On permutable meromorphic functions

Journal Item

On permutable meromorphic functions

J. W. Osborne, D. J. Sixsmith

To Phil Rippon on the occasion of his 65th birthday

Abstract. We study the class $\mathcal{M}$ of functions meromorphic outside a countable closed set of essential singularities. We show that if a function in $\mathcal{M}$, with at least one essential singularity, permutes with a non-constant rational map g, then g is a Möbius map that is not conjugate to an irrational rotation. For a given function $f \in \mathcal{M}$ which is not a Möbius map, we show that the set of functions in $\mathcal{M}$ that permute with f is countably infinite. Finally, we show that there exist transcendental meromorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that, among functions meromorphic in the plane, f permutes only with itself and with the identity map.

Mathematics Subject Classification (2010). Primary 30D05; Secondary 30D30.

1. Introduction

If f and g are meromorphic functions, then, in general, $f \circ g$ is not meromorphic. In view of this, we let $\mathcal{M}$ be the class of functions f with the following property; there is a countable closed set $S(f) \subset \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that f is meromorphic in $\widehat{\mathbb{C}} \setminus S(f)$, and $S(f)$ is the set of essential singularities of f. All functions that are meromorphic in $\mathbb{C}$ lie in $\mathcal{M}$, which is closed under composition. The dynamics of functions in the class $\mathcal{M}$ was considered by Bolsch [8, 9].

In this short paper we extend some results of Baker and Iyer on permutable entire functions to permutable functions in the class $\mathcal{M}$. Here, if f and g are functions defined on a subset of $\widehat{\mathbb{C}}$, then we say that these functions are *permutable*, or that f *permutes* with g, if

$$f \circ g = g \circ f. \tag{1}$$

The second author was supported by Engineering and Physical Sciences Research Council grant EP/J022160/1.

We take (1) to mean that, for all $z \in \widehat{\mathbb{C}}$, either $f(g(z)) = g(f(z))$, or else both $f(g(z))$ and $g(f(z))$ are undefined.

The case where f is transcendental entire and g is a polynomial was considered by Baker [1] and independently by Iyer [12]. They proved the following.

Theorem A. *Suppose that g is a non-constant polynomial. Then there exists a transcendental entire function f that permutes with g if and only if g is an affine map of the form*

$$g(z) = ze^{2\pi im/n} + c, \quad \textit{for some } m, n \in \mathbb{Z},\ c \in \mathbb{C}. \tag{2}$$

Throughout this paper, when we refer to a transcendental meromorphic function, we mean a function meromorphic in $\mathbb{C}$ and with an essential singularity at infinity. It is straightforward to use results in [11] to generalise Theorem A to transcendental meromorphic functions. As this result is not stated in [11], we give it here.

Theorem 1. *Suppose that g is a non-constant rational map. Then there exists a transcendental meromorphic function f that permutes with g if and only if g is an affine map of the form (2).*

Our main generalisation of Theorem A is to functions in the class $\mathcal{M}$. It is natural to state these results in terms of conjugacies. For suppose that f and g are permuting elements of $\mathcal{M}$, and that g is *conjugate* to a map $G \in \mathcal{M}$; in other words, there is a Möbius map L such that $G = L^{-1} \circ g \circ L$. Then F and G are permuting elements of $\mathcal{M}$, where $F := L^{-1} \circ f \circ L$.

It is well-known that a Möbius map M is conjugate either to the map $z \mapsto z + 1$ (in which case M is called *parabolic*), or to the map $z \mapsto kz$, for some $k \in \mathbb{C} \setminus \{0\}$. In the second case, if $k = e^{i\theta}$, for θ rational (resp. irrational), then we say that M is conjugate to a *rational rotation* (resp. *irrational rotation*).

Theorem 2. *Suppose that g is a non-constant rational map. Then there exists a function $f \in \mathcal{M}$, with $S(f) \neq \emptyset$, and that permutes with g, if and only if g is a Möbius map that is not conjugate to an irrational rotation.*

Baker [2, 3] also proved the following result.

Theorem B. *If an entire function f is not a polynomial of degree less than 2, then the set of all entire functions that permute with f is countably infinite.*

We show that a result of Bergweiler and Hinkkanen on semiconjugation of entire functions [7, Theorem 3] can readily be extended to the class $\mathcal{M}$; see Section 3 for details. In particular, this yields the following analogue of Theorem B.

Theorem 3. *If a function $f \in \mathcal{M}$ is not a Möbius map, then the set of all elements of $\mathcal{M}$ that permute with f is countably infinite.*

If f is an entire function that is not a polynomial of degree less than two, then the iterates of f form a countably infinite set of entire functions that permute with f. An analogous remark holds if $f \in \mathcal{M}$. However, if f is a transcendental meromorphic function that is not entire then, in general, the iterates of f are not meromorphic in $\mathbb{C}$. Indeed, suppose we define a function meromorphic in $\mathbb{C}$ to be *minimally permuting* if, among such functions, it permutes only with itself and with the identity map. Then we have the following.

Theorem 4. *There exist transcendental meromorphic functions that are minimally permuting.*

The organisation of this paper is as follows. We give the proofs of Theorems 1 and 2 in Section 2, and the proof of Theorem 3 in Section 3. Then, in Section 4, we prove Theorem 4 by giving two examples of minimally permuting transcendental meromorphic functions.

2. Proofs of Theorems 1 and 2

We first show that Theorem 1 follows easily from results of Goldstein [11].

Proof of Theorem 1. If g is an affine map of the form (2), then it follows from Theorem A that there is a transcendental entire function that permutes with g. Thus it suffices to prove the 'only if' direction of Theorem 1.

Suppose, then, that g is a non-constant rational map and that f is a transcendental meromorphic function that permutes with g. Since there exists $\zeta \in \widehat{\mathbb{C}}$ such that $g(\zeta) = \infty$, it follows that $f(g(\zeta))$ is undefined, so $g(f(\zeta))$ is also undefined by (1). Since g is rational, we deduce that $\zeta = \infty$. Thus infinity is the only pole of g, which is therefore a polynomial. It follows by [11, Theorem 2] that g is an affine map. The result then follows by [11, Theorem 11]. □

We now give the proof of Theorem 2, our main generalisation of Theorem A. The proof uses certain ideas from iteration theory. We denote the *iterates* of the function f by $f^n := \underbrace{f \circ f \circ ... \circ f}_{n \text{ times}}$, for $n \in \mathbb{N}$. The *Julia set* of a rational map g of degree at least 2 is defined to be the set of points in $\widehat{\mathbb{C}}$ with no neighbourhood in which the iterates of g form a normal family. We refer to [5, 6, 13], for example, for the properties of this set and an introduction to complex dynamics.

Proof of Theorem 2. If g is a Möbius map that is either parabolic or conjugate to a rational rotation, then it follows from Theorem A that there is an element of $\mathcal{M}$, conjugate to a transcendental entire function, that permutes with g.

If g is a Möbius map that is neither parabolic nor conjugate to a rotation, then it is conjugate to a map of the form $z \mapsto \lambda z$, for some $\lambda \in \mathbb{C}$ with $|\lambda| \neq 0, 1$. Without loss of generality, we can assume that g is of this form.

We construct a function $f \in \mathcal{M}$ that permutes with g as follows. Replacing λ with $1/\lambda$ if necessary, we assume that $|\lambda| > 1$. Let $h(z)$ be the function $h(z) := z^2(1-z)^{-2}$, and let

$$f(z) := \sum_{k\in\mathbb{Z}} \lambda^{-k} h(\lambda^k z). \tag{3}$$

We claim that the (double) series in (3) defines a function which is meromorphic in $\widehat{\mathbb{C}}\setminus\{0,\infty\}$, but in no larger domain in $\widehat{\mathbb{C}}$. In particular $f \in \mathcal{M}$, but f is not a transcendental meromorphic function. Since f permutes with the maps $z \mapsto \lambda z$ and $z \mapsto z/\lambda$, this will complete the first part of the proof.

To prove the claim, suppose that $z \in \widehat{\mathbb{C}} \setminus \{0,\infty\}$. Let U be a neighbourhood of z sufficiently small that $\overline{U} \subset \widehat{\mathbb{C}} \setminus \{0,\infty\}$. Since $|\lambda| > 1$, it can be seen that there exists $k_0 \in \mathbb{N}$ such that, for all $w \in U$, we have

$$|\lambda^{-k} h(\lambda^k w)| = \left| \frac{\lambda^k w^2}{(1-\lambda^k w)^2} \right| < \begin{cases} 2|\lambda|^{-k}, & \text{for } k \geq k_0, \\ 2|w|^2|\lambda|^k, & \text{for } k \leq -k_0. \end{cases}$$

It follows by the Weierstrass M–test that f restricted to U can be written as the sum of two analytic functions and a rational function, and so f is meromorphic in U. Hence f is indeed meromorphic in $\widehat{\mathbb{C}} \setminus \{0,\infty\}$, since z was arbitrary. Moreover, the poles of f are easily seen to be the points λ^k, for $k \in \mathbb{Z}$. Since these points accumulate on $\{0,\infty\}$, it follows that f cannot be meromorphic in a neighbourhood of either zero or infinity. This completes the proof of our claim.

It remains to prove the 'only if' direction of Theorem 2. Suppose that g is any non-constant rational map and that $f \in \mathcal{M}$ permutes with g. We consider separately the cases where $S(f)$ has one, two, or more than two points.

First suppose that $S(f)$ is a singleton. Without loss of generality, we can assume that $S(f) = \{\infty\}$. Then f is a transcendental meromorphic function, so it follows from Theorem 1 that g is an affine map of the form (2), and thus is a Möbius map that is not conjugate to an irrational rotation.

Suppose next that $S(f)$ has two elements. If there exists $\zeta \in \widehat{\mathbb{C}} \setminus S(f)$ such that $f(\zeta) \in S(f)$, then $S(f^2)$ has more than two elements, so we can replace f with f^2. It follows that we can assume that both elements of $S(f)$ are omitted values, and without loss of generality take $S(f) = \{0,\infty\}$. Thus f is a *transcendental self-map of the punctured plane*. It was pointed out by Rådström [14] that such maps are necessarily of the form

$$f(z) := z^k \exp(f_1(z) + f_2(1/z)),$$

where $k \in \mathbb{Z}$ and f_1, f_2 are entire functions. If $|z|$ is large, then f behaves like the transcendental entire function $F(z) := z^k \exp(f_1(z))$. It is straightforward to show that the techniques of [1, 12] can also be applied in this situation with the same result; we omit the detail.

Finally, consider the case where $S(f)$ has at least three elements. We first show that any rational map g that permutes with f is a Möbius map.

Suppose, by way of contradiction, that the degree of g is at least two. We deduce by Picard's great theorem and [5, Theorem 4.1.2] that there exists $\alpha \in \widehat{\mathbb{C}} \setminus S(f)$ such that $f(\alpha) \in S(f)$ and the set $\bigcup_{k\geq 0} g^{-k}(\alpha)$ is infinite.

Suppose that $k \geq 0$ and that $\zeta \in g^{-k}(\alpha)$. Since f^2 and g^k permute, it follows that $g^k(f^2(\zeta))$ is undefined, and so $\zeta \in f^{-1}(S(f)) \cup S(f)$. We deduce that

$$\bigcup_{k\geq 0} g^{-k}(\alpha) \subset f^{-1}(S(f)) \cup S(f) = S(f^2). \tag{4}$$

It follows from (4), and [5, Theorem 4.2.7] that $\overline{S(f^2)}$ contains the Julia set of g. Since [5, Theorem 4.2.4] the Julia set of g is uncountable and $S(f^2)$ is closed and countable, this is a contradiction, so it follows that g is a Möbius map.

To complete the proof, it is sufficient to show that g is not of the form $z \mapsto e^{i\theta}z$, where θ is irrational. If this holds, then $e^{i\theta}f(z) = f(e^{i\theta}z)$, for $z \in \widehat{\mathbb{C}} \setminus S(f)$. Differentiating, we obtain a non-constant function $H \in \mathcal{M}$ with the property that $H(z) = H(e^{i\theta}z)$, for $z \in \widehat{\mathbb{C}} \setminus S(f)$. Now choose a point $\xi \in \mathbb{C} \setminus (\{0\} \cup S(f))$. Observe that

$$H(e^{ik\theta}\xi) = H(\xi), \quad \text{for } k \in \mathbb{Z}.$$

Since θ is irrational, the points $e^{ik\theta}\xi$, for $k \in \mathbb{N}$, accumulate on the whole circle $\{w : |w| = |\xi|\}$. This is a contradiction, since all these points are elements of $H^{-1}(H(\xi))$, which can accumulate only on $S(f)$, and $S(f)$ is countable. □

3. Proof of Theorem 3

In [7, Theorem 3], Bergweiler and Hinkkanen gave the following result on semiconjugation of entire functions, which is a generalisation of Theorem B.

Theorem C. *Let f and h be entire functions such that f is not a Möbius map, and h is not the identity map. Then there are only countably many entire functions g such that*

$$h \circ g = g \circ f. \tag{5}$$

The method of proof of Theorem C can readily be adapted to give the corresponding result for functions in the class $\mathcal{M}$:

Theorem 5. *Let $f, h \in \mathcal{M}$ be such that f is not a Möbius map, and h is not the identity map. Then there are only countably many $g \in \mathcal{M}$ such that (5) holds.*

Bergweiler and Hinkkanen's proof of Theorem C uses the facts that, if $n \in \mathbb{N}$ and f is entire but is not a Möbius map, then f^n is also entire, and the repelling periodic points of f are dense in the Julia set $J(f)$, which is a non-empty perfect set. Here a point $\zeta \in \widehat{\mathbb{C}}$ is called *periodic* if there exists $p \in \mathbb{N}$ such that $f^p(\zeta) = \zeta$, and it is also called *repelling* if $|f^p(\zeta)| > 1$.

It was shown in [8] that, if $f \in \mathcal{M}$ is not a Möbius map, then the Julia set $J(f)$ is a non-empty perfect set and that the repelling periodic points of f are dense in $J(f)$. In this case $J(f)$ is the set of points in $\widehat{\mathbb{C}}$ at which either some iterate of f is not defined, or the iterates $\{f^n : n \in \mathbb{N}\}$ are all defined but do not form a normal family.

These results on the properties of the Julia set for functions in $\mathcal{M}$ are all that is needed to adapt the proof of Theorem C and so prove Theorem 5. Clearly, Theorem 3 then follows immediately. For completeness, we give a brief proof of Theorem 5 using Bergweiler and Hinkkanen's method.

Proof of Theorem 5. We will define a countable collection of subsets of $\mathcal{M}$, denoted by $(P_{i,j,k})$, for $i, j, k \in \mathbb{N}$, and show that:

(i) every non-constant $g \in \mathcal{M}$ that satisfies (5) lies in $P_{i,j,k}$ for some $i, j, k \in \mathbb{N}$, and
(ii) for each $i, j, k \in \mathbb{N}$, the set $P_{i,j,k}$ contains at most one element.

Since there are at most countably many constant functions g satisfying (5) (because if $g \equiv c$ then c is a fixed point of h), it is easy to see that Theorem 5 then follows.

To define the sets $P_{i,j,k}$, we consider the indices i, j, k in turn.

- For some $p \in \mathbb{N}$, f^p has a repelling fixed point, ξ say. Thus we can construct a nested sequence of disks $(D_i)_{i\in\mathbb{N}}$, centred at ξ, in which f^p is defined and univalent, with the radius of D_i tending to 0 as $i \to \infty$, and such that a univalent branch F of f^{-p} maps each D_i into a relatively compact subset of itself, with $F^n(z) \to \xi$ uniformly as $n \to \infty$ for $z \in D_i$. Since $\xi \in J(f)$, it follows from the properties of $J(f)$ described above that, for each $i \in \mathbb{N}$, we can choose $a_i \in D_i \setminus \{\xi\}$ and $p_i \geq 1$ such that $f^{p_i}(a_i) = a_i$.
- Now let $(\eta_j)_{j\in\mathbb{N}}$ be an enumeration of the repelling fixed points of h^p. Then we argue similarly that, for each $j \in \mathbb{N}$, there is a disk K_j centred at η_j in which h^p is defined and univalent, and such that a univalent branch H_j of h^{-p} defined on K_j fixes η_j and maps K_j into a relatively compact subset of itself, with $H_j^n(z) \to \eta_j$ uniformly as $n \to \infty$ for $z \in K_j$.
- Finally, let $(b_k)_{k\in\mathbb{N}}$ be an enumeration of all the periodic points of h.

Note that, for simplicity, we have assumed here that h has infinitely many repelling fixed points and periodic points, but the argument remains valid in cases where there are only finitely many.

For each $i, j, k \in \mathbb{N}$, we now define $P_{i,j,k}$ to be the set of all non-constant $g \in \mathcal{M}$ that satisfy (5) and are such that

$$g(\xi) = \eta_j, \quad g(D_i) \subset K_j \quad \text{and} \quad g(a_i) = b_k \in K_j.$$

To see that property (i) holds, suppose that $g \in \mathcal{M}$ is non-constant. Note that since $\xi = f^p(\xi)$ it follows from (5) that $g(\xi) \in \mathbb{C}$ is a fixed point of h^p. Moreover, by a calculation, $g(\xi)$ is a *repelling* fixed point of h^p, so there exists $j \in \mathbb{N}$ such that $g(\xi) = \eta_j$. By the continuity of g, there also exists

$i \in \mathbb{N}$ sufficiently large that $g(\overline{D}_i) \subset K_j$. Now $a_i = f^{p_i}(a_i)$, and by (5) we have $g(a_i) = g(f^{p_i}(a_i)) = h^{p_i}(g(a_i))$, so $g(a_i) \in K_j$ is a fixed point of h^{p_i}; in other words, there exists $k \in \mathbb{N}$ such that $g(a_i) = b_k$. Thus $g \in P_{i,j,k}$.

Finally we show that property (ii) also holds. Fix $i, j, k \in \mathbb{N}$, and assume that $g, \tilde{g} \in P_{i,j,k}$. Define $a_{i,n} = F^n(a_i) \in D_i$ and $b_{k,n} = H_j^n(b_k) \in K_j$, for $n \in \mathbb{N} \cup \{0\}$. We claim that $g(a_{i,n}) = b_{k,n}$, for $n \in \mathbb{N} \cup \{0\}$. This is certainly true for $n = 0$, so suppose it is true for $0 \le n \le m-1$, for some $m \ge 1$. The point $z = a_{i,m}$ is the unique solution in D_i of the equation $f^p(z) = a_{i,m-1}$, so

$$b_{k,m-1} = g(a_{i,m-1}) = g(f^p(a_{i,m})).$$

Hence, by (5), we have $b_{k,m-1} = h^p(g(a_{i,m}))$. Also, the point $w = b_{k,m}$ is the unique solution in K_j of the equation $h^p(w) = b_{k,m-1}$. We deduce that $g(a_{i,m}) = b_{k,m}$, which proves the claim.

The same argument can be applied to $\tilde{g}$, so we have $g(a_{i,n}) = \tilde{g}(a_{i,n})$ for all $n \in \mathbb{N}$. Since g and $\tilde{g}$ are meromorphic, and $\lim_{n\to\infty} a_{i,n} = \xi$ is finite, we have $g \equiv \tilde{g}$ by the identity principle. Thus $P_{i,j,k}$ indeed contains at most one element. □

4. Proof of Theorem 4

In this section we prove Theorem 4 by giving two examples of transcendental meromorphic functions that are minimally permuting. The function in the first example has a simple form and the proof uses only elementary arguments. The second example involves a more complicated function and the proof uses a result from value distribution theory, but is very short. It seems worthwhile to give both examples.

For the first example, let f be given by

$$f(z) := \frac{1}{z}e^z + z.$$

This function has no fixed points, and so permutes with no constant functions. Moreover, by an application of Theorem 1 and an elementary calculation, it can be shown that f permutes with no rational maps apart from the identity map.

Let g be a transcendental meromorphic function that permutes with f; we need to show that $g = f$. Note that zero is the only pole of f, and it is of order one. It follows from (1) that zero is the only pole of g. Let $m \in \mathbb{N}$ denote the order of this pole of g.

Let ζ be a zero of f, and let its order be $n \in \mathbb{N}$. Then ζ is a pole of $g \circ f$ of order mn. It follows from (1) that ζ is a pole of $f \circ g$ of order mn, and so ζ is a zero of g of order mn. Similarly, let ζ' be a zero of g, and let its order be $n' \in \mathbb{N}$. Then ζ' is a pole of $f \circ g$ of order n'. It follows from (1) that ζ' is a pole of $g \circ f$ of order n', and so ζ' is a zero of f of order n'/m.

We deduce that $g/(f)^m$ is a meromorphic function in $\mathbb{C}$ with no poles or zeros, and thus there exists an entire function H, with no zeros, such that

$$g(z) = f(z)^m H(z), \quad \text{for } z \in \mathbb{C}. \tag{6}$$

We now show that $m = 1$ and $H(0) = 1$. To achieve this we consider the zeros of f of large modulus. These points are the solutions of $e^z = -z^2$. Suppose that $z = x + iy$ is such a point. Since $e^x = |e^z| = |z|^2 = x^2 + y^2$, it follows first that x is large and positive, and then that y is close to $\pm e^{x/2}$. Thus $\arg z$ is close to $\pm\pi/2$ and y is close to a large positive or negative even multiple of π. We deduce that, for large positive or negative values of n, there are zeros of f close to the points $\zeta_n := 2\log(2|n|\pi) + 2n\pi i$; we label the corresponding zeros z_n. It can be shown that

$$z_n - \zeta_n \to 0 \text{ as } |n| \to \infty. \tag{7}$$

Now there is a neighbourhood of infinity, U say, in which f has an inverse branch, F say, that maps U to a neighbourhood of the origin. Since $f(z) = 1/z + 1 + O(|z|)$ as $z \to 0$, we have that $F(z) = 1/z + 1/z^2 + O(|z|^{-3})$ as $z \to \infty$. Let $n_0 \in \mathbb{N}$ be sufficiently large that $z_n \in U$, for $|n| \geq n_0$. Then, by (1) and (6), we have

$$0 = g(z_n) = g(f(F(z_n))) = f(g(F(z_n))) = f(z_n^m H(F(z_n))), \quad \text{for } |n| \geq n_0. \tag{8}$$

In other words, $z_n^m H(F(z_n))$ is a zero of f, for each n such that $|n| \geq n_0$. Hence for each such n there exists $p_n \in \mathbb{Z}$ such that

$$z_{p_n} = z_n^m H(F(z_n)). \tag{9}$$

Now

$$H(F(z)) = H(0) + H'(0)/z + O(|z|^{-2}) \text{ as } z \to \infty. \tag{10}$$

We deduce, by (7) and (10), that

$$\zeta_{p_n} \sim \zeta_n^m H(0) \text{ as } |n| \to \infty. \tag{11}$$

It is easy to see that $|p_n| \to \infty$ as $|n| \to \infty$. Since $\arg \zeta_n \to \pm\pi/2$ as $|n| \to \infty$, it follows from (11) that $H(0) = \pm i^{1-m}|H(0)|$. Therefore, as $|n| \to \infty$,

$$\begin{aligned} \zeta_{p_n} \sim \zeta_n^m H(0) &= (2n\pi i)^m \left(1 - \frac{i\log(2|n|\pi)}{n\pi}\right)^m H(0) \\ &\sim \pm(2n\pi)^m \left(\frac{m\log(2|n|\pi)}{n\pi} + i\right) |H(0)|. \end{aligned}$$

Since ζ_{p_n} also satisfies $\operatorname{Im}\zeta_{p_n} = \pm\exp(\operatorname{Re}\zeta_{p_n}/2)$, it follows first that $m = 1$, then that $\operatorname{Im}\zeta_{p_n}$ and $\operatorname{Im}\zeta_n$ have the same sign, and finally that $H(0) = 1$.

Thus, for $|n|$ sufficiently large, we have $p_n = n$ and hence $H(F(z_n)) = 1$, by (9). Since the points $F(z_n)$ accumulate at the origin, and $H(0) = 1$, it follows by the identity principle that $H(z) \equiv 1$, and this completes the proof that f is minimally permuting.

For the second example, let $q \in \mathbb{N}$ be greater than 16, and let $p_1, p_2, \ldots, p_q$ be the zeros of the polynomial given in [4, Theorem 1] (the same result was proved independently in [10]). We then set

$$h(z) := \frac{e^z}{\prod_{j=1}^{q}(z - p_j)} + z.$$

Then h has no fixed points, and it can be shown using Theorem 1 that h permutes with no rational maps apart from the identity. We denote the set of poles of h by $S := \{p_1, \ldots, p_q\}$.

Suppose that g is a transcendental meromorphic function that permutes with h. Then S is also the set of poles of g by (1). Suppose that $z \in g^{-1}(S)$. Then (1) also gives that $g(h(z)) = \infty$, so $z \in h^{-1}(S)$, and we deduce that $g^{-1}(S) = h^{-1}(S)$. It then follows from [4, Theorem 1] that $h = g$. Thus h is indeed minimally permuting.

Acknowledgment.

The authors are grateful to the referee for helpful comments.

References

[1] Baker, I. N. Zusammensetzungen ganzer Funktionen. *Math. Z. 69* (1958), 121–163.

[2] Baker, I. N. Permutable entire functions. *Math. Z. 79* (1962), 243–249.

[3] Baker, I. N. Repulsive fixpoints of entire functions. *Math. Z. 104* (1968), 252–256.

[4] Bartels, S. Meromorphic functions sharing a set with 17 elements ignoring multiplicities. *Complex Variables Theory Appl. 39*, 1 (1999), 85–92.

[5] Beardon, A. *Iteration of rational functions.* Springer, 1991.

[6] Bergweiler, W. Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N.S.) 29*, 2 (1993), 151–188.

[7] Bergweiler, W., and Hinkkanen, A. On semiconjugation of entire functions. *Math. Proc. Cambridge Philos. Soc. 126*, 3 (1999), 565–574.

[8] Bolsch, A. Repulsive periodic points of meromorphic functions. *Complex Variables Theory Appl. 31*, 1 (1996), 75–79.

[9] Bolsch, A. Periodic Fatou components of meromorphic functions. *Bull. Lond. Math. Soc. 31*, 5 (1999), 543–555.

[10] Fang, M., and Guo, H. On unique range sets for meromorphic or entire functions. *Acta Math. Sinica (N.S.) 14*, 4 (1998), 569–576.

[11] Goldstein, R. On certain compositions of functions of a complex variable. *Aequationes Math. 4* (1970), 103–126.

[12] Iyer, V. G. On permutable integral functions. *J. London Math. Soc. 34* (1959), 141–144.

[13] Milnor, J. *Dynamics in one complex variable*, third ed., vol. 160 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2006.

[14] Rådström, H. On the iteration of analytic functions. *Math. Scand. 1* (1953), 85–92.

J. W. Osborne, D. J. Sixsmith
Department of Mathematics and Statistics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UK
e-mail: john.osborne@open.ac.uk, david.sixsmith@open.ac.uk