Investigations on families of probability measures depending on several parameters.

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Investigations on Families of Probability Measures Depending on Several Parameters

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Declaration

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Abstract

In this thesis we discuss how to extend the notion of one-parameter Feller semigroups and one-parameter Markov processes to several time parameters. Following a summary of the most important preliminaries we discuss N-parameter Feller semigroups and address the difficulty of extending the generator of one-parameter semigroups to multiparameters semigroups. In particular we extend the differential equation associated with the generator in the classical case to several parameters. Finally, we investigate families of operators depending on several parameters which go beyond operator semigroups and construct associated processes. For these families of operators the equality $\mathrm{T}_s \circ \mathrm{T}_t = \mathrm{T}_{s+t}$, $s, t \in \mathbb{R}_+^N$, which is typical for operator semigroups, does no longer hold, and consequently the associated processes are time-inhomogeneous. However, using a transform of variables we link these operator families with operator semigroups.

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Contents

Introduction

The aim of this thesis is to extend the notion of thoroughly studied one-parameter convolution semigroups of (sub-)probability measures and semigroups of operators to N-parameter convolution semigroups of (sub-) probability measures and semigroups of operators, respectively, i.e. to objects indexed by a parameter $t \in \mathbb{R}^N$. On the one hand this extension to N-dimensional parameters is interesting in itself, on the other hand we are particularly interested in families of probability measures which give rise to stochastic processes. We will mainly concentrate on the analytic point of view. One-parameter convolution semigroups of (sub-)probability measures are associated to (one-parameter) Lévy-processes. The analytic counterpart are strongly continuous one-parameter semigroups of translation invariant operators on C_∞. More generally, positivity preserving, strongly continuous contraction semigroups on C_∞, i.e. Feller semigroups, are associated with (one-parameter) Feller processes. We aim to extend this correspondence to the N-parameter case. Certain stochastic processes with N-dimensional (time-)parameter t, in literature also called random fields, can be described by $N-$parameter convolution semigroups of (sub-)probability measures or $N-$parameter semigroups of operators. When extending stochastic processes or semigroups to parameters of N dimensions the main difficulty, which arises from the extension to a multidimensional index, is the lack of a total order on the parameter set. The notion of the generator of an N-parameter semigroup of operators, for example, cannot be defined as smoothly as in the one-parameter case. Another problem, which we do not attempt to tackle is the definition of multiparameter càdlàg-property. Yet there exists a partial order on $\mathbb{R}^N$ of which we may make use. Together with a commutation property this will allow us to define nice processes indexed by subsets of $\mathbb{R}^N$. Moreover, we will construct vaguely continuous N-parameter families of probability measures - not fulfilling an N-parameter semigroup property - which give rise to multiparameter stochastic processes, which - to the best of our knowledge - have not yet been discussed by other authors before.

Summing up, the processes considered in Chapter 2 are both time- and space-homogenous Markov processes whereas the processes in Chapter 3 are space-homogenous and, in particular, time-inhomogeneous processes. If one wishes to drop space-homogeneity the characteristic exponent needs to be substituted by an $x-$dependent symbol. Symbols of this kind are briefly introduced in the first chapter.

Multiparameter processes have been investigated by many authors before. We mention just a few: R. Cairoli [5] looked at the (direct) product of two Markov processes and compared its properties with those of the factor processes, in [7] with J. B. Walsh they studied stochastic integrals in the plane, i.e. of processes with a two-dimensional continuous parameter. Moreover, in a book with R. C. Dalang [6] they provide a systematic exposition of the theory of optimal stopping for multiparameter stochastic processes with discrete parameter spaces. E. B. Dynkin [8] solved a type of Dirichlet problem for the product of the infinitesimal generators of several diffusion processes. In [17] G. Mazziotto developed a potential theory for two-parameter Markov processes with regular trajectories and in [10] F. Hirsch and S. Song introduce a Skrorokhod topology which allows them to define the notion of complete $N-$parameter symmetric Markov processes. O. E. Barndorff-Nielsen, J. Pedersen, and K.-I. Sato [2] analyse multivariate subordination of multiparameter processes. The latter comes very close to some parts of this thesis and will be focused on in the due text. The book [16] by D. Khoshnevisan presents a comprehensible account on the theory of multiparameter stochastic processes, in which both the discrete time-parameter case and the continuous time-parameter case is discussed. Moreover, two different notions of multiparameter martingales are introduced and it is proved that these two are equivalent under a commutation hypothesis; examples of multiparameter processes are presented and a potential theory for these processes is developed. Finally we mention paper [18] of J. Pedersen and K.-I. Sato who investigated stochastic processes indexed by a cone, more general than $\mathbb{R}_+^N$.

We briefly want to describe the content in some detail. In the first chapter we give preliminaries, which are necessary to sooth the reading of Chapters 2 and 3. We start by defining the partial order $\preceq$ on multidimensional parameter sets which substitutes the total order. We state a theorem on tensor product spaces and define the Fourier transform which is an important tool to find the correspondence between convolution semigroups and continuous negative definite functions, in addition we discuss the convolution theorem. Moreover, we state a representation theorem for continuous negative definite functions, the Lévy-Khinchin formula. We define convolution semigroups of measures which are supported on the positive half-line and subordination of convolution semigroups as well as semigroups of operators. We introduce the notion of the generator of a (one-parameter) semigroup of operators and state the Hille-Yosida-Ray theorem, which characterises generators of Feller semigroups. Finally, we list some special functions which are needed later in the text.

In Chapter 2 we extend the notion of convolution semigroups of probability measures and semigroups of operators to the case of an N-dimensional parameter. The first section focuses on N-parameter convolution semigroups. After the definition an intuitive example is given. Then we prove the continuity of the N-parameter convolution semigroups with respect to (time-) parameter t. The first theorem of this section establishes a one-to-one correspondence between N continuous negative definite functions and $N-$parameter convolution semigroups. The second theorem validates subordination in the sense of Bochner for $N-$parameter convolution semigroups. The following Section 2.2 is devoted to $N-$parameter semigroups of operators. The definition is complemented by two examples which also illustrate that $N-$parameter convolution semigroups of measures give rise to N-parameter semigroups of operators. Continuity with respect to the (time-)parameter t is shown and decomposition of $N-$parameter semigroups into its marginal (one-parameter) semigroups is proven. Concluding this section we address the problem of generalising the generator by looking at the evolution equation associated to the generator of a one-parameter semigroup of operators. We extend it to an N-parameter evolution equation which is associated to the composition of the generators of the marginal semigroups of the N-parameter semigroup and analyse the domain of this operator. In the subsequent Section 2.3 subordination (in the sense of Bochner) is defined for N-parameter semigroups of operators. We emphasize that the subordinate semigroup may be a indexed by an M-dimensional parameter and both $M < N$ and $M > N$ (as well as $M = N$) are possible. Moreover, we describe the generator substitute of the subordinate semigroup. The concluding section of this chapter contains examples of multiparameter processes and draws attention to the third example of this section which illustrates the limitations following from the semigroup property.

In Chapter 3 drop the restrictive semigroup property and focus, more generally, on vaguely continuous families of probability measures and operators. We are especially interested in the processes which correspond to these families. The first section contains a case study of a two-parameter family of probability measures. Since it does not possess the semigroup property, subordination as described in Chapter 2 is not feasible, instead we use iterated subordination "by hand". In addition we construct two kinds of processes which are associated to this family of probability measures. In Section 3.2 we look at other examples of two-parameter families of measures with the aim of finding a convolution property or commuting structure as known from the semigroup case discussed in Chapter 2. Indeed we obtain such a structure on a curvilinear net and are also able to transform the families of probability measures into convolution semigroups of probability measures.

Chapter 1

Preliminaries

In this chapter we introduce necessary notations and definitions and state theorems needed in the due text. Moreover, the reader will be made familiar with some basic facts of one-parameter semigroups which, in the subsequent chapter, will serve as a basis for the introduction and analysis of N-parameter semigroups. Where it seems helpful we give references for the reader to find proofs to the theorems and further properties.

1.1 Notation

We use the following notation for vectors in $\mathbb{R}^N$ to handle N-dimensional parameters in a comfortable manner. First, $\mathrm{e}_j := (0, \ldots, 1, \ldots, 0)$ defines the j-th canonical basis vector of $\mathbb{R}^N$. Further let $s, t \in \mathbb{R}^N$ then we define $s \curlywedge t := (s_1 \wedge t_1, \ldots, s_N \wedge t_N)$ where for $a, b \in \mathbb{R}$ we set $a \wedge b = \min(a, b)$. Furthermore, $s \succeq t$ is defined to hold if and only if $s_j \geq t_j$ holds for all $j \in \{1, \ldots, N\}$. In an analogous way we define $s \succ t$, $s \preceq t$ and $s \prec t$.
$\mathrm{B}^n_\delta(c)$ defines the open ball of radius δ centered at c in the space $\mathbb{R}^n$.

Among others we will need the following function spaces. ($\Omega \subset \mathbb{R}^n$ is an open set)

$\mathrm{C}(\Omega)$	continuous functions on Ω
$\mathrm{C}_0(\Omega)$	continuous functions with compact support in Ω
$\mathrm{C}^m(\Omega)$	m-times continuously differentiable functions on Ω
$\mathrm{C}^\infty(\Omega)$	$\bigcap_{m\in\mathbb{N}} \mathrm{C}^m(\Omega)$
$\mathrm{C}_0^\infty(\Omega)$	$\mathrm{C}_0(G) \cap \mathrm{C}^\infty(\Omega)$
$\mathrm{B}(\Omega)$	Borel measurable functions on Ω

Definition 1.1.1. *Let $\Omega_j, j = 1,2,$ be open sets in $\mathbb{R}^{n_j}$ and $\mathrm{u}_j \in C(\Omega_j)$. Then we define on $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{n_1+n_2}$ the function $\mathrm{u}_1 \otimes \mathrm{u}_2$ by*

$$(\mathrm{u}_1 \otimes \mathrm{u}_2)(x_1, x_2) = \mathrm{u}_1(x_1) \cdot \mathrm{u}_2(x_2), \qquad \textit{for all } x_j \in \Omega_j,$$

*and we call $\mathrm{u}_1 \otimes \mathrm{u}_2$ the **tensor product** of u_1 and u_2. Furthermore, let $C_0^\infty(\Omega_1) \otimes C_0^\infty(\Omega_2)$ denote the set of all finite linear combinations of the form*

$$\sum_{k=1}^{m} \mathrm{u}_k^{(1)} \otimes \mathrm{u}_k^{(2)},$$

*with $\mathrm{u}_k^{(j)} \in \mathrm{C}_0^\infty(\Omega_j), k = 1, \ldots, m,\ m \in \mathbb{N},\ j = 1,2$. We call $\mathrm{C}_0^\infty(\Omega_1) \otimes \mathrm{C}_0^\infty(\Omega_2)$ the **algebraic tensor product** of $\mathrm{C}_0^\infty(\Omega_1)$ and $\mathrm{C}_0^\infty(\Omega_2)$.*

The importance of the tensor product for our considerations lies in the following lemma.

Lemma 1.1.2. *A dense subset of $\mathrm{C}_0^\infty(\Omega_1 \times \Omega_2)$ is given by $\mathrm{C}_0^\infty(\Omega_1) \otimes \mathrm{C}_0^\infty(\Omega_2)$.*

See §14 in [13] for a proof of this lemma and more properties of tensor product spaces.

When introducing convolution semigroups of measures we will need the space $\mathrm{M}_b^+(\mathbb{R}^n)$ of positive bounded measures on $\mathbb{R}^n$ and for a measure $\mu \in \mathrm{M}_b^+(\mathbb{R}^n)$ its total mass is defined to be the non-negative value $\|\mu\| = \mu(\mathbb{R}^n)$. Now, we can define the convolution of measures.

Definition 1.1.3. *Let $\mu_j \in \mathrm{M}_b^+(\mathbb{R}^n), j = 1,2,$ and define the mapping $\mathrm{A}_2 : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n, (x_1, x_2) \longmapsto x_1 + x_2$. The image of $\mu_1 \otimes \mu_2$, i.e. the product* *measure of μ_1 and μ_2, under the mapping* A_2 *is called the convolution of* μ_1 *and* μ_2 *and is denoted by*

$$\mu_1 * \mu_2 := \mathrm{A}_2(\mu_1 \otimes \mu_2).$$

It follows that

$$(\mu_1 * \mu_2)(\mathrm{A}) = \int \mu_1(\mathrm{A} - y)\, \mu_2(\mathrm{d}y).$$

Moreover, the convolution of two integrable functions $\mathrm{f}, \mathrm{g} : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is defined by*

$$\mathrm{f} * \mathrm{g}(x) = \int_{\mathbb{R}^n} \mathrm{f}(x - y)\mathrm{g}(y)\, \mathrm{d}y.$$

Definition 1.1.4. *Let μ be a Borel measure on $\mathbb{R}^n$. We define the support of μ as the complement of the largest open set G, such that $\mu(G) = 0$, and we will denote it by* $\operatorname{supp}\mu$.

1.2 One-Parameter Convolution Semigroups

In this chapter we will summarize definitions and theorems arising when analysing convolution semigroups and semigroups of operators depending on one parameter. Since our aim is to investigate multi-parameter convolution and operator semigroups we sometimes speak of one- or multi-dimensional semigroups with reference to the parameter set. We omit proofs, however, refer the reader to [14] by N.Jacob and [4] by C.Berg and G.Forst.

First we introduce the Schwartz space:

Definition 1.2.1. *The* ***Schwartz space*** $\mathcal{S}(\mathbb{R}^n)$ *consists of all functions* $\mathrm{u} \in \mathrm{C}^\infty(\mathbb{R}^n)$ *such that for all $m_1, m_2 \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0$*

$$\mathrm{p}_{m_1,m_2}(\mathrm{u}) := \sup_{x \in \mathbb{R}^n} \left[(1 + |x|^2)^{m_1/2} \sum_{|\alpha| \leq m_2} |\partial^\alpha \mathrm{u}(x)| \right] < \infty.$$

The family $(\mathrm{p}_{m_1,m_2})_{m_1,m_2 \in \mathbb{N}}$ is a family of separating semi-norms on $\mathcal{S}(\mathbb{R}^n)$ and the Schwartz space equipped with the topology induced by these semi-norms is a Fréchet space. Before introducing the Fourier transform we mention that $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n) \subset L^p, 1 \leq p \leq \infty$, as a dense subspace.

Definition 1.2.2. ***A.*** *The* ***Fourier transform*** *of* $u \in \mathcal{S}(\mathrm{R}^n)$ *is defined by*

$$\hat{u}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx,$$

instead of $\hat{u}$ *we sometimes write* $F(u)$ *or* $F_{x \mapsto \xi}(u)$.
B. *The* ***Fourier transform of a measure*** μ *on* $\mathbb{R}^n$ *is defined by the following integral (if it exists):*

$$\hat{\mu}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} \mu(dx).$$

It is easy to see that the Fourier transform extends from $\mathcal{S}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

Theorem 1.2.3. *The Fourier transform is a linear operator from* $\mathcal{S}(\mathbb{R}^n)$ *into itself which is continuous and bijective and has a continuous inverse given by:*

$$\mathcal{F}^{-1}(u)(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} u(\xi) d\xi, \quad \textit{for all } u \in \mathcal{S}(\mathbb{R}^n).$$

The estimate in the following theorem is often called Lemma of Riemann-Lebesgue:

Theorem 1.2.4. *The Fourier transform is a linear operator which maps* $(L^1(\mathbb{R}^n), \|.\|_{L^1})$ *continuously into* $(C_\infty(\mathbb{R}^n), \|.\|_\infty)$. *In particular, the following inequality holds true for all* $u \in L^1(\mathbb{R}^n)$:

$$\|\hat{u}\|_\infty \leq (2\pi)^{-\frac{n}{2}} \|u\|_{L^1}.$$

For the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{M}_b^+(\mathbb{R}^n)$ we have the following convolution theorem:

Theorem 1.2.5. ***A.*** *For* $u, v \in \mathcal{S}(\mathbb{R}^n)$ *it holds*

$$(u \cdot v)^\wedge(\xi) = (2\pi)^{-\frac{n}{2}} (\hat{u} * \hat{v})(\xi) \tag{1.1}$$

and

$$(u * v)^\wedge(\xi) = (2\pi)^{\frac{n}{2}} \hat{u}(\xi) \cdot \hat{v}(\xi). \tag{1.2}$$

B. *For $\mu, \nu \in \mathcal{M}_b^+(\mathbb{R}^n)$ we have the following equality*

$$(\mu * \nu)^\wedge(\xi) = (2\pi)^{\frac{n}{2}} \hat{\mu}(\xi) \cdot \hat{\nu}(\xi), \tag{1.3}$$

as well as for $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ and $\nu \in \mathcal{M}_b^+(\mathbb{R}^m)$

$$(\mu \otimes \nu)^\wedge (\xi, \eta) = \hat{\mu}(\xi) \cdot \hat{\nu}(\eta), \tag{1.4}$$

for $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$.

To characterise bounded Borel measures on $\mathbb{R}^n$ and convolution semigroups of measures we introduce positive definite functions:

Definition 1.2.6. *A function $\mathrm{u} : \mathbb{R}^n \longrightarrow \mathbb{C}$ is said to be positive definite if for any fixed $k \in \mathbb{N}$, vectors $\xi^1, \dots, \xi^k \in \mathbb{R}^n$ and all $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ it holds*

$$\sum_{j,l=1}^{k} \mathrm{u}(\xi^j - \xi^l)\lambda_j \overline{\lambda_l} \geq 0.$$

Now, we can state Bochner's theorem:

Theorem 1.2.7. *A function $\mathrm{u} : \mathbb{R}^n \longrightarrow \mathbb{C}$ is the Fourier transform of a measure $\mu \in \mathrm{M}_b^+(\mathbb{R}^n)$ with total mass $\|\mu\|$, if and only if the following three conditions are fulfilled*

$$\begin{aligned} &(i) && \mathrm{u} \textit{ is continuous,} \\ &(ii) && \mathrm{u}(0) = \hat{\mu}(0) = (2\pi)^{-\frac{n}{2}} \|\mu\|, \\ &(iii) && \mathrm{u} \textit{ is positive definite.} \end{aligned} \tag{1.5}$$

Remark. *We found a one-to-one correspondence between positive bounded Borel measures and continuous positive definite functions; especially, for a probability measure the corresponding continuous positive definite function attains the value $(2\pi)^{-\frac{n}{2}}$ at the origin. Note that this correspondence can be extended to a correspondence respecting natural topologies on $\mathrm{M}_b^+(\mathbb{R}^n)$ and on $\mathrm{C}_b(\mathbb{R}^n)$, respectively.*

Definition 1.2.8. *Let $(\mu_\nu)_{\nu\in\mathbb{N}} \subset \mathrm{M}_b^+(\mathbb{R}^n)$ be a sequence of bounded positive measures on $\mathbb{R}^n$ and $\mu_0 \in \mathrm{M}_b^+(\mathbb{R}^n)$.*

***A.** Then μ_ν is said to converge weakly to μ_0 as ν tends to infinity, if for all* $\mathrm{u} \in \mathrm{C}_b(\mathbb{R}^n;\mathbb{R})$

$$\lim_{\nu\longrightarrow\infty}\int_{\mathbb{R}^n}\mathrm{u}(x)\,\mu_\nu(\mathrm{d}x) = \int_{\mathbb{R}^n}\mathrm{u}(x)\,\mu_0(\mathrm{d}x). \tag{1.6}$$

***B.** Moreover, μ_ν is said to converge vaguely to μ_0 as $\nu \longrightarrow \infty$, if equality (1.6) holds for all* $\mathrm{u} \in \mathrm{C}_0(\mathbb{R}^n;\mathbb{R})$.

Remark. *It is obvious, that weak convergence implies vague convergence, since $C_0(\mathbb{R}^n;\mathbb{R}) \subset C_b(\mathbb{R}^n;\mathbb{R})$.*

Definition 1.2.9. *A family $(\mu_t)_{t\geq 0}$ of bounded Borel measures on $\mathbb{R}^n$ is called a **convolution semigroup** on $\mathbb{R}^n$ if the following conditions are satisfied:*

$$\begin{array}{lll}
(i) & \mu_t(\mathbb{R}^n) \leq 1, & \text{for all } t \geq 0,\\
(ii) & \mu_s * \mu_t = \mu_{s+t}, & \text{for all } s,t \geq 0,\\
 & \text{and } \mu_0 = \varepsilon_0, & \\
(iii) & \mu_t \longrightarrow \varepsilon_0 \text{ vaguely as } t \longrightarrow 0. &
\end{array}$$

Remark. *The vague convergence of a convolution semigroup for $t \longrightarrow 0$ already implies weak convergence to ε_0 for $t \longrightarrow 0$ as the following theorem states.*

Theorem 1.2.10. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of positive bounded measures on $\mathbb{R}^n$ which converges vaguely to $\mu \in \mathcal{M}_b^+(\mathbb{R}^n)$ and satisfies $\lim_{n\to\infty}\mu_n(\mathbb{R}^n) = \mu(\mathbb{R}^n)$ also converges weakly to the same limit μ.*

Definition 1.2.11. *A function $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ satisfying*

$$\begin{array}{ll}
(i) & \psi(0) \geq 0\\
(ii) & \xi \longmapsto (2\pi)^{-\frac{n}{2}}\mathrm{e}^{-t\psi(\xi)} \text{ is positive definite for all } t \geq 0,
\end{array}$$

*is called **negative definite**.*

For continuous negative definite functions we have the following representation theorem:

Theorem 1.2.12 (Lévy-Khinchin). *Every continuous negative definite function $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ has a representation of the form*

$$\psi(\xi) = c + \mathrm{i}(d \cdot \xi) + \mathrm{q}(x) + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \mathrm{e}^{-\mathrm{i}x \cdot \xi} - \frac{\mathrm{i}x \cdot \xi}{1 + |x|^2}\right) \frac{1 + |x|}{|x|^2}\, \mu(\mathrm{d}x) \tag{1.7}$$

with a non-negative constant $c \geq 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form q, *and a finite Borel measure μ on $\mathbb{R}^n \setminus \{0\}$. The quadruple (c, d, q, μ) determines the function ψ uniquely. Conversely, any such quadruple defines a continuous negative definite function by equation (1.7).*

Using the correspondence between positive definite functions and positive Borel measures we now establish a one-to-one correspondence between continuous negative definite functions and convolution semigroups on $\mathbb{R}^n$:

Theorem 1.2.13. *Let $(\mu_t)_{\mu \geq 0}$ be a convolution semigroup on $\mathbb{R}^n$, then there exists a unique continuous negative definite function $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$, such that*

$$\hat{\mu_t}(\xi) = (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-t\psi(\xi)}, \tag{1.8}$$

for all $t \geq 0$ and $\xi \in \mathbb{R}^n$. The converse also holds, to every continuous negative definite function, there exists a convolution semigroup such that (1.8) is satisfied.

To investigate subordination of convolution semigroups we introduce convolution semigroups of measures which are supported on the positive half-line and again we find a one-to-one correspondence between convolution semigroups which are supported on $[0, \infty)$ and a class of functions, called Bernstein functions and this correspondence will be established using the Laplace transform.

Definition 1.2.14. *Let $(\nu_t)_{t \geq 0}$ be a convolution semigroup of measures on $\mathbb{R}$. It is said to be* ***supported by*** $[0, \infty)$, *if* $\operatorname{supp} \nu_t \subset [0, \infty)$ *for all $t \geq 0$.*

Definition 1.2.15. *A real-valued function* $\mathrm{f} \in \mathrm{C}^\infty((0,\infty))$ *is called a **Bernstein function** if*

$$\mathrm{f} \geq 0 \text{ and } (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathrm{f}(x) \leq 0, \quad \text{for all } k \in \mathbb{N}.$$

Definition 1.2.16. ***A.*** *The **Laplace transform of a function*** $\mathrm{u} \in \mathrm{L}^1(\mathbb{R}), \operatorname{supp} \mathrm{u} \subset [0,\infty)$, *is defined by*

$$\mathcal{L}(\mathrm{u})(z) := \int_0^\infty \mathrm{e}^{-zt}\mathrm{u}(t)\,\mathrm{d}t, \quad \text{for all } z \in \mathbb{C}.$$

B. *Moreover, we define the **Laplace transform of a measure*** μ *on* $\mathbb{R}$ *which fulfils* $\operatorname{supp}\mu \subset \mathbb{R}$ *as well as* $\int_0^\infty \mathrm{e}^{-xs}\,\mu(\mathrm{d}s) < \infty$ *by*

$$\mathcal{L}(\mu)(z) := \int_0^\infty \mathrm{e}^{-zt}\,\mu(\mathrm{d}t).$$

Remark. *Only the real-part of* $z = x + \mathrm{i}y$, *for* $x, y \in \mathbb{R}$ *is relevant for the convergence of the Laplace transform of a function . Furthermore, there exists an* x_0, *called the **abscissa of convergence**, such that the Laplace transform exists for* $z \in \mathbb{C}$ *with* $\operatorname{Re} z > x_0$ *and diverges for all* $z \in \mathbb{C}$ *with* $\operatorname{Re} z < x_0$, *where* x_0 *may also assume the values* $-\infty$ *or* $+\infty$.

Theorem 1.2.17. *Let* $\mathrm{f} : (0,\infty) \longrightarrow \mathbb{R}$ *be a Bernstein function. Then there exists a uniquely defined convolution semigroup* $(\nu_t)_{t\geq 0}$ *supported on* $[0,\infty)$ *such that*

$$\mathcal{L}(\nu_t)(x) = \mathrm{e}^{-t\mathrm{f}(x)} \tag{1.9}$$

holds for all $x > 0$ *and* $t \geq 0$. *Conversely, for a convolution semigroup* $(\nu_t)_{t\geq 0}$, *which is supported on the positive half-line, there exists a unique Bernstein function such that (1.9) holds.*

Bernstein functions have a modified Lévy-Khinchin representation of the following form:

Theorem 1.2.18. *For every Bernstein function f, there exist constants* $a, b \geq 0$ *and a measure* μ *on* $(0, \infty)$ *satisfying*

$$\int_{0+}^{\infty} \frac{s}{1+s}\, \mu(\mathrm{d}s) < \infty$$

such that

$$\mathrm{f}(x) = a + bx + \int_{0+}^{\infty} (1 - \mathrm{e}^{-xs})\, \mu(\mathrm{d}s), \quad x > 0. \tag{1.10}$$

The triple (a, b, μ) *is uniquely determined by f. Conversely, each such triple defines by (1.10) a Bernstein function.*

Remark 1.2.19. ***A.*** *Since the Laplace transform of a probability measure*

$$\int_{0}^{\infty} \mathrm{e}^{-zx} \nu_t(\mathrm{d}x)$$

is well defined for $z \in \mathbb{C}$ *with* $\mathrm{Re} z \geq 0$*, we extend the domain of Bernstein functions to the complex half-plane* $\mathrm{Re} z \geq 0$ *and only in the light of this extension Lemma 1.2.20 makes sense.*
B. *Using the relation between Fourier transform and Laplace transform we find*

$$(2\pi)^{\frac{1}{2}} \mathrm{F}(\nu_t)(y) = \mathcal{L}(\nu_t)(\mathrm{i}y) = \mathrm{e}^{-\mathrm{f}(\mathrm{i}y)}, \quad y \in \mathbb{R}$$

and hence, remembering Theorem 1.2.13, it becomes apparent that for a Bernstein function f *the function* $x \longmapsto \mathrm{f}(\mathrm{i}x)$ *is negative definite.*

Lemma 1.2.20. *Let f be a Bernstein function and* ψ *a continuous negative definite function. Then the function* $f \circ \psi$ *is also continuous and negative definite.*

Now, we have all tools to introduce subordination in the sense of Bochner for convolution semigroups.

Theorem 1.2.21. *Let* ψ *be a continuous negative definite function on* $\mathbb{R}^n$ *associated with the convolution semigroup* $(\mu_t)_{t\geq 0}$*. Furthermore, let a Bernstein function* f *with associated convolution semigroup* $(\nu_t)_{t\geq 0}$ *be given. Then the* *convolution semigroup* $(\mu_t^{\mathrm{f}})_{t\geq 0}$ *associated with the continuous negative definite function* $f\circ\psi$ *is given by*

$$\int_{\mathbb{R}^n}\phi(x)\,\mu_t^{\mathrm{f}}(\mathrm{d}x)=\int_0^\infty\int_{\mathbb{R}^n}\phi(x)\,\mu_s(\mathrm{d}x)\,\nu_t(\mathrm{d}s),\qquad \phi\in \mathrm{C}_0(\mathbb{R}^n).$$

Definition 1.2.22. *The semigroup* $(\mu_t^{\mathrm{f}})_{t\geq 0}$ *is called the* ***semigroup subordinate (in the sense of Bochner)*** *to* $(\mu_t)_{t\geq 0}$ *with respect to* $(\nu_t)_{t\geq 0}$.

1.3 (One-Parameter) Operator Semigroups

Now, we extend our considerations to semigroups of operators. It is an extension since every convolution semigroup defines an operator semigroup as will be shown in this section. As in the previous sections we do not prove the theorems stated and refer the reader to [14].
We start with:

Definition 1.3.1. *Let* $(\mathrm{X},||.||_{\mathrm{X}})$ *be a real or complex Banach space, then a one-parameter family* $(\mathrm{T}_t)_{t\geq 0}$ *of bounded linear operators* $\mathrm{T}_t:\mathrm{X}\longrightarrow\mathrm{X}$, $t\geq 0$, *is called a* ***strongly continuous one-parameter contraction semigroup of operators*** *if*

$$\begin{aligned}
&(i)\quad \mathrm{T}_0=\mathrm{id}\ \textit{and}\ \mathrm{T}_{s+t}=\mathrm{T}_s\circ\mathrm{T}_t\ \textit{for all}\ s,t\geq 0,\\
&(ii)\quad \lim_{t\to 0}||\mathrm{T}_t\mathrm{u}-\mathrm{u}||_{\mathrm{X}}=0,\\
&(iii)\quad ||\mathrm{T}_t||\leq 1,\ \textit{for all}\ t\geq 0,
\end{aligned}$$

where $||.||$ *denotes the operator norm.*
If, instead of condition (iii), we only know that inequality

$$(iii')\ ||\mathrm{T}_t||\leq C,\ \textit{for all}\ t\geq 0,$$

holds for an arbitrarily fixed $C>0$, *then we call* $(\mathrm{T}_t)_{t\geq 0}$ *a* ***strongly continuous one-parameter semigroup***.

Furthermore, we define a special class of semigroups of operators:

Definition 1.3.2. *Let $(\mathrm{T}_t)_{t\geq 0}$ be a strongly continuous contraction one-parameter operator semigroup on $\mathrm{X} = \mathrm{C}_\infty$ and assume that for all $t \geq 0$ the operators $\mathrm{T}_t : \mathrm{C}_\infty \longrightarrow \mathrm{C}_\infty$ are positivity preserving, i.e. $\mathrm{u} \geq 0$ implies $\mathrm{T}_t\mathrm{u} \geq 0, t \geq 0$, then we call $(\mathrm{T}_t)_{t\geq 0}$ a* ***Feller semigroup****.*

Lemma 1.3.3. *For a strongly continuous one-parameter semigroup $(\mathrm{T}_t)_{t\geq 0}$ on $(\mathrm{X}, \|.\|_\mathrm{X})$ we have the following estimate:*

$$\|\mathrm{T}_t\| \leq \mathrm{M}_\omega \mathrm{e}^{\omega t},$$

for all $t \geq 0$, with constants $\omega \geq 0$ and $\mathrm{M}_\omega \geq 1$.

Due to the following theorem we are able to define subordination of one-parameter contraction operator semigroups:

Theorem 1.3.4. *Let $(\mathrm{T}_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on the Banach space $(\mathrm{X}, \|.\|_\mathrm{X})$ and $(\nu_t)_{t\geq 0}$ be a positively supported convolution semigroup associated with the Bernstein function f. Then we define $\mathrm{T}_t^\mathrm{f}\mathrm{u}$, for $\mathrm{u} \in \mathrm{X}$, by the Bochner integral*

$$\mathrm{T}_f^\mathrm{f}\mathrm{u} = \int_0^\infty \mathrm{T}_s\mathrm{u}\, \nu_t(\mathrm{d}s).$$

The integral is well-defined and $(\mathrm{T}_t^\mathrm{f})_{t\geq 0}$ is a strongly continuous contraction semigroup on X.

The semigroup $(\mathrm{T}_t^\mathrm{f})_{t\geq 0}$ is called the subordinate to $(\mathrm{T}_t)_{t\geq 0}$ with respect to f or $(\nu_t)_{t\geq 0}$.

For a strongly continuous (one-parameter) contraction semigroup $(\mathrm{T}_t)_{t\geq 0}$ we define an operator A called the generator of $(\mathrm{T}_t)_{t\geq 0}$.

Definition 1.3.5. *Let $(\mathrm{T}_t)_{t\geq 0}$ be a strongly continuous (one-parameter) contraction semigroup on a Banach space $(\mathrm{X}, \|.\|_\mathrm{X})$, then we define its* ***generator*** *by*

$$\mathrm{Au} = \lim_{t\to 0} \frac{\mathrm{T}_t\mathrm{u} - \mathrm{u}}{t} \qquad \textit{as strong limit,} \tag{1.11}$$

for all

$$\mathrm{u} \in \mathrm{D(A)} := \left\{ \mathrm{u} \in \mathrm{X} : \lim_{t \to 0} \frac{\mathrm{T}_t \mathrm{u} - \mathrm{u}}{t} \text{ exists as strong limit} \right\}.$$

From the general theory of strongly continuous one-parameter semigroups we need the following result:

Lemma 1.3.6. *Let* $(\mathrm{T}_t)_{t \geq 0}$ *be a strongly continuous semigroup on the Banach space* $(\mathrm{X}, \|.\|_{\mathrm{X}})$ *and denote its generator by* A *with domain* $\mathrm{D(A)} \subset \mathrm{X}$.
A. *For any* $\mathrm{u} \in \mathrm{X}$ *and* $t \geq 0$ *it follows* $\int_0^t \mathrm{T}_s \mathrm{u} \, \mathrm{d}s \in \mathrm{D(A)}$.
B. *For* $\mathrm{u} \in \mathrm{D(A)}$ *and* $t \geq 0$ *we have* $\mathrm{T}_t \mathrm{u} \in \mathrm{D(A)}$, *i.e.* $\mathrm{D(A)}$ *is invariant under* T_t, *and*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{T}_t \, \mathrm{u} = \mathrm{A} \mathrm{T}_t \, \mathrm{u} = \mathrm{T}_t \mathrm{A} \, \mathrm{u}.$$

To characterise generators of Feller semigroups we need the following two definitions

Definition 1.3.7. *A linear operator* $\mathrm{A} : \mathrm{D(A)} \longrightarrow \mathrm{B}(\mathbb{R}^n; \mathbb{R}), \mathrm{D(A)} \subset \mathrm{B}(\mathbb{R}^n; \mathbb{R})$, *is said to satisfy the* ***positive maximum principle*** *if for* $\mathrm{u} \in \mathrm{D(A)}$ *and some* $x_0 \in \mathbb{R}^n$ *it holds* $\mathrm{u}(x_0) = \sup_{x \in \mathbb{R}^n} \mathrm{u}(x) \geq 0$ *then it follows:*

$$\mathrm{Au}(x_0) \leq 0.$$

Definition 1.3.8. *Let* $\mathrm{q} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ *be a measurable, locally bounded function for which* $\xi \longmapsto \mathrm{q}(x, \xi)$ *is continuous negative definite for all* $x \in \mathbb{R}^n$. *Then we define a* ***pseudo-differential operator*** *for* $\mathrm{u} \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ *by*

$$\mathrm{q}(x, \mathrm{D})\mathrm{u}(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{q}(x, \xi) \hat{u}(\xi) \mathrm{d}\xi,$$

and we call the function q *the* ***symbol*** *of the pseudo-differential operator* $\mathrm{q}(x, \mathrm{D})$. *Since* $\mathrm{q}(x, .)$ *is a continuous negative definite function we call* $\mathrm{q}(x, \mathrm{D})$ *a pseudodifferential operator with continuous negative definite symbol.*

The following result due to Courrège gives an important representation theorem for operators which satisfy the positive maximum principle.

Theorem 1.3.9 (Courrège). *Let* $\mathrm{A} : \mathrm{C}_0^\infty(\mathbb{R}^n;\mathbb{R}) \longrightarrow \mathrm{C}(\mathbb{R}^n;\mathbb{R})$ *be a linear operator satisfying the positive maximum principle. Then there exists a pseudo-differential operator with continuous negative definite symbol* $\mathrm{q}(x,\xi)$ *such that*

$$\mathrm{A} = -\mathrm{q}(x, \mathrm{D}). \tag{1.12}$$

In the following characterisation theorem for generators of Feller semigroups the positive maximum principle plays a central rôle and together with Theorem 1.3.9 it interlinks pseudodifferential operators with generators of Feller semigroups.

Theorem 1.3.10 (Hille-Yosida-Ray). *A linear operator* $(\mathrm{A}, \mathrm{D}(\mathrm{A}))$ *on* $\mathrm{C}_\infty(\mathbb{R}^n;\mathbb{R})$, $\mathrm{D}(\mathrm{A}) \subset \mathrm{C}_\infty(\mathbb{R}^n;\mathbb{R})$, *is closable and its closure is the generator of a Feller semigroup if and only if the three following conditions are fulfilled:*

(*i*) $\mathrm{D}(\mathrm{A}) \subset \mathrm{C}_\infty(\mathbb{R}^n;\mathbb{R})$ *is dense,*

(*ii*) $(\mathrm{A}, \mathrm{D}(\mathrm{A}))$ *satisfies the positive maximum principle,*

(*iii*) $\mathrm{R}(\lambda - \mathrm{A})$ *is dense in* $\mathrm{C}_\infty(\mathbb{R}^n;\mathbb{R})$ *for some* $\lambda > 0$.

Here $\mathrm{R}(\lambda - \mathrm{A})$ *defines the range of the operator* $\lambda \cdot id - \mathrm{A}$.

There exist more general versions of the previous theorem, however, since we will be handling Feller semigroups this form is the best choice for our purpose. In order to prove that a given operator extends to a generator of a Feller semigroup we have to verify conditions (i)-(iii) in the Hille-Yosida-Ray theorem. Courrège's theorem helps tackling (ii), (i) is a question of starting with a good domain. Condition (iii) is crucial, it requires to solve the equation $(\lambda - \mathrm{A})\mathrm{u} = \mathrm{f}$ for f in a dense set C_∞. For certain operators we may use Hoh's symbolic calculus for this problem. The idea is to reduce manipulation of (pseudo-differential) operators to calculations on the level of symbols, i.e. functions. In our presentation we follow §2.4 of [15] but also refer to [11] and [12] by W.Hoh.

Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous negative definite function with Lévy-Khinchin representation

$$\psi(\xi) = c + \mathrm{Q}(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} (1 - \cos x \cdot \xi)\nu(\mathrm{d}x).$$

Here $c \geq 0$ is a non-negative constant, Q a symmetric positive semidefinite quadratic form and let ν be a measure whose absolute moments for $2 \leq l \leq k$ we assume to exist, i.e.

$$\mathrm{M}_l := \int_{\mathbb{R}^n\setminus\{0\}} |x|^l \ \nu(\mathrm{d}x) < \infty, \quad 2 \leq l \leq k. \tag{1.13}$$

Then we know that ψ is of class $\mathrm{C}^k(\mathbb{R}^n;\mathbb{R})$. Moreover, for $a \in \mathbb{N}_0^n, |\alpha| \leq k$ the estimates

$$\left|\partial_\xi^\alpha \psi(\xi)\right| \leq c_{|\alpha|} \cdot \begin{cases} \psi(\xi), & \alpha \ \ = 0 \\ \psi^{\frac{1}{2}}(\xi), & |\alpha| \ = 1 \\ 1, & |\alpha| \ = 2 \end{cases},$$

hold for a constant $c_{|\alpha|}$, only depending on $|\alpha|$.

Lemma 1.3.11. *Let the negative definite function ψ satisfy (1.13) for all $k \in \mathbb{N}$. Then for all $m \in \mathbb{R}$ and all $\alpha \in \mathbb{N}_0^n$ we have the estimate*

$$\left|\partial_\xi^\alpha (1 + \psi(\xi))^{\frac{m}{2}}\right| \leq c_{|\alpha|} \cdot (1 + \psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}} \tag{1.14}$$

Here and in the following we use the function $\rho : \mathbb{N}_0 \Longrightarrow \mathbb{N}_0, \rho(k) = k \wedge 2$.

Definition 1.3.12. *The class Λ is the set of continuous negative definite functions $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying (1.14) for $m = 2$ and all $\alpha \in \mathbb{N}_0^n$.*

Definition 1.3.13. *For $m \in \mathbb{R}$ and $\psi \in \Lambda$ a function $\mathrm{q} \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C})$ is called a symbol of the class $S_\rho^{m,\psi}(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}_0^n$ there are constants $c_{\alpha,\beta} \geq 0$ such that*

$$\left|\partial_\xi^\alpha \partial_x^\beta \mathrm{q}(x,\xi)\right| \leq c_{\alpha,\beta} \cdot (1 + \psi(\xi))^{\frac{m-\rho(|\alpha|)}{2}} \tag{1.15}$$

holds for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. The order of the symbol is m. If instead of (1.15) q satisfies the weaker inequality

$$\left|\partial_\xi^\alpha \partial_x^\beta \mathrm{q}(x,\xi)\right| \leq \tilde{c}_{\alpha,\beta} \cdot (1 + \psi(\xi))^{\frac{m}{2}},$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, $x, \xi \in \mathbb{R}^n$, *and some constants* $\tilde{c}_{\alpha,\beta} \geq 0$, *then* q *is called a symbol of the class* $\mathrm{S}_0^{m,\psi}$. *It is obvious, that* $\mathrm{S}_\rho^{m,\psi} \subset \mathrm{S}^{m,\psi}$.

Now, we extend the definition of pseudo-differential operators and introduce certain operator classes:

Definition 1.3.14. *Let* $\mathrm{q} \in \mathrm{S}_\rho^{m,\psi}$ *or* $\mathrm{q} \in \mathrm{S}_0^{m,\psi}$. *On* $\mathrm{S}(\mathbb{R}^n)$ *we define the pseudo-differential operator* $\mathrm{q}(x, \mathrm{D})$ *by*

$$\mathrm{q}(x, \mathrm{D})\mathrm{u}(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{q}(x, \xi)\hat{\mathrm{u}}(\xi)\, \mathrm{d}\xi.$$

The classes of these operators are denoted by $\Psi_\rho^{m,\psi}(\mathbb{R}^n)$ *and* $\Psi_0^{m,\psi}(\mathbb{R}^n)$, *respectively.*

Theorem 1.3.15. *The operator* $\mathrm{q}(x, \mathrm{D}) \in \Psi_0^{m,\psi}$ *maps* $\mathcal{S}(\mathbb{R}^n)$ *continuously into itself.*

Further we need to introduce double symbols:

Definition 1.3.16. *For* $\psi \in \Lambda$ *and* $m, m' \in \mathbb{R}$ *the class* $\mathrm{S}_0^{m,m',\psi}(\mathbb{R}^n)$ *of double symbols of order m and m' consists of all* C^∞*-functions* $\mathrm{q} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ *satisfying*

$$\left|\partial_\xi^\alpha \partial_x^\beta \partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} \mathrm{q}(x, \xi; x', \xi')\right| \leq c_{\alpha,\beta,\alpha',\beta'} (1 + \psi(\xi))^{\frac{m}{2}} (1 + \psi(\xi))^{\frac{m'}{2}},$$

for all $\alpha, \beta, \alpha', \beta' \in \mathbb{N}_0^n$. *For* $\mathrm{q} \in \mathrm{S}_0^{m,m',\psi}(\mathbb{R}^n)$ *we define on* $\mathcal{S}(\mathbb{R}^n)$ *the operator*

$$\begin{aligned} &\mathrm{q}(x, \mathrm{D}_x; x', \mathrm{D}_{x'})\mathrm{u}(x) \\ &\quad = (2\pi)^{-\frac{3n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x-x')\cdot\xi + \mathrm{i}x'\cdot\xi'} q(x, \xi; x', \xi')\hat{\mathrm{u}}(\xi')\mathrm{d}\xi' \mathrm{d}x' \mathrm{d}\xi. \quad (1.16) \end{aligned}$$

Theorem 1.3.17. *Let* $\psi \in \Lambda$ *and* $\mathrm{q} \in \mathrm{S}_0^{m,m',\psi}(\mathbb{R}^n)$. *Then the iterated integral (1.16) exists for* $\mathrm{u} \in \mathcal{S}(\mathbb{R}^n)$ *and defines a pseudo-differential operator in the class* $\Psi_0^{m+m',\psi}(\mathbb{R}^n)$. *Moreover,*

$$\mathrm{q_L}(x, \xi) := \ Os - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\eta} \mathrm{q}(x, \xi + \eta, x + y, \xi) \ \mathrm{d}y\, \mathrm{d}\eta \qquad (1.17)$$

is a symbol in $\mathrm{S}_0^{m+m',\psi}(\mathbb{R}^n)$ *and*

$$\mathrm{q}(x,\mathrm{D}_x;x',\mathrm{D}_{x'})\mathrm{u} = \mathrm{q}_{\mathrm{L}}(x,\mathrm{D}_x)\mathrm{u}.$$

The symbol q_{L} *is called the simplified symbol of* $\mathrm{q}(x,\xi;x',\xi')$.

Remark. *The integral in (1.17) is an oscillatory integral which in this case is defined by:*

$$\lim_{\varepsilon\longrightarrow 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}y\cdot\eta}\chi(\varepsilon y,\varepsilon\eta)\mathrm{q}(x,\xi+\eta,x+y,\xi)\ \mathrm{d}y\,\mathrm{d}\eta,$$

where $\chi\in\mathcal{S}(\mathbb{R}^n,\mathbb{R}^n)$ *and* $\chi(0,0)=1$.

One important consequence of Theorem 1.3.17 is that for $\mathrm{q}_j\in\mathrm{S}_0^{m_j,\psi}(\mathbb{R}^n)$, $j=1,2$, we know that $\mathrm{q}_1(x,\mathrm{D})\circ\mathrm{q}_2(x,\mathrm{D})\in\Psi_0^{m_1,m_2,\psi}(\mathbb{R}^n)$ holds.

Lemma 1.3.18. *For* $\psi\in\Lambda$ *and* $\mathrm{q}\in\mathrm{S}_0^{m,m',\psi}(\mathbb{R}^n)$ *such that*

$$\partial_\xi^\alpha\mathrm{q}(x,\xi;x',\xi')\in\mathrm{S}_0^{m-\rho(|\alpha|),m',\psi}(\mathbb{R}^n),\quad \textit{for all } \alpha\in\mathbb{N}_0^n,$$

it holds

$$\mathrm{q}_{\mathrm{L}}(x,\xi)-\sum_{|\alpha|<N}\frac{1}{\alpha!}\mathrm{q}_\alpha(x,\xi)\in\mathrm{S}_0^{m+m'-\rho(N),\psi}(\mathbb{R}^n);$$

here q_{L} *denotes the simplified symbol of* q *and*

$$\mathrm{q}_\alpha(x,\xi)=\mathrm{D}_{x'}^\alpha\partial_\xi^\alpha\mathrm{q}(x,\xi;x',\xi')\big|_{\substack{x'=x\\ \xi'=\xi}}\in\mathrm{S}_0^{m+m'-\rho(|\alpha|),\psi}(\mathbb{R}^n).$$

Finally we want to point out that under certain conditions on the symbol of the operator $-\mathrm{q}(x,\mathrm{D})$, the operator extends to the generator of a Feller semigroup. For the proof of the following theorem we refer to Chapter 2.6 in [15].

Theorem 1.3.19. *Let* $\psi:\mathbb{R}^n\longrightarrow\mathbb{R}$ *be a continuous negative definite function in the class* Λ *which satisfies a minimum growth condition, i.e.*

$$\psi(\xi)\geq c|\xi|^r$$

for some $c, r > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi|$ large. Furthermore, let $\mathrm{q}(x,\xi)$ be a continuous negative definite symbol of class $\mathrm{S}_\rho^{2,\psi}(\mathbb{R}^n)$ and

$$\mathrm{q}(x,\xi) \geq \delta\,(1+\psi(\xi))\,,$$

for some $\delta > 0$ and all $\xi \in \mathbb{R}^n$, $|\xi|$ large.
Then $-\mathrm{q}(x,\mathrm{D})$ with domain $\mathrm{C}_0^\infty(\mathbb{R}^n)$ is closable in C_∞ and the closure is the generator of a Feller semigroup.

1.4 Other Preliminaries

This final preliminary section is a collection of a few results which cannot be assigned any of the previous sections.
Our first result is about functional equations. From Theorem 5.3, page 216 of [1] we deduce:

Lemma 1.4.1. *The general continuous nonvanishing solution of*

$$\mathrm{f}(x)\cdot\mathrm{f}(y) = \mathrm{f}(x+y), \tag{1.18}$$

for $x, y \in \mathbb{R}^n$, is

$$\mathrm{f}(x) = \mathrm{e}^{c\cdot x},$$

for an arbitrary constant vector $c \in \mathbb{C}^n$.

The following result is taken from page 146 in [9].

Lemma 1.4.2. *The Laplace transform of*

$$\mathrm{f}(q) = q^{\nu-1}\mathrm{e}^{\frac{-t^2}{8}},$$

where $Re t > 0$, is given by

$$\mathcal{L}(\mathrm{f})(p) = \Gamma(t)\cdot 2^t\cdot \mathrm{e}^{\frac{p^2}{8}}\cdot \mathrm{D}_{-t}(2p),$$

where D_t *is the parabolic cylinder function given by:*

$$\begin{aligned} D_t(z) &= 2^{\frac{1}{2}t+\frac{1}{4}} \left(\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}t\right)} {}_1\mathrm{F}_1\left(\frac{1}{2}-\frac{1}{2}t; \frac{3}{2}; \frac{1}{2}z^2\right) z^{\frac{1}{4}} \mathrm{e}^{-\frac{1}{4}z^2} \right. \\ &\qquad \left. + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}t\right)} {}_1\mathrm{F}_1\left(-\frac{1}{2}t; \frac{1}{2}; \frac{1}{2}z^2\right) z^{-\frac{1}{4}} \mathrm{e}^{-\frac{1}{4}z^2} \right), \end{aligned}$$

here ${}_1\mathrm{F}_1$ *denotes the confluent hypergeometric function, see below.*

Since we will need it in another context, we give the definition of the generalised hyperbolic function with indices m and n. The confluent hypergeometric function is a generalised hypergeometric function with indices $m = 1$, $n = 1$.

Definition 1.4.3. *The generalised hypergeometric function is given by:*

$$ {}_m\mathrm{F}_n(a_1, \ldots, a_m; b_1, \ldots, b_n; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{(b_1)_k \cdots (b_n)_k} \frac{z^k}{k!}, $$

with $(a)_k = a \cdot (a+1) \cdot \ldots \cdot (a+k-1)$ *for* $k \in \mathbb{N}$.

Chapter 2

N-Parameter Semigroups of Operators

2.1 N-Parameter Convolution Semigroups

In this section we introduce N-parameter convolution semigroups of measures which will be used to construct examples of N-parameter operator semigroups. Further we define subordination of multi-parameter semigroups of operators. Many of our statements are quite analogous to the one-parameter case and in our presentation we often follow N. Jacob [14], Chapter 3.6., where the one-parameter case is treated. As source for the multi-parameter case we refer to D. Khoshnevisan [16].

Definition 2.1.1. *By definition an* ***N-parameter convolution semigroup*** $(\mu_t)_{t\succeq 0}, t \in \mathbb{R}^N_+$, *on* $\mathbb{R}^n$ *is a family of sub-probability measures satisfying for all* $s, t \succeq 0$:

$$\begin{aligned} &(i) && \mu_t(\mathbb{R}^n) \leq 1, \\ &(ii) && \mu_s * \mu_t = \mu_{s+t}, \\ & && \textit{and } \mu_0 = \varepsilon_0, \\ &(iii) && \mu_t \longrightarrow \varepsilon_0, \end{aligned}$$

where the convergence in (iii) is meant in the vague sense and for $t \longrightarrow 0$.

Example 2.1.2. *Let $(\mu_s)_{s\geq 0}$ and $(\nu_t)_{t\geq 0}$ be two one-parameter convolution semigroups of sub-probability measures on $\mathbb{R}^n$, then*

$$\eta_{s,t} := \mu_s \otimes \nu_t \qquad \text{for all } s,t \geq 0$$

defines a two-parameter convolution semigroup on $\mathbb{R}^{2n}$.

Proof. Obviously $\eta_{s,t}(\mathbb{R}^{2n}) \leq 1$ is fulfilled. Moreover, for $\xi = (\xi_1, \xi_2)$ and $\xi_1, \xi_2 \in \mathbb{R}^n$ as well as arbitrary $s_1, s_2, t_1, t_2 \in \mathbb{R}_+$ we find

$$\begin{aligned}
\mathrm{F}(\eta_{s_1,t_1} * \eta_{s_2,t_2})(\xi) &= (2\pi)^n \hat{\eta}_{s_1,t_1}(\xi)\hat{\eta}_{s_2,t_2}(\xi) \\
&= (2\pi)^n \hat{\mu}_{s_1}(\xi_1)\hat{\nu}_{t_1}(\xi_2)\hat{\mu}_{s_2}(\xi_1)\hat{\nu}_{t_2}(\xi_2) \\
&= (\mu_{s_1} * \mu_{s_2})^\wedge(\xi_1) \cdot (\nu_{t_1} * \nu_{t_2})^\wedge(\xi_1) \\
&= (\mu_{s_1+s_2} \otimes \nu_{t_1+t_2})^\wedge(\xi_1, \xi_2) \\
&= \hat{\eta}_{s_1+s_2,t_1+t_2}(\xi),
\end{aligned}$$

where we applied (1.3) and (1.4) of Theorem 1.2.5. Moreover, $\eta_{0,0} = \varepsilon_0$, thus, the second property (ii) of a convolution semigroup is proven. It remains to prove, that $\eta_{s,t} \longrightarrow \varepsilon_0$ vaguely as $s,t \longrightarrow 0$. For this let $\phi^{(1)}$ and $\phi^{(2)}$ be elements of $\mathrm{C}_0(\mathbb{R}^n)$. For $\phi(x) := \phi^{(1)}(x_1) \cdot \phi^{(2)}(x_2)$ with $x = (x_1, x_2) \in \mathbb{R}^{2n}$, $x_1, x_2 \in \mathbb{R}^n$, we obtain

$$\int_{\mathbb{R}^{2n}} \phi \, d\eta_{s,t} = \int_{\mathbb{R}^n} \phi^{(1)} d\mu_s \cdot \int_{\mathbb{R}^n} \phi^{(2)} d\nu_t. \tag{2.1}$$

Since $(\mu_s)_{s\geq 0}$ and $(\nu_t)_{t\geq 0}$ are convolution semigroups on $\mathbb{R}^n$, μ_s and ν_t converge vaguely to ε_0 as s and t tend to zero, respectively, we find that the product of the integrals on the right hand side of (2.1) converges to $\phi^{(1)}(0) \cdot \phi^{(2)}(0)$, respectively. Thus, the integral on the left hand side converges to $\phi(0)$ as $s,t \longrightarrow 0$. Using the density of $\mathrm{C}_0(\mathbb{R}^n) \otimes \mathrm{C}_0(\mathbb{R}^n)$ in $\mathrm{C}_0(\mathbb{R}^{2n})$ we obtain the result. $\square$

Remark. *For convolution semigroups $(\mu_s)_{s\geq 0}$ on $\mathbb{R}^{n_1}$ and $(\nu_t)_{t\geq 0}$ on $\mathbb{R}^{n_2}$, $n_1, n_2 \in \mathbb{N}$, the product $(\mu_s \otimes \nu_t)_{s,t}$ is a two-parameter convolution semigroup $\eta_{s,t}$ on $\mathbb{R}^{n_1+n_2}$.*

Example 2.1.3. ***A.*** *Let* $(\mu_s^{(1)})_{s\geq 0}$ *and* $(\mu_t^{(2)})_{t\geq 0}$ *be Brownian semigroups on* $\mathbb{R}$, *then their product* $(\mu_s^{(1)}\otimes\mu_t^{(2)})_{s,t\geq 0}$ *is given by*

$$\begin{aligned}\left(\mu_s^{(1)}\otimes\mu_t^{(2)}\right)(\mathrm{d}x_1,\mathrm{d}x_2) &= \mu_s^{(1)}(\mathrm{d}x_1)\cdot\mu_t^{(2)}(\mathrm{d}x_2)\\ &= \frac{1}{(4\pi s)^{1/2}}\mathrm{e}^{-\frac{x_1^2}{4s}}\mathrm{d}x_1\cdot\frac{1}{(4\pi t)^{1/2}}\mathrm{e}^{-\frac{x_2^2}{4t}}\mathrm{d}x_2\\ &= \frac{1}{4\pi(st)^{1/2}}\mathrm{e}^{-\frac{x_1^2}{4s}-\frac{x_2^2}{4t}}\,\mathrm{d}x_1\,\mathrm{d}x_2,\end{aligned}$$

for all $s,t\geq 0$.

B. *Now, let* $(\mu_s^{(1)})_{s\geq 0}$ *and* $(\mu_t^{(2)})_{t\geq 0}$ *be* Γ*-semigroups, their product is given by*

$$\begin{aligned}\left(\mu_s^{(1)}\otimes\mu_t^{(2)}\right)(\mathrm{d}x_1,\mathrm{d}x_2) &= \mu_s^{(1)}(\mathrm{d}x_1)\cdot\mu_t^{(2)}(\mathrm{d}x_2)\\ &= \chi_{(0,\infty)}(x_1)\cdot\frac{1}{\Gamma(s)}x^{s-1}\mathrm{e}^{-x_1}\,\mathrm{d}x_1\\ &\quad\cdot\chi_{(0,\infty)}(x_2)\cdot\frac{1}{\Gamma(t)}x^{t-1}\mathrm{e}^{-x_2}\,\mathrm{d}x_2\\ &= \chi_{(0,\infty)^2}(x_1,x_2)\cdot\frac{1}{\Gamma(s+t)}x^{s+t-2}\mathrm{e}^{-x_1-x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2,\end{aligned}$$

for all $s,t\geq 0$.

Lemma 2.1.4. *Let* $(\mu_t)_{t\in\mathbb{R}_+^N}$ *be an N-parameter convolution semigroup on* $\mathbb{R}^n$. *Then the mapping* $t\mapsto\mu_t$ *is continuous at* 0 *with respect to the Bernoulli topology.*

Proof. For $\phi\in C_0(\mathbb{R}^n)$, $0\leq\phi\leq 1$ and $\phi(0)=1$, we find by Definition 2.1.1

$$1=\phi(0)=\lim_{|t|\to 0}\int_{\mathbb{R}^n}\phi\,\mathrm{d}\mu_t\leq\liminf_{|t|\to 0}\mu_t(\mathbb{R}^n)\leq\limsup_{|t|\to 0}\mu_t(\mathbb{R}^n)\leq 1,$$

hence,

$$\lim_{|t|\to 0}\mu_t(\mathbb{R}^n)=\varepsilon_0(\mathbb{R}^n),$$

and since now vague convergence to ε_0 implies weak convergence to ε_0 for $t\longrightarrow 0$, see Theorem 1.2.10, the lemma is proved. ☐

Lemma 2.1.5. *Let* $(\mu_t)_{t\succeq 0}$ *be an N-parameter convolution semigroup on* $\mathbb{R}^n$. *Then the mapping* $t\longmapsto\mu_t$ *is continuous from* $\mathbb{R}_+^N$ *to* $\mathcal{M}_b^+(\mathbb{R}^n)$ *equipped with the Bernoulli topology.*

Proof. For $s, t \in \mathbb{R}^N_+$ and $\xi \in \mathbb{R}^n$ we get

$$\begin{aligned}
|\hat{\mu}_s(\xi) - \hat{\mu}_t(\xi)| &= |\hat{\mu}_{s_1,\dots,s_N}(\xi) - \hat{\mu}_{t_1,s_2,\dots,s_N}(\xi) - \\
&\quad \hat{\mu}_{t_1,t_2,s_3,\dots,s_N}(\xi) + \hat{\mu}_{t_s,t_2,s_3,\dots,s_N}(\xi) - \dots \\
&\quad \dots + \hat{\mu}_{t_1,\dots,t_{N-1},s_N}(\xi) - \hat{\mu}_{t_1,\dots,t_N}(\xi)| \\
&\leq |\hat{\mu}_{s_1\wedge t_1,s_2,\dots,s_N}(\xi)| \left|\hat{\mu}_{|t_1-s_1|,0,\dots,0}(\xi) - (2\pi)^{-\frac{n}{2}}\right| + \dots \\
&\quad \dots + \left|\hat{\mu}_{t_1,\dots,t_{N-1},s_N\wedge t_N}(\xi)\right| \left|\hat{\mu}_{0,\dots,0,|t_N-s_N|} - (2\pi)^{-\frac{n}{2}}\right| \qquad (2.2)
\end{aligned}$$

The right hand side of (2.2) tends to zero as $|t-s|$ tends to zero, moreover, the mapping $\mu \longmapsto \hat{\mu}$ is bicontinuous, compare Theorem 1.2.4, hence the lemma is proved. □

Theorem 2.1.6. *For an N-parameter convolution semigroup $(\mu_t)_{t\in\mathbb{R}^N_+}$ on $\mathbb{R}^n$ there exist continuous negative definite functions $\psi_1, \psi_2, \dots, \psi_N : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that*

$$\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t_1\psi_1(\xi)-\dots-t_N\psi_N(\xi)} \qquad (2.3)$$

holds for all $\xi \in \mathbb{R}^n$ and $t \succeq 0$.

Proof. Let $s, t \in \mathbb{R}^N_+$ then

$$\mu_s * \mu_t = \mu_{s+t},$$

hence, by the convolution theorem

$$(2\pi)^{-\frac{n}{2}} \hat{\mu}_s(\xi) \cdot \hat{\mu}_t(\xi) = \hat{\mu}_{s+t}(\xi). \qquad (2.4)$$

We define for each $\xi \in \mathbb{R}^n$ the function

$$\phi_\xi(t) := (2\pi)^{\frac{n}{2}} \hat{\mu}_t(\xi), \qquad t \succeq 0,$$

which, by the previous Lemma 2.1.5, is continuous in t and by (2.4) it holds:

$$\phi_\xi(s) \cdot \phi_\xi(t) = \phi_\xi(s+t), \qquad s, t \succeq 0,$$

hence ϕ_ξ fulfils a generalised Cauchy Functional Equation. By Lemma 1.4.1 it has a representation of the form:

$$\phi_\xi(t) = e^{-t_1\psi_1(\xi)-\dots-t_N\psi_N(\xi)},$$

with complex numbers $\psi_1(\xi), \ldots, \psi_N(\xi)$ depending only on ξ.
It remains to prove that $\psi_1, \ldots, \psi_N$ are continuous negative definite functions in $\xi \in \mathbb{R}^n$. For this let $j \in \{1, \ldots, N\}$ and $t_j^* = (0, \ldots, t_j, \ldots, 0)$, $t_j \in \mathbb{R}_+$. We find that $(\mu_{t_j^*})_{t_j \geq 0}$ is a one-parameter convolution semigroup and

$$\hat{\mu}_{t_j^*}(\xi) = (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-t_j \psi_j(\xi)},$$

which is the Fourier transform of a probability measure, hence by Schoenberg's theorem a continuous positive definite function in ξ, herewith ψ_j is a continuous negative definite function for all $j = 1, \ldots, N$ and the theorem is proved. □

Remark. ***A.*** *An N-parameter convolution semigroup $(\mu_t)_{t \in \mathbb{R}^N}$ has a representation as a convolution of N one-parameter semigroups. By definition*

$$\mu_t = \mu_{t_1 \cdot e_1} * \mu_{t_2 \cdot e_2} * \ldots * \mu_{t_N \cdot e_N}$$

(remember e_j denotes the j-th unit vector). Now upon defining for all $j \in (1, \ldots, N)$ the one-parameter semigroup

$$\mu_{t_j}^{(j)} := \mu_{t_j \cdot e_j}$$

we arrive at

$$\mu_t = \mu_{t_1}^{(1)} * \mu_{t_2}^{(2)} * \ldots * \mu_{t_N}^{(N)}. \tag{2.5}$$

Conversely, given N one-parameter convolution semigroups $(\mu_{t_j})_{t_j \geq 0}$, $j = 1, \ldots, N$, equation (2.5) defines an N-parameter convolution semigroup. In terms of the Fourier transform, given arbitrary continuous negative definite functions $\psi_1, \ldots, \psi_N$ attaining value zero at the origin an N-parameter convolution semigroup is defined by (2.3). ***B.*** *Finally, we show that the product of convolution semigroups can be expressed by the convolution of convolution semigroups. Let $(\mu_{t_1}^{(1)})_{t_1 \geq 0}, \ldots, (\mu_{t_n}^{(n)})_{t_n \geq 0}$ be one-parameter convolution semigroups on $\mathbb{R}$ and define the one-parameter convolution semigroups $(\tilde{\mu}_{t_1}^{(1)})_{t_1 \geq 0}, \ldots, (\tilde{\mu}_{t_n}^{(n)})_{t_n \geq 0}$ on $\mathbb{R}^n$ by*

$$\tilde{\mu}_{t_j}^{(j)}(\mathrm{d}x) := \mu_{t_j}^{(j)}(\mathrm{d}x_j),$$

for all $t_j \geq 0$ $x \in \mathbb{R}^n$ *and* $j \in \{1, \dots, n\}$, *then it holds*

$$\mu_{t_1}^{(1)} \otimes \mu_{t_2}^{(2)} \otimes \dots \otimes \mu_{t_n}^{(n)} = \tilde{\mu}_{t_1}^{(1)} * \tilde{\mu}_{t_2}^{(2)} * \dots * \tilde{\mu}_{t_n}^{(n)}.$$

Next we want to extend subordination to multiparameter convolution semigroups.

Theorem 2.1.7. *Let* $(\mu_s)_{s \in \mathbb{R}_+^N}$ *be an arbitrary N-parameter convolution semigroup on* $\mathbb{R}^n$ *and let* $(\eta_t)_{t \in \mathbb{R}_+^M}$ *be an M-parameter convolution semigroup supported on* $\mathbb{R}_+^N$. *Then the integral*

$$\nu_t = \int_{[0,\infty)^N} \mu_s \eta_t(\mathrm{d}s) \tag{2.6}$$

defines an M-parameter convolution semigroup on $\mathbb{R}^n$.

Proof. Since for $t \in \mathbb{R}_+^M$ and $\phi \in \mathrm{C}_0(\mathbb{R}^n)$ the mapping

$$\phi \longmapsto \int_{[0,\infty)^N} \int_{\mathbb{R}^n} \phi(x)\, \mu_s(\mathrm{d}x) \eta_t(\mathrm{d}s)$$

is positive and linear, there exists a measure ν_t on $\mathbb{R}^n$ such that

$$\nu_t = \int_{[0,\infty)^N} \mu_s \eta_t(\mathrm{d}s) \qquad \text{vaguely.}$$

Obviously, ν_t is a (sub-)probability measure for all $t \in \mathbb{R}_+^M$ and we find for $t_1, t_2 \in \mathbb{R}_+^M$

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(x)\, (\nu_{t_1} * \nu_{t_2})(\mathrm{d}x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(x+y)\, \nu_{t_1}(\mathrm{d}x) \nu_{t_2}(\mathrm{d}y) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \phi(x+y)\, \mu_p(\mathrm{d}x) \mu_r(\mathrm{d}y) \eta_{t_1}(\mathrm{d}p) \eta_{t_2}(\mathrm{d}r) \\
&= \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}^n} \phi(x)\, (\mu_p * \mu_r)(\mathrm{d}x) \eta_{t_1}(\mathrm{d}p) \eta_{t_2}(\mathrm{d}r) \\
&= \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} \int_{\mathbb{R}^n} \phi(x)\, \mu_{p+r}(\mathrm{d}x) \eta_{t_1}(\mathrm{d}p) \eta_{t_2}(\mathrm{d}r) \\
&= \int_{\mathbb{R}_+^N} \int_{\mathbb{R}^n} \phi(x)\, \mu_p(\mathrm{d}x) \eta_{t_1+t_2}(\mathrm{d}p) \\
&= \int_{\mathbb{R}^n} \phi(x) \nu_{t_1+t_2}(\mathrm{d}x),
\end{aligned}$$

thus $(\nu_t)_{t\in\mathbb{R}_+^M}$ fulfills the semigroup property. Moreover, $(\mu_s)_{s\in\mathbb{R}_+^N}$ and $(\eta_t)_{t\in\mathbb{R}_+^M}$ converge vaguely to ε_0 for $s,t\geq 0$ and so does $(\nu_t)_{t\in\mathbb{R}_+^M}$, hence is an M-parameter convolution semigroup. $\square$

The semigroup we constructed in Theorem 2.1.7 leads to

Definition 2.1.8. *The M-parameter convolution semigroup defined in 2.6 is called the* ***multiparameter subordinate convolution semigroup*** *of* $(\mu_s)_{s\in\mathbb{R}_+^N}$ *with respect to* $(\eta_t)_{t\in\mathbb{R}_+^M}$.

The following example illustrates subordination.

Example 2.1.9. *Let* $(\mu_s)_{s\in\mathbb{R}_+^2}$ *be a* ***Product Brownian semigroup****, i.e.*

$$\mu_{s_1,s_2}(\mathrm{d}x)=\frac{1}{(4\pi s_1)^{1/2}}\mathrm{e}^{-\frac{x_1^2}{4s_1}}\cdot\frac{1}{(4\pi s_2)^{1/2}}\mathrm{e}^{-\frac{x_2^2}{4s_2}}\,\mathrm{d}x_1\mathrm{d}x_2.$$

We subordinate $(\mu_s)_{s\in\mathbb{R}_+^2}$ *with the* ***Product*** Γ***-semigroup*** $(\eta_t)_{t\in\mathbb{R}_+^2}$ *which is given by*

$$\eta_{t_1,t_2}(\mathrm{d}x)=\chi_{(0,\infty)}(x_1)\cdot\frac{1}{\Gamma(t_1)}x_1^{t_1-1}\mathrm{e}^{-x_1}\cdot\chi_{(0,\infty)}(x_2)\frac{1}{\Gamma(t_2)}x_2^{t_2-1}\mathrm{e}^{x_2}\,\mathrm{d}x_1\mathrm{d}x_2.$$

We find

$$\begin{aligned}\int_{\mathbb{R}_+^2}\mu_s\,\eta_t(\mathrm{d}s) &= \int_{\mathbb{R}_+^2}\frac{1}{4\pi(s_1s_2)^{1/2}}\mathrm{e}^{-\frac{x_1^2}{4s_1}-\frac{x_2^2}{4s_2}}\cdot\frac{1}{\Gamma(t_1)\Gamma(t_2)}s_1^{t_1-1}s_2^{t_2-1}\mathrm{e}^{-s_1-s_2}\,\mathrm{d}s\\ &= \frac{1}{\Gamma(t_1)\Gamma(t_2)}\int_0^\infty\frac{1}{(4\pi s_1)^{1/2}}\mathrm{e}^{-\frac{x_1^2}{4s_1}}s_1^{t_1-1}\mathrm{e}^{-s_1}\,\mathrm{d}s_1\\ &\quad\cdot\int_0^\infty\frac{1}{(4\pi s_2)^{1/2}}\mathrm{e}^{-\frac{x_2^2}{4s_2}}s_2^{t_2-1}\mathrm{e}^{-s_2}\,\mathrm{d}s_2\\ &= \frac{1}{\pi\Gamma(t_1)\Gamma(t_2)}\frac{x_1^2}{4}^{\frac{t_1}{2}-\frac{1}{4}}\mathrm{K}_{\frac{t_1}{2}-\frac{1}{2}}(x_1)\cdot\frac{x_2^2}{4}^{\frac{t_2}{2}-\frac{1}{4}}\mathrm{K}_{\frac{t_2}{2}-\frac{1}{2}}(x_2).\end{aligned}$$

The latter equality follows with (17) on page 313 in [9] and K *denotes the modified Bessel function of the third kind.*

2.2 N-Parameter Operator Semigroups

Many properties of N-parameter operator semigroups are similar to those of one-parameter semigroups, which have been discussed in many monographs, we refer to [14]. When extending the notion of strongly continuous operator semigroups to the N-parameter case we encounter one main difficulty. This is the extension of the notion of a generator. We tackle this problem to a certain degree by investigating the differential equation which is associated to the generator of a one-parameter semigroup and extending it to a partial differential equation for the N-parameter case.

Let $(X, \|.\|_X)$ be a real or complex Banach space.

Definition 2.2.1. ***A.*** *An N-parameter family* $(\mathrm{T}_t)_{t\succeq 0}$, $t \in \mathbb{R}_+^N$, *of bounded linear operators* $\mathrm{T}_t : X \longrightarrow X$ *is called an* ***N-parameter semigroup of operators****, if*

$$\mathrm{T}_0 = id$$

and

$$\mathrm{T}_{s+t} = \mathrm{T}_s \circ \mathrm{T}_t$$

holds for all $s, t \in \mathbb{R}_+^N$.

B. *We call* $(\mathrm{T}_t)_{t\succeq 0}$ ***strongly continuous*** *if*

$$\lim_{t\to 0} \|\mathrm{T}_t \mathrm{u} - \mathrm{u}\|_X = 0$$

holds for all $x \in X$.

C. *The semigroup* $(\mathrm{T}_t)_{t\succeq 0}$ *is a* ***contraction*** *semigroup, if*

$$\|\mathrm{T}_t\| \leq 1$$

holds for all $t \succeq 0$, *i.e. each operator* T_t *is a contraction. Here* $\|.\|$ *denotes the operator norm* $\|.\|_{X,X}$.

Example 2.2.2. *Let* A *and* B *be bounded operators on* X *such that* $[\mathrm{A}, \mathrm{B}] := \mathrm{AB} - \mathrm{BA} = 0$, *and define for all* $t = (t_1, t_2) \in \mathbb{R}_+^2$

$$\mathrm{T}_t \mathrm{u} := e^{t_1 \mathrm{A}} \circ e^{t_2 \mathrm{B}} \mathrm{u} = \sum_{k_1=0}^{\infty} \frac{(t_1 \mathrm{A})^{k_1}}{k_1!} \circ \sum_{k_2=0}^{\infty} \frac{(t_2 \mathrm{B})^{k_2}}{k_2!} \mathrm{u}. \tag{2.7}$$

For $t \succeq 0$ we find

$$\begin{aligned}\left\|e^{t_1\mathrm{A}}\circ e^{t_2\mathrm{B}}\right\| &\leq \left\|e^{t_1\mathrm{A}}\right\|\cdot\left\|e^{t_2\mathrm{B}}\right\| \leq \sum_{k_1=0}^{\infty}\frac{t_1^{k_1}}{k_1!}\left\|\mathrm{A}^{k_1}\right\|\sum_{k_2=0}^{\infty}\frac{t_2^{k_2}}{k_2!}\left\|\mathrm{B}^{k_2}\right\| &(2.8)\\ &\leq \sum_{k_1=0}^{\infty}\frac{t_1^{k_1}}{k_1!}\left\|\mathrm{A}\right\|^{k_1}\sum_{k_2=0}^{\infty}\frac{t_2^{k_2}}{k_2!}\left\|\mathrm{B}\right\|^{k_2} &(2.9)\\ &= e^{t_1\|\mathrm{A}\|}\cdot e^{t_2\|\mathrm{B}\|}<\infty. &(2.10)\end{aligned}$$

Hence, the sum converges uniformly in X which allows us to change the order of summation, and using $[\mathrm{A},\mathrm{B}]=0$ we obtain the semigroup property of $(\mathrm{T}_t)_{t\succeq 0}$. For $s,t\in\mathbb{R}^2_+$

$$\begin{aligned}\mathrm{T}_s\circ\mathrm{T}_t\mathrm{u} &= e^{s_1\mathrm{A}}\circ e^{s_2\mathrm{B}}\circ e^{t_1\mathrm{A}}\circ e^{t_2\mathrm{B}}\mathrm{u} &(2.11)\\ &= e^{(s_1+t_1)\mathrm{A}}\circ e^{(s_2+t_2)\mathrm{B}}\mathrm{u} &(2.12)\\ &= \mathrm{T}_{s+t}\mathrm{u}, &(2.13)\end{aligned}$$

as well as $e^{0\mathrm{A}}\circ e^{0\mathrm{B}}\mathrm{u}=\mathrm{u}$.
Furthermore, we have uniform continuity of the family $(\mathrm{T}_t)_{t\in\mathbb{R}^2_+}$ as $t\longrightarrow 0$, i.e.

$$\lim_{t\to 0}\left\|e^{t_1\mathrm{A}}\circ e^{t_2\mathrm{B}}-id\right\|=0$$

implying strong continuity, i.e. $\lim_{t\to 0}\left\|e^{t_1\mathrm{A}}\circ e^{t_2\mathrm{B}}u-u\right\|_X=0$. Hence, $(\mathrm{T}_t)_{t\in\mathbb{R}^2_+}$ as defined in (2.7) is a strongly continuous two-parameter semigroup on $(X,\|.\|_X)$. Moreover, if for $t_1,t_2\geq 0$ the operators $\mathrm{e}^{t_1\mathrm{A}}$ and $\mathrm{e}^{t_2\mathrm{B}}$ are contractions, then $(\mathrm{T}_t)_{t\in\mathbb{R}^2_+}$ is a contraction semigroup.

Example 2.2.3. *Let $(\mu_{t_1})_{t_1\geq 0}$ and $(\nu_{t_2})_{t_2\geq 0}$ be two convolution semigroups of probability measures on $\mathbb{R}^n$. On the Banach space $(\mathrm{C}_\infty(\mathbb{R}^{2n}),\|\cdot\|_\infty)$ we define for all $t\in\mathbb{R}^2_+$ the operator*

$$\mathrm{T}_t\mathrm{u}(x):=\int_{\mathbb{R}^{2n}}\mathrm{u}(x-y)(\mu_{t_1}\otimes\nu_{t_2})(\mathrm{d}y). \tag{2.14}$$

We claim $(\mathrm{T}_t)_{t\succeq 0}$ is a strongly continuous contraction semigroup.
First, since $\mathrm{u}\in\mathrm{C}_\infty(\mathbb{R}^{2n})$ is bounded, we find

$$|\mathrm{T}_t\mathrm{u}(x)|\leq\int_{\mathbb{R}^{2n}}|\mathrm{u}(x-y)|(\mu_{t_1}\otimes\nu_{t_2})(\mathrm{d}y)\leq\|\mathrm{u}\|_\infty(\mu_{t_1}\otimes\nu_{t_2})(\mathbb{R}^{2n})$$

but $(\mu_{t_1} \otimes \nu_{t_2})(\mathbb{R}^{2n}) \leq 1$, *which implies*

$$\sup_{x \in \mathbb{R}^{2n}} |\mathrm{T}_t \mathrm{u}(x)| \leq \|\mathrm{u}\|_\infty < \infty, \tag{2.15}$$

i.e. T_t *is defined on* $\mathrm{C}_\infty(\mathbb{R}^{2n})$, *for all* $t \succeq 0$, *and* $\mathrm{T}_t\mathrm{u}$ *is a bounded function. We show that* $\mathrm{T}_t\mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^{2n})$. *In fact, for* $\mathrm{u} \in \mathcal{S}(\mathbb{R}^{2n})$ *we find using Theorem 1.2.5 and Theorem 1.2.13 that*

$$\begin{aligned}
(\mathrm{T}_t\mathrm{u})^\wedge(\xi) &= (2\pi)^n\ \hat{\mathrm{u}}(\xi)\ (\mu_{t_1} \otimes \nu_{t_2})^\wedge(\xi) \\
&= \hat{\mathrm{u}}(\xi)\ (2\pi)^{n/2}\hat{\mu}_{t_1}(\xi_1)\ (2\pi)^{n/2}\hat{\nu}_{t_2}(\xi_2) \\
&= \hat{\mathrm{u}}(\xi)\mathrm{e}^{-t_1\cdot\phi(\xi_1)}\mathrm{e}^{-t_2\psi(\xi_2)},
\end{aligned} \tag{2.16}$$

where $\xi = (\xi_1, \xi_2)$ *with* $\xi_1, \xi_2 \in \mathbb{R}^n$ *and* $\phi, \psi : \mathbb{R}^n \to \mathbb{C}$ *are the continuous negative definite functions associated with the convolution semigroups* $(\mu_{t_1})_{t_1 \geq 0}$ *and* $(\nu_{t_2})_{t_2 \geq 0}$, *respectively. Since* $\hat{\mu}_{t_1} = (2\pi)^{-n/2}\mathrm{e}^{-t_1\phi}$ *and* $\hat{\nu}_{t_2} = (2\pi)^{-n/2}\mathrm{e}^{-t_2\psi}$ *(2.16) implies that* $(\mathrm{T}_t\mathrm{u})^\wedge \in \mathrm{L}^1(\mathbb{R}^{2n})$ *for* $\mathrm{u} \in \mathcal{S}(\mathbb{R}^{2n})$, *and the Riemann-Lebesgue-Lemma, Theorem 1.2.4, implies* $\mathrm{T}_t\mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^{2n})$. *Using the density of* $\mathcal{S}(\mathbb{R}^{2n})$ *in* $\mathrm{C}_\infty(\mathbb{R}^{2n})$ *and (2.15) we find that* T_t *is a contraction on* $\mathrm{C}_\infty(\mathbb{R}^{2n})$ *for all* $t \in \mathbb{R}^{2n}_+$. *Furthermore, we find using Example 2.1.2*

$$\begin{aligned}
\mathrm{T}_s \circ \mathrm{T}_t\mathrm{u}(x) &= \int_{\mathbb{R}^{2n}} \left\{ \int_{\mathbb{R}^{2n}} \mathrm{u}(x - y - z)(\mu_{t_1} \otimes \nu_{t_2})(\mathrm{d}y) \right\} (\mu_{s_1} \otimes \nu_{s_2})(\mathrm{d}z) \\
&= \int_{\mathbb{R}^{2n}} \mathrm{u}(x - z)\big((\mu_{s_1} \otimes \nu_{s_2}) * (\mu_{t_1} \otimes \nu_{t_2})\big)(\mathrm{d}z) \\
&= \int_{\mathbb{R}^{2n}} \mathrm{u}(x - z)(\mu_{s_1+t_1} \otimes \nu_{s_2+t_2})(\mathrm{d}z) \\
&= \mathrm{T}_{s_1+t_1, s_2+t_2}\mathrm{u}(x) \\
&= \mathrm{T}_{s+t}\mathrm{u}(x).
\end{aligned}$$

Obviously, we have $\mathrm{T}_{(0,0)}\mathrm{u} = \mathrm{u}$, *since* $\mu_0 = \varepsilon_0$ *and* $\nu_0 = \varepsilon_0$.
Finally, we prove that $(\mathrm{T}_t)_{t \succeq 0}$ *is strongly continuous for* $t \to 0$. *First, note that any function in* $\mathrm{C}_\infty(\mathbb{R}^{2n})$ *is uniformly continuous. Hence, for* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|\mathrm{u}(x) - \mathrm{u}(x - y)| < \varepsilon \qquad \textit{for all} \quad |x - y| < \delta.$$

The continuity of $(\mu_{t_1} \otimes \nu_{t_2})_{t \succeq 0}$ *with respect to the Bernoulli topology implies, see Lemma 2.1.4, that*

$$\lim_{t \to 0} (\mu_{t_1} \otimes \nu_{t_2}) (\mathrm{B}_\delta^{2n}(0)) = \varepsilon_0(\mathrm{B}_\delta^{2n}(0)) = 1,$$

i.e. it exists $r > 0$ *such that*

$$(\mu_{t_1} \otimes \nu_{t_2}) (\mathrm{B}_\delta^c(0)) < \varepsilon \text{ and } 1 - (\mu_{t_1} \otimes \nu_{t_2}) (\mathbb{R}^{2n}) < \varepsilon \quad \text{for all } t \in [0, r)^2,$$

where $\mathrm{B}_\delta^c(0)$ *denotes the complement of the open ball* $\mathrm{B}_\delta^{2n}(0)$.

Now, we find

$$\begin{aligned}
|\mathrm{T}_t\mathrm{u}(x) - \mathrm{u}(x)| &\leq \left| \int_{\mathbb{R}^{2n}} [\mathrm{u}(x-y) - \mathrm{u}(x)](\mu_{t_1} \otimes \nu_{t_2})(\mathrm{d}y) \right| \\
&\quad + |\mathrm{u}(x)|(1 - (\mu_{t_1} \otimes \nu_{t_2})(\mathbb{R}^{2n})) \\
&\leq \int_{\mathrm{B}_\delta(0)} |\mathrm{u}(x-y) - \mathrm{u}(x)|(\mu_{t_1} \otimes \nu_{t_2})(\mathrm{d}y) \\
&\quad + \int_{\mathrm{B}_\delta^c(0)} |\mathrm{u}(x-y) - \mathrm{u}(x)|(\mu_{t_1} \otimes \nu_{t_2})(\mathrm{d}y) \\
&\quad + \|\mathrm{u}\|_\infty(1 - (\mu_{t_1} \otimes \nu_{t_2})(\mathbb{R}^{2n})) \\
&\leq \varepsilon \cdot (\mu_{t_1} \otimes \nu_{t_2})(\mathrm{B}_\delta(0)) + 2 \cdot \|\mathrm{u}\|_\infty (\mu_{t_1} \otimes \nu_{t_2})(\mathrm{B}_\delta^c(0)) \\
&\quad + \|\mathrm{u}\|_\infty (1 - (\mu_{t_1} \otimes \nu_{t_2})(\mathbb{R}^{2n})) \\
&\leq \varepsilon + 2 \cdot \varepsilon \|\mathrm{u}\|_\infty + \varepsilon \|\mathrm{u}\|_\infty \\
&= \varepsilon(1 + 3 \|\mathrm{u}\|_\infty)
\end{aligned}$$

implying that $(\mathrm{T}_t)_{t \succeq 0}$ *is strongly continuous as* $t \to 0$. *Note also that* $(\mathrm{T}_t)_{t \succeq 0}$ *is positivity preserving, i.e.* $\mathrm{u} \geq 0$ *yields* $\mathrm{T}_t\mathrm{u} \geq 0$. □

Remark. *Clearly, Example 2.2.3 extends easily to the N-parameter case.*

Definition 2.2.4. *Let* $(\mathrm{T}_t)_{t \succeq 0}$ *be a strongly continuous contraction N-parameter semigroup on* $(\mathrm{C}_\infty(\mathbb{R}^n), \|.\|_\infty)$ *which is positivity preserving. Then* $(\mathrm{T}_t)_{t \succeq 0}$ *is called an* ***N-parameter Feller semigroup****.*

Remark. *Since for* $\mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^{2n};\mathbb{R})$ *the function* $\mathrm{T}_t\mathrm{u}$ *defined in Example 2.2.3 by (2.14) is a real-valued function, it follows that* $(\mathrm{T}_t)_{t\succeq 0}$ *is a Feller semigroup.*

Example 2.2.5. *Let* $(\mu_{t_1} \otimes \nu_{t_2})_{t\succeq 0}$, $t = (t_1, t_2)$, *be as in Example 2.2.3. For* $\mathrm{u} \in \mathcal{S}(\mathbb{R}^{2n})$ *we define as before*

$$\mathrm{T}_t\mathrm{u}(x) = \int_{\mathbb{R}^{2n}} \mathrm{u}(x-y)(\mu_{t_1} \otimes \nu_{t_2})(\mathrm{d}y) \tag{2.17}$$

and we obtain for $\xi = (\xi_1, \xi_2)$ *with* $\xi_1, \xi_2 \in \mathbb{R}^n$

$$(\mathrm{T}_t\mathrm{u})^\wedge(\xi) = \mathrm{e}^{-t_1\phi(\xi_1)-t_2\psi(\xi_2)}\hat{\mathrm{u}}(\xi) \tag{2.18}$$

where $\phi, \psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ *are the continuous negative definite functions associated with the one-parameter convolution semigroups* $(\mu_{t_1})_{t_1\geq 0}$ *and* $(\nu_{t_2})_{t_2\geq 0}$, *respectively.*

Now, Plancherel's theorem, see Corollary 3.2.17 in [14], implies

$$\|\mathrm{T}_t\mathrm{u}\|_0 = \|(\mathrm{T}_t\mathrm{u})^\wedge\|_0 \leq \|\mathrm{u}\|_0,$$

where $\|.\|_0$ *denotes the norm in* $\mathrm{L}^2(\mathbb{R}^{2n})$.
Since $\mathcal{S}(\mathbb{R}^{2n})$ *is dense in* $\mathrm{L}^2(\mathbb{R}^{2n})$, *it follows that each of the operators* T_t *has an extension to* $\mathrm{L}^2(\mathbb{R}^{2n})$ *and that these extensions are contractions. We denote this extension once again by* $(\mathrm{T}_t)_{t\succeq 0}$. *Moreover, we find*

$$\begin{aligned} \|\mathrm{T}_t\mathrm{u} - \mathrm{u}\|_0^2 &= \int_{\mathbb{R}^{2n}} |\mathrm{e}^{-t_1\phi(\xi_1)-t_2\psi(\xi_2)}\hat{\mathrm{u}}(\xi) - \hat{\mathrm{u}}(\xi)|^2\mathrm{d}\xi & (2.19)\\ &= \int_{\mathbb{R}^{2n}} |\mathrm{e}^{-t_1\phi(\xi_1)-t_2\psi(\xi_2)} - 1|^2|\hat{\mathrm{u}}(\xi)|^2\mathrm{d}\xi \xrightarrow{|t|\to 0} 0, & (2.20) \end{aligned}$$

implying the strong continuity of $(\mathrm{T}_t)_{t\succeq 0}$ *as* $|t| \to 0$. *From (2.18) it is obvious that* $(\mathrm{T}_t)_{t\succeq 0}$ *is a semigroup, hence, it gives a strongly continuous contraction semigroup on* $\mathrm{L}^2(\mathbb{R}^{2n})$. *Now, let* $\mathrm{u} \in \mathrm{L}^1(\mathbb{R}^{2n}) \cap \mathrm{L}^2(\mathbb{R}^{2n})$, *then (2.17) makes sense as a Lebesgue integral and we find for* $0 \leq \mathrm{u} \leq 1$ *(a.e.) that* $0 \leq \mathrm{T}_t\mathrm{u} \leq 1$ *(a.e.). As before the operator* T_t *maps real-valued functions onto real-valued functions.*

Definition 2.2.6. *A. Let* $(\mathrm{T}_t)_{t\succeq 0}$ *be a strongly continuous N-parameter contraction semigroup on* $\mathrm{L}^p(\mathbb{R}^n;\mathbb{R}), 1 \le p < \infty$. *We call* $(\mathrm{T}_t)_{t\succeq 0}$ *a sub-Markovian semigroup on* L^p*, if for* $u \in \mathrm{L}^p(\mathbb{R}^n;\mathbb{R})$ *such that* $0 \le u \le 1$ *(a.e.) it follows that* $0 \le \mathrm{T}_t\mathrm{u} \le 1$ *(a.e.). B. Let* $(\mathrm{T}_t)_{t\succeq 0}$ *be a strongly continuous contraction semigroup on* $\mathrm{L}^p(\mathbb{R}^n), 1 \le p < \infty$, *or on* $\mathrm{C}_\infty(\mathbb{R}^n)$. *We call* $(\mathrm{T}_t)_{t\succeq 0}$ *symmetric, if for all* $\mathrm{u},\mathrm{v} \in \mathrm{L}^p(\mathbb{R}^n) \cap \mathrm{L}^2(\mathbb{R}^n)$ *or* $\mathrm{u},\mathrm{v} \in \mathrm{C}_\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, *respectively, we have*

$$(\mathrm{T}_t\mathrm{u},\mathrm{v})_0 = (\mathrm{u},\mathrm{T}_t\mathrm{v})_0 .$$

Remark. *The semigroups constructed in Example 2.2.5 are sub-Markovian semigroups on* $\mathrm{L}^2(\mathbb{R}^{2n};\mathbb{R})$. *Furthermore, with the definitions given there, we find*

$$\hat{\mu}_{t_1}(\xi_1) = (2\pi)^{-n/2}\mathrm{e}^{-t_1\phi(\xi_1)}$$

and

$$\hat{\nu}_{t_2}(\xi_2) = (2\pi)^{-n/2}\mathrm{e}^{-t_2\psi(\xi_2)},$$

where $\xi_1,\xi_2 \in \mathbb{R}^n$.
We get for $\mathrm{u},\mathrm{v} \in \mathrm{L}^2(\mathbb{R}^{2n})$ *and real-valued continuous negative definite* ϕ *and* ψ

$$\begin{aligned}
(\mathrm{T}_t\mathrm{u},\mathrm{v})_0 &= ((\mathrm{T}_t\mathrm{u})^\wedge,\hat{\mathrm{v}})_0 && (2.21)\\
&= \int_{\mathbb{R}^{2n}} \mathrm{e}^{-t_1\phi(\xi_1)-t_2\psi(\xi_2)}\hat{\mathrm{u}}(\xi)\overline{\hat{\mathrm{v}}(\xi)}\mathrm{d}\xi && (2.22)\\
&= \int_{\mathbb{R}^{2n}} \hat{\mathrm{u}}(\xi)\overline{\mathrm{e}^{-t_1\phi(\xi_1)-t_2\psi(\xi_2)}\hat{\mathrm{v}}(\xi_2)}\mathrm{d}\xi && (2.23)\\
&= (\mathrm{u},\mathrm{T}_t\mathrm{v})_0 . && (2.24)
\end{aligned}$$

Hence, in this case the semigroups constructed in Example 2.2.5 are symmetric on $\mathrm{L}^2(\mathbb{R}^{2n})$. *Conversely, the same calculation gives that for symmetric semigroups on* $\mathrm{L}^2(\mathbb{R}^{2n})$ *given by (2.17) the continuous negative definite functions* ϕ *and* ψ *must be real-valued.*

We introduce the following notation of marginal semigroups to handle and analyse N-parameter semigroups of operators in a more comfortable way.

Definition 2.2.7. *Let $(\mathrm{T}_t)_{t\succeq 0}$ be an N-parameter semigroup of operators, for $j = 1, \dots, N$ we define the j-th marginal (1-parameter) semigroup $\left(\mathrm{T}^{(j)}_{t_j}\right)_{t_j \geq 0}$ by*

$$\mathrm{T}^{(j)}_{t_j} := \mathrm{T}_{t_j \cdot \mathrm{e}_j} \qquad \text{for all } t_j \geq 0.$$

Remark. *Note that by the very definition of an N-parameter semigroup of operators and its marginal semigroup it follows for $t_i, t_j \geq 0$ and $i \neq j$ that*

$$\mathrm{T}_{t_i \cdot \mathrm{e}_i + t_j \mathrm{e}_j} = \mathrm{T}_{t_i \cdot \mathrm{e}_i} \circ \mathrm{T}_{t_j \cdot \mathrm{e}_j} = \mathrm{T}_{t_j \cdot \mathrm{e}_j} \circ \mathrm{T}_{t_i \cdot \mathrm{e}_i},$$

which yields

$$[\mathrm{T}^{(i)}_{t_i}, \mathrm{T}^{(j)}_{t_j}] := \mathrm{T}^{(i)}_{t_i} \circ \mathrm{T}^{(j)}_{t_j} - \mathrm{T}^{(j)}_{t_j} \circ \mathrm{T}^{(i)}_{t_i} = 0,$$

i.e. the marginal semigroups are mutually commutating.
Moreover for each marginal semigroup $\left(\mathrm{T}^{(j)}_{t_j}\right)_{t_j \geq 0}$ we can define a generator $A^{(j)}$ with domain $\mathrm{D}\left(A^{(j)}\right)$ according to Definition 1.3.5.
In the proof of Lemma 2.2.8 we make use of the following decomposition of $(\mathrm{T}_t)_{t\succeq 0}$ into its marginal semigroups. For $t = (t_1, \dots, t_N)$ we have

$$\mathrm{T}_t = \mathrm{T}^{(1)}_{t_1} \circ \mathrm{T}^{(2)}_{t_2} \circ \ldots \circ \mathrm{T}^{(N)}_{t_N}.$$

Lemma 2.2.8. *Let $(T_t)_{t\succeq 0}$ be a strongly continuous N-parameter semigroup on $(X, \|.\|_X)$. Then there exist constants $\omega \in \mathbb{R}^N_+$ and $M_\omega \geq 1$ such that*

$$\|T_t\|_{X,X} \leq M_\omega \, \mathrm{e}^{\omega \cdot t}$$

Proof. Using the decomposition of $(\mathrm{T}_t)_{t\succeq 0}$ we find

$$\begin{aligned} \|\mathrm{T}_t\| &= \|\mathrm{T}^{(1)}_{t_1} \circ \mathrm{T}^{(2)}_{t_2} \circ \ldots \circ \mathrm{T}^{(N)}_{t_N}\| \\ &\leq \|\mathrm{T}^{(1)}_{t_1}\| \cdot \| T^{(2)}_{t_2}\| \cdot \ldots \cdot \|\mathrm{T}^{(N)}_{t_N}\|, \end{aligned}$$

where $\|.\|$ denotes the operator norm $\|.\|_{\mathrm{X,X}}$. By Lemma 1.3.3 there exist constants $\omega = (\omega_1, \dots, \omega_N) \succeq 0$ and $\mathrm{M}_{\omega_1}, \dots, \mathrm{M}_{\omega_N} \geq 1$ such that

$$\|\mathrm{T}_t\| \leq \mathrm{M}_\omega \mathrm{e}^{\omega \cdot t},$$

where $M_\omega := \mathrm{M}_{\omega_1} \cdot \ldots \cdot \mathrm{M}_{\omega_N}$. □

Corollary 2.2.9. *Let $(T_t)_{t\succeq 0}$ be a strongly continuous N-parameter semigroup on $(X, \|.\|_X)$. For any $u \in X$ the mapping $t \rightarrow \mathrm{T}_t\mathrm{u}$ is continuous from $\mathbb{R}_+^N$ to X.*

Proof. Let $t, h \in \mathbb{R}_+^N$ and $-t \preceq h$ be fixed, then we find using Lemma 2.2.8

$$\begin{aligned}\|\mathrm{T}_{t+h}\mathrm{u} - \mathrm{T}_t\mathrm{u}\|_X &\leq \left\|\mathrm{T}_{t\curlywedge(t+h)}\right\|_{X,X} \left\|\mathrm{T}_{t+h-(t\curlywedge(t+h))}\mathrm{u} - \mathrm{T}_{t-(t\curlywedge(t+h))}\mathrm{u}\right\|_X \\ &\leq \left\|\mathrm{T}_{t\curlywedge(t+h)}\right\|_{X,X} \left(\left\|\mathrm{T}_{t+h-(t\curlywedge(t+h))}\mathrm{u} - \mathrm{u}\right\|_X \right. \\ &\quad \left. + \left\|\mathrm{T}_{t-(t\curlywedge(t+h))}\mathrm{u} - \mathrm{u}\right\|_X\right),\end{aligned}$$

implying the continuity. □

By definition the generator A of a strongly continuous one-parameter semigroup of operators on a Banach space $(X, \|.\|_X)$, see Definition 1.3.5, is given by:

$$\mathrm{Au} := \lim_{t\to 0} \frac{\mathrm{T}_t\mathrm{u} - \mathrm{u}}{t} \qquad \text{as strong limit}$$

with domain

$$\mathrm{D(A)} := \left\{\mathrm{u} \in X : \lim_{t\to 0} \frac{1}{t}(\mathrm{T}_t\mathrm{u} - \mathrm{u}) \text{ exists as strong limit}\right\}.$$

Moreover, a solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{f}(x,t) = \mathrm{Af}(x,t)$$

is given by

$$\mathrm{f}(x,t) = \mathrm{T}_t\mathrm{u}(x)$$

for $u \in \mathrm{D(A)}$.

However, this definition cannot easily be extended to the N-parameter case as we may see by investigating the 'generator' of a 2-paramater semigroup of operators, starting with an analysis of the derivatives $\frac{\partial}{\partial t_j}\mathrm{T}_t$, for $j = 1, 2$, and $\frac{\partial^2}{\partial_{t_1}\partial_{t_2}}\mathrm{T}_t$.
It is well-known from the one-parameter case, that for $j = 1, \ldots, N$ it holds:

$$\frac{\partial}{\partial_{t_j}}\mathrm{T}_t\mathrm{u} \;=\; \mathrm{A}^{(j)}\mathrm{T}_t\mathrm{u}. \tag{2.25}$$

However, equality (2.25) does only make sense for $\mathrm{u} \in X$ with $\mathrm{u}, \mathrm{T}_t\mathrm{u} \in \mathrm{D}(\mathrm{A}^{(j)})$. Therefore, to investigate $\frac{\partial^N}{\partial_{t_1}\ldots\partial_{t_N}}\mathrm{T}_t$, we first give some properties of the domain $\mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)})$ of the operator $\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}$.

Lemma 2.2.10. *Let $(\mathrm{T}_t)_{t\succeq 0}$ be an strongly continuous N-parameter semigroup of operators. Its marginal semigroups of operators are defined by $\left(\mathrm{T}_{t_j}^{(j)}\right)_{t_j\geq 0} = \left(\mathrm{T}_{t_j \cdot e_j}\right)_{t_j\geq 0}$, $j = 1, ..., N$; their generators be $(\mathrm{A}^{(j)}, \mathrm{D}(\mathrm{A}^{(j)}))$, respectively. Then we have (i) $\mathrm{u} \in \mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}) \Longrightarrow \mathrm{T}_t\mathrm{u} \in \mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)})$ for all $t \succeq 0$, i.e. $\mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)})$ is invariant under T_t for all $t \succeq 0$, (ii) $[\mathrm{A}^{(i)}, \mathrm{A}^{(j)}] = 0$ for all $i, j = 1, ..., N$, i.e. $\mathrm{A}^{(1)}, ..., \mathrm{A}^{(N)}$ commutate mutually, and (iii) for all permutations $\pi : \{1, ..., N\} \longrightarrow \{1, ..., N\}$ we get*

$$\mathrm{D}\left(\mathrm{A}^{(1)} \circ \mathrm{A}^{(2)} \circ \ldots \circ \mathrm{A}^{(N)}\right) = \mathrm{D}\left(\mathrm{A}^{(\pi(1))} \circ \mathrm{A}^{(\pi(2))} \circ \ldots \circ \mathrm{A}^{(\pi(N))}\right).$$

Proof. (i) Using Lemma 1.3.6, which states the invariance of $\mathrm{D}(\mathrm{A}^{(j)})$ under $\mathrm{T}_{t_j}^{(j)}$ for $j = 1, ..., N$, $t_j \geq 0$, and the commutating property of marginal semigroups of operators, we obtain

$$\begin{aligned}
&\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \circ \mathrm{T}_{t_N}^{(N)}\mathrm{u} = \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N-1)} \circ \mathrm{T}_{t_N}^{(N)} \circ \mathrm{A}^{(N)}\mathrm{u} \\
&= \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N-2)} \left[\lim_{t_{N-1}\to 0} \frac{1}{t_{N-1}} \left(\mathrm{T}_{t_{N-1}}^{(N-1)} \circ \mathrm{T}_{t_N}^{(N)} \circ \mathrm{A}^{(N)}\mathrm{u} - \mathrm{T}_{t_N}^{(N)} \circ \mathrm{A}^{(N)}\mathrm{u}\right)\right] \\
&= \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N-2)} \left[\lim_{t_{N-1}\to 0} \frac{1}{t_{N-1}} \left(\mathrm{T}_{t_N}^{(N)} \circ \mathrm{T}_{t_{N-1}}^{(N-1)} \circ \mathrm{A}^{(N)}\mathrm{u} - \mathrm{T}_{t_N}^{(N)} \circ \mathrm{A}^{(N)}\mathrm{u}\right)\right].
\end{aligned}$$

Since $\mathrm{T}_{t_N}^{(N)}$ is continuous from X to X we can interchange the limit and $\mathrm{T}_{t_N}^{(N)}$; we get

$$\begin{aligned}
&\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N-2)} \circ \mathrm{T}_{t_N}^{(N)} \left[\lim_{t_{N-1}\to 0} \frac{1}{t_{N-1}} \left(\mathrm{T}_{t_{N-1}}^{(N-1)} \circ \mathrm{A}^{(N)}\mathrm{u} - \mathrm{A}^{(N)}\mathrm{u}\right)\right] \\
&= \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N-2)} \circ \mathrm{T}_{t_N}^{(N)} \circ \mathrm{A}^{(N-1)} \circ \mathrm{A}^{(N)}\mathrm{u},
\end{aligned}$$

for the last equation we used that $\mathrm{u} \in \mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)})$, thus in particular $\mathrm{u} \in \mathrm{D}(\mathrm{A}^{(N-1)} \circ \mathrm{A}^{(N)})$. Repeating this procedure (N-2)-times we obtain

$$\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \circ \mathrm{T}_{t_N}^{(N)}\mathrm{u} = \mathrm{T}_{t_N}^{(N)} \circ \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\mathrm{u}.$$

Since $u \in D(A^{(1)} \circ ... \circ A^{(N)})$ it follows $A^{(1)} \circ ... \circ A^{(N)}u \in X$, thus the right hand side is well defined and so is the left hand side. Analogously, we see for an arbitrary t_j with $j = 1, ..., N$ that

$$A^{(1)} \circ ... \circ A^{(N)} \circ T^{(j)}_{t_j} u = T^{(j)}_{t_j} \circ A^{(1)} \circ ... \circ A^{(N)} u.$$

Combining this result for all marginal semigroups we finally obtain

$$A^{(1)} \circ ... \circ A^{(N)} \circ T_t u = T_t \circ A^{(1)} \circ ... \circ A^{(N)} u,$$

and especially $u \in D(A^{(1)} \circ ... \circ A^{(N)})$ implies $T_t u \in D(A^{(1)} \circ ... \circ A^{(N)})$, i.e. $D(A^{(1)} \circ ... \circ A^{(N)})$ is invariant under T_t.
(ii) $u \in D(A^{(i)} \circ A^{(j)})$, then

$$\begin{aligned} A^{(i)} \circ A^{(j)} u &= \lim_{t_i \to 0} \frac{1}{t_i} \left[T^{(i)}_{t_i} \circ A^{(j)} u - A^{(j)} u \right] \\ &= \lim_{t_i \to 0} \lim_{t_j \to 0} \frac{1}{t_i t_j} \left[T^{(i)}_{t_i} \left(T^{(j)}_{t_j} u - u \right) - \left(T^{(j)}_{t_j} u - u \right) \right] \\ &= \lim_{t_j \to 0} \lim_{t_i \to 0} \frac{1}{t_i t_j} \left[T^{(j)}_{t_j} \left(T^{(i)}_{t_i} u - u \right) - \left(T^{(i)}_{t_i} u - u \right) \right] \\ &= A^{(j)} \circ A^{(i)} u, \end{aligned}$$

where we can change the order of limits since the marginal semigroups are commutating and we have proven that $A^{(i)}$ and $A^{(j)}$ commutate in case both $A^{(i)} \circ A^{(j)}$ and $A^{(j)} \circ A^{(i)}$ are defined.
(iii) From part (ii) we see that in case $A^{(i)} \circ A^{(j)} u$ is defined for some $u \in X$ then, due to $\left[T^{(i)}_{t_i}, T^{(j)}_{t_j} \right] = 0$, we can reformulate this as $A^{(j)} \circ A^{(i)} u$, which therefore is also defined. Part (iii) is just a consequence a finite number of applications of this property. □

The previous lemma enables us to solve a partial differential equation which resembles the differential equation associated to the generator of a one-parameter semigroup. The following partial differential equation illustrates that to some extend the operator $A^{(1)} \circ A^{(2)} \circ \ldots \circ A^{(N)}$ may be considered as the multiparameter equivalent to the generator A of a one-parameter semigroup.

Lemma 2.2.11. *For $u \in D(A^{(1)} \circ ... \circ A^{(N)})$ and $(T_t)_{t \succeq 0}$ as above we have*

$$\frac{\partial^N}{\partial_{t_1} ... \partial_{t_N}} T_t u = A^{(1)} \circ ... \circ A^{(N)} \circ T_t u.$$

Proof. Let $\mathrm{u} \in \mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)})$, then, in particular, u is in the domain of the generator $\mathrm{A}^{(N)}$ of the N-th marginal semigroup of $(\mathrm{T}_t)_{t \succeq 0}$, and we have

$$\frac{\partial}{\partial t_N} \mathrm{T}_{t_N}^{(N)} \mathrm{u} = \mathrm{A}^{(N)} \mathrm{T}_{t_N}^{(N)} \mathrm{u}.$$

Now define

$$\tilde{\mathrm{u}}(x) := \mathrm{A}^{(N)} \mathrm{T}_{t_N}^{(N)} \mathrm{u}.$$

The $\tilde{\mathrm{u}} \in \mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N-1)})$, since $\mathrm{u} \in \mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)})$ and $\mathrm{D}(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)})$ is invariant under $\mathrm{T}_{t_N}^{(N)}$. We obtain

$$\frac{\partial}{\partial t_{N-1}} \mathrm{T}_{t_{N-1}}^{(N-1)} \tilde{\mathrm{u}} = \mathrm{A}^{(N-1)} \mathrm{T}_{t_{N-1}}^{(N-1)} \tilde{\mathrm{u}},$$

and by substituting $\tilde{\mathrm{u}}$

$$\frac{\partial}{\partial t_{N-1}} \mathrm{T}_{t_{N-1}}^{(N-1)} \frac{\partial}{\partial t_N} \mathrm{T}_{t_N}^{(N)} \mathrm{u} = \mathrm{A}^{(N-1)} \mathrm{T}_{t_{N-1}}^{(N-1)} \mathrm{A}^{(N)} \mathrm{T}_{t_N}^{(N)} \mathrm{u}.$$

Interchanging $\mathrm{T}_{t_{N-1}}^{(N-1)}$ and the derivative $\frac{\partial}{\partial t_N}$ as well as using the commuting property as shown in the proof of Lemma 2.2.10 leads to

$$\frac{\partial^2}{\partial t_{N-1} \partial t_N} \mathrm{T}_{t_{N-1}}^{(N-1)} \mathrm{T}_{t_N}^{(N)} \mathrm{u} = \mathrm{A}^{(N-1)} \circ \mathrm{A}^{(N)} \circ \mathrm{T}_{t_{N-1}}^{(N-1)} \circ \mathrm{T}_{t_N}^{(N)} \mathrm{u}.$$

Iterating these steps another $(N-2)$-times we arrive at

$$\frac{\partial^N}{\partial t_1 \cdot \ldots \cdot \partial t_N} \mathrm{T}_{t_1}^{(1)} \cdots \mathrm{T}_{t_N}^{(N)} \mathrm{u} = \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \circ \mathrm{T}_{t_1}^{(1)} \circ \ldots \circ \mathrm{T}_{t_N}^{(N)} \mathrm{u},$$

which completes the proof. □

Lemma 2.2.12. *Let $(\mathrm{T}_t)_{t \succeq 0}$ be a strongly continuous N-parameter semigroup on the Banach space $(X, \|.\|_X)$ with commutating marginal semigroups $\left(T_{t_j}^{(j)}\right)_{t_j \geq 0}$, $j = 1, \ldots, N$, whose generators are given by $\mathrm{A}^{(1)}, \ldots, \mathrm{A}^{(N)}$, respectively. Then the domain $\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ is dense* *in* $\mathrm{D}\left(\mathrm{A}^{(i)}\right)$, *for all* $i = 1, \ldots, N$, *and* $\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}$ *is a closed operator. Furthermore,* $(\mathrm{T}_t)_{t \succeq 0}$ *is a strongly continuous N-parameter semigroup on* $\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ *when* $\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ *is equipped with the graph norm* $\|\mathrm{u}\|_{\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}, X} = \left\|\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u}\right\|_X + \|\mathrm{u}\|_X$.

Proof. We give the proof of density in $\mathrm{D}\left(\mathrm{A}^{(1)}\right)$, the other cases follow by permutation. Assume $u \in \mathrm{D}\left(\mathrm{A}^{(1)}\right)$, thus $\mathrm{A}^{(1)}\mathrm{u} \in X$. With Fubini's theorem and the commutating property of the marginal semigroups we get for $t \in \mathbb{R}_+^{N-1}$

$$
\begin{aligned}
\mathrm{v}_t &:= \int_0^{t_N} \mathrm{T}_{s_N}^{(N)} \ldots \int_0^{t_2} \mathrm{T}_{s_2}^{(2)} \circ \mathrm{A}^{(1)} \mathrm{u} \; \mathrm{d}s_2 \ldots \mathrm{d}s_N \\
&= \int_0^{t_{\pi(N)}} \mathrm{T}_{s_{\pi(N)}}^{(\pi(N))} \ldots \int_0^{t_{\pi(2)}} \mathrm{T}_{s_{\pi(2)}}^{(\pi(2))} \circ \mathrm{A}^{(1)} \mathrm{u} \; \mathrm{d}s_{\pi(2)} \ldots \mathrm{d}s_{\pi(N)},
\end{aligned}
$$

where $\pi : \{2, \ldots, N\} \to \{2, \ldots, N\}$ is an arbitrary permutation. Lemma 1.3.6 then gives $\mathrm{v}_t \in \mathrm{D}\left(\mathrm{A}^{(2)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ and for

$$
\mathrm{u}_t := \frac{1}{t_N} \int_0^{t_N} \mathrm{T}_{s_N}^{(N)} \ldots \frac{1}{t_2} \int_0^{t_2} \mathrm{T}_{s_2}^{(2)} \mathrm{u} \; \mathrm{d}s_2 \ldots \mathrm{d}s_N,
$$

with $t_j > 0$, for all $j = 2, \ldots, N$, we get $\mathrm{u}_t \in \mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$, where we again used the commutating property of the generators of the marginal semigroups. Since for $t_i \to 0$ it holds $\frac{1}{t_i} \int_0^{t_i} \mathrm{T}_{s_i}^{(i)} \mathrm{w} \, \mathrm{d}s_i \to \mathrm{w}$ (strongly), for all $\mathrm{w} \in X$, we finally obtain $\mathrm{u}_t \to u$ strongly as $t \to 0$, which gives the density of $D\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ in $\mathrm{D}\left(\mathrm{A}^{(1)}\right)$.

We want to show that $A^{(1)} \circ \ldots \circ A^{(N)}$ is a closed operator. For this we let $(\mathrm{u}_\nu)_{\nu \geq 0} \subset D\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ be a sequence converging to $\mathrm{u} \in X$ as $\nu \to \infty$ and $\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u}_\nu \to \mathrm{g} \in X$ as $\nu \to \infty$. We need to prove that $\mathrm{u} \in D\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$and it holds $\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u} = \mathrm{g}$. From Lemma 2.2.11 we obtain for all $t \succeq 0$

$$
\begin{aligned}
&\sum_{s_1 \in \{0; t_1\}, \ldots, s_N \in \{0; t_N\}} \left((-1)^N \prod_{j=1}^N (-1)^{s_j} \mathrm{T}_s \mathrm{u}_\nu \right) \\
&= \int_0^{t_N} \frac{\partial}{\partial_{s_N}} \cdots \left(\int_0^{t_2} \frac{\partial}{\partial_{s_2}} \left(\int_0^{t_1} \frac{\partial}{\partial_{s_1}} \mathrm{T}_s \mathrm{u}_\nu \; \mathrm{d}s_1 \right) \mathrm{d}s_2 \right) \ldots \mathrm{d}s_N
\end{aligned}
$$

and for $\nu \to \infty$, $t_1, \ldots, t_N > 0$ we find

$$\frac{1}{t_1 \cdot \ldots \cdot t_N} \sum_{s_1 \in \{0;t_1\},\ldots,s_N \in \{0;t_N\}} \left((-1)^N \prod_{j=1}^{N} (-1)^{s_j} \mathrm{T}_s \mathrm{u} \right)$$
$$= \frac{1}{t_1 \cdot \ldots \cdot t_N} \int_0^t \mathrm{T}_s \mathrm{A}^{(N)} \circ \ldots \circ \mathrm{A}^{(1)} \mathrm{u} \, \mathrm{d}(s) = \frac{1}{t_1 \cdot \ldots \cdot t_N} \int_0^t \mathrm{T}_s \mathrm{g} \, \mathrm{d}s.$$

As $t \to 0$ we find that the limit on the right hand side exists, hence so does the limit on the left hand side, herewith $\mathrm{u} \in \mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ and $\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u} = \mathrm{g}$, implying that $\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}$ is closed.

Next we want to prove that $(\mathrm{T}_t)_{t \succeq 0}$ is a strongly continuous contraction semigroup on $\left(\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right), \|.\|_{\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}, X}\right)$. The semigroup property on $\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ is obvious since $\mathrm{T}_t \mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right) \subset \mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$, and from

$$\begin{aligned} \|\mathrm{T}_t \mathrm{u} - \mathrm{u}\|_{\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}, X} &= \left\|\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \circ \mathrm{T}_t \mathrm{u} - \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u}\right\|_X \\ &\quad + \|T_t \mathrm{u} - \mathrm{u}\|_X \\ &= \left\|\mathrm{T}_t \circ \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u} - \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u}\right\|_X \\ &\quad + \|T_t \mathrm{u} - \mathrm{u}\|_X \end{aligned}$$

follows the strong continuity on $\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ since $(\mathrm{T}_t)_{t \succeq 0}$ is strongly continuous on X. □

Remark 2.2.13. *Since* $\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ *is dense in* $\mathrm{D}\left(A^{(1)}\right)$, *which again is dense in* X, *see Corollary 4.1.15 from [14], we get immediately, that* $\mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ *is a dense subset of* X. *Furthermore, the proof can easily be extended to show that* $\mathrm{D}\left(\left(\mathrm{A}^{(1)}\right)^{i_1} \circ \ldots \circ \left(\mathrm{A}^{(N)}\right)^{i_N}\right)$ *with* $i_1, \ldots, i_N \in \mathbb{N}$ *is dense in* $\mathrm{D}\left(\mathrm{A}^{(j)}\right)$.

Proposition 2.2.14. *Let* $(\mathrm{T}_t)_{t \succeq 0}$ *be a strongly continuous N-parameter semigroup on the Banach space* $(X, \|.\|_X)$ *with strongly continuous marginal semigroups* $\left(\mathrm{T}_{t_j}^{(j)}\right)_{t_j \geq 0}$ *and generators* $\left(\mathrm{A}^{(j)}, \mathrm{D}\left(\mathrm{A}^{(j)}\right)\right)$, $j = 1, \ldots, N$, *respectively. Suppose* $Y \subset \mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ *is dense in* X *and invariant under*

T_t, *for all* $t \succeq 0$, *i.e.* $\mathrm{T}_t Y \subset Y$. *Then* Y *is a core for* $\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}$, *i.e.* $\overline{\mathrm{Y}}^{\|\cdot\|_{\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)},X}} = \mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$.

Proof. If Y is dense in X, then for all $\mathrm{u} \in \mathrm{D}\left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)}\right)$ there exists a sequence $(\mathrm{u}_\nu)_{\nu\in\mathbb{N}} \subset Y$ with $\|\mathrm{u}_\nu - \mathrm{u}\|_X \to 0$ as $\nu \to \infty$. By Lemma 2.2.12 and Corollary 2.2.9 the mapping $t \mapsto \mathrm{T}_t \mathrm{u}_\nu$ is continuous with respect to $\|.\|_{\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)},X}$ and it follows that

$$\int_0^t \mathrm{T}_s \mathrm{u}_\nu \, \mathrm{d}s \in \overline{\mathrm{Y}}^{\|\cdot\|_{\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)},X}}.$$

Moreover, by the strong continuity and the commutating property we find

$$\begin{aligned}
&\left\| \int_0^t \mathrm{T}_s \mathrm{u}_\nu \, \mathrm{d}s - \int_0^t \mathrm{T}_s \mathrm{u} \, \mathrm{d}s \right\|_{\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)},X} \\
\leq\ & \left\| \int_0^t \mathrm{T}_s \left(\mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} (\mathrm{u}_\nu - \mathrm{u})\right) \, \mathrm{d}s \right\|_X + \|\mathrm{T}_t \mathrm{u}_\nu - \mathrm{u}_\nu - \mathrm{T}_t \mathrm{u} + \mathrm{u}\|_X
\end{aligned}$$

and the right hand side goes to zero for $\nu \longrightarrow \infty$. Thus,

$$\int_0^t \mathrm{T}_s \mathrm{u} \, \mathrm{d}s \in \overline{Y}^{\|\cdot\|_{\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)},X}}.$$

Finally we get

$$\begin{aligned}
&\left\| \int_0^t \mathrm{T}_s \mathrm{u} \, \mathrm{d}s - \mathrm{u} \right\|_{\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)},X} \\
\leq\ & \left\| \frac{1}{t_1}\left(T^{(1)}_{t_1} - \mathrm{id}\right) \circ \ldots \circ \frac{1}{t_N}\left(T^{(N)}_{t_N} - \mathrm{id}\right) \mathrm{u} - \mathrm{A}^{(1)} \circ \ldots \circ \mathrm{A}^{(N)} \mathrm{u} \right\|_X \\
&+ \left\| \frac{1}{t_1 \cdot \ldots \cdot t_N} \int_0^t \mathrm{T}_s \mathrm{u} \, \mathrm{d}s - \mathrm{u} \right\|_X
\end{aligned}$$

which tends to 0 for $t \to 0$, hence $\mathrm{u} \in \overline{Y}^{\|\cdot\|_{\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)},X}}$. □

2.3 Subordination for Operator Semigroups

Now we analyse the behaviour of N-parameter operator semigroups under subordination by an M-parameter convolution semigroup. The subordinating

convolution semigroup has its support in $[0,\infty)^N$. Here we also stress that by performing this subordination an N-parameter semigroup is transformed into an M-parameter semigroup.
We start with

Theorem 2.3.1. *Let $(T_t)_{t\succeq 0}, t \in \mathbb{R}_+^N$, be a strongly continuous N-parameter contraction semigroup on a Banach space $(X, \|.\|_X)$ and $(\eta_s)_{s\succeq 0}, s \in \mathbb{R}_+^M$, be an M-parameter convolution semigroup on $\mathbb{R}^N$ with* $\operatorname{supp}\eta_s \subset [0,\infty)^N$, *for all $s \in \mathbb{R}_+^M$. We define using the **N-dimensional Bochner integral** for all $u \in X$:*

$$T_s^\eta u := \int_{\mathbb{R}_+^N} T_t u\, \eta_s(dt). \tag{2.26}$$

Then the integral is well-defined and $(T_s^\eta)_{s\succeq 0}$ is a strongly continuous M-parameter contraction semigroup on X.

Proof. We find

$$\|T_s^\eta u\|_X \leq \int_{\mathbb{R}_+^N} \|T_t u\|_X\, \eta_s(dt) \leq \eta_s(\mathbb{R}_+^N)\, \|u\|_X \leq \|u\|_X\,,$$

thus, for $s \succeq 0$, T_s^η is a contraction on X. Moreover, using the semigroup property of $(T_t)_{t\succeq 0}$ and $(\eta_s)_{s\succeq 0}$, respectively, we find

$$\begin{aligned}
T_{r+s}^\eta u &= \int_{\mathbb{R}_+^N} T_p u\, \eta_{r+s}(dp) \\
&= \int_{\mathbb{R}_+^N} T_p u (\eta_r * \eta_s)(dp) \\
&= \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} T_{p+q} u\, \eta_r(dq) \eta_s(dp) \\
&= \int_{\mathbb{R}_+^N} \int_{\mathbb{R}_+^N} (T_p \circ T_q)\, u\, \eta_r(dq) \eta_s(dp) \\
&= \int_{\mathbb{R}_+^N} T_p(T_r^\eta u)\, \eta_s(dp) \\
&= (T_s^\eta \circ T_r^\eta)\, u,
\end{aligned}$$

thus, $(T_s^\eta)_{s\succeq 0}$ is a semigroup on X.
Finally, we show that $(T_s^\eta)_{s\succeq 0}$ is strongly continuous on X. For this note

that the function $t \mapsto \|T_t u - u\|_X$ is continuous and bounded on $\mathbb{R}^N_+$ and $\lim_{t\to 0} \|T_t u - u\|_X = 0$. Furthermore, the convolution semigroup $(\eta_s)_{s\succeq 0}$ tends to ε_0 in the Bernoulli topology as $s \to 0$, see Lemma 2.1.4.
Since

$$\|T^\eta_s u - u\|_X \leq \int_{\mathbb{R}^N_+} \|T_t u - u\|_X \,\eta_s(dt) + (1 - \eta_s(\mathbb{R}^N_+))\|u\|_X,$$

it is sufficient to prove for any function $v \in \mathrm{C}_b(\mathbb{R}^N_+), v(0) = 0$ that

$$\lim_{s\to 0} \int_{\mathbb{R}^N_+} v(t)\,\eta_s(dt) = 0$$

which follows from the definition of the Bernoulli topology and the definition of ε_0. Thus the theorem is proved. $\square$

Definition 2.3.2. *The semigroup $(T^\eta_s)_{s\succeq 0}$ as defined in (2.26) is called the* ***subordinate (in the sense of Bochner)*** *to $(T_t)_{t\succeq 0}$ with respect to the convolution semigroup $(\eta_s)_{s\succeq 0}$.*

Remark. *Since the convolution semigroup $(\eta_s)_{s\succeq 0}$ has the representation*

$$\eta_s = \eta^{(1)}_{s_1} * \ldots * \eta^{(M)}_{s_M},$$

one gets instantly that

$$(\mathrm{T}^\eta)^{(j)}_{s_j} = \mathrm{T}^{\eta_j}_{s_j},$$

for $s_j \geq 0$ and $j \in 1, \ldots, M$, and $(\mathrm{T}^{\eta_j}_{s_j})_{s_j \geq 0}$ is obviously the j-th marginal semigroup of $(\mathrm{T}^\eta_s)_{s\succeq 0}$, compare Definition 2.2.7.

Corollary 2.3.3. *Let $(T_t)_{t\succeq 0}$ be either an N-parameter Feller semigroup on $\mathrm{C}_\infty(\mathbb{R}^n;\mathbb{R})$ or an N-parameter sub-Markovian semigroup on $L^p(\mathbb{R}^n;\mathbb{R}), 1 \leq p \leq \infty$, further let $(\eta_s)_{s\succeq 0}$ be an M-parameter convolution semigroup on $\mathbb{R}^N_+$ with $\operatorname{supp}\eta_s \subset \mathbb{R}^N_+$, i.e. with positive support. Then $(T^\eta_s)_{s\succeq 0}$ is an M-parameter Feller or sub-Markovian semigroup, respectively.*

Lemma 2.3.4. *For the derivatives of the subordinate $(T^\eta_s)_{s\succeq 0}$ we have:*

$$\frac{\partial}{\partial_{s_j}} T^\eta_s u = A^{\eta,(j)} \circ T^\eta_s u$$

and

$$\frac{\partial^M}{\partial_{s_1}\ldots\partial_{s_M}} T^\eta_s u = A^{\eta,(1)} \circ \ldots \circ A^{\eta,(M)} \circ T^\eta_s u,$$

where $A^{\eta,(j)}$ is the generator of the j-th marginal semigroup of operators $(T^{\eta,(j)}_{s_j})_{s_j \geq 0}$.

Proof. We start with the partial derivative

$$\begin{aligned}
\frac{\partial}{\partial_{s_j}} T^\eta_s u &= \lim_{h\to 0} \frac{1}{h}\left(\int_0^\infty T_t u(x)\eta_{s+h\cdot e_j}(dt) - \int_0^\infty T_t u(x)\eta_s(dt)\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_0^\infty T_t u(x)(\eta_s * \eta_{h\cdot e_j})(dt) - \int_0^\infty T_t u(x)\eta_s(dt)\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_0^\infty \int_0^\infty T_{t+q} u(x)\eta_{h\cdot e_j}(dq)\eta_s(dt) - \int_0^\infty T_t u(x)\eta_s(dt)\right) \\
&= T^\eta_t \left(\lim_{h\to 0}(T^{\eta,(j)}_{h\cdot e_j} u(x) - u(x))\right) \\
&= T^\eta_s \circ A^{\eta,(j)} u(x) = A^{\eta,(j)} \circ T^\eta_s u(x),
\end{aligned}$$

where $e_j = (0,\ldots,0,1,0,\ldots,0)$ is the j-th unit vector in $\mathbb{R}^M$.
For the second partial derivative $\frac{\partial^2}{\partial_{s_i}\partial_{s_j}} T^\eta_s u$ with $i \neq j$ we find

$$\begin{aligned}
\frac{\partial^2}{\partial_{s_i}\partial_{s_j}} T^\eta_s u(x) &= \frac{\partial}{\partial_{s_i}}\left(A^{\eta,(j)} \circ T^\eta_j u(x)\right) \\
&= A^{\eta,(j)} \circ \frac{\partial}{\partial_{s_i}} T^\eta_j u(x) \\
&= A^{\eta,(j)} \circ A^{\eta,(i)} \circ T^\eta_s u(x).
\end{aligned}$$

Applying this calculations another (M-2)-times in an analogous manner leads to

$$\frac{\partial^M}{\partial_{s_1}\ldots\partial_{s_M}} T^\eta_s u = A^{\eta,(1)} \circ \ldots \circ A^{\eta,(M)} \circ T^\eta_s u.$$

□

Proposition 2.3.5. *Let $(\mathrm{T}_t)_{t \succeq 0}$ be a strongly continuous N-parameter contraction semigroup on a Banach space $(X, \|.\|_X)$ with marginal semigroups $\left(\mathrm{T}^{(j)}_{t_j}\right)_{t_j \geq 0}$ and their generators $\left(\mathrm{A}^{(j)}, \mathrm{D}\left(\mathrm{A}^{(j)}\right)\right)$, $j = 1,\ldots,N$. Further, let $(\eta_s)_{s \succeq 0}$ be an M-parameter convolution semigroup with positive support, i.e.*

$\operatorname{supp}\eta_s \subset \mathbb{R}_+^N$, *for all* $s \succeq 0$. *Where* $(\eta_s)_{s\succeq 0}$ *may be represented as the product, see the remark following Theorem 2.1.6, of M one-parameter convolution semigroups* $\left(\eta_{s_j}^{(j)}\right)_{s_j\geq 0}$, *for* $j = 1,\ldots,M$, *each with support on the positive half-line* $[0,\infty)$. *Thus we can associate* $(\eta_s)_{s\succeq 0}$ *with M Bernstein functions* $\mathrm{f}^{(1)},\ldots,\mathrm{f}^{(M)}$, *through its association with* $(\eta_{t_1}^{(1)})_{t_1\geq 0},\ldots,(\eta_{t_M}^{(M)})_{t_M\geq 0}$, *respectively.*
Then $\mathrm{D}\left(\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)}\right)$ *is a core for* $\mathrm{D}\left(\mathrm{A}^{(1),\mathrm{f}^{(1)}}\circ\ldots\circ\mathrm{A}^{(M),f^{(M)}}\right)$.

Proof. Recall that Remark 2.2.13 states that $\mathrm{D}\left(\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\right)$, $i_1,\ldots,i_N \in \mathbb{N}$ is dense in X. In order to apply Proposition 2.2.14 we have to show that $\mathrm{D}\left(\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\right)$ is invariant under T_t^η for $t\succeq 0$. For this take $\mathrm{u}\in D\left(\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\right)$ and consider the sequence

$$\left(\int_{(0,\ldots,0)}^{(n,\ldots,n)} \mathrm{T}_s\mathrm{u}\ \eta_t(\mathrm{d}s)\right)_{n\in\mathbb{N}}.$$

This sequence converges to $\mathrm{T}_t^\eta\mathrm{u}$ and the closedness of the operator $\mathrm{A}^{(1)}\circ\ldots\circ\mathrm{A}^{(N)}$ gives

$$\begin{aligned}
&\left\|\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\int_{(m,\ldots,m)}^{(n,\ldots,n)} \mathrm{T}_s\mathrm{u}\ \eta_t(\mathrm{d}s)\right\|_X \\
= &\left\|\int_{(m,\ldots,m)}^{(n,\ldots,n)} \mathrm{T}_s\circ\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\mathrm{u}\ \eta_t(\mathrm{d}s)\right\|_X \\
\leq &\int_{(m,\ldots,m)}^{(n,\ldots,n)} \|\mathrm{T}_s\|\ \eta_t(\mathrm{d}s)\left\|\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\mathrm{u}\right\|_X \\
\leq &\ \eta_t\left([m,n]^N\right)\cdot\left\|\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\right\|_X,
\end{aligned}$$

implying that $\left(\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\int_{(0,\ldots,0)}^{(n,\ldots,n)}\mathrm{T}_s\mathrm{u}\ \eta_t(\mathrm{d}s)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space X, and the closedness of $\left(\mathrm{A}^{(1)}\right)\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)$ yields that $\mathrm{T}_t^\eta\mathrm{D}\left(\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\right)\subset\mathrm{D}\left(\left(\mathrm{A}^{(1)}\right)^{i_1}\circ\ldots\circ\left(\mathrm{A}^{(N)}\right)^{i_N}\right)$. □

We once again draw attention to the fact that the dimension of the parameter of the subordinated semigroup $(\mathrm{T}_t^\eta)_{t\succeq 0}$ may differ from the dimension of the parameter of semigroup $(\mathrm{T}_s)_{s\succeq 0}$. Now we give an example in which subordination reduces the dimension of the (time-)parameter.

Example 2.3.6. *Let $(\mu_s)_{s\succeq 0}$, $s \in \mathbb{R}^2_+$, be a two-parameter convolution semigroup on $\mathbb{R}^n$ and define for all $s \succeq 0$ the integral operator*

$$\mathrm{T}_s\mathrm{u}(x) := \int_0^\infty \mathrm{u}(y-x)\mu_s(\mathrm{d}y),$$

for $\mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^n)$. Moreover, let $(\eta_t)_{t\geq 0}$ be a one-parameter convolution semigroup with support in $\mathbb{R}^2_+$ and representation $\eta_t(\mathrm{d}s_1, \mathrm{d}s_2) = \eta_t^{(1)}(\mathrm{d}s_1) \otimes \eta_t^{(2)}(\mathrm{d}s_2)$, for all $t \geq 0$, with two (one-parameter) convolution semigroups $\left(\eta_t^{(1)}\right)_{t\geq 0}$ and $\left(\eta_t^{(2)}\right)_{t\geq 0}$. Then there are Bernstein functions f_1 and f_2 such that:

$$\left(\mathrm{T}_t^\eta \mathrm{u}\right)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{e}^{-t\mathrm{f}_1(\psi_1(\xi)) - t\mathrm{f}_2(\psi_2(\xi))}\, \hat{\mathrm{u}}(\xi) \mathrm{d}\,\xi.$$

Proof. Using Theorem 2.1.6 we find the existence of continuous negative definite functions ψ_1 and ψ_2 such that

$$\begin{aligned}
\left(\mathrm{T}_t^\eta \mathrm{u}\right)^\wedge(\xi) &= \int_{\mathbb{R}^2_+} \mathrm{e}^{-s_1\psi_1(\xi) - s_2\psi_2(\xi)} \hat{\mathrm{u}}(\xi)\ \eta_t(\mathrm{d}s) \\
&= \int_0^\infty \mathrm{e}^{-s_1\psi_1(\xi)}\ \eta_t^{(1)}(\mathrm{d}s_1) \cdot \int_0^\infty \mathrm{e}^{-s_2\psi_2(\xi)} \eta_t^{(2)}(\mathrm{d}s_2)\ \cdot \hat{\mathrm{u}}(\xi) \\
&= \mathrm{e}^{-t\mathrm{f}_1(\psi(\xi))} \cdot \mathrm{e}^{-t\mathrm{f}_2(\psi_2(\xi))} \hat{\mathrm{u}}(\xi), \qquad (2.27)
\end{aligned}$$

where the existence of the Bernstein functions f_1 and f_2 follows from Theorem 1.2.17. Applying the inverse Fourier transform to (2.27) we obtain

$$\mathrm{T}_t^\eta \mathrm{u}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{e}^{-t\mathrm{f}_1(\psi_1(\xi)) - t\mathrm{f}_2(\psi_2(\xi))} \hat{\mathrm{u}(\xi)}\ \mathrm{d}\,\xi.$$

□

The dimension of the parameter may as well be increased by performing subordination as the following example illustrates:

Example 2.3.7. *Let $(\mu_s)_{s\geq 0}$ be a one-parameter convolution semigroup on $\mathbb{R}^n$ with corresponding continuous negative definite function ψ and define for all $s \geq 0$ the operator*

$$\mathrm{T}_s\,\mathrm{u}(x) = \int_{\mathbb{R}^n} \mathrm{u}(y-x)\,\mu_s(\mathrm{d}y),$$

for $\mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^n)$. *Moreover, let* $(\eta_t)_{t\succeq 0}$, $t \in \mathbb{R}^2_+$, *be a two-parameter convolution semigroup on* $\mathbb{R}$ *with support on the positive half-line. Then the subordinate to* $(\mathrm{T}_s)_{s\geq 0}$ *with respect to* $(\eta_t)_{t\succeq 0}$ *is a two-parameter semigroup of operators and has representation*

$$\mathrm{T}^\eta_t \mathrm{u}(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{e}^{-t_1 \mathrm{f}_1(\psi(\xi)) - t_2 \mathrm{f}_2(\psi(\xi))} \hat{\mathrm{u}}(\xi)\, \mathrm{d}\,\xi,$$

for the Bernstein functions f_1 *and* f_2 *which are associated to* $(\eta_t)_{t\succeq 0}$.

2.4 Examples of N-Parameter Semigroup

Finally, in this section we give some examples of multiparameter semigroups of operators. The first two examples fit smoothly into the category of N-parameter semigroups. The third example, however, unveils how restrictive the semigroup property is.

The first example is taken from [16], Ex.11.2.2.

Example 2.4.1 (Product Feller Semigroup). *Let* $\left(\mathrm{T}^{(1)}_{t_1}\right)_{t_1\geq 0}$ *and* $\left(\mathrm{T}^{(2)}_{t_2}\right)_{t_2\geq 0}$ *be one-parameter Feller semigroups on* $\mathrm{C}_\infty(\mathbb{R}^n)$. *Then we define a two-parameter semigroup on* $(\mathrm{C}_\infty(\mathbb{R}^{2n}), \|.\|_\infty)$ *by defining it for* $\mathrm{f}(x_1, x_2) = \mathrm{f}_1(x_1) \cdot \mathrm{f}_2(x_2)$, $\mathrm{f}_1, \mathrm{f}_2 \in \mathrm{C}_\infty(\mathbb{R}^n)$:

$$\mathrm{T}_t \mathrm{f}(x) = \mathrm{T}^{(1)}_{t_1} \mathrm{f}(x_1) \cdot \mathrm{T}^{(2)}_{t_2} \mathrm{f}(x_2),$$

where we used the notation $t = (t_1, t_2)$, $t_1, t_2 \in \mathbb{R}_+$ *and* $x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{R}^n$. *Using a density argument it becomes evident that* T_t, $t \succeq 0$, *extends to an operator on* $\mathrm{C}_\infty(\mathbb{R}^{2n})$ *and one can prove that* $(\mathrm{T}_t)_{t\succeq 0}$ *is a two-parameter Feller semigroup. Note that* $\left(\mathrm{T}^{(1)}_{t_1}\right)_{t_1\geq 0}$ *and* $\left(\mathrm{T}^{(2)}_{t_2}\right)_{t_2\geq 0}$ *do not define the marginal semigroups of two-parameter semigroup. (Surely, this example may be extended to the N-parameter case.)*

The next example is taken from Example 11.2.3 in [16].

Example 2.4.2 (Additive Feller Semigroup). *Again starting with two one-parameter semigroups* $\left(\mathrm{T}^{(1)}_{t_1}\right)_{t_1\geq 0}$ *and* $\left(\mathrm{T}^{(2)}_{t_2}\right)_{t_2\geq 0}$ *on* $\mathrm{C}_\infty(\mathbb{R}^n)$ *we now define a two-parameter family of operators* $(\mathrm{T}_t)_{t\succeq 0}$ *on* $\mathrm{C}_\infty(\mathbb{R}^n)$ *by*

$$\mathrm{T}_t\mathrm{u}(x) = \mathrm{T}_{t_1} \circ \mathrm{T}_{t_2}\mathrm{u}(x), \qquad\qquad \mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^n), t = (t_1, t_2).$$

Since we want the two-parameter family of operators to be a semigroup, additionally we need pose a commuting condition on the one-parameter semigroups, namely $\left[\mathrm{T}^{(1)}_{t_1}, \mathrm{T}^{(2)}_{t_2}\right] = 0$, *for all* $t_1, t_2 \in \mathbb{R}_+$. *Now it is easy to prove that* $(\mathrm{T}_t)_{t\succeq 0}$ *is a Feller semigroup on* $\mathrm{C}_\infty(\mathbb{R}^n)$.

Now we show that the semigroup property is, indeed, a very restrictive property:

Example 2.4.3. *Let* $\mathrm{q}_1(x, \mathrm{D}_x)$ *and* $\mathrm{q}_2(x, \mathrm{D}_x)$ *be pseudo-differential operators on* $\mathcal{S}(\mathbb{R}^2)$ *which are generators of the Feller semigroups* $(\mathrm{T}^{(1)}_t)_{t\geq 0}$ *and* $(\mathrm{T}^{(2)}_t)_{t\geq 0}$, *respectively. We now define the following two-parameter family of operators on* $\mathcal{S}(\mathbb{R}^2)$

$$\mathrm{T}_t := \mathrm{T}_{t_1} \circ \mathrm{T}_{t_2} \quad , t = (t_1, t_2) \in \mathbb{R}^2. \tag{2.28}$$

We want $(\mathrm{T}_t)_{t\succeq 0}$ *to be a two-parameter semigroup of operators. Thus, according to Lemma 2.2.10 the generators need to fulfil:*

$$[\mathrm{q}_1(x, \mathrm{D}_x), \mathrm{q}_2(x, \mathrm{D}_x)] = 0. \tag{2.29}$$

On the other hand if condition (2.29) holds, then we know that the family defined in (2.28) is a 2-parameter Feller semigroup since the marginal semigroups are Feller. However, in general (2.29) is not fulfilled, but one needs further conditions on $\mathrm{q}_1(x, \mathrm{D})$ *and* $\mathrm{q}_2(x, \mathrm{D})$, *for example the following: if*

$$\mathrm{q}_1(x, \mathrm{D}_x) = \tilde{\mathrm{q}}_1(x_1, \mathrm{D}_{x_1})$$
$$\mathrm{q}_2(x, \mathrm{D}_x) = \tilde{\mathrm{q}}_2(x_2, \mathrm{D}_{x_2})$$

where $\tilde{\mathrm{q}}_1(x_1, \mathrm{D}_{x_1})$ *and* $\tilde{\mathrm{q}}_2(x_2, \mathrm{D}_{x_2})$ *are generators of (1-parameter) Feller semigroups on* $\mathrm{C}_\infty(\mathbb{R})$.

This condition is very restrictive and motivates the considerations in the next Chapter 3.

2.5 N-Parameter Markov Processes

In this section we briefly give the definition of an N-parameter stochastic process following the presentation in Chapter 11 in [16]. (Although there the definition is given in a more general context, we consider $\mathbb{R}^n$-valued stochastic processes.) Furthermore, we interlink the theory developed there with our results.

Definition 2.5.1. *An N-parameter, $\mathbb{R}^n$-valued stochastic process $(\mathrm{X}_t)_{t\in\mathbb{R}^N_+}$ is said to be a* ***multiparameter Markov process*** *if there exists an N-parameter filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^N_+}$ and a family of operator $(\mathrm{T}_t)_{t\in\mathbb{R}^N_+}$, such that for all $x \in \mathbb{R}^n$, there exists a probability measure P_x for which holds:*

(i) $(\mathrm{X}_t)_{t\in\mathbb{R}^N_+}$ is $(\mathcal{F}_t)_{t\in\mathbb{R}^N_+}$-adapted,
(ii) for all $\in \mathbb{R}^n$ $(\mathrm{X}_t)_{t\in\mathbb{R}^N_+}$ has P_x-almost surely right continuous paths,
(iii) for all $t \in \mathbb{R}^N_+$, $\mathcal{F}_t$ is P_x-complete for all $x \in \mathbb{R}^n$, and $(\mathcal{F}_t)_{t\in\mathbb{R}^N_+}$ is a commuting σ-field with respect to all measures P_x, $x \in \mathbb{R}^n$,
(iv) for all $s, t \in \mathbb{R}^N_+$ and $\mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^n)$, it holds for all $x \in \mathbb{R}^n$

$$\mathbb{E}_x[\mathrm{u}(\mathrm{X}_{s+t})|\mathcal{F}_s] = \mathrm{T}_t\mathrm{u}(\mathrm{X}_s), \qquad \mathrm{P}_x - a.s.,$$

(v) for all $x \in \mathbb{R}^n$ it holds $\mathrm{P}_x(\mathrm{X}_0 = x) = 1$.

Furthermore $(\mathrm{X}_t)_{t\in\mathbb{R}^N_+}$ is called an N-parameter Feller process, if: (i) for all $t \in \mathbb{R}^N_+$, $\mathrm{T}_t : \mathrm{C}_\infty(\mathbb{R}^n) \longrightarrow \mathrm{C}_\infty$,
(ii) for each $\mathrm{u} \in \mathrm{C}_\infty(\mathbb{R}^n)$,

$$\lim_{t\to 0} \|\mathrm{T}_t\mathrm{u} - \mathrm{u}\|_\infty = 0.$$

Moreover, it is shown that $(\mathrm{T}_t)_{t\in\mathbb{R}^N_+}$ is a semigroup. It follows that the process $(\mathrm{X}_t)_{t\in\mathbb{R}^N_+}$ is Feller, if the semigroup $(\mathrm{T}_t)_{t\in\mathbb{R}^N_+}$ is Feller, as defined in Definition 2.2.4.

Chapter 3

Beyond N-parameter Semigroups

In this chapter we will analyse multiparameter families of probability measures leading to families of operators which no longer satisfy the semigroup property introduced and investigated in the previous chapter. In the first section we will construct such a two-parameter family of measures which we will then subordinate (in the sense of Bochner) to derive therefrom more families of this kind and to outline the differences to subordination in the "nice" case of commuting families, i.e. semigroups. Moreover, we will show one way of how to obtain a stochastic process associated with such a family of probability measures. In section two we then will consider yet another multiparameter family which does not commute in the sense of Chapter 2. However, we will investigate the structure of these operators in dependence of curves in the parameter space which will lead us to some interesting observations.

3.1 A Case Study

In the following we present two-parameter families of probability measures no longer satisfying the semigroup property introduced earlier. For this let

$\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a continuous negative definite function with $\psi(0) = 0$. We consider

$$\phi_{s,t}(\xi) := (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-st\psi(\xi)},$$

which is for $s, t \geq 0$ a continuous positive definite function, i.e. by Bochner's theorem, Theorem 1.2.7 each $\phi_{s,t}$ is the Fourier transform of a measure on $\mathbb{R}^n$. Obviously, we also have $\phi_{s,t}(0) = 1$ for all $s, t \in \mathbb{R}_+$, hence each associated measure is a probability measures and we can now define the two-parameter family $(\mu_{s,t})_{s,t\geq 0}$ of probability measures by

$$\hat{\mu}_{s,t}(\xi) := (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-st\psi(\xi)}. \tag{3.1}$$

Remark 3.1.1. *Fix $s = s_0 > 0$, then $(\mu_{s_0,t})_{t\geq 0}$ is a (one-parameter) convolution semigroup in t associated with the continuous negative definite function $s_0\psi$ and for $s_0 = 0, t \geq 0$ we find that $\mu_{s_0,t}$ is the Dirac measure. The same holds, of course, for a fixed parameter t_0 and a "running" parameter s. However, $(\phi_{s,t})_{s,t\geq 0}$ is not a two-parameter convolution semigroup.*

This case study of the family of measures defined in (3.1) is divided into two parts. The first of which is an investigation of the behaviour under subordination of this family and in the second part we show how one can construct processes from such a family of probability measures.

The nice results for subordination obtained in Section 2.3 are highly dependent on the semigroup property, hence here we cannot expect to get the same results, since we are not able to make use of the convolution theorem, instead we perform the subordination of $(\mu_{s,t})_{s,t\geq 0}$ step by step with two particular convolution semigroups supported on the positive half-line. First we subordinate with respect to parameter s and then with respect to parameter t. For the primary step we consider the one-parameter family of measures $(\mu_{s,t})_{s\geq 0}$ where t remains fixed greater or equal zero. We subordinate $(\mu_{s,t})_{s\geq 0}$ with respect to the one-sided stable semigroup of order $\frac{1}{2}$, which is denoted by $(\sigma_s^{\frac{1}{2}})_{s\geq 0}$ and is associated to the Bernstein function $\mathrm{f}(x) = x^{\frac{1}{2}}$. Let $(\nu_{s,t})_{s\geq 0}$ be the subordinate of $(\mu_{s,t})_{s\geq 0}$ with respect to $(\sigma_s^{\frac{1}{2}})_{s\geq 0}$, then we get for all

$s \geq 0$

$$\hat{\nu}_{s,t}(\xi) = (2\pi)^{-\frac{1}{2}} e^{-st^{\frac{1}{2}}[\psi(\xi)]^{\frac{1}{2}}}. \tag{3.2}$$

Performing this subordination for all $t \geq 0$ we get a new two-parameter family of probability measures described by (3.2).

Now for each $s \geq 0$, fixed, we subordinate $(\nu_{s,t})_{t\geq 0}$ with respect to the Γ-semigroup which is given by

$$\eta_t(\mathrm{d}x) = \chi_{(0,\infty)}(x)\frac{1}{\Gamma(t)}(x)^{t-1}e^{-x}\lambda^{(1)}(\mathrm{d}x),$$

(here $\lambda^{(1)}$ denotes the one-dimensional Lebesgue-measure) which is associated to the Bernstein function $g(x) = \log(1+x)$. We denote the new family of probability measures by $(\tau_{s,t})_{s,t\geq 0}$. It is given by

$$\hat{\tau}_{s,t}(\xi) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-sp^{1/2}[\psi(\xi)]^{1/2}} \cdot \frac{1}{\Gamma(t)} \cdot p^{t-1} \cdot e^{-p}\,\mathrm{d}p.$$

Now, substituting $q = p^{1/2}$ we obtain

$$\hat{\tau}_{s,t}(\xi) = (2\pi)^{-\frac{1}{2}} \int_0^\infty e^{-sq[\psi(\xi)]^{1/2}} \cdot \frac{2}{\Gamma(t)} \cdot q^{2t-1} \cdot e^{-q^2}\,\mathrm{d}q.$$

Using Lemma 1.4.2 we get

$$\hat{\tau}_{s,t}(\xi) = (2\pi)^{-\frac{1}{2}}\frac{\Gamma(2t)\cdot 2^{1-t}}{\Gamma(t)} \cdot e^{\frac{1}{8}s^2\psi(\xi)} \cdot \mathrm{D}_{-2t}(\frac{1}{\sqrt{2}} \cdot s \cdot [\psi(\xi)]^{1/2}).$$

Here D_ν denotes the parabolic cylinder function. We transform $\hat{\tau}_{s,t}$ to:

$$\begin{aligned} \hat{\tau}_{s,t}(\xi) \;&=\; (2\pi)^{-\frac{1}{2}}\frac{-\Gamma(2t)}{(\Gamma(t))^2}\cdot 2^{1-2t}\cdot[\psi(\xi)]^{(1/2)}\cdot\sqrt{\pi}\cdot s\cdot {}_1\mathrm{F}_1\left(\frac{1}{2}+t;\frac{3}{2};\frac{1}{4}s^2\psi(\xi)\right) \\ &\quad +(2\pi)^{-\frac{1}{2}}\frac{\Gamma(2t)\cdot 2^{1-2t}}{\Gamma(t)\Gamma(t+\frac{1}{2})}\cdot\sqrt{\pi}\cdot {}_1\mathrm{F}_1\left(t;\frac{1}{2};\frac{1}{4}s^2\psi(\xi)\right), \end{aligned}$$

where ${}_1\mathrm{F}_1$ is the confluent hypergeometric function. Table 3.1 gives $\hat{\tau}_{s,t}$, when $(\mu_{s,t})_{s,t\geq 0}$ is subordinated in the first step by $\mathrm{f}(x) = x^{1/n}$.

$f(x) = x^{(\cdot)}$	$(2\pi)^{\frac{1}{2}}\hat{\tau}_{s,t}$
$\frac{1}{1}$	$\mathrm{e}^{-t\log(1+s\cdot\psi(\xi)}$
$\frac{1}{2}$	$\frac{\mathrm{e}^{s^2\psi/4}}{\Gamma(t)}\left(\Gamma(t)\ {}_1\mathrm{F}_1\left(\frac{1}{2}-t;\frac{1}{2};-\frac{s^2\psi}{4}\right) - s\sqrt{\psi}\Gamma(\frac{1}{2}+t)\ {}_1\mathrm{F}_1\left(1-t;\frac{3}{2};-\frac{s^2\psi}{4}\right)\right)$
$\frac{1}{3}$	$\frac{1}{2\Gamma(t)}\Big(2\Gamma(t)\ {}_1\mathrm{F}_2(t;\frac{1}{3},\frac{2}{3};-\frac{s^3\psi}{27}) - 2s\psi^{1/3}\Gamma(\frac{1}{3}+t)\ {}_1\mathrm{F}_2(\frac{1}{3}+t;\frac{2}{3},\frac{4}{3};-\frac{s^3\psi}{27})$ $+s^2\psi^{2/3}\Gamma(\frac{2}{3}+t)\ {}_1\mathrm{F}_2(\frac{2}{3}+t;\frac{4}{3},\frac{5}{3};-\frac{s^3\psi}{27})\Big)$
$\frac{1}{4}$	${}_1\mathrm{F}_3(t;\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{s^4\psi}{256}) - \frac{\Gamma(\frac{1}{4}+t)}{\Gamma(t)}s\psi^{1/4}\ {}_1\mathrm{F}_3(\frac{1}{4}+t;\frac{1}{2},\frac{3}{4},\frac{5}{4};\frac{s^4\psi}{256}) + \frac{\Gamma(\frac{1}{2}+t)}{2\Gamma(t)}s^2\psi^{1/2}\ {}_1\mathrm{F}_3(\frac{1}{2}+t;\frac{3}{4},\frac{5}{4},\frac{3}{2};\frac{s^4\psi}{256})+$ $\frac{\Gamma(\frac{3}{4}+t)}{6\Gamma(t)}s^3\psi^{3/4}\ {}_1\mathrm{F}_3(\frac{3}{4}+t;\frac{5}{4},\frac{3}{2},\frac{7}{4};\frac{s^4\psi}{256})$
$\frac{1}{5}$	${}_1\mathrm{F}_4(t;\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};-\frac{s^5\psi}{3125}) - \frac{\Gamma(\frac{1}{5}+t)}{\Gamma(t)}s\psi^{1/5}\ {}_1\mathrm{F}_4(\frac{1}{5}+t;\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{6}{5};-\frac{s^5\psi}{3125})$ $+\frac{\Gamma(\frac{2}{5}+t)}{2\Gamma(t)}s^2\psi^{2/5}\ {}_1\mathrm{F}_4(\frac{2}{5}+t;\frac{3}{5},\frac{4}{5},\frac{6}{5},\frac{7}{5};-\frac{s^5\psi}{3125}) + \frac{\Gamma(\frac{3}{5}+t)}{6\Gamma(t)}s^3\psi^{3/5}\ {}_1\mathrm{F}_4(\frac{3}{5}+t;\frac{4}{5},\frac{6}{5},\frac{7}{5},\frac{8}{5};-\frac{s^5\psi}{3125})$ $+\frac{\Gamma(\frac{4}{5}+t)}{24\Gamma(t)}s^5\psi\ {}_1\mathrm{F}_4(\frac{4}{5}+t;\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5};-\frac{s^5\psi}{3125})$
$\frac{1}{6}$	$\frac{1}{\sqrt{2\pi}\Gamma(t)}\Big(\Gamma(t)\ {}_1\mathrm{F}_5(t;\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},\frac{\psi(\xi)s^6}{46656}) - \frac{\psi^{\frac{1}{6}}(\xi)s}{120}\Big(120\Gamma\left(\frac{1}{6}+t\right)\ {}_1\mathrm{F}_5(\frac{1}{6}+t;\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},\frac{7}{6};\frac{\psi(\xi)s^6}{46656})$ $+\psi^{\frac{1}{6}}(\xi)s\Big(-60\,\Gamma(\frac{1}{3}+t){}_1\mathrm{F}_5(\frac{1}{3}+t;\frac{1}{2},\frac{2}{3},\frac{5}{6},\frac{7}{6},\frac{4}{3};\frac{\psi(\xi)s^6}{46656} + 20\psi^{\frac{1}{6}}s\Gamma(\frac{1}{2}+t){}_1\mathrm{F}_5(\frac{1}{2}+t;\frac{2}{3},\frac{5}{6},\frac{7}{6},\frac{4}{3},\frac{3}{2};\frac{\psi(\xi)s^6}{46656})$ $-5\psi^{\frac{1}{3}}(\xi)s^2\Gamma(\frac{2}{3}+t){}_1\mathrm{F}_5(\frac{2}{3}+t;\frac{5}{6},\frac{7}{6},\frac{4}{3},\frac{3}{2},\frac{5}{3};\frac{\psi(\xi)s^6}{46656}) + \psi^{\frac{1}{2}}(\xi)s^3\Gamma(\frac{5}{6}+t){}_1\mathrm{F}_5(\frac{5}{6}+t;\frac{7}{6},\frac{4}{3},\frac{3}{2},\frac{5}{3},\frac{11}{6};\frac{\psi(\xi)s^6}{46656})\Big)\Big)\Big)$

Table 3.1: Subordination 1 (calculated with Mathematica)

For the second example we start again with the two-parameter family defined by 3.1 and subordinate $(\mu_{s,t})_{s,t\geq 0}$ for each fixed $t \geq 0$ with respect to $(\sigma_s^{\frac{1}{2}})_{s\geq 0}$, obtaining the same $(\nu_{s,t})_{s,t\geq 0}$ as in the previous example. In the second step we subordinate $(\nu_{s,t})_{s,t\geq 0}$ for each fixed $s \geq 0$ with respect to the one-sided stable semigroup of order $\frac{1}{2}$ and denote the resulting family by $(\tau_{s,t})_{s,t\geq 0}$. We get

$$\begin{aligned}\hat{\tau}_{s,t}(\xi) &= \int_0^\infty \mathrm{e}^{-s\sqrt{p}[\psi(\xi)]^{1/2}} \cdot \frac{1}{\sqrt{4\pi}} \cdot t \cdot p^{-\frac{3}{2}} \cdot \mathrm{e}^{-\frac{t^2}{4p}}\, \mathrm{d}p \\ &= \frac{t}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-s[\psi(\xi)]^{1/2}p} \cdot p^{-2} \cdot e^{-\frac{t^2}{4p^2}} \mathrm{d}p,\end{aligned}$$

in the last step we substituted $q = \sqrt{t}$. Using the Meijer-G-function we can represent $\hat{\tau}_{r,p}$ as follows

$$\hat{\tau}_{s,t}(\xi) = \frac{s \cdot [\psi(\xi)]^{1/2}}{4\pi} \, \mathrm{G}_{0,3}^{3,0}\left(\begin{array}{cccc} - & - & - & \frac{1}{16}t^2 s^2 \psi \\ -\frac{1}{2} & 0 & 0 & \end{array} \right)$$

Table 3.2 gives $\hat{\tau}_{s,t}$ for $(\mu_{s,t})_{s,t\geq 0}$ subordinated by $\mathrm{f}(x) = x^{\frac{1}{n}}$ in the first step. Having analysed particular cases of subordination of the family $(\mu_{s,t})_{s,t\geq 0}$ we, now, show two constructions of different processes associated to $(\mu_{s,t})_{s,t\geq 0}$. The first one is merely a field of random variables, but nevertheless by definition a stochastic process.

Recall that

$$\hat{\mu}_{s,t}(\xi) = (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-st\psi(\xi)}.$$

Since the function ψ is continuous negative definite and $\psi(0) = 0$, hence by Bochner's theorem $\mu_{s,t}$ is a probability measure on $(\Omega_{s,t}, \mathcal{A}_{s,t})$ for all $s, t \geq 0$, where $\Omega_{s,t} = \mathbb{R}^n$ and $\mathcal{A}_{s,t} = \mathcal{B}^{(n)}$. Defining the product probability space by

$$(\Omega, \mathcal{A}, \mathrm{P}) := \bigotimes_{(s,t)\in\mathbb{R}_+^2} (\Omega_{s,t}, \mathcal{A}_{s,t}, \mu_{s,t})$$

and using Corollary 9.5, p.62 in [3], one gets immediately the existence of an independent family of random variables $(X_{s,t})_{s,t\in\mathbb{R}_+}$ over the defined product

$f(x) = x^{(\cdot)}$	$\hat{\tau}_{s,t}$
$\frac{1}{2}$	$\frac{ts}{4\pi} \times \sqrt{\psi}\ \mathrm{G}^{3,0}_{0,3}\left(\begin{matrix} \\ -\frac{1}{2} \quad 0 \quad 0 \end{matrix} \ \frac{1}{16}t^2s^2\psi\right)$
$\frac{1}{3}$	$\frac{1}{4\pi}\ {}_0F_3\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};\frac{1}{108}t^2s^3\psi\right) - 2\sqrt{3}s\sqrt[3]{t^2\psi}\,\Gamma\left(\frac{1}{3}\right)^2\ {}_0F_3\left(\frac{2}{3},\frac{5}{6},\frac{4}{3};\frac{1}{108}t^2s^3\psi\right)$ $+8\pi\sqrt{t^2s^3\psi}\ {}_0F_3\left(\frac{5}{6},\frac{7}{6},\frac{3}{2};\frac{1}{108}t^2s^3\psi\right) + 3\sqrt{\pi}(2t^2s^3\psi)^{(2/3)}\Gamma\left(\frac{5}{6}\right)\ {}_0F_3\left(\frac{7}{6},\frac{4}{3},\frac{5}{3};\frac{1}{108}t^2s^3\psi\right)$
$\frac{1}{4}$	$\frac{ts^2\sqrt{\psi}}{32\sqrt{2}\pi^2} \times \mathrm{G}^{5,0}_{0,5}\left(\begin{matrix} \\ -\frac{1}{2} \quad -\frac{1}{4} \quad 0 \quad 0 \quad \frac{1}{4} \end{matrix} ; \frac{1}{1024}t^2s^4\psi\right)$
$\frac{1}{5}$	$\frac{5t}{2\sqrt{\pi}}\left(\frac{2}{5}\sqrt{\frac{\pi}{t^2}}\ {}_0F_5\left(\frac{1}{5},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{4}{5};\frac{t^2s^5\psi}{12500}\right) - t^{-3/5}\sqrt{\frac{10}{5\pi+\sqrt{5}\pi}}s\psi^{1/5}\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{6}{5}\right)\ {}_0F_5\left(\frac{2}{5},\frac{3}{5},\frac{7}{10},\frac{4}{5},\frac{6}{5};\frac{t^2s^5\psi}{12500}\right)\right.$ $+\frac{1}{\sqrt{10(5-\sqrt{5})\pi}}\frac{s^2\psi^{2/5}}{t^{1/5}}\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{2}{5}\right)\ {}_0F_5\left(\frac{3}{5},\frac{4}{5},\frac{9}{10},\frac{6}{5},\frac{7}{5};\frac{t^2s^5\psi}{12500}\right) - \frac{8\sqrt{\pi s^5\psi}}{15}\ {}_0F_5\left(\frac{7}{10},\frac{9}{10},\frac{11}{10},\frac{13}{10},\frac{3}{2};\frac{t^2s^5\psi}{12500}\right)$ $\left.\frac{s^3\psi^{3/5}}{30}\left(\frac{t}{2}\right)^{1/5}\Gamma(-\frac{1}{10})\ {}_0F_5\left(\frac{4}{5},\frac{11}{10},\frac{6}{5},\frac{7}{5},\frac{8}{5};\frac{t^2s^5\psi}{12500}\right)\frac{s^4\psi^{4/5}}{120}\left(\frac{t}{2}\right)^{3/5}\Gamma\left(-\frac{3}{10}\right)\ {}_0F_5\left(\frac{6}{5},\frac{13}{10},\frac{7}{5},\frac{8}{5},\frac{9}{5};\frac{t^2s^5\psi}{12500}\right)\right)$
$\frac{1}{6}$	$\frac{ts^3\sqrt{\psi}}{576\sqrt{3}\pi^3} \times \mathrm{G}^{7,0}_{0,7}\left(\begin{matrix} \\ -\frac{1}{2} \quad -\frac{1}{3} \quad -\frac{1}{6} \quad 0 \quad 0 \quad \frac{1}{6} \quad \frac{1}{3} \end{matrix} \ \frac{1}{186624}t^2s^6\psi\right)$
$\frac{1}{8}$	$\frac{ts^4\sqrt{\psi}}{32768\pi^4} \times \mathrm{G}^{9,0}_{0,9}\left(\begin{matrix} \\ -\frac{1}{2} \quad -\frac{3}{8} \quad -\frac{1}{4} \quad -\frac{1}{8} \quad 0 \quad 0 \quad \frac{1}{8} \quad \frac{1}{4} \quad \frac{3}{8} \end{matrix} \ \frac{1}{67108864}t^2s^8\psi\right)$

Table 3.2: Subordination 2 (calculated with Mathematica)

probability space $(\Omega, \mathcal{A}, \mathrm{P})$. The state spaces are equal, hence $(X_{s,t})_{s,t\in\mathbb{R}_+}$ is a stochastic process. Due to the independence we shall not expect any further interesting properties of $(\mathrm{X}_{s,t})_{s,t\in\mathbb{R}_+}$. The opposite is the case for the second process which is associated to $(\mu_{s,t})_{s,t\geq 0}$.

For this let $\gamma = (\gamma_1, \gamma_2) : \mathbb{R}_+ \longrightarrow \mathbb{R}^2_+$ be a continuous, $\preceq$-monotone increasing curve starting at the origin, i.e. $\gamma(0) = (0,0)$. We construct a process along γ, i.e. associated with $(\mu_{s,t})_{(s,t)\in\gamma([0,\infty))}$, by defining a projective family of probability measures.

Let $u_1, \ldots, u_k \in \gamma$ with

$$u_1 \leq u_2 \leq \ldots \leq u_k$$

and

$$\{u_1, u_2, \ldots, u_k\} = \mathrm{K} \in \mathcal{H},$$

for the set $\mathcal{H}$ of all finite subset of $\gamma \subset \mathbb{R}^2_+$.

For K we define the measure $\mathrm{P_K}$ on $(\mathbb{R}^n)^k$ by

$$\begin{aligned}&\mathrm{P_K}(\mathrm{A}_1 \times \mathrm{A}_2 \times \ldots \times A_k) = \\ &= \int_{\mathbb{R}^n}\int_{\mathrm{A}_1}\cdots\int_{\mathrm{A}_k} \mathrm{p}(u_{k-1}, u_k, x_{k-1}, \mathrm{d}x_k)\cdot\ldots\cdot\mathrm{p}(u_1, u_2, x_1, \mathrm{d}x_2)\cdot\mathrm{p}(0, u_1, x, \mathrm{d}x_1)\nu(\mathrm{d}x),\end{aligned}$$

for all $\mathrm{A}_1, \ldots, \mathrm{A}_k \in \mathcal{B}(\mathbb{R}^n)$.

Here

$$\mathrm{p}(u_j, u_{j+1}, x_j, \mathrm{A}) = \mu_{u_j,u_{j+1}}(\mathrm{A} - x_j),$$

for all $\mathrm{A} \in \mathcal{B}^{(n)}$. Where $\mu_{u_j,u_{j+1}}(\mathrm{A} - x_j)$ is defined by:

$$\hat{\mu}_{u_j,u_{j+1}}(\xi) = (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-(\gamma_1(u_{j+1})\gamma_2(u_{j+1}) - \gamma_1(u_j)\gamma_2(u_j))\psi(\xi)},$$

ν is the initial distribution.

This family is projective, and applying Kolomogorov's theorem it follows the existence of a probability measure P on $\mathcal{B}^{(n)}$ satisfying

$$\pi_{\mathrm{K}}(\mathrm{P}) = \mathrm{P_K}, \text{ for all } \mathrm{K} \in \mathcal{H},$$

and the existence of a stochastic process with state space $\mathbb{R}^n$, whose finite-dimensional distributions are given by $(\mathrm{P_K})_{\mathrm{K}\in\mathcal{H}}$.

More general, we can consider $(2\pi)^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{k}(t,\xi)}$, where we have to assume, that for fixed $t \in \mathbb{R}_+^N$ $k(t,.)$ is a continuous negative definite function and for all $t \succeq s$ and $\xi \in \mathbb{R}^n$

$$\mathrm{k}(t,\xi) \geq \mathrm{k}(s,\xi).$$

3.2 Commuting Structure on a Curvilinear Net

Previously we have constructed a multiparameter family of measures and one corresponding multiparameter process which does in general not satisfy the semigroup property or is not time-parameter homogeneous, respectively. However, in the case of $s = 0$ or $t = 0$ the measures were degenerated, moreover, we did not obtain any nice structure, meaning, we did not find the commuting property which is common to all semigroups mentioned in Chapter 2. Now we want to give an example of a 2-parameter family of measures - defined by its Fourier transform - which is not a convolution semigroup, and we will establish structural property similar to the commuting property in the sense, that the family of measures will be commuting on a skewed net which we will obtain by transformation of variables in the parameter space.

We start with the family $(\mu_{s,t})_{s,t\geq 0}$ of probability measures, which we define by its Fourier transform:

$$\hat{\mu}_{s,t}(\xi) := (2\pi)^{-\frac{n}{2}} e^{-(s+t+t\cdot\arctan(s))\cdot\psi(\xi)} \tag{3.3}$$

where ψ is a continuous negative definite function, $\xi \in \mathrm{R}^n$, and $s,t \in \mathrm{R}_+$. For $s_0 = 0$ $(\mu_{s_0,t})_{t\geq 0}$ and for $t_0 = 0$ $(\mu_{s,t_0})_{s\geq 0}$ are convolution semigroups. However, for any other fixed $s_0 > 0$ and $t_0 > 0$ we find that $(\mu_{s_0,t})_{t\geq 0}$ and $(\mu_{s,t_0})_{s\geq 0}$, respectively, are no convolution semigroups and consequently $(\mu_{s,t})_{s,t\geq 0}$ is not a two-parameter convolution semigroup. One can also argue, that

$$\hat{\mu}_{s_0,t_0} \cdot \hat{\mu}_{s_1,t_1} \neq \hat{\mu}_{s_0+s_1,t_0+t_1} \tag{3.4}$$

and hence

$$\mu_{s_0,t_0} * \mu_{s_1,t_1} \neq \mu_{s_0+s_1,t_0+t_1}. \tag{3.5}$$

Although $(\mu_{s,t})_{s,t\geq 0}$ is not a convolution semigroup for $s,t \geq 0$ each $\mu_{s,t}$ is a probability measure and on the Schwartz space we can define the following family of operators. Let $s,t \in \mathbb{R}_+$ and $\mathrm{u} \in \mathcal{S}(\mathbb{R}^n)$, then

$$\mathrm{T}_{s,t}\mathrm{u}(x) = (2\pi)^{-\frac{n}{2}} \int e^{\mathrm{i}x\cdot\xi} e^{-(s+t+t\arctan(s))\cdot\psi(\xi))}\hat{\mathrm{u}}(\xi)\mathrm{d}\xi \tag{3.6}$$

is a well-defined 2-parameter family of operators, since $0 \leq \mathrm{e}^{-(s+t+t\cdot\arctan(s))\cdot\psi(\xi)} \leq 1$, but it is not a semigroup of operators with respect to the two-dimensional parameter vector (s,t). We want to find a two-parameter convolution semigroup $(\eta_{\sigma,\tau})_{\sigma,\tau\geq 0}$ such that

$$\mu_{s,t} = \eta_{\sigma,\tau},$$

for all $s,t \geq 0$ and $\sigma = \mathrm{g}_1(s,t)$, $\tau = \mathrm{g}_2(s,t)$ for some mapping

$$g : (s,t) \longmapsto (\sigma,\tau) = g(s,t) = (g_1(s,t), g_2(s,t)).$$

To find a convolution semigroup $(\eta_{\sigma,\tau})_{\sigma,\tau\geq 0}$ with this property we define the following transformation of variables

$$\begin{aligned} \sigma &:= \sigma(s,t) = s \\ \tau &:= \tau(s,t) = [1+\arctan(s)]\cdot t \end{aligned}$$

hence

$$\begin{aligned} s &= \sigma \\ t &= \tfrac{\tau}{1+\arctan(\sigma)}. \end{aligned}$$

and

$$\det\begin{pmatrix} \frac{\partial\sigma}{\partial s} & \frac{\partial\sigma}{\partial t} \\ \frac{\partial\tau}{\partial s} & \frac{\partial\tau}{\partial t} \end{pmatrix} = 1+\arctan(s),$$

hence determinant of the Jacobian is non-zero for all $s, t \geq 0$ and herewith the transformation is one-to-one for all $s, t \in \mathbb{R}_+$. We emphasise that $(\eta_{\sigma,\tau})_{\sigma,\tau\geq 0}$ has the Fourier transform

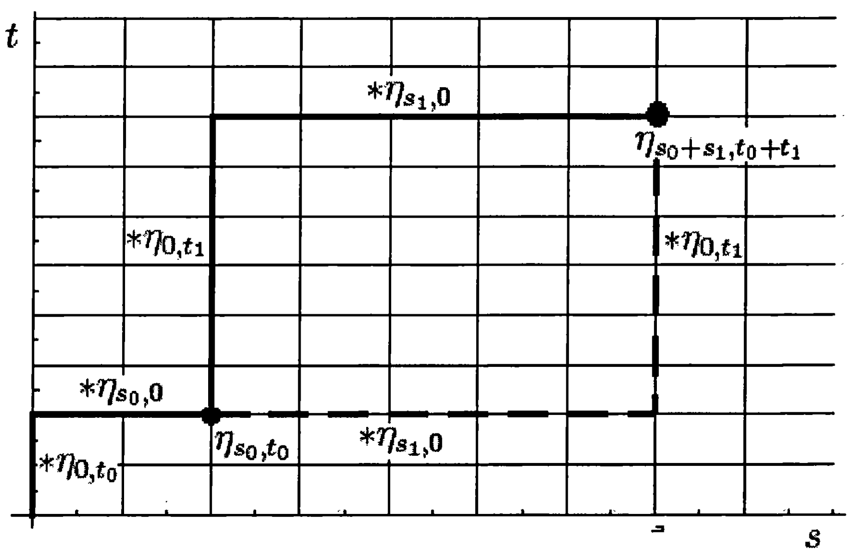
$$\hat{\eta}_{\sigma,\tau}(\xi) = (2\pi)^{\frac{n}{2}} \mathrm{e}^{-\sigma\psi(\xi)-\tau\psi(\xi)},$$

for all $\sigma, \tau \geq 0$ and $\xi \in \mathbb{R}$, and hence is a convolution semigroup while $(\mu_{s,t})_{s,t\geq 0}$ is not. However, the semigroup property of $(\eta_{\sigma,\tau})_{\sigma,\tau\geq 0}$ is reflected on $(\mu_{s,t})_{s,t\geq 0}$ as the following equation shows

$$\mu_{s_0,t_0} * \mu_{s_1,t_1} = \eta_{\sigma_0,\tau_0} * \eta_{\sigma_1,\tau_1} = \eta_{\sigma_0+\sigma_1,\tau_0+\tau_1} = \mu_{\mathrm{g}^{-1}(\sigma_0+\sigma_1,\tau_0+\tau_1)},$$

where $\sigma_0 = \mathrm{g}_1(s_0, t_0)$, $\tau_0 = \mathrm{g}_2(s_0, t_0)$, $\sigma_1 = g_1(s_1, t_1)$, $\tau_1 = \mathrm{g}_2(s_1, t_1)$. Herewith we find that the semigroup property of $(\mu_{\sigma,\tau})_{\sigma,\tau\geq 0}$ reflects on $(\mu_{s,t})_{s,t\geq 0}$ as a "curvilinear convolution additivity", more specifically, a convolution additivity on the pre-image of the parallels of the coordinate axes of the (σ, τ)-coordinate system. To visualise this let us consider the convolution:

$$\eta_{0,0} * \eta_{0,t_0} * \eta_{s_0,0} * \eta_{0,t_1} * \eta_{s_1,0} = \eta_{s_0+s_1,t_0+t_1}$$



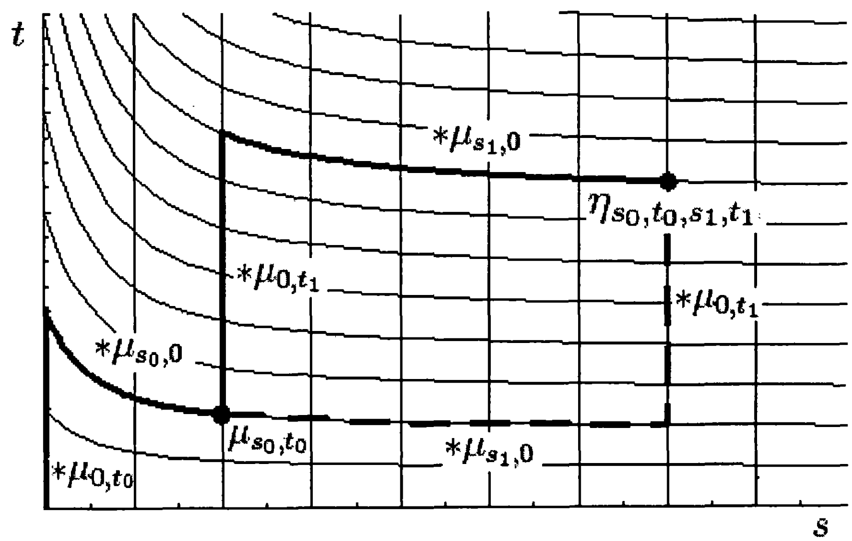
We find for all $s, t \geq 0$ that

$$\eta_{s,0} = \mu_{s,0} \quad \text{and} \quad \eta_{0,t} = \mu_{0,t}.$$

Thus

$$\mu_{0,0} * \mu_{0,t_0} * \mu_{s_0,0} * \mu_{0,t_1} * \mu_{s_1,0} = \eta_{s_0+s_1,t_0+t_1} = \mu_{g^{-1}(s_0+s_1,t_0+t_1)}.$$

The curvilinear convolution additivity becomes apparent in the following graphic, where $\hat{\eta}(\xi) = (2\pi)^{\frac{n}{2}} e^{-(s_1+t_1+(t_0+t_1)\arctan(s_0+s_1)-t_0\arctan(s_0))\psi(\xi)}$.



The convolution is additive on a curvilinear net. Especially notice that the path from (s_0, t_0) to $(s_0 + s_1, t_0 + t_1)$ does not effect the distribution of $(X_{s_0+s_1,t_0+t_1} - X_{s_0,t_0})$.

We now investigate how the partial differential equation associated to $\mathrm{T}_{s,t}$, see (3.6), behaves under the variable transform. We perform the transform $(s,t) \longmapsto (\sigma, \tau)$ on the Fourier-transformed differential equation and expect the resulting equation to be associated with $\mathrm{T}_{\sigma,\tau}$.

$$\frac{\partial^2 \mathrm{u}}{\partial s \partial t} - \frac{\partial^2}{\partial s \partial t}\left(e^{-(s+t+t\arctan(s))\psi}\mathrm{u}\right) = 0, \tag{3.7}$$

$$\frac{\partial^2 \mathrm{u}}{\partial s \partial t} + \frac{1}{1+s^2}\psi \mathrm{u} - \frac{(1+t+s^2)(1+\arctan(s))}{1+s^2}\psi^2 \mathrm{u} = 0. \tag{3.8}$$

Now performing the above-defined transformation of variables we obtain

$$
\begin{aligned}
(1+\arctan(\sigma))\frac{\partial^2 v}{\partial\sigma\partial\tau} + \frac{\tau}{1+\sigma^2}\cdot\frac{\partial^2 v}{\partial\tau^2} + \frac{1}{1+\sigma^2}\cdot\frac{\partial v}{\partial\tau} + \frac{1}{1+\sigma^2}\psi v \\
-\frac{(1+\sigma^2)(1+\arctan(\sigma))+\tau}{1+\sigma^2}\psi^2 v = 0. \qquad (3.9)
\end{aligned}
$$

Where in the latter equation v is defined such that

$$
\mathrm{v}(\sigma,\tau) = \mathrm{v}(\sigma(s,t),\tau(s,t)) = \mathrm{u}(s,t).
$$

Now we assume that there exists a solution of the form

$$
v(\sigma,\tau) = \mathrm{e}^{-\sigma\psi}\cdot \mathrm{w}(\tau;\psi), \qquad (3.10)
$$

and in the previous differential equation we substitute v by the term on the right hand side of (3.10) and we get:

$$
\begin{aligned}
0 = \mathrm{e}^{-\sigma\psi}\Bigg(-(1+\arctan(\sigma))\psi\frac{\partial \mathrm{w}}{\partial\tau} + \frac{\tau}{1+\sigma^2}\cdot\frac{\partial^2 \mathrm{w}}{\partial\tau^2} + \frac{1}{1+\sigma^2}\frac{\partial \mathrm{w}}{\partial\tau} + \\
+\frac{1}{1+\sigma^2}\psi \mathrm{w} - \frac{(1+\sigma^2)(1+\arctan\sigma)+\tau}{1+\sigma^2}\psi^2 \mathrm{w}\Bigg). \qquad (3.11)
\end{aligned}
$$

This reduces the partial differential equation to the ordinary differential equation:

$$
\begin{aligned}
0 = -(1+\arctan(\sigma))\psi\frac{\partial \mathrm{w}}{\partial\tau} + \frac{\tau}{1+\sigma^2}\cdot\frac{\partial^2 \mathrm{w}}{\sigma\tau^2} + \frac{1}{1+\sigma^2}\frac{\partial \mathrm{w}}{\partial\tau} \\
+\frac{1}{1+\sigma^2}\psi \mathrm{w} - \frac{(1+\sigma^2)(1+\arctan\sigma)+\tau}{1+\sigma^2}\psi^2 \mathrm{w}.
\end{aligned}
$$

The in our case interesting w which solves this equation is

$$
\mathrm{w} = \mathrm{e}^{-\psi\tau}.
$$

Hence equation (3.7) is solved by

$$
v(\sigma,\tau) = \mathrm{e}^{-\sigma\psi}\cdot \mathrm{e}^{-\tau\psi}. \qquad (3.12)
$$

The other possible solutions $v_1(\sigma,\tau) = \mathrm{e}^{\sigma\psi}\cdot \mathrm{e}^{\tau\psi}$, $v_2(\sigma,\tau) = \mathrm{e}^{-\sigma\psi}\cdot \mathrm{e}^{\tau\psi}$, and $v_3(\sigma,\tau) = \mathrm{e}^{\sigma\psi}\cdot \mathrm{e}^{-\tau\psi}$ are, in this context, of little interest, since they are not positive definite and do not correspond to convolution semigroups.

Now, more generally, we consider $(\mu_{s,t})_{s,t\geq 0}$ with

$$\hat{\mu}_{s,t}(\xi) = (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-\mathrm{k}(s,t)\cdot\psi(\xi)} \tag{3.13}$$

for all $s,t \geq 0$ and $\xi \in \mathbb{R}^n$, where $\mathrm{k} : \mathrm{D} \longrightarrow \mathbb{R}$, $\mathrm{D} \subset \mathbb{R}^2_+$, is a C^1-function and ψ is a continuous negative definite function from $\mathbb{R}^n$ to $\mathbb{R}$. Then by Bochner's theorem $\mu_{s,t}$ is a sub-probability measure for all pairs (s,t) for which $\mathrm{k}(s,t) \geq 0$ holds. Therefore we define

$$\mathrm{D}_0 := \{(s,t) \in \mathrm{D} : \mathrm{k}(s,t) \geq 0\}\,. \tag{3.14}$$

We are aiming at constructing a process with an oriented parameter set $\gamma \subset \mathrm{D}_0$ following Kolmogorov's theorem. Thus we need to find a suitable order $\preceq_*$ on the parameter set D_0 or on γ, such that

$$(\gamma, \preceq_*) \tag{3.15}$$

is a totally order set.

By suitable we mean that for any two pairs $(s_1,t_1),(s_2,t_2) \in \gamma$ the following implication holds:

$$(s_1,t_1) \preceq_* (s_2,t_2) \qquad \Longrightarrow \qquad \mu_{s_2,t_2} = \mu_{s_1,t_1} * \eta, \tag{3.16}$$

where $\eta = \eta_{s_1,t_1;s_2,t_2}$ is a probability measure (on $\mathbb{R}^n$). Desirable is also the reverse implication of (3.16).

Remark. *In case* $\mathrm{k}(s,t) = s \cdot t$ *we find*

$$(s_1,t_1) \preceq (s_2,t_2) \Longrightarrow \mathrm{k}(s_1,t_1) \leq \mathrm{k}(s_2,t_2), \tag{3.17}$$

where $\preceq$ *denotes the partial order defined in Section 1.1. For pairs* (s_1,t_1) *and* (s_2,t_2)*, such that* $(s_1,t_1) \npreceq (s_2,t_2)$ *and* $(s_1,t_1) \nsucceq (s_2,t_2)$*, we cannot draw any conclusion. Hence the partial order* $\preceq$ *is insufficient in this case.*

Using (3.13) we rewrite (3.16)

$$(s_1,t_1) \preceq_* (s_2,t_2) \Longrightarrow \mathrm{e}^{-\mathrm{k}(s_2,t_2)\psi(\xi)} = \mathrm{e}^{-\mathrm{k}(s_1,t_1)\psi(\xi)}\,(2\pi)^{\frac{n}{2}}\hat{\eta}(\xi), \tag{3.18}$$

and get

$$\hat{\eta}(\xi) = (2\pi)^{-\frac{n}{2}}\mathrm{e}^{-(\mathrm{k}(s_2,t_2)-\mathrm{k}(s_1,t_1))\psi(\xi)}. \tag{3.19}$$

Hence, for η to be a probability measure it is necessary and sufficient to assume that $\mathrm{k}(s_2,t_2)-\mathrm{k}(s_1,t_1)$ is positive, and (3.16) can finally be reformulated as

$$(s_1,t_1) \preceq_* (s_2,t_2) \Longrightarrow \mathrm{k}(s_1,t_1) \leq \mathrm{k}(s_2,t_2). \tag{3.20}$$

Vice versa, if $(s_1,t_1),(s_2,t_2) \in \mathbb{R}^2_+$ are such that

$$0 \leq \mathrm{k}(s_1,t_1) \leq \mathrm{k}(s_2,t_2), \tag{3.21}$$

then there exists an η such that

$$\mu_{s_2,t_2} = \mu_{s_1,t_1} * \eta \tag{3.22}$$

where η is defined by (3.19).

In the multiparameter case of Chapter 2, admissible successive time-parameter points for a process, which has reached time-parameter $p \in \mathbb{R}^N_+$, are exactly those of the subset $\{r : p \preceq r\}$, i.e. the positive cone with vertex p.

Now assume that there exists a bicontinuous and injective map I from D to $\mathbb{R}^2_+$, $\mathrm{I} : (s,t) \longmapsto (\sigma,\tau)$ such that

$$k(s,t) = |\mathrm{I}(s,t)|^2 = \sigma + \tau.$$

Moreover, we define a convolution semigroup $(\nu_{\sigma,\tau})_{\sigma,\tau\geq 0}$ of probability measures by

$$\hat{\nu}_{\sigma,\tau}(\xi) = (2\pi)^{-\frac{n}{2}}\mathrm{e}^{-(\sigma+\tau)\psi(\xi)},$$

for all $\sigma,\tau \geq 0$. Then for $(s_1,t_1) \in \mathbb{R}^2_+$ and $(\sigma_1,\tau_1) = \mathrm{I}(s_1,t_1)$ we have

$$\mu_{s_1,t_1} = \nu_{\sigma_1,\tau_1}.$$

Admissible successive points with respect to (σ_1, τ_1) are those in the cone

$$\mathrm{C_I}(\sigma_1, \tau_1) := \left\{(\sigma, \tau) \in \mathbb{R}^2_+ : (\sigma_1, \tau_1) \preceq (\sigma, \tau)\right\},$$

thus we get the following set $\mathrm{C}(s_1, t_1)$ of admissible successive points for (s_1, t_1) with respect to $(\mu_{s,t})_{s,t\geq 0}$ in form of the pre-image of $\mathrm{C_I}(\sigma_1, \tau_1)$ under the map I:

$$\mathrm{C}(s_1, t_1) := \mathrm{I}^{-1}\left(\mathrm{C_I}(\sigma_1, \tau_1)\right) = \left\{(s, t) \in \mathrm{D} : \mathrm{I}(s, t) \geq \mathrm{I}(s_1, t_1)\right\}.$$

If $(s_2, t_2) \in \mathrm{C}(s_1, t_1)$ then we denote it by

$$(s_2, t_2) \preceq_{\mathrm{I}} (s_1, t_1). \tag{3.23}$$

This relation satisfies (3.20), hence suffices our needs.

Remark. *There may be pairs $(s_1, t_1), (s_2, t_2) \in \mathbb{R}^2_+$ with both $(s_1, t_1) \preceq_{\mathrm{I}} (s_2, t_2)$ and $(s_1, t_1) \succeq_{\mathrm{I}} (s_2, t_2)$. This, however, will not effect the Kolmogorov construction of a process.*
If one chooses the function $\mathrm{k}(s, t)$ to be strictly monotone increasing $\preceq_{\mathrm{I}}$ becomes a total order, but since this would exclude curves γ^* which are gripped by $\preceq_{\mathrm{I}}$, we do not for strict monotonicity.

Now define for all $c \geq 0$ a set D_c by

$$\mathrm{D}_c = \left\{(s, t) \in \mathrm{D} : \mathrm{k}(s, t) \geq c\right\}. \tag{3.24}$$

Furthermore, let γ^* be an injective C^1-function from $[0, \infty)$ onto γ. Then γ^* is called a parameterisation of γ. If in addition γ^* is increasing with respect to $\preceq_{\mathrm{I}}$, then it is called **I-admissible**.
We conclude that for a I-admissible parameterisation γ^* of γ, analogously to Chapter 3.1, there exists a projective family associated to the family of probability measures $(\mu_{s,t})_{(s,t)\in\gamma}$. Thus by Kolmogorov's theorem there exists a process associated to $(\mu_{s,t})_{(s,t)\in\gamma}$.

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