$$

since $\tau^{-1} \in \mathbb{N}$. We conclude that $1 \in \{e^{\lambda t} : \lambda \in \sigma_p(A)\}$. $\square$

The above spectral mapping theorem readily implies the following important corollary which will be used in Part III.

Corollary I.2.34. *Let $A : \mathcal{H} \supset \mathrm{dom}(A) \to \mathcal{H}$ generate the strongly continuous semigroup $(T(t))_{t\geq 0}$ and assume that $(T(t))_{t\geq 0}$ is eventually compact. Then*

(i) The spectrum of A consists of isolated points in $\mathbb{C}$ and $\sigma(A) = \sigma_p(A)$;

(ii) One has $\sigma(T(t)) \setminus \{0\} = \{e^{\lambda t} : \lambda \in \sigma(A)\}$ for all $t \geq 0$

Proof. Let $t_0 > 0$ such that $T(t_0)$ is compact. Then $\sigma(T(t_0)) = \sigma_p(T(t_0))$ by the spectral theory of compact operators. The above spectral inclusion and spectral mapping theorems now give the identities

$$\begin{aligned} \left\{e^{\mu t} : \mu \in \sigma(A)\right\} &\subset \sigma(T(t)) \setminus \{0\} \\ &= \sigma_p(T(t)) \setminus \{0\} \\ &= \left\{e^{\lambda t} : \lambda \in \sigma_p(A)\right\} \\ &\subset \left\{e^{\mu t} : \mu \in \sigma(A)\right\} \end{aligned}$$

for all $t \geq t_0$. We conclude that $\{e^{\lambda t} : \lambda \in \sigma_p(A)\} = \{e^{\mu t} : \mu \in \sigma(A)\}$ for all $t \geq t_0$ which implies $\sigma(A) = \sigma_p(A)$. $\square$

I.3. Convergent Sequences of Unbounded Operators

Consider a sequence $(A_n)_{n\in\mathbb{N}}$ of closed operators $A_n : \mathcal{H} \supset \mathrm{dom}(A_n) \to \mathcal{H}$. It is already familiar from the theory of bounded operators on Banach spaces that different notions of convergence have to be studied (e.g. strong convergence versus convergence in operator norm). However, in the situation of unbounded operators neither strong convergence nor operator norm convergence can a priori be defined in a meaningful way. The former is ill-defined because the domains of the A_n may depend on n, while the latter fails simply because $\|A_n\|_{\mathcal{L}(\mathcal{H})}$ does not exist. The solution to this problem is to not consider the operators A_n directly, but rather study their *resolvents*. In this way, the question of convergence of unbounded operators is reduced to a question about *bounded* operators which behave in a much more tame way. The drawback is, of course, that the resolvents of the A_n must *exist*, that is, there has to be a $\lambda \in \mathbb{C}$ such that $\lambda \in \rho(A_n)$ for all n. To ensure that this is always the case, we restrict our attention to m-accretive operators (cf. Definition I.2.10). The results in this section are classical and versions of them can be found in [RS80, Kat95].

Definition I.3.1. Let $A : \mathrm{dom}(A) \to \mathcal{H}$ and $A_n : \mathrm{dom}(A_n) \to \mathcal{H}$ be m-accretive for all $n \in \mathbb{N}$. We say that $(A_n)_{n\in\mathbb{N}}$ converges to A

(i) in the *strong resolvent sense* if $(\mathrm{id} + A_n)^{-1}x \to (\mathrm{id} + A)^{-1}x$ for all $x \in \mathcal{H}$,

(ii) in the *norm resolvent sense* if $\left\|(\mathrm{id} + A_n)^{-1} - (\mathrm{id} + A)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \to 0$.

The following two propositions demonstrate that this is a reasonable definition.

Proposition I.3.2. *If $(B_n)_{n\in\mathbb{N}}$ is a sequence of bounded operators $B_n : \mathcal{H} \to \mathcal{H}$, then $B_n \to B$ in norm resolvent sense if and only if $\|B_n - B\|_{\mathcal{L}(\mathcal{H})} \to 0$.*

Proof. It is easy to see that for any z with $\mathrm{Re}\, z < 0$ the formulas

$$(z - B)^{-1} - (z - B_n)^{-1} = (z - B)^{-1} (B_n - B) (z - B_n)^{-1} \tag{I.27}$$

$$B_n - B = (B_n - z) \left((z - B)^{-1} - (z - B_n)^{-1}\right) (B - z) \tag{I.28}$$

hold from which the assertion follows immediately. □

Proposition I.3.3. *One has $A_n \to A$ in norm resolvent sense if and only if $\big\|(\lambda + A_n)^{-1} - (\lambda + A)^{-1}\big\|_{\mathcal{L}(\mathcal{H})} \to 0$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < 0$.*

Proof. Assume that $\left\|(\mathrm{id} + A_n)^{-1} - (\mathrm{id} + A)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \to 0$ and let $\lambda \in \mathbb{C} \setminus \{-1\}$ with $\mathrm{Re}\,\lambda < 0$. A simple computation shows that we have the following identity for the resolvent at λ

$$(\lambda - A)^{-1} = -(\lambda + 1)^{-1} - (\lambda + 1)^{-2} \left(\tfrac{1}{\lambda+1} - (\mathrm{id} + A)^{-1}\right)^{-1} \tag{I.29}$$

with an analogous identity for A_n. For notational convenience, let us define $B := (\mathrm{id} + A)^{-1}$, $B_n := (\mathrm{id} + A_n)^{-1}$ and $z := \frac{1}{\lambda+1}$. Equation (I.29) applied to A and A_n yields for their difference

$$\begin{aligned}\left\|(\lambda - A)^{-1} - (\lambda - A_n)^{-1}\right\| &= |\lambda + 1|^{-2} \left\|(z - B_n)^{-1} - (z - B)^{-1}\right\| \\ &\leq |\lambda + 1|^{-2} \left\|(z - B)^{-1}\right\| \left\|(z - B_n)^{-1}\right\| \|B_n - B\|,\end{aligned}$$

where we have used eq. (I.28) in the second line. The right-hand side of the above equation converges to 0 because $\|B_n - B\| \to 0$ by assumption and $\left\|(z-B)^{-1}\right\|$ remains bounded since $(\lambda+1)^{-1}$ has a fixed distance from $\sigma(B_n)$ for all n.

□

Next we show that the concept of norm-resolvent convergence is not only reasonable but actually very useful in spectral analysis.

Theorem I.3.4. *Let A_n be a sequence of m-accretive operators converging to A in norm resolvent sense. Then*

(i) for every compact $K \subset \rho(A)$ there exists $N \in \mathbb{N}$ such that $K \subset \rho(A_n)$ for all $n > N$.

(ii) For any $U \subset \mathbb{C}$ such that $U \subset \rho(A_n)$ for almost all n one has $U \subset \rho(A)$.

Proof. We first prove (i). W.l.o.g. we may assume that K lies in the right half plane. Let $K \subset \rho(A)$ be compact. For $\lambda \in K$ denote $z := \frac{1}{1+\lambda}$ and note that

$$\left\|\left(z-(1+A)^{-1}\right)-\left(z-(1+A_n)^{-1}\right)\right\| = \left\|(1+A)^{-1}-(1+A_n)^{-1}\right\| \qquad \text{(I.30)}$$

Since $\lambda \in \rho(A)$ we have $z \in \rho((1+A)^{-1})$ by Theorem I.1.14 and $\left(z-(1+A)^{-1}\right)$ is boundedly invertible. Since the set of invertible operators is open in $\mathcal{L}(\mathcal{H})$, eq. (I.30) implies that $\left(z-(1+A_n)^{-1}\right)$ is boundedly invertible for n large enough and we conclude that $z \in \rho((1+A_n)^{-1})$. Since the resolvent set is open, it follows immediately that $w \in \rho((1+A_n)^{-1})$ for all w in an open neighbourhood of z (which can be chosen independent of n by convergence of $\|(\lambda - A_n)^{-1}\|$). Applying Theorem I.1.14 again we conclude that an open neighbourhood of λ is contained in $\rho(A_n)$ for all sufficiently large n.

This procedure yields an open covering $\{U_\lambda\}_{\lambda\in K}$ of K such that for each λ there exists $n_\lambda \in \mathbb{N}$ such that $U_\lambda \subset \rho(A_n)$ for all $n > n_\lambda$. By compactness of K we can extract finitely many $U_{\lambda_1}, \ldots, U_{\lambda_m}$ such that $K \subset \bigcup_{k=1}^m U_{\lambda_k}$ which implies that $K \subset \rho(A_n)$ for all $n > \max\{n_{\lambda_1}, \ldots, n_{\lambda_m}\}$.

Assertion (ii) follows by an analogous argument. □

Corollary I.3.5. *If $A_n \to A$ in norm resolvent sense and $\lambda \in \sigma(A)$ then there exists a sequence (λ_n) such that $\lambda_n \in \sigma(A_n)$ for all n and $\lambda_n \to \lambda$.*

Proof. We argue by contradiction. Assume that there were no such sequence (λ_n). Then there exists an ε-neighbourhood $B_\varepsilon(\lambda)$ with $B_\varepsilon(\lambda) \subset \rho(A_n)$ for all n. By Theorem

I.3.4 (ii) we would have $B_\varepsilon(\lambda) \subset \rho(A)$ and thus $\lambda \in \rho(A)$ which contradicts our assumption. □

Corollary I.3.6. *Every bounded sequence* (λ_n) *with* $\lambda_n \in \sigma(A_n)$ *for all* n *has an accumulation point in* $\sigma(A)$.

Proof. Proof by contradiction. Assume that no accumulation point in $\sigma(A)$ exists. Then we can extract a subsequence (λ_{n_k}) such that the compact set $K := \overline{\{\lambda_{n_k} : k \in \mathbb{N}\}}$ is contained in $\rho(A)$. By Theorem I.3.4 (i) we would have $K \subset \rho(A_{n_k})$ for large k contradicting the assumption that $\lambda_n \in \sigma(A_n)$ for all n. □

Theorem I.3.4 implies that for every compact $L \subset \mathbb{C}$ the sets $L \cap \sigma(A_\varepsilon)$ converge to $L \cap \sigma(A)$ in the *Hausdorff sense* (see e.g. [RW98]):

Definition I.3.7. Let $M, N \subset \mathbb{C}$ be two nonempty subsets. The *Hausdorff distance* between M and N is defined as

$$\begin{aligned} d_H(M,N) &:= \max\left\{ \sup_{x\in M} \inf_{y\in N} |x-y| \,,\, \sup_{y\in N} \inf_{x\in M} |x-y| \right\} \\ &= \inf\bigl\{\varepsilon > 0 : M \subset U_\varepsilon(N) \text{ and } N \subset U_\varepsilon(M)\bigr\}, \end{aligned}$$

where $U_\varepsilon(\cdot)$ denotes the ε-neighbourhood of a set. A sequence of sets $(M_n) \subset \mathbb{C}$ is said to converge to $M \subset \mathbb{C}$ in the Hausdorff sense, if $d_H(M_n, M) \to 0$ as $n \to \infty$.

Indeed, let $L \subset \mathbb{C}$ be compact and $\varepsilon > 0$. Put $K_\varepsilon := L \setminus U_\varepsilon(\sigma(A))$. If $A_n \to A$ in the norm resolvent sense, Theorem I.3.4 (i) states that $K_\varepsilon \subset \rho(A_n)$ for almost all n. Hence, for almost all n we have $L \cap \sigma(A_n) \subset L \cap U_\varepsilon(\sigma(A))$. An analogous argument using Theorem I.3.4 (ii) shows that $L \cap \sigma(A) \subset L \cap U_\varepsilon(\sigma(A_n))$ for almost all n, which concludes the proof.

II. Introduction and Previous Work

The previous sections have shown the relevance of norm-resolvent estimates for both pure mathematics and applications. We have already seen two contexts in which these estimates are particularly relevant: the generation of strongly continuous semigroups (cf. Theorem I.2.8) and the convergence of spectra (cf. Theorem I.3.4).

This thesis studies two mathematical problems which illustrate the importance of norm-resolvent estimates in these two contexts. We will first demonstrate the amount of information contained in the resolvent norm in the context of non-selfadjoint operators and then take a more general point of view and consider *sequences* of operators and norm-resolvent convergence.

II.1. Pseudospectra

We have seen in Section I.1 that if A is a *selfadjoint* operator, the spectrum of A contains a great deal of information about A, such as (cf. Theorems I.2.8, I.1.20 and Corollary I.1.21)

- Does A generate a one-parameter semigroup?
- Large t-behaviour of $\|e^{-tA}\|$,
- Norm of the resolvent $\|(z-A)^{-1}\|$ for arbitrary $z \in \rho(A)$,
- Location of $\sigma(A+V)$ if V is a bounded perturbation.

In addition, if A has compact resolvent, the eigenvectors of A form a basis, by the spectral theorem for compact operators and Theorem I.1.14.

For *non-selfadjoint* (NSA) operators, however, *none* of the above properties can, in general, be deduced from the spectrum. This demonstrates that for NSA operators the spectrum by itself contains very little information about A. Due to the lack of the Spectral Theorem, the spectral theory of such operators is quite rich and yields interesting phenomena. NSA operators have to be carefully controlled and failure to do so can lead to undesired outcomes [Gre12]. The following example provides an informative illustration of this fact. For $c \in \mathbb{R}$ consider the non-normal differential operator

$$H_c = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3 + cx^2 \tag{II.1}$$

on its maximal domain $\mathrm{dom}(H_c) = \{\phi \in L^2(\mathbb{R}) : H_c\phi \in L^2(\mathbb{R})\}$. A numerical plot of the spectrum of H_c is shown in Figure II.1.

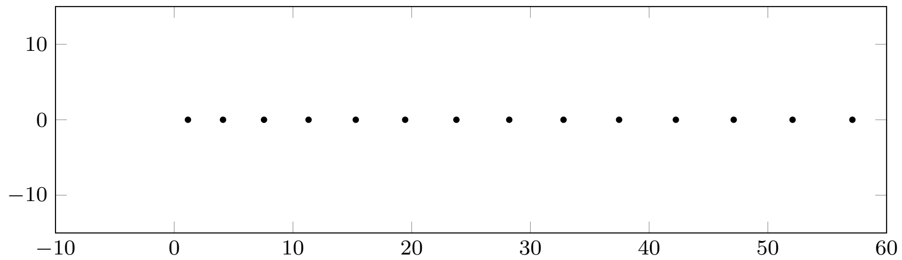


Figure II.1.: The spectrum of H_c for $c = 1$, obtained in MATLAB using the EigTool package and a modified code from [Tre01, TE05].

It was shown in [DDT01] that the spectrum of H_c is indeed real and positive. Moreover, H_c is closed and has compact resolvent [CGM80, Mez01] so the spectrum is also discrete. On the other hand, Novák and Krejčiřík have obtained the following result

Theorem II.1.1 ([Nov14])**.** *The operator H_c has the following properties:*

(i) The eigenfunctions of H_c do not form a (Schauder) basis in $L^2(\mathbb{R})$.

(ii) $-iH_c$ does not generate a bounded semigroup.

(iii) H_c is not similar to a self-adjoint operator via bounded and boundedly invertible transformations.

This theorem makes it clear that H_c is very different from a selfadjoint operator even though its spectrum looks well-behaved.

The above considerations motivate the definition of a *finer* indicator than the spectrum for non-selfadjoint operators.

Definition II.1.2. For any closed operator A and $\varepsilon > 0$ the set

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{ z \in \rho(A) : \|(z - A)^{-1}\| > \tfrac{1}{\varepsilon} \right\}$$

is called the ε-*pseudospectrum* of A.

By Corollary I.1.16, the ε-pseudospectrum always contains an ε-neighbourhood of the spectrum. Moreover, Corollary I.1.21 shows that the ε-pseudospectrum of a selfadjoint operator is always equal to the set $\{z \in \mathbb{C} : \operatorname{dist}(\sigma(A), z) < \varepsilon\}$. In particular, the spectrum and the pseudospectrum contain the same amount of information about the operator in the selfadjoint case. As we will see, in the non-selfadjoint case the pseudospectrum contains significantly more information about the operator than the spectrum. We begin with a theorem concerning bounded perturbations. The proof we present here is taken from [TE05].

Theorem II.1.3. *Let A be a closed operator on $\mathcal{H}$. One has*

$$\sigma_\varepsilon(A) = \bigcup_{\|V\|_{\mathcal{L}(\mathcal{H})} \leq \varepsilon} \sigma(A + V).$$

Proof. We first prove the inclusion $\sigma_\varepsilon(A) \supset \bigcup_{\|V\|_{\mathcal{L}(\mathcal{H})} \leq \varepsilon} \sigma(A + V)$. Let $\lambda \in \mathbb{C} \setminus \sigma_\varepsilon(A)$ and V be bounded with $\|V\| < \varepsilon$. Then we can write

$$\lambda - (A + V) = (\mathrm{id} - V(\lambda - A)^{-1})(\lambda - A).$$

By assumption on V we have $\|V(\lambda - A)^{-1}\| < \varepsilon\|(\lambda - A)^{-1}\| \leq 1$ and hence $\mathrm{id} - V(\lambda - A)^{-1}$ is invertible by means of the Neumann series. We conclude that $\lambda \notin \sigma(A + V)$.

To prove the converse inclusion, let $\lambda \in \sigma_\varepsilon(A)$. By definition of the operator norm, there exists $x \in \mathcal{H}$ with $\|x\| = 1$ such that $\|(\lambda - A)^{-1}x\| > \frac{1}{\varepsilon}$, or equivalently, there exists $y \in \operatorname{dom}(A)$ such that $\|y\| = 1$ and $\|(\lambda - A)y\| < \varepsilon$. By the Hahn-Banach theorem there exists an operator $V \in \mathcal{L}(\mathcal{H})$ such that $V(y) = -(\lambda - A)y$ and $\|V\| = \|(\lambda - A)y\| < \varepsilon$. By construction, $\ker(\lambda - A - V) \neq \emptyset$ and thus $\lambda \in \sigma(A+V)$. $\square$

This theorem shows that the spectra of slightly perturbed operators must always be contained in the pseudospectrum. Consequently, if the ε-pseudospectrum of an operator A is large, a perturbation V with $\|V\| < \varepsilon$ might alter the spectrum of A dramatically, while of $\sigma_\varepsilon(A)$ is small, the spectrum of A is stable under such perturbations. This general picture even extends beyond bounded perturbations as demonstrated by the following classical theorem which we quote without proof.

Theorem II.1.4 ([Kat95, Th. IV.3.17])**.** *Let A be a closed operator in $\mathcal{H}$ and let B be an operator such that* $\operatorname{dom}(B) \supset \operatorname{dom}(A)$ *and* $\|Bx\| \leq a\|x\| + b\|Ax\|$ *for all* $x \in \operatorname{dom}(A)$ *with* $a > 0$ *and* $b \in (0, 1)$*. If there exists* $z \in \rho(A)$ *such that*

$$a\left\|(z - A)^{-1}\right\| + b\left\|A\,(z - A)^{-1}\right\| < 1 \tag{II.2}$$

then $S := A + B$ is closed and $z \in \rho(S)$ with

$$\left\|(z-S)^{-1}\right\| \leq \frac{\|(z-A)^{-1}\|}{1 - a\|(z-A)^{-1}\| - b\|A\,(z-A)^{-1}\|}. \tag{II.3}$$

□

Remark II.1.5. Operators B as in Theorem II.1.4 are said to be *relatively bounded* with respect to A and the number b is called its *relative bound.*

Numerical approximation of spectra. Formulas (II.2) and (II.3) clearly demonstrate the significance of the knowledge of $\|(z-A)^{-1}\|$. To illustrate this point, suppose that A is some differential operator and we would like to find a reasonable numerical approximation for $\sigma(A)$. Common methods typically discretise the domain on which A operates on a certain length scale h which leads to a finite-dimensional matrix S_h expected to approximate A. The spectrum of S_h can be readily computed by matrix factorisation methods. But clearly, passing from A to S_h constitutes a perturbation and a-priori it is not at all clear whether $\sigma(S_h)$ will be a good approximation of $\sigma(A)$, even if h is small, unless information about $\sigma_\varepsilon(A)$ is known. Thus, the pseudospectrum is an essential tool in assessing the reliability of such methods.

For $c = 0$ it was shown by Krejčiřík and Siegl [KSTV15] that the pseudospectrum of the operator H_c always contains an unbounded component. More precisely, they showed that for every $\delta > 0$ there exist constants $C_1, C_2 > 0$ such that for all $\varepsilon > 0$

$$\sigma_\varepsilon(H_0) \supset \left\{ z \in \mathbb{C} : |z| \geq C_1,\ |\arg z| < \left(\frac{\pi}{2} - \delta\right),\ |z| \geq C_2 \left(\log \frac{1}{\varepsilon}\right)^{6/5} \right\}. \tag{II.4}$$

This shows that the large eigenvalues of H_0 are highly unstable under small perturbations. A similar result for $c = 1$ was shown by Novák in [Nov14] and is easily extended to arbitrary $c > 0$. Figure II.2 shows a numerical computation of the pseudospectrum of H_1.

Equation (II.4) and Figure II.2 make it clear that for every fixed ε the pseudospectrum of H_c contains a whole sector in the complex plane for $c > 0$. Moreover, the opening angle of the sector can be chosen arbitrarily close to π provided that a ball of sufficiently large radius around 0 is removed. In particular, large eigenvalues are very unstable under small perturbations.

On the other hand, Figure II.2 suggests that the unbounded component of the

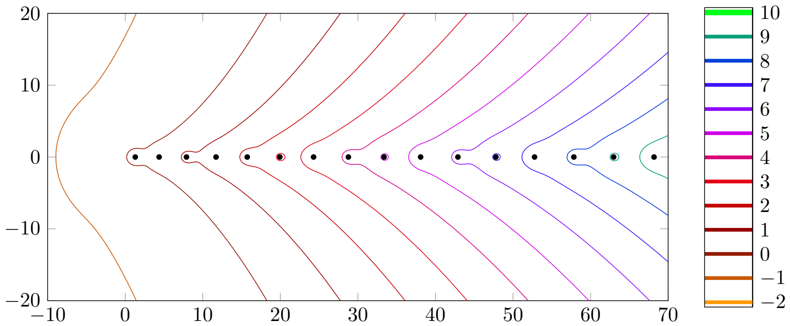


Figure II.2.: Numerical plot of the lines of constant resolvent norm of H_1 also obtained using the EigTool package and a modified code from [Tre01, TE05]. The colour bar shows the values of $\log_{10}(\|(\lambda - H_1)^{-1}\|)$.

pseudospectrum escapes towards $+\infty$ as $\varepsilon \to 0$. All of this suggests that the *lower* eigenvalues of H_c should indeed be stable (for $c > 0$) under small perturbations of H_c, despite the above results.

It should be noted that the operator H_c was first considered in the works of Bender et al. who studied it in the context of non-Hermitian Quantum Mechanics (see e.g. [BB98, BBM99, Ben07]). This theory is inspired by the desire to relax the condition of self-adjointness which is commonly imposed on quantum mechanical observables. Instead, a weaker condition known as $\mathcal{PT}$ symmetry is assumed: an operator H is called $\mathcal{PT}$ symmetric if $H\mathcal{PT} = \mathcal{PT}H$, where $\mathcal{P}\psi(x) = \psi(-x)$ and $\mathcal{T}\psi(x) = \overline{\psi(x)}$. Under certain additional assumptions, the spectrum of a $\mathcal{PT}$ symmetric operator can indeed be shown to be real [Mos02]. In this thesis we will not be concerned with the physical relevance of non-Hermitian Quantum Mechanics, but focus on the underlying mathematics whose applications extend beyond quantum theory.

Other examples of Schrödinger operators exhibit a similar behaviour. The so-called complex harmonic oscillator (or Davies oscillator) $-\frac{d^2}{dx^2} + ix^2$ on $L^2(\mathbb{R}^d)$ has been studied in [Dav00, Dav99]. It has a discrete spectrum and its ε-pseudospectrum contains an unbounded component for every $\varepsilon > 0$. An upper bound on the pseudospectrum has been found by Boulton [Bou02].

In Part III we will study a class of non-normal Schrödinger operators containing the operators H_c, $(c > 0)$. More precisely, we will prove an upper bound on the pseudospectrum of the operator $H = -\Delta + V$, where $\operatorname{Re} V(x) \geq c|x|^2 - b$ for some $c, b > 0$ on $L^2(\mathbb{R}^d)$, which complements the results of [KS12, Nov14]. Our method of proof is based on ideas from [Bou02].

II.2. Norm-Resolvent Convergence in Homogenisation

We have seen above that norm resolvent estimates give essential information about the quality of numerical estimates for the spectrum of an operator.

In certain applications however, numerical approximations are not feasible in the first place. In such situations, norm-resolvent estimates may be used to prove that an *effective model* with virtually the same physical properties may be considered instead. A popular field of research in which the above paradigm has been applied successfully for decades is the theory of *homogenisation* of which we will now give a brief introduction.

Suppose we are given a material with mechanical properties alternating on a fine length scale ε (e.g. a crystal, which has a fine periodic structure). Studying the physics of such media will involve the consideration of differential equations whose coefficients oscillate on a length scale ε. In the simplest (interesting) case, one is led to a scalar second order equation of the form

$$\begin{cases} A_\varepsilon u := -\nabla \cdot (a_\varepsilon \nabla u_\varepsilon) & = f \quad \text{in } \Omega \\ \qquad\qquad\qquad\quad u_\varepsilon & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{II.5}$$

where Ω denotes the region of space occupied by the periodic medium, $f \in L^2(\Omega)$ and $a_\varepsilon(x) = a\big(\frac{x}{\varepsilon}\big)$, where $a \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is a matrix valued function of period $Y \in \mathbb{R}^d$ such that $a(x)$ is symmetric for almost all x and there exists $\alpha > 0$ such that $\xi \cdot a(x)\xi \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and almost all x (cf. Figure II.3).

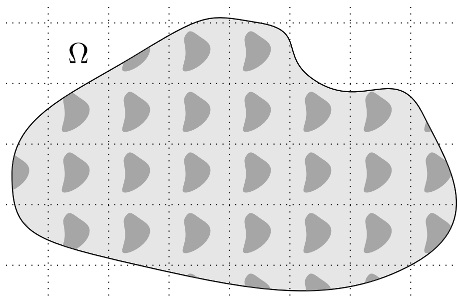


Figure II.3.: Sketch of the periodic medium in the domain Ω. The varying shades of grey indicate varying values of $a_\varepsilon(x)$

If we assume Ω to be bounded, problem (II.5) is easily seen to possess a unique weak solution u_ε by virtue of the Poincaré inequality and the Lax-Milgram theorem. However, if the period ε of the coefficients is much smaller than the spatial extent of the object Ω, this solution will oscillate on a very fine length scale (this is illustrated in Figure II.4 for a simple 1-dimensional problem). For such functions numerical approximation is not feasible, because e.g. in a finite element setting the triangulation of Ω would have to be finer than ε in order to resolve the oscillations of u which quickly becomes too computationally expensive. An idea to circumvent this problem is to "average out" the fine oscillations of u_ε while retaining its macroscopic behaviour. The result is expected to be a function varying on a finite length scale which can be resolved numerically. This process is known as *homogenisation*.

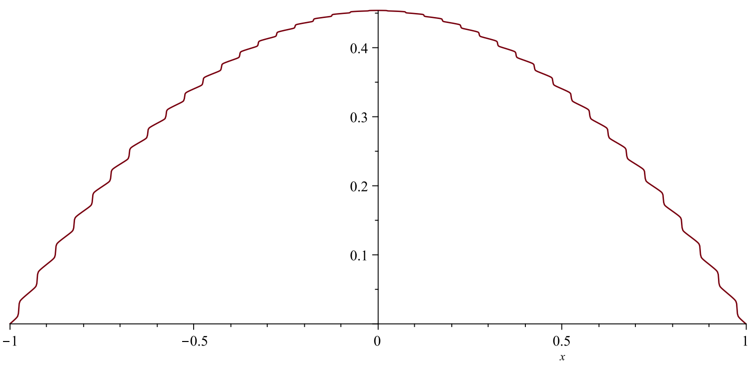


Figure II.4.: Plot of the real part of the solution to the equation $\frac{d}{dx}\left(e^{\frac{2\pi i x}{\varepsilon}} + 1.1\right)\frac{du_\varepsilon}{dx} = 1$ for $\varepsilon = 0.05$. We can clearly observe two features: (i) u oscillates on a length scale of order ≈ 0.05 and (ii) Besides the oscillations there exists a global shape describing a "macroscopic behaviour".

In the abstract framework of eq. (II.5), the idea of homogenisation leads to the following questions.

(i) Does the sequence of solutions (u_ε) converge to a unique limit u in L^2?

(ii) If so, does u satisfy any reasonable boundary value problem that can be computed from (II.5)?

If the answer to both of the above questions turns out to be affirmative, one refers to the limit problem satisfied by u as the *homogenised* problem.

For didactic purposes, let us investigate this question in the one-dimensional setting, i.e. let $\Omega = (a, b) \subset \mathbb{R}$ and assume that u_ε is a weak solution of (II.5). The variational formulation of (II.5) reads

$$\int_a^b a_\varepsilon u_\varepsilon' \varphi_\varepsilon' \, dx = \int_a^b \varphi f \, dx \qquad \text{for all } \varphi \in H_0^1((a,b))$$

Plugging in $\varphi = u_\varepsilon$ and using Poincaré's inequality and our assumptions on a immediately yields

$$\|u_\varepsilon\|_{H^1((a,b))} \leq C\|f\|_{L^2((a,b))} \tag{II.6}$$

for some $C > 0$. Hence there exists $u \in H_0^1((a,b))$ such that $u_\varepsilon \rightharpoonup u$ in H^1. Moreover, it is easy to see using periodicity that $a_\varepsilon \overset{*}{\rightharpoonup} \langle a \rangle$ in $L^\infty((a,b))$, where

$$\langle a \rangle = \frac{1}{Y}\int_0^Y a(y)\,dy$$

denotes the mean value of a. A crude guess for the homogenised equation might be that u satisfy $\frac{d}{dx}\left(\langle a \rangle \frac{du}{dx}\right) = f$, but this is *not correct* in general, as we show now. To this end, denote by $p_\varepsilon := a_\varepsilon u_\varepsilon'$ the *flux* of u_ε and note that we have

$$\begin{aligned}\|p_\varepsilon\|_{L^2((a,b))}^2 &\leq \|a\|_{L^\infty}\|u_\varepsilon'\|_{L^2((a,b))}^2, \qquad \|p_\varepsilon'\|_{L^2((a,b))}^2 = \|f\|_{L^2((a,b))}^2.\\ &\leq C\|f\|_{L^2((a,b))}^2\end{aligned}$$

Hence, p_ε is bounded in $H^1((a,b))$ by (II.6). Using the Rellich-Kondrachov theorem, we conclude that for a subsequence

$$p_\varepsilon \to p \qquad \text{in } L^2((a,b))$$

for some $p \in H^1((a,b))$. Combining our results we see that

$$a_\varepsilon^{-1}p_\varepsilon \rightharpoonup \left\langle a^{-1}\right\rangle p \qquad \text{weakly in } L^2((a,b)).$$

But on the other hand we also have $a_\varepsilon^{-1}p_\varepsilon = u_\varepsilon' \rightharpoonup u'$ weakly in $L^2((a,b))$, since $u_\varepsilon \rightharpoonup u$ in $H^1((a,b))$. We conclude that

$$\frac{du}{dx} = \left\langle a^{-1}\right\rangle p.$$

Finally, we note that $p' = f$, which follows from the definition of p_ε. We obtain the homogenised problem

$$Au := \frac{d}{dx}\left(\left\langle a^{-1}\right\rangle^{-1}\frac{du}{dx}\right) = f, \tag{II.7}$$

with the *homogenised coefficient matrix* $\left\langle a^{-1}\right\rangle^{-1}$ (which is 1×1 in our case). We conclude that questions (i), (ii) can be answered in the affirmative in the 1-dimensional case and that the "averaged" solution u is a good approximation for u_ε in the sense that $\|u_\varepsilon - u\|_{L^2((a,b))} \to 0$ as $\varepsilon \to 0$ (cf. Figure II.5).

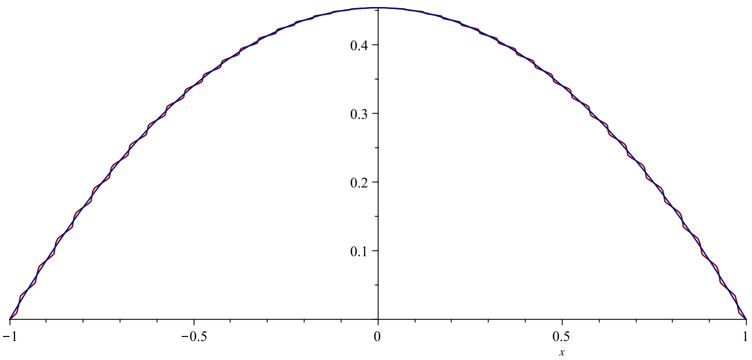


Figure II.5.: Plot of the real part of the solution u_ε from Figure II.4, together with the homogenised solution u, which displays the macroscopic behaviour of u_ε

A physical understanding of the homogenised coefficient $\langle a^{-1}\rangle^{-1}$ can be gained by the following interpretation: Equation (II.5) models the diffusion of particles in an inhomogeneous medium with diffusion constant a_ε (that is, a_ε is constant in time, but depends on space). Assume that there are enough diffusing particles around to be described by our deterministic model. For simplicity, let us further assume that a_ε alternates between two constant values, i.e.

$$a(x) = \begin{cases} \alpha_1, & \text{for } x \in [0, q) \\ \alpha_2, & \text{for } x \in (q, 1), \end{cases}$$

where $q \in (0,1)$, $\alpha_2 > \alpha_1 > 0$ and a_ε is extended to $\mathbb{R}$ by periodicity. This choice represents diffusion inside a long tube filled with periodically alternating media (e.g. water and honey). In order to find the effective diffusion constant for small ε, recall that the physical definition of the diffusion constant in a *homogeneous* medium is $D := \frac{\ell v_T}{3}$, where v_T is the mean thermal velocity and ℓ is the mean free path of the particles. Now suppose we let our particles diffuse for some time T. We have a decomposition $T = T_1 + T_2$, where

- $T_1 \sim \frac{1}{\ell_1}$ is the mean time that particles spend in water, where $a_\varepsilon(x) \equiv \alpha_1$ and
- $T_2 \sim \frac{1}{\ell_2} > T_1$ is the mean time that particles spend in honey, where $a_\varepsilon(x) \equiv \alpha_2$.

Obviously, the time to traverse a given distance $s \gg \varepsilon$ will be proportional to the weighted mean

$$\begin{aligned}\overline{T} := qT_1 + (1-q)T_2 &\sim \frac{q}{\ell_1} + \frac{1-q}{\ell_2}\\ &\sim \frac{q}{\alpha_1} + \frac{1-q}{\alpha_2}\\ &= \left\langle a^{-1}\right\rangle.\end{aligned}$$

Hence for the effective diffusion constant $\overline{D}$ of the particles for small ε we obtain the relation $\overline{D} \sim \overline{\ell} \sim \overline{T}^{-1} \sim \left\langle a^{-1}\right\rangle^{-1}$.

Convergence theorems like the above can be obtained in much more general situations (cf. the classical textbook [PBL78] from which the above discussion was taken). But note that in the above we have only shown *strong* convergence (rather than operator norm convergence). Indeed, the statement $u_\varepsilon \xrightarrow{L^2} u$ can be reformulated in operator-theoretic terms as

$$A_\varepsilon^{-1} f \to A^{-1} f \qquad \text{for all } f \in L^2((a,b)).$$

This is not enough to answer certain questions of physical interest, e.g. whether $\sigma(A)$ is a good approximation for $\sigma(A_\varepsilon)$, or whether the decay rate of e^{-tA} approximates that of e^{-tA_ε}. To address these questions, norm resolvent estimates are necessary (cf. Theorem I.3.4). In fact, the question of norm resolvent convergence in the situation of classical homogenisation described so far has been addressed in previous works, most notably by Birman and Suslina [BS03, BS06] (see also the references therein). In these two works, the authors develop and apply operator-theoretic recipes to obtain norm-resolvent estimates in many physically relevant PDE, including acoustic equations, linear elasticity and Maxwell's equations.

However, there exist mathematically interesting homogenisation problems which cannot be tackled by the above methods. One class of such problems is given by *high contrast* homogenisation in which the condition $a_\varepsilon \geq \alpha > 0$ fails to be true uniformly in ε (clearly, the proof shown above breaks down in this case). Homogenisation results in high contrast media have been obtained by [Zhi00] who proved strong resolvent convergence for the equation $-\nabla \cdot (a_\varepsilon(x)u(x)\nabla) = f$, where $a_\varepsilon(x) = a_1\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 a_0\left(\frac{x}{\varepsilon}\right)$ and a_1, a_0 are periodic and $a_1(y) + a_2(y)$ is uniformly elliptic. Clearly, these assumptions allow high contrast in the limit $\varepsilon \to 0$. Later, the authors of [KS18] and [CC16] extended these results.

Another class of examples which do not fall in the category of classical homogenisation are problems in which the domain Ω depends on ε and becomes singular in the limit $\varepsilon \to 0$. Homogenisation problems of this type have been studied e.g. in [Zhi00, Pas06] (for Neumann boundary conditions) and in [MK64, CM97, RT75] (for Dirichlet boundary conditions). It is this field field in which the present thesis makes a contribution.

The crushed ice problem Consider a container filled with some medium of nonzero heat conductance occupying a domain $\Omega \subset \mathbb{R}^d$. We are interested in the efficiency of cooling the medium by adding crushed ice to the container. This problem has been posed and studied in [Rau75]. In order to obtain a well-defined mathematical problem, we make the following idealising assumptions:

(i) The ice cubes in the container are spherically shaped objects* $B_r(i)$ sitting at the vertices i of a periodic lattice $2\varepsilon\mathbb{Z}^d \cap \Omega$,

(ii) the ice does not melt and remains at temperature 0 throughout the cooling process.

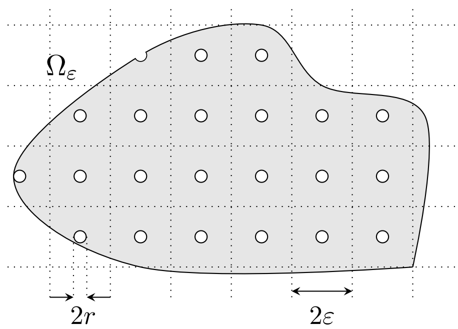


Figure II.6.: Sketch of the crushed ice problem

The above situation is modelled by the heat equation

$$\begin{cases} \partial_t u_{\varepsilon,r} &= \Delta u_{\varepsilon,r} \qquad \text{in } \Omega \setminus \bigcup_{i\in\varepsilon\mathbb{Z}^d\cap\Omega} B_r(i) \\ u_{\varepsilon,r} &= 0 \qquad \text{on } \partial\Omega \cup \bigcup_{i\in\varepsilon\mathbb{Z}^d\cap\Omega} \partial B_r(i), \end{cases}$$

where we have assumed for convenience that $\partial\Omega$ is held at temperature 0. We pose the question to what extent crushing the ice (that is, decreasing the size of the $B_r(i)$

*Sincere apologies for implying that cubes are spheres.

and increasing their number) accelerates the cooling process. It is clear from intuition that reducing the radius r of the balls and their distance ε simultaneously in such a way that $\varepsilon^{-n}|B_r(i)|$ remains constant should make the cooling more efficient. Indeed, this process keeps the total mass of the ice constant while increasing its surface area which enhances thermal contact.

On the other hand, keeping the distance ε between the ice cubes fixed and letting $r \to 0$ will surely diminish the cooling effect. We immediately are led to the following question: *What happens at intermediate scalings?* More precisely, what are the convergence properties of the solution $u_{\varepsilon,r_\varepsilon}$ if $\frac{r_\varepsilon}{\varepsilon} \to 0$ at various rates as $\varepsilon \to 0$?

These are in fact classical questions which have been addressed in several works starting from the 1960s. We quote two theorems about the *stationary* situation which illustrate the above discussion. With the notation from above, let the radius of the ice cubes be of the form $r_\varepsilon := C\varepsilon^\alpha$ for some $C > 0$ and $\alpha > 1$. Furthermore, for notational convenience, denote by $T_\varepsilon := \bigcup_{i \in \varepsilon\mathbb{Z}^d \cap \Omega} B_{r_\varepsilon}(i)$ the set of holes. Then one has the following

Theorem II.2.1 ([Rau75, RT75]). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$ be a bounded domain, let $f \in L^2(\Omega)$ and $u_\varepsilon : \Omega \setminus T_\varepsilon \to \mathbb{R}$ be the solution of*

$$\begin{cases} -\Delta u_\varepsilon &= f \qquad \text{in } \Omega \setminus T_\varepsilon \\ \quad u_\varepsilon &= 0 \qquad \text{on } \partial(\Omega \setminus T_\varepsilon). \end{cases}$$

Then

(i) if $\alpha > \frac{d}{d-2}$, then $u_\varepsilon \to u$ strongly in $H^1(\Omega)$, where u solves the Dirichlet problem in Ω:

$$\begin{cases} -\Delta u &= f \qquad \text{in } \Omega \\ \quad u &= 0 \qquad \text{on } \partial\Omega; \end{cases}$$

(ii) if $\alpha < \frac{d}{d-2}$, then $u \to 0$ strongly in $L^2(\Omega)$.

This theorem confirms our intuitive expectation, but makes no statement about the borderline case $\alpha = \frac{d}{d-2}$, where the transition between "infinitely effective cooling" in the limit and "no cooling at all" happens. Indeed, this case has a mathematically interesting solution which was found by [MK64] and extended in [CM97].

Theorem II.2.2 ([MK64, CM97]). *Let Ω and u_ε be as in Theorem II.2.1 with $r_\varepsilon = C\varepsilon^{\frac{d}{d-2}}$. Then $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$, where u solves*

$$\begin{cases} (-\Delta+\mu)u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

with $\mu = C^{d-2}\frac{(d-2)|\partial B_1(0)|}{2^d} > 0$.

Remark II.2.3. Several comments are in order.

(i) The actual time-dependent problem has been considered in [MK74], while [Rau75, RT75] have proven estimates on the lowest eigenvalue of $-\Delta$ on $\Omega \setminus T_\varepsilon$.

(ii) We note that the restriction $d \geq 3$ is not essential and we have omitted the case $d = 2$ merely for cosmetic reasons. The analogous result in the 2-dimensional case can be found in [CM97].

(iii) An analogous result to Theorem II.2.2 in the case of Robin boundary conditions on the holes has been found in [Kai85, Kai89].

Theorem II.2.2 shows that at least in the case of a bounded domain, there exists a reasonable limit operator which is *not* equal to merely the Laplacian, but shifted by a positive constant. In other words, cooling becomes more efficient in this case, but only by a finite rate constant μ.

However, convergence has only been shown in the *strong* (or pointwise) sense. Indeed, Theorem II.2.2 states that for *fixed* $f \in L^2(\Omega)$ one has $u_\varepsilon \to u$ weakly in $H^1(\Omega)$ and thus strongly in $L^2(\Omega)$, by the Rellich-Kondrachov theorem. As we have argued above, this is not enough to prove e.g. convergence of the spectrum of the operator.

Norm resolvent convergence in perforated domains has been studied previously in a number of publications (cf. [Pas06, BCD16] and the references therein). However, previous results have only covered the subcritical case $\alpha = 1$ and their methods of proof do not extend to the critical case $\alpha = \frac{d}{d-2}$.

In Part IV we will investigate the question of norm-resolvent convergence in the situation of Theorem II.2.2. Our results will apply not only to the Dirichlet problem, but to any of Dirichlet, Neumann or Robin boundary conditions with a complex parameter $\alpha \in \mathbb{C}$. Furthermore, our results extend to *unbounded* domains Ω. Note that in the case of Robin boundary conditions the corresponding operator can be non-selfadjoint.

III. Norm-Resolvent Estimates for a Class of Non-Selfadjoint Schrödinger Operators.

III.1. The Operator of Interest and Main Results

Unless otherwise stated, the notation $L^2(\mathbb{R}^d)$ will always denote $L^2(\mathbb{R}^d, \mathbb{C})$. The same convention holds for other function spaces. Motivated by the examples in the introduction, we are going to investigate Schrödinger Operators with growing real parts.

III.1.1. Definition of the Operator

To begin with, let us quote results by [BST17] and [EE87] which allow the rigorous definition of a large class of Schrödinger operators.*

Proposition III.1.1 ([BST17, EE87]). *Let $V \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ be a function such that*

(i) $\operatorname{Re} V \geq 0$

(ii) There exist $a, b' > 0$ such that $|\nabla V|^2 \leq a + b'|V|^2$

(iii) V is unbounded at infinity: $|V(x)| \to \infty \quad as \quad |x| \to \infty$

Then we have the following.

1. *The minimal operator*

$$H_{min} := -\Delta + V, \quad \mathcal{D}(H_{min}) := C_0^\infty(\mathbb{R}^d) \tag{III.1}$$

is closable on $L^2(\mathbb{R}^d)$ with closure

$$T = -\Delta + V, \quad \operatorname{dom}(T) = H^2(\mathbb{R}^d) \cap \{\psi \in L^2(\mathbb{R}^d) : Vf \in L^2(\mathbb{R}^d)\};$$

*The original proposition in [BST17] in fact allows even more general potentials than the one we state here.

2. *T is m-accretive;*

3. *The resolvent of T is compact.*

Using the above proposition, let us define an operator H on $L^2(\mathbb{R}^d)$ as follows.

Definition III.1.2. Let $V : \mathbb{R}^d \to \mathbb{C}$ satisfy the conditions of Prop III.1.1 and assume in addition that there exist constants $c, b > 0$ such that

$$\operatorname{Re} V(x) \geq c|x|^2 - b. \tag{III.2}$$

We denote by H the linear operator $H : \operatorname{dom}(H) \to L^2(\mathbb{R}^d)$ as the closure of

$$H_{\min} := -\Delta + V \quad \text{on} \quad C_0^\infty(\mathbb{R}^d).$$

according to Proposition III.1.1.

III.1.2. Main Results

From now on, unless otherwise stated, H will denote the operator defined in Definition III.1.2. Our first result is the following.

Lemma III.1.3. *The one-parameter semigroup generated by $-H$ is immediately compact (i.e. e^{-tH} is a compact operator for every $t > 0$).*

This is used to prove our main theorem

Theorem III.1.4. *Let H be defined as in Definition III.1.2. Then for every $\delta, R > 0$ there exists an $\varepsilon > 0$ such that*

$$\sigma_\varepsilon(H) \subset \{z : \operatorname{Re} z \geq R\} \cup \bigcup_{\lambda \in \sigma(H)} \{z : |z - \lambda| < \delta\}. \tag{III.3}$$

We immediately obtain the following corollary about the so-called harmonic oscillator with imaginary cubic potential.

Corollary III.1.5. *Let*

$$H_c = -\frac{d^2}{dx^2} + ix^3 + cx^2$$

for some $c > 0$ be defined on $\operatorname{dom}(H_c) = H^2(\mathbb{R}) \cap \{\psi \in L^2(\mathbb{R}) : x^3\psi \in L^2(\mathbb{R})\} \subset L^2(\mathbb{R})$. *Then one has the inclusion* (III.3) *for the pseudospectrum of H_c.*

We remark that the inclusion (III.3) is optimal in the sense that the unbounded component of the pseudospectrum cannot be contained in a sector of opening angle less than π as the discussion following equation (II.4) shows.

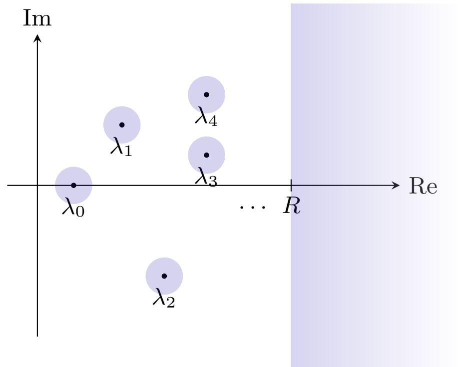


Figure III.1.: The pseudospectrum of H is contained in sets of the above shape.

Moreover, Theorem III.1.4 can be seen as complementary to the results of [Nov14]. Indeed, while it was shown there that there always exist infinitely many eigenvalues which are highly unstable under bounded perturbations, our result shows that the *lower* eigenvalues (that is, those with small real part) *do* remain stable if the perturbation is small enough in norm.

The method of proof of Theorem III.1.4 is inspired by ideas in [Bou02] and based on estimates of the semigroup generated by $-H$.

III.2. Proof of Theorem III.1.4

In this section we will first prove Lemma III.1.3 and then use it to prove Theorem III.1.4. Throughout this section, H denotes the operator defined in Definition III.1.2 and we will make frequent use of properties 1., 2., 3. of Proposition III.1.1 without further reference.

III.2.1. Proof of Lemma III.1.3

It is well-known (cf. Theorem I.2.4) that for all $\phi_0 \in L^2(\mathbb{R}^d)$ the semigroup generated by $-H$ is nothing but the solution operator to the initial value problem

$$\begin{cases} \partial_t \phi &= -H\phi \\ \phi(0) &= \phi_0. \end{cases} \tag{III.4}$$

In this section we will show that the operator e^{-tH} is compact on $L^2(\mathbb{R}^d)$ for $t > 0$. The first step will be to turn (III.4) into a coupled system of real equations and then using the results of [DL11].

Rewriting the equation as a system. We will use the fact that $L^2(\mathbb{R}^d, \mathbb{C})$ is canonically isomorphic to $L^2(\mathbb{R}^d, \mathbb{R}^2)$. In the following we will denote this isomorphism by $U : L^2(\mathbb{R}^d, \mathbb{C}) \to L^2(\mathbb{R}^d, \mathbb{R}^2)$.

Now, let us write $\phi(x) = f_1(x) + i f_2(x)$. A straightforward calculation shows that (III.4) is equivalent to the system

$$\begin{cases} \partial_t f_1 &= \Delta f_1 + \operatorname{Im}(V) f_2 - \operatorname{Re}(V) f_1 \\ \partial_t f_2 &= \Delta f_2 - \operatorname{Im}(V) f_1 - \operatorname{Re}(V) f_2 \end{cases} \tag{III.5}$$

which we will write as

$$\begin{aligned} \partial_t \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= [\Delta + Q(x)] \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ &= -UHU^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \end{aligned}$$

where $Q(x) = \begin{pmatrix} -\operatorname{Re} V(x) & \operatorname{Im} V(x) \\ -\operatorname{Im} V(x) & -\operatorname{Re} V(x) \end{pmatrix}$. Along the lines of [DL11] we define $\kappa(x) := -c|x|^2 + b$(with c, b from Definition III.1.2) which satisfies the estimate

$$\langle Q(x)\xi, \xi \rangle \leq \kappa(x) \|\xi\|^2 \qquad \forall \xi \in \mathbb{R}^2, \tag{III.6}$$

according to our assumptions about V. We also define the scalar differential operator[†]

$$\hat{H}_{2\kappa} := -\Delta - 2\kappa(x) \quad \text{on} \quad L^2(\mathbb{R}^d, \mathbb{R}). \tag{III.7}$$

The operators $-UHU^{-1}$ and $-\hat{H}_{2\kappa}$ satisfy Hypothesis 2.1 of [DL11] enabling us to prove the following lemma by following the proof of [DL11, Prop. 2.4].

Lemma III.2.1. *Let $\boldsymbol{f}^0 \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^2)$. There exists a unique classical solution to the initial value problem* [(III.5), $\boldsymbol{f}(0, \cdot) = \boldsymbol{f}^0$] *and one has*

$$|\boldsymbol{f}(t, \cdot)|^2 \leq e^{-t\hat{H}_{2\kappa}} \big(|f^0|^2 \big), \quad t \geq 0. \tag{III.8}$$

Proof. This proof uses the local Hölder continuity of V. By [DL11, Th. 2.6] there exists a unique classical solution $\boldsymbol{f} = (f_1, f_2)$ for our choice of initial condition. Let us now multiply the first equation of (III.5) by f_1 and the second by f_2 and add the

[†]More precisely, $\hat{H}_{2\kappa}$ should be regarded as the L^2-closure of the operator initially defined on the space $C_0^\infty(\mathbb{R})$.

resulting equations. We obtain

$$\frac{1}{2}\partial_t|\boldsymbol{f}|^2 = \boldsymbol{f}\cdot\Delta\boldsymbol{f} - \operatorname{Re}(V)|\boldsymbol{f}|^2.$$

Using the product rule this may be rewritten as

$$\begin{aligned}\partial_t|\boldsymbol{f}|^2 &= (\Delta - 2\operatorname{Re}V)|\boldsymbol{f}|^2 - 2|\nabla\boldsymbol{f}|^2\\ &= \big(\Delta{+}2\kappa(x) - 2W(x)\big)|\boldsymbol{f}|^2 - 2|\nabla\boldsymbol{f}|^2\\ &= -\hat{H}_{2\kappa}\big(|\boldsymbol{f}|^2\big) - 2\big(W(x)|\boldsymbol{f}|^2 + |\nabla\boldsymbol{f}|^2\big),\end{aligned}$$

where we have defined $W(x) := \operatorname{Re}V(x) + \kappa(x) \geq 0$. Now, define $w := |\boldsymbol{f}|^2 - e^{-t\hat{H}_{2\kappa}}\big(|\boldsymbol{f}^0|^2\big)$. We obviously have $w(0,\cdot) = 0$ and from the above calculation we obtain

$$(\partial_t - \Delta{-}2\kappa(x))w \leq 0, \quad t > 0.$$

Thus applying the maximum principle [DL11, Prop. 2.3 (ii)] we obtain $w \leq 0$. ☐

The operator $\hat{H}_{2\kappa}$. Regarded as an operator on $L^2(\mathbb{R}^d,\mathbb{R})$, the operator $\hat{H}_{2\kappa}$ is of course nothing but the harmonic oscillator with frequency $\omega = \sqrt{8c}$, shifted by the constant $-2b$. Its negative is well-known to generate a one-parameter semigroup $e^{-t\hat{H}_{2\kappa}}$ which can be represented by the Mehler kernel

$$\begin{aligned}\big(e^{-t\hat{H}_{2\kappa}}g\big)(t,x) &= e^{2td}\Big(\frac{2\pi}{\omega}\sinh(2\omega t)\Big)^{-\frac{1}{2}}\int e^{-\frac{\omega}{2}\frac{\cosh(2\omega t)(|x|^2+|y|^2)-2x\cdot y}{\sinh(2\omega t)}}g(y)\,dy\\ &=: \int K(t,x,y)g(y)\,dy\end{aligned}$$

(cf. [Dav80, Chapter 7.2]).

Lemma III.2.2. *Let $t > 0$ and $0 < \alpha \leq \cosh(2\omega t){-}1$ and define $\mu(x) := e^{-\frac{\alpha\omega}{2\sinh(2\omega t)}|x|^2}$. Then*

$$|K(t,x,y)| \leq C_{t,\omega}\,\mu(x)\mu(y), \tag{III.9}$$

where $C_{t,\omega}$ depends only on t and ω.

Proof. We only have to check that $-\alpha(|x|^2+|y|^2) \geq -\cosh(2\omega t)(|x|^2+|y|^2) - 2x\cdot y$. This follows immediately from the assumption on α. Note that $\cosh(2\omega t) - 1 > 0$ for $t > 0$, so such an α exists. ☐

Note that this lemma implies that $e^{-t\hat{H}_{2\kappa}}$ is a Hilbert-Schmidt operator.

Compactness of e^{-tH}. The following lemma states that a cut-off version of e^{-tH} converges in norm to e^{-tH}.

Lemma III.2.3. *Let $t > 0$ and $\theta_n \in C_c(\mathbb{R}^d)$ such that $\chi_{B_{r_n}(0)} \leq \theta_n \leq \chi_{B_{2r_n}(0)}$, where r_n is defined such that*

$$\sup_{x \in \mathbb{R}^d \setminus B_{r_n}(0)} \big(\mu(x)\big) < \frac{1}{n^2} \tag{III.10}$$

(where μ was defined in Lemma III.2.2) and define the operator $R_n(t)$ by

$$R_n(t)\boldsymbol{f} := (Ue^{-\frac{t}{2}H}U^{-1})\big(\theta_n(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}\big).$$

Then

$$\|Ue^{-tH}U^{-1} - R_n(t)\|_{\mathcal{L}(L^2(\mathbb{R}^d,\mathbb{R}^2))} \to 0 \qquad (n \to \infty). \tag{III.11}$$

Proof. Let $n \in \mathbb{N}$ and $\boldsymbol{f} \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^2)$ and compute

$$\begin{aligned} |Ue^{-tH}U^{-1}\boldsymbol{f}(x) - R_n(t)\boldsymbol{f}(x)|^2 &\leq e^{-t\hat{H}_{2\kappa}}\big(|Ue^{-\frac{t}{2}H}U^{-1}\boldsymbol{f} - \theta_n(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}|^2\big)(x) \\ &= \int K\big(\tfrac{t}{2}, x, y\big)\big|(1-\theta_n(y))(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}(y)\big|^2\,dy \end{aligned}$$

where we have used Lemma III.2.1 in the first line. Now integrate both sides over x.

$$\begin{aligned} \|Ue^{-tH}U^{-1}\boldsymbol{f} - R_n(t)\boldsymbol{f}\|_{L^2}^2 &\leq \iint K\big(\tfrac{t}{2}, x, y\big)\big|(1-\theta_n(y))(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}(y)\big|^2\,dxdy \\ &\leq C\iint \mu(x)\mu(y)|1-\theta_n(y)|^2\,|(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}(y)|^2\,dxdy \\ &\leq C\left(\int\mu(x)dx\right)\|\mu(y)(1-\theta_n(y))^2\|_\infty\int|(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}(y)|^2dy \\ &\leq C'\Big(\sup_{y\in\mathbb{R}^d\setminus B_{r_n}}\mu(y)\Big)\,\|(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}\|_{L^2}^2 \\ &\leq \frac{M}{n^2}\|(Ue^{-\frac{t}{2}H}U^{-1})\boldsymbol{f}\|_{L^2}^2 \end{aligned}$$

for some $M > 0$. Using the unitarity of U and the fact that $e^{-\frac{t}{2}H}$ is a bounded operator on $L^2(\mathbb{R}^d, \mathbb{C})$ we finally arrive at

$$\|Ue^{-tH}U^{-1}\boldsymbol{f} - R_n(t)\boldsymbol{f}\|_{L^2(\mathbb{R}^d,\mathbb{R}^2)}^2 \leq \left(\frac{M}{n^2}\|e^{-\frac{t}{2}H}\|^2\right)\|\boldsymbol{f}\|_{L^2(\mathbb{R}^d,\mathbb{R}^2)}^2. \tag{III.12}$$

By density of $C_0^\infty(\mathbb{R}^d, \mathbb{R}^2)$ we conclude that this inequality is valid for all $\boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{R}^2)$.

This immediately yields

$$\|Ue^{-tH}U^{-1} - R_n(t)\|_{\mathcal{L}(L^2(\mathbb{R}^d,\mathbb{R}^2))} \leq \frac{L}{n} \tag{III.13}$$

for some $L > 0$. $\square$

We can now use Lemma III.2.3 to prove Lemma III.1.3. By closedness of the set of compact operators and Lemma III.2.3 we only have to show that $R_n(\tau)$ is compact for every n. Since furthermore $Ue^{-\frac{\tau}{2}H}U^{-1}$ is a bounded operator on $L^2(\mathbb{R}^d,\mathbb{C})$, we only show that $T_n(\tau) := \theta_n Ue^{-\frac{\tau}{2}H}U^{-1}$ is compact. This will be established in several steps:

Step 1: Pass to a bounded domain by suitably cutting off the solution $\boldsymbol{f}$ of (III.5). The cut function $\boldsymbol{u}$ will satisfy the inhomogeneous equation

$$\partial_t \boldsymbol{u} + H\boldsymbol{u} = \boldsymbol{g}_n \tag{III.14}$$

with $\boldsymbol{g}_n \in H^{-1}$.

Step 2: Use Galerkin approximation to obtain the estimate

$$\|\boldsymbol{u}\|^2_{L^2((0,1);H^1_0)} \leq C\|\boldsymbol{g}_n\|^2_{L^2((0,1);H^{-1})} \tag{III.15}$$

Step 3: Cut off again to improve the estimate to

$$\|\boldsymbol{v}\|^2_{L^\infty((0,1);H^1_0)} \leq C\|\boldsymbol{h}_n\|^2_{L^2((0,1);L^2)} \tag{III.16}$$

Step 4: Conclude that

$$\|\boldsymbol{u}(1)\|_{H^1_0} \leq C\|\boldsymbol{f}^0\|_{L^2}. \tag{III.17}$$

Let us begin with the details.

Step 1

Let $\boldsymbol{f}$ be a solution of (III.5) and let $\psi \in C^\infty([0,1])$ with

$$\psi(0) = 0, \qquad \psi\big|_{[\frac{1}{2},1]} \equiv 1$$

and $\eta_n \in C_c^\infty(\mathbb{R}^d)$ such that

$$\chi_{B_{2r_n}(0)} \leq \eta_n \leq \chi_{B_{4r_n}(0)}.$$

Now define

$$\boldsymbol{u} := \psi(t)\eta_n(x)\boldsymbol{f}(t,x). \tag{III.18}$$

A straightforward calculation shows that $\boldsymbol{u}$ satisfies the equation

$$\partial_t \boldsymbol{u} + H\boldsymbol{u} = \boldsymbol{g}_n, \tag{III.19}$$

where $\boldsymbol{g}_n = \eta_n(\partial_t\psi)\boldsymbol{u} - \psi(\Delta\eta_n)\boldsymbol{u} - 2\psi\nabla\eta_n \cdot \nabla\boldsymbol{u}$. Since $\boldsymbol{g}_n$ contains a spatial derivative of the L^2-function $\boldsymbol{u}$ we only have $\boldsymbol{g}_n(t,\cdot) \in H^{-1}(B_{4r_n}(0);\mathbb{R}^2)$.

Let us denote $\Omega := B_{4r_n}(0)$. Note that we have chosen η_n and ψ such that $\boldsymbol{u}$ has the boundary values

$$\begin{cases} \boldsymbol{u}(x,t) & = 0, \quad \forall x \in \partial\Omega,\ t > 0 \\ \boldsymbol{u}(x,0) & = 0, \quad \forall x \in \overline{\Omega} \end{cases} \tag{III.20}$$

Step 2

Notation. In this step we will apply Galerkin approximation to the system (III.19) with boundary conditions (III.20) to obtain an estimate for $\|\boldsymbol{u}\|^2_{L^2((0,1);H_0^1)}$. We follow a standard procedure presented in many PDE textbooks. First, let us introduce the notation

$$a(\boldsymbol{w},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{w} \cdot \nabla\boldsymbol{v}\,dx + \int_\Omega (V\boldsymbol{w}) \cdot \boldsymbol{v}\,dx \tag{III.21}$$

for $\boldsymbol{w},\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^2)$, where "$\cdot$" denotes the scalar product in $\mathbb{R}^2$. Then, we have

$$a(\boldsymbol{v},\boldsymbol{v}) = \|\boldsymbol{v}\|^2_{H_0^1} + c\||x|\boldsymbol{v}\|^2_{L^2} \tag{III.22}$$

$$|a(\boldsymbol{w},\boldsymbol{v})| \leq C\|\boldsymbol{w}\|_{H_0^1}\|\boldsymbol{v}\|_{H_0^1}. \tag{III.23}$$

Now we choose a set $\{w_j\}$ of eigenfunctions of Δ which forms an orthonormal basis of $L^2(\Omega)$ and of $H_0^1(\Omega)$. The eigenvalue corresponding to w_j will be denoted λ_j. We have that

(a) $\boldsymbol{v} = (v^1, v^2) \in L^2(\Omega;\mathbb{R}^2)$ if and only if $v^\alpha = \sum_{j=1}^\infty c_j^\alpha w_j$ for a sequence (c_j) with $\sum_{j=1}^\infty |c_j^\alpha|^2 < \infty$ for $\alpha = 1, 2$.

(b) $\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^2)$ if in addition $\sum_{j=1}^\infty \lambda_j |c_j^\alpha|^2 < \infty$ for $\alpha = 1, 2$.

Furthermore, we denote $E_N = \operatorname{span}(w_1, ..., w_N)$.

Construction of approximate solution.

Definition III.2.4. A function $\boldsymbol{u}_N : [0,1] \to E_N \times E_N$ is called an *approximate solution* to the initial value problem (III.19), (III.20) if

(i) $\boldsymbol{u}_N \in L^2((0,1); E_N \times E_N)$ and $\partial_t \boldsymbol{u}_N \in L^2((0,1); E_N \times E_N)$

(ii) for all $\boldsymbol{v} \in E_N \times E_N$ one has

$$\int_\Omega (\partial_t \boldsymbol{u}_N) \cdot \boldsymbol{v} + a(\boldsymbol{u}_N, \boldsymbol{v}) = \langle \boldsymbol{g}_n, \boldsymbol{v} \rangle \tag{III.24}$$

pointwise a.e. in $t \in (0,1)$, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between H^{-1} and H_0^1.

(iii) $\boldsymbol{u}_N(0) = 0$

Now, expand the components as $u_N^\alpha(t,x) = \sum_{j=1}^N c_j^\alpha(t) w_j(x)$, plug this into (III.19) and test the resulting equation with $(w_k, 0)$ and $(0, w_k)$, respectively. We get

$$\frac{dc_k^\alpha}{dt} + \sum_j c_j^\alpha \langle \nabla w_j, \nabla w_k \rangle_{L^2} + \sum_{j,\beta} \left\langle V^{\alpha\beta} c_j^\beta w_j, w_k \right\rangle_{L^2} = \langle g_n^\alpha, w_k \rangle \tag{III.25}$$

$$\Leftrightarrow \qquad \frac{dc_k^\alpha}{dt} + \sum_{j,\beta} A_{j,k}^{\alpha\beta} c_j^\beta = g_k^\alpha \tag{III.26}$$

for $\alpha \in \{1,2\}$, $k \in \{1, ..., N\}$, where $A_{j,k}^{\alpha\beta} = \langle \nabla w_j, \nabla w_k \rangle_{L^2} \delta^{\alpha\beta} + \left\langle V^{\alpha\beta} w_j, w_k \right\rangle_{L^2}$ and $g_k^\alpha = \langle g_n^\alpha, w_k \rangle_{H^{-1}, H_0^1}$.

Lemma III.2.5. *The system of ODEs* (III.26) *has a unique solution* $\boldsymbol{c} = (c^1, c^2) \in C([0,1]; \mathbb{R}^{2N})$ *with* $\boldsymbol{c}(0) = 0$.

Proof. This is a standard application of Banach's fixed point theorem. Note that $\|A_{jk}^{\alpha\beta}\|_\infty \le |\lambda_j|^2 + \|V\|_{L^\infty(\Omega)}$. □

Lemma III.2.5 gives us an approximate solution $\boldsymbol{u}_N \in C([0,1]; E_N \times E_N)$. Note that we have

$$\frac{d\boldsymbol{c}}{dt} = -A\boldsymbol{c} + (g_k^\alpha) \in L^2((0,1); \mathbb{R}^{2N}) \;\; \Rightarrow \;\; \partial_t \boldsymbol{u}_N \in L^2((0,1); E_N \times E_N)$$

Proposition III.2.6. *For every $N \in \mathbb{N}$ this approximate solution satisfies*

$$\|\boldsymbol{u}_N\|_{L^\infty((0,1);L^2)} + \|\boldsymbol{u}_N\|_{L^2((0,1);H_0^1)} + \|\partial_t \boldsymbol{u}_N\|_{L^2((0,1);H^{-1})} \leq C\|\boldsymbol{g}_n\|_{L^2((0,1);H^{-1})}. \tag{III.27}$$

Proof. Take $\boldsymbol{v} = \boldsymbol{u}_N$ in (III.24):

$$\begin{aligned}
&\langle \partial_t \boldsymbol{u}_N, \boldsymbol{u}_N \rangle_{L^2} + a(\boldsymbol{u}_N, \boldsymbol{u}_N) = \langle \boldsymbol{g}_n, \boldsymbol{u}_N \rangle \\
\Leftrightarrow\ & \frac{1}{2}\partial_t \int_\Omega |\boldsymbol{u}_N|^2 + \|\boldsymbol{u}_N\|_{H_0^1}^2 + c_1\|x\boldsymbol{u}_N\|_{L^2}^2 = \langle \boldsymbol{g}_n, \boldsymbol{u}_N \rangle \\
\Rightarrow\ & \frac{1}{2}\partial_t \|\boldsymbol{u}_N\|_{L^2}^2 + \|\boldsymbol{u}_N\|_{H_0^1}^2 \leq \|\boldsymbol{g}_n\|_{H^{-1}}\|\boldsymbol{u}_N\|_{H_0^1}.
\end{aligned}$$

Integrating this inequality from 0 to t, we get

$$\begin{aligned}
\frac{1}{2}\big(\|\boldsymbol{u}_N(t)\|_{L^2}^2 - \|\boldsymbol{u}_N(0)\|_{L^2}^2\big) + \|\boldsymbol{u}_N\|_{L^2((0,t);H_0^1)}^2 &\leq \int_0^t \|\boldsymbol{g}_n\|_{H^{-1}}\|\boldsymbol{u}_N\|_{H_0^1}\, ds \\
&\leq \left(\int_0^t \|\boldsymbol{g}_n\|_{H^{-1}}^2\right)^{\frac{1}{2}} \left(\int_0^t \|\boldsymbol{u}_N\|_{H_0^1}^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\|\boldsymbol{g}_n\|_{L^2((0,t);H^{-1})}^2 + \frac{1}{2}\|\boldsymbol{u}_N\|_{L^2((0,t);H_0^1)}^2 \\
\Rightarrow \qquad \|\boldsymbol{u}_N(t)\|_{L^2}^2 + \|\boldsymbol{u}_N\|_{L^2((0,t);H_0^1)}^2 &\leq \|\boldsymbol{g}_n\|_{L^2((0,t);H^{-1})}^2
\end{aligned}$$

Taking the supremum over $t \in (0,1)$ we get

$$\|\boldsymbol{u}_N\|_{L^\infty((0,1);L^2)}^2 + \|\boldsymbol{u}_N\|_{L^2((0,1);H_0^1)}^2 \leq \|\boldsymbol{g}_n\|_{L^2((0,1);H^{-1})}^2 \tag{III.28}$$

To estimate the time derivative, note that since $\partial_t \boldsymbol{u}_N \in E_N \times E_N$:

$$\|\partial_t \boldsymbol{u}_N(t)\|_{H^{-1}} = \sup_{v \in E_N \times E_N \setminus \{0\}} \frac{\langle \partial_t \boldsymbol{u}_N, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|_{H_0^1}}.$$

Furthermore,

$$\begin{aligned}
\langle \partial_t \boldsymbol{u}_N, \boldsymbol{v} \rangle &\overset{\text{(III.24)}}{=} \langle \boldsymbol{g}_n, \boldsymbol{v} \rangle - a(\boldsymbol{u}_N, \boldsymbol{v}) \\
&\leq |\langle \boldsymbol{g}_n, \boldsymbol{v} \rangle| + |a(\boldsymbol{u}_N, \boldsymbol{v})| \\
&\overset{\text{(III.23)}}{\leq} \big(\|\boldsymbol{g}_n\|_{H^{-1}} + C\|\boldsymbol{u}_N\|_{H_0^1}\big)\|\boldsymbol{v}\|_{H_0^1}
\end{aligned}$$

This shows that we have

$$\|\partial_t \boldsymbol{u}_N\|^2_{H^{-1}} \leq C\big(\|\boldsymbol{u}_N\|^2_{H^1_0} + \|\boldsymbol{g}_n\|^2_{H^{-1}}\big) \tag{III.29}$$

for some new constant C. Integrate this with respect to t and use (III.28). □

Convergence of approximate solutions. Proposition III.2.6 implies that

$$(\boldsymbol{u}_N) \text{ is bounded in } L^2((0,1); H^1_0)$$
$$(\partial_t \boldsymbol{u}_N) \text{ is bounded in } L^2((0,1); H^{-1}).$$

From the Banach-Alaoglu theorem it follows that there exists a subsequence (which we again denote by $(\boldsymbol{u}_N)$) with

$$\boldsymbol{u}_N \rightharpoonup \boldsymbol{u} \text{ in } L^2((0,1); H^1_0) \quad \text{and} \quad \partial_t \boldsymbol{u}_N \overset{*}{\rightharpoonup} \partial_t \boldsymbol{u} \text{ in } L^2((0,1); H^{-1})$$

Let $\varphi \in C_c^\infty(0,1)$, $\boldsymbol{w} \in E_M \times E_M$ and take $\boldsymbol{v} = \varphi \boldsymbol{w}$ in (III.24):

$$\int_0^1 \big[\langle \partial_t \boldsymbol{u}_N, \varphi \boldsymbol{w}\rangle_{L^2} + a(\boldsymbol{u}_N, \varphi \boldsymbol{w})\big]\, dt = \int_0^1 \langle \boldsymbol{g}_n, \varphi \boldsymbol{w}\rangle dt. \tag{III.30}$$

Now, take $N \to \infty$ on both sides. Then

- $\displaystyle\int_0^1 (\partial_t \boldsymbol{u}_N, \varphi \boldsymbol{w})_{L^2} dt \to \int_0^1 (\partial_t \boldsymbol{u}, \varphi \boldsymbol{w})_{L^2} dt$ because of weak* convergence in H^{-1}
- From (III.23) it follows that $\boldsymbol{u} \mapsto \int_0^1 a(\boldsymbol{u}, \varphi \boldsymbol{w}) dt$ is a continuous linear form on $L^2((0,1); H^1_0)$ and so we have

$$\int_0^1 a(\boldsymbol{u}_N, \varphi \boldsymbol{w}) dt \to \int_0^1 a(\boldsymbol{u}, \varphi \boldsymbol{w}) dt.$$

Thus, (III.30) becomes

$$\int_0^1 \varphi \big[(\partial_t \boldsymbol{u}, \boldsymbol{w})_{L^2} + a(\boldsymbol{u}, \boldsymbol{w})\big]\, dt = \int_0^t \varphi \langle \boldsymbol{g}_n, \boldsymbol{w}\rangle dt \tag{III.31}$$

and since this holds for every $\varphi \in C_c^\infty(0,1)$ this implies

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{w}\rangle_{L^2} + a(\boldsymbol{u}, \boldsymbol{w}) = \langle \boldsymbol{g}_n, \boldsymbol{w}\rangle \qquad \forall \boldsymbol{w} \in E_M \times E_M. \tag{III.32}$$

Since $\bigcup_{M \in \mathbb{N}} E_M \times E_M$ is dense in $H^1_0(\Omega; \mathbb{R}^2)$, this holds for all $\boldsymbol{w} \in H^1_0(\Omega; \mathbb{R}^2)$.

Initial condition Let now $\varphi \in C^\infty(0,1)$, $\varphi(0) = 1$, $\varphi(1) = 0$. Partial integration in (III.32) gives

$$\langle \boldsymbol{u}(0), \boldsymbol{w}\rangle_{L^2} = \int_0^1 \partial_t\varphi \, \langle \boldsymbol{u}, \boldsymbol{w}\rangle_{L^2} \, dt + \int_0^1 \varphi\big[\langle \boldsymbol{g}_n, \boldsymbol{w}\rangle - a(\boldsymbol{u}, \boldsymbol{w})\big] \, dt. \tag{III.33}$$

Similarly,

$$0 = \int_0^1 \partial_t\varphi \, \langle \boldsymbol{u}_N, \boldsymbol{w}\rangle_{L^2} \, dt + \int_0^1 \varphi\big[\langle \boldsymbol{g}_n, \boldsymbol{w}\rangle - a(\boldsymbol{u}_N, \boldsymbol{w})\big] \, dt. \tag{III.34}$$

for $\boldsymbol{w} \in E_M \times E_M$ and $N > M$. Letting $N \to \infty$ we may conclude that $\langle \boldsymbol{u}(0), \boldsymbol{w}\rangle_{L^2} = 0$ for any $\boldsymbol{w} \in L^2((0,1); H_0^1)$ and thus $\boldsymbol{u}(0) = 0$.

Uniqueness Let $\boldsymbol{u}_1, \boldsymbol{u}_2$ obey (III.19) and set $\boldsymbol{u}_3 = \boldsymbol{u}_1 - \boldsymbol{u}_2$. Then

$$\begin{aligned}
&\langle \partial_t \boldsymbol{u}_3, \boldsymbol{u}_3\rangle_{L^2} + a(\boldsymbol{u}_3, \boldsymbol{u}_3) = 0\\
\Leftrightarrow\ &\langle \partial_t \boldsymbol{u}_3, \boldsymbol{u}_3\rangle_{L^2} + \|\boldsymbol{u}_3\|_{H_0^1}^2 + c_1\|x\boldsymbol{u}_3\|_{L^2}^2 = 0\\
\Rightarrow\ &\langle \partial_t \boldsymbol{u}_3, \boldsymbol{u}_3\rangle_{L^2} \le 0\\
\Leftrightarrow\ &\frac{d}{dt}\|\boldsymbol{u}_3\|_{L^2}^2 \le 0
\end{aligned}$$

Together with $\boldsymbol{u}_3(0) = 0$ this implies $\boldsymbol{u}_3 = 0$ and so $\boldsymbol{u}_1 = \boldsymbol{u}_2$

Step 3

Recall that we had $\boldsymbol{u}(t,x) = \psi(t)\eta_n(x)\boldsymbol{f}(t,x)$ and $\psi|_{[\frac12,1]} \equiv 1$, $\eta_n|_{B_{2r_n}(0)} \equiv 1$, so we have

$$\boldsymbol{u} = \boldsymbol{f} \text{ on } [\tfrac12, 1] \times \overline{B_{2r_n}(0)}$$

and thus

$$t \mapsto \boldsymbol{f}(t,\cdot) \in L^2\big((0,1); H^1(B_{2r_n}(0))\big).$$

Now let us cut off again by choosing new functions $\phi \in C^\infty([\frac12, 1])$ with

$$\phi(\tfrac12) = 0, \quad \phi(1) = 1$$

and θ_n with

$$\chi_{B_{r_n}(0)} \le \theta_n \le \chi_{B_{2r_n}(0)}$$

(cf. Lemma III.2.3) and put

$$\boldsymbol{v}(t,x) := \phi(t)\theta_n(x)\boldsymbol{u}(t,x) \tag{III.35}$$

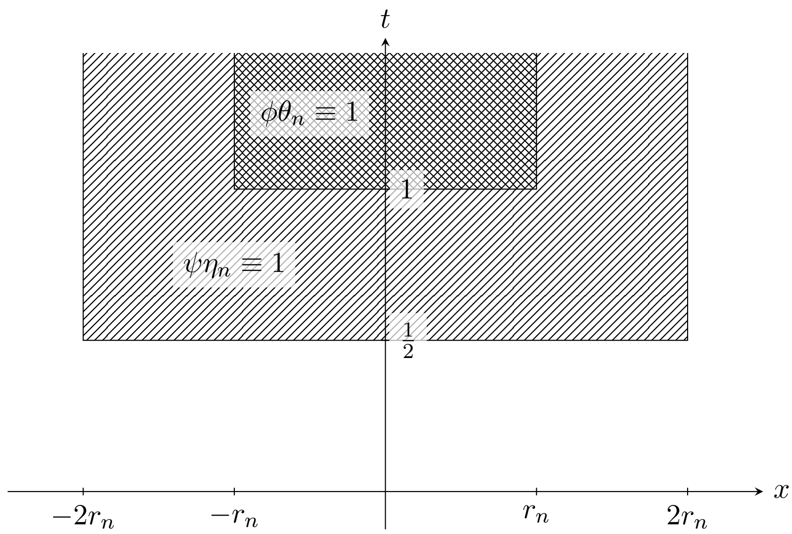


Figure III.2.: The cutting process in the x-t plane

Note that we have $\boldsymbol{v} = \phi\theta_n\boldsymbol{f}$ wherever $\phi\theta_n \neq 0$. The same calculation as at the beginning of Step 1 shows that $\boldsymbol{v}$ satisfies the boundary value problem

$$\begin{cases} \partial_t\boldsymbol{v} + H\boldsymbol{v} &= \boldsymbol{h}_n \\ \boldsymbol{v}(\tfrac{1}{2},x) &= 0 \\ \boldsymbol{v}(t,x) &= 0, \qquad \text{for } x \in \partial B_{2r_n}(0),\ t > \tfrac{1}{2}, \end{cases} \tag{III.36}$$

where $\boldsymbol{h}_n = \phi\theta_n\boldsymbol{g}_n + \theta_n(\partial_t\phi)\boldsymbol{f} - \phi(\Delta\theta_n)\boldsymbol{f} - 2\phi\nabla\theta_n\cdot\nabla\boldsymbol{f}$. Note that we now have $\boldsymbol{h}_n \in L^2\big((0,1);L^2(B_{2r_n}(0))\big)$, in contrast to before, when we had $\boldsymbol{g}_n(t,\cdot) \in H^{-1}(B_{4r_n}(0))$.

Denote $\tilde\Omega := B_{2r_n}(0)$. A modification of Theorem 5, Chapter 7.1 in [Eva98] (checked assumptions in this theorem; the condition $\boldsymbol{h}_n \in L^2\big((0,1);L^2(B_{2r_n}(0))\big)$ is sufficient.) gives that

$$\begin{aligned} \boldsymbol{v} &\in L^2\big((0,1);H^2(\tilde\Omega;\mathbb{R}^2)\big) \cap L^\infty\big((0,1);H^1_0(\tilde\Omega;\mathbb{R}^2)\big) \\ \partial_t\boldsymbol{v} &\in L^2\big((0,1);L^2(\tilde\Omega;\mathbb{R}^2)\big) \end{aligned}$$

and

$$\|\boldsymbol{v}\|_{L^2((0,1);H^2)} + \|\boldsymbol{v}\|_{L^\infty((0,1);H_0^1)} \leq C\|\boldsymbol{h}_n\|_{L^2((0,1);L^2)} \tag{III.37}$$

Finally, note that

$$\begin{aligned} \boldsymbol{v}(1,x) &= \theta_n(x)\boldsymbol{u}(1,x) \\ &= \theta_n(x)\Big(Ue^{-H}U^{-1}\boldsymbol{f}^0\Big)(x) \end{aligned}$$

so we obtain

$$\big\|\theta_n\big(Ue^{-H}U^{-1}\boldsymbol{f}^0\big)\big\|_{H_0^1(\tilde{\Omega};\mathbb{R}^2)} \leq C\|\boldsymbol{h}_n\|_{L^2((0,1);L^2(\tilde{\Omega};\mathbb{R}^2))} \tag{III.38}$$

Step 4

In this last step we will estimate the right-hand side of (III.38) by the L^2-norm of the initial condition $\boldsymbol{f}^0$. Constants C may change from line.

By definition of $\boldsymbol{h}_n$ we have

$$\|\boldsymbol{h}_n\|_{L^2((0,1);L^2(\tilde{\Omega};\mathbb{R}^2))} \leq C\|\boldsymbol{u}\|_{L^2((0,1);H_0^1(\tilde{\Omega};\mathbb{R}^2))}.$$

Using (III.27) (which also holds for $\boldsymbol{u}$) we get

$$\begin{aligned} \|\boldsymbol{h}_n\|_{L^2((0,1);L^2(\tilde{\Omega};\mathbb{R}^2))} &\leq C\|\boldsymbol{g}\|_{L^2((0,1);H^{-1}(\tilde{\Omega};\mathbb{R}^2))} \\ &\leq C\|\boldsymbol{g}\|_{L^\infty((0,1);H^{-1}(\tilde{\Omega};\mathbb{R}^2))} \\ &= C\|(\partial_t\psi)\boldsymbol{f}\|_{L^\infty((0,1);H^{-1}(\tilde{\Omega};\mathbb{R}^2))} \\ &\leq C\|\boldsymbol{f}\|_{L^\infty((0,1);H^{-1}(\tilde{\Omega};\mathbb{R}^2))} \\ &\leq C\|\boldsymbol{f}(0)\|_{H^{-1}(\tilde{\Omega};\mathbb{R}^2)} \end{aligned}$$

since the operator $Ue^{-H}U^{-1}$ is bounded and $\|e^{-H}\| \leq 1$. Recalling our initial condition $\boldsymbol{u}(0) = \boldsymbol{f}^0$, we thus get

$$\begin{aligned} \|\boldsymbol{h}_n\|_{L^2((0,1);L^2(\tilde{\Omega};\mathbb{R}^2))} &\leq C\|\boldsymbol{f}^0\|_{H^{-1}(\tilde{\Omega};\mathbb{R}^2)} \\ &\leq C\|\boldsymbol{f}^0\|_{L^2(\tilde{\Omega};\mathbb{R}^2)} \\ &\leq C\|\boldsymbol{f}^0\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)}. \end{aligned}$$

Using this in (III.38), we finally arrive at

$$\big\|\theta_n\big(Ue^{-H}U^{-1}\boldsymbol{f}^0\big)\big\|_{H_0^1(\tilde{\Omega};\mathbb{R}^2)} \leq C\|\boldsymbol{f}^0\|_{L^2(\mathbb{R}^d;\mathbb{R}^2)}. \tag{III.39}$$

Thus, the image of the unit ball in $L^2(\mathbb{R}^d;\mathbb{R}^2)$ under $\theta_n Ue^{-H}U^{-1}$ is bounded in $H_0^1(\tilde{\Omega};\mathbb{R}^2)$. By the compact embedding $H_0^1(\tilde{\Omega};\mathbb{R}^2) \hookrightarrow L^2(\tilde{\Omega};\mathbb{R}^2)$ we conclude that

$$\big\{\theta_n\big(Ue^{-H}U^{-1}\boldsymbol{f}^0\big) : \|\boldsymbol{f}^0\|_{L^2(\mathbb{R};\mathbb{R}^2)} \leq 1\big\} \tag{III.40}$$

is precompact in $L^2(\tilde{\Omega};\mathbb{R}^2)$ (and thus in $L^2(\mathbb{R}^d;\mathbb{R}^2)$). Thus the operator

$$\theta_n Ue^{-H}U^{-1} : L^2(\mathbb{R}^d;\mathbb{R}^2) \to L^2(\mathbb{R}^d;\mathbb{R}^2)$$

is compact which completes the proof of Lemma III.1.3.

□

Corollary III.2.7. *The semigroup e^{-tH} is immediately norm-continuous.*

Proof. This follows from the above compactness result, together with Lemma I.2.18.
□

Note that by applying Lemma I.2.17, we obtain the following bound on the spectrum of H:

Corollary III.2.8. *Let $b \in \mathbb{R}$. Then the set*

$$\{\lambda \in \sigma(H) : \operatorname{Re}\lambda \leq b\}$$

is bounded.

III.2.2. Bound on the Pseudospectrum

Recall from the introductory sections that by Proposition I.2.28, the large-t behaviour of a strongly continuous semigroup $T(t)$ with generator A is determined by the spectrum of $T(t)$. However, as we noted, it is not necessarily determined by the spectrum of its generator, A, since $\sigma(T(t))$ might be larger than $e^{\sigma(A)}$ for generic semigroups. This issue vanishes for eventually compact semigroups, as we have seen in Corollary I.2.34.

Moreover, recall from Corollary I.2.7 that if $(T(t))_{t\geq 0}$ is a one-parameter semigroup with $\|T(t)\| \leq Me^{at}$ for all $t \geq 0$, then

$$\|(z - A)^{-1}\| \leq \frac{M}{\operatorname{Re} z - a} \qquad \forall z : \operatorname{Re} z > a. \tag{III.41}$$

Note that in the following we will be dealing with *accretive* operators, rather than dissipative ones, i.e. their *negative* generates a semigroup. The reader should be aware of the corresponding sign changes.

Example: The imaginary Airy Operator. The theorems mentioned above can be used to estimate the pseudospectra of m-accretive operators. As an illustrative example, let us treat the imaginary Airy operator defined as

$$H_{Ai} = -\frac{d^2}{dx^2} + ix \quad \text{on} \quad \operatorname{dom}(H_{Ai}) = \{\phi \in L^2(\mathbb{R}) \mid -\phi'' + ix\phi \in L^2(\mathbb{R})\}. \tag{III.42}$$

This operator is m-accretive, and thus generates a one-parameter semigroup. Using the Fourier transform one can show that [Dav07]

$$\|e^{-tH_{Ai}}\| = e^{-\frac{t^3}{12}}, \tag{III.43}$$

which, together with the Hille-Yosida theorem, implies that $\sigma(H_{Ai}) = \emptyset$. Let now $a > 0$. Choosing $M_a = \sup_{t\geq 0}\left(e^{at-\frac{t^3}{12}}\right)$ we have

$$e^{-\frac{t^3}{12}} \leq M_a e^{-at}$$

and so

$$\|e^{-tH_{Ai}}\| \leq M_a e^{-at}. \tag{III.44}$$

Thus Corollary I.2.7 tells us that

$$\|(z - H_{Ai})^{-1}\| \leq \frac{M_a}{a - \operatorname{Re} z} \qquad \forall z : \operatorname{Re} z < a. \tag{III.45}$$

(note that the generator of the semigroup is not H_{Ai} but $-H_{Ai}$). In particular, we have for (say) $\operatorname{Re} z < a - 1$ that

$$\|(z - H_{Ai})^{-1}\| \leq M_a. \tag{III.46}$$

This shows that for $\varepsilon < \frac{1}{M_a}$ the set $\{z \,|\, \operatorname{Re} z < a-1\}$ does not intersect the ε-pseudospectrum. In more suggestive terms: The ε-pseudospectrum wanders off towards $+\infty$ as we decrease ε.

A simple calculation shows that $M_a = \sup_{t\geq 0}\left(e^{at-\frac{t^3}{12}}\right) = e^{\frac{4}{3}a^{3/2}}$. This even enables us to estimate how fast the pseudospectrum moves with decreasing ε. To this end, let $z \in \sigma_\varepsilon(H_{Ai})$ for some fixed $\varepsilon > 0$. Then by (III.46) we have

$$\begin{aligned} \frac{1}{\varepsilon} &\leq \|(z - H_{Ai})^{-1}\| \\ &\leq e^{\frac{4}{3}(\operatorname{Re} z+1)^{3/2}} \\ &\leq e^{w(\operatorname{Re} z)^{3/2}} \end{aligned}$$

for some $w > 0$ and $\operatorname{Re} z$ large enough. This inequality immediately leads to

$$\operatorname{Re} z \geq w^{-1}\left(\log\frac{1}{\varepsilon}\right)^{2/3}, \tag{III.47}$$

with w independent of ε. This shows that indeed every point in the ε-pseudospectrum moves towards $+\infty$ at a rate of at least $\left(\log\frac{1}{\varepsilon}\right)^{2/3}$.

Let us compare this to the results of [KSTV15]. Using semiclassical techniques the authors showed that there exist constants $C_1, C_2 > 0$ such that for all $\varepsilon > 0$

$$\sigma_\varepsilon(H_{Ai}) \supset \left\{z : \operatorname{Re}(z) \geq C_1,\ \operatorname{Re}(z) \geq C_2\Big(\log\frac{1}{\varepsilon}\Big)^{2/3}\right\}.$$

Equation (III.47) confirms that the scaling found in [KSTV15] is in fact optimal. The same result has previously been obtained in [Bor13] using a different method of proof.

Note that together with the observation that $\|(H_{Ai} - z)^{-1}\|$ is independent of $\operatorname{Im}(z)$ (see [Dav07, Problem 9.1.10]) the pseudospectrum of H_{Ai} is (essentially) completely characterised: it consists of half-planes moving towards $+\infty$ with asymptotic velocity $\left(\log\frac{1}{\varepsilon}\right)^{2/3}$.

The General Case: A First Estimate. Let us now turn back to the operator $H = -\Delta + V$ of Definition III.1.2. To conclude the proof of Theorem III.1.4 we will need several lemmas which will be established next. By Corollary I.2.34 we know that

$$\sigma(e^{-tH}) = \{0\} \cup \{e^{-t\lambda} \,|\, \lambda \in \sigma(H)\}. \tag{III.48}$$

Let us denote the eigenvalues of H by λ_j such that $\operatorname{Re}\lambda_j \leq \operatorname{Re}\lambda_i$ for $j < i$ (and we *do not* count multiplicities). Thus, λ_0 denotes an eigenvalue with minimal real part. In fact, up to now we could have $\operatorname{Re}\lambda_0 = -b$. We will account for this problem below in Lemma III.2.9. With this notation, we obtain from eq. (III.48) that

$$r(e^{-tH}) = e^{-t\operatorname{Re}\lambda_0}, \tag{III.49}$$

Thus by Proposition I.2.28 we have

$$-\operatorname{Re}\lambda_0 = \lim_{t\to\infty} t^{-1}\log\|e^{-tH}\|. \tag{III.50}$$

In other words, we have that for every $\alpha < \operatorname{Re}\lambda_0$

$$\lim_{t\to\infty} e^{\alpha t}\|e^{-tH}\| = 0. \tag{III.51}$$

Let such an $\alpha < \operatorname{Re}\lambda_0$ be fixed and choose t_α such that $e^{\alpha t}\|e^{-tH}\| < 1$ for all $t > t_\alpha$. On the whole we have

$$\begin{aligned} &\|e^{-tH}\| < e^{-\alpha t} && \forall t > t_\alpha \\ &\|e^{-tH}\| \le 1 && \forall t > 0 \quad (\text{since } e^{-tH} \text{ is a contraction semigroup}), \end{aligned}$$

so we finally arrive at

$$\|e^{-tH}\| \le M_\alpha e^{-\alpha t} \qquad \forall t > 0, \tag{III.52}$$

with $M_\alpha = e^{\alpha t_\alpha}$.

We are now in the position to proceed as for the imaginary Airy operator. Corollary I.2.7 tells us that

$$\|(z-H)^{-1}\| \le \frac{M_\alpha}{\alpha - \operatorname{Re} z} \qquad \forall z : \operatorname{Re} z < \alpha. \tag{III.53}$$

Note, however, that this time we cannot simply let $\alpha \to +\infty$ since we are restricted to $\alpha < \operatorname{Re}\lambda_0$.

Pushing the Pseudospectrum Towards Infinity. Let $Q_n = \frac{1}{2\pi i}\oint_\gamma (H-z)^{-1}dz$ denote the Riesz projection associated with H, where γ encloses only the n-th eigenvalue λ_n (which is possible since the spectrum of H is discrete). Moreover, define $P_m := \sum_{n=0}^m Q_n$. Then each of the operators Q_n, P_m commutes with the resolvent of H.

Since H has compact resolvent, we have that $\dim(\operatorname{Ran} Q_n) < \infty \quad \forall n$. For each $m \in$

$\mathbb{N}$ the space $L^2(\mathbb{R}^d)$ decomposes into a direct sum of closed, H-invariant subspaces[‡]

$$L^2(\mathbb{R}^d) = \operatorname{Ran} Q_0 \oplus \cdots \oplus \operatorname{Ran} Q_m \oplus \operatorname{Ran}(I - P_m) \tag{III.54}$$

Because e^{-tH} commutes with the resolvent of H, each of the above subspaces is invariant under e^{-tH} and hence the generator of $e^{-tH}|_{\operatorname{Ran} Q_n}$ is $-H|_{\operatorname{Ran} Q_n}$. The same is true for $\operatorname{Ran}(I - P_m)$.

Since the spectrum of $H|_{\operatorname{Ran}(I-P_m)}$ is $\{\lambda_n : n > m\}$ (and since the restriction of a compact operator is compact), applying Corollary I.2.34 again gives

$$\sigma\big(e^{-tH}|_{\operatorname{Ran}(I-P_m)}\big) = \{0\} \cup \{e^{-t\lambda_n}\}_{n=m+1}^{\infty}. \tag{III.55}$$

Lemma III.2.9. *For all $z \in \rho(H)$, one has*

$$\|(H-z)^{-1}\| \le C \left(\sum_{n=0}^{m} \|(H|_{\operatorname{Ran} Q_n} - z)^{-1}\| + \|(H|_{\operatorname{Ran}(I-P_m)} - z)^{-1}\| \right) \tag{III.56}$$

where C depends only on $\|Q_n\|$ ($n \le m$).

Proof. Let $z \in \rho(H)$ and $\xi, \psi \in L^2(\mathbb{R}^d)$ such that $(H - z)\xi = \psi$ and $\|\psi\| = 1$. We want to estimate $\|\xi\|$. To do this, note that by surjectivity of $(H - z)$ we have

$$L^2(\mathbb{R}^d) = \left(\bigoplus_{n=0}^{m} \operatorname{Ran}(H|_{\operatorname{Ran}(Q_n)} - z) \right) + \operatorname{Ran}(H|_{\operatorname{Ran}(I-P_m)} - z). \tag{III.57}$$

Note that the first term on the right hand side is actually equal to $\bigoplus_{n=0}^{m} \operatorname{Ran} Q_n$, since $\operatorname{Ran} Q_n$ is H-invariant.

Claim: We have $\operatorname{Ran}(I - P_m) = \operatorname{Ran}(H|_{\operatorname{Ran}(I-P_m)} - z)$.

Proof of Claim: Since the Q_n commute with H, we have

$$\begin{aligned} \operatorname{Ran}(H|_{\operatorname{Ran}(I-P_m)} - z) &= \operatorname{Ran}\big((H|_{\operatorname{Ran}(I-P_m)} - z)(I - P_m)\big) \\ &= \operatorname{Ran}\big((I - P_m)(H|_{\operatorname{Ran}(I-P_m)} - z)\big) \\ &\subset \operatorname{Ran}(I - P_m). \end{aligned}$$

Now, suppose there was a $0 \ne \phi \in \operatorname{Ran}(I - P_m) \setminus \operatorname{Ran}(H|_{\operatorname{Ran}(I-P_m)} - z)$. Since (III.54) is a direct sum ϕ cannot have any components in $\bigoplus_{n=0}^{m} \operatorname{Ran} Q_n$. But

[‡] H-invariance follows from the fact that the Q_n commute with H and closedness of $\operatorname{Ran}(I - P_m)$ follows from the Fredholm alternative.

then $\phi \notin \operatorname{Ran}(H-z)$, by (III.57), which contradicts surjectivity.

Now, decompose

$$\begin{aligned}\psi &= \sum_{n=1}^{m} Q_n\psi + (I-P_m)\psi \\ &=: \sum_{n=1}^{m} \psi_n + \tilde{\psi}.\end{aligned}$$

Choose $\xi_n \in \operatorname{Ran} Q_n$ such that $(H-z)\xi_n = \psi_n$ and $\tilde{\xi} \in \operatorname{Ran}(I-P_m)$ such that $(H-z)\tilde{\xi} = \tilde{\psi}$ (which is possible since $\operatorname{Ran}(I-P_m) = \operatorname{Ran}(H|_{\operatorname{Ran}(I-P_m)} - z)$). But now it is clear that

$$\begin{aligned}\|\xi_n\| &\le \|(H|_{\operatorname{Ran} Q_n} - z)^{-1}\|\|\psi_n\| &&\le \|(H|_{\operatorname{Ran} Q_n} - z)^{-1}\|\|Q_n\|\|\psi\| \\ \|\tilde{\xi}\| &\le \|(H|_{\operatorname{Ran}(I-P_m)} - z)^{-1}\|\|\tilde{\psi}\| &&\le \|(H|_{\operatorname{Ran}(I-P_m)} - z)^{-1}\|\|(I-P_m)\|\|\psi\|\end{aligned}$$

Finally, using the triangle inequality we obtain

$$\begin{aligned}\|\xi\| &\le \sum_{n=1}^{m} \|\xi_n\| + \|\tilde{\xi}\| \\ &\le \left(\sum_{n=0}^{m} \|Q_n\|\|(H|_{\operatorname{Ran} Q_n} - z)^{-1}\| + \|(I-P_m)\|\|(H|_{\operatorname{Ran}(I-P_m)} - z)^{-1}\|\right)\|\psi\| \\ &\le \left(1 + \sum_{n=0}^{m} \|Q_n\|\right)\left(\|(H|_{\operatorname{Ran} Q_n} - z)^{-1}\| + \|(H|_{\operatorname{Ran}(I-P_m)} - z)^{-1}\|\right)\end{aligned}$$

which concludes the proof. $\square$

We are finally able to complete the proof of Theorem III.1.4. In (III.56) the first term on the right hand side is nothing but a sum of the resolvents of matrices (cf. Theorem I.1.18). These are well-known to decay in norm at infinity. In fact, a simple calculation shows that one has $\|(T-\lambda)^{-1}\| \le \left(|\lambda| - \|T\|\right)^{-1}$ as $|\lambda| \to \infty$. As a consequence, the ε-pseudospectra of $(H|_{\operatorname{Ran} Q_n} - z)^{-1}$ are contained in discs around the λ_n for ε small enough.

For the second term we can use (III.55) in Proposition I.2.28 and Corollary I.2.7 to obtain an estimate similar to (III.53), but with $\alpha < \operatorname{Re}\lambda_{m+1}$ instead. By Corollary III.2.8 we necessarily have $\operatorname{Re}\lambda_n \to \infty$ as $n \to \infty$. Thus we obtain a bound on $\|(H-\lambda)^{-1}\|$ on vertical lines with arbitrarily large real part and the proof of Theorem III.1.4 is completed.

III.3. Potentials with vanishing or negative real part

It is natural to ask whether the condition $\operatorname{Re} V(x) \geq c|x|^2 - b$ can be relaxed. In this section we will discuss two examples giving hints as to what might or might not be possible. First, we will consider an example of a Schrödinger operator with $\operatorname{Re} V = 0$ which still satisfies the inclusion (III.3). Second, we will show that in the case $\operatorname{Re} V(x) \leq -c|x|^2$ one can not expect any inclusion of the form (III.3).

III.3.1. Example: The Imaginary Cubic Oscillator

In this section we consider the operator

$$H_B = -\frac{d^2}{dx^2} + ix^3 \quad \text{on} \quad L^2(\mathbb{R}), \tag{III.58}$$

defined in the sense of Proposition III.1.1. H_B is sometimes called the imaginary cubic oscillator, or the Bender oscillator. We immediately obtain closedness of H_B, compactness of its resolvent and m-accrevity from Proposition III.1.1. Moreover, it is known [DDT01, Shi02] that the spectrum of H_B is entirely real and positive which enables us to number the eigenvalues λ_i of H_B such that $\lambda_i \leq \lambda_j$ for $i \leq j$ and $\lambda_0 > 0$. In this section, we will prove the following result about H_B.

Theorem III.3.1. *For the pseudospectrum of H_B the inclusion* (III.3) *holds and in addition there exists a $C > 0$ such that for every $\delta > 0$ there is an $\varepsilon > 0$ such that*

$$\sigma_\varepsilon(H_B) \subset \left\{ z : \operatorname{Re} z \geq C \left(\log \frac{1}{\varepsilon} \right)^{6/5} \right\} \cup \bigcup_{\lambda \in \sigma(H_B)} \{ z : |z - \lambda| < \delta \}. \tag{III.59}$$

In particular, apart from disks around the eigenvalues, the ε-pseudospectrum is contained in the half plane $\left\{ \operatorname{Re} z \geq C \left(\log \frac{1}{\varepsilon} \right)^{6/5} \right\}$.

Proof. As in the previous section we want to estimate $\|e^{-tH_B}|_{\operatorname{Ran}(I-P_m)}\|$ for $m \in \mathbb{N}$. We know that the eigenfunctions of H_B form a complete set in $L^2(\mathbb{R})$ and the algebraic eigenspaces are one-dimensional [KS12, Tai06]. Thus, we can use Lemma 3.1 of [Dav05]:

Lemma III.3.2 ([Dav05])**.** *Let $T(t)$ be a strongly continuous semigroup and $\{\psi_n\}_{n=1}^\infty$ a complete set of linearly independent vectors. Let $T_n(t)$ denote the restriction of $T(t)$ to* $\operatorname{span}\{\psi_1, \ldots, \psi_n\}$. *Then*

$$\|T(t)\| = \lim_{n \to \infty} \|T_n(t)\| \tag{III.60}$$

for all $t \geq 0$.

From now on, let $\{\psi_n\}_{n=1}^{\infty}$ denote the set of eigenvectors of H_B and let $V_m^n := \operatorname{span}\{\psi_m, \ldots, \psi_n\} = \bigoplus_{k=m}^{n} \operatorname{Ran}(Q_k)$. The Lemma now implies

$$\|e^{-tH_B}|_{\operatorname{Ran}(I-P_{m-1})}\| = \lim_{n\to\infty} \|e^{-tH_B}|_{V_m^n}\|.$$

The analytic functional calculus (see [TL80, Ch.V.]) shows that $\sum_{k=m}^{n} Q_k$ is a projection again and thus we have $\psi = \sum_{i=m}^{n} Q_i\psi$ for every $\psi \in V_m^n$ which we can use as follows.

$$\begin{aligned}
\|e^{-tH_B}|_{\operatorname{Ran}(I-P_m)}\psi\| &= \lim_{n\to\infty} \|e^{-tH_B}|_{V_m^n}\psi\| \\
&= \lim_{n\to\infty} \left\| \sum_{k=m}^{n} e^{-t\lambda_k} Q_k\psi \right\| \\
&\leq \lim_{n\to\infty} \sum_{k=m}^{n} e^{-t\lambda_k} \|Q_k\| \|\psi\| \\
&= \left(\sum_{k=m}^{\infty} e^{-t\lambda_k} \|Q_k\| \right) \|\psi\|
\end{aligned}$$

so we obtain

$$\|e^{-tH_B}|_{\operatorname{Ran}(I-P_m)}\| \leq \sum_{k=m}^{\infty} e^{-t\lambda_k} \|Q_k\|. \tag{III.61}$$

In [Hen14b] it was shown that $\lim_{k\to\infty} \frac{\log\|Q_k\|}{k} = \frac{\pi}{\sqrt{3}}$. Accordingly, for every $\mu > \frac{\pi}{\sqrt{3}}$ there exists a $C > 0$ such that

$$\|Q_k\| \leq Ce^{\mu k}. \tag{III.62}$$

In particular, choosing $\mu = 2$, we obtain $\|Q_k\| \leq Ce^{2k}$ for some $C > 0$.

On the other hand, it is well-known from [Sib75] that

$$\lambda_k \geq ck^{6/5}. \tag{III.63}$$

Combining these two facts, we arrive at

$$\begin{aligned}
\|e^{-tH_B}|_{\operatorname{Ran}(I-P_m)}\| &\leq \sum_{k=m}^{\infty} e^{-tck^{6/5}} Ce^{2k} \\
&= C \sum_{k=m}^{\infty} e^{-tck^{6/5}+2k}
\end{aligned}$$

Clearly, there exists a k_0 such that $\frac{1}{2}tck^{6/5} > 2k$ for all $k > k_0$ and k_0 is independent of t as long as (say) $t \geq 1$. So we can decompose

$$\|e^{-tH_B}|_{\mathrm{Ran}(I-P_{m-1})}\| \leq C \sum_{k=m}^{k_0} e^{-tck^{6/5}+2k} + C \sum_{k=k_0+1}^{\infty} e^{-\frac{c}{2}tk^{6/5}}$$

Since k_0 is independent of m and t, the first term in this estimate is only present as long as $m < k_0$.

Since we are interested in asymptotics, let us assume $m > k_0 \geq 1$ from now on. Our task is thus to estimate the second term in the above inequality. This is easily done by using $\lfloor x+1 \rfloor \geq x$ for all $x > 0$ and calculating

$$\begin{aligned}
\sum_{k=m}^{\infty} e^{-\frac{c}{2}t(k+1)^{6/5}} &\leq \int_m^{\infty} e^{-\frac{c}{2}tx^{6/5}}dx \\
&\leq \int_m^{\infty} \left(\tfrac{6}{5}x^{1/5}\right)e^{-\frac{c}{2}tx^{6/5}}dx \\
&= \tfrac{2}{ct}\left[-e^{-\frac{c}{2}tx^{6/5}}\right]_m^{\infty} \\
&= \tfrac{2}{ct}e^{-\frac{c}{2}tm^{6/5}}
\end{aligned}$$

This finally shows our main ingredient

Lemma III.3.3. *There exist constants $k_0, M, \omega > 0$ such that*

$$\|e^{-tH_B}|_{\mathrm{Ran}(I-P_{m-1})}\| \leq Me^{-\omega m^{\frac{6}{5}}t}$$

for all $m > k_0$, $t \geq 1$.

This immediately leads to[§]

$$\|(H_B|_{\mathrm{Ran}(I-P_{m-1})} - z)^{-1}\| \leq \frac{\tilde{M}}{\omega m^{\frac{6}{5}} - \operatorname{Re} z} \tag{III.64}$$

for all $\operatorname{Re} z < \omega m^{\frac{6}{5}}$, where $\tilde{M}, \omega$ are independent of m. On the whole, the resolvent of H_B is estimated by (see the proof of Lemma III.2.9)

$$\left\|(H_B-z)^{-1}\right\| \leq \left(1+\sum_{k=1}^{m}\|Q_k\|\right)\left(\sum_{k=1}^{m}\left\|(H_B|_{\mathrm{Ran}\,Q_k}-z)^{-1}\right\| + \left\|(H_B|_{\mathrm{Ran}(I-P_m)}-z)^{-1}\right\|\right)$$

[§]Since we only know that $\|e^{-tH_B}\|$ is bounded by 1 between $t=0$ and $t=1$, we might need to increase M to obtain (III.64).

$$\leq \Big(1 + \sum_{k=1}^{m} \|Q_k\|\Big)\Big(\sum_{k=1}^{m} \frac{1}{|\lambda_k - z|} + \frac{\tilde{M}}{\omega(m+1)^{\frac{6}{5}} - \operatorname{Re} z}\Big)$$

The first summand in the second factor gives the discs around the eigenvalues in (III.3), the second gives the half-plane. If we keep the distance of $\operatorname{Re}(z)$ to $\omega(m+1)^{6/5}$ constant, the second factor on the right-hand side stays bounded as $m \to \infty$. Since the first factor grows as $e^{(\text{constant})\cdot m}$, we have

$$\|(H_B - z)^{-1}\| \leq C e^{C'(\operatorname{Re} z)^{5/6}} \tag{III.65}$$

uniformly in z as long as $\operatorname{dist}(z, \sigma(H_B))$ is bounded below by a positive constant.

Keeping this in mind, suppose now that $z \in \sigma_\varepsilon(H_B) \cap \{\operatorname{dist}(z, \sigma(H_B)) > 1\}$. We deduce

$$\log\left(\frac{1}{\varepsilon}\right) \leq \log \|(H_B - z)^{-1}\| \leq C''(\operatorname{Re} z)^{5/6}$$
$$\Leftrightarrow \left(\log \frac{1}{\varepsilon}\right)^{6/5} \leq C'' \operatorname{Re} z$$

Together with the complementary estimate in (II.4) this proves the scaling in (III.59).
$\square$

Let us compare Theorem III.3.1 to the results of [KSTV15]. As noted in the introduction, it was shown there that for every $\delta > 0$ there exist constants $C_1, C_2 > 0$ such that for all $\varepsilon > 0$

$$\sigma_\varepsilon(H_B) \supset \left\{ z \in \mathbb{C} : |z| \geq C_1,\ |\arg z| < \left(\frac{\pi}{2} - \delta\right),\ |z| \geq C_2 \left(\log \frac{1}{\varepsilon}\right)^{6/5} \right\}.$$

Clearly, we have found the same scaling in (III.59). Thus, Theorem III.3.1 shows that the scaling (II.4) obtained in [KSTV15] is sharp.

Moreover, we obtain as a byproduct the following two statements about the semigroup and the resolvent of H_B.

Corollary III.3.4. *The semigroup e^{-tH_B} is immediately differentiable.*

Corollary III.3.5. *The resolvent norm of H_B satisfies*

$$\lim_{r \to \infty} \|(H_B - s - ir)^{-1}\| = 0 \tag{III.66}$$

for all $s \in \mathbb{R}$.

Proof. By [EN00, Cor. II.4.15] and the estimate (III.64) the semigroups $e^{-tH_B}|_{\mathrm{Ran}(I-P_m)}$ are immediately differentiable for every m and hence immediately norm-continuous. By [EN00, Cor. II.4.19] one has

$$\lim_{r\to\infty} \left\| \left(H_B|_{\mathrm{Ran}(I-P_{m-1})} - (s+ir)\right)^{-1} \right\| \to 0 \quad \forall s < \omega m^{\frac{6}{5}}.$$

Together with the estimate (III.56) the assertion follows. □

Notice that the strategy of the proof of Theorem III.3.1 also applies to more general classes of operators. The essential ingredients were the knowledge of the norms of the spectral projections, together with the fact that these norms are asymptotically small compared to $e^{-t\lambda_k}$. Examples of operators satisfying these conditions are considered in [Hen14a, MSV17].

III.3.2. Counterexample: An Operator with Negative Real Part

Let us again consider the operator H_c from (II.1), but now let $c < 0$. This operator can be defined rigorously using [BST17, Prop 2.4] and is still well-behaved in the sense that it is closed and its resolvent is compact. Moreover, its spectrum is still real and positive [Shi02, Cor. 3]. However, as we will show, its pseudospectrum is not well-behaved at all. In fact, H_c does not even generate a one-parameter semigroup in this case.

Theorem III.3.6. *For H_c, $c < 0$ no inclusion of the type* (III.3) *is possible. More precisely, for every $C, R, M > 0$ there exists $z \in \mathbb{C}$ such that $\operatorname{Re} z < -R$, $|z| > M$ and*

$$\|(H_c - z)^{-1}\| \geq C. \tag{III.67}$$

In particular, H_c does not generate a one-parameter semigroup.

Proof. We will use Theorem 3.1 and Lemma 4.1 of [Nov14]. Similarly to their strategy, let us define the unitary transformation

$$(\mathcal{U}\psi)(x) := \tau^{1/2}\psi(\tau x),$$

with $\tau > 0$. This transformation takes H_c to its semiclassical analogue

$$H_c^h := \tau^{-3}\mathcal{U}H_c\mathcal{U}^{-1} = -h^2\frac{d^2}{dx^2} + ix^3 - ch^{2/5}x^2,$$

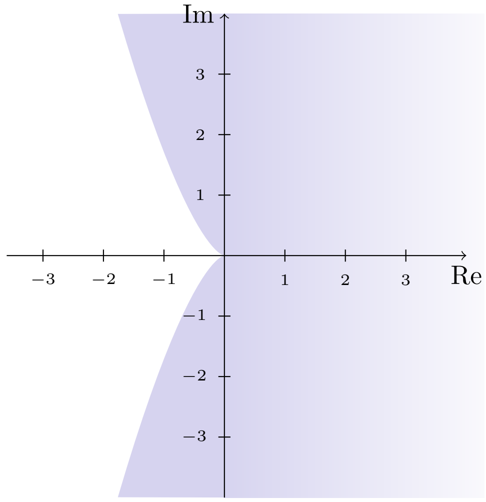


Figure III.3.: The semiclassical pseudospectrum of H_c^h. The boundary curve approaches the imaginary axis as $h \to 0$.

where $h = \tau^{-5/2}$. The *semiclassical pseudospectrum* (cf. (3.2) in [Nov14]) for this operator is the set (cf. Figure III.3)

$$\Lambda_h = \{\xi^2 + ix^3 - ch^{2/5}x^2 : \xi, x \neq 0\}.$$

We obviously have $i \in \Lambda_h$ for every $h > 0$ (remember that $c < 0$). By [Nov14, Theorem 3.1] and the unitarity of $\mathcal{U}$ there exists a $C > 0$ such that

$$\begin{aligned}\|(H_c - i\tau^3)^{-1}\| &= \tau^{-3}\|(H_c^h - i)^{-1}\| \\ &\geq h^{6/5}C^{1/h}\end{aligned}$$

Sending $\tau = h^{-2/5} \to \infty$, we see that the resolvent norm of H_c diverges exponentially on the imaginary axis.

To show divergence on vertical lines with strictly negative real part we may shift H_c by a real constant and then apply the above procedure. More precisely, let $\alpha > 0$ and consider the operator $H_c + \alpha$. Its semiclassical analogue is

$$\tau^{-3}\mathcal{U}(H_c + \alpha)\mathcal{U}^{-1} = H_c^h + h^{6/5}\alpha$$

and its semiclassical pseudospectrum

$$\Lambda_h = \{\xi^2 + ix^3 - ch^{2/5}x^2 + h^{6/5}\alpha : \xi, x \neq 0\}$$

is shifted to the right by $h^{6/5}\alpha$. Its boundary curve intersects the imaginary axis when $-ch^{2/5}x^2 + h^{6/5}\alpha = 0$ the solution of which is $h^{2/5}\left(\frac{\alpha}{c}\right)^{1/2}$. Since this tends to 0 as $h \to 0$ one can always find $h_0 > 0$ such that $i \in \Lambda_h$ for all $h < h_0$. This enables us to apply the above procedure for the shifted operator and obtain again exponential divergence on the imaginary axis. $\square$

Remark: Given the above lower estimate of $\|(H_c - z)^{-1}\|$, let us mention that it is still possible to obtain weaker upper bounds on the resolvent norm of H_c. Boegli, Siegl and Tretter have shown in [BST17] that for a very general class of Schroedinger operators, including H, H_c and H_B, the resolvent norm always decays in a sector in the complex plane which opens to the *left*.

In other words, operators such as H_c are still sectorial in the sense of [Haa06] (but not in the sense of Definition I.2.10). In particular, there exists an analytic functional calculus for these operators which, in turn, yields the existence e.g. of fractional powers of H_c.

IV. Norm-Resolvent Convergence in Perforated Domains

In this part we study the following homogenisation problems labelled by $\iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}$ ("D" for Dirichlet, "N" for Neumann, and "α" for Robin). Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be open (bounded or unbounded) We make the following further assumptions on Ω:

Dirichlet case: $\partial\Omega$ is uniformly C^2 (cf. [AF03, Definition 4.10]) and there exists $\delta > 0$ such that for all $y \in \mathbb{R}^d \setminus \Omega$ there exists a ball B with radius δ such that $y \in B$ and $B \cap \Omega = \emptyset$, i.e. the complement of Ω does not become "too narrow".

Neumann and Robin case: $\partial\Omega$ is of class C^2 and Ω is translation invariant, i.e. for every $j \in \mathbb{Z}^d$ one has $\Omega + j = \Omega$.

Note that the interesting special case $\Omega = \mathbb{R}^d$ satisfies all the above assumptions. Let $\alpha \in \mathbb{C} \setminus \{0\}$, $\mathrm{Re}(\alpha) \geq 0$ and denote $\Omega_\varepsilon := \Omega \setminus \bigcup_{i \in L_\varepsilon} B_{r_\varepsilon}(i)$ where $\varepsilon \in (0,1)$, $B_{r_\varepsilon}(i)$ is the ball of radius

$$r_\varepsilon^{\mathrm{D}} = \begin{cases} \varepsilon^{d/(d-2)}, & d \geq 3, \\ \mathrm{e}^{-1/\varepsilon^2}, & d = 2, \end{cases} \qquad r_\varepsilon^{\mathrm{N}} = o(\varepsilon) \quad (\varepsilon \to 0), \qquad r_\varepsilon^{\alpha} = \varepsilon^{d/(d-1)}. \tag{IV.1}$$

centered at the point $i \in L_\varepsilon$, and

$$L_\varepsilon := \{ i \in 2\varepsilon\mathbb{Z}^d : \mathrm{dist}(i, \partial\Omega) > \varepsilon \}. \tag{IV.2}$$

(cf. Figure IV.1). Consider the boundary value problems

$$\begin{cases} (-\Delta + 1)u^\varepsilon &= f \text{ in } \Omega_\varepsilon, \\ u^\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{Dir}$$

$$\begin{cases} (-\Delta+1)u^\varepsilon &= f \text{ in } \Omega_\varepsilon, \\ \partial_\nu u^\varepsilon &= 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{Neu}$$

$$\begin{cases} (-\Delta+1)u^\varepsilon &= f \text{ in } \Omega_\varepsilon, \\ \partial_\nu u^\varepsilon + \alpha u &= 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{Rob}$$

i.e. the resolvent problem for the Laplacian, subject to the Dirichlet, Neumann and Robin boundary conditions, respectively. It is easy to see, using the Lax-Milgram theorem, that for all $\varepsilon \in (0,1)$ each of these problems has a unique weak solution u^ε. It is a classical question, which we refer to as the homogenisation problem, whether the family of solutions to (Dir), (Neu), (Rob), obtained by varying the parameter ε, converges in the sense of the L^2-norm to a function $u \in L^2(\Omega)$ as $\varepsilon \to 0$ and whether the limit function u solves, in a reasonable sense, some PDE whose form is independent of the right-hand side datum f.

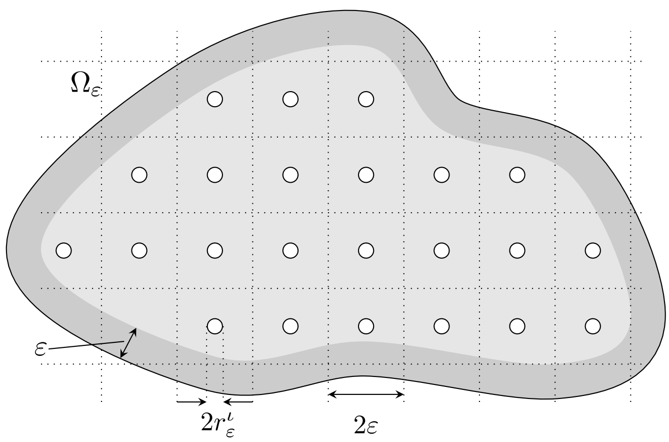


Figure IV.1.: Sketch of the perforated domain with an ε-neighbourhood of the boundary in which there are no holes.

Homogenisation problems of this type have been studied extensively for a long time [CM97, RT75, MK64, Kai85, Zhi00, Pas06, BCD16]. For example, results by Marchenko-Khruslov and Kaizu give a positive answer to the previous question for all three choices of boundary conditions at least in the case of *bounded* domains. In fact, they showed that the solutions of (Dir), (Rob), (Neu) converge strongly in $L^2(\Omega)$ to the solution $u \in H^1(\Omega)$ of $(-\Delta + 1 + \mu_\iota)u = f$, where

$$\mu_\iota = \begin{cases} \frac{\pi}{2}, & \iota = D,\ d = 2, \\ \frac{(d-2)S_d}{2^d}, & \iota = D,\ d \geq 3, \\ 0, & \iota = N, \\ \frac{\alpha S_d}{2^d}, & \iota = \alpha \end{cases} \tag{IV.3}$$

and S_d denotes the surface area of the unit ball in $\mathbb{R}^d$.

In this article we attempt to improve this result in two directions. First, we show the above convergence not only in the strong sense, but in the *norm resolvent sense* (that is, the right-hand side f is allowed to depend on ε). Second, our result is then extended to unbounded domains Ω. As a corollary, we obtain a statement about the convergence of the spectra of the perforated domain problems (Dir), (Neu), (Rob) as $\varepsilon \to 0$.

This part is organised as follows. In section IV.1 we review concepts of convergence on varying Hilbert spaces, in Section IV.2 we will briefly give a more precise formulation of the problem and include previous results. In Section IV.3 we will state our main result and its implications. Sections IV.4, IV.5 and IV.6 contain the proof of the main theorem and in Section IV.7 we consider implications of our main theorem on the semigroup generated by the Robin Laplacian.

IV.1. Convergence of Operators on Varying Spaces

This preliminary section is intended to deal with the technical complication presented by the fact that the spaces $L^2(\Omega_\varepsilon)$ in which the operators act depend on ε. Due to this issue the notion of norm resolvent convergence is ill-defined a priori. On the other hand, convergence of the *spectra* does not depend on the domains of the operators and it is a legitimate question whether the spectra of the perforated domain operators converge to the spectra of the limit operators $-\Delta + 1 + \mu_\iota$.

In the following we will review the results of [MNP13] who introduced an extended notion of norm resolvent convergence for operators A_ε with varying domains. In order to make sense of this, one needs to introduce identification operators between the domains of the A_ε. In short, the result we are going to prove states that if these identification operators satisfy a set of reasonable conditions, then a notion of norm resolvent convergence can be defined which implies spectral convergence. We use the notation and conventions from Part I.

Let $\mathcal{H}_\varepsilon, \mathcal{H}$ be Hilbert spaces and $A : \mathcal{H} \supset \operatorname{dom}(A) \to \mathcal{H}$ be m-accretive and for $\varepsilon > 0$ and let $A_\varepsilon : \mathcal{H}_\varepsilon \supset \operatorname{dom}(A_\varepsilon) \to \mathcal{H}_\varepsilon$ be a sequence of m-accretive operators. Let us denote $\mathcal{V}_\varepsilon := \big(\mathcal{H}_\varepsilon, \|\cdot\|_{A_\varepsilon}\big)$ and $\mathcal{V} := \big(\mathcal{H}, \|\cdot\|_A\big)$, where $\|\cdot\|_A$ denotes the norm generated by the sesquilinear form of A, that is, $\|u\|_{\mathcal{V}}^2 := \|u\|_A^2 := \|u\|_{\mathcal{H}}^2 + \operatorname{Re}\langle Au, u\rangle_{\mathcal{H}}$ (analogously for $\|\cdot\|_{\mathcal{V}_\varepsilon}$). By m-accretivity of the operators involved we have $-1 \in \rho(A_\varepsilon)$ for all $\varepsilon > 0$ and $-1 \in \rho(A)$ and the operator norms $\big\|(1+A)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}$ are finite. Indeed, we have

Lemma IV.1.1. *For $z \in \rho(A)$ one has*

$$\big\|(z-A)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}^2 \leq \Big(1 + |1+z|\big\|(z-A)^{-1}\big\|_{\mathcal{L}(\mathcal{H})}\Big)^2. \tag{IV.4}$$

Proof. Let $z \in \rho(A)$. Then

$$\begin{aligned}\big\|(z-A)^{-1}u\big\|_{\mathcal{V}}^2 &\leq \big|\big\langle (A+\mathrm{id})(z-A)^{-1}u, (z-A)^{-1}u\big\rangle_{\mathcal{H}}\big| \\ &= \big|\big\langle (1+z)(z-A)^{-1}u - u, (z-A)^{-1}u\big\rangle_{\mathcal{H}}\big| \\ &\leq \big(|1+z|\big\|(z-A)^{-1}u\big\|_{\mathcal{H}} + \|u\|_{\mathcal{H}}\big)\,\big\|(z-A)^{-1}u\big\|_{\mathcal{H}},\end{aligned}$$

hence

$$\begin{aligned}\big\|(z-A)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}^2 &\leq \Big(1 + |1+z|\big\|(z-A)^{-1}\big\|_{\mathcal{L}(\mathcal{H})}\Big)\,\big\|(z-A)^{-1}\big\|_{\mathcal{L}(\mathcal{H})} \\ &\leq \Big(1 + |1+z|\big\|(z-A)^{-1}\big\|_{\mathcal{L}(\mathcal{H})}\Big)^2.\end{aligned}$$

□

Definition IV.1.2. Assume that there exist operators $J_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}$ and $I_\varepsilon : \mathcal{H} \to \mathcal{H}_\varepsilon$ such that

(i) $I_\varepsilon J_\varepsilon = \mathrm{id}_{\mathcal{H}_\varepsilon}$,

(ii) $\|J_\varepsilon I_\varepsilon - \mathrm{id}_{\mathcal{H}}\|_{\mathcal{L}(\mathcal{V},\mathcal{H})} \to 0$ as $\varepsilon \to 0$,

(iii) $\|I_\varepsilon\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_\varepsilon)}, \|J_\varepsilon\|_{\mathcal{L}(\mathcal{H}_\varepsilon,\mathcal{H})} \leq M$ for some $M > 0$ uniformly in ε,

(iv) $\big\|J_\varepsilon(\mathrm{id}_{\mathcal{H}_\varepsilon} + A_\varepsilon)^{-1} - (\mathrm{id}_{\mathcal{H}} + A)^{-1}J_\varepsilon\big\|_{\mathcal{L}(\mathcal{H}_\varepsilon,\mathcal{H})} \to 0$ as $\varepsilon \to 0$.

Then we say that the sequence (A_ε) converges to A in the *norm resolvent sense.*

Note that if $\mathcal{H}_\varepsilon \equiv \mathcal{H}$ for all $\varepsilon > 0$ and $I_\varepsilon = J_\varepsilon = \mathrm{id}_\mathcal{H}$ for all $\varepsilon > 0$, this definition reduces to the classical definition I.3.1. In order to demonstrate the usefulness of this definition, let us give an exposition of the proof in [MNP13] showing that this notion of norm resolvent convergence implies spectral convergence. This turns out to be considerably more difficult than the classical proof; mainly because the I_ε, J_ε are not necessarily invertible.

Lemma IV.1.3. *If $A_\varepsilon \to A$ in norm resolvent sense, then*

$$\left\|(\mathrm{id}_{\mathcal{H}_\varepsilon} + A_\varepsilon)^{-1} I_\varepsilon - I_\varepsilon(\mathrm{id}_\mathcal{H} + A)^{-1}\right\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_\varepsilon)} \to 0 \tag{IV.5}$$

if I_ε is as in Definition IV.1.2.

Proof. For notational convenience, denote $R_\varepsilon := (\mathrm{id}_{\mathcal{H}_\varepsilon} + A_\varepsilon)^{-1}$ and $R := (\mathrm{id}_\mathcal{H} + A)^{-1}$. A quick calculation shows that

$$\begin{aligned} R_\varepsilon I_\varepsilon - I_\varepsilon R &= I_\varepsilon(J_\varepsilon R_\varepsilon - R J_\varepsilon) I_\varepsilon - (I_\varepsilon J_\varepsilon - \mathrm{id}_{\mathcal{H}_\varepsilon}) R_\varepsilon I_\varepsilon \\ &= I_\varepsilon(J_\varepsilon R_\varepsilon - R J_\varepsilon) I_\varepsilon, \end{aligned}$$

by (i) of Definition IV.1.2. Hence

$$\begin{aligned} \|R_\varepsilon I_\varepsilon - I_\varepsilon R\|_{\mathcal{L}(\mathcal{H},\mathcal{H}_\varepsilon)} &\le \|I_\varepsilon\|^2_{\mathcal{L}(\mathcal{H},\mathcal{H}_\varepsilon)} \|J_\varepsilon R_\varepsilon - R J_\varepsilon\|_{\mathcal{L}(\mathcal{H}_\varepsilon,\mathcal{H})} \\ &\to 0 \end{aligned}$$

as $\varepsilon \to 0$, by (iii) and (iv) of Definition IV.1.2. $\square$

Lemma IV.1.4 ([MNP13])**.** *For every $l, r > 0$ there exist $\delta > 0$ and $L > 0$ such that if*

$$\left\|J_\varepsilon(\mathrm{id}_{\mathcal{H}_\varepsilon} + A_\varepsilon)^{-1} - (\mathrm{id}_\mathcal{H} + A)^{-1} J_\varepsilon\right\|_{\mathcal{L}(\mathcal{H}_\varepsilon,\mathcal{H})} < \delta$$

and $z \in \rho(A_\varepsilon) \cap \rho(A) \cap B_r(0)$ and $\|(z-A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le l$, then $\|(z-A_\varepsilon)^{-1}\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} \le L$.

The useful point in this lemma is that L does not depend on z as long as $z \in \rho(A_\varepsilon) \cap \rho(A) \cap B_r(0)$ and $\|(z-A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le l$.

Proof. As above, we use the shorthand notation $R_\varepsilon(z) := (z - A_\varepsilon)^{-1}$ and $R(z) := (z-A)^{-1}$. For $z \in \rho(A_\varepsilon) \cap \rho(A) \cap B_r(0)$ define

$$V(z) := J_\varepsilon R_\varepsilon(z) - R(z) J_\varepsilon.$$

The resolvent identity can be used to show that

$$\big(R(-1) - R(z)\big) J_\varepsilon R_\varepsilon(z) R_\varepsilon(-1) = R(z) R(-1) J_\varepsilon \big(R_\varepsilon(-1) - R_\varepsilon(z)\big)$$

which implies

$$R(-1) V(z) R_\varepsilon(-1) = R(z) V(-1) R_\varepsilon(z)$$

or

$$\begin{aligned} V(z) &= (\mathrm{id}_{\mathcal{H}} + A) R(z) V(-1) R_\varepsilon(z) (\mathrm{id}_{\mathcal{H}_\varepsilon} + A_\varepsilon) \\ &= \big(\mathrm{id}_{\mathcal{H}} - (1+z) R(z)\big) V(-1) \big(\mathrm{id}_{\mathcal{H}_\varepsilon} - (1+z) R_\varepsilon(z)\big) \end{aligned}$$

on $\mathrm{dom}(A_\varepsilon)$ and thus on $\mathcal{H}_\varepsilon$ by density. Using our assumptions we deduce that

$$\|V(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon, \mathcal{H})} \leq \delta\big(1 + |1+z| l\big)\big(1 + |1+z| \|R_\varepsilon(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)}\big). \tag{IV.6}$$

Now, use $I_\varepsilon J_\varepsilon = \mathrm{id}_{\mathcal{H}_\varepsilon}$ to write

$$R_\varepsilon(z) = I_\varepsilon\big(J_\varepsilon R_\varepsilon(z) - R(z) J_\varepsilon\big) + I_\varepsilon R(z) J_\varepsilon. \tag{IV.7}$$

This representation, together with (IV.6) shows that

$$\begin{aligned} \|R_\varepsilon(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} &\leq \|I_\varepsilon\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}_\varepsilon)} \|V(z)]\|_{\mathcal{L}(\mathcal{H}_\varepsilon, \mathcal{H})} + \|I_\varepsilon\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}_\varepsilon)} \|J_\varepsilon\|_{\mathcal{L}(\mathcal{H}_\varepsilon, \mathcal{H})} \|R(z)\|_{\mathcal{L}(\mathcal{H})} \\ &\leq M\delta\big(1 + |1+z| l\big)\big(1 + |1+z| \|R_\varepsilon(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)}\big) + M^2 l \\ &\leq \delta M (1 + |1+z| l) |1+z| \|R_\varepsilon(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} + \delta M (1 + |1+z| l) + M^2 l \\ &\leq \delta M (1 + (1+r) l)(1+r) \|R_\varepsilon(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} + \delta M (1 + (1+r) l) + M^2 l \end{aligned}$$

Thus, if we choose $\delta < \frac{1}{M(1+(1+r)l)(1+r)}$, we obtain the estimate

$$\begin{aligned} \|R_\varepsilon(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} &\leq \frac{\delta M (1 + (1+r) l) + M^2 l}{1 - \delta M (1 + (1+r) l)(1+r)} \qquad &\text{(IV.8)} \\ &=: L &\text{(IV.9)} \end{aligned}$$

uniformly for $z \in \rho(A_\varepsilon) \cap \rho(A) \cap B_r(0)$. □

Theorem IV.1.5 ([MNP13]). *Let $A_\varepsilon : \mathcal{H}_\varepsilon \supset \mathrm{dom}(A_\varepsilon) \to \mathcal{H}_\varepsilon$ converge to $A : \mathcal{H} \supset \mathrm{dom}(A) \to \mathcal{H}$ in norm-resolvent sense. Then for every compact, connected $K \subset \rho(A)$*

such that $K \cap \rho(A_\varepsilon) \neq \emptyset$ for ε small enough there exists $\varepsilon_0 > 0$ such that $K \subset \rho(A_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. We use the notation from the previous proof. Let $K \subset \rho(A)$ be compact and choose $r > 0$ such that $K \subset B_r(0)$. Denote

$$l := \sup_{z\in K} \|R(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} < \infty$$

and choose $\delta > 0$ as in Lemma IV.1.4 and $\varepsilon_0 > 0$ such that $\left\| J_\varepsilon(\mathrm{id}_{\mathcal{H}_\varepsilon} + A_\varepsilon)^{-1} - (\mathrm{id}_{\mathcal{H}} + A)^{-1} J_\varepsilon \right\|_{\mathcal{L}(\mathcal{H}_\varepsilon, \mathcal{H})} < \delta$ for all $\varepsilon \in (0, \varepsilon_0)$, which is possible by norm resolvent convergence. Let $K_\varepsilon := \rho(A_\varepsilon) \cap K$, which is non-empty by assumption and by definition relatively open in K.

We will show that K_ε is also relatively closed in K which by connectedness of K implies $K_\varepsilon = K$. To this end, let (z_n) be a sequence in K_ε converging to $z \in K$. By Lemma IV.1.4, the sequence $\left(\|R_\varepsilon(z_n)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)}\right)_{n\in\mathbb{N}}$ is bounded. Finally, using Corollary I.1.16, we conclude that $z \in \rho(A_\varepsilon)$. Hence, K_ε is closed in K and the proof is completed. □

Using an analogous reasoning as in the previous proof, one can show

Theorem IV.1.6 ([MNP13])**.** *If $A_\varepsilon \to A$ in norm resolvent sense, then for every compact, connected $K \subset \mathbb{C}$ such that $K \subset \rho(A_\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0)$ and $K \cap \rho(A) \neq \emptyset$ one has $K \subset \rho(A)$.*

Sketch of proof. Since the proof is largely analogous to that of Theorem IV.1.5, we only sketch the idea. As in equation (IV.7), write

$$R(z) = J_\varepsilon(I_\varepsilon R(z) - R_\varepsilon(z) I_\varepsilon) + (\mathrm{id}_{\mathcal{H}} - J_\varepsilon I_\varepsilon) R(z) + J_\varepsilon R_\varepsilon(z) I_\varepsilon.$$

In order to estimate $\|R(z)\|_{\mathcal{L}(\mathcal{H})}$ by $\|R_\varepsilon(z)\|_{\mathcal{L}(\mathcal{H}_\varepsilon)}$, as in (IV.8), we can proceed as in the proof of Lemma IV.1.4, but we will have to estimate $\|(\mathrm{id}_{\mathcal{H}} - J_\varepsilon I_\varepsilon) R(z)\|_{(\mathcal{H})}$. This is easily done by noting that

$$\|(\mathrm{id}_{\mathcal{H}} - J_\varepsilon I_\varepsilon) R(z)\|_{(\mathcal{H})} \leq \|\mathrm{id}_{\mathcal{H}} - J_\varepsilon I_\varepsilon\|_{\mathcal{L}(\mathcal{V},\mathcal{H})} \|R(z)\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}$$

and applying (ii) of Definition IV.1.2 and (IV.4).

The proof of spectral convergence now follows that of Theorem IV.1.5 verbatim, exchanging the roles of A and A_ε. □

As in Section I.3, we readily obtain the following

Corollary IV.1.7. *Let $A_\varepsilon : \mathcal{H}_\varepsilon \supset \mathrm{dom}(A_\varepsilon) \to \mathcal{H}_\varepsilon$ converge to $A : \mathcal{H} \supset \mathrm{dom}(A) \to \mathcal{H}$ in norm-resolvent sense. Then for every compact $K \subset \mathbb{C}$, one has $K \cap \sigma(A_\varepsilon) \to K \cap \sigma(A)$ in Hausdorff sense (cf. Definition I.3.7).*

IV.2. Geometric Setting and Previous Results

As above, assume $d \geq 2$, and let

$$T_\varepsilon := \bigcup_{i\in L_\varepsilon} T_i^\varepsilon, \qquad T_i^\varepsilon := B_{r_\varepsilon^\iota}(i), \quad i \in L_\varepsilon,$$

with r_ε^ι, L_ε as in (IV.1), (IV.2). Denote $\Omega_\varepsilon := \Omega \setminus T_\varepsilon$. We also denote $B_i^\varepsilon := B_\varepsilon(i)$ and $P_i^\varepsilon := \varepsilon[-1,1]^d + i$ for $i \in L_\varepsilon$. Constants independent of ε will be denoted C and may change from line to line. Note that our assumptions on Ω ensure that the set $\{\phi|_\Omega : \phi \in C_0^\infty(\mathbb{R}^d)\}$ is dense in $H^1(\Omega)$ (cf. [Bre10, Cor. 9.8]).

Moreover, we define the identification operators

$$J_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega), \qquad J_\varepsilon f(x) = \begin{cases} f(x), & x \in \Omega_\varepsilon, \\ 0, & x \in \Omega \setminus \Omega_\varepsilon \end{cases} \tag{IV.10}$$

$$I_\varepsilon : L^2(\Omega) \to L^2(\Omega_\varepsilon), \qquad I_\varepsilon g(x) = g|_{\Omega_\varepsilon} \tag{IV.11}$$

$$\mathcal{T}_\varepsilon : H^1(\Omega_\varepsilon) \to H^1(\Omega), \qquad \mathcal{T}_\varepsilon u = \begin{cases} u & \text{in } \Omega_\varepsilon, \\ v & \text{in } T_\varepsilon, \end{cases} \tag{IV.12}$$

where v is the harmonic extension of u into the holes, i.e.

$$\begin{cases} \Delta v &= 0 \text{ in } T_\varepsilon, \\ \quad v &= u \text{ on } \partial T_\varepsilon. \end{cases} \tag{IV.13}$$

The above definitions are in fact useful in the context of norm resolvent convergence, as the following lemma shows.

Lemma IV.2.1. *Denote $\mathcal{H}_\varepsilon := L^2(\Omega_\varepsilon)$ and $\mathcal{H} := L^2(\Omega)$ and $\mathcal{V} := H^1(\Omega)$. The operators $I_\varepsilon, J_\varepsilon$ defined in* (IV.10), (IV.11) *satisfy* (i) *and* (ii) *of Definition IV.1.2.*

Proof. It is clear that $I_\varepsilon J_\varepsilon = \mathrm{id}_{L^2(\Omega_\varepsilon)}$. To prove that $\|\mathrm{id}_\mathcal{H} - J_\varepsilon I_\varepsilon\|_{\mathcal{L}(H^1(\Omega),L^2(\Omega))} \to 0$, let $f \in H^1(\Omega)$. Then $\|f - J_\varepsilon I_\varepsilon f\|_{L^2(\Omega)} = \|f\|_{L^2(T_\varepsilon)}$. To show that this quantity converges to 0 uniformly in f, denote $Q_k := [0,1)^d + k$ for $k \in \mathbb{Z}^d$ a cube shifted by k, so that

$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}^d} Q_k$. Then we have

$$\begin{aligned}\|f\|^2_{L^2(T_\varepsilon)} &= \sum_{k\in\mathbb{Z}^d} \|f\|^2_{L^2(Q_k\cap T_\varepsilon)} \\ &\le \sum_{k\in\mathbb{Z}^d} \|1\|^2_{L^{2p}(Q_k\cap T_\varepsilon)} \|f\|^2_{L^{2q}(Q_k\cap T_\varepsilon)}\end{aligned}$$

for $p, q > 1$ with $p^{-1} + q^{-1} = 1$, by Hölder's inequality. Since $f \in H^1(\Omega)$, we can use the Gagliardo-Sobolev-Nierenberg inequality to conclude (for $q = 2^*$, the Sobolev conjugate exponent) that

$$\begin{aligned}\|f\|^2_{L^2(T_\varepsilon)} &\le \|1\|^2_{L^{2p}(Q_0\cap T_\varepsilon)} \sum_{k\in\mathbb{Z}^d} \|f\|^2_{L^{2q}(Q_k\cap T_\varepsilon)} \\ \|f\|^2_{L^2(T_\varepsilon)} &\le \|1\|^2_{L^{2p}(Q_0\cap T_\varepsilon)} \sum_{k\in\mathbb{Z}^d} \|f\|^2_{L^{2q}(Q_k)} \\ &\le \|1\|^2_{L^{2p}(Q_0\cap T_\varepsilon)} \sum_{k\in\mathbb{Z}^d} C\|f\|^2_{H^1(Q_k)} \\ &= |Q_0 \cap T_\varepsilon|^{1/p} C\|f\|^2_{H^1(\Omega)}\end{aligned}$$

with some suitable $p > 0$. Since $|Q_0 \cap T_\varepsilon| \to 0$ as $\varepsilon \to 0$ (cf. the definition of r^ι_ε, (IV.1)), the desired convergence follows. □

Lemma IV.2.2. *The harmonic extension operator $\mathcal{T}_\varepsilon$ satisfies*

(i) $\limsup_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon\|_{\mathcal{L}(H^1(\Omega_\varepsilon), H^1(\Omega))} < \infty$.

(ii) There exists $C > 0$ such that $\|\mathcal{T}_\varepsilon w\|_{H^1(P^\varepsilon_i)} \le C\|w\|_{H^1(P^\varepsilon_i)}$ for all $w \in H^1(\Omega_\varepsilon)$ and $i \in L_\varepsilon$.

(iii) For any sequence w_ε such that $\limsup_{\varepsilon\to 0} \|w_\varepsilon\|_{H^1(\Omega_\varepsilon)} < \infty$ *one has* $\|\mathcal{T}_\varepsilon w_\varepsilon - J_\varepsilon w_\varepsilon\|_{L^2(\Omega)} \to 0$.

Proof. See [Kai85], [RT75, p. 40]. □

In the above geometric setting, we will study the linear operators A^ι_ε, $\iota = \mathrm{D}, \mathrm{N}, \alpha$ in $L^2(\Omega_\varepsilon)$, defined by the differential expression $-\Delta + 1$, with (dense) domains

$$\begin{aligned}\mathcal{D}(A^{\mathrm{D}}_\varepsilon) &= H^1_0(\Omega_\varepsilon) \cap H^2(\Omega_\varepsilon), \\ \mathcal{D}(A^{\mathrm{N}}_\varepsilon) &= \{u \in H^2(\Omega_\varepsilon) : \partial_\nu u = 0 \text{ on } \partial\Omega_\varepsilon\}, \\ \mathcal{D}(A^{\alpha}_\varepsilon) &= \{u \in H^2(\Omega_\varepsilon) : \partial_\nu u + \alpha u = 0 \text{ on } \partial\Omega_\varepsilon\},\end{aligned}$$

respectively, and the linear operators A^ι in $L^2(\Omega_\varepsilon)$ defined by the expression $-\Delta + 1 + \mu_\iota$, with domains

$$\begin{aligned}\mathcal{D}(A^{\mathrm{D}}) &= H_0^1(\Omega) \cap H^2(\Omega),\\ \mathcal{D}(A^{\mathrm{N}}) &= \big\{u \in H^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\big\},\\ \mathcal{D}(A^{\alpha}) &= \big\{u \in H^2(\Omega) : \partial_\nu u + \alpha u = 0 \text{ on } \partial\Omega\big\},\end{aligned}$$

respectively, where μ_ι, $\iota = \mathrm{D}, \mathrm{N}, \alpha$, are defined in (IV.3).

Remark IV.2.3. In the case when $d \geq 3$ one has the characterisation

$$\mu_{\mathrm{D}} = \frac{1}{2^d} \inf\left\{ \int_{\mathbb{R}^d \setminus B_1(0)} |\nabla u|^2, \quad u \in H^1(\mathbb{R}^d),\ u = 1 \text{ on } B_1(0)\right\}. \tag{IV.14}$$

Note that the factor $1/2^d$ arises from the fact that the unit cell is of size 2ε.

Using the notation above, we recall the following classical results.

Theorem IV.2.4 ([MK64, CM97])**.** *Let $\Omega \subset \mathbb{R}^d$ be open (bounded or unbounded). Suppose that $f \in L^2(\Omega)$, and let u^ε and $\tilde{u}$ be the solutions to*

$$\begin{aligned}(-\Delta + 1)u^\varepsilon &= f, \qquad & u^\varepsilon &\in H_0^1(\Omega_\varepsilon),\\ (-\Delta + 1 + \mu_{\mathrm{D}})\tilde{u} &= f, \qquad & \tilde{u} &\in H_0^1(\Omega).\end{aligned}$$

Then $J_\varepsilon u^\varepsilon \xrightharpoonup{\varepsilon\to 0} \tilde{u}$ in $H_0^1(\Omega)$.

Theorem IV.2.5 ([Kai85])**.** *Let $\Omega \subset \mathbb{R}^d$ be open (bounded or unbounded), and suppose that $\partial\Omega$ is smooth. Suppose also that $f \in L^2(\Omega)$, and let u^ε and $\tilde{u}$ be the solutions to*

$$\begin{aligned}(-\Delta + 1)u^\varepsilon &= f, \qquad & u^\varepsilon &\in \mathcal{D}(A_\varepsilon^{\alpha,\mathrm{N}}),\\ (-\Delta + 1 + \mu_{\alpha,\mathrm{N}})\tilde{u} &= f, \qquad & \tilde{u} &\in \mathcal{D}(A^{\alpha,\mathrm{N}}).\end{aligned}$$

Then one has

$$\mathcal{T}_\varepsilon u^\varepsilon \xrightharpoonup{\varepsilon\to 0} \tilde{u} \quad \text{in } H^1(\Omega).$$

Proof of Theorems IV.2.4 and IV.2.5. The results are obtained by following the proofs of [CM97, Thm 2.2], [Kai85, Thm 2]. Note that the weak convergence in $H^1(\Omega)$ is immediately obtained also for unbounded domains (and complex α). □

An important ingredient in the proofs are auxiliary functions $w_\varepsilon^\iota \in W^{1,\infty}(\mathbb{R}^d)$ defined, for each $\varepsilon \in (0,1)$, as the solution to the problems

$$w_\varepsilon^{\mathrm{N}} \equiv 1, \qquad \begin{cases} w_\varepsilon^{\mathrm{D}} = 0 & \text{in } T_i^\varepsilon, \\ \Delta w_\varepsilon^{\mathrm{D}} = 0 & \text{in } B_i^\varepsilon \setminus T_i^\varepsilon, \\ w_\varepsilon^{\mathrm{D}} = 1 & \text{in } P_i^\varepsilon \setminus B_i^\varepsilon, \\ w_\varepsilon^{\mathrm{D}} & \text{continuous}, \end{cases} \qquad \begin{cases} \partial_\nu w_\varepsilon^\alpha + \alpha w_\varepsilon^\alpha = 0 & \text{on } \partial T_i^\varepsilon, \\ \Delta w_\varepsilon^\alpha = 0 & \text{in } B_i^\varepsilon \setminus T_i^\varepsilon, \\ w_\varepsilon^\alpha = 1 & \text{in } P_i^\varepsilon \setminus B_i^\varepsilon, \\ w_\varepsilon^\alpha & \text{continuous}, \end{cases} \tag{IV.15}$$

used as a test function in the weak formulation of the problems (Dir), (Neu), (Rob).

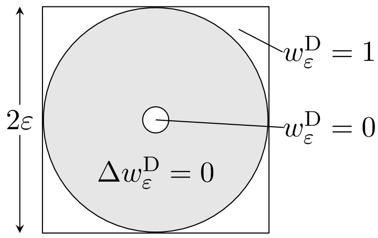


Figure IV.2.: Sketch of the auxiliary function $w_\varepsilon^{\mathrm{D}}$ in the Dirichlet case.

These functions were used in [CM97, Kai85] as test functions to prove strong convergence of solutions. They are "optimal" in the sense that they minimise the energy in annular regions around the holes. In the Dirichlet case, the function $w_\varepsilon^{\mathrm{D}}$ is nothing but the potential for the capacity $\operatorname{cap}\big(B_\varepsilon(i); B_{r_\varepsilon^{\mathrm{D}}}(i)\big)$. It can be shown that one has the convergences

$$\left.\begin{aligned} \mathcal{T}_\varepsilon w_\varepsilon^\alpha &\rightharpoonup 1 \\ w_\varepsilon^{\mathrm{D}} &\rightharpoonup 1 \end{aligned}\right\} \quad \text{weakly in } H^1(\Omega) \tag{IV.16}$$

$$-\nabla \cdot (\chi_{\Omega_\varepsilon} \nabla w_\varepsilon^\alpha) + \alpha w_\varepsilon^\alpha \delta_{\partial T_\varepsilon} \to \mu_\alpha \quad \text{strongly in } W^{-1,\infty}(\Omega) \tag{IV.17}$$

$$\begin{aligned} -\Delta w_\varepsilon^{\mathrm{D}} &= \mu_\varepsilon + \nu_\varepsilon, && \text{where } \nu_\varepsilon \text{ vanishes on } H_0^1(\Omega_\varepsilon) \text{ and} \\ \mu_\varepsilon &\to \mu_{\mathrm{D}} && \text{strongly in } W_{\mathrm{loc}}^{-1,\infty}(\Omega) \end{aligned} \tag{IV.18}$$

as $\varepsilon \to 0$, where $\delta_{\partial T_\varepsilon}$ denotes the Dirac measure on the boundary of the holes (for a proof of the above facts, see [CM97, Lemma 2.3] and [Kai85, Section 3]).

IV.3. Main results

In what follows we prove the following claim.

Theorem IV.3.1. *Let $J_\varepsilon, A^\iota_\varepsilon, A^\iota$ be defined as in the previous section. Then for $\iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}$ one has*

$$\left\| J_\varepsilon (A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1} J_\varepsilon \right\|_{\mathcal{L}(L^2(\Omega_\varepsilon),\, L^2(\Omega))} \to 0 \qquad (\varepsilon \to 0),$$

that is, the operator sequence A^ι_ε converges to A^ι in the norm-resolvent sense.

This theorem implies that for solutions u^ι_ε of (Dir), (Neu), (Rob) and the corresponding "limit functions" $\overline{u}^\iota_\varepsilon = (A^\iota)^{-1} J_\varepsilon f$ there is an error estimate which is independent of the right hand side datum f. More precisely: There exists a function $a(\varepsilon)$ with $a(\varepsilon) \to 0$ for $\varepsilon \to 0$ such that $\|J_\varepsilon u^\iota_\varepsilon - \overline{u}^\iota_\varepsilon\|_{L^2(\Omega)} \leq a(\varepsilon) \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ for any uniformly bounded family (f_ε) with $f_\varepsilon \in L^2(\Omega_\varepsilon)$ $\forall \varepsilon > 0$.

Applying Corollary IV.1.7, we immediately obtain the following important consequence of the above theorem.

Corollary IV.3.2. *For all compact $K \subset \mathbb{C}$, one has $\sigma(A^\iota_\varepsilon) \cap K \xrightarrow{\varepsilon \to 0} \sigma(A^\iota) \cap K$ in the Hausdorff sense.*

In particular, this corollary shows that (if $\mathrm{Re}(\mu_\iota) > 0$) a spectral gap opens for A^ι_ε between 0 and $\mathrm{Re}(\mu_\iota)$.

Remark IV.3.3. We note that our assumption on the spherical shape of the holes was made only for the sake of definiteness, and our results easily generalise to more general geometries as detailed in [CM97, Th. 2.7]. Moreover, our results are also valid for more general elliptic operators $\mathrm{div}(A\nabla)$ with continuous coefficients A (cf. [CM97]).

IV.4. Uniformity with respect to the right-hand side

In this section we prove that the result of Theorems IV.2.4, IV.2.5 hold in a strengthened form, namely, uniformly with respect to the right-hand side f. More precisely, the following holds.

Theorem IV.4.1. *Suppose that $\varepsilon_n \searrow 0$, $f_n \in L^2(\Omega_{\varepsilon_n})$, $n \in \mathbb{N}$, with $\|f_n\|_{L^2(\Omega_{\varepsilon_n})} \leq 1$, and let u^ι_n and $\tilde{u}^\iota_n$ be the solutions to the problems ($\iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}$)*

$$(-\Delta + 1) u^\iota_n = f_n, \qquad u^\iota_n \in \mathcal{D}(A^\iota_{\varepsilon_n}), \tag{IV.19}$$

$$(-\Delta + 1 + \mu_\iota) \tilde{u}^\iota_n = J_{\varepsilon_n} f_n, \qquad \tilde{u}^\iota_n \in \mathcal{D}(A^\iota). \tag{IV.20}$$

Then for every bounded, open $K \subset \Omega$ one has

$$J_{\varepsilon_n} u^\iota_n - \tilde{u}^\iota_n \to 0 \qquad \text{strongly in } L^2(K),$$

$$J_{\varepsilon_n}\nabla u_n^\iota - \nabla\tilde{u}_n^\iota \to 0 \qquad \textit{weakly in } L^2(K),$$

for $\iota \in \{\mathrm{D},\mathrm{N},\alpha\}$.

Proof. We have the following *a priori* estimates (note Lemma IV.2.2):

$$\begin{aligned}
\|\mathcal{T}_{\varepsilon_n} u_n^{\alpha,\mathrm{N}}\|_{H^1(\Omega)} &\le C\|J_{\varepsilon_n} f_n\|_{L^2(\Omega)},\\
\|J_{\varepsilon_n} u_n^{\mathrm{D}}\|_{H^1(\Omega)} &\le C\|J_{\varepsilon_n} f_n\|_{L^2(\Omega)},\\
\|\tilde{u}_n^\iota\|_{H^1(\Omega)} &\le C\|J_{\varepsilon_n} f_n\|_{L^2(\Omega)} \quad \forall\iota \in \{\mathrm{D},\mathrm{N},\alpha\}.
\end{aligned}$$

Thus, there exists a subsequence (still indexed by n) and $u^\iota, \tilde{u}^\iota \in H^1(\Omega)$ such that

$$\left.\begin{aligned}
J_{\varepsilon_n} u_n^{\mathrm{D}} &\xrightharpoonup{n\to\infty} u^{\mathrm{D}}\\
\mathcal{T}_{\varepsilon_n} u_n^{\alpha,\mathrm{N}} &\xrightharpoonup{n\to\infty} u^{\alpha,\mathrm{N}}\\
\tilde{u}_n^\iota &\xrightharpoonup{k\to\infty} \tilde{u}^\iota, \quad \iota \in \{\mathrm{D},\mathrm{N},\alpha\}
\end{aligned}\right\} \text{weakly in } H^1(\Omega). \tag{IV.21}$$

Note that that for every bounded $K \subset \Omega$ the convergence statements (IV.21) are strong in $L^2(K)$. In particular, employing Lemma IV.2.2 (i), (iii) we immediately obtain

$$J_{\varepsilon_n} u_n^\iota \to u^\iota \qquad \text{strongly in } L^2(K), \tag{IV.22}$$

$$J_{\varepsilon_n}\nabla u_n^\iota \rightharpoonup \nabla u^\iota \qquad \text{weakly in } L^2(K). \tag{IV.23}$$

for all $\iota \in \{\mathrm{D},\mathrm{N},\alpha\}$. Next, choose a further subsequence (still indexed by n) such that also $J_{\varepsilon_n} f_n \xrightharpoonup{n\to\infty} f$ weakly in $L^2(\Omega)$, where the limit f may depend on the choice of subsequence.

Dirichlet and Neumann case. We restrict ourselves to the Dirichlet and Neumann problems first and comment on the Robin problem at the end of the proof. Consider the weak formulations of the problem (IV.20), *i.e.*

$$\int_\Omega \overline{\nabla\tilde{u}_n^\iota}\nabla\phi + (1+\mu_\iota)\int_\Omega \overline{\tilde{u}_n^\iota}\phi = \int_\Omega \overline{f_n}\phi,$$

where $\phi \in C_0^\infty(\Omega)$ for $\iota = \mathrm{D}$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ for $\iota = \mathrm{N}$. Letting $n \to \infty$ and using the convergencies (IV.22),(IV.23) (with $K = \Omega \cap \operatorname{supp}\phi$) we obtain

$$\int_\Omega \overline{\nabla \tilde{u}^\iota}\, \nabla\phi + (1+\mu_\iota) \int_\Omega \overline{\tilde{u}^\iota}\phi = \int_\Omega \overline{f}\phi.$$

Next consider the weak formulation of (IV.19),where we choose the test function $w^\iota_{\varepsilon_n}\phi$:

$$\int_{\Omega_{\varepsilon_n}} \overline{\nabla u^\iota_n} \nabla\big(w^\iota_{\varepsilon_n}\phi\big) + \int_{\Omega_{\varepsilon_n}} \overline{u^\iota_n} w^\iota_{\varepsilon_n}\phi = \int_{\Omega_{\varepsilon_n}} \overline{f_n} w^\iota_{\varepsilon_n}\phi,$$

where again $\phi \in C_0^\infty(\Omega)$ for $\iota = \mathrm{D}$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ for $\iota = \mathrm{N}$. It follows from the results of [CM97, Kai85] (cf. (IV.16)-(IV.18)) that the left and right-hand side of this equation converge to

$$\int_\Omega \big(\overline{\nabla u^\iota} \nabla\phi + (1+\mu_\iota)\overline{u^\iota}\phi\big) \quad \text{and} \quad \int_\Omega \overline{f}\phi,$$

respectively. Thus, we obtain

$$\int_\Omega \big(\overline{\nabla u^\iota}\, \nabla\phi + (1+\mu_\iota)\overline{u^\iota}\phi\big) = \int_\Omega \overline{f}\phi,$$

and hence u^ι and $\tilde{u}^\iota$ are weak solutions to the same equation. Uniqueness of solutions (for all $\iota \in \{\mathrm{D}, \mathrm{N}\}$) implies $\tilde{u}^\iota = u^\iota$, which shows the assertion for the chosen subsequence.

Finally, applying the above reasoning to every subsequence of $(J_{\varepsilon_n} u^\iota_n - \tilde{u}^\iota_n)$ yields the result for the whole sequence.

Robin case. In the Robin case, the above proof remains valid in the interior of Ω_ε, but convergence of the boundary terms

$$\int_{\partial T_{\varepsilon_n}} w^\iota_{\varepsilon_n} \overline{u^\iota_n}\phi \qquad \text{and} \qquad \int_{\partial\Omega} \overline{u^\iota_n}\phi$$

has to be shown. Convergence of the second term follows since $u^\iota_n \rightharpoonup u^\iota$ in $L^2(\partial\Omega)$, while convergence of the first term follows from (IV.17). For details, see [Kai85]. □

Corollary IV.4.2. *If the domain Ω is bounded, one has*

$$\big\| J_\varepsilon (A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1} J_\varepsilon \big\|_{\mathcal{L}(L^2(\Omega_\varepsilon),\, L^2(\Omega))} \to 0 \qquad (\varepsilon \to 0)$$

for $\iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}$, i.e., Theorem IV.3.1 holds in that case of bounded Ω.

Proof. Since Ω is bounded, the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is compact, thus the sequence $J_{\varepsilon_n} u_n^\iota - \tilde{u}_n^\iota$ from the previous proof has a subsequence converging to 0 strongly in $L^2(\Omega)$. Since this can be done for every subsequence of $(J_{\varepsilon_n} u_n^\iota - \tilde{u}_n^\iota)$, the whole sequence converges to 0.

Now, choose a sequence $f_n \in L^2(\Omega_{\varepsilon_n})$, $\|f_n\|_{L^2(\Omega_\varepsilon)} \leq 1$, such that

$$\sup_{\substack{f \in L^2(\Omega_{\varepsilon_n}) \\ \|f\| \leq 1}} \left\| (J_{\varepsilon_n} (A_\varepsilon^\iota)^{-1} - (A^\iota)^{-1} J_{\varepsilon_n}) f \right\|_{L^2(\Omega_\varepsilon)} - \frac{1}{n} < \left\| (J_{\varepsilon_n} (A_{\varepsilon_n}^\iota)^{-1} - (A^\iota)^{-1} J_{\varepsilon_n}) f_n \right\|_{L^2(\Omega_{\varepsilon_n})}.$$

By the above, the right-hand side of this inequality converges to zero, which implies the claim. □

Remark IV.4.3. We note that the conclusion of Theorem IV.4.1 remains true if we replace the lattice L_ε on which the holes are situated by a lattice L_ε^*, which is "shifted of order ε", i.e. $L_\varepsilon^* = L_\varepsilon + y_\varepsilon$ with $\mathbb{R}^d \ni y_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, it is straightforward to prove that the convergences (IV.16)-(IV.18) are still valid for the shifted auxiliary functions $w_\varepsilon^{\iota *} := w_\varepsilon^\iota(\,\cdot\, + y_\varepsilon)$. Replacing w_ε^ι by $w_\varepsilon^{\iota *}$ in the proof of Theorem IV.4.1 yields the desired result.

For more details in the Dirichlet case, see the proof of Lemma IV.6.1 (*cf. Claim 3* there).

Treating unbounded domains requires further effort. Since we lack compact embeddings in this case, we will have to take advantage of the sufficiently rapid decay of solutions to $(-\Delta + 1)u = f$ and a decomposition of the right hand side with a bound on the interactions.

IV.5. Exponential decay of solutions

We begin with a general result which we assume is classical, but include for the sake of completeness. Let $U \subset \mathbb{R}^d$ open satisfying the strong local Lipschitz condition, $\lambda > \frac{1}{2}$ and consider the problems (*cf.* (Dir), (Neu), (Rob))

$$\begin{cases} (-\Delta + \lambda) u^\alpha &= f \quad \text{in } U, \\ \partial_\nu u^\alpha + \alpha u^\alpha &= 0 \quad \text{on } \partial U; \end{cases} \tag{IV.24}$$

$$\begin{cases} (-\Delta+\lambda)u^{\mathrm{N}} &= f \quad \text{in } U, \\ \partial_\nu u^{\mathrm{N}} &= 0 \quad \text{on } \partial U; \end{cases} \tag{IV.25}$$

$$\begin{cases} (-\Delta+\lambda)u^{\mathrm{D}} &= f \quad \text{in } U, \\ u^{\mathrm{D}} &= 0 \quad \text{on } \partial U. \end{cases} \tag{IV.26}$$

Let $x_0 \in \mathbb{R}^d$, and define the function $\omega(x) = \cosh(|x-x_0|)$. Then the following statement holds.

Proposition IV.5.1. *Let $f \in L^2(U)$, $\operatorname{supp}(f)$ compact. Then each of the problems* (IV.24)–(IV.26) *has a unique weak solution $u^\iota \in H^1(U)$ satisfying*

$$\int_U |u^\iota|^2 \omega \, dx \le M \int_U |f|^2 \omega \, dx \tag{IV.27}$$

$$\int_U |\nabla u^\iota|^2 \omega \, dx \le M \int_U |f|^2 \omega \, dx, \tag{IV.28}$$

where $M := \max\left\{2, (\lambda - \frac{1}{2})^{-1}\right\}$.

We postpone the proof, in order to introduce some notation and prove auxiliary results. First, let us denote $d\mu := \omega dx$ and introduce the weighted Sobolev spaces $\mathcal{H} := W^{1,2}(U;\omega)$, $\mathcal{H}_0 := W^{1,2}_0(U;\omega)$ with scalar product

$$\langle u, v \rangle_{\mathcal{H}} = \int_U uv \, d\mu + \int_U \nabla u \cdot \nabla v \, d\mu.$$

Moreover, let $\lambda > \frac{1}{2}$ and define the sesquilinear forms

$$a^\alpha(u,v) := \int_U (\overline{\nabla u} \cdot \nabla v + \lambda \overline{u} v) \, d\mu + \int_U v \overline{\nabla u} \cdot \frac{\nabla \omega}{\omega} \, d\mu + \alpha \int_{\partial U} \overline{u} v \, \omega \, dS \qquad \text{on } \mathcal{H}, \tag{IV.29}$$

$$a^{\mathrm{N}}(u,v) := \int_U (\overline{\nabla u} \cdot \nabla v + \lambda \overline{u} v) \, d\mu + \int_U v \overline{\nabla u} \cdot \frac{\nabla \omega}{\omega} \, d\mu \qquad \text{on } \mathcal{H}, \tag{IV.30}$$

$$a^{\mathrm{D}}(u,v) := \int_U (\overline{\nabla u} \cdot \nabla v + \lambda \overline{u} v) \, d\mu + \int_U v \overline{\nabla u} \cdot \frac{\nabla \omega}{\omega} \, d\mu \qquad \text{on } \mathcal{H}_0. \tag{IV.31}$$

Lemma IV.5.2. *For $\lambda > \frac{1}{2}$ and $\iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}$, the form a^ι is continuous and coercive on $\mathcal{H}$ (on $\mathcal{H}_0$ in the case $\iota = \mathrm{D}$).*

Proof. We will only treat the Robin case here, the other cases being analogous. Denote by I the second term in (IV.29) and note that ω was chosen so that $|\nabla\omega| \leq \omega$. By Hölder's inequality with respect to μ one has

$$|I| \leq \underbrace{\left\| \frac{\nabla\omega}{\omega} \right\|_\infty}_{\leq 1} \|\nabla u\|_{L^2(\mu)} \|v\|_{L^2(\mu)} \leq \frac{1}{2}\|\nabla u\|^2_{L^2(\mu)} + \frac{1}{2}\|v\|^2_{L^2(\mu)},$$

and thus

$$\begin{aligned} \left|a(u,u)\right| &\geq \|\nabla u\|^2_{L^2(\mu)} + \lambda\|u\|^2_{L^2(\mu)} + |\alpha|\left\|\omega^{1/2}u\right\|^2_{L^2(\partial U)} + I \\ &\geq \|\nabla u\|^2_{L^2(\mu)} + \lambda\|u\|^2_{L^2(\mu)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\mu)} - \frac{1}{2}\|u\|^2_{L^2(\mu)} \\ &= \frac{1}{2}\|\nabla u\|^2_{L^2(\mu)} + \left(\lambda - \frac{1}{2}\right)\|u\|^2_{L^2(\mu)}, \end{aligned}$$

which shows coercivity in $\mathcal{H}$. Continuity follows by estimating the boundary term. By the trace theorem [DiB16, Prop. IX.18.1] we have, for each $\delta > 0$,

$$\int_{\partial U} |u|^2\omega\,dx \leq 2\delta\|\nabla(\omega^{1/2}u)\|^2_{L^2(U)} + \frac{C}{\delta}\|\omega^{1/2}u\|^2_{L^2(U)}. \tag{IV.32}$$

The first term can be estimated using the special choice of ω :

$$\begin{aligned} \|\nabla(\omega^{1/2}u)\|^2_{L^2(U)} &= \int_U \left|\omega^{1/2}\nabla u + \frac{1}{2}u\frac{\nabla\omega}{\omega^{1/2}}\right|^2 dx \\ &\leq 2\int_U \omega|\nabla u|^2\,dx + \frac{1}{2}\int_U |u|^2\frac{|\nabla\omega|^2}{\omega}\,dx \\ &\leq 2\|\nabla u\|_{L^2(\mu)} + 2\left\|\frac{\nabla\omega}{\omega}\right\|^2_\infty \int_U |u|^2\omega\,dx \\ &\leq 2\|\nabla u\|^2_{H^1(\mu)}. \end{aligned} \tag{IV.33}$$

The desired continuity now follows immediately by combining (IV.32) and (IV.33). □

Lemma IV.5.3. *Let $f \in L^2(U)$, $\iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}$, and suppose that* $\operatorname{supp}(f)$ *compact.*

Then the problem

$$a^\iota(u,v) = \int_U \overline{f} v \, d\mu \qquad \forall v \in \mathcal{H} \tag{IV.34}$$

has a solution in $\mathcal{H}$.

Proof. By Hölder inequality, one has

$$\left| \int_U \overline{f} v \, d\mu \right| \leq \|f\|_{L^2(\mu)} \|v\|_{L^2(\mu)} \leq \|\omega\|_{L^\infty(\operatorname{supp} f)} \|f\|_{L^2(U)} \|v\|_{L^2(\mu)},$$

so $f \in \mathcal{H}'$. The assertion now follows from Lemma IV.5.2 and the Lax-Milgram theorem for complex, non-symmetric sesquilinear forms [TL80, Thm. VI.1.4]. □

Proof of Proposition IV.5.1. Again we focus on the Robin case, the other cases being analogous. Denote by u the solution obtained from Prop. IV.5.3. Then $u \in H^1(U)$, since $\mathcal{H} \subset H^1(U)$. Moreover, let $\phi \in C_0^\infty(\mathbb{R}^d)$ be arbitrary and decompose it as $\phi = \omega\psi$. Then $\psi \in C_0^\infty(\mathbb{R}^d) \subset \mathcal{H}$ and one has

$$\begin{aligned}
\int_U \overline{\nabla u} \cdot \nabla\phi \, dx + \lambda \int_U \overline{u}\phi \, dx + \alpha \int_{\partial U} \overline{u}\phi \, dS \\
&= \int_U \overline{\nabla u} \cdot \big(\omega\nabla\psi + \psi\nabla\omega\big) \, dx + \lambda \int_U \overline{u}\psi\omega \, dx + \alpha \int_{\partial U} \overline{u}\psi\omega \, dS \\
&= a^\alpha(u,\psi) \\
&= \int_U \overline{f}\psi \, d\mu \\
&= \int_U \overline{f}\phi \, dx.
\end{aligned}$$

Thus, the function u solves the problem

$$\int_U \overline{\nabla u} \cdot \nabla\phi \, dx + \lambda \int_U \overline{u}\phi \, dx + \alpha \int_{\partial U} \overline{u}\phi \, dS = \int_U \overline{f}\phi \, dx \qquad \forall \phi \in C_0^\infty(\mathbb{R}^d). \tag{IV.35}$$

Uniqueness of solutions and density of $C_0^\infty(\mathbb{R}^d)$ in $H^1(U)$ implies that u is the weak solution in $H^1(U)$ to the Robin problem (IV.24).

The estimates (IV.27), (IV.28) follow from the coercivity of a^ι. □

IV.6. Decomposition of the right-hand side

In this section we prove norm resolvent convergence in the case of unbounded Ω. We conclude the proof of Theorem IV.3.1 by decomposing the domain into cubes Q_i, writing $f = \sum_i f\chi_{Q_i}$ and then applying the above results to each term $f\chi_{Q_i}$. The following lemma shows uniform convergence with respect to the position of the cubes.

Lemma IV.6.1. *Let* $\varepsilon_n \searrow 0$ *and* $f_n \in L^2(\Omega_{\varepsilon_n})$, $n \in \mathbb{N}$, *be such that* $\|J_{\varepsilon_n} f_n\|_{L^2(\Omega)} \leq 1$ *and* $\operatorname{supp}(f_n) \subset Q_{i_n}$, *where* $Q_{i_n} = [0,1]^d + i_n$ *with* $i_n \in \mathbb{Z}^d$. *Let* $u_n^\iota, \tilde{u}_n^\iota$ *be the solutions to the problems*

$$A_{\varepsilon_n}^\iota u_n^\iota = f_n, \qquad A^\iota \tilde{u}_n^\iota = J_{\varepsilon_n} f_n, \qquad n \in \mathbb{N}, \qquad \iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}. \tag{IV.36}$$

Then $\|J_{\varepsilon_n} u_n^\iota - \tilde{u}_n^\iota\|_{L^2(\Omega)} \to 0$ *for all* $\iota \in \{\mathrm{D}, \mathrm{N}, \alpha\}$.

Proof. The idea of the proof is to use translation invariance, in order to shift $\operatorname{supp}(f_n)$ back near zero for every n, and then use the Fréchet-Kolmogorov compactness criterion to obtain a convergent subsequence of $(J_{\varepsilon_n} u_n^\iota - \tilde{u}_n^\iota)$; Theorem IV.4.1 will identify its limit as zero. In order not to overburden notation we omit the index ι.

We now carry out the outlined strategy. We set, for $i \in \mathbb{N}$,

$$u_n^*(x) := u_n(x + i_n), \qquad \tilde{u}_n^*(x) := \tilde{u}_n(x + i_n), \qquad f_n^*(x) := f_n(x + i_n).$$

These functions still solve the problems (IV.36) with f_n replaced by f_n^* and Ω replaced by $\Omega - i_n$. The new sequence f_n^* has the nice property that $\operatorname{supp}(f_n^*) \subset [0,1]^d$ for all n. In the following we consider $J_{\varepsilon_n} u_n^*$, $\tilde{u}_n^*$, f_n^* as elements of $L^2(\mathbb{R}^d)$ that are zero outside $\Omega - i_n$. We will now show that $\tilde{u}_n^* - J_{\varepsilon_n} u_n^*$ converges to zero in $L^2(\mathbb{R}^d)$. To this end, consider the bounded set

$$\mathcal{F} := \{\tilde{u}_n^* - J_{\varepsilon_n} u_n^* : n \in \mathbb{N}\} \subset L^2(\mathbb{R}^d). \tag{IV.37}$$

Claim: $\mathcal{F}$ is precompact in $L^2(\mathbb{R}^d)$.

We postpone the proof of this claim to Lemma IV.6.2. We immediately obtain that $(\tilde{u}_n^* - J_{\varepsilon_n} u_n^*)$ has a convergent subsequence in $L^2(\mathbb{R}^d)$. In the remainder of the proof we distinguish the Dirichlet case from the Neumann and Robin cases.

Neumann and Robin case. By translation invariance of Ω, all quantities with asterisks are still in $H^1(\Omega)$ with Neumann, resp. Robin boundary conditions. In addition, by ε-periodicity there exists a null sequence $(y_n) \subset \mathbb{R}^d$ such that $L_\varepsilon + i_n = L_\varepsilon + y_n$ for all n. Therefore, Theorem IV.4.1 and Remark IV.4.3 can be applied to conclude that $\|\tilde{u}_n^* - J_{\varepsilon_n} u_n^*\|_{L^2(K)} \to 0$ for every bounded $K \subset \mathbb{R}^d$ which identifies the limit of the subsequence as zero. Arguing as above for all subsequences of $(\tilde{u}_n - J_{\varepsilon_n} u_n)$, we conclude that $\tilde{u}_n - J_{\varepsilon_n} u_n \to 0$ in $L^2(\Omega)$.

Dirichlet case. We know that a subsequence of $\tilde{u}_n^* - J_{\varepsilon_n} u_n^*$ is convergent in $L^2(\mathbb{R}^d)$. The limit is denoted $h^* \in L^2(\mathbb{R}^d)$. Since the sequence $(\tilde{u}_n^* - J_{\varepsilon_n} u_n^*)$ is also bounded in $H^1(\mathbb{R}^d)$, there exists a subsequence (still indexed by n) converging weakly in $H^1(\mathbb{R}^d)$. This weak limit must coincide with h^*. Therefore, $h^* \in H^1(\mathbb{R}^d)$. The goal is to prove $h^* = 0$. Define the set

$$\Omega^* := \{x \in \mathbb{R}^d \,|\, \exists \varepsilon > 0 \,:\, B_\varepsilon(x) \subset (\Omega - i_n) \text{ for almost all } n\}.$$

Clearly, Ω^* is open. The idea is to show that

(I) outside Ω^*, h^* is identically zero and

(II) inside Ω^*, h^* is harmonic with zero boundary values (hence zero).

Proof of (I):

Claim 1: Let $\eta > 0$. There exists a $\delta > 0$ such that for every $x \in \mathbb{R}^d \setminus \Omega^*$ there exists a ball B_x with radius δ such that

(i) $\operatorname{dist}(x, B_x) < \eta$

(ii) $h^* = 0$ on B_x.

Proof. Let $x \in \mathbb{R}^d \setminus \Omega^*$ and $\eta > 0$. By definition of Ω^*, we have

$$B_\eta(x) \cap (\mathbb{R}^d \setminus (\Omega - i_n)) \neq \emptyset$$

for infinitely many n. Choose a sequence (y_k) with $y_k \in B_\eta(x) \cap (\mathbb{R}^d \setminus (\Omega - i_{n_k}))$ for all k. (in the following, we relabel $n_k \to n$). Then, by the assumption on Ω, there exists a sequence of balls B_n with radius δ and $y_n \in B_n$ and $B_n \subset \mathbb{R}^d \setminus (\Omega - i_n)$ for all n.

Now let $\phi \in C_0^\infty(B_\delta(0))$ and define $\phi_n := \phi(\,\cdot\, + c_n)$, where c_n denotes the centre of B_n. The sequence (c_n) is bounded in $\mathbb{R}^d$ and therefore has a convergent subsequence $c_{n_k} \to c_\infty$. The corresponding subsequence ϕ_{n_k} then converges in $L^2(\mathbb{R}^d)$ to a limit ϕ_∞, which has the form $\phi_\infty = \phi(\,\cdot\, + c_\infty) \in C_0^\infty(B_\infty)$ for the δ-ball B_∞ with centre c_∞ (this follows e.g. from dominated convergence).

Since $\tilde{u}_n^* - J_{\varepsilon_n} u_n^* \equiv 0$ on B_n for all n, we obtain

$$\begin{aligned} 0 &= \lim_{k\to\infty} \int_{\mathbb{R}^d} (\tilde{u}_{n_k}^* - J_{\varepsilon_{n_k}} u_{n_k}^*)\phi_{n_k}\, dx \\ &= \int_{B_\infty} h^* \phi_\infty\, dx. \end{aligned}$$

Since the function $\phi \in C_0^\infty(B_\delta(0))$ was arbitrary, we conclude that the equation

$$\int_{B_\infty} h^* \varphi\, dx = 0$$

holds for all $\varphi \in C_0^\infty(B_\infty)$ and hence $h = 0$ on B_∞. This proves the claim.

From Claim 1 it follows that $h^* = 0$ on $\mathbb{R}^d \setminus \Omega^*$ as the next assertion shows.

Claim 2: We have $h^* = 0$ on $\mathbb{R}^d \setminus \Omega^*$.

Proof. Let $\eta > 0$ and take a lattice $L_\eta := \eta \cdot \mathbb{Z}^N$. Then choose for every $k \in L_\eta \setminus \Omega^*$ a ball B_k of radius δ as in Claim 1. The union of all B_k will not cover all of $\mathbb{R}^d \setminus \Omega^*$, but we can do the following: Let $K \subset \mathbb{R}^d \setminus \Omega^*$ be compact. Then

$$\left| K \setminus \bigcup_{k \in L_\eta} B_k \right| \to 0 \qquad \text{as } \eta \to 0$$

For $m \in \mathbb{N}$ define the set

$$S_{>m} := \left\{ x \in K \,\middle|\, |h^*(x)| > m \right\}$$

and compute

$$\begin{aligned} \int_{K \setminus S_{>m}} |h^*|^2\, dx &\le m^2 \left| K \setminus \bigcup_{k\in L_\eta} B_k \right| \\ &\to 0 \qquad (\eta \to 0) \end{aligned}$$

hence $h^* = 0$ on $K \setminus S_{>m}$. Since m was arbitrary, we immediately obtain

$$h^* = 0 \quad \text{on} \quad K \setminus \bigcap_{m\in\mathbb{N}} S_{>m}.$$

But $\bigcap_{m\in\mathbb{N}} S_{>m}$ has measure zero, hence $h^* = 0$ almost everywhere on K.

This concludes the proof of (I).

Proof of (II): Let $\phi \in C_0^\infty(\Omega^*)$. Then for every $x \in \operatorname{supp}(\phi)$ there exists $\varepsilon = \varepsilon(x) > 0$ such that

$$B_{\varepsilon(x)}(x) \subset \Omega - i_n \quad \text{for almost all } n \in \mathbb{N}$$

(by definition of Ω^*). These $B_{\varepsilon(x)}(x)$ cover $\operatorname{supp}(\phi)$. Hence, there is a finite subcovering $\{B_{\varepsilon_1}(x_1), \dots, B_{\varepsilon_\nu}(x_\nu)\}$. In other words, there exists $n_0 \in \mathbb{N}$ such that $\operatorname{supp}(\phi) \subset (\Omega - i_n)$ for all $n > n_0$. Hence, we can write down

$$\int_{\mathbb{R}^d} \nabla \tilde{u}_n^* \cdot \nabla \phi \, dx + (1+\mu) \int_{\mathbb{R}^d} \tilde{u}_n^* \phi \, dx = \int_{\mathbb{R}^d} f_n^* \phi \, dx \tag{IV.38}$$

$$\int_{\mathbb{R}^d} \nabla (J_{\varepsilon_n} u_n^*) \cdot \nabla (w_{\varepsilon_n}^* \phi) \, dx + \int_{\mathbb{R}^d} J_{\varepsilon_n} u_n^* \, w_{\varepsilon_n}^* \phi \, dx = \int_{\mathbb{R}^d} f_n^* w_{\varepsilon_n}^* \phi \, dx, \tag{IV.39}$$

where $w_\varepsilon^*(x) = w_\varepsilon(x + i_n)$. By H^1-boundedness, $(\tilde{u}_n^*)$ and $(J_{\varepsilon_n} u_n^*)$ have convergent subsequences. We denote the limits $\tilde{u}^*$ and u^*, respectively. Clearly, we have $h^* = \tilde{u}^* - u^*$. Furthermore, one can assume $f_n \rightharpoonup f$ in L^2 for some f. Convergence of every term in (IV.38) is immediate. Convergence in (IV.39) is treated by

Claim 3: For $\phi \in C_0^\infty(\Omega^*)$, we have

$$\int_{\mathbb{R}^d} \nabla (J_{\varepsilon_n} u_n^*) \cdot \nabla (w_{\varepsilon_n}^* \phi) \, dx \to \mu^{\mathrm{D}} \int_{\mathbb{R}^d} u^* \phi \, dx \tag{IV.40}$$

$$\int_{\mathbb{R}^d} J_{\varepsilon_n} u_n^* \, w_{\varepsilon_n}^* \phi \, dx \to \int_{\mathbb{R}^d} u^* \phi \, dx \tag{IV.41}$$

$$\int_{\mathbb{R}^d} f_n^* w_{\varepsilon_n}^* \phi \, dx \to \int_{\mathbb{R}^d} f \phi \, dx. \tag{IV.42}$$

Proof. Let $\phi \in C_0^\infty(\Omega^*)$ and denote $K := \operatorname{supp}(\phi)$. We first show that $w_{\varepsilon_n}^* \rightharpoonup 1$ in $H^1(K)$. First, note that $w_{\varepsilon_n}^*$ is bounded in $H^1(K)$, so there exists a weakly convergent subsequence $w_{\varepsilon_n}^* \rightharpoonup 1$. By ε-periodicity of $w_{\varepsilon_n}^* = w_\varepsilon(\,\cdot\, + i_n)$, there exists a null sequence $(x_n) \subset \mathbb{R}^d$ with $w_{\varepsilon_n}^* = w_\varepsilon(\,\cdot\, + x_n)$. Choose an open ball B such that $K \subset B$ and let n be large enough such that $K - x_n \subset B$. Then compute

$$\begin{aligned}
\int_K |w_{\varepsilon_n}^*(x) - 1|^2 \, dx &= \int_K |w_{\varepsilon_n}(x + x_n) - 1|^2 \, dx \\
&= \int_{K - x_n} |w_{\varepsilon_n}(x) - 1|^2 \, dx \\
&\le \int_B |w_{\varepsilon_n}(x) - 1|^2 \, dx
\end{aligned}$$

$$\to 0$$

as $n \to \infty$, since the unshifted function satisfies $w_\varepsilon \xrightarrow{\varepsilon \to 0} 1$ on bounded sets. The convergence $w^*_{\varepsilon_n} \rightharpoonup 1$ proves (IV.41) and (IV.42).

To prove (IV.40), we closely follow [CM97]. We have

$$\int_{\mathbb{R}^d} \nabla(J_{\varepsilon_n} u^*_n) \cdot \nabla(w^*_{\varepsilon_n} \phi)\, dx = \left\langle -\Delta w^*_{\varepsilon_n}, \phi\, J_{\varepsilon_n} u^*_n \right\rangle - \int_{\mathbb{R}^d} u^*_n \nabla\phi \cdot \nabla w^*_{\varepsilon_n}\, dx.$$

The last term converges to 0, since $w_{\varepsilon_n} \rightharpoonup 1$ in $H^1(K)$ and u^*_n converges strongly in L^2. The first term on the right-hand side is proportional to

$$\left\langle \sum_{k=1}^{\nu_n} \varepsilon_n \delta_{\partial(U_k^{\varepsilon_n} + i_n)}\, ,\, \phi\, J_{\varepsilon_n} u^*_n \right\rangle,$$

where ν_n denotes the number of holes in K and $U_k^{\varepsilon_n}$ denotes the ball of radius ε centered on the k-th hole (see [CM97, eq. (2.6)]). Since $\phi J_{\varepsilon_n} u^*_n$ is weakly convergent in $W^{1,1}(K)$, the assertion will be proved if we show that

$$\sum_{k=1}^{\nu_n} \varepsilon_n \delta_{\partial(U_k^{\varepsilon_n} + i_n)} \to \frac{|\partial B_1(0)|}{2^d} \qquad \text{strongly in } W^{-1,\infty}_{\mathrm{loc}}(\mathbb{R}^d).$$

To this end, introduce the auxiliary function $q^*_{\varepsilon_n}$, defined as the solution of

$$\begin{cases} -\Delta q^*_{\varepsilon_n} = d & \text{in } U_k^{\varepsilon_n} + i_n \\ \partial_\nu q^*_{\varepsilon_n} = \varepsilon & \text{on } \partial(U_k^{\varepsilon_n} + i_n) \\ q^*_{\varepsilon_n} = 0 & \text{on } \partial(U_k^{\varepsilon_n} + i_n). \end{cases}$$

Extend this function by zero to the cube of edge length ε centered at $U_k^{\varepsilon_n} + i_n$ and then to all of $\mathbb{R}^d$ by periodicity. This yields a function with $\|\nabla q^*_{\varepsilon_n}\|_\infty < \varepsilon$, hence

$$q^*_{\varepsilon_n} \to 0 \quad \text{in } W^{1,\infty}(\mathbb{R}^d). \tag{IV.43}$$

Denote $\chi^n_U := \chi_{\bigcup_k (U_k^{\varepsilon_n} + i_n)}$. Then

$$-\Delta q^*_{\varepsilon_n} = d\chi^n_U + \sum_{k=1}^{d} \varepsilon_n \delta_{\partial(U_k^{\varepsilon_n} + i_n)}.$$

It follows from (IV.43) that $-\Delta q^*_{\varepsilon_n} \to 0$ strongly in $W^{-1,\infty}(\mathbb{R}^d)$, so the claim is proved if we can show that $\chi^n_U \overset{*}{\rightharpoonup} \frac{|\partial B_1(0)|}{d 2^d}$ weakly* in $L^\infty(\mathbb{R}^d)$ (and hence strongly in $W^{-1,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$). As above, choose a sequence $(y_n) \subset \mathbb{R}^d$ with $y_n \to 0$ such that $\bigcup_k (U^{\varepsilon_n}_k + i_n) = \bigcup_k (U^{\varepsilon_n}_k + y_n)$. We have for $f \in L^1(\mathbb{R}^d)$

$$\begin{aligned}\langle \chi^n_U, f\rangle &= \int_{\bigcup_k (U^{\varepsilon_n}_k + y_n)} f(x)\,dx \\ &= \int_{\bigcup_k U^{\varepsilon_n}_k} f(x+y_n)\,dx \\ &= \int_{\mathbb{R}^d} \chi_{\bigcup_k U^{\varepsilon_n}_k} \cdot f(x+y_n)\,dx\end{aligned}$$

The characteristic function in this last integral is known to converge to $\frac{|\partial B_1(0)|}{d\,2^d}$ weakly* in $L^\infty(\mathbb{R}^d)$ (cf. [CM97], proof of Lemma 2.3), while the sequence $f(\cdot + y_n)$ converges to f strongly in $L^1(\mathbb{R}^d)$ (this follows by smooth approximation). Thus, we obtain

$$\langle \chi^n_U, f\rangle \to \frac{|\partial B_1(0)|}{d\,2^d} \int_{\mathbb{R}^d} f\,dx.$$

Hence $\chi^n_U \overset{*}{\rightharpoonup} \frac{|\partial B_1(0)|}{d\,2^d}$ weakly* in $L^\infty(\mathbb{R}^d)$ and the lemma is proved.

Conclusion. Claim 3, together with eqs. (IV.38), (IV.39) immediately yield

$$\int_{\Omega^*} \nabla h^* \cdot \nabla\phi\,dx + (1+\mu)\int_{\Omega^*} h^*\phi = 0. \tag{IV.44}$$

for $\phi \in C^\infty_0(\Omega^*)$. We know from (I) that $h^* \in H^1(\mathbb{R}^d)$ and that $h^* = 0$ outside Ω^*. Hence, we have $h|_{\Omega^*} \in H^1_0(\Omega^*)$ and uniqueness of solution of equation (IV.44) implies that $h^* = 0$ on Ω^*. Hence $h^* \equiv 0$ in $L^2(\mathbb{R}^d)$.

Arguing as above for all subsequences of $(\tilde{u}_n - J_{\varepsilon_n} u_n)$, we conclude that $\tilde{u}_n - J_{\varepsilon_n} u_n \to 0$ in $L^2(\Omega)$. □

Lemma IV.6.2. *The set $\mathcal{F}$ defined in* (IV.37) *is precompact in* $L^2(\mathbb{R}^d)$.

Proof. We will use the notation and conventions from the previous proof and distinguish between the Dirichlet case and the Robin/Neumann cases.

Dirichlet case. Step 1: We have

$$\sup_n \big\|\tau_h(\tilde{u}_n^* - J_{\varepsilon_n} u_n^*) - (\tilde{u}_n^* - J_{\varepsilon_n} u_n^*)\big\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } h \to 0 \qquad \forall n \in \mathbb{N},$$

where τ_h denotes the operator of translation by h. Indeed, the standard regularity theory implies

$$\begin{aligned}\big\|\tau_h(\tilde{u}_n^* - J_{\varepsilon_n} u_n^*) - (\tilde{u}_n^* - J_{\varepsilon_n} u_n^*)\big\|_{L^2(\mathbb{R}^d)} &\le \big\|\nabla(\tilde{u}_n^* - J_{\varepsilon_n} u_n^*)\big\|_{L^2(\mathbb{R}^d)} |h| \\ &\le C\|f_n\|_{L^2(\Omega)}|h|.\end{aligned}$$

Step 2: Notice that

$$\sup_n \|\tilde{u}_n^* - J_{\varepsilon_n} u_n^*\|_{L^2(\mathbb{R}^d \setminus B_R(0))} \to 0 \quad \text{as } R \to \infty,$$

due to the following estimate in which we set $\omega_0(x) := \cosh(|x|)$.

$$\begin{aligned}\|\tilde{u}_n^* - J_{\varepsilon_n} u_n^*\|^2_{L^2(\mathbb{R}^d \setminus B_R(0))} &\le 2\|\tilde{u}_n^* \omega_0 \omega_0^{-1}\|^2_{L^2(\Omega \setminus B_R(0))} + 2\|J_\varepsilon u_n^* \omega_0 \omega_0^{-1}\|^2_{L^2((\mathbb{R}^d \setminus B_R(0))} \\ &\le 4M\|f_n^* \omega_0\|^2_{L^2(\mathbb{R}^d)} \|\omega_0^{-1}\|^2_{L^\infty(\mathbb{R}^d \setminus B_R(0))} \\ &\overset{\text{Prop. IV.5.1}}{\le} C\|J_{\varepsilon_n} f_n\|^2_{L^2(\Omega)} \exp(-R).\end{aligned}$$

which completes Step 2. Applying the Fréchet-Kolmogorov theorem yields the precompactness of $\mathcal{F}$.

Neumann and Robin case. Here the strategy is the same, but matters are complicated by the fact that $J_{\varepsilon_n} u_n^*$ is not in $H^1(\mathbb{R}^d)$. To show that $\mathcal{F}$ is precompact, we decompose elements in $\mathcal{F}$ as

$$\tilde{u}_n^* - J_{\varepsilon_n} u_n^* = (\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*) + (\mathcal{T}_{\varepsilon_n} - J_{\varepsilon_n}) u_n^*,$$

define $\mathcal{F}_1 := \{\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^* : n \in \mathbb{N}\}$, $\mathcal{F}_2 := \{(\mathcal{T}_{\varepsilon_n} - J_{\varepsilon_n}) u_n^* : n \in \mathbb{N}\}$ and show that $\mathcal{F}_1$ and $\mathcal{F}_2$ are precompact in $L^2(\mathbb{R}^d)$. We will begin by showing that $\mathcal{F}_1$ is precompact. To this end, denote by $\mathcal{E} : H^1(\Omega) \to H^1(\mathbb{R}^d)$ an extension operator satisfying $\mathcal{E}u|_\Omega = u$ and $\|\mathcal{E}u\|_{H^1(\mathbb{R}^d)} \le C\|u\|_{H^1(\Omega)}$ for all $u \in H^1(\Omega)$ [AF03, Theorem 5.24].

Clearly, for every $\xi \in \mathbb{R}^d$ the operators $\mathcal{E}_\xi : H^1(\Omega - \xi) \to H^1(\mathbb{R}^d)$ defined by $\mathcal{E}_\xi u :=$

$\tau_\xi \mathcal{E} \tau_{-\xi} u$ satisfy $\|\mathcal{E}_\xi\|_{\mathcal{L}(H^1(\Omega-\xi),H^1(\mathbb{R}^d))} = \|\mathcal{E}\|_{\mathcal{L}(H^1(\Omega),H^1(\mathbb{R}^d))}$. We start by proving that

$$\sup_n \left\|\tau_h \mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*) - \mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*)\right\|_2 \to 0 \quad \text{as } h \to 0$$

This readily follows from the estimate

$$\begin{aligned}
\left\|\tau_h \mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*) - \mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*)\right\|_{L^2(\mathbb{R}^d)} &\le \left\|\nabla \mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*)\right\|_{L^2(\mathbb{R}^d)} |h| \\
&\le C\|\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*\|_{H^1(\Omega+i_n)} |h| \\
&\le C\|J_{\varepsilon_n} f_n^*\|_{L^2(\Omega+i_n)} |h| \\
&\le C|h|.
\end{aligned}$$

Next we prove that

$$\sup_n \left\|\mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*)\right\|_{L^2(\mathbb{R}^d \setminus B_R(0))} \to 0 \quad \text{as } R \to \infty.$$

Indeed, notice first that

$$\begin{aligned}
\left\|\mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*)\right\|^2_{L^2(\mathbb{R}^d \setminus B_R(0))} &\le C\left(\|\tilde{u}_n^*\|^2_{L^2((\Omega+i_n)\setminus B_R(0))} + \|\mathcal{T}_{\varepsilon_n} u_n^*\|^2_{L^2((\Omega_{\varepsilon_n}+i_n)\setminus B_R(0))}\right) \\
&= C\left(\|\tilde{u}_n\|^2_{L^2(\Omega\setminus B_R(i_n))} + \|\mathcal{T}_{\varepsilon_n} u_n\|^2_{L^2((\Omega_{\varepsilon_n})\setminus B_R(i_n))}\right),
\end{aligned} \tag{IV.45}$$

To treat the two terms on the right-hand side we apply Lemma IV.2.2 (ii) and Proposition IV.5.1 with $\omega_{i_n}(x) = \cosh(|x - i_n|)$ as follows. For the second term in (IV.45), we obtain

$$\begin{aligned}
\|\mathcal{T}_{\varepsilon_n} u_n\|_{L^2(\Omega_{\varepsilon_n}\setminus B_R(i_n))} &\le C\big(\|u_n\|_{L^2(\Omega\setminus B_{R/2}(i_n))} + \|\nabla u_n\|_{L^2(\Omega\setminus B_{R/2}(i_n))}\big) \\
&\le \left\|\omega_{i_n}^{1/2}\omega_{i_n}^{-1/2} u_n\right\|_{L^2(\Omega\setminus B_{R/2}(i_n))} + \left\|\omega_{i_n}^{1/2}\omega_{i_n}^{-1/2}\nabla u_n\right\|_{L^2(\Omega\setminus B_{R/2}(i_n))} \\
&\le C\Big(\left\|\omega_{i_n}^{1/2} u_n\right\|_{L^2(\Omega\setminus B_{R/2}(i_n))} \\
&\qquad + \left\|\omega_n^{1/2}\nabla u_n\right\|_{L^2(\Omega\setminus B_{R/2}(i_n))}\Big)\|\omega_{i_n}^{-1/2}\|_{L^\infty(\Omega\setminus B_{R/2}(i_n))} \\
&\le CM\left\|f_n\omega_{i_n}^{1/2}\right\|_{L^2(\Omega)} \exp(-R/3) \\
&\le 2CM \exp(-R/3),
\end{aligned}$$

where we used the fact that ω_{i_n} is bounded by 2 on $\operatorname{supp} f_n$. With an analogous calculation for the first term in (IV.45), we finally find

$$\left\|\mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*)\right\|_{L^2(\mathbb{R}^d \setminus B_R(0))} \leq C \exp(-R/3), \tag{IV.46}$$

with C independent of n. Applying the Fréchet-Kolmogorov theorem yields the precompactness of the set $\{\mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*) : n \in \mathbb{N}\}$. Finally, noting that $\mathcal{F}_1 = \{\mathcal{E}_{i_n}(\tilde{u}_n^* - \mathcal{T}_{\varepsilon_n} u_n^*) : n \in \mathbb{N}\} \cdot \chi_\Omega$ and that multiplication by χ_Ω is a bounded operator on $L^2(\mathbb{R}^d)$ we obtain precompactness of $\mathcal{F}_1$.

To prove precompactness of $\mathcal{F}_2$, first note that by Lemma IV.2.2 (iii) for any $\delta > 0$ there exists a n_0 such that

$$\left\|(J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^*\right\|_2 < \delta \quad \forall n > n_0.$$

Let us fix arbitrary $\delta > 0$ and n_0 as above. It remains to estimate the terms

$$\left\|\tau_h (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^* - (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^*\right\|_{L^2(\mathbb{R}^d)}, \qquad n \leq n_0,$$

but these are only finitely many, which clearly converge to zero individually as $h \to 0$, and hence

$$\sup_{n \leq n_0} \left\|\tau_h (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^* - (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^*\right\|_2 \to 0 \quad \text{ as } h \to 0$$

Altogether we have shown that

$$\begin{aligned}
&\sup_n \left\|\tau_h (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^* - (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^*\right\|_{L^2(\mathbb{R}^d)} \\
&\qquad \leq \max\left\{\sup_{n \leq n_0} \left\|\tau_h (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^* - (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^*\right\|_2, 2\delta\right\} \\
&\qquad \xrightarrow{h \to 0} 2\delta.
\end{aligned}$$

Since $\delta > 0$ was arbitrary we finally get

$$\lim_{h \to 0} \sup_{n \in \mathbb{N}} \left\|\tau_h (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^* - (J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n}) u_n^*\right\|_{L^2(\mathbb{R}^d)} = 0.$$

This completes the first Fréchet-Kolmogorov-condition. The proof of the second condition

$$\sup_n \big\|(J_{\varepsilon_n} - \mathcal{T}_{\varepsilon_n})u_n^*\big\|_{L^2(\mathbb{R}^d\setminus B_R(0))} \to 0 \quad \text{as } R\to\infty$$

is analogous to the case of $\mathcal{F}_1$. Applying the Fréchet-Kolmogorov theorem yields precompactness of $\mathcal{F}_2$ and completes the proof. □

Corollary IV.6.3. *There exists δ_ε with $\delta_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0$ such that*

$$\big\|(J_\varepsilon(A^\iota)^{-1} - (A^\iota_\varepsilon)^{-1}J_\varepsilon)(f\chi_{Q_i\cap\Omega_\varepsilon})\big\|_{L^2(\Omega)} \le \delta_\varepsilon \|f\chi_{Q_i}\|_{L^2(\Omega)}$$

for all $f\in L^2(\Omega)$ and $i\in\mathbb{Z}^d$.

Proof. We argue by contradiction. Suppose that there is no such function δ_ε. Then there exist sequences ε_n, f_n, i_n with $\|f_n\|_{L^2(\Omega)} = 1$ such that $\|(J_\varepsilon(A^\iota)^{-1} - (A^\iota_{\varepsilon_n})^{-1}J_\varepsilon)\cdot(f_n\chi_{Q_{i_n}\cap\Omega_{\varepsilon_n}})\|_{L^2(\Omega)}$ does not converge to zero, which is a contradiction to Lemma IV.6.1. □

In order to finalise the decomposition, we require the following two lemmas.

Lemma IV.6.4. *Suppose that $f\in L^2(\Omega_\varepsilon)$, and denote*

$$u_i := \big(J_\varepsilon(A^\iota)^{-1} - (A^\iota_\varepsilon)^{-1}J_\varepsilon\big)(f\chi_{Q_i\cap\Omega_\varepsilon}), \quad i\in\mathbb{Z}^d.$$

Then one has

$$\big|\langle u_i, u_j\rangle_{L^2(\Omega)}\big| \le Ce^{-|i-j|/2}\|f\chi_{Q_i}\|_{L^2(\Omega)}\|f\chi_{Q_j}\|_{L^2(\Omega)} \tag{IV.47}$$

for all $i,j\in\mathbb{Z}^d$ with $i\neq j$, where $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ denotes the standard inner product in $L^2(\Omega)$.

Proof. For convenience we write $f_i := f\chi_{Q_i}$, $i\in\mathbb{Z}^d$. Denote $\omega_i(x) = \cosh(|x-i|)$ and note that by Proposition IV.5.1 we have $\|\omega_i^{1/2}u_i\|_{L^2(\Omega)} \le C\|f_i\omega_i^{1/2}\|_{L^2(\Omega)}$. The statement of the lemma is a consequence of the following estimate:

$$\begin{aligned}
\big|\langle u_i, u_j\rangle_{L^2(\Omega)}\big| &\le \int_\Omega |u_i(x)||u_j(x)|\,dx \\
&= \int_\Omega \big(|u_i(x)|\omega_i^{1/2}\big)\big(|u_j(x)|\omega_j^{1/2}\big)\omega_i^{-1/2}\omega_j^{-1/2}\,dx \\
&\le \big\|u_i\omega_i^{1/2}\big\|_{L^2(\Omega)}\big\|u_j\omega_j^{1/2}\big\|_{L^2(\Omega)}\big\|\omega_i^{-1/2}\omega_j^{-1/2}\big\|_{L^\infty(\Omega)}
\end{aligned}$$

$$\le C\|f_i\omega_i^{1/2}\|_{L^2(\Omega)}\,\|f_j\omega_j^{1/2}\|_{L^2(\Omega)}\,\|\omega_0^{-1/2}\omega_{j-i}^{-1/2}\|_{L^\infty(\Omega)}$$
$$\le C\|f_i\|_{L^2(\Omega)}\|f_j\|_{L^2(\Omega)}e^{-|i-j|/2},$$

where we use the fact that $\operatorname{supp}(f_i)\subset Q_i$ and $\omega_i|_{Q_i}\le 2$. □

Lemma IV.6.5. *Suppose that $f\in C_0^\infty(\Omega_\varepsilon)$ and let $u_i:=(J_\varepsilon(A_\varepsilon^\iota)^{-1}-(A^\iota)^{-1}J_\varepsilon)(f\chi_{Q_i})$, $i\in\mathbb{Z}^d$. Then for every $n>1$ one has the inequality*

$$\left\|\sum_{m=1}^N u_{i_m}\right\|^2_{L^2(\Omega)}\le C\left(n^3\sum_{m=1}^N\|u_{i_m}\|^2_{L^2(\Omega)}+\|f\|_{L^2(\Omega_\varepsilon)}e^{-n/3}\right),\tag{IV.48}$$

where N is the number of cubes such that $Q_{i_k}\cap\operatorname{supp}(f)\neq\emptyset$, and C,n do not depend on N.

Proof.

$$\begin{aligned}
\left\|\sum_{m=1}^N u_{i_m}\right\|^2_{L^2(\Omega)} &\le \sum_{m,p=1}^N\langle u_{i_m},u_{j_p}\rangle_{L^2(\Omega)}\\
&=\sum_{k=0}^\infty\left(\sum_{|i-j|\in[k,k+1)}\langle u_i,u_j\rangle_{L^2(\Omega)}\right)\\
&\le\sum_{k=0}^n\left(\sum_{|i-j|\in[k,k+1)}\|u_i\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)}\right)+\sum_{k=n}^\infty\left(\sum_{|i-j|\in[k,k+1)}\langle u_i,u_j\rangle_{L^2(\Omega)}\right)\\
&\le\sum_{k=0}^n\sum_{|i-j|\in[k,k+1)}\left(\frac{\|u_i\|^2_{L^2(\Omega)}}{2}+\frac{\|u_j\|^2_{L^2(\Omega)}}{2}\right)\\
&\qquad\qquad+\sum_{k=n}^\infty\left(\sum_{|i-j|\in[k,k+1)}\langle u_i,u_j\rangle_{L^2(\Omega)}\right)\\
&\le\sum_{k=0}^n\sum_{m=1}^N\left(\|u_{i_m}\|^2_{L^2(\Omega)}\sum_{\{j:|i_m-j|\in[k,k+1)\}}1\right)+\sum_{k=n}^\infty\left(\sum_{|i-j|\in[k,k+1)}\langle u_i,u_j\rangle_{L^2(\Omega)}\right)\\
&\le C\sum_{k=1}^n k^2\sum_{m=1}^N\|u_{i_m}\|^2_{L^2(\Omega)}+\sum_{k=n}^\infty\left(\sum_{|i-j|\in[k,k+1)}\langle u_i,u_j\rangle_{L^2(\Omega)}\right)\\
&\le Cn^3\sum_{m=1}^N\|u_{i_m}\|^2_{L^2(\Omega)}+\sum_{k=n}^\infty\left(\sum_{|i-j|\in[k,k+1)}\langle u_i,u_j\rangle_{L^2(\Omega)}\right).
\end{aligned}\tag{IV.49}$$

We now study the last term of (IV.49). It follows from Lemma IV.6.4 that

$$\big|\langle u_i, u_j\rangle_{L^2(\Omega)}\big| \le C\|f_i\|_{L^2(\Omega)}\|f_j\|_{L^2(\Omega)}e^{-\frac{1}{2}|i-j|}.$$

Using this fact and fixing k for the moment, we obtain

$$\begin{aligned}
\left|\sum_{|i-j|\in[k,k+1)}\langle u_i, u_j\rangle_{L^2(\Omega)}\right| &\le C\sum_{|i-j|\in[k,k+1)}\|f_i\|_{L^2(\Omega)}\|f_j\|_{L^2(\Omega)}e^{-|i-j|/2}\\
&\le C\sum_{|i-j|\in[k,k+1)}\left(\frac{\|f_i\|^2_{L^2(\Omega)}}{2}+\frac{\|f_j\|^2_{L^2(\Omega)}}{2}\right)e^{-|i-j|/2}\\
&\le C\sum_{m=1}^{N}\|f_{i_m}\|^2_{L^2(\Omega)}k^2e^{-k/2}\\
&= C\|f\|^2_{L^2(\Omega)}k^2e^{-k/2}\\
&\le C\|f\|^2_{L^2(\Omega)}e^{-k/3}.
\end{aligned}$$

Summing this inequality from $k = n$ to infinity concludes the proof. □

Combining the above lemmas, we have the following quantitative statement.

Proposition IV.6.6. *Suppose that $f \in C_0^\infty(\Omega_\varepsilon)$. Then for every $n \in \mathbb{N}$,*

$$\big\|(J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)f\big\|^2_{L^2(\Omega)} \le C\big(n^3\delta_\varepsilon^2 + e^{-n/3}\big)\|f\|^2_{L^2(\Omega)}$$

for some $C > 0$, where δ_ε was defined in Corollary IV.6.3.

Proof. We denote $u_i^\varepsilon := (J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)(f\chi_{Q_i})$, $i \in \mathbb{R}^d$, and estimate

$$\begin{aligned}
\big\|(J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)f\big\|^2_{L^2(\Omega)} &= \left\|\sum_{m=1}^{N}u^\varepsilon_{i_m}\right\|^2_{L^2(\Omega)}\\
&\overset{\text{Lemma IV.6.5}}{\le} C\left(n^3\sum_{m=1}^{N}\|u^\varepsilon_{i_m}\|^2_{L^2(\Omega)} + e^{-n/3}\|f\|_{L^2(\Omega_\varepsilon)}\right)\\
&\overset{\text{Cor. IV.6.3}}{\le} C\left(n^3\delta_\varepsilon^2\sum_{m=1}^{N}\|f_{i_m}\|^2_{L^2(\Omega_\varepsilon)} + e^{-n/3}\|f\|_{L^2(\Omega_\varepsilon)}\right)\\
&= C\left(n^3\delta_\varepsilon^2 + e^{-n/3}\right)\|f\|^2_{L^2(\Omega)}.
\end{aligned}$$

□

Proof of Theorem IV.3.1. Let $g \in L^2(\Omega_\varepsilon)$ with $\|g\|_{L^2(\Omega_\varepsilon)} \leq 1$. Fix $\delta > 0$ and choose $f \in C_0^\infty(\Omega_\varepsilon)$ such that $\|g - f\|^2_{L^2(\Omega_\varepsilon)} < \delta$ and choose $n \in \mathbb{N}$ such that $e^{-n/3} \leq \delta$. Now compute

$$\begin{aligned}
\big\|(J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)g\big\|^2_{L^2(\Omega)} &\leq 2\big\|(J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)f\big\|^2_{L^2(\Omega)} \\
&\qquad + 2\big\|(J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)(g-f)\big\|^2_{L^2(\Omega)} \\
&\leq C\Big(\big(n^3\delta_\varepsilon^2 + e^{-n/3}\big)\|f\|^2_{L^2(\Omega_\varepsilon)} \\
&\qquad + \underbrace{\big\|J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon\big\|^2}_{\text{bounded}}\|g-f\|^2_{L^2(\Omega_\varepsilon)}\Big) \\
&\leq C(n^3\delta_\varepsilon^2 + \delta)\|g\|^2_{L^2(\Omega_\varepsilon)} + C\delta,
\end{aligned}$$

hence

$$\sup_{\|g\|_{L^2(\Omega_\varepsilon)}\leq 1}\big\|(J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)g\big\|^2_{L^2(\Omega)} \leq Cn^3\delta_\varepsilon^2 + C\delta + C\delta,$$

and therefore

$$\limsup_{\varepsilon\to 0}\big\|(J_\varepsilon(A^\iota_\varepsilon)^{-1} - (A^\iota)^{-1}J_\varepsilon)\big\|^2_{\mathcal{L}(L^2(\Omega_\varepsilon),L^2(\Omega))} \leq C\delta.$$

Since $\delta > 0$ is arbitrary, the result follows. $\square$

IV.7. Behaviour of the Semigroup

In this section we want to give an application of Theorem IV.3.1. In particular, we focus on the non-selfadjoint operator A_α and study the large-time behaviour of its semigroup. In order to do this, we shall first study the numerical range of the Robin Laplacians more closely. In the remainder of this section, unless otherwise stated, the symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will denote the L^2 (operator-) norm and scalar product, respectively, and the symbol Σ_θ denotes a sector of half-angle θ in the complex plane.

IV.7.1. Decay of $\mathrm{e}^{-t(A^\alpha - \mathrm{id})}$

Let $\alpha \in \mathbb{C}$ and assume $\operatorname{Re}\alpha > 0$. We want to study the decay properties of the heat semigroup $e^{t(\Delta-\mu_\alpha)}$. To this end, let us denote by $B^\alpha := A^\alpha - \mathrm{id}$ the Robin Laplacian on Ω. It is our goal to derive estimates on the numerical range of B^α. Let

$u \in \mathcal{D}(B^\alpha) = \mathcal{D}(A^\alpha)$ and assume that $\|u\|_{L^2(\Omega)} = 1$. Notice that

$$\begin{aligned}\langle B^\alpha u, u\rangle &= \int_\Omega |\nabla u|^2\, dx + \mu_\alpha \int_\Omega |u|^2\, dx + \alpha \int_{\partial\Omega} |u|^2\, dS \\ &= \|\nabla u\|^2 + \mu_\alpha + \alpha \|u\|^2_{L^2(\partial\Omega)},\end{aligned}$$

and therefore

$$\begin{aligned}\operatorname{Re}\big\langle B^\alpha u, u\big\rangle &\geq \operatorname{Re}\mu_\alpha + \operatorname{Re}\alpha \|u\|^2_{L^2(\partial\Omega)}, \\ |\operatorname{Im}\langle B^\alpha u, u\rangle| &\leq |\operatorname{Im}\mu_\alpha| + |\operatorname{Im}\alpha| \|u\|^2_{L^2(\partial\Omega)}.\end{aligned}$$

Now, let $\lambda \in (0, \operatorname{Re}\mu_\alpha)$ and compute

$$\begin{aligned}\big|\operatorname{Im}\big\langle (B^\alpha - \lambda)u, u\big\rangle\big| &\leq |\operatorname{Im}\mu_\alpha| + |\operatorname{Im}\alpha| \|u\|^2_{L^2(\partial\Omega)} \\ &= \frac{|\operatorname{Im}\mu_\alpha|}{\operatorname{Re}\mu_\alpha} \operatorname{Re}\mu_\alpha + \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}\alpha} \operatorname{Re}\alpha \|u\|^2_{L^2(\partial\Omega)}. \end{aligned} \tag{IV.50}$$

Recall from (IV.3) that $\mu_\alpha = \alpha S_d / 2^d$ and hence $|\operatorname{Im}\mu_\alpha| / \operatorname{Re}\mu_\alpha = |\operatorname{Im}\alpha| / \operatorname{Re}\alpha$. Combining this with (IV.50), we obtain

$$\begin{aligned}\big|\operatorname{Im}\langle (B^\alpha - \lambda)u, u\rangle\big| &\leq \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}\alpha} \Big(\operatorname{Re}\mu_\alpha + \operatorname{Re}\alpha \|u\|^2_{L^2(\partial\Omega)}\Big) \\ &\leq \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}\alpha} \big(\operatorname{Re}\big\langle (B^\alpha - \lambda)u, u\big\rangle + \lambda\big) \\ &\leq \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}\alpha - \frac{\lambda}{2^{-d} S_d}} \operatorname{Re}\big\langle (B^\alpha - \lambda)u, u\big\rangle.\end{aligned}$$

Using Theorem I.2.21, the next statement follows.

Proposition IV.7.1. *The operator $-(B^\alpha - \lambda)$ generates a bounded analytic semigroup in the sector $\Sigma_{\frac{\pi}{2} - \theta_\lambda}$, where*

$$\theta_\lambda = \arctan\left(\frac{|\operatorname{Im}\alpha|}{\operatorname{Re}\alpha - \frac{\lambda}{2^{-d} S_d}}\right).$$

Equivalently, $-B^\alpha$ generates an analytic semigroup with

$$\big\|e^{-zB^\alpha}\big\| \leq e^{-\lambda z} \qquad \forall z \in \Sigma_{\frac{\pi}{2} - \theta_\lambda}.$$

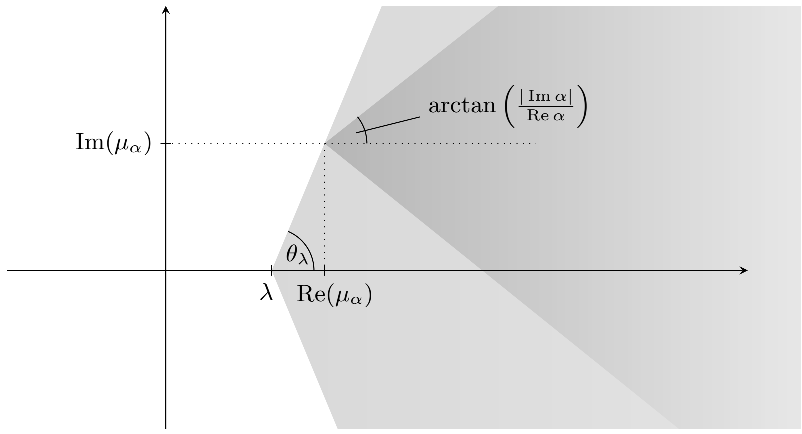


Figure IV.3.: The sector of decay and angle θ_λ for B^α.

IV.7.2. Decay of $e^{-t(A^\alpha_\varepsilon - \mathrm{id})}$

In this section we denote $B^\alpha_\varepsilon := A^\alpha_\varepsilon - \mathrm{id}$. By calculations analogous to the above, we have

$$\left|\operatorname{Im}\langle B^\alpha_\varepsilon u, u\rangle\right| \le \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}\alpha} \operatorname{Re}\langle B^\alpha_\varepsilon u, u\rangle,$$

that is, B^α_ε is sectorial with sector Σ_{θ_0}, where $\theta_0 = \arctan(|\operatorname{Im}\alpha|/\operatorname{Re}\alpha)$, and hence generates a bounded analytic semigroup in the sector $\Sigma_{\frac{\pi}{2}-\theta_0}$. In this subsection we improve this *a priori* result using spectral convergence. To this end, let $\delta > 0$ and define the compact set

$$K_\delta := \left\{x + iy : x \in [0, \operatorname{Re}\mu_\alpha],\, y \in \left[-|\operatorname{Im}\mu_\alpha|, |\operatorname{Im}\mu_\alpha|\right]\right\}.$$

Note that then $\Sigma_{\theta_0} \cap \{\operatorname{Re} z \le \operatorname{Re}\mu_\alpha - \delta\} \subset K_\delta$. By [EE87, Th. III.2.3] one has $K_\delta \subset \rho(B^\alpha)$ for every $\delta > 0$. Applying Corollary IV.3.2 we see that for every $\delta > 0$ there exists a $\varepsilon_0 > 0$ such that $K_\delta \subset \rho(B^\alpha_\varepsilon)$ for all $\varepsilon < \varepsilon_0$.

In particular we have shown that the resolvent norm $\|(B^\alpha_\varepsilon - z)^{-1}\|$ is bounded on $\Sigma_{\theta_0} \cap \{\operatorname{Re} z \le \operatorname{Re}\mu_\alpha - \delta\}$. By a trivial calculation analogous to the previous subsection this leads to the following statement.

Lemma IV.7.2. *For every $\lambda \in (0, \operatorname{Re}\mu_\alpha - \delta)$ one has*

$$\sigma(B^\alpha_\varepsilon - \lambda) \subset \Sigma_{\theta^\delta_\lambda}, \qquad \theta^\delta_\lambda = \arctan\left(\frac{|\operatorname{Im}\mu_\alpha|}{\operatorname{Re}\mu_\alpha - \lambda - \delta}\right).$$

Furthermore, we obtain the following lemma.

Lemma IV.7.3. *For every* $\lambda \in (0, \operatorname{Re}\mu_\alpha - \delta)$ *one has* $\mathbb{C} \setminus \Sigma_{\theta_\lambda^\delta} \subset \rho(B_\varepsilon^\alpha - \lambda)$ *and there exists a* $M = M(\lambda, \delta) > 0$ *such that*

$$\left\|(B_\varepsilon^\alpha - \lambda - z)^{-1}\right\| \leq \frac{M}{|z|} \qquad \forall z \in \mathbb{C} \setminus \Sigma_{\theta_\lambda^\delta}.$$

Proof. This is obtained by combining Lemma IV.7.2 with the following two facts:

$$|\operatorname{Im}\langle B_\varepsilon^\alpha u, u\rangle| \leq \frac{|\operatorname{Im}\alpha|}{\operatorname{Re}\alpha} \operatorname{Re}\langle B_\varepsilon^\alpha u, u\rangle, \qquad \left\|(B_\varepsilon^\alpha - z)^{-1}\right\| \leq C \quad \text{on } K_\delta.$$

□

By the theory of analytic semigroups (*cf.* Section I.2.4), we immediately obtain the following corollary.

Corollary IV.7.4. *For all* $\lambda \in (0, \operatorname{Re}\mu_\alpha - \delta)$, *the operator* $B_\varepsilon^\alpha - \lambda$ *generates a bounded analytic semigroup in the sector* $\Sigma_{\frac{\pi}{2} - \theta_\lambda^\delta}$.

This yields the main result of this section, as follows.

Theorem IV.7.5. *For every* $\delta > 0$ *there exists* $\varepsilon_0 > 0$ *such that for every* $\lambda \in (0, \operatorname{Re}\mu_\alpha - \delta)$ *there exists* $M > 0$ *such that*

$$\left\|e^{-zB_\varepsilon^\alpha}\right\| \leq Me^{-\lambda \operatorname{Re} z} \qquad \forall z \in \Sigma_{\theta_\lambda^\delta}, \quad \varepsilon \in (0, \varepsilon_0).$$

Remark IV.7.6. It is straightforward to repeat the above proof for the case of Dirichlet boundary conditions to obtain an analogous result for $\left\|e^{-t(A^{\mathrm{D}} - \mathrm{id})}\right\|$. Here, the selfadjointness of A^{D} allows us to choose the half-angle θ arbitrarily close to $\pi/2$.

V. Conclusion

Non-Selfadjoint Schrödinger Operators: We have shown that for $\operatorname{Re} V \geq c|x|^2$ the unbounded component of the pseudospectrum of $H = -\Delta + V$ moves towards $+\infty$ as $\varepsilon \to 0$. We note that this result holds for arbitrary imaginary part of the potential.

For a similar operator with $\operatorname{Re} V = 0$ we were able to give a precise scaling for how fast this happens. To obtain this scaling the knowledge of the norms of the Riesz projections was crucial.

Let us remark that an analogous result to Theorem IV.3.1 trivially holds for operators which are m-sectorial (in the sense of [Kat95]). This is due to the fact that the resolvent norm decays outside the numerical range. This includes e.g. the Bender oscillator $-\frac{d^2}{dx^2} - (ix)^\nu$, $2 < \nu < 4$ (cf. [Mez01] for a precise definition). The conclusion of Theorem IV.3.1 holds for H if $2 < \varepsilon \leq 3$. Furthermore, by semiclassical methods, the conclusion of Theorem III.3.6 holds if $3 < \varepsilon < 4$.

More generally, Schrödinger Operators with a potential whose range is contained in a sector belong to the above category (cf. [BST17, Prop. 2.2] for a precise study).

A number of open questions remain.

- To the authors' knowledge the norms of the Riesz projections of the harmonic oscillator with imaginary cubic potential have not been computed yet, but we strongly suspect that the scaling $\|Q_k\| \sim e^{\omega k}$ (which holds for the Bender oscillator) is also true in this case.

- Furthermore, we have seen that the resolvent norm of the Bender oscillator H_B goes to zero on vertical lines in the complex plane. However, we do not know the rate of the decay. Clearly, there exists no $C > 0$ such that

  $$\|(H_c - s - ir)^{-1}\| \leq \frac{C}{|r|}, \quad \forall s \in \mathbb{R}$$

  because this would imply that H_B generates an *analytic* semigroup (which is false by (II.4)). The question remains exactly how slow the decay is. The answer could be used to confirm the results of [Bor13] who computed the asymptotic shape of the level sets of the resolvent norm.

- Finally, there is the obvious question as to whether the central assumption $\operatorname{Re} V \geq c|x|^2 - d$ can be relaxed. It is not obvious how to generalise our method of proof to potentials which do not satisfy this lower bound. Indeed, our compactness proof of the semigroup heavily relied on the fundamental solution of the harmonic oscillator. However, the examples of the imaginary cubic oscillator and the imaginary airy operator suggest that the lower bound on $\operatorname{Re} V$ is not essential. It seems likely to the authors that under suitable conditions on $\operatorname{Im} V$ the semigroup of $-\Delta + V$ will be compact even for $\operatorname{Re} V = 0$. This issue has been partially addressed in [KS17]

Perforated Domains: We have shown norm-resolvent convergence in the classical perforated domain problem with Dirichlet boundary conditions which has the interesting implication of spectral convergence (Cor. IV.3.2). Some questions remain open and will be addressed in the future. While the norm $\|J_\varepsilon A_\varepsilon^{-1} - A^{-1} J_\varepsilon\|_{\mathcal{L}(L^2(\Omega_\varepsilon), L^2(\Omega))}$ converges to 0, it is not clear from our method of proof what the rate of convergence is. It would be desirable to obtain a precise convergence rate. In the case of Dirichlet boundary conditions an explicit convergence rate has been found by [KP18].

Another interesting question is whether in the case $\Omega = \mathbb{R}^d$ there exist gaps in the spectrum of A_ε and how these depend on ε. The existence of spectral gaps has been confirmed in two dimensions [NRT12], but to the authors' knowledge the higher-dimensional case is still open.

Bibliography

[AF03] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[BB98] C. M. Bender and S. Boettcher. Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry. *Physical Review Letters*, (80):5243–5246, 1998.

[BBM99] C. M. Bender, S. Boettcher, and P. Meisinger. PT Symmetric Quantum Mechanics. *J.Math.Phys.*, 40:2201–2229, 1999.

[BCD16] D. Borisov, G. Cardone, and T. Durante. Homogenization and norm-resolvent convergence for elliptic operators in a strip perforated along a curve. *Proc. Roy. Soc. Edinburgh Sect. A*, 146(6):1115–1158, 2016.

[Ben07] C. M. Bender. Making Sense of Non-Hermitian Hamiltonians. *Rept.Prog.Phys.*, 70:947, 2007.

[BK08] D. Borisov and D. Krejčiřík. PT-Symmetric Waveguides. *Integral Equations and Operator Theory*, 62(4):489–515, 2008.

[Bor13] W. Bordeaux Montrieux. Estimation de Resolvante et Construction de Quasimode Pres du Bord du Pseudospectre. *ArXiv e-prints, 1301.3102v1*, 2013.

[Bou02] L. S. Boulton. Non-Self-Adjoint Harmonic Oscillator, Compact Semigroups and Pseudospectra. *J. Oper. Theo.*, (47):413–429, 2002.

[Bre10] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations (Universitext)*. Springer, 2011 edition, 2010.

[BS03] M. S. Birman and T. A. Suslina. Periodic second-order differential operators. Threshold properties and averaging. *Algebra i Analiz*, 15(5):1–108, 2003.

[BS06] M. S. Birman and T. A. Suslina. Averaging of periodic differential operators taking a corrector into account. Approximation of solutions in the Sobolev class $H^2(\mathbb{R}^d)$. *Algebra i Analiz*, 18(6):1–130, 2006.

[BST17] S. Bögli, P. Siegl, and C. Tretter. Approximations of spectra of Schrödinger operators with complex potentials on $\mathbb{R}^d$. *Comm. Partial Differential Equations*, 42(7):1001–1041, 2017.

[CC16] K. D. Cherednichenko and S. Cooper. Resolvent estimates for high-contrast elliptic problems with periodic coefficients. *Arch. Ration. Mech. Anal.*, 219(3):1061–1086, 2016.

[CGM80] E. Caliceti, S. Graffi, and M. Maioli. Perturbation Theory of Odd Anharmonic Oscillators. *Comm. Math. Phys.*, 75(1):51–66, 1980.

[CM97] D. Cioranescu and F. Murat. A strange term coming from nowhere. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 45–93. Birkhäuser Boston, Boston, MA, 1997.

[Dav80] E. B. Davies. *One-Parameter Semigroups (L.M.S. Monographs)*. Academic Pr, 1980.

[Dav99] E. B. Davies. Pseudospectra, the harmonic oscillator and complex resonances. *Proc. Roy. Soc. London A*, 455:585–599, 1999.

[Dav00] E. B. Davies. Pseudospectra of differential operators. *J. Operator Theory*, 43(2):243–262, 2000.

[Dav05] E. B. Davies. Semigroup Growth Bounds. *J. Oper. Theo.*, 53(2):225–249, 2005.

[Dav07] E. B. Davies. *Linear Operators and their Spectra (Cambridge Studies in Advanced Mathematics)*. Cambridge University Press, first edition, 2007.

[DDT01] P. Dorey, C. Dunning, and R. Tateo. Spectral Equivalences, Bethe Ansatz Equations, and Reality Properties in PT-Symmetric Quantum Mechanics. *Journal of Physics A: Mathematical and General*, 34(28):5679, 2001.

[DiB16] E. DiBenedetto. *Real Analysis*. Birkhäuser Advanced Texts Basler Lehrbücher. Springer New York, 2016.

[DK13] M. Demuth and W. Kirsch, editors. *Mathematical Physics, Spectral Theory and Stochastic Analysis (Operator Theory: Advances and Applications / Advances in Partial Differential Equations)*. Birkhaeuser, 2013 edition, 2013.

[DL11] S. Delmonte and L. Lorenzi. On a Class of Weakly Coupled Systems of Elliptic Operators with Unbounded Coefficients. *Milan Journal of Mathematics*, 79(2):689–727, 2011.

[EE87] D. E. Edmunds and W. D. Evans. *Spectral Theory and Differential Operators (Oxford Mathematical Monographs)*. Oxford University Press, 1987.

[EN00] K. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Graduate Texts in Mathematics. Springer, 2000.

[Eva98] L. C. Evans. *Partial Differential Equations (Graduate Studies in Mathematics, Vol. 19)*. Amer Mathematical Society, 1st edition, 1998.

[GG15] I. Giordanelli and G. M. Graf. The real spectrum of the imaginary cubic oscillator: an expository proof. *Ann. Henri Poincaré*, 16(1):99–112, 2015.

[GGK90] I. Gohberg, S. Goldberg, and M. A. Kaashoek. *Classes of Linear Operators Vol. 1*. Operator Theory. Birkhäuser Verlag, 1990.

[Gre12] G. Greenwald. NSA collecting phone records of millions of verizon customers daily. *https://www.theguardian.com/world/2013/jun/06/nsa-phone-records-verizon-court-order*, 2012.

[Haa06] M. Haase. *The Functional Calculus for Sectorial Operators*. Operator Theory: Advances and Applications. Birkhäuser Verlag, 2006.

[Hen14a] R. Henry. Spectral instability for even non-selfadjoint anharmonic oscillators. *Journal of Spectral Theory*, 4(2):349–364, 2014.

[Hen14b] R. Henry. Spectral Projections of the Complex Cubic Oscillator. *Annales Henri Poincaré*, 15:2025–2043, 2014.

[Kai85] S. Kaizu. The Robin problems on domains with many tiny holes. *Proc. Japan Acad. Ser. A Math. Sci.*, 61(2):39–42, 1985.

[Kai89] S. Kaizu. The Poisson equation with semilinear boundary conditions in domains with many tiny holes. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36(1):43–86, 1989.

[Kat95] T. Kato. *Perturbation Theory for Linear Operators (Classics in Mathematics)*. Springer, reprint of the corr. 2nd edition, 1995.

[KP18] A. Khrabustovskyi and O. Post. Operator estimates for the crushed ice problem. *Asymptotic Analysis (accepted)*, 2018.

[KS12] D. Krejčiřík and P. Siegl. On the Metric Operator for the Imaginary Cubic Oscillator. *Phys. Rev. D 86, 121702(R)*, 86(12):121702, 2012.

[KS17] D. Krejčiřík and P. Siegl. Pseudomodes for Schroedinger operators with complex potentials. *ArXiv e-prints*, May 2017.

[KS18] I. V. Kamotski and V. P. Smyshlyaev. Two-scale homogenization for a general class of high contrast pde systems with periodic coefficients. *Applicable Analysis*, pages 1–27, 2018.

[KSTV15] D. Krejčiřík, P. Siegl, M. Tater, and J. Viola. Pseudospectra in non-Hermitian quantum mechanics. *J. Math. Phys.*, 56(10):103513, 32, 2015.

[Mez01] G. A. Mezincescu. The operator $p^2 - (ix)^\nu$ on $L^2(\mathbb{R})$ (reply to Comment by Bender and Wang). *J. Phys. A: Math. Gen.*, (34):3329–3332, 2001.

[MK64] V. A. Marcenko and E. Y. Khruslov. Boundary-value problems with fine-grained boundary. *Mat. Sb. (N.S.)*, 65 (107):458–472, 1964.

[MK74] V. A. Marchenko and E. Y. Khruslov. *Boundary-value problems in domains with a fine-grained boundary*. Izdat. "Naukova Dumka", Kiev, 1974.

[MNP13] D. Mugnolo, R. Nittka, and O. Post. Norm convergence of sectorial operators on varying Hilbert spaces. *Oper. Matrices*, 7(4):955–995, 2013.

[Mos02] A. Mostafazadeh. Pseudo-Hermiticity versus PT symmetry: the necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian. *J. Math. Phys.*, 43(1):205–214, 2002.

[MSV17] B. Mityagin, P. Siegl, and J. Viola. Differential operators admitting various rates of spectral projection growth. *J. Funct. Anal.*, 272(8):3129–3175, 2017.

[Nov14] R. Novák. On the Pseudospectrum of the Harmonic Oscillator with Imaginary Cubic Potential. *Int. J. Theor. Phys.*, 2014.

[NRT12] S. A. Nazarov, K. Ruotsalainen, and J. Taskinen. Spectral gaps in the Dirichlet and Neumann problems on the plane perforated by a double-periodic family of circular holes. *J. Math. Sci. (N.Y.)*, 181(2):164–222, 2012. Problems in mathematical analysis. No. 63.

[Pas06] S. E. Pastukhova. Some estimates from homogenized elasticity problems. *Doklady Mathematics*, 73(1):102–106, 2006.

[PBL78] G. Papanicolau, A. Bensoussan, and J. Lions. *Asymptotic Analysis for Periodic Structures.* Studies in Mathematics and its Applications. Elsevier Science, 1978.

[Rau75] J. Rauch. Lecture #2. The mathematical theory of crushed ice. In J. A. Goldstein, editor, *Partial Differential Equations and Related Topics*, pages 370–379, Berlin, Heidelberg, 1975. Springer Berlin Heidelberg.

[RS78] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV: Analysis of Operators.* Academic Press, Inc., 1978.

[RS80] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis.* Academic Press, Inc., 1980.

[RT75] J. Rauch and M. Taylor. Potential and scattering theory on wildly perturbed domains. *J. Funct. Anal.*, 18:27–59, 1975.

[RW98] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1998.

[Shi02] K. Shin. On the Reality of the Eigenvalues for a Class of Symmetric Oscillators. *Communications in Mathematical Physics*, 229:543–564, 2002.

[Sib75] Y. Sibuya. *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient.* Number Bd. 18 in Global theory of a second order linear ordinary differential equation with a polynomial coefficient. North-Holland Publishing Company, 1975.

[Tai05] T. D. Tai. Remarks on the PT-Pseudo-Norm in PT-Symmetric Quantum Mechanics. *Journal of Physics A: Mathematical and General*, 38(16):3665, 2005.

[Tai06] T. D. Tai. On the Simpleness of Zeros of Stokes Multipliers. *Journal of Differential Equations*, 223(2):351 – 366, 2006.

[TE05] L. N. Trefethen and M. Embree. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators.* Princeton University Press, 2005.

[TL80] A. Taylor and D. Lay. *Introduction to Functional Analysis.* Wiley, 1980.

[Tre01] L. N. Trefethen. *Spectral Methods in MATLAB (Software, Environments, Tools).* SIAM: Society for Industrial and Applied Mathematics, 2001.

[Wer08] D. Werner. *Funktionalanalysis (Springer-Lehrbuch) (German Edition).* Springer, 6th edition, 2008.

[Zhi00] V. V. Zhikov. On an extension and an application of the two-scale convergence method. *Mat. Sb.*, 191(7):31–72, 2000.