Lumps, rational maps and monopoles.

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Lumps, Rational Maps and Monopoles

by

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To Maya and Rosanna

One thing only I know, and that is that I know nothing.

Socrates

469 – 399 BC

Abstract

We study two models of topological solitons in which there exists a correspondence between static solitons and rational maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$. In both cases, the domain of the rational map can be identified as a two-sphere with the round metric.

The first of these is the $\mathbb{C}P^1$ model on $S^2 \times \mathbb{R}$, in which lump configurations may be described directly in terms of a rational map. We parametrise the moduli space of static two-lumps using natural group actions and consider low energy dynamics using the geodesic approximation, identifying a number of geodesic submanifolds. The metric is found explicity for one of these submanifolds and the corresponding geodesics, which describe the "scattering" of lumps, are discussed. In particular there is a four dimensional submanifold which exhibits qualitatively similar features to the low energy scattering of monopoles, including right angle scattering.

The second model is the $SU(2)$ Yang-Mills-Higgs model in the Prasad-Sommerfield limit in which static solutions are described by the Bogomol'nyi equations. We discuss the rational map and the metric introduced by Jarvis and show how solutions of the linear system corresponding to the Bogomol'nyi equations give rise to solutions of the Jarvis equation in a simple way. Since the linear system is covariant under the action of the Galilean group, this gives a procedure for finding the solution of the Jarvis equations corresponding to a translated monopole. We carry out this procedure explicitly for a single monopole to obtain solutions corresponding to a monopole with arbitrary position in the Jarvis gauge, from which the Jarvis rational map is obtained. We introduce the dual rational map and give an argument relating the spectral lines through the origin to the Jarvis rational map and its dual. A novel functional relation obeyed by the metric is presented. We show how the one-monopole solution to the linear system in the Jarvis gauge justifies the charge one inverse scattering ansatz and show that the seed solution depends on the Jarvis rational map and its dual. We also discuss the effect of infinitesimal translations on solutions to the Jarvis equation and the possible relationship between the Jarvis rational map and centre of a monopole.

Declarations

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

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This thesis is the result of my own investigations, except where otherwise stated. Sources are acknowledged by reference to a bibliography.

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Contents

Chapter 1

Introduction

Many physical theories of the natural world are described by non-linear differential equations. A surprising fact is that some of these theories admit solutions whose energy density is localised and which behave in many respects like smooth extended particles. Such solutions are known as *solitons.* Solitons have found a wide range of uses in describing phenomena in many branches of physics and have been observed, for example, in condensed matter system and in fluids.

Solitons have appeared in many theories of high energy particle physics, and new uses for them and their parameter, or *moduli* spaces, continue to be found. In some supersymmetric quantum field theories, solitons occur as states of the theory which may be related by a *duality* to the fundamental fields. Such dualities have become a powerful tool for probing the strongly-coupled region of such theories. Perhaps more importantly for us, though, is the fact that the study of solitons has turned into an extremely rich field of research for mathematicians and physicists alike.

In this thesis, we will be interested in two models of solitons in particular; those describing magnetic monopoles and sigma model lumps. These are examples of topological solitons which occur in theories in which the space of vacua is a manifold with non-trivial topology. Solitons are solutions which interpolate between different vacua and thus pick up a topological classification.

The theories of SU(2) magnetic monopoles in Minkowski space and lumps in the $\mathbb{C}P^1$ model on $\mathbb{R}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$ share a number of interesting characteristics. The first of these is the fact that the solitons of both theories fall into topological sectors classified by the second homotopy group of the two-sphere $\Pi_2(S^2)$. The second is that there are correspondences between static solitons and rational maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$. In the case of lumps, static configurations may be described directly as rational maps. For monopoles, the correspondence of the fields of the monopole with a rational map is much less explicit.

It should be pointed out that there is more than one correspondence of monopoles to rational maps. The first of these was found by Donaldson from a study of Nahm's equations [41]. In this case, the rational map is defined in terms of a complex coordinate parametrising the plane orthogonal to a fixed direction. The second, the Jarvis rational map [34], was discovered relatively recently and is defined in terms of the complex coordinate on the Riemann sphere describing the direction from a fixed point.

Given a monopole, the definition of a rational map is relatively straightforward and is defined in terms of solutions to a particular differential operator along straight lines. This is the scattering operator introduced by Hitchin [25] and used to define the spectral curve associated to a monopole. Hurtubise [42] showed that the Donaldson rational map can be defined in terms of scattering along lines in a fixed direction, while the Jarvis rational map can be defined in terms of scattering on half-lines emanating from a point in $\mathbb{R}^3$.

However, the construction of the fields of the monopole given a rational map is much less explicit. Jarvis has given an argument based on a heat flow for both the Donaldson [53, 54] and the Jarvis rational maps [34], in which the asymptotic fields are given in terms of the rational map. The result of the heat flow is a solution to the Bogomol'nyi equations, unique up to gauge equivalence, which reproduces the required rational map, thus proving the correspondence in each case. So far, attempts to make this construction more explicit have been limited to trying to clarify the asymptotic conditions on the fields in terms of the Jarvis rational map [47], although we will have cause to question these results in Chapter 5. The heat flow method has also been used by Ioannidou and Sutcliffe to generate monopole solutions numerically using the Jarvis map as input [48].

The n-monopole moduli space has a group of isometries which includes the Galilean group of $\mathbb{R}^3$. However, choosing a fixed direction or a fixed point means that only that part of the Galilean group which fixes this acts in a simple way on the relevant rational map. If we choose a fixed direction, then this is preserved by translations and rotations about this axis, therefore this subgroup has a prescribed action on the Donaldson rational map [55], while the action of rotations which change the direction is, in general, unknown. On the other hand, a fixed point is preserved by rotations about this point but not by translations. Hence the action of rotations on the Jarvis map is well understood while the action of translations is not. This fact has been used to prove the existence of monopoles with certain discrete rotational symmetries [49].

Ioannidou and Sutcliffe suggest that the inverse scattering method may be useful in making the correspondence between a monopole and its Jarvis rational map more explicit. This method was first applied to monopoles by Forgács, Horváth and Palla [44–46] who were able to generate the general two-monopole and axially symmetric n-monopoles in this way, obtaining solutions in singular gauges. The difficulty with this method is that there are a large number of arbitrary functions which are used as input and these must be chosen in such a way as to ensure the resulting solution is smooth. It would be nice to be able to determine these functions in terms of the Jarvis rational map, say.

The inverse scattering method makes use of the linear system for the Bogomol'nyi equations which corresponds to a zero curvature condition or "Lax pair" for two operators depending on an extra parameter. Solutions to the linear system may be translated or rotated to obtain other solutions, so the full Galilean group has a straightforward action. Additionally, we show that, given any solution to the linear system, the gauge transformation which takes us to a gauge in which the Jarvis equation holds is simply found by evaluating the solution at a value of the parameter depending on the complex coordinate z. If the solution also obeys a particular conjugate relation, then we can obtain a solution in the Jarvis gauge described by a unimodular Hermitian metric from which the Jarvis rational map may be obtained.

The moduli space approximation has become a useful tool for studying the low energy dynamics of theories of solitons. This was first proposed by Manton to describe the low-energy dynamics of magnetic monopoles [18]. Atiyah and Hitchin found the metric on the two-monopole moduli space [32] and used it to describe the scattering of monopoles. In general, the difficulty in applying this method to monopoles lies in the fact that it is very difficult to write down explicit multimonopole field configurations.

$\mathbb{C}P^1$ sigma models in 2+1 dimensions have proven to be especially amenable to the moduli space approximation since static field configurations can be written down explicitly, hence the calculation of the metric is simply a matter of performing the kinetic energy integral. Furthermore, since, for the $\mathbb{C}P^1$ model on $\mathbb{R}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$, the static solutions are described explicitly as rational maps, the metrics we obtain depend explicitly on the parameters appearing in these maps. Thus we can think of this as a model for describing moduli space metrics directly in terms of rational maps which could perhaps be extended to monopole moduli spaces.

The metric has been found for one- and two-lumps in flat space [27,28] on which some components of the metric are unbounded. Recently, the focus has shifted to working on spacetimes where space is compact but time remains in $\mathbb{R}$, which avoids this problem. So far two-lumps on a two-torus [30] and a single lump on the two-sphere [29] have been studied. We will extend this programme by considering two-lumps on the two-sphere. Rather than try to compute the metric on the full 10-dimensional moduli space of two-lumps and then discuss behaviour on some geodesic submanifolds, we take the simpler approach of first restricting to a geodesic submanifold on which it is possible to use isometries to find a metric dependent on a single parameter.

Whilst the analogy between the rational map description of monopoles and lumps has motivated our study of lumps, the metrics we obtain cannot be the same as those for monopoles. This is because the moduli space of $\mathbb{C}P^1$ lumps on an arbitrary compact Riemann surface is not geodesically complete with respect to the metric obtained using the geodesic approximation [31], whereas monopole moduli spaces are [56–58]. In practice, this means that, in this approximation, lumps can shrink to zero size in finite time. So while the similarities may be qualitatively interesting, some modification of the $\mathbb{C}P^1$ model must be necessary if we hope to reproduce monopole moduli space metrics in this way. Having said this, the "scattering" of lumps we find shows qualitatively similar behaviour to the scattering of monopoles, including right-angle scattering and "dyon" pair production.

The plan of the thesis is as follows. The following chapter gives an introduction to the models we study and the rational maps they have in common. Chapter 3 describes our work on two-lumps on $S^2 \times \mathbb{R}$, parametrising the space of static two-lumps in terms of group actions and identifying geodesic submanifolds. The metric is calculated on one of these submanifolds to describe the low-energy "scattering" of lumps. Plots showing the potential energy density of various two-lump configurations, including those of three interesting geodesic motions, are included. The MATHEMATICA code used to generate these plots is given in an appendix.

The rest of the thesis concerns BPS monopoles. In Chapter 4, we introduce the linear system for the Bogomol'nyi equations and show how a solution to the linear system gives rise to a solution of the Jarvis equation in the Jarvis gauge. We calculate the solution to the linear system corresponding to the BPS monopole at the origin and, by translating this solution, obtain a solution in the Jarvis gauge corresponding to a single monopole with arbitrary position from which the Higgs field and rational map are calculated.

In Chapter 5, we introduce the spectral curve of a monopole and give an argument relating the spectral lines through the origin to the Jarvis rational map. We also tentatively introduce a functional condition on the metric. We discuss the asymptotic conditions on the metric and the Higgs field in the Jarvis gauge, finding some disagreement with the analysis of Ioannidou and Sutcliffe [47]. This chapter also contains some of our first work on the Jarvis rational map. Firstly, by considering translational zero modes of the Bogomol'nyi equations, we show how an infinitessimal translation acts on the solution to the Jarvis equation and give the example of translating the spherically symmetric one-monopole. Secondly, we discuss the centre of a monopole and how it may be related to the Jarvis rational map.

In Chapter 6, we give the charge one inverse scattering argument of Forgács, Horváth and Palla and show how the solution found in Chapter 4 reproduces the charge one ansatz. We use this to try and start a course on constructing higher charge monopoles with the inverse scattering ansatz using the Jarvis rational map as input. Lastly, we summarize our work and point to extensions in Chapter 7.

Chapter 2

Lumps, Rational Maps and Monopoles

We said in the previous chapter that magnetic monopoles and sigma-model lumps are two examples of what may be called topological solitons. The models in which they occur are different in terms of the number of dimensions and the number or fields involved, but both share a classification in terms of the homotopy of maps from the two-sphere S^2 to itself. Both also share the fact that a bound on the energy can be found within a given topological sector, and static solutions which obtain, or *saturate*, this bound satisfy a first-order differential equation. In the case of monopoles this also involves taking a particular limit of the theory.

Our real interest lies in the fact that there is a correspondence with spaces of rational maps for both lumps and magnetic monopoles satisfying the first order equations. In the models considered, these rational maps are holomorphic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$.

The purpose of this chapter is to introduce the models which we will study in the remainder of this thesis and the pertinent results on rational maps.

2.1 Lumps

We begin by describing the simpler of the two models, the $\mathbb{C}P^1$ model in its guise as the $O(3)$ model on $\mathbb{R}^2 \times \mathbb{R}$. A discussion of the degree of a map leads us to recast the $O(3)$ model in terms of a complex field living in $\mathbb{C}P^1$, thereby obtaining the $\mathbb{C}P^1$ model on $\mathbb{R}^2 \times \mathbb{R}$. Using the natural metric on $S^2 \times \mathbb{R}$, we can obtain the Lagrangian for the $\mathbb{C}P^1$ model on the sphere which is the model studied in Chapter 3. We also discuss the geodesic approximation which will be our tool for studying the low-energy dynamics of lumps in this model.

2.1.1 The $O(3)$ Model

The $O(3)$ model can be considered the continuum limit of a model of a planar isotropic ferromagnet where the field represents the direction of the magnetic domain at a given point, all domains being of the same magnetic strength. Our general reference for the study of lumps is the book by Rajaraman [1].

The model consists of three real scalar fields ϕ_a, $a = 1 \ldots 3$, which we write as a vector $\boldsymbol{\phi}$, subject to the constraint

$$\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1. \tag{2.1}$$

In other words, $\boldsymbol{\phi}$ is constrained to take values in the unit two-sphere, in sigma-model terms, the *target space* of the model.

The fields live in a 2+1-dimensional spacetime with the action

$$S = \int dt \int d^2x \left[\frac{1}{2}\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} + \lambda(\mathbf{x}, t)(\boldsymbol{\phi} \cdot \boldsymbol{\phi} - 1)\right]. \tag{2.2}$$

This action is clearly invariant under the global $O(3)$ transformation

$$\boldsymbol{\phi} \to M\boldsymbol{\phi}, \qquad M \in O(3). \tag{2.3}$$

The constraint (2.1) is imposed by means of the Lagrange multiplier λ which can be thought of as an auxiliary field. Varying $\boldsymbol{\phi}$ gives the equation of motion

$$\partial_\mu \partial^\mu \boldsymbol{\phi} + \lambda \boldsymbol{\phi} = 0, \tag{2.4}$$

whilst demanding that the action is stationary under a variation of λ recovers the constraint (2.1). We can eliminate λ by taking the scalar product of $\boldsymbol{\phi}$ with the equation of motion (2.4) obtaining

$$\lambda = -\boldsymbol{\phi} \cdot \partial_\mu \partial^\mu \boldsymbol{\phi}. \tag{2.5}$$

Thus the equation of motion is finally

$$\partial_\mu \partial^\mu \boldsymbol{\phi} - (\boldsymbol{\phi} \cdot \partial_\mu \partial^\mu \boldsymbol{\phi})\boldsymbol{\phi} = 0. \tag{2.6}$$

We now look for finite-energy static solutions. These will be the *lumps*, the name given to the solitons of this theory. The energy of a static solution is

$$E = \frac{1}{2}\int d^2x \, \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi}. \tag{2.7}$$

For this to be finite we require

$$\partial_i \boldsymbol{\phi} \sim o(r^{-1}) \text{ as } r \to \infty \tag{2.8}$$

so that

$$\lim_{r\to\infty} \boldsymbol{\phi}(\mathbf{x}) = \boldsymbol{\phi}_0, \tag{2.9}$$

independent of the direction. Since $\boldsymbol{\phi}$ is single-valued at infinity, we can consider the fields to be defined on $\mathbb{R}^2 \cup \{\infty\}$ which we can identify with S^2 via stereographic projection. Thus, just from the constraint of finite energy, we find that static configurations can be identified with continuous maps from S^2 to S^2.

$\boldsymbol{\phi}(\mathbf{x}) = \boldsymbol{\phi}_0$, a constant unit-length vector, is a zero-energy solution to the equations of motion. Thus the space of vacua form the two-sphere S^2, which we can also identify with the coset space $O(3)/O(2)$.

Just as continuous maps from S^1 to S^1 are characterised by an integer winding number, continuous maps from S^2 to S^2 fall into different *homotopy classes*, each labelled by a integer which is the *degree* of the map. These integers are properly seen to be elements of the homotopy groups $\Pi_1(S^1)$ and $\Pi_2(S^2)$ respectively, both of which are isomorphic to $\mathbb{Z}$.

The degree of the map $\boldsymbol{\phi}$ can be written as follows

$$Q = \frac{1}{8\pi}\int d^2x\, \epsilon_{ij}\boldsymbol{\phi}\cdot(\partial_i\boldsymbol{\phi}\times\partial_j\boldsymbol{\phi}). \tag{2.10}$$

To show that this is an integer we will introduce some new coordinates. We can define a complex field from $\boldsymbol{\phi} \in S^2$ using stereographic projection.

$$u = \frac{\phi_1 + i\phi_2}{1-\phi_3} \qquad \boldsymbol{\phi} = \left(\frac{u+\bar{u}}{1+u\bar{u}}, -i\frac{u-\bar{u}}{1+u\bar{u}}, \frac{u\bar{u}-1}{1+u\bar{u}}\right). \tag{2.11}$$

We will also introduce a complex coordinate on $\mathbb{R}^2 \cup \{\infty\}$

$$z = x_1 + ix_2. \tag{2.12}$$

For reference we list the derivatives of $\boldsymbol{\phi}$ with respect to u and $\bar{u}$ and their properties

$$\begin{aligned}\frac{\partial\boldsymbol{\phi}}{\partial u} &= \left(\frac{1-\bar{u}^2}{(1+u\bar{u})^2}, -i\frac{1+\bar{u}^2}{(1+u\bar{u})^2}, \frac{2\bar{u}}{(1+u\bar{u})^2}\right)\\ \frac{\partial\boldsymbol{\phi}}{\partial \bar{u}} &= \left(\frac{1-u^2}{(1+u\bar{u})^2}, i\frac{1+u^2}{(1+u\bar{u})^2}, \frac{2u}{(1+u\bar{u})^2}\right),\end{aligned}$$

$$\begin{aligned} \frac{\partial \boldsymbol{\phi}}{\partial u} \times \frac{\partial \boldsymbol{\phi}}{\partial \bar{u}} &= -\frac{2i}{(1+u\bar{u})^2}\boldsymbol{\phi} \\ \frac{\partial \boldsymbol{\phi}}{\partial u} \cdot \frac{\partial \boldsymbol{\phi}}{\partial \bar{u}} &= \frac{2}{(1+u\bar{u})^2} \\ \frac{\partial \boldsymbol{\phi}}{\partial u} \cdot \frac{\partial \boldsymbol{\phi}}{\partial u} &= \frac{\partial \boldsymbol{\phi}}{\partial \bar{u}} \cdot \frac{\partial \boldsymbol{\phi}}{\partial \bar{u}} = 0. \end{aligned} \tag{2.13}$$

Using these the integrand in (2.10) is

$$\epsilon_{ij}\boldsymbol{\phi} \cdot (\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi}) = -4i\boldsymbol{\phi} \cdot (\partial_z \boldsymbol{\phi} \times \partial_{\bar{z}} \boldsymbol{\phi}) \tag{2.14}$$

$$\begin{aligned} &= -4i\boldsymbol{\phi} \cdot \frac{\partial \boldsymbol{\phi}}{\partial u} \times \frac{\partial \boldsymbol{\phi}}{\partial \bar{u}} (\partial_z u\, \partial_{\bar{z}} \bar{u} - \partial_z \bar{u}\, \partial_{\bar{z}} u) \\ &= 8\frac{\partial_z u\, \partial_{\bar{z}} \bar{u} - \partial_z \bar{u}\, \partial_{\bar{z}} u}{(1+u\bar{u})^2}. \end{aligned} \tag{2.15}$$

Hence we obtain

$$\begin{aligned} Q &= -\frac{1}{4\pi}\int 2i\, dz\, d\bar{z}\, \frac{\partial_z u\, \partial_{\bar{z}} \bar{u} - \partial_z \bar{u}\, \partial_{\bar{z}} u}{(1+u\bar{u})^2} \\ &= -\frac{1}{4\pi}\int 2i\frac{\partial_z u\, \partial_{\bar{z}} \bar{u} - \partial_z \bar{u}\, \partial_{\bar{z}} u}{(1+u\bar{u})^2} dz \wedge d\bar{z} \\ &= -\frac{1}{4\pi}\int \left(\frac{2i\, du \wedge d\bar{u}}{(1+u\bar{u})^2}\right)^*, \end{aligned} \tag{2.16}$$

where $*$ denotes the pullback under the map $z \to u(z, \bar{z})$. The integrand is the pullback of the volume form on the target space so evaluating the integral gives 4π times the number of times u covers the target space S^2 as $\mathbf{x}$ covers $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$. Thus the topological charge Q is an integer.

2.1.2 The $\mathbb{C}P^1$ Model

We can rewrite the $O(3)$ model action in terms of the stereographic field u to obtain the $\mathbb{C}P^1$ model action. Since $\boldsymbol{\phi}$ given by (2.11) automatically satisfies $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1$, we can reformulate the constrained dynamics of $\boldsymbol{\phi}$ in terms of the unconstrained dynamics of u.

The action is

$$\begin{aligned} S &= -2\int dt \int d^2x\, \frac{\partial_\mu u\, \partial^\mu \bar{u}}{(1+u\bar{u})^2} \\ &= 2\int dt \left[\int \frac{i}{2} dz\, d\bar{z}\, \frac{\dot{u}\dot{\bar{u}}}{(1+u\bar{u})^2} - \int i\, dz\, d\bar{z}\, \frac{\partial_z u\, \partial_{\bar{z}} \bar{u} + \partial_{\bar{z}} u\, \partial_z \bar{u}}{(1+u\bar{u})^2}\right], \end{aligned} \tag{2.17}$$

where $\dot{}$ denotes $\partial/\partial t$.

The equations of motion (2.6) for the field ϕ in terms of the field u and its conjugate $\bar{u}$ are

$$\partial_\mu \partial^\mu u - \frac{2\bar{u}}{1+u\bar{u}} \partial_\mu u \partial^\mu u = 0, \tag{2.18}$$

and its complex conjugate. These are the same as the Euler-Lagrange equations of motion for the action (2.17). Thus the equation governing a static solution is

$$\partial_z \partial_{\bar{z}} u - \frac{2\bar{u}}{1+u\bar{u}} \partial_z u \, \partial_{\bar{z}} u = 0 \tag{2.19}$$

The energy of a static solution is

$$E = \int 2i \, dz \, d\bar{z} \, \frac{\partial_z u \, \partial_{\bar{z}} \bar{u} + \partial_{\bar{z}} u \, \partial_z \bar{u}}{(1+u\bar{u})^2}. \tag{2.20}$$

Following Belavin and Polyakov [2] we obtain a bound on the energy by writing

$$\begin{aligned} E &= \int i \, dz \, d\bar{z} \left| \frac{2\partial_z u}{1+u\bar{u}} \right|^2 - \int 2i \, dz \, d\bar{z} \, \frac{\partial_z u \, \partial_{\bar{z}} \bar{u} - \partial_z \bar{u} \, \partial_{\bar{z}} u}{(1+u\bar{u})^2} \\ &= \int i \, dz \, d\bar{z} \left| \frac{2\partial_{\bar{z}} u}{1+u\bar{u}} \right|^2 + \int 2i \, dz \, d\bar{z} \, \frac{\partial_z u \, \partial_{\bar{z}} \bar{u} - \partial_z \bar{u} \, \partial_{\bar{z}} u}{(1+u\bar{u})^2}. \end{aligned} \tag{2.21}$$

The integrals on the far right are $\pm 4\pi Q$ where Q is the degree (2.16), giving the bound

$$E \geq 4\pi |Q|. \tag{2.22}$$

Thus the energy of static lumps within a given topological sector has a minimum.

Solutions which saturate the bound must satisfy

$$\begin{aligned} \partial_z u = 0 \qquad & Q \geq 0 \\ \partial_{\bar{z}} u = 0 \qquad & Q \leq 0. \end{aligned} \tag{2.23}$$

It is straightforward to see that solutions satisfying (2.23) are solutions to the full second order equation (2.19). In the case of the $\mathbb{C}P^1$ model, these are all the finite-energy solutions of (2.19) [3]. However, in models where the target space is $\mathbb{C}P^N$ or a higher dimensional Grassmannian manifold, there are static solutions to the second order equation (2.19) which are not solutions of the first order equations [5].

Restricting our attention to the solutions of (2.23) with negative Q, we see that configurations saturating the bound are holomorphic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$. These are *rational maps*, which are maps of the form

$$u(z) = \frac{p(z)}{q(z)}, \tag{2.24}$$

where p and q are polynomials in z. These will be described in more detail below.

A general nonlinear sigma model describes fields $\sigma^i(x_\mu)$ defined on some spacetime manifold M of dimension D with metric $g_{\mu\nu}$, taking values in a target space manifold N with metric γ_{ij}. The action of the sigma model is constructed in a natural way from the metrics on the spacetime and target space manifolds as follows

$$S = -\int \sqrt{-\det g_{\alpha\beta}}\, d^D x\, \partial_\mu \sigma^i \partial_\nu \sigma^j \gamma_{ij} g^{\mu\nu}. \tag{2.25}$$

We are assuming that the spacetime metric is Minkowskian with signature $(-1, 1, 1, \ldots, 1)$ so that $\sqrt{-\det g_{\alpha\beta}}\, d^D x$ is the spacetime measure.

We illustrate the construction in the case of the $\mathbb{C}P^1$ model on $\mathbb{R}^2 \times \mathbb{R}$. The metric on $\mathbb{R}^2 \times \mathbb{R}$ is

$$g = dz\, d\bar{z} - dt^2, \tag{2.26}$$

with measure $i\, dt\, dz\, d\bar{z}/2$. The metric on $\mathbb{C}P^1$ is

$$\gamma = \frac{2}{(1+u\bar{u})^2} du\, d\bar{u}. \tag{2.27}$$

Substituting this into (2.25) recovers the action (2.17).

One straightforward consequence of the form of the action (2.25) is that symmetries of the manifolds M and N, in other words isometries of the metrics on these manifolds, are symmetries of the action.

2.1.3 The $\mathbb{C}P^1$ Model on the Sphere

To obtain the action for the $\mathbb{C}P^1$ model on $S^2 \times \mathbb{R}$, we simply use the metric on this space

$$g = \frac{2}{(1+z\bar{z})^2} dz\, d\bar{z} - dt^2, \tag{2.28}$$

where now z is identified with the coordinate on the Riemann sphere. The corresponding measure is

$$\sqrt{-\det g_{\alpha\beta}}\, dz\, d\bar{z}\, dt = \frac{i\, dt\, dz\, d\bar{z}}{(1+z\bar{z})^2}. \tag{2.29}$$

The action we obtain from (2.25) is

$$S = \int \frac{i\, dt\, dz\, d\bar{z}}{(1+z\bar{z})^2} \left[\frac{\dot{u}\dot{\bar{u}}}{(1+u\bar{u})^2} - (1+z\bar{z})^2 \frac{\partial_z u\, \partial_{\bar{z}} \bar{u} + \partial_z \bar{u}\, \partial_{\bar{z}} u}{(1+u\bar{u})^2} \right]. \tag{2.30}$$

Therefore the Lagrangian of the model is

$$\begin{aligned} L &= T - V \\ V &= \int i\, dz\, d\bar{z}\, \frac{\partial_z u\, \partial_{\bar{z}} \bar{u} + \partial_z \bar{u}\, \partial_{\bar{z}} u}{(1+u\bar{u})^2} \\ T &= \int \frac{i\, dz\, d\bar{z}}{(1+z\bar{z})^2} \frac{\dot{u}\dot{\bar{u}}}{(1+u\bar{u})^2}. \end{aligned} \tag{2.31}$$

It is interesting to note that the potential energy functional is the same as that for flat space. This can be understood as follows. For a 2+1 dimensional spacetime metric of the form $g = \tilde{g}_{IJ} dx^I\, dx^J - dt^2$, the potential energy is

$$V = \int \sqrt{\det \tilde{g}_{KL}}\, d^2x\, \frac{\partial_I u\, \partial_J \bar{u}}{(1+u\bar{u})^2} \tilde{g}^{IJ}, \tag{2.32}$$

and this is invariant under a conformal transformation which rescales the metric

$$\tilde{g}_{IJ} \to \frac{\tilde{g}_{IJ}}{f(x^K)} \tag{2.33}$$

The metrics on $\mathbb{R}^2 \cup \{\infty\}$ and S^2 are conformally equivalent

$$\tilde{g} = dz\, d\bar{z} \to \frac{2\, dz\, d\bar{z}}{(1+z\bar{z})^2}. \tag{2.34}$$

Since the potential energy term is exactly the same as that in flat space, the argument that allowed us to put a bound on the energy of static configurations again tells us that such configurations are described by rational maps. However, the kinetic energy functional is clearly different and therefore we expect the low-energy dynamics of lumps to be markedly different to that in flat space. It is also important to note that, while the potential energy of a rational map configuration is the same, the potential energy *density* is not because the measure on the two sphere is

$$\frac{2i\, dz\, d\bar{z}}{(1+z\bar{z})^2}. \tag{2.35}$$

Therefore the potential energy density is

$$\mathcal{E} = (1+z\bar{z})^2 \frac{\partial_z u\, \partial_{\bar{z}} \bar{u} + \partial_z \bar{u}\, \partial_{\bar{z}} u}{(1+u\bar{u})^2}. \tag{2.36}$$

There is also a conceptual difference in the way that maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ occur in the model in flat space and on the two-sphere. In flat space, finiteness of energy forces the field to tend to a constant value asymptotically which we can identify with the value of the field at infinity. During evolution, this constant asymptotic value must be imposed as a boundary condition on the field. On the two-sphere, configurations are maps from $S^2 \cong \mathbb{C}P^1$ to $\mathbb{C}P^1$ and the energy is finite because space is compact. Since S^2 is closed, there are no boundary conditions on the field.

2.1.4 The Geodesic Approximation

The idea of approximating the low-energy dynamics of solitons by dynamics on the moduli space of static solitons was introduced by Manton in the context of monopoles [18]. The idea is that, since static solutions minimise the potential energy in a given topological sector, they form a "valley floor" in the space of configurations. If we consider slowly moving solitons whose initial motion is tangential to the valley floor, then the oscillations up the sides of the valley will be small, and it is hoped that ignoring these oscillations by restricting to motion along the valley floor will be a valid approximation to the true low-energy dynamics. The validity of the approximation has now been proved for monopoles [19], and although this is not yet the case for $\mathbb{C}P^1$ lumps, it provides hope that this is simply a matter of providing the correct mathematical analysis.

In the case of lumps, the static solutions are the holomorphic (or anti-holomorphic) configurations described by rational maps which depend on parameters, or *moduli*, a^i. Letting these moduli depend on time $a^i = a^i(t)$, we approximate the configuration at a given time by $u = u(a^i(t))$. Substituting this into the Lagrangian (2.31), we obtain

$$\begin{aligned} L &= \underbrace{\int \frac{i\,dz\,d\bar{z}}{(1+z\bar{z})^2} \frac{\frac{\partial u}{\partial a^i}\frac{\partial u}{\partial a^j}}{(1+u\bar{u})^2}}_{g_{ij}} \dot{a}^i\dot{a}^j - 4\pi n \\ &= \qquad\qquad g_{ij} \qquad\qquad \dot{a}^i\dot{a}^j - 4\pi n, \end{aligned} \tag{2.37}$$

where g_{ij} is the metric on the moduli space and, since the potential energy is constant in this approximation, the low-energy dynamics we obtain from this effective Lagrangian is that of geodesic motion with respect to the metric g.

2.2 Rational Maps

We have seen that configurations in the $\mathbb{C}P^1$ model which minimise the energy in a topological sector correspond to holomorphic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$. We will see below that, for monopoles which obey first order equations, there is also a correspondence with such maps, although the correspondence between the fields of the monopole and the map is much less direct.

As we stated above, holomorphic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ are rational maps

$$u(z) = \frac{p(z)}{q(z)}, \tag{2.38}$$

where p and q are polynomials with no non-constant common factor.

Rational maps have a degree which is the number of times u covers $\mathbb{C}P^1$ as z covers it once. To calculate this we find the number of times u equals some constant value c.

Setting $u = c$ in (2.38) leads to the equation

$$p(z) - cq(z) = 0. \tag{2.39}$$

so the number of solutions, and hence the degree of the map, is equal to the maximum of the degrees of the polynomials p and q. For a rational map of degree n, the topological charge Q is

$$Q = -\frac{1}{4\pi}\int 2i\,dz\,d\bar{z}\frac{\partial_z u \partial_{\bar{z}}\bar{u}}{(1+u\bar{u})^2} = -\frac{n}{4\pi}\int \frac{2i\,du\,d\bar{u}}{(1+u\bar{u})^2} = -n. \tag{2.40}$$

The degree of the map is related to the number of zeros of the potential energy density by the Riemann-Hurwitz relation. The general form of this relation concerns holomorphic maps between compact Riemann surfaces and is as follows.

Consider a holomorphic map $f : M \to N$ where M has genus g and N has genus γ. The degree of the map, n, is defined as the number of times f covers N over the domain M. Branch points, or *ramification points*, are those points around which the map looks like $(z-a)^m$ where $m > 1$, and the branching number at a ramification point, $b_f(p)$, is defined to be $m-1$.

Define the total branching number

$$B = \sum_{p \in M} b_f(p), \tag{2.41}$$

then the Riemann-Hurwitz relation states that

$$g = n(\gamma - 1) + 1 + \frac{B}{2}. \tag{2.42}$$

For a rational map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ which has genus zero we therefore have

$$B = 2(n-1). \tag{2.43}$$

The potential energy density on the sphere for a holomorphic map $u(z)$ is

$$\mathcal{E} = (1+z\bar{z})^2 \frac{\partial_z u\,\partial_{\bar{z}}\bar{u}}{(1+u\bar{u})^2}. \tag{2.44}$$

At points where $\partial_z u$ has a zero of multiplicity m, the energy density has a zero of multiplicity $2m$. Points where $\partial_z u = 0$ are the ramification points of the map since $\frac{d}{dz}(z-a)^m = m(z-a)^{m-1}$ is zero at $z = a$. Hence the relation (2.43) tells us that a one-lump has nowhere vanishing potential energy density, whereas a two-lump has two points (counted with multiplicity) where it vanishes. Of course, since the factor $(1+z\bar{z})^2$ is always positive, the same is true for the flat-space potential energy density.

We can specify an arbitrary rational map of degree n by giving the $2n+2$ complex coefficients of the polynomials p and q. Since the rational map is unchanged if we multiply p and q by some factor, a rational map of degree n depends on $2n+1$ complex parameters. We have to be careful if we just write down a map by choosing these coefficients since there may be a non-constant common factor between p and q, in which case the degree of the rational map is less than the maximum of the degrees of p and q.

The condition that p and q have no non-constant common factor can be expressed in terms of their *resultant* $R(p,q)$, which is zero if and only if they have one or more roots in common. We can obtain the resultant in terms of p and q as follows: Suppose that p and q have a common factor

$$p(z) = (z-a)\tilde{p}(z) \qquad q(z) = (z-a)\tilde{q}(z), \tag{2.45}$$

where $\tilde{p}$ and $\tilde{q}$ have degree at most $n-1$. Then

$$(z-a)\tilde{p}(z)\tilde{q}(z) = \tilde{q}(z)p(z) = \tilde{p}(z)q(z). \tag{2.46}$$

If we write

$$\tilde{q}(z) = \sum_{i=0}^{n-1} a^i z^i \qquad \tilde{p}(z) = \sum_{i=0}^{n-1} b^i z^i, \tag{2.47}$$

then (2.46) becomes

$$\sum_{i=0}^{n-1} a^i z^i p(z) - \sum_{i=0}^{n-1} b^i z^i q(z) = 0. \tag{2.48}$$

In other words, the polynomials

$$p(z),\ zp(z),\ \ldots,\ z^{n-1}p(z),\ q(z),\ zq(z),\ \ldots,\ z^{n-1}q(z), \tag{2.49}$$

are linearly dependent.

Thus the resultant is given by the following determinant

$$R(p,q) = \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_n & 0 & 0 \\ 0 & p_0 & p_1 & \cdots & p_{n-1} & p_n & 0 \\ 0 & 0 & p_0 & \cdots & p_{n-2} & p_{n-1} & p_n \\ q_0 & q_1 & q_2 & \cdots & q_n & 0 & 0 \\ 0 & q_0 & q_1 & \cdots & q_{n-1} & q_n & 0 \\ 0 & 0 & q_0 & \cdots & q_{n-2} & q_{n-1} & q_n \end{vmatrix}, \tag{2.50}$$

which vanishes precisely when the polynomials (2.49) are linearly dependent and therefore p and q have a common factor.

Knowing the resultant provides a natural way of parametrising the space of degree n rational maps, up to some discrete degeneracy. The freedom to multiply p and q by a non-zero constant allows us to set the resultant to a non-zero constant value. Under a scaling $p \to cp$, $q \to cq$ the resultant is multiplied by c^{2n}, and so this fixes the freedom up to a residual $\mathbb{Z}_{2n}$. Thus the set of polynomials p and q of degree n such that the resultant is a constant is a $2n$-fold cover of the set of rational maps of degree n.

2.3 Magnetic Monopoles

The magnetic monopoles we study occur in $SU(2)$ Yang-Mills-Higgs theory in a limit in which solutions obey a first order differential equation. These are so-called BPS monopoles and it is these monopoles for which the correspondence with rational maps holds. The natural place to start, though, is with the Dirac monopole which occurs in a $U(1)$ electromagnetic theory. Since the asymptotic Higgs field breaks $SU(2)$ down to $U(1)$, the asymptotic behaviour of $SU(2)$ monopoles may be related to that of the Dirac monopole.

2.3.1 The Dirac Monopole

A discussion of magnetic monopoles naturally begins with the Dirac monopole [9, 10]. These are the simplest example of magnetic monopoles occuring in a U(1) gauge theory. In addition, they will turn out to be a good long-range approximation to $SU(2)$ monopoles. The main reference for this section is the review by Goddard and Olive [7], as well as Rajaraman [1], Coleman [6] and Figueroa-O'Farrill [20].

Rather than begin with a Lagrangian, we start with the equations of motion, which are, of course, Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} = \rho \qquad & \nabla \times \mathbf{B} - \dot{\mathbf{E}} = \mathbf{j} \\ \nabla \cdot \mathbf{B} = \sigma \qquad & \nabla \times \mathbf{E} + \dot{\mathbf{B}} = \mathbf{k}. \end{aligned} \tag{2.51}$$

It will be noticed that as well as the relativistic electric current $(\rho, \mathbf{j})$, we have introduced the corresponding magnetic current $(\sigma, \mathbf{k})$. There is a good reason that we didn't derive these equations from a Lagrangian, which is that the the electric and magnetic currents couple to gauge potentials which are non-local with respect to one another.

Instead, the introduction of a magnetic current may be motivated by noticing that the sourceless equations are invariant under the *duality transformation*

$$\mathbf{E} \to \mathbf{B} \qquad \mathbf{B} \to -\mathbf{E}. \tag{2.52}$$

This is a symmetry of the equations with sources if they also transform as

$$j^\mu \to k^\mu \qquad k^\mu \to -j^\mu \tag{2.53}$$

This simple example of *electromagnetic duality* has motivated the search for examples of dualities in non-abelian gauge theories. These dualities occur in the quantum field theory and allow strongly coupled regions to be explored since the dual theory is often weakly coupled.

A magnetic monopole of charge g sitting at the origin has $(\sigma, \mathbf{k}) = (4\pi g\, \delta^3(\mathbf{x}), \mathbf{0})$ and gives rise to the magnetic field

$$\mathbf{B} = \frac{g\mathbf{x}}{r^3}. \tag{2.54}$$

We would like to write $\mathbf{B}$ in terms of a real vector potential A_i

$$B_i = -\epsilon_i{}^{jk}\partial_j A_k. \tag{2.55}$$

In general, it is possible to find a gauge field or vector potential that gives rise to a particular magnetic field only in a contractible region. Since the magnetic field is not defined at the origin, we have to remove this point which results in space being non-contractible. The remedy is to find different gauge fields in different contractible regions. Since they must give rise to the same magnetic field, they must differ by a gauge transformation on the overlap.

It will be useful here to introduce a set of spherical polar coordinates

$$\mathbf{x} = r\left(\frac{z+\bar{z}}{1+z\bar{z}}, -i\frac{z-\bar{z}}{1+z\bar{z}}, \frac{z\bar{z}-1}{1+z\bar{z}}\right). \tag{2.56}$$

It should be recognised from (2.11) that z is a complex coordinate obtained by stereographic projection. In terms of conventional spherical polar coordinates, $z = e^{i\phi}/\tan\frac{\theta}{2}$.

We will work with gauge fields in these coordinates defined by

$$A_r = \frac{\partial x_1}{\partial r}A_1 + \frac{\partial x_2}{\partial r}A_2 + \frac{\partial x_3}{\partial r}A_3, \tag{2.57}$$

with similar expressions for A_z and $A_{\bar{z}}$.

We will define two gauges, one, A_i^S, defined everywhere apart from the half-line $z = \infty$ and the other, A_i^N defined everywhere apart from the half-line $z = 0$. These lines are what may be referred to as "Dirac strings", which to us are just gauge singularities. Explicitly we have

$$\begin{aligned} A_r^S &= 0 \qquad A_z^S = -\frac{ig\bar{z}}{2(1+z\bar{z})} \qquad A_{\bar{z}}^S = \frac{igz}{2(1+z\bar{z})} \\ A_r^N &= 0 \qquad A_z^N = \frac{ig}{2z(1+z\bar{z})} \qquad A_{\bar{z}}^N = -\frac{ig}{2\bar{z}(1+z\bar{z})}. \end{aligned} \tag{2.58}$$

Using $\epsilon_r{}^{z\bar{z}} = i(1+z\bar{z})^2/2r^2$, the only non-zero component of the magnetic field is found to be

$$\begin{aligned} B_r &= -\frac{i(1+z\bar{z})^2}{2r^2}(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) \\ &= \frac{g}{r^2}. \end{aligned} \tag{2.59}$$

On the overlap A_i^S and A_i^N are related by the gauge transformation

$$\left(\frac{z}{\bar{z}}\right)^{\frac{g}{2}} = e^{ig\phi}. \tag{2.60}$$

This gauge transformation is only well-defined if g is an integer, which means that the magnetic charge is quantised.

2.4 The 't Hooft-Polyakov Monopole

We now come to the model which is the home of the monopoles we will study: the $SU(2)$ Yang-Mills-Higgs or Georgi-Glashow model [11]. Here we will use more modern notation, taking Lie-algebra valued, anti-Hermitian fields.

2.4.1 The $SU(2)$ Yang-Mills-Higgs Model

The theory consists of gauge fields A_μ and an adjoint Higgs field Φ, both taking values in the Lie algebra $su(2)$. The norm of an element of the Lie algebra T is given by

$$\parallel T \parallel^2 = -\frac{1}{2}\mathrm{tr}(T^2) \tag{2.61}$$

Taking an orthonormal basis T^a with the commutation relations

$$[T^a, T^b] = -2\epsilon^{abc}T^c, \tag{2.62}$$

we can write, for instance, $\Phi = \phi^a T^a$.

As usual we have the covariant derivative

$$D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]. \tag{2.63}$$

The field strength tensor is defined in terms of the commutator of covariant derivatives

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2.64}$$

Under a gauge transformation g, the Higgs and gauge fields transform as follows

$$\begin{aligned} \Phi &\to g^{-1}\Phi g \\ A_\mu &\to g^{-1}A_\mu g + g^{-1}\partial_\mu g, \end{aligned} \tag{2.65}$$

so the covariant derivative and field strength are conjugated by g

$$\begin{aligned} D_\mu\Phi &\to g^{-1}D_\mu\Phi g \\ F_{\mu\nu} &\to g^{-1}F_{\mu\nu}g. \end{aligned} \tag{2.66}$$

The Lagrangian density consists of a Lorentz-invariant gauge self-coupling, a minimal coupling to the Higgs field and a symmetry breaking potential

$$\mathcal{L} = \frac{1}{8}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}\mathrm{tr}(D_\mu\Phi D^\mu\Phi) - \frac{1}{8}\lambda(\| \Phi \|^2 - 1)^2. \tag{2.67}$$

This is clearly invariant under the gauge transformation (2.65), (2.66).

The equations of motion are found by varying the action $S = \int \mathcal{L}\, d^4x$ with respect to the fields A_μ and Φ to obtain

$$D_\nu F^{\mu\nu} = [\Phi, D^\mu\Phi] \tag{2.68}$$

$$D_\mu D^\mu\Phi = -\lambda\Phi(\| \Phi \|^2 - 1). \tag{2.69}$$

We can also define the non-abelian electric and magnetic fields

$$E^i = -F^{0i}, \qquad B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}. \tag{2.70}$$

2.4.2 The Yang-Mills-Higgs Energy-Momentum Tensor

The best way to derive a symmetric gauge-invariant energy-momentum tensor is to write the action on a general curved spacetime manifold, explicitly in terms of the metric tensor $g_{\mu\nu}$ and its inverse. We will take the metric to have signature $(-1, +1, +1, +1)$. Note that it is not necessary to introduce a connection on spacetime in this case, since the non-abelian field strength is a two-form and hence independent of the metric while the Higgs field is a spacetime scalar. Varying the action with respect to the metric gives us the energy-momentum tensor on this curved manifold, and evaluating this when $g_{\mu\nu}$ is the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, gives the energy-momentum tensor for our model.

The action is

$$S = \int \mathcal{L}\, \sqrt{-\det g_{\mu\nu}}\, d^4x \tag{2.71}$$

where

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{4}\mathrm{tr}(g^{\alpha\beta}g^{\gamma\delta}F_{\alpha\gamma}F_{\beta\delta}) - \frac{1}{2}\mathrm{tr}(g^{\alpha\beta}D_\alpha\Phi D_\beta\Phi) - \frac{1}{4}\lambda(\parallel\Phi\parallel^2 - 1)^2\right] \tag{2.72}$$

The variation in S comes from the variation of the inverse of the metric $g^{\alpha\beta}$ and the variation of the measure $\sqrt{-\det g_{\mu\nu}}\, d^4x$.

The energy-momentum tensor $T^{\alpha\beta}$ is defined (see [24]) by

$$\frac{\partial S}{\partial g_{\alpha\beta}} = -\frac{1}{2}\int T^{\alpha\beta}\sqrt{-\det g_{\mu\nu}} d^4x = \int\left[\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} + \frac{1}{2}\mathcal{L}\, g^{\alpha\beta}\right]\sqrt{-\det g_{\mu\nu}}\, d^4x, \tag{2.73}$$

where we have used the standard result that

$$\frac{\partial}{\partial g_{\alpha\beta}}\det g_{\mu\nu} = \det g_{\mu\nu}\; g^{\alpha\beta}. \tag{2.74}$$

Plugging in (2.72) gives us the energy-momentum tensor for the model

$$T^{\alpha\beta} = -\frac{1}{2}\mathrm{tr}\left(\frac{1}{4}g^{\alpha\beta}F^{\gamma\delta}F_{\gamma\delta} - F^{\alpha\gamma}F^{\beta}{}_{\gamma} + D^\alpha\Phi D^\beta\Phi - \frac{1}{2}g^{\alpha\beta}D^\gamma\Phi D_\gamma\Phi\right) + \frac{1}{8}\lambda(\parallel\Phi\parallel^2 - 1)^2 g^{\alpha\beta}. \tag{2.75}$$

Evaluating this when $g_{\mu\nu} = \eta_{\mu\nu}$ we obtain the energy

$$T_{00} = -\frac{1}{4}\int \mathrm{tr}(D_i\Phi D_i\Phi + B_iB_i + E_iE_i + D_0\Phi D_0\Phi) + V(\Phi)\, d^3x. \tag{2.76}$$

where $V(\Phi)$ is the potential $\frac{1}{8}\lambda(\parallel\Phi\parallel^2 - 1)^2$.

For finite energy we require that each of these terms decays at least as fast as r^{-2}. In particular the condition that $V(\Phi) \to 0$ implies that the asymptotic Higgs field Φ^∞ satisfies

$$\parallel\Phi\parallel^2 = 1. \tag{2.77}$$

Choosing an orthonormal basis for the Lie algebra T^a with $\Phi = \phi^a T^a$, this means that $\phi^a\phi^a = 1$, so the Higgs field lives on a two-sphere.

In terms of symmetry breaking, the fact that the Higgs has a non-zero value means that the full symmetry group $SU(2)$ is broken to the exact subgroup which fixes the Higgs field at each point which is simply the group $U(1)$ generated by the Higgs field itself. We can think of this residual $U(1)$ as being electromagnetic and define the electromagnetic field strength as

$$F^{\mathrm{em}}_{\mu\nu} = -\frac{1}{2}\mathrm{tr}(\Phi F_{\mu\nu}). \tag{2.78}$$

Similarly we can define the electric and magnetic charges, q and g, of a solution as follows. The electric charge is

$$\begin{aligned} q &= \int_{S^2_\infty} E_i^{\mathrm{em}}\, dS_i \\ &= -\frac{1}{2}\int_{S^2_\infty} \mathrm{tr}(\Phi E_i)\, dS_i \\ &= -\frac{1}{2}\int \partial_i\, \mathrm{tr}(\Phi E_i)\, d^3x. \end{aligned} \tag{2.79}$$

For a static solution, ie. one for which $D_0\Phi = 0$ and $D_0 E_i = 0$, the equation of motion (2.68) implies

$$D_i F_{0i} = D_i E_i = [\Phi, D_0\Phi] = 0, \tag{2.80}$$

and hence we can write the electric charge as

$$q = -\frac{1}{2}\int \mathrm{tr}(D_i\Phi E_i)\, d^3x. \tag{2.81}$$

Similarly, using the Bianchi identity $D_i B_i \equiv 0$, the magnetic charge is

$$g = \int_{S^2_\infty} B_i^{\mathrm{em}}\, dS_i \tag{2.82}$$

$$= -\frac{1}{2}\int \mathrm{tr}(D_i\Phi B_i)\, d^3x. \tag{2.83}$$

2.4.3 The Topology of Magnetic Charge

The condition of finite energy (2.76) implies that asymptotically the Higgs field and its covariant derivative must satisfy

$$-\frac{1}{2}\mathrm{tr}(\Phi^2) \to 1 \tag{2.84}$$

$$D_i\Phi \to 0. \tag{2.85}$$

We will refer to fields satisfying these conditions as being in the *Higgs vacuum.*

The magnetic charge of a monopole is given in (2.82) by an integral over the sphere at infinity, on which the fields are in the Higgs vacuum. We will use this fact to compute the magnetic charge of the monopole in two different ways. The first of these shows that the magnetic charge is proportional to the degree of the map from S^2 to S^2 defined by the Higgs field. The expression for the degree is just the same as the expression (2.10) obtained for the $O(3)$ model field $\boldsymbol{\phi}$. The degree of this map corresponds to an element of the homotopy group $\Pi_2(S^2)$.

The second method of calculating the degree uses the fact that there are gauges in which the Higgs field looks like a Dirac monopole (2.58). In such a gauge, the topology is no longer contained in the Higgs field, but in the overlap between different gauges, which is an element of the residual symmetry subgroup H corresponding to an element of $\Pi_1(H)$. This is an example of the isomorphism between the homotopy groups $\Pi_2(G/H)$ and $\Pi_1(H)$ when G is simply-connected.

Initially we will work in a gauge in which the Higgs and gauge fields, and therefore their Cartesian derivatives, are smooth. We will take the liberty of writing them as real vectors $\boldsymbol{\phi}$, $\mathbf{A}_i$ where

$$\Phi = \boldsymbol{\phi}^a T^a \qquad A_i = \mathbf{A}_i^a T^a. \tag{2.86}$$

T^a are the orthonormal basis of $su(2)$ generators satisfying (2.62). Thus we can rewrite the Higgs vacuum conditions (2.84), (2.85) in terms of standard $\mathbb{R}^3$ vector algebra

$$\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1 \tag{2.87}$$

$$\partial_i \boldsymbol{\phi} = 2\mathbf{A}_i \times \boldsymbol{\phi} \tag{2.88}$$

Decomposing $\mathbf{A}_i$ into parts orthogonal and parallel to $\boldsymbol{\phi}$ and using the above equations gives

$$\mathbf{A}_i = \frac{1}{2}\boldsymbol{\phi} \times \partial_i \boldsymbol{\phi} + \boldsymbol{\phi}\, a_i, \tag{2.89}$$

where a_i is a smooth field defined by this equation.

Computing the non-abelian field strength we find

$$\begin{aligned} \mathbf{F}_{ij} &= \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i - 2\mathbf{A}_i \times \mathbf{A}_j \\ &= \partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi} - \frac{1}{2}\boldsymbol{\phi} \cdot (\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi})\, \boldsymbol{\phi} + (\partial_i a_j - \partial_j a_i)\, \boldsymbol{\phi}. \end{aligned} \tag{2.90}$$

The corresponding electromagnetic field strength is

$$F_{ij}^{\mathrm{em}} = \boldsymbol{\phi} \cdot \mathbf{F}_{ij} = \frac{1}{2}\boldsymbol{\phi} \cdot (\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi}) + \partial_i a_j - \partial_j a_i. \tag{2.91}$$

Now the magnetic charge (2.83) is

$$g = \int_{S_\infty^2} B_i^{\mathrm{em}} dS_i. \tag{2.92}$$

Putting in the field strength (2.91) we obtain

$$g = -\frac{1}{2}\int_{S^2_\infty} (\epsilon_{ijk}\boldsymbol{\phi}\cdot(\partial_j\boldsymbol{\phi}\times\partial_k\boldsymbol{\phi}) + 2\epsilon_{ijk}\partial_j a_k)\, dS_i. \tag{2.93}$$

The term involving a vanishes using Stokes' theorem since $\partial_j a_k$ is smooth and S^2_∞ is closed. We recognise the term involving $\boldsymbol{\phi}$ to be -4π times the degree of the map from S^2 to S^2 defined for the $O(3)$ model (2.10), and therefore the magnetic charge is 4π times an integer. We have therefore shown that the magnetic charge of a monopole is governed by the homotopy class, $\Pi_2(S_2) = \mathbb{Z}$, of the asymptotic Higgs field.

Now we can use (2.85) in another way. Denote the value of the Higgs field at the north pole of the sphere Φ^N. We return to Lie algebra-valued fields from now on. Consider a path C from the north pole to a point and define the following path ordered exponential along this path

$$g = \mathrm{P}\exp\int_C \vec{A}\cdot\vec{dl}. \tag{2.94}$$

From the properties of the path ordered exponential we know that this is a unitary gauge transformation and that it obeys

$$\partial_i g = A_i g. \tag{2.95}$$

If we gauge transform by this group element we obtain a new Higgs field $\Phi' = g^{-1}\Phi g$ which obeys

$$\partial_i\Phi' = g^{-1}(D_i\Phi)g = 0. \tag{2.96}$$

In other words Φ' takes the constant value Φ^N along the path.

It is not possible to define a gauge transformation in this way that covers the whole sphere, since a sphere is not contractible, so we will define two gauge transformations, one valid on the "northern" hemisphere, g^N, and the other on the "southern", g^S. We obtain these gauge transformations by specifying the path along which we integrate to get to a particular point.

To get g^N we take paths from the north pole along great circles. Clearly this will be well-defined everywhere apart from the south pole. To obtain g^S we firstly take a fixed great circle (corresponding to a fixed group element) from the north pole to the south pole, then to reach a particular point we take the great circle from the south pole.

We can consider the overlap of these gauge transformations on the equator. Since the transformed Higgs is Φ^N in both cases it must be that g^N and g^S differ by an element in the Cartan subalgebra that preserves Φ^N. ie.

$$g^S = g^N h. \tag{2.97}$$

Furthermore, since the original gauge field A_i and the paths vary smoothly, g^N and g^S are continuous and equal at the point on the equator which lies along the fixed path. Hence h is a continuous function from the equator to the stabiliser of Φ^N, ie. a map from S^1 to $U(1)$.

The expression for the magnetic charge is gauge invariant, hence we can work it out in the gauges just described by splitting up the integration into northern and southern hemispheres.

The gauge fields on the two hemispheres are

$$\begin{aligned} A_i^N &= \Phi^N a_i^N \\ A_i^S &= \Phi^N a_i^S, \end{aligned} \tag{2.98}$$

with field strengths

$$\begin{aligned} F_{ij}^N &= \Phi^N(\partial_i a_j^N - \partial_j a_i^N) \\ F_{ij}^S &= \Phi^N(\partial_i a_j^S - \partial_j a_i^S). \end{aligned} \tag{2.99}$$

Now we use Stokes' theorem to compute the magnetic charge

$$\begin{aligned} g &= -\int_{H^N} \epsilon_{ijk}\partial_j a_k^N dS_i - \int_{H^S} \epsilon_{ijk}\partial_j a_k^S dS_i \\ &= \int_E (a_i^N - a_i^S)\, dl_i \\ &= \int_E h^{-1}\partial_i h\, dl_i. \end{aligned} \tag{2.100}$$

E denotes the equator of the sphere in an anticlockwise direction. The last expression is the Jacobian of the map h from the equator to $U(1)$. In other words the result is the winding number of this map which is an element of the homotopy class $\Pi_1(S_1)$.

To sum up, we can either work in a smooth gauge in which the topological information is contained in the Higgs field and the gauge fields are topologically trivial, or we can work in gauges on patches in which the Higgs field is constant and the topological information is contained in the gauge transformation between the patches.

2.4.4 The Hedgehog Ansatz

The vacuum solution

$$\Phi = \hat{\mathbf{n}} \qquad A_\mu = 0, \tag{2.101}$$

where $\hat{\mathbf{n}}$ is a constant unit vector, is symmetric under Galilean transformations. In particular, since it is constant on the sphere at infinity, the degree of the Higgs field, and hence the magnetic charge, is zero. In order to find a solution with magnetic charge one, 't Hooft [12] and Polyakov [13] (independently) looked for a solution with the maximum amount of symmetry for the Higgs field at infinity to have degree 1. This is given by a map where the Higgs field points in the same direction in internal space as the direction of the point from the centre of the monopole, so the field is visualised as pointing directly outwards. The ansatz is

$$\Phi = \frac{x^i T^i}{2r^2} H(r) \qquad A_i = -\epsilon_{ijk} \frac{x^j T^k}{r^2}(1 - K(r)) \qquad A_0 = 0. \tag{2.102}$$

For the resulting monopole to be well-defined at the origin and be in the Higgs vacuum at spatial infinity we require that

$$\begin{aligned} K - 1 = o(r) \qquad & H = o(r) \qquad \text{as } r \to 0 \\ K \to 0 \qquad & |H| \to 2r \qquad \text{as } r \to \infty. \end{aligned} \tag{2.103}$$

It is easily verified using (2.93) that this gives rise to a monopole with magnetic charge $\pm 4\pi$ when $H(r) \to \pm 2r$ respectively.

We will rewrite the ansatz in terms of the coordinates (2.56), representing the $su(2)$ generators T^i in terms of the Pauli matrices as $i\tau^i$. We obtain

$$\begin{aligned} \Phi &= \frac{iH(r)}{2r}(\mathbb{I} - 2\mathbb{P}) \\ A_r &= 0 \\ A_z &= -(1 - K(r))\, \partial_z \mathbb{P} \\ A_{\bar{z}} &= (1 - K(r))\, \partial_{\bar{z}} \mathbb{P}, \end{aligned} \tag{2.104}$$

where

$$\begin{aligned} \mathbb{P} &= \frac{1}{1 + z\bar{z}} \begin{pmatrix} -1 \\ z \end{pmatrix} \begin{pmatrix} -1 & \bar{z} \end{pmatrix} \\ \partial_z \mathbb{P} &= \frac{1}{(1 + z\bar{z})^2} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} \begin{pmatrix} -1 & \bar{z} \end{pmatrix} \\ \partial_{\bar{z}} \mathbb{P} &= \frac{1}{(1 + z\bar{z})^2} \begin{pmatrix} -1 \\ z \end{pmatrix} \begin{pmatrix} z & 1 \end{pmatrix}. \end{aligned} \tag{2.105}$$

We will use the ansatz in this form to find a one-monopole solution below.

The hedgehog ansatz is spherically symmetric in the sense that a spatial rotation of the fields can be undone by an equivalent gauge rotation, ie. the fields are invariant under a diagonal subgroup of $SO(3)^{\text{spatial}} \times SO(3)^{\text{gauge}}$.

2.4.5 BPS Monopoles

An interesting limit of the $SU(2)$ Yang-Mills-Higgs model was proposed by Prasad and Sommerfield [14]. They suggested sending the coupling constant λ to zero whilst retaining the condition that $\phi^{\infty a}$ lives on the unit two-sphere. In this limit we find solutions satisfying first order differential equations that minimise the energy in a given topological sector in an analogous way to lumps, following the famous argument of Bogomol'nyi [15] From now on, it is assumed that we have taken the Prasad-Sommerfield limit.

The energy (2.76) of a solution in this limit is

$$E = -\frac{1}{4}\int \mathrm{tr}(D_i\Phi D_i\Phi + B_iB_i + E_iE_i + D_0\Phi D_0\Phi)\, d^3x. \tag{2.106}$$

We will consider the energy of a static solution for which the fields A_μ and Φ are independent of time. Introducing an arbitrary angle θ, we can write

$$E = -\frac{1}{4}\int \mathrm{tr}(E_i - (D_i\Phi)\sin\theta)^2\, d^3x - \frac{1}{4}\int \mathrm{tr}(B_i - (D_i\Phi)\cos\theta)^2\, d^3x \\ - \frac{\sin\theta}{2}\int \mathrm{tr}(E_iD_i\Phi) - \frac{\cos\theta}{2}\int \mathrm{tr}(B_iD_i\Phi). \tag{2.107}$$

Clearly the first two terms are positive whilst the last two terms are the electric and magnetic charges (2.81) and (2.83), therefore we have

$$E \geq q\sin\theta + g\cos\theta. \tag{2.108}$$

The most stringent bound is obtained by choosing $\tan\theta = q/g$ so that

$$E \geq (q^2 + g^2)^{\frac{1}{2}}. \tag{2.109}$$

Of course, we may identify the energy in the rest frame with the mass of the monopole and so we have arrived at the Bogomol'nyi bound on the mass of a monopole or dyon.

A purely magnetic solution which saturates the bound $M \geq |g|$ must satisfy the *Bogomol'nyi equations*

$$B_i = \pm D_i\Phi \qquad \text{for } g \gtrless 0. \tag{2.110}$$

It can be checked that solutions to the Bogomol'nyi equation satisfy the full second-order equations of motion (2.68) and (2.69). Monopoles which satisfy the Bogomol'nyi equations in the Prasad-Sommerfield limit are now referred to as BPS monopoles, although in fact Prasad and Sommerfield found their solution using the second order equations of motion. The solution they found, the BPS monopole, corresponds to a spherically symmetric monopole of charge 1 sitting at the origin and will be described below.

The Bogomol'nyi equations $B_i = D_i\Phi$, rewritten in terms of the coordinates (2.56), take on a special form [34]

$$[D_r - i\Phi, D_{\bar{z}}] = 0 \tag{2.111}$$
$$[D_r + i\Phi, D_z] = 0 \tag{2.112}$$
$$2iD_r\Phi = \frac{(1+z\bar{z})^2}{r^2}[D_z, D_{\bar{z}}]. \tag{2.113}$$

Of course, in a unitary gauge, (2.112) is just the Hermitian conjugate of (2.111).

It is the appearance of the commutators (2.111) and (2.112) which enable us to define a rational map associated to a monopole. Before we do this, however, we will present the explicit one-monopole solution first discovered by Prasad and Sommerfield.

2.4.6 The BPS Monopole

We will find an explicit, smooth, one-monopole solution of the Bogomol'nyi equations (2.111), (2.112), (2.113) using the hedgehog ansatz (2.104). Firstly we need the following properties of the projector $\mathbb{P}$ and its derivatives:

$$\begin{aligned}
\partial_{\bar{z}}\partial_z\mathbb{P} &= \frac{1}{(1+z\bar{z})^2}(\mathbb{I} - 2\mathbb{P}) \\
[(\mathbb{I} - 2\mathbb{P}), \partial_z\mathbb{P}] &= 2\partial_z\mathbb{P} \\
[(\mathbb{I} - 2\mathbb{P}), \partial_{\bar{z}}\mathbb{P}] &= -2\partial_{\bar{z}}\mathbb{P} \\
[\partial_z\mathbb{P}, \partial_{\bar{z}}\mathbb{P}] &= \frac{1}{(1+z\bar{z})^2}(\mathbb{I} - 2\mathbb{P})
\end{aligned} \tag{2.114}$$

If we think of $2\mathbb{P} - \mathbb{I}$ as being an element of the Cartan subalgebra of $su(2)$, then $\partial_z\mathbb{P}$ and $\partial_{\bar{z}}\mathbb{P}$ are proportional to the step operators.

Now plugging in the ansatz (2.104), the equations (2.111) and (2.113) give the following equations for $H(r)$ and $K(r)$

$$\begin{aligned}
(r\partial_r K - HK)\,\partial_{\bar{z}}\mathbb{P} &= 0 \\
(r\partial_r H - H + 1 - K^2)(\mathbb{I} - 2\mathbb{P}) &= 0.
\end{aligned} \tag{2.115}$$

If we substitute

$$H(r) = 1 + rh(r) \qquad K(r) = rk(r), \tag{2.116}$$

we obtain

$$\partial_r k - hk = 0 \qquad \partial_r h - k^2 = 0, \tag{2.117}$$

which implies that $h\partial_r h - k\partial_r k = 0$ and hence $\partial_r(h^2 - k^2) = 0$. The solution with the correct boundary conditions is

$$h(r) = -2\coth 2r \qquad k(r) = 2\,\mathrm{cosech}\,2r. \tag{2.118}$$

We have thus obtained a smooth monopole solution

$$\begin{aligned} \Phi &= i\left(\frac{1}{2r} - \coth 2r\right)(\mathbb{I} - 2\mathbb{P}) \\ A_r &= 0 \\ A_z &= -(1 - 2r\,\mathrm{cosech}\,2r)\,\partial_z\mathbb{P} \\ A_{\bar{z}} &= (1 - 2r\,\mathrm{cosech}\,2r)\,\partial_{\bar{z}}\mathbb{P}. \end{aligned} \tag{2.119}$$

2.5 The Jarvis Rational Map

There are two crucial properties of BPS monopoles that allow us to define the Jarvis correspondence of n-monopoles with rational maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ of degree n. These are the Bogomol'nyi equations (2.111), (2.113) and the finite energy boundary conditions, which we will write in a more precise form. In fact, all we need to define the Jarvis rational map corresponding to a monopole is the zero curvature condition (2.111)

$$[D_r - i\Phi, D_{\bar{z}}] = 0, \tag{2.120}$$

and the finite energy boundary conditions. It is only necessary to consider the other equation (2.113) to show that a rational map uniquely specifies a monopole up to gauge transformations.

Another way of looking at the zero curvature condition (2.120) is that it means that is possible to find a gauge in which this commutator is *trivialised*, in other words, a gauge in which $A_r - i\Phi = A_{\bar{z}} = 0$ so that

$$[D_r - i\Phi, D_{\bar{z}}] = [\partial_r, \partial_{\bar{z}}] = 0. \tag{2.121}$$

Since these combinations of fields are not anti-Hermitian, the gauge transformation which achieves this trivialisation is necessarily non-unitary.

To define the Jarvis rational map [34], we start in a unitary gauge in which the Higgs and Cartesian gauge fields are smooth and well-defined on the whole of $\mathbb{R}^3$. We will specify the finite energy boundary conditions by stating the asymptotic behaviour of the fields in unitary gauges in which the Higgs field points in a fixed direction. These gauges are not well-defined on the whole of $\mathbb{R}^3$. This is essentially a corollary to the expression of the magnetic charge in terms of $\Pi_1(U(1))$ (2.100), when the monopole obeys the Bogomol'nyi equations. We found gauge transformations g^S $(z \neq \infty)$ and g^N $(z \neq 0)$ which take us to gauges in which the Higgs field points in a fixed direction. The gauge fields in the Higgs vacuum are then proportional to the Higgs field and therefore also point in this direction. From the radial magnetic field

$$B_r^{\rm em} \sim \frac{n}{r^2} T_3, \tag{2.122}$$

we deduce that A_z and $A_{\bar{z}}$ are the same as those of the Dirac monopole (2.58). By performing an r-dependent gauge transformation in the fixed direction to set $A_r = 0$ and using the radial Bogomol'nyi equation $B_r = D_r \Phi$, we deduce that

$$\Phi \sim i\left(1 - \frac{n}{2r}\right) T_3. \tag{2.123}$$

Thus we have asymptotic expressions for the fields in which the degree of the map appears explicitly

$$\begin{aligned} \Phi^S &= i\left(1 - \frac{n}{2r}\right) T_3 + O(\frac{1}{r^2}) \\ A_r^S &= O(\frac{1}{r^2}) \\ A_z^S &= \frac{n\bar{z}}{2(1+z\bar{z})} T_3 + O(\frac{1}{r}) \\ A_{\bar{z}}^S &= -\frac{nz}{2(1+z\bar{z})} T_3 + O(\frac{1}{r}), \end{aligned} \tag{2.124}$$

and

$$\begin{aligned} \Phi^N &= i\left(1 - \frac{n}{2r}\right) T_3 + O(\frac{1}{r^2}) \\ A_r^N &= O(\frac{1}{r^2}) \\ A_z^N &= -\frac{n}{2z(1+z\bar{z})} T_3 + O(\frac{1}{r}) \\ A_{\bar{z}}^N &= \frac{n}{2\bar{z}(1+z\bar{z})} T_3 + O(\frac{1}{r}), \end{aligned} \tag{2.125}$$

where $T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We will refer to these as the southern and northern gauges. Asymptotically they are related by the gauge transformation

$$A_i^N = A_i^S + g^{-1}\partial_i g \qquad \text{where} \qquad g = \begin{pmatrix} (\frac{z}{\bar{z}})^{\frac{n}{2}} & 0 \\ 0 & (\frac{\bar{z}}{z})^{\frac{n}{2}} \end{pmatrix}. \tag{2.126}$$

To define the rational map, we consider the *scattering equation* introduced by Hitchin [25] along half-lines from the origin

$$(D_r - i\Phi)\mathbf{s} = 0, \tag{2.127}$$

for a complex doublet $\mathbf{s}$. In the southern gauge this equation is of the form

$$\left\{\partial_r + \left(1 - \frac{n}{2r}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + C(r)\right\}\mathbf{s} = 0, \tag{2.128}$$

where $C(r)$ is an integrable matrix satisfying

$$\int_0^\infty |C(r)|dr < \infty. \tag{2.129}$$

The idea is to show that there is a basis of solutions $\mathbf{s}_1$, $\mathbf{s}_2$, which, as $r \to \infty$, satisfy

$$r^{-\frac{n}{2}}e^r\mathbf{s}_1 \to \begin{pmatrix}1\\0\end{pmatrix} \qquad r^{\frac{n}{2}}e^{-r}\mathbf{s}_2 \to \begin{pmatrix}0\\1\end{pmatrix}. \tag{2.130}$$

We follow the method of Coddington and Levinson [26]. The first step is to regard $-C(r)\mathbf{s}$ as a forcing term. We find the Green function $G(u,r)$ satisfying

$$\frac{dG}{dr} - \begin{pmatrix}-1+\frac{n}{2r} & 0\\ 0 & 1-\frac{n}{2r}\end{pmatrix}G(u,r) = \delta(r-u), \tag{2.131}$$

together with certain boundary conditions. In the case given above we have

$$G(u,r) = -\begin{pmatrix}e^{r-u}\left(\frac{u}{r}\right)^{\frac{n}{2}} & 0\\ 0 & 0\end{pmatrix} \qquad u > r$$
$$\begin{pmatrix}0 & 0\\ 0 & e^{u-r}\left(\frac{r}{u}\right)^{\frac{n}{2}}\end{pmatrix} \qquad u \leq r. \tag{2.132}$$

The solution is then given by

$$\mathbf{s}_j(r) = \mathbf{y}_j(r) + \int_{r_0}^\infty G(u,r)C(u)\mathbf{s}_j(u)du, \tag{2.133}$$

where $\mathbf{y}_j(r)$ is a solution to the scattering equation without the forcing term, in this case

$$\mathbf{y}_1(r) = r^{\frac{n}{2}}e^{-r} \qquad \text{or} \qquad \mathbf{y}_2(r) = r^{-\frac{n}{2}}e^r. \tag{2.134}$$

The constant r_0 is chosen so that

$$\int_{r_0}^\infty |C(u)|\,du < \frac{1}{2}. \tag{2.135}$$

We will write (2.133) in the form of a mapping T

$$\mathbf{s}_j(r) = \mathbf{y}_j(r) + (T\mathbf{s}_j)(r). \tag{2.136}$$

We can obtain a solution to this iteratively as follows :

Given a trial solution $\mathbf{y}_j^{(i)}$ define $\mathbf{y}_j^{(i+1)}$ by

$$\mathbf{y}_j^{(i+1)}(r) = \mathbf{y}_j(r) + \left(T\mathbf{y}_j^{(i)}\right)(r). \tag{2.137}$$

Choosing $\mathbf{y}_j^{(0)}(r) = 0$ gives $\mathbf{y}_j^{(1)}(r) = \mathbf{y}_j(r)$ so clearly

$$\left|\mathbf{y}_j^{(1)}(r) - \mathbf{y}_j^{(0)}(r)\right| = |\mathbf{y}_j(r)|. \tag{2.138}$$

Now (2.137) gives us the formula

$$\mathbf{y}_j^{(i+1)}(r) - \mathbf{y}_j^{(i)}(r) = T\left(\mathbf{y}_j^{(i)} - \mathbf{y}_j^{(i-1)}\right)(r). \tag{2.139}$$

Writing out the right-hand side in full we have

$$\begin{aligned}\mathbf{y}_j^{(i+1)}(r) - \mathbf{y}_j^{(i)}(r) = &\int_{r_0}^{r} \begin{pmatrix} 0 & 0 \\ 0 & e^{u-r}\left(\frac{r}{u}\right)^{\frac{n}{2}} \end{pmatrix} C(u)\left(\mathbf{y}_j^{(i)}(u) - \mathbf{y}_j^{(i-1)}(u)\right) du \\ &- \int_{r}^{\infty} \begin{pmatrix} e^{r-u}\left(\frac{u}{r}\right)^{\frac{n}{2}} & 0 \\ 0 & 0 \end{pmatrix} C(u)\left(\mathbf{y}_j^{(i)}(u) - \mathbf{y}_j^{(i-1)}(u)\right) du. \end{aligned} \tag{2.140}$$

Taking the norm of both sides we use either

$$e^{u-r}\left(\frac{r}{u}\right)^{\frac{n}{2}} \leq 1 \leq e^{r-u}\left(\frac{u}{r}\right)^{\frac{n}{2}} \qquad u \leq r \tag{2.141}$$

$$e^{r-u}\left(\frac{u}{r}\right)^{\frac{n}{2}} \leq 1 \leq e^{u-r}\left(\frac{r}{u}\right)^{\frac{n}{2}} \qquad r \leq u, \tag{2.142}$$

to give us two inequalities

$$\left|\mathbf{y}_j^{(i+1)}(r) - \mathbf{y}_j^{(i)}(r)\right| \leq \int_{r_0}^{\infty} |C(u)| e^{r-u}\left(\frac{u}{r}\right)^{\frac{n}{2}} \left|\mathbf{y}_j^{(i)}(u) - \mathbf{y}_j^{(i-1)}(u)\right| du, \tag{2.143}$$

and

$$\left|\mathbf{y}_j^{(i+1)}(r) - \mathbf{y}_j^{(i)}(r)\right| \leq \int_{r_0}^{\infty} |C(u)| e^{u-r}\left(\frac{r}{u}\right)^{\frac{n}{2}} \left|\mathbf{y}_j^{(i)}(u) - \mathbf{y}_j^{(i-1)}(u)\right| du. \tag{2.144}$$

The claim is that

$$\left|\mathbf{y}_j^{(i+1)}(r) - \mathbf{y}_j^{(i)}(r)\right| \leq \left(\frac{1}{2}\right)^{i} \mathbf{y}_j(r), \tag{2.145}$$

where $\mathbf{y}_j(r)$ is either $\mathbf{y}_1(r) = r^{\frac{n}{2}}e^{-r}$ or $\mathbf{y}_2(r) = r^{-\frac{n}{2}}e^{r}$, and hence the series tends to a unique solution with the given asymptotic properties. For $\mathbf{y}_1(r)$ the proof follows by induction using (2.143), and for $\mathbf{y}_2$ the proof follows similarly using (2.144).

Further analysis shows that, as $r \to \infty$

$$\begin{aligned} r^{-\frac{n}{2}}e^{r}\int_{r_0}^{\infty} G(u,r)C(u)\mathbf{s}_1(u)du \to 0 \\ r^{\frac{n}{2}}e^{-r}\int_{r_0}^{\infty} G(u,r)C(u)\mathbf{s}_2(u)du \to 0, \end{aligned} \tag{2.146}$$

so the solutions defined by (2.133) have the claimed asymptotic behaviour (2.130).

We can write the solutions (2.130) $\mathbf{s}_1$ and $\mathbf{s}_2$ in the form of a matrix

$$S = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{pmatrix}. \tag{2.147}$$

The Wronskian of $\mathbf{s}_1$ and $\mathbf{s}_2$ is the determinant of this matrix which is independent of r since

$$\partial_r \det S = \mathrm{tr}(S^{-1}\partial_r S) = -\mathrm{tr}(S^{-1}(A_r - i\Phi)S) = 0. \tag{2.148}$$

The asymptotic behaviour (2.130) therefore implies that $\det S = 1$.

In the $SU(2)$ case we can write the Wronskian as

$$W(\mathbf{s}_1, \mathbf{s}_2) = \mathbf{s}_1^t J \mathbf{s}_2 \qquad \text{where} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.149}$$

Now the commutator $[D_r - i\Phi, D_{\bar{z}}] = 0$ implies that $D_{\bar{z}}\mathbf{s}$ are also solutions of the scattering equation and therefore

$$D_{\bar{z}}\begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{pmatrix} A(z,\bar{z}) = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{pmatrix}\begin{pmatrix} a_{11}(z,\bar{z}) & a_{12}(z,\bar{z}) \\ a_{21}(z,\bar{z}) & a_{22}(z,\bar{z}) \end{pmatrix}. \tag{2.150}$$

If we take a different basis of solutions

$$S' = SM(z,\bar{z}), \tag{2.151}$$

then we find

$$D_{\bar{z}}S' = S'A'(z,\bar{z}) = S'(M^{-1}AM + M^{-1}\partial_{\bar{z}}M), \tag{2.152}$$

so A transforms like a gauge field.

In the southern gauge we can take the following basis of solutions to the scattering equation

$$S^S = \begin{pmatrix} \mathbf{s}_1^S & \mathbf{s}_2^S \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{pmatrix} \begin{pmatrix} (1+z\bar{z})^{\frac{n}{2}} & 0 \\ 0 & (1+z\bar{z})^{-\frac{n}{2}} \end{pmatrix}. \tag{2.153}$$

Then, using the aymptotic form of $A_{\bar{z}}$ in the southern gauge (2.124), we find that

$$D_{\bar{z}} S^S = S^S A^S = S^S O(\frac{1}{r}). \tag{2.154}$$

Since A^S is independent of r, and $\mathbf{s}_1$ decays faster than $\mathbf{s}_2$, it must be that

$$A^S = \begin{pmatrix} 0 & a_{12}^S \\ 0 & 0 \end{pmatrix}. \tag{2.155}$$

In particular

$$D_{\bar{z}} \mathbf{s}_1^S = 0. \tag{2.156}$$

Similarly, in the northern gauge the scattering equation is also of the form (2.128) but with a different integrable matrix $C(r)$. The same argument gives us solutions $\mathbf{s}_1$ and $\mathbf{s}_2$ satisfying

$$r^{-\frac{n}{2}} e^r \mathbf{s}_1 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad r^{\frac{n}{2}} e^{-r} \mathbf{s}_2 \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.157}$$

Now defining the basis

$$S^N = \begin{pmatrix} \mathbf{s}_1^N & \mathbf{s}_2^N \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 \end{pmatrix} \begin{pmatrix} (1+\frac{1}{z\bar{z}})^{\frac{n}{2}} & 0 \\ 0 & (1+\frac{1}{z\bar{z}})^{-\frac{n}{2}} \end{pmatrix}, \tag{2.158}$$

we find similarly

$$D_{\bar{z}} S^N = S^N A^N \qquad \text{where} \qquad A^N = \begin{pmatrix} 0 & a_{12}^N \\ 0 & 0 \end{pmatrix}, \tag{2.159}$$

and $D_{\bar{z}} \mathbf{s}_1^N = 0$.

If the unitary gauge transformations which take us from the original, smooth gauge to the southern and northern gauges are g^S and g^N then we have solutions $g^S \mathbf{s}_1^S$ and $g^N \mathbf{s}_1^N$ in the original smooth gauge which satisfy

$$(D_r - i\Phi) g^S \mathbf{s}_1^S = D_{\bar{z}} g^S \mathbf{s}_1^S = 0 \qquad (D_r - i\Phi) g^N \mathbf{s}_1^N = D_{\bar{z}} g^N \mathbf{s}_1^N = 0. \tag{2.160}$$

Since both solutions decay as $r \to \infty$, this implies that $g^N \mathbf{s}_1^N = f(z) g^S \mathbf{s}_1^S$.

We can find $f(z)$ from the Wronskian $W(\mathbf{s}_1^S, \mathbf{s}_2^N)$ which is again independent of r. Asymptotically we have

$$\mathbf{s}_1^S \to e^{-r}\left(\frac{r}{1+z\bar{z}}\right)^{\frac{n}{2}} g_\infty^S \begin{pmatrix}1\\0\end{pmatrix} \qquad \mathbf{s}_2^N \to e^{r}\left(\frac{rz\bar{z}}{1+z\bar{z}}\right)^{-\frac{n}{2}} g_\infty^N \begin{pmatrix}0\\1\end{pmatrix}. \tag{2.161}$$

Now, using the fact that, for a matrix M in $SL(2,\mathbb{C})$, $J^{-1}M^tJ = M^{-1}$, we find

$$\begin{aligned} f(z) &= W(\mathbf{s}_1^S, \mathbf{s}_2^N) = (\mathbf{s}_1^S)^t J \mathbf{s}_2^N \\ &= (z\bar{z})^{-\frac{n}{2}} \begin{pmatrix}1 & 0\end{pmatrix} J g_\infty^{S-1} g_\infty^N \begin{pmatrix}0\\1\end{pmatrix} \\ &= (z\bar{z})^{-\frac{n}{2}} \begin{pmatrix}0 & 1\end{pmatrix} \begin{pmatrix}(\frac{z}{\bar{z}})^{\frac{n}{2}} & 0\\ 0 & (\frac{\bar{z}}{z})^{\frac{n}{2}}\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \\ &= \frac{1}{z^n}. \end{aligned} \tag{2.162}$$

The original gauge is one in which the Euclidean gauge fields are well-defined and the gauge field $A_{\bar{z}} = \frac{\partial x^i}{\partial \bar{z}} A_i$ is zero at the origin. Given that $D_{\bar{z}} g^S \mathbf{s}_1^S = 0$, this means that $g^S \mathbf{s}_1^S$ is holomorphic at the origin.

Writing

$$g^S \mathbf{s}_1^S \bigg|_{r=0} = \begin{pmatrix}p(z)\\q(z)\end{pmatrix}, \tag{2.163}$$

which is defined for $z \neq \infty$, the Jarvis rational map is defined to be

$$q(z)/p(z). \tag{2.164}$$

Similarly

$$g^N \mathbf{s}_1^N \bigg|_{r=0} = \begin{pmatrix}\tilde{p}(1/z)\\ \tilde{q}(1/z)\end{pmatrix} = \frac{1}{z^n}\begin{pmatrix}p(z)\\q(z)\end{pmatrix}, \tag{2.165}$$

is defined for $z \neq 0$ and gives the same rational map on the overlap. If p and q are polynomials in z, and $\tilde{p}$ and $\tilde{q}$ are polynomials in $1/z$, then it must be that the maximum of the degrees of $p(z)$ and $q(z)$ is n. Therefore $q(z)/p(z) = \tilde{q}(1/z)/\tilde{p}(1/z)$ is a rational map of degree n.

We can characterize $g^S \mathbf{s}_1^S$ and $g^N \mathbf{s}_1^N$ as being solutions of the scattering equation which decay as $r \to \infty$. Any decaying solution of the scattering equation will be of the form $\mathbf{s}_1' = f(z,\bar{z}) g^S \mathbf{s}_1^S$ and thus evaluating this at the origin and taking the ratio gives the same rational map:

$$\mathbf{s}_1'(\mathbf{0}) = f(z,\bar{z})\begin{pmatrix}p(z)\\q(z)\end{pmatrix} \to \frac{q(z)}{p(z)}. \tag{2.166}$$

In general, such a solution will satisfy $D_{\bar{z}}\mathbf{s}_1' = \partial_{\bar{z}} \log f(z,\bar{z})\mathbf{s}_1'$.

If we begin in a different smooth unitary gauge, related to the first by a gauge transformation u, then the solutions to the scattering equation will be $ug^S\mathbf{s}_1^S$ and $ug^S\mathbf{s}_2^S$. Evaluating the decaying solution at the origin gives

$$u(0)\begin{pmatrix}p(z)\\ q(z)\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\begin{pmatrix}p(z)\\ q(z)\end{pmatrix}, \tag{2.167}$$

which results in the $SU(2)$ Möbius action on the rational map

$$\frac{q(z)}{p(z)} \to \frac{\bar{\alpha}q(z) - \bar{\beta}p(z)}{\beta q(z) + \alpha p(z)}. \tag{2.168}$$

Thus the correspondence between a monopole and a rational map is defined modulo constant $SU(2)$ transformations which correspond to the choice of unitary gauge at the origin.

2.5.1 The Jarvis Gauge and the Metric

A consequence of the equation $[D_r - i\Phi, D_{\bar{z}}] = 0$ is that we can transform to a non-unitary gauge in which $A_r - i\Phi = A_{\bar{z}} = 0$ and the other fields $A_r + i\Phi$ and A_z can be written in terms of a Hermitian matrix. In particular, there is a gauge, unique up to constant $SU(2)$ gauge transformations, in which this Hermitian matrix is well-defined on $\mathbb{R}^3$ and equals the identity at the origin. This gauge is referred to as the *Jarvis gauge* and the Hermitian matrix function $\mathcal{H}$ from which the non-zero components of the fields may be obtained is termed the *metric* associated to the monopole.

In the previous section we found a basis of solutions to the scattering equation $\mathbf{s}_1^S$, $\mathbf{s}_2^S$ satisfying

$$D_{\bar{z}}S^S = S^S\begin{pmatrix}0 & a_{12}^S(z,\bar{z})\\ 0 & 0\end{pmatrix} \qquad \text{where} \qquad S^S = \begin{pmatrix}\mathbf{s}_1^S & \mathbf{s}_2^S\end{pmatrix}. \tag{2.169}$$

If we change to a new basis

$$\hat{S}^S = S^S V^S \qquad V^S = \begin{pmatrix}1 & v^S(z,\bar{z})\\ 0 & 1\end{pmatrix}, \tag{2.170}$$

then, using (2.152), we find

$$D_{\bar{z}}\hat{S}^S = \hat{S}^S\begin{pmatrix}0 & a_{12}^S + \partial_{\bar{z}}v^S\\ 0 & 0\end{pmatrix}. \tag{2.171}$$

So, by taking

$$v^S(z,\bar{z}) = -\int d\bar{z}\, a^S_{12}(z,\bar{z}), \tag{2.172}$$

we obtain a basis $\hat{S}^S$ defined for $z \neq \infty$ satisfying

$$(D_r - i\Phi)\hat{S}^S = D_{\bar{z}}\hat{S}^S = 0. \tag{2.173}$$

The same carries through for S^N to obtain a basis defined for $z \neq 0$ satisfying

$$(D_r - i\Phi)\hat{S}^N = D_{\bar{z}}\hat{S}^N = 0. \tag{2.174}$$

As before, if we evaluate $g^S\hat{S}^S$ and $g^N\hat{S}^N$ at the origin we obtain matrices which are holomorphic on $z \neq \infty$ and $z \neq 0$ respectively.

$$g^S\hat{S}^S\Big|_{r=0} = F^S(z) \qquad g^N\hat{S}^N\Big|_{r=0} = F^N(z). \tag{2.175}$$

Defining

$$a^S = g^S\hat{S}^S F^{S-1} \qquad a^N = g^N\hat{S}^N F^{N-1}, \tag{2.176}$$

corresponds to a change of basis. It is easy to see that $a^{N-1}a^S$ is independent of r and $\bar{z}$ and is equal to the identity at the origin, therefore we can define

$$a = \begin{cases} a^S & z \neq \infty \\ a^N & z \neq 0 \end{cases}, \tag{2.177}$$

which is well-defined on the whole of $\mathbb{R}^3$.

The matrix a also corresponds to a basis which satisfies $(D_r - i\Phi)a = D_{\bar{z}}a = 0$. Therefore the fields in the original unitary gauge are

$$A_r - i\Phi = -(\partial_r a)a^{-1} \qquad A_{\bar{z}} = -(\partial_{\bar{z}} a)a^{-1}. \tag{2.178}$$

Since the fields are anti-Hermitian in this gauge the other fields are

$$A_r + i\Phi = a^{\dagger-1}(\partial_r a^\dagger) \qquad A_z = a^{\dagger-1}(\partial_z a^\dagger). \tag{2.179}$$

Now if we gauge transform by a we end up in a non-unitary gauge in which

$$\begin{aligned} A_r - i\Phi &= 0 & A_{\bar{z}} &= 0 \\ A_r + i\Phi &= \mathcal{H}^{-1}\partial_r\mathcal{H} & A_z &= \mathcal{H}^{-1}\partial_z\mathcal{H}. \end{aligned} \tag{2.180}$$

where $\mathcal{H} = a^\dagger a$. This is the metric, a unimodular, Hermitian matrix function defined on $\mathbb{R}^3$ and satisfying $\mathcal{H}(\mathbf{0}) = \mathbb{I}$. Note that $\mathcal{H}$ is independent of the unitary gauge in which the monopole is originally defined. The condition that $\mathcal{H}(\mathbf{0}) = \mathbb{I}$ and that $\mathcal{H}$ is Hermitian means that this gauge is defined up to conjugation by a constant $SU(2)$ matrix. It is continuous at the origin and smooth away from the origin.

In this gauge, the Bogomol'nyi equations (2.111) and (2.112) are trivially true and the third equation (2.113) is equivalent to the *Jarvis equation* for the metric $\mathcal{H}$

$$\partial_r(\mathcal{H}^{-1}\partial_r\mathcal{H}) + \frac{(1+z\bar{z})^2}{r^2}\partial_{\bar{z}}(\mathcal{H}^{-1}\partial_z\mathcal{H}) = 0. \tag{2.181}$$

As pointed out in [47], following a suggestion of Manton, we can determine the rational map from the metric as follows. In the original unitary gauge we have a decaying solution to the scattering equation $g^N \mathbf{s}_1^N$ and a gauge transformation a which takes us to the Jarvis gauge in which $D_r - i\Phi = \partial_r$ and $D_{\bar{z}} = \partial_{\bar{z}}$. In a non-unitary gauge, it doesn't make sense to talk about the decaying solution so instead we recognise that this is the solution which tends to an eigenvector of $-i\Phi^\infty$ with eigenvalue $+1$. Hence $a^{-1}g^N\mathbf{s}_1^N$ is the r-independent solution to the scattering equation in the Jarvis gauge, or equivalently an eigenvector of $-i\Phi^\infty$ in this gauge with eigenvalue $+1$. Since $a^{-1}(r=0) = \mathbb{I}$, evaluating this at the origin gives the rational map as before.

2.5.2 An Example: The BPS Monopole

It will be instructive to go through the procedure that gives rise to the Jarvis rational map and the function $\mathcal{H}$ for the BPS monopole.

We found the fields of the BPS monopole situated at the origin above

$$\begin{aligned}
\Phi &= i\left(\frac{1}{2r} - \coth 2r\right)\frac{1}{1+z\bar{z}}\begin{pmatrix} z\bar{z}-1 & 2\bar{z} \\ 2z & 1-z\bar{z}\end{pmatrix} \\
A_z &= \left(\frac{2r}{\sinh 2r} - 1\right)\frac{1}{(1+z\bar{z})^2}\begin{pmatrix} -\bar{z} & \bar{z}^2 \\ -1 & \bar{z}\end{pmatrix} \\
A_{\bar{z}} &= \left(\frac{2r}{\sinh 2r} - 1\right)\frac{1}{(1+z\bar{z})^2}\begin{pmatrix} z & 1 \\ -z^2 & -z\end{pmatrix} \\
A_r &= 0.
\end{aligned} \tag{2.182}$$

This corresponds to a smooth Cartesian gauge.

The eigenvectors of Φ are $(1,-z)^t$ and $(\bar{z},1)^t$ so the gauge transformation g^S which takes us to the Dirac string gauge excluding the line $z=\infty$ is

$$g^S = \frac{1}{\sqrt{1+z\bar{z}}}\begin{pmatrix} 1 & \bar{z} \\ -z & 1\end{pmatrix}. \tag{2.183}$$

In this gauge we have

$$\begin{aligned}
\Phi^S &= \frac{i}{2}\left(\coth 2r - \frac{1}{2r}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
A^S_z &= \frac{\bar{z}}{2(1+z\bar{z})}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \frac{2r}{\sinh 2r(1+z\bar{z})}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\
A^S_{\bar{z}} &= -\frac{z}{2(1+z\bar{z})}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{2r}{\sinh 2r(1+z\bar{z})}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\\
A^S_r &= 0.
\end{aligned} \tag{2.184}$$

The solutions to the scattering equation $(D_r - i\Phi)\mathrm{s} = 0$ with the prescribed asymptotic behaviour are

$$\mathrm{s}^S_0 = \sqrt{\frac{2r(1+z\bar{z})}{\sinh 2r}}\begin{pmatrix}1\\ 0\end{pmatrix} \qquad \mathrm{s}^S_1 = \sqrt{\frac{\sinh 2r}{2r(1+z\bar{z})}}\begin{pmatrix}0\\ 1\end{pmatrix}, \tag{2.185}$$

and these satisfy

$$D_{\bar{z}}\left(\mathrm{s}^S_1 \ \mathrm{s}^S_2\right) = \left(\mathrm{s}^S_1 \ \mathrm{s}^S_2\right)\begin{pmatrix}0 & \frac{1}{(1+z\bar{z})^2}\\ 0 & 0\end{pmatrix}, \tag{2.186}$$

so clearly $D_{\bar{z}}\left(\mathrm{s}_1 - \frac{\bar{z}}{1+z\bar{z}}\mathrm{s}_0\right) = 0$.

In other words, starting from a smooth gauge, the gauge transformation which trivialises the commutator $[D_r - i\Phi, D_{\bar{z}}]$ is

$$\tilde{g}^S = \frac{1}{\sqrt{1+z\bar{z}}}\begin{pmatrix}1 & \bar{z}\\ -z & 1\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2r(1+z\bar{z})}{\sinh 2r}} & 0\\ 0 & \sqrt{\frac{\sinh 2r}{2r(1+z\bar{z})}}\end{pmatrix}\begin{pmatrix}1 & -\frac{\bar{z}}{1+z\bar{z}}\\ 0 & 1\end{pmatrix}. \tag{2.187}$$

Evaluating this at the origin we obtain a holomorphic representative of the rational map in this patch

$$F^S(z) = \tilde{g}^S(\mathbf{0}) = \begin{pmatrix}1 & 0\\ -z & 1\end{pmatrix}, \tag{2.188}$$

corresponding to the rational map $-1/z$. It should be noted that this is a coset representative in the coset space $SL(2,\mathbb{C})/B$, where B is the group of unimodular upper triangular matrices. This space is isomorphic to $\mathbb{C}P^1$, but it can also be identified with a complex *flag manifold* [21–23].

If we work on the other hemisphere we take the gauge transformation

$$g^N = \sqrt{\frac{z\bar{z}}{1+z\bar{z}}}\begin{pmatrix}\frac{1}{z} & 1\\ -1 & \frac{1}{\bar{z}}\end{pmatrix} \tag{2.189}$$

so the fields are

$$\begin{aligned}
\Phi^N &= \Phi^S \\
A_z^N &= -\frac{1}{2z(1+z\bar{z})}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \frac{2r\bar{z}}{z\sinh 2r(1+z\bar{z})}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \\
A_{\bar{z}}^N &= \frac{1}{2\bar{z}(1+z\bar{z})}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{2rz}{\bar{z}\sinh 2r(1+z\bar{z})}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \\
A_r^N &= 0.
\end{aligned} \tag{2.190}$$

The solutions to the scattering equation in this gauge are

$$\mathrm{s}_0^N = \sqrt{\frac{2r(1+z\bar{z})}{z\bar{z}\sinh 2r}}\begin{pmatrix}1\\0\end{pmatrix} \qquad \mathrm{s}_1^N = \sqrt{\frac{z\bar{z}\sinh 2r}{2r(1+z\bar{z})}}\begin{pmatrix}0\\1\end{pmatrix}, \tag{2.191}$$

and

$$D_{\bar{z}}\begin{pmatrix}\mathrm{s}_1^N & \mathrm{s}_2^N\end{pmatrix} = \begin{pmatrix}\mathrm{s}_1^N & \mathrm{s}_2^N\end{pmatrix}\begin{pmatrix}0 & \frac{z^2}{(1+z\bar{z})^2}\\ 0 & 0\end{pmatrix}, \tag{2.192}$$

ie. $D_{\bar{z}}\left(\mathrm{s}_1 + \frac{z}{1+z\bar{z}}\mathrm{s}_0\right) = 0$.

Therefore the whole gauge transformation is

$$\tilde{g}^N = \sqrt{\frac{z\bar{z}}{1+z\bar{z}}}\begin{pmatrix}\frac{1}{z} & 1\\ -1 & \frac{1}{\bar{z}}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2r(1+z\bar{z})}{z\bar{z}\sinh 2r}} & 0\\ 0 & \sqrt{\frac{z\bar{z}\sinh 2r}{2r(1+z\bar{z})}}\end{pmatrix}\begin{pmatrix}1 & \frac{z}{1+z\bar{z}}\\ 0 & 1\end{pmatrix}. \tag{2.193}$$

This gives us a holomorphic representative on the northern hemisphere

$$F^N(z) = \tilde{g}^N(\mathbf{0}) = \begin{pmatrix}\frac{1}{z} & 1\\ -1 & 0\end{pmatrix}. \tag{2.194}$$

The difference between the two final results is a holomorphic function on the overlap

$$\tilde{g}^{N-1}\tilde{g}^S = \begin{pmatrix}z & -1\\ 0 & \frac{1}{z}\end{pmatrix}, \tag{2.195}$$

which is related to the degree of the rational map.

We calculate $\mathcal{H}$ from $\tilde{g}^N$ or $\tilde{g}^S$ as follows

$$a^S = \tilde{g}^S F^S(z)^{-1} \qquad a^N = \tilde{g}^N F^N(z)^{-1}, \tag{2.196}$$

finding

$$\mathcal{H} = \frac{2r}{\sinh 2r}\mathbb{P} + \frac{\sinh 2r}{2r}(\mathbb{I} - \mathbb{P}), \tag{2.197}$$

where $\mathbb{P}$ is the projector introduced in the hedgehog ansatz (2.104)

$$\mathbb{P} = \frac{1}{1+z\bar{z}}\begin{pmatrix}-1\\ z\end{pmatrix}\begin{pmatrix}-1 & \bar{z}\end{pmatrix}. \tag{2.198}$$

2.6 Summary

We have given a basic introduction to the two models studied in the remainder of this thesis. In particular we described static $\mathbb{C}P^1$ lumps on the sphere as rational maps and defined the metric on the moduli space which we can use to approximate low energy soliton dynamics. We derived the Bogomol'nyi equations which describe static monopoles in the Prasad-Sommerfield limit and saw that they take on a special form in spherical polar coordinates involving the complex coordinate on the Riemann sphere. This allowed us to define the Jarvis rational map and a metric from which the fields may be calculated in the non-unitary Jarvis gauge. The metric satisfies the Jarvis equation, which is equivalent to the Bogomol'nyi equations in this gauge. We tried to make the definition of the Jarvis rational map as explicit as possible and have provided the detailed example of the one-monopole. Thus we have now layed the foundations for the work which follows.

Chapter 3

Two-Lumps on the Sphere

In this chapter we will use the geodesic approximation introduced in Chapter 2 to study the low energy dynamics of two-lumps on $S^2 \times \mathbb{R}$. The low energy dynamics of a single lump on $S^2 \times \mathbb{R}$ has already been studied by Speight [29] and yields some non-trivial geodesic motion. One interesting feature of the scattering of two solitons in many models is that solitons which collide head-on emerge at right angles to the initial direction of motion. We will find that this is the case here too, and in fact we can find a submanifold on which geodesic motion is remarkably similar to that of two monopoles.

Since the space on which the lumps live is compact, we are using the term scattering loosely. Instead of a motion in which the solitons move in from spatial infinity, interact, and return to spatial infinity, our lumps start from localised, "zero size" lumps, interact, and return to zero size lumps. Thus our asymptotic states may be identified with zero size lumps. We will also find that such a process takes place within a finite time, which is a consequence of the compactness of space.

The work in section 3.5 was carried out in collaboration with Sanjeev Shukla.

3.1 Degree Two Rational Maps

We begin by describing the space of degree two rational maps, which is a ten-dimensional manifold isomorphic to the moduli space of two-lumps. Six of these dimensions can be identified with the rotational symmetries of the target space and spatial two-spheres, leaving four non-trivial parameters. By studying the potential energy density we can see how these parameters affect the solution.

Our aim is to use the two natural $SL(2,\mathbb{C})$ actions on rational maps which preserve the degree, one on the spatial variable z and the other on the target space u, to parametrise degree 2 maps. We will denote them $SL(2,\mathbb{C})^{\mathrm{S}}$ and $SL(2,\mathbb{C})^{\mathrm{T}}$ respectively. These group actions contain the $SU(2)$ rotational isometries of the spatial and target space two-spheres.

We can represent a general degree two rational map as a 3×2 matrix

$$M = \begin{pmatrix} p^1 & p^2 & p^3 \\ q^1 & q^2 & q^3 \end{pmatrix} \to \frac{p(z)}{q(z)} = \frac{p^1 l_1(z) + p^2 l_2(z) + p^3 l_3(z)}{q^1 l_1(z) + q^2 l_2(z) + q^3 l_3(z)}, \tag{3.1}$$

where $l_i(z)$, $i = 1 \ldots 3$ are a basis of degree two polynomials. Multiplying M by a non-zero constant

$$M \to c\,M, \tag{3.2}$$

leaves the rational map invariant so we must identify matrices which differ in this way.

A convenient basis of polynomials is found by considering the generators of spatial $SL(2,\mathbb{C})$ transformations

$$L_1 = \frac{i}{2}(z^2-1)\frac{d}{dz} \qquad L_2 = -\frac{1}{2}(z^2+1)\frac{d}{dz} \qquad L_3 = -iz\frac{d}{dz}, \tag{3.3}$$

which obey the commutation relations

$$[L_1, L_2] = -\epsilon_{ijk} L_k. \tag{3.4}$$

Under an $SL(2,\mathbb{C})^{\mathrm{S}}$ transformation

$$z \to \frac{Az+B}{Cz+D} \qquad \text{with} \qquad AD - BC = 1, \tag{3.5}$$

these operators transform under the adjoint action

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} \to S \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(A^2-B^2-C^2+D^2) & -\frac{i}{2}(A^2+B^2-C^2-D^2) & CD-AB \\ \frac{i}{2}(A^2-B^2+C^2-D^2) & \frac{1}{2}(A^2+B^2+C^2+D^2) & -i(AB+CD) \\ BD-AC & i(AC+BD) & AD+BC \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}, \tag{3.6}$$

defining S, a matrix in $SO(3,\mathbb{C}) = \{S \in \mathrm{SL(3,\mathbb{C})} \,|\, S^t S = \mathbb{I}\}$.

We therefore take the following basis of degree two polynomials

$$l_1(z) = \frac{i}{2}(z^2-1) \qquad l_2(z) = -\frac{1}{2}(z^2+1) \qquad l_3(z) = -iz. \tag{3.7}$$

The polynomials transform like the operators L_i up to a factor of $(Cz+D)^2$ which appears on the denominator. However, we can cancel this factor from the top and bottom of the rational map and therefore represent the $SL(2,\mathbb{C})^S$ action (3.5) on the matrix M as right multiplication by S

$$M \to MS. \tag{3.8}$$

The $SL(2,\mathbb{C})^{\mathrm{T}}$ action on the target space

$$\frac{p(z)}{q(z)} \to \frac{Ep(z)+Fq(z)}{Gp(z)+Hq(z)} \qquad \text{with} \qquad EH-FG=1, \tag{3.9}$$

acts simply as left multiplication on M

$$M \to TM \qquad \text{where} \qquad T = \begin{pmatrix} E & F \\ G & H \end{pmatrix}. \tag{3.10}$$

There is an $SO(3,\mathbb{C})^{\mathrm{S}}$ vector which we can associate to the matrix M which occurs naturally when we consider the zeros of the potential energy density (2.36)

$$\mathcal{E} = (1+z\bar{z})^2 \frac{(q\partial_z p - p\partial_z q)(\bar{q}\partial_{\bar{z}}\bar{p} - \bar{p}\partial_{\bar{z}}\bar{q})}{(p\bar{p}+q\bar{q})^2}. \tag{3.11}$$

Using the commutation relations (3.4) we find

$$q\partial_z p - p\partial_z q = -\epsilon_{ijk} q^i p^j l^k = v^k l^k, \tag{3.12}$$

defining $v^i = \epsilon_{ijk} p^j q^k$, which is a vector under the $SO(3,\mathbb{C})^{\mathrm{S}}$ action (3.8) and a scalar under the $SL(2,\mathbb{C})^{\mathrm{T}}$ action (3.10). The zeros of the potential energy density occur at the points $(v^3 \pm \sqrt{vv^t})/(v^1+iv^2)$.

We will partially fix the freedom (3.2) by fixing the resultant of the map $p(z)/q(z)$. In the process this ensures that M really does correspond to a degree two map. For the map (3.1), it turns out that the resultant can be written in terms of the vector v. Plugging the map into the formula (2.50) we find

$$R(p,q) = -vv^t/4, \tag{3.13}$$

which is invariant under both $SL(2,\mathbb{C})^{\mathrm{T}}$ and $SO(3,\mathbb{C})^{\mathrm{S}}$. We will fix $R(p,q)$ to be 1, so that v lives in the set $S^2_{\mathbb{C}} = \{v \in \mathbb{C}^3 | vv^t = -4\}$. This fixes the matrix M up to a $\mathbb{Z}_4$ freedom.

We can identify $S^2_{\mathbb{C}}$ with the action of $SO(3)$ on a vector depending on a real parameter. Let us write $v = x + iy$ where x and y are real vectors. Then the condition $vv^t = -4$ implies

$$\begin{aligned} |y|^2 - |x|^2 &= 4 \\ x \cdot y &= 0. \end{aligned} \tag{3.14}$$

So x and y are orthogonal vectors and thus using an $SO(3)$ transformation we can choose

$$\begin{aligned} x &= (0, 2\tan\zeta, 0) \\ y &= (0, 0, 2\sec\zeta), \end{aligned} \tag{3.15}$$

where $\zeta \in [0, \frac{\pi}{2}]$.

When $\zeta \neq 0$, $SO(3)$ acts transitively on v and we have an identification of $S^2_{\mathbb{C}}$ with

$$(0, 2\tan\zeta, 2i\sec\zeta)R \qquad R \in SO(3). \tag{3.16}$$

When $\zeta = 0$ there is an $SO(2)$ subgroup of $SO(3)$ which fixes v, and the space becomes isomorphic to $SO(3)/SO(2) \cong S^2$.

A matrix which gives rise to the vector $v = (0, 2\tan\zeta, 2i\sec\zeta)$ is of the form $TZ(\zeta)$ where $T \in SL(2,\mathbb{C})$ and

$$Z(\zeta) = \begin{pmatrix} -i & -\sec\zeta & -i\tan\zeta \\ i & -\sec\zeta & -i\tan\zeta \end{pmatrix}. \tag{3.17}$$

Thus the target space $SL(2,\mathbb{C})$ is precisely the group of transformations of M which fixes a given v. An element of $SL(2,\mathbb{C})$ can be uniquely decomposed into the product of a unitary matrix $U \in SU(2)$ and a positive definite unimodular Hermitian matrix H, so we arrive at the parametrisation

$$M = UHZ(\zeta)R. \tag{3.18}$$

U and R correspond to rotations of the target space and spatial two-spheres, leaving four non-trivial parameters contained in the matrix H and ζ.

There is a remaining $\mathbb{Z}_4$ freedom in M, generated by $M \to iM$, which corresponds to the fact that

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} Z(\zeta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = iZ(\zeta). \tag{3.19}$$

If we wish, we can remove this freedom by identifying

$$U \leftrightarrow U \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad H \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} H \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{3.20}$$

Note that this identifies v and $-v$ so $v \in S^2_{\mathbb{C}}/\mathbb{Z}_2$. When $\zeta = 0$, $S^2_{\mathbb{C}}/\mathbb{Z}_2 = S^2/\mathbb{Z}_2 = \mathbb{R}P(2)$.

When $\zeta = 0$, the subgroup of $SO(3)$ which fixes the vector $v = (0, 0, 2i)$ is

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.21}$$

Since this fixes v, it must coincide with a target space transformation. In this case, we find that right multiplication by R is equivalent to left multiplication by a $U(1)$ matrix

$$\begin{pmatrix} -i & -1 & 0 \\ i & -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} -i & -1 & 0 \\ i & -1 & 0 \end{pmatrix}. \tag{3.22}$$

Therefore this action is equivalent to

$$
\begin{aligned}
H &\to \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} H \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \\
U &\to U \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} .
\end{aligned} \tag{3.23}
$$

If H is not diagonal, we can fix this freedom by choosing H to be real. If H is diagonal, then the configuration is axially symmetric and a spatial rotation about the axis of symmetry is equivalent to a target space rotation.

3.2 Some Features of Two-Lumps

We will plot the potential energy density (3.11) for various ranges of the four non-trivial parameters. In the figures, which were generated using MATHEMATICA, the potential energy density of a two-lump configuration is represented as the height of a point above a sphere of fixed radius.

As noted in the previous section, the numerator of the potential energy density is invariant under a target space $SL(2,\mathbb{C})^{\mathrm{T}}$ transformation and therefore depends only on ζ. The denominator is only invariant under a target space $SU(2)$ transformation, and so depends on ζ and the Hermitian matrix H.

We consider the map described by M in (3.18) when U and R are the identity. The corresponding vector $v = (0, 2\tan\zeta, 2i\sec\zeta)$ gives rise to zeros of the potential energy density at $\tan(\zeta/2)$ and $\cot(\zeta/2)$. We will describe the potential energy density for three directions in the space of unimodular Hermitian matrices.

We begin by choosing,

$$
H = \begin{pmatrix} \cosh\frac{\lambda}{2} & \sinh\frac{\lambda}{2} \\ \sinh\frac{\lambda}{2} & \cosh\frac{\lambda}{2} \end{pmatrix} . \tag{3.24}
$$

Firstly we discuss the energy density when $\zeta = 0$ and the zeros are at $z = 0$ and $z = \infty$, which describes an interesting one-dimensional space of configurations. When λ is large and negative, the lumps are localised opposite one another on the 1-axis. As λ tends to zero, the configuration approaches the map $u = z^2$ which describes a particularly symmetric two-lump configuration which is axially symmetric about the 3-axis and symmetric under reflection in the 1-2 plane. The energy density is concentrated in a ring around the equator of the two-sphere, imagining the points $z = 0$ and $z = \infty$ as the poles.

As λ increases from 0, the lumps become more and more localised opposite each other on the 2-axis. We will find geodesics in this submanifold later on which give an analogue of the right-angle scattering found in the plane. Increasing ζ moves the zeros towards the point $z = 1$ and in the process squashes the energy density so that it becomes more concentrated near $z = 1$ and less concentrated around the opposite point $z = -1$. This is shown in figure 3.1 for $\zeta = 0$ and $\zeta = \pi/8$.

Now consider

$$H = \begin{pmatrix} \cosh\frac{\lambda}{2} & -i\sinh\frac{\lambda}{2} \\ i\sinh\frac{\lambda}{2} & \cosh\frac{\lambda}{2} \end{pmatrix}. \tag{3.25}$$

When $\zeta = 0$ we can perform a rotation $z \to e^{-i\pi/4}z$ and unitary transformation $u \to iu$ to obtain the same configuration as given by (3.24). For non-zero ζ this is not the case and the configurations cannot be related by an isometry. These configurations are shown in figure 3.2. Again the effect of increasing ζ is seen to be to squash the density towards $z = 1$.

Lastly, when $\zeta = 0$

$$H = \begin{pmatrix} e^{\frac{\lambda}{2}} & 0 \\ 0 & e^{-\frac{\lambda}{2}} \end{pmatrix}, \tag{3.26}$$

describes the axially symmetric maps $u = e^{\lambda}z^2$. When $\lambda = 0$, this is the axially symmetric ring around the equator discussed above. As λ increases, the ring moves up the sphere, similarily decreasing λ moves the ring down the sphere as illustrated in figure 3.3. If we think of this as a ring moving up and down an axis, then giving ζ a positive value has the effect of "bending" the axis towards $z = 1$.

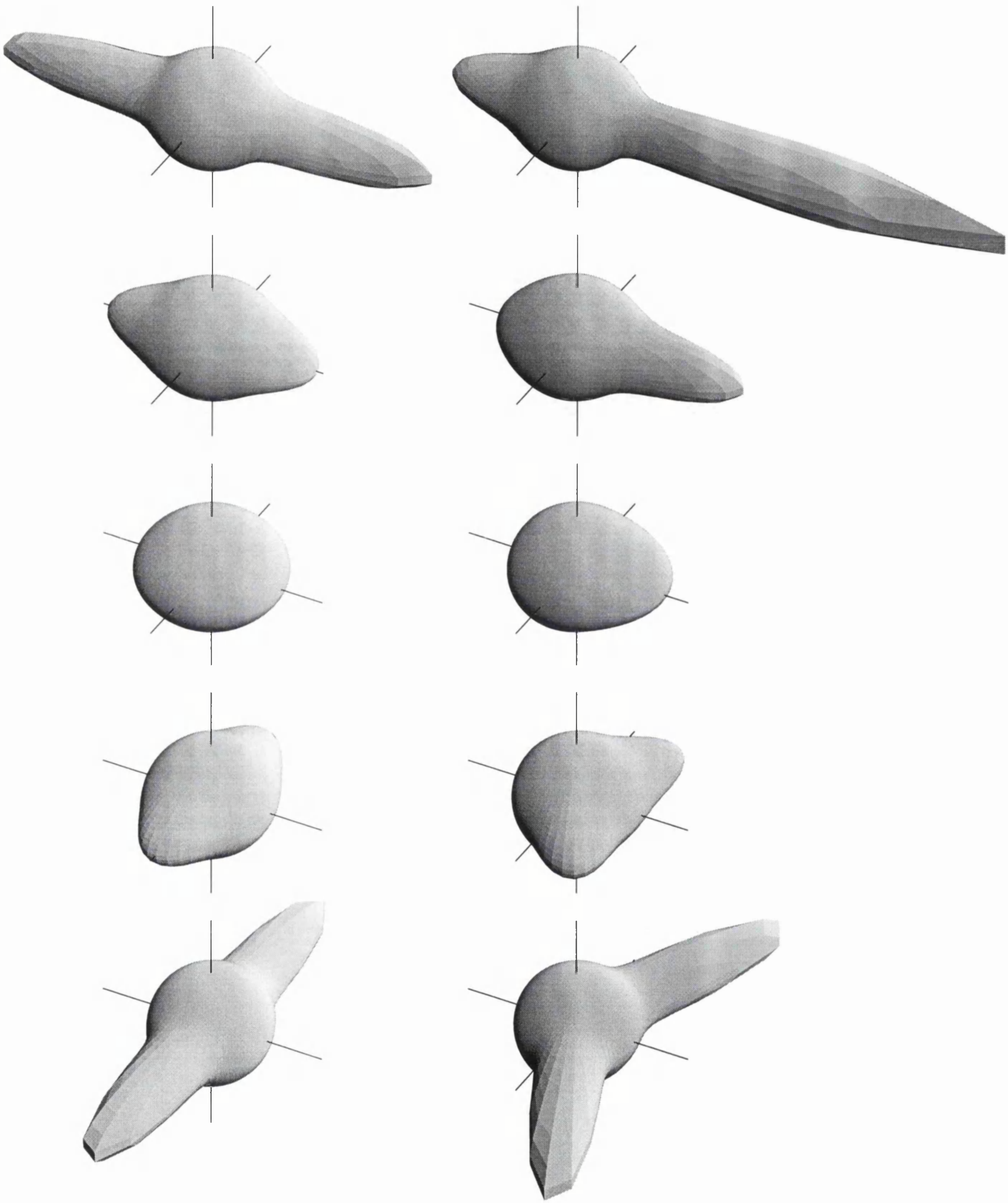


Figure 3.1: The potential energy density for H given by (3.24) with λ running from -1.2 to $+1.2$. The plots on the left are when $\zeta = 0$ and the energy density has zeros at $\theta = 0$ and $\theta = \pi$. Those on the right have $\zeta = \pi/8$ and corresponding zeros at $\theta = \pi/8$ and $\theta = 7\pi/8$.

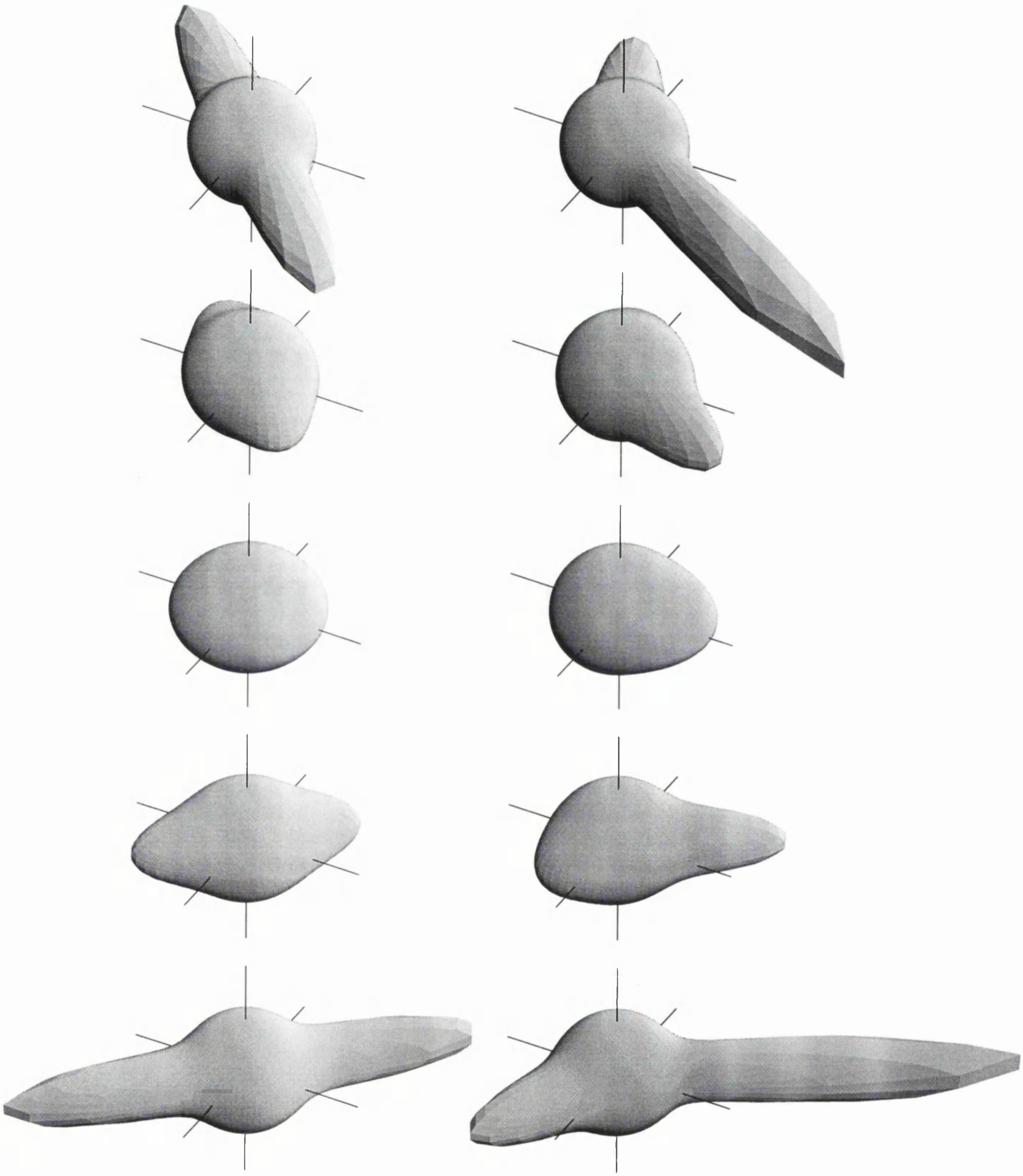


Figure 3.2: The potential energy density for H given by (3.25) with λ running from -1.2 to $+1.2$. Again, the plots on the left are when the zeros are at $\theta = 0$ and $\theta = \pi$ and those on the right at $\theta = \pi/8$ and $\theta = 7\pi/8$.

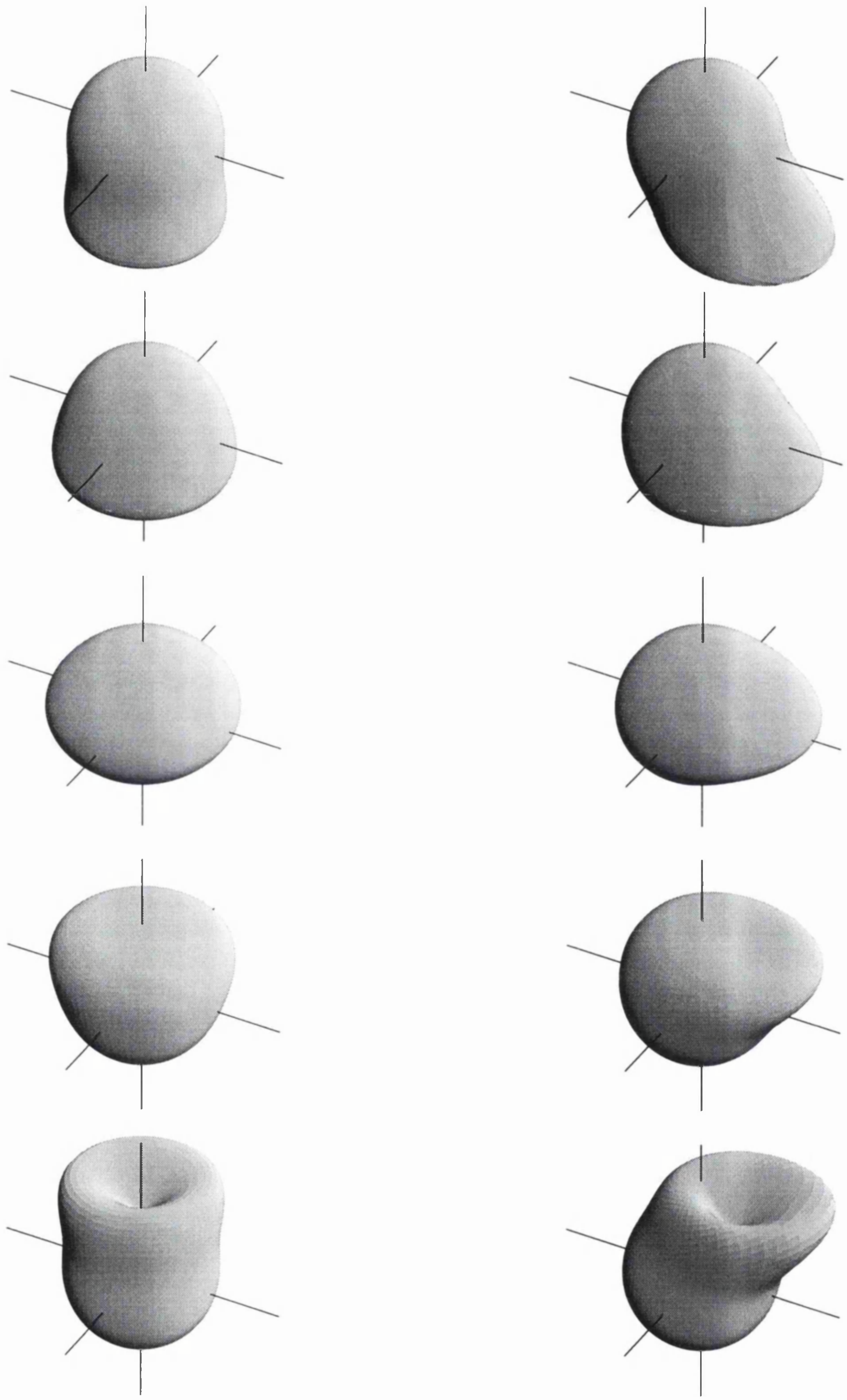


Figure 3.3: The potential energy density for H defined by (3.26) with λ running from -2 to $+2$. The plots on the left are for $\zeta = 0$ and those on the right for $\zeta = \pi/8$.

3.3 Isometries of the Moduli Space Metric

As pointed out in [29], isometries of the spatial and target space two-spheres induce isometries of the moduli space metric. We can see this by writing the moduli space metric (2.37) in a coordinate independent way

$$g = \int \frac{dz\, d\bar{z}}{(1+z\bar{z})^2} \frac{du\, d\bar{u}}{(1+u\bar{u})^2}. \tag{3.27}$$

Since the metric is constructed from the product of the measure on the spatial and target space two-spheres, it is invariant under isometries of the underlying spaces.

Isometries of the spatial two-sphere are of the form

$$z' = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}} \qquad \text{or} \qquad z' = \frac{\alpha \bar{z} + \beta}{-\bar{\beta}\bar{z} + \bar{\alpha}} \qquad \text{where} \qquad \alpha\bar{\alpha} + \beta\bar{\beta} = 1. \tag{3.28}$$

It is convenient to envisage this two-sphere as being the unit two-sphere centred at the origin of $\mathbb{R}^3$, in which case these transformations correspond to rotations about some axis composed with reflections in some plane. We will talk about the 1-, 2- and 3-axes, the 1-2 plane etc., where the 1-axis is the line through $z = \pm 1$, the 2-axis is the line through $z = \pm i$ and the 3-axis is the line through $z = 0$ and $z = \infty$.

Similarly, target space isometries are transformations of the form

$$u' = \frac{\gamma u + \delta}{-\bar{\delta} u + \bar{\gamma}} \qquad \text{or} \qquad u' = \frac{\gamma \bar{u} + \delta}{-\bar{\delta}\bar{u} + \bar{\gamma}} \qquad \text{with} \qquad \gamma\bar{\gamma} + \delta\bar{\delta} = 1. \tag{3.29}$$

We must be slightly careful in that isometries correspond to involutions of the moduli space which consists of holomorphic functions of a particular degree. Therefore isometries which take z to a function of $\bar{z}$ must be composed with a target space isometry which takes u to a function of $\bar{u}$.

As an example, consider sending $z \to -1/\bar{z}$ composed with $u \to 1/\bar{u}$ so that

$$u(a^i, z) \to \frac{1}{\overline{u(a^i, -1/\bar{z})}} = u(b^i, z) \tag{3.30}$$

This is clearly a holomorphic function of z and so defines an isometric map in the parameter space $a^i \to b^i(a^j)$.

3.4 Geodesic Submanifolds

Finding the metric on the full 10-dimensional space of two-lumps would be a daunting task, since there are 4 parameters which are not accounted for by the spatial and target space isometries. To make life easier, we look for *geodesic submanifolds* on which the metric can be written in terms of functions of a fewer number of parameters.

The essential feature of a geodesic submanifold is that geodesic motion with respect to the metric on the full moduli space which starts from a point in the submanifold and is initially tangential to it, remains within this submanifold. In this case the motion is simply geodesic motion within the submanifold with respect to the metric restricted to the submanifold.

A simple way to find geodesic submanifolds is to look for fixed point sets of discrete isometries. If the initial motion is tangential to the fixed point set, then by the uniqueness of solutions to the geodesic equation, motion must remain within the fixed point set. If it were to leave this space, we could use the isometry to find two solutions corresponding to geodesic motion.

If an isometry fixes a configuration, it must map the zeros of the potential energy density into each other. By rotating our configuration we can take the zeros to be at $z = \tan(\zeta/2)$ and $z = \cot(\zeta/2)$. When $\zeta \neq 0$, the isometries which map the zeros into one another are generated by reflections in the 1-2 and 1-3 planes. This group is therefore isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the other non-trivial element being a rotation through π about the 1-axis. When $\zeta = 0$, the zeros are diametrically opposite one another at $z = 0$ and $z = \infty$, and the group of isometries is enlarged to include composition with arbitrary rotations about the 3-axis.

We will enumerate the different submanifolds that may be found as fixed point sets of a discrete isometry which fixes the configuration $u = z^2$. It would be nice to prove that any isometry which fixes a configuration fixes one which can be obtained by rotating this configuration in the target space and spatial two-spheres, for then we would have an exhaustive list of submanifolds which are fixed point sets of a single discrete isometry. Of course, we can also look for sets of configurations which are fixed by more than one discrete isometry and these will also be geodesic submanifolds.

We consider the four cases corresponding to the group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ isometries which are symetries of the points $z = \tan(\zeta/2)$ and $z = \cot(\zeta/2)$ composed with a rotation through a fixed angle θ about the 3-axis. Thus, when $\theta = 0$ we expect to find submanifolds on which ζ is arbitrary, and when $\theta \neq 0$ we must have $\zeta = 0$ so that the zeros are diametrically opposite.

(1). Rotation through θ about the 3-axis

Consider a rotation through a fixed angle θ about the 3-axis

$$z \to e^{i\theta} z \qquad u \to e^{-2i\theta} u. \tag{3.31}$$

If the rational map corresponding to the matrix $M = TZ(\zeta)R$ is fixed by this then

$$TZ(\zeta)R = c\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} TZ(\zeta)R \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.32}$$

for some non-zero constant c. We can use (3.22) to rewrite $Z(0)$ as

$$Z(0) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} Z(0) \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.33}$$

and then look for subgroups of $SU(2)^{\mathrm{T}}$ and $SO(3)^{S}$ satisfying

$$\begin{aligned} T &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} T \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \\ R &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} R \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.34}$$

There are three cases:

(a) $\theta = 0$
The isometry is the identity and we get the whole space of degree 2 maps with arbitrary ζ.

(b) $\theta = \pi$
$\zeta = 0,\ T = UH \in SL(2,\mathbb{C}),\ R \in \mathfrak{R}(0,\infty)$
We can fix H to be real so this space depends on 2 non-isometric parameters.

(c) $\theta \neq 0, \pi$
$\zeta = 0,\ U \in \mathfrak{U}(0,\infty),\ H = \begin{pmatrix} e^{\frac{\lambda}{2}} & 0 \\ 0 & e^{-\frac{\lambda}{2}} \end{pmatrix},\ R \in \mathfrak{R}(0,\infty)$
These configurations are axially symmetric about the 3-axis.

Here $\mathfrak{R}(0,\infty)$ is the $SO(2)$ subgroup which fixes the points $z = 0, \infty$, ie. rotations about the axis through these points. Similarly $\mathfrak{U}(0,\infty)$ is the $U(1)$ subgroup which fixes $u = 0, \infty$.

(2). Rotation through π about the 1-axis composed with (1)

This isometry is of the form

$$z \to \frac{e^{i\theta}}{z} \qquad u \to \frac{e^{-2i\theta}}{u}. \tag{3.35}$$

For arbitrary ζ we use

$$iZ(\zeta) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} Z(\zeta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.36}$$

and for $\theta \neq 0$ and $\zeta = 0$ additionally (3.33) to impose

$$\begin{aligned} T &= \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} T \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \\ R &= \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} R \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.37}$$

The isometries fix the points $z = \pm e^{\frac{i\theta}{2}}$, $u = \pm e^{-i\theta}$ so for a given θ, $U \in \mathfrak{U}(e^{i\theta}, -e^{-i\theta})$ and $R \in \mathfrak{R}(e^{\frac{i\theta}{2}}, -e^{\frac{i\theta}{2}})$. H must be of the form

$$\begin{pmatrix} \cosh\frac{\lambda}{2} & e^{-i\theta}\sinh\frac{\lambda}{2} \\ e^{i\theta}\sinh\frac{\lambda}{2} & \cosh\frac{\lambda}{2} \end{pmatrix}. \tag{3.38}$$

There are two cases

(a) $\theta = 0$
$\zeta \in [0, \frac{\pi}{2}]$, so this space depends on two non-isometric parameters

(b) $\theta \neq 0$
$\zeta = 0$, so there is only one non-isometric parameter.

(3). Reflection in the 1-3 plane composed with (1)

The isometry is

$$z \to e^{i\theta}\bar{z} \qquad u \to e^{-2i\theta}\bar{u}. \tag{3.39}$$

Under $z \to \bar{z}$, $u \to \bar{u}$, the polynomials transform as follows

$$l_1 \to -l_1 \qquad l_2 \to l_2 \qquad l_3 \to -l_3. \tag{3.40}$$

We can use

$$Z(\zeta) = \overline{Z(\zeta)} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.41}$$

and, for $\theta \neq 0$, (3.33) so that, in terms of $M = TZ(\zeta)R$ satisfying

$$TZ(\zeta)R = c\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \bar{T}\overline{Z(\zeta)}R \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3.42}$$

there are subgroups satisfying

$$\begin{aligned} T &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \bar{T} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \\ R &= \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} R \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned} \tag{3.43}$$

We use the fact that

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{U} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.44}$$

to find that $U \in \mathfrak{U}(ie^{-i\theta}, -ie^{-i\theta})$, whilst $R \in \mathfrak{R}(ie^{\frac{\theta}{2}}, -ie^{\frac{\theta}{2}})$. These are just the rotations about the axes fixed by the target space and spatial isometries. H must now be of the form

$$\begin{pmatrix} e^{\frac{\mu}{2}}\cosh\frac{\lambda}{2} & \sinh\frac{\lambda}{2}\, e^{-i\theta} \\ \sinh\frac{\lambda}{2}\, e^{i\theta} & e^{-\frac{\mu}{2}}\cosh\frac{\lambda}{2} \end{pmatrix}. \tag{3.45}$$

Again there are two cases

(a) $\theta = 0$

$\zeta \in [0, \frac{\pi}{2}]$ and there are three non-isometric parameters.

(b) $\theta \neq 0$

$\zeta = 0$ and there are two non-isometric parameters.

(4). Reflection in the 1-2 plane composed with (1)

This isometry is

$$z \to \frac{e^{i\theta}}{\bar{z}} \qquad u \to \frac{e^{-2i\theta}}{\bar{u}}, \tag{3.46}$$

Using (3.41), (3.19) and (3.33) we need

$$\begin{aligned} T &= \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \bar{T} \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \\ R &= \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} R \begin{pmatrix} -\cos\theta & -\sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.47}$$

There are four cases to consider:

(a) $\theta = 0$
$\zeta \in [0, \frac{\pi}{2}]$, $U \in \mathfrak{U}(0,\infty)$, $R \in \mathfrak{R}(0,\infty)$. H is of the form

$$\begin{pmatrix} \cosh\frac{\lambda}{2} & e^{\frac{\mu}{2}}\sinh\frac{\lambda}{2} \\ e^{-\frac{\mu}{2}}\sinh\frac{\lambda}{2} & \cosh\frac{\lambda}{2} \end{pmatrix}. \tag{3.48}$$

(b) $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$,
$\zeta = 0$. In this case $U \in SU(2)$ but $H = \mathbb{I}$. $R \in \mathfrak{R}(0,\infty)$.

(c) $\theta = \pi$
This corresponds to the antipodal map $z \to -1/\bar{z}$ which sends a point on the spatial two-sphere to the point diametrically opposite. $\zeta = 0$, $U \in \mathfrak{U}(0,\infty)$, $R \in SO(3)$. H is as in case (a) except now we can rotate so that $\mu = 0$.

(d) Otherwise
$\zeta = 0$, $U \in \mathfrak{U}(0,\infty)$, $H = \mathbb{I}$, $R \in \mathfrak{R}(0,\infty)$.
This is the most trivial case depending only on parameters which correspond to isometries.

3.5 The Scattering Metric

The antipodally symmetric space found in case (4c) above is special since it includes the action of the full group of spatial rotations. It is a five-dimensional submanifold depending on one non-isometric parameter. In the picture in which the lumps live on a spatial two-sphere embedded at the origin of $\mathbb{R}^3$, the antipodal symmetry means that these configurations have their centre of mass at the origin. We will think of geodesic motion on this space as being analogous to scattering of particles in their centre of mass frame.

To compute the metric on this space, it is convenient to reparametrise u as follows

$$u(z) = e^{i\chi}\frac{z'^2 + \tan\frac{\psi}{2}}{z'^2\tan\frac{\psi}{2} + 1}. \tag{3.49}$$

Here $z = R \odot z'$ where $\odot$ denotes the Möbius action of the $SU(2)$ matrix R on z'. The configuration has 1-, 2- and 3-body axes defined through the points $z' = \pm 1$, $z' = \pm i$ and $z' = 0, \infty$ respectively.

We can write R in terms of Euler angles $\beta \in [0, 4\pi]$, $\gamma \in [0, 2\pi]$, $\alpha \in [0, \pi]$

$$R = \begin{pmatrix} e^{\frac{i\gamma}{2}} & 0 \\ 0 & e^{-\frac{i\gamma}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{i\beta}{2}} & 0 \\ 0 & e^{-\frac{i\beta}{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\frac{\alpha}{2}\, e^{i\frac{\beta+\gamma}{2}} & \sin\frac{\alpha}{2}\, e^{i\frac{\beta-\gamma}{2}} \\ -\sin\frac{\alpha}{2}\, e^{-i\frac{\beta-\gamma}{2}} & \cos\frac{\alpha}{2}\, e^{-i\frac{\beta+\gamma}{2}} \end{pmatrix}, \tag{3.50}$$

so that

$$z' = e^{-i\gamma} \frac{z - e^{i\beta}\tan\frac{\alpha}{2}}{z\tan\frac{\alpha}{2} + e^{i\beta}}, \tag{3.51}$$

where we identify R and $-R$ so that $\beta \in [0, 2\pi]$.

We can describe this configuration as follows: The parameter ψ controls the size of the lumps which are diametrically opposite one another. As ψ tends to $-\pi/2$, the lumps shrink to become infinitely tall, zero-size lumps on the 1-body axis. As ψ approaches $+\pi/2$, the configuration similarly tends to infinitely tall, zero-size lumps on the 2-body axis. In general the configuration does not have an axis of symmetry but, when $\psi = 0$, the two lumps form an axially symmetric ring localised around the plane containing the 1- and 2-body axes. In this case there is a coordinate singularity, since rotating around the axis of symmetry is equivalent to a target space $U(1)$ transformation, so the coordinates χ and -2γ are identified.

Recall that a component of the moduli space metric g_{ab} is of the form

$$g_{ab} = \int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2} \frac{\partial_a u \partial_b \bar{u} + \partial_b u \partial_a \bar{u}}{2(1+u\bar{u})^2}. \tag{3.52}$$

Since the measure is invariant under the rotation $z \to z' = R^{-1} \odot z$, we can evaluate the integrals at $z' = z$.

Taking an infinitesimal rotation

$$R = 1 + \frac{a}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \frac{b}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{c}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{3.53}$$

so that

$$z' = R^{-1} \odot z = \begin{pmatrix} 1 - \frac{ic}{2} & -\frac{b+ia}{2} \\ \frac{b-ia}{2} & 1 + \frac{ic}{2} \end{pmatrix} \odot z = \frac{(2-ic)z - (b+ia)}{(b-ia)z + (2+ic)}, \tag{3.54}$$

we find

$$\partial_a z' \,|_{a=b=c=0} = \frac{i}{2}(z^2 - 1)$$
$$\partial_b z' \,|_{a=b=c=0} = -\frac{1}{2}(z^2 + 1)$$
$$\partial_b z' \,|_{a=b=c=0} = -iz. \tag{3.55}$$

For the calculation we will use the chain rule $\partial_a u = \partial_{z'} u \, \partial_a z'$. The other derivatives we require are

$$\begin{aligned}
\partial_\psi u &= e^{i\delta} \frac{(1 - z'^4) \sec^2 \frac{\psi}{2}}{2(1 + z'^2 \tan \frac{\psi}{2})^2} \\
\partial_{z'} u &= e^{i\delta} \frac{2z'(1 - \tan^2 \frac{\psi}{2})}{(1 + z'^2 \tan \frac{\psi}{2})^2} \\
\partial_\delta u &= i e^{i\delta} \frac{z'^4 \tan \frac{\psi}{2} + z'^2 \sec \frac{\psi}{2}^2 + \tan \frac{\psi}{2}}{(1 + z'^2 \tan \frac{\psi}{2})^2}.
\end{aligned} \tag{3.56}$$

The metric we obtain can be written in terms of the left-invariant one-forms defined by

$$R^{-1} dR = \sigma_i \left(\frac{i\tau^i}{2} \right), \tag{3.57}$$

where τ^i are the Pauli matrices, obtaining

$$\begin{aligned}
\sigma_1 &= -\sin\gamma \, d\alpha + \cos\gamma \, \sin\alpha \, d\beta \\
\sigma_2 &= \cos\gamma \, d\alpha + \sin\gamma \, \sin\alpha \, d\beta \\
\sigma_3 &= \cos\alpha \, d\beta + d\gamma.
\end{aligned} \tag{3.58}$$

Comparing with (3.53) when $z' = z$, we identify $\sigma^1 = da$, $\sigma^2 = db$ and $\sigma^3 = dc$.

All off-diagonal components of the metric apart from $g_{c\delta}$ can be seen to vanish using simple symmetry arguments. For example, consider

$$\begin{aligned}
g_{\psi a} &= \int \frac{i \, dz \, d\bar{z}}{2(1 + z\bar{z})^2} \frac{\partial_\psi u \partial_a \bar{u} + \partial_a u \partial_\psi \bar{u}}{2(1 + u\bar{u})^2} \\
&= \int \frac{i \, dz \, d\bar{z}}{2(1 + z\bar{z})^2} \frac{i(1 - z^2)(1 - \bar{z})^2 (z - \bar{z})(z\bar{z} - 1) \cos\psi}{4(1 + (z^2 + \bar{z}^2) \sin\psi + z^2 \bar{z}^2)^2}.
\end{aligned} \tag{3.59}$$

The measure $i \, dz \, d\bar{z}/(1 + z\bar{z})^2$ is symmetric under $z \to -z$, while the rest of the integrand is antisymmetric, and so this integral vanishes.

Similarly

$$g_{\psi c} = \int \frac{i \, dz \, d\bar{z}}{2(1 + z\bar{z})^2} \frac{i(\bar{z}^2 - z^2)(1 + z^2 \bar{z}^2) \cos\psi}{2(1 + (z^2 + \bar{z}^2) \sin\psi + z^2 \bar{z}^2)^2}, \tag{3.60}$$

vanishes because the integrand is antisymmetric under $z \to iz$.

We will present the calculation of the component of the metric $g_{\psi\psi}$ in detail. The calculations for the other non-zero components are given in outline.

$$g_{\psi\psi} = \int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2} \frac{\partial_\psi u\,\partial_\psi \bar{u}}{(1+u\bar{u})^2}$$
$$= \int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2} \frac{(1-z^4)(1-\bar{z}^4)}{4(1+(z^2+\bar{z}^2)\sin\psi + z^2\bar{z}^2)}. \tag{3.61}$$

The integrand is a function of z^2, $\bar{z}^2$ and $z\bar{z}$ so we can substitute

$$z^2 = (\tan\frac{\phi}{2})x \qquad \bar{z}^2 = (\tan\frac{\phi}{2})/x, \tag{3.62}$$

where x is a complex variable living on the unit circle. The integral is now over

$$\int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2} = \frac{1}{2}\int_0^\pi \frac{d\phi}{1+\sin\phi} \oint \frac{dx}{2ix}. \tag{3.63}$$

We have taken into account that x goes twice around the unit circle as $\arg z$ goes around once, so the contour integral is performed once around the unit circle.

Firstly we use partial fractions to obtain

$$g_{\psi\psi} = \frac{1}{2}\int_0^\pi \frac{d\phi}{1+\sin\phi}\oint \frac{dx}{2i}\Bigg(-\frac{1}{4\sin\psi^2}\frac{1}{x} + \frac{1}{\sin\psi^3\sin\phi}\frac{1}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)} - \frac{\cos\psi^2}{\sin\psi^4\sin\phi^2}\frac{x}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)^2}\Bigg). \tag{3.64}$$

Now we evaluate the three contour integrals. The first is simply

$$\frac{1}{2i}\oint\frac{dx}{x} = \pi. \tag{3.65}$$

The quadratic $x^2 + 2x/(\sin\phi\sin\psi) + 1$ has roots

$$x = -\frac{1}{\sin\phi\sin\psi}\left(1 \pm \sqrt{1-\sin^2\phi\sin^2\psi}\right), \tag{3.66}$$

so we find

$$\frac{1}{2i}\oint \frac{dx}{x^2+\frac{2}{\sin\phi\sin\psi}x+1} = \frac{\pi\sin\psi\sin\phi}{2\sqrt{1-\sin^2\phi\sin^2\psi}}. \tag{3.67}$$

The residue of $f(x)/(x-\alpha)^2$ at $x=\alpha$ is $f'(\alpha)$, assuming that $f(x)$ is well-defined at this point. For the function $f(x) = x/(x-\beta)^2$ we find

$$f'(\alpha) = -\frac{\alpha+\beta}{(\alpha-\beta)^3}. \tag{3.68}$$

Therefore the third contour integral is

$$\frac{1}{2i}\oint \frac{x\,dx}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)^2} = \frac{\pi\sin^2\psi\sin^2\phi}{4(1-\sin^2\phi\sin^2\psi)^2}. \tag{3.69}$$

Therefore the result of the contour integration for $g_{\psi\psi}$ is

$$g_{\psi\psi} = \frac{\pi}{8}\int_0^\pi \frac{d\phi}{1+\sin\phi}\left(-\operatorname{cosec}^2\psi + \frac{2\operatorname{cosec}^2\psi}{\sqrt{1-\sin^2\phi\sin^2\psi}} - \frac{\cot^2\psi}{(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}}\right). \tag{3.70}$$

Since the integrand depends on ϕ only through $\sin\phi$, we can take twice the value of the integral from 0 to $\pi/2$. We are then in a position to express $g_{\psi\psi}$ in terms of complete elliptic integrals.

In this case, the results we require are

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\sin\phi}\frac{1}{\sqrt{1-\sin^2\phi\sin^2\psi}} &= 1-(\mathrm{B}(\sin\psi)-1)\tan^2\psi \\ \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\sin\phi}\frac{1}{(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}} &= \frac{1-(\mathrm{E}(\sin\psi)+\mathrm{B}(\sin\psi)-2)\tan^2\psi}{\cos^2\psi},\end{aligned} \tag{3.71}$$

where E and B are the standard elliptic integrals [35]

$$\mathrm{E}(\sin\psi) = \int_0^{\frac{\pi}{2}} d\phi\sqrt{1-\sin^2\phi\sin^2\psi} \tag{3.72}$$

$$\mathrm{B}(\sin\psi) = \int_0^{\frac{\pi}{2}} d\phi\frac{\cos^2\phi}{\sqrt{1-\sin^2\phi\sin^2\psi}}. \tag{3.73}$$

It will be useful to define

$$\epsilon(\psi) = \mathrm{E}(\sin\psi)-1 \qquad \beta(\psi) = \mathrm{B}(\sin\psi)-1. \tag{3.74}$$

Expressing (3.70) in terms of these functions gives us the result

$$g_{\psi\psi} = \frac{\pi(\epsilon(\psi)-\beta(\psi))}{4\cos^2\psi}. \tag{3.75}$$

The calculation of the other components of the metric follows the same method. In each case the integrand is a function of z^2, $\bar{z}^2$ and $z\bar{z}$, so we can use the substitution

(3.63). We list the following integrals:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin\phi\, d\phi}{(1+\sin\phi)\sqrt{1-\sin^2\phi\sin^2\psi}} &= \frac{\epsilon(\psi)}{\cos^2\psi} \\
\int_0^{\frac{\pi}{2}} \frac{\sin\phi\, d\phi}{(1+\sin\phi)(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}} &= \frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^4\psi} \\
\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi\, d\phi}{(1+\sin\phi)(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}} &= \frac{1}{\cos^2\psi} - \frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^4\psi} \\
\int_0^{\frac{\pi}{2}} \frac{\cos^2\phi\, d\phi}{(1+\sin\phi)(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}} &= \frac{\epsilon(\psi)}{\cos^2\psi}.
\end{aligned} \tag{3.76}$$

The calculations are as follows:

$$\begin{aligned}
g_{aa} &= \int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2}\frac{z\bar{z}(1-z^2)(1-\bar{z}^2)\cos^2\psi}{(1+(z^2\bar{z}^2)\sin\psi+z^2\bar{z}^2)^2} \\
&= -\frac{\cot^2\psi}{4}\int_0^{\pi}\frac{d\phi}{1+\sin\phi} \\
&\quad \oint\frac{dx}{2i}\left[\frac{1}{x^2+\frac{2}{\sin\phi\sin\psi}x+1} - \frac{2(1+\sin\psi)}{\sin\phi\sin\psi}\frac{x}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)^2}\right] \\
&= -\frac{\cot^2\psi}{2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{1+\sin\phi}\left[\frac{\sin\phi}{\sqrt{1-\sin^2\phi\sin^2\psi}} - \frac{\sin\psi(1+\sin\psi)\sin\phi}{(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}}\right] \\
&= \frac{\pi}{2}\left(\frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi} + \sin\psi\,\frac{\beta(\psi)+\epsilon(\psi)}{\cos^2\psi}\right).
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
g_{bb} &= \int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2}\frac{z\bar{z}(1+z^2)(1+\bar{z}^2)\cos^2\psi}{(1+(z^2\bar{z}^2)\sin\psi+z^2\bar{z}^2)^2} \\
&= \frac{\cot^2\psi}{4}\int_0^{\pi}\frac{d\phi}{1+\sin\phi} \\
&\quad \oint\frac{dx}{2i}\left[\frac{1}{x^2+\frac{2}{\sin\phi\sin\psi}x+1} - \frac{2(1-\sin\psi)}{\sin\phi\sin\psi}\frac{x}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)^2}\right] \\
&= \frac{\cot^2\psi}{2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{1+\sin\phi}\left[\frac{\sin\phi}{\sqrt{1-\sin^2\phi\sin^2\psi}} - \frac{\sin\psi(1-\sin\psi)\sin\phi}{(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}}\right] \\
&= \frac{\pi}{2}\left(\frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi} - \sin\psi\frac{\beta(\psi)+\epsilon(\psi)}{\cos^2\psi}\right).
\end{aligned} \tag{3.78}$$

$$
\begin{aligned}
g_{cc} &= \int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2} \frac{4z^2\bar{z}^2\cos^2\psi}{(1+(z^2\bar{z}^2)\sin\psi + z^2\bar{z}^2)^2} \\
&= \frac{\cot^2\psi}{4}\int_0^{\pi} \frac{d\phi}{1+\sin\phi} \oint \frac{dx}{2i} \frac{4x}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)^2} \\
&= \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\sin\phi} \frac{\cos^2\psi\sin^2\phi}{(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}} \\
&= \pi\left(1 - \frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi}\right).
\end{aligned} \tag{3.79}
$$

$$
\begin{aligned}
g_{\chi\chi} &= \int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2} \frac{(2z^2+(1+z^4)\sin\psi)(2\bar{z}^2+(1+\bar{z}^4)\sin\psi)}{4(1+(z^2\bar{z}^2)\sin\psi+z^2\bar{z}^2)^2} \\
&= \frac{1}{4}\int_0^{\pi} \frac{d\phi}{1+\sin\phi} \oint \frac{dx}{2i}\left[\frac{1}{4x} - \frac{x\cot^2\psi\cot^2\phi}{4(x^2+\frac{2}{\sin\phi\sin\psi}x+1)^2}\right] \\
&= \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\sin\phi}\left[\frac{1}{4} - \frac{\cos^2\psi\cos^2\phi}{4(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}}\right] \\
&= \frac{\pi}{4}\left(1-\epsilon(\psi)\right).
\end{aligned} \tag{3.80}
$$

$$
\begin{aligned}
g_{c\chi} &= -\int \frac{i\,dz\,d\bar{z}}{2(1+z\bar{z})^2} \frac{\cos\psi(4z^2\bar{z}^2+(z^2+\bar{z}^2)(1+z^2\bar{z}^2)\sin\psi)}{2(1+(z^2\bar{z}^2)\sin\psi+z^2\bar{z}^2)^2} \\
&= -\frac{\cot\psi}{4}\int_0^{\pi} \frac{d\phi}{1+\sin\phi} \oint \frac{dx}{2i}\left[\frac{\operatorname{cosec}\phi}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)} - \frac{2x\operatorname{cosec}\psi\cot^2\psi}{(x^2+\frac{2}{\sin\phi\sin\psi}x+1)^2}\right] \\
&= -\frac{\cos^2\psi}{2}\int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\sin\phi}\left[\frac{1}{\sqrt{1-\sin^2\phi\sin^2\psi}} - \frac{\cos^2\phi}{(1-\sin^2\phi\sin^2\psi)^{\frac{3}{2}}}\right] \\
&= \frac{\pi\cos\psi}{2}\left(\frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi} - 1\right).
\end{aligned} \tag{3.81}
$$

So finally we obtain the metric

$$
g = F(\psi)\,d\psi^2 + A(\psi)\,\sigma_1^2 + B(\psi)\,\sigma_2^2 + C(\psi)\,\sigma_3^2 + 2D(\psi)\,\sigma_3\,d\chi + E(\psi)\,d\chi^2 \tag{3.82}
$$

where

$$\begin{aligned}
F(\psi) &= \pi\,\frac{\epsilon(\psi)-\beta(\psi)}{4\cos^2\psi} \\
A(\psi) &= \pi\left(\frac{1}{2}\frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi}+\frac{\sin\psi}{2}\frac{\beta(\psi)+\epsilon(\psi)}{\cos^2\psi}\right) \\
B(\psi) &= \pi\left(\frac{1}{2}\frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi}-\frac{\sin\psi}{2}\frac{\beta(\psi)+\epsilon(\psi)}{\cos^2\psi}\right) \\
C(\psi) &= \pi\left(1-\frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi}\right) \\
D(\psi) &= \pi\,\frac{\cos\psi}{2}\left(\frac{\epsilon(\psi)+\beta(\psi)\sin^2\psi}{\cos^2\psi}-1\right) \\
E(\psi) &= \pi\,\frac{1-\epsilon(\psi)}{4}
\end{aligned} \tag{3.83}$$

There is a coordinate singularity when $\psi = 0$, corresponding to axially symmetric configurations, in which case we have the identification $\chi = -2\gamma$.

We can check that the metric has sensible limits as $\psi \to \pm\pi/2$, in which case the lumps shrink to zero size, and $\psi \to 0$, in which case the lumps describe an axially symmetric ring. We use the following formulae for the elliptic integrals $E(\sin\psi)$ and $B(\sin\psi)$ in these limits [36]. For $k^2 = \sin^2\psi \approx 1$ there is an expansion in terms of $1 - k^2 = \cos^2\psi$:

$$\begin{aligned}
E(\sin\psi) &= 1+\frac{1}{2}\left(\Lambda-\frac{1}{2}\right)\cos^2\psi+O(\cos^4\psi) \\
B(\sin\psi) &= 1-\frac{1}{2}\left(\Lambda-\frac{3}{2}\right)\cos^2\psi+O(\cos^4\psi)
\end{aligned} \tag{3.84}$$

where $\Lambda = \log(4/\cos\psi)$.

Near $\psi = 0$ we have $\sin^2\psi \ll 1$ and

$$\begin{aligned}
E(\sin\psi) &= \frac{\pi}{2}-\frac{\pi}{8}\sin^2\psi+O(\sin^4\psi) \\
B(\sin\psi) &= \frac{\pi}{2}+\frac{\pi}{8}\sin^2\psi+O(\sin^4\psi).
\end{aligned} \tag{3.85}$$

The limits of the functions A through to F are tabulated below.

ψ	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	0
A	0	$\frac{1}{2}$	$\frac{1}{2}\left(\frac{\pi}{2}-1\right)$
B	$\frac{1}{2}$	0	$\frac{1}{2}\left(\frac{\pi}{2}-1\right)$
C	$\frac{1}{2}$	$\frac{1}{2}$	$2-\frac{\pi}{2}$
D	0	0	$\frac{\pi}{4}-1$
E	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}\left(2-\frac{\pi}{2}\right)$
F	$\frac{1}{4}(\Lambda-1)$	$\frac{1}{4}(\Lambda-1)$	$\frac{\pi}{16}$

In the zero-size limit we obtain the moments of inertia corresponding to two antipodally opposite point particles on the 1-body axis for $\psi = -\pi/2$ and on the 2-body axis for $\psi = \pi/2$. When $\psi = 0$ the moments of inertia about the 1- and 2-body axes are equal as expected for a ring. Note that Λ is divergent in the limit $\cos\psi \to 0$ but by choosing a different coordinate, for instance by setting $\sin\psi = \tanh r$, we can see that this is just a coordinate singularity.

3.6 Low Energy Dynamics

Here we will describe the dynamics associated to geodesics on this submanifold. The effective Lagrangian describing low-energy dynamics is

$$L = \int dt\, \left[F(\psi)\dot\psi^2 + A(\psi)L_1^2 + B(\psi)L_2^2 + C(\psi)L_3^2 + 2D(\psi)L_3\dot\chi + E(\psi)\dot\chi^2\right], \qquad (3.86)$$

where L_i, are the angular velocities about the body axes

$$\begin{aligned} L_1 &= -\sin\gamma\,\dot\alpha + \cos\gamma\,\sin\alpha\,\dot\beta \\ L_2 &= \cos\gamma\,\dot\alpha + \sin\gamma\,\sin\alpha\,\dot\beta \\ L_3 &= \cos\alpha\,\dot\beta + \dot\gamma \end{aligned} \qquad (3.87)$$

The equation of motion for $\dot\psi$ is found straightforwardly.

$$\frac{d}{dt}[2F(\psi)\dot\psi] = F'(\psi)\dot\psi^2 + A'(\psi)L_1^2 + B'(\psi)L_2^2 + C'(\psi)L_3^2 + 2D'(\psi)L_3\dot\chi + E'(\psi)\dot\chi^2. \qquad (3.88)$$

The $\dot\gamma$ equation is also straightforward using $\frac{\partial L_1}{\partial\gamma} = -L_2$ and $\frac{\partial L_2}{\partial\gamma} = L_1$, and is

$$\frac{d}{dt}[C(\psi)L_3 + D(\psi)\dot\chi] = L_1L_2(B(\psi) - A(\psi)). \qquad (3.89)$$

We find the other two equations of motion by choosing different Euler angles

$$R = \begin{pmatrix} \cos\frac{\gamma}{2} & i\sin\frac{\gamma}{2} \\ i\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & i\sin\frac{\beta}{2} \\ i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}. \tag{3.90}$$

In this case the angular velocities are

$$\begin{aligned} L_1 &= \cos\alpha\,\dot{\beta} + \dot{\gamma} \\ L_2 &= -\sin\gamma\,\dot{\alpha} + \cos\gamma\,\sin\alpha\,\dot{\beta} \\ L_3 &= \cos\gamma\,\dot{\alpha} + \sin\gamma\,\sin\alpha\,\dot{\beta}, \end{aligned} \tag{3.91}$$

and the $\dot{\gamma}$ equation of motion is

$$\frac{d}{dt}[A(\psi)L_1] = L_2L_3(C(\psi) - B(\psi)) + D(\psi)L_2\dot{\chi}. \tag{3.92}$$

Similarly for

$$R = \begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \\ -\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}., \tag{3.93}$$

we have angular velocities

$$\begin{aligned} L_1 &= \cos\gamma\,\dot{\alpha} + \sin\gamma\,\sin\alpha\,\dot{\beta} \\ L_2 &= \cos\alpha\,\dot{\beta} + \dot{\gamma} \\ L_3 &= -\sin\gamma\,\dot{\alpha} + \cos\gamma\,\sin\alpha\,\dot{\beta}, \end{aligned} \tag{3.94}$$

and the $\dot{\gamma}$ equation is

$$\frac{d}{dt}[B(\psi)L_2] = L_3L_1(A(\psi) - C(\psi)) - D(\psi)L_1\dot{\chi}. \tag{3.95}$$

The $\dot{\chi}$ equation gives us the conserved quantity corresponding to the global $U(1)$ transformation

$$\frac{d}{dt}[D(\psi)L_3 + E(\psi)\dot{\chi}] = 0. \tag{3.96}$$

It is consistent to fix our attention on geodesics that are orthogonal to this isometry, ie. those for which $D(\psi)L_3 + E(\psi)\dot{\chi} = 0$, since this equation implies that geodesics which begin orthogonal to this "vertical" direction remain orthogonal throughout their motion.

If we substitute $\dot{\chi} = -\frac{D}{E}L_3$ into (3.88), (3.89), (3.92) and (3.95), we obtain the equations of motion on the space orthogonal to the target space isometry

$$\frac{d}{dt}[2F(\psi)\dot{\psi}] = F'(\psi)\dot{\psi}^2 + A'(\psi)L_1^2 + B'(\psi)L_2^2 + \tilde{C}'(\psi)L_3^2 \tag{3.97}$$

$$\frac{d}{dt}[\tilde{C}(\psi)L_3] = L_1L_2(B(\psi) - A(\psi)) \tag{3.98}$$

$$\frac{d}{dt}[A(\psi)L_1] = L_2L_3(\tilde{C}(\psi) - B(\psi)) \tag{3.99}$$

$$\frac{d}{dt}[B(\psi)L_2] = L_3L_1(A(\psi) - \tilde{C}(\psi)), \tag{3.100}$$

where $\tilde{C}(\psi) = C(\psi) - D(\psi)^2/E(\psi)$. These may be obtained as the equations of motion of the reduced Lagrangian

$$L = \int dt \left[F(\psi)\dot{\psi}^2 + A(\psi)L_1^2 + B(\psi)L_2^2 + \tilde{C}(\psi)L_3^2\right], \tag{3.101}$$

which comes from the four dimensional metric $g = F(\psi)dr^2 + A(\psi)\sigma_1^2 + B(\psi)\sigma_2^2 + \tilde{C}(\psi)\sigma_3^2$. This way of reducing to a four dimensional metric was suggested to us by Conor Houghton.

We will concentrate on the dynamics on this space, following analysis similar to that of Gibbons and Manton in the case of monopoles [33]. Lagrangians of this form are familiar from the dynamics of rigid bodies, except that here the mass and principal moments of inertia are functions of ψ.

3.6.1 Rotation about the 1-Body Axis

There are three special cases of motion which we will consider, corresponding to non-zero angular momentum about one of the body axes of the configuration. The first case we will take is when there is angular momentum about the 1-body axis. The equations of motion (3.98) and (3.100) imply that if $L_2 = L_3 = 0$ initially, then they remain zero throughout the motion. Equation (3.99) implies that $A(\psi)L_1 = M_1$ is the conserved angular momentum about the first body axis.

Integrating (3.97) gives us the conserved energy of the motion

$$T = F(\psi)\dot{\psi}^2 + \frac{{M_1}^2}{A(\psi)}. \tag{3.102}$$

This system is that of a particle with position-dependent mass moving in a potential $1/A(\psi)$. As such, the qualitative features of the dynamics depend on the shape of the functions $F(\psi)$ and $1/A(\psi)$ which are shown in figures 3.4 and 3.5.

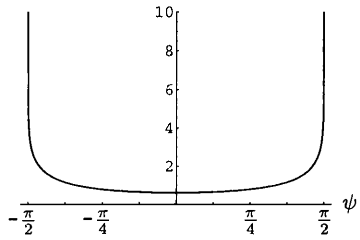


Figure 3.4: $F(\psi)$

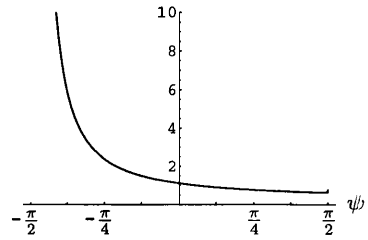


Figure 3.5: $1/A(\psi)$

Since $1/A(\psi)$ is monotonically decreasing, a configuration starting off with a negative value of ψ, corresponding to lumps on the 1-body axis, necessarily passes through the axially symmetric configuration $\psi = 0$ and results in lumps on the 2-body axis.

Using the Euler angles (3.90) with $\alpha = \beta = 0$ we have

$$z' = \frac{z - i\tan\frac{\gamma}{2}}{-iz\tan\frac{\gamma}{2} + 1} \qquad L_1 = \dot{\gamma}. \tag{3.103}$$

To find the geodesic motion, the equations

$$2F(\psi)\ddot{\psi} + F'(\psi)\dot{\psi}^2 = \frac{{M_1}^2 A'(\psi)}{A(\psi)^2} \qquad \dot{\gamma} = \frac{M_1}{A(\psi)}, \tag{3.104}$$

were solved numerically using MATHEMATICA for $\psi(t)$ and $\gamma(t)$ with initial data

$$\dot{\psi}(0) = 0.08 \qquad \psi(0) = -\pi/2 + 0.4 \qquad \gamma(0) = 0. \tag{3.105}$$

The constant M_1 was taken to have the value 0.04. The resulting functions $\psi(t)$ and $\gamma(t)$ are shown in figures 3.6 and 3.7.

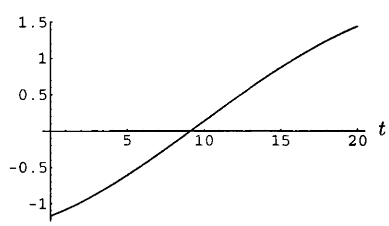


Figure 3.6: $\psi(t)$

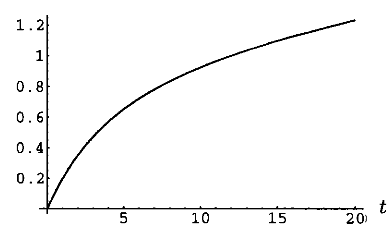


Figure 3.7: $\gamma(t)$

The evolution of the potential energy density and the body axes of the configuration during this geodesic motion is shown in figure 3.8. We can see that it corresponds to a form of right-angle scattering in which lumps starting on the 1-body axis end up on the 2-body axis. If the angular momentum M_1 is zero then planar right-angle scattering results.

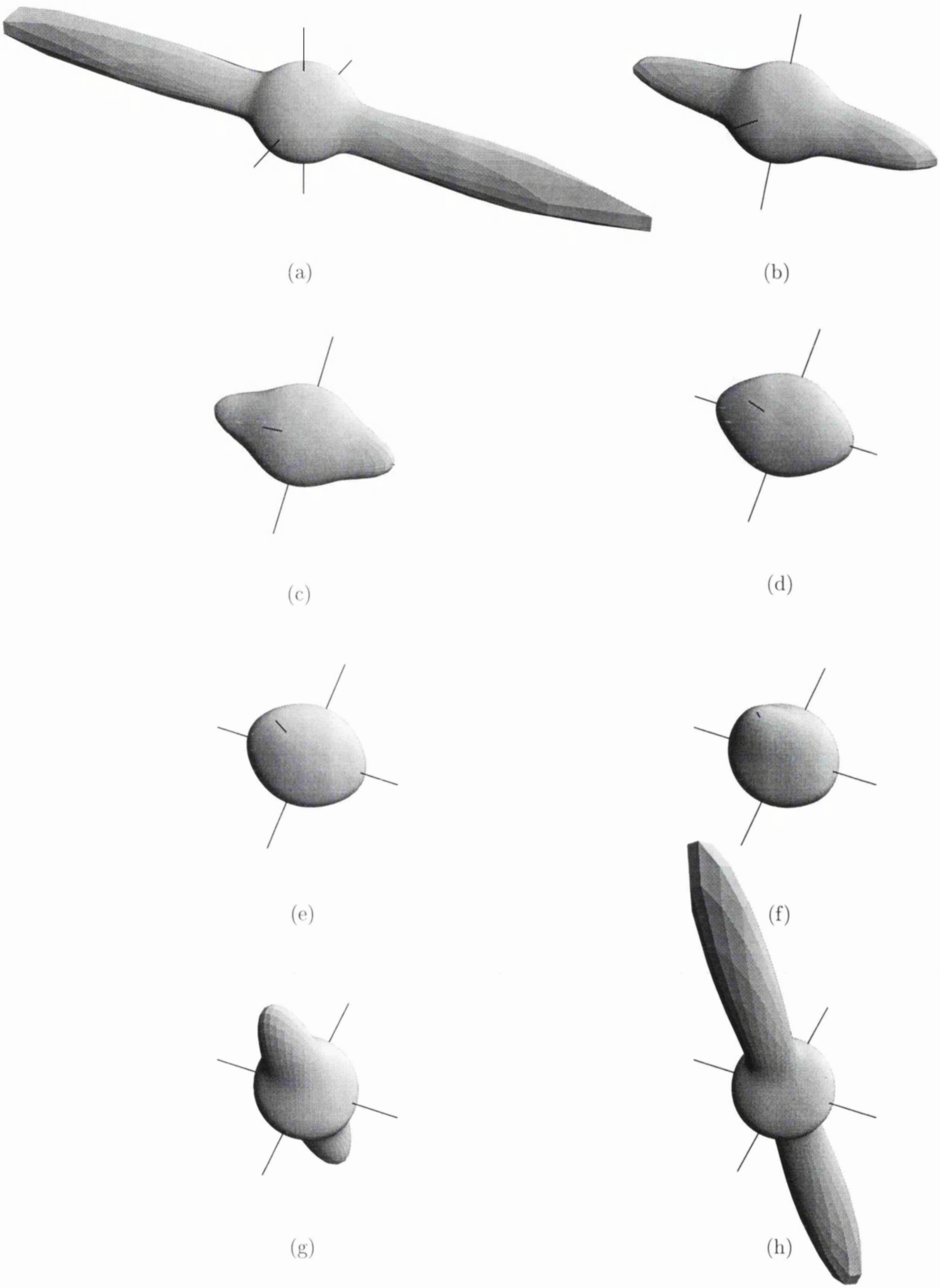


Figure 3.8: Geodesic motion with non-zero angular momentum about the 1-body axis.

3.6.2 Rotation about the 2-Body Axis

The next case we consider is when there is angular momentum about the second body axis. In this case, $L_1 = L_3 = 0$, and the conserved energy and angular momentum are

$$T = F(\psi)\dot{\psi}^2 + \frac{{M_2}^2}{B(\psi)} \qquad B(\psi)L_2 = M_2. \tag{3.106}$$

The potential $1/B(\psi)$ is shown in figure 3.9 below. We recognise that $B(\psi) = A(-\psi)$ and so the geodesics we obtain are just rotations through $\frac{\pi}{2}$ of those considered above.

In this case, if we start with ψ negative we must also end up with ψ negative. If the initial energy is less than ${M_2}^2/A(0)$ then the motion does not pass through the axially symmetric configuration. If it is greater than this and $\dot{\psi}$ is positive, then the configuration passes through the axially symmetric ring to become lumps on the 2-body axis before passing back through the ring and reverting to lumps on the 1-body axis.

Using the Euler angles (3.93) with $\alpha = \beta = 0$ we have

$$z' = \frac{z - \tan\frac{\gamma}{2}}{\tan\frac{\gamma}{2}z + 1} \qquad L_2 = \dot{\gamma}. \tag{3.107}$$

The equations

$$2F(\psi)\ddot{\psi} + F'(\psi)\dot{\psi}^2 = \frac{{M_2}^2B'(\psi)}{B(\psi)^2} \qquad \dot{\gamma} = \frac{M_2}{B(\psi)}, \tag{3.108}$$

were solved numerically for $\psi(t)$ and $\gamma(t)$ with initial data

$$\dot{\psi}(0) = 0.2 \qquad \psi(0) = -\pi/2 + 0.4 \qquad \gamma(0) = 0. \tag{3.109}$$

The constant M_1 was taken to have the value 0.19. Figures 3.10 and 3.11 show the result of this integration.

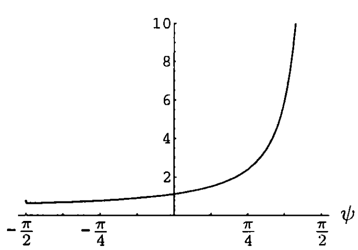


Figure 3.9: $1/B(\psi)$

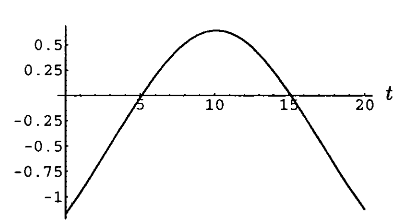


Figure 3.10: $\psi(t)$

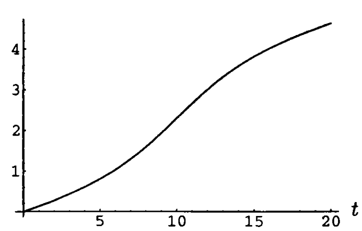


Figure 3.11: $\gamma(t)$

Plots of the potential energy density during this motion are shown in figure 3.12 below. In this case the lumps pass through the axially symmetric configuration to become lumps on the 2-body axis before returning to lumps on the 1-body axis. This motion is analagous to the dyon "pair production" seen in two-monopole scattering.

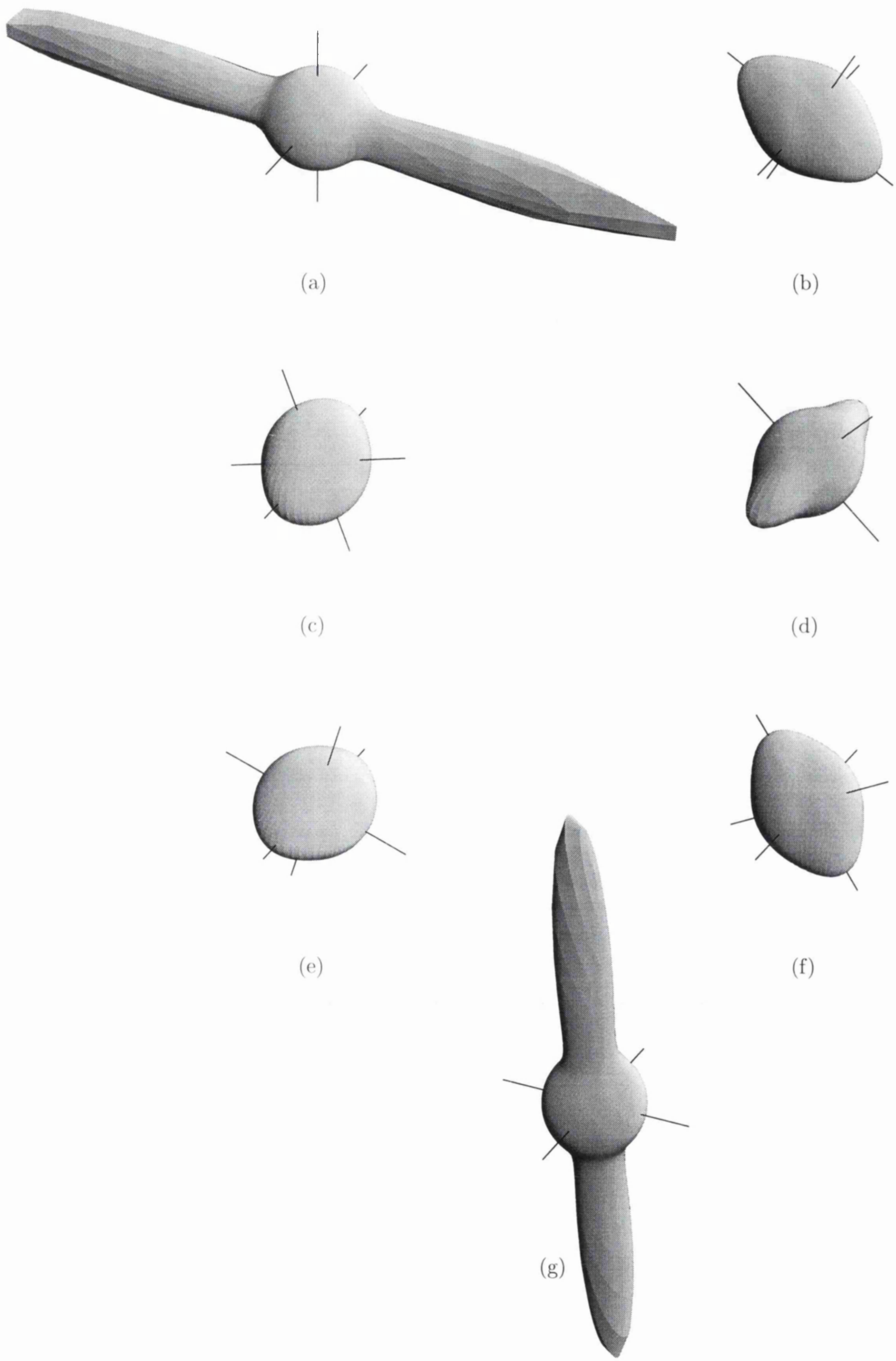


Figure 3.12: Geodesic motion with non-zero angular momentum about the 2-body axis.

3.6.3 Rotation about the 3-Body Axis

The last case we consider is when there is angular momentum about the third body axis. The conserved energy and angular momentum are

$$T = F(\psi)\dot{\psi}^2 + \frac{{M_3}^2}{\tilde{C}(\psi)} \qquad \tilde{C}(\psi)L_3 = M_3, \tag{3.110}$$

where the potential $1/\tilde{C}(\psi)$ is shown in figure 3.13 below.

In this case we can see that the configuration cannot pass through $\psi = 0$ and so the lumps lie on the 1-body axis throughout the motion, this axis rotating about the spatial 3-axis.

Using the Euler angles (3.90) with $\alpha = \beta = 0$ we have

$$z' = e^{-i\gamma} z \qquad L_3 = \dot{\gamma}. \tag{3.111}$$

The equations

$$2F(\psi)\ddot{\psi} + F'(\psi)\dot{\psi}^2 = \frac{{M_3}^2\tilde{C}'(\psi)}{\tilde{C}(\psi)^2} \qquad \dot{\gamma} = \frac{M_3}{\tilde{C}(\psi)}, \tag{3.112}$$

were solved numerically for $\psi(t)$ and $\gamma(t)$ with initial data

$$\psi'(0) = 0.1 \qquad \psi(0) = -\pi/2 + 0.4 \qquad \gamma(0) = 0. \tag{3.113}$$

The constant M_1 was taken to have the value 0.02. The functions $\psi(t)$ and $\gamma(t)$ are shown in figures 3.14 and 3.15 below.

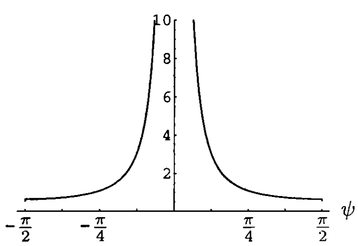


Figure 3.13: $1/\tilde{C}(\psi)$

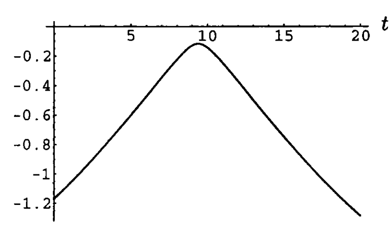


Figure 3.14: $\psi(t)$

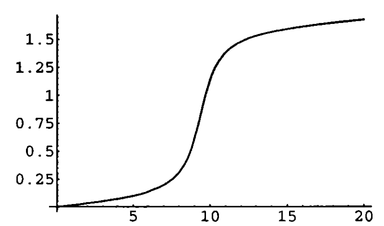
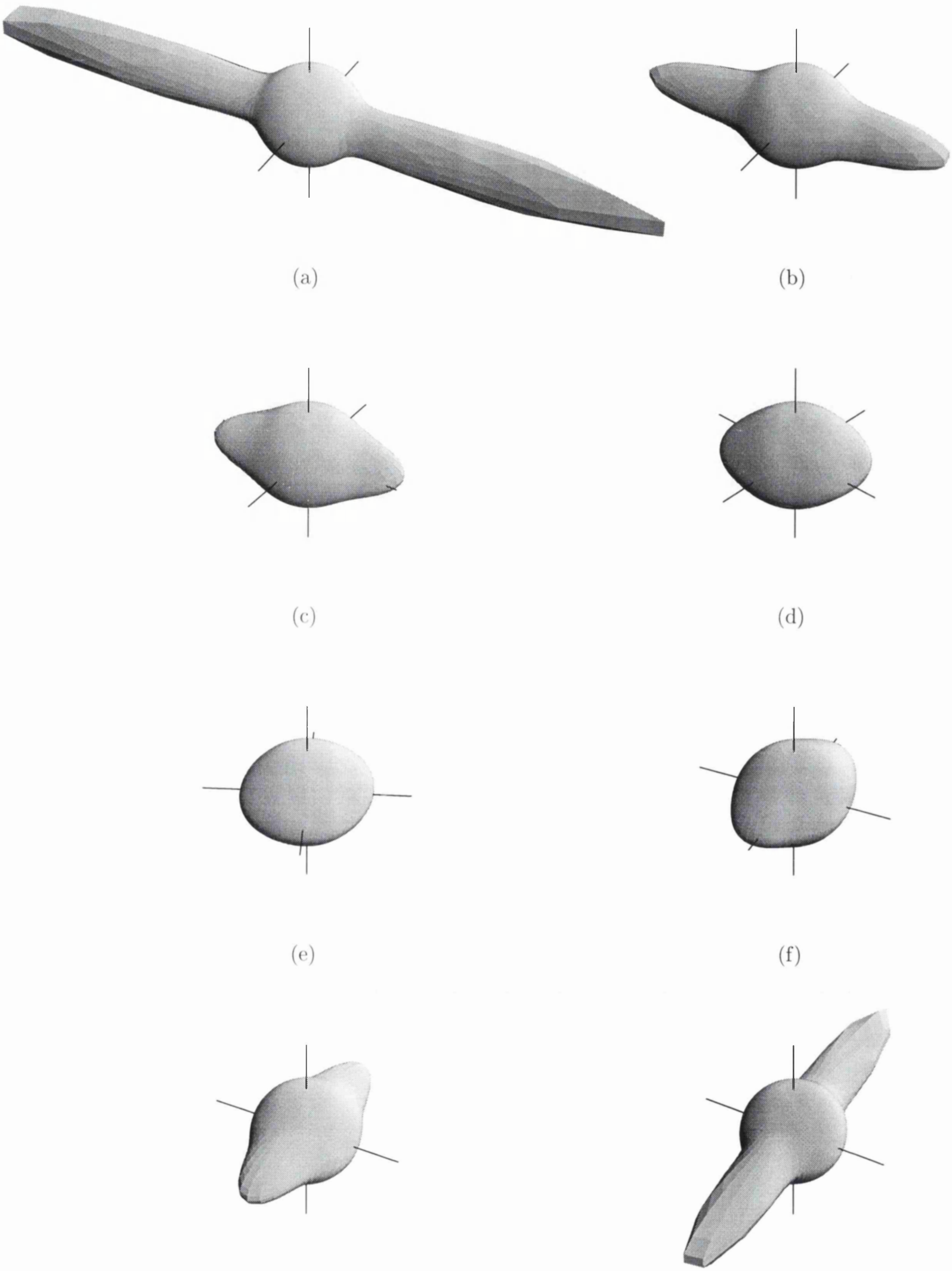


Figure 3.15: $\gamma(t)$

The motion is shown on the following page in figure 3.16.



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 3.16: Geodesic motion with non-zero angular momentum about the 3-body axis.

3.7 Comments

The scattering of two monopoles described by the Atiyah-Hitchin metric shows qualitatively similar behaviour to that of two-lumps on the sphere described above. It is tempting to think that a modification of the model may be possible so that the moduli space metrics coincide. This would be equivalent to directly relating the monopole moduli space metrics with the Jarvis rational map. Of course, to do this, one would somehow have to ensure hyper-Kählerity of the resulting metric.

One feature of dynamics of lumps in the geodesic approximation that would have to be overcome is that they can shrink to zero size in finite time. This is equivalent to the statement that the moduli space is geodesically incomplete with respect to the moduli space metric. Sadun and Speight have shown that this is the case for lumps on any compact Riemann surface [31]. They do this by constructing a Cauchy sequence of maps which does not converge. Geodesic completeness is equivalent to completeness with respect to the metric given by the geodesic distance between two configurations.

There are two other submanifolds which depend on a single non-isometric parameter, found in cases (1c) and (2b) above. It would be interesting to compute the metric on these spaces and thus describe some different features of the low energy dynamics of two-lumps. It may also be possible to compute the metric on some spaces depending on two non-trivial parameters, obtaining a result in terms of complete elliptic integrals of the third kind.

Chapter 4

One Monopole in the Jarvis Gauge

We discussed in the introduction the fact that the definition of the Jarvis rational map "breaks" the Galilean group down to the group of rotations around the origin which have a well-defined action on the map. Similarly, the Jarvis equation, introduced in Chapter 2, is defined in a gauge in which $A_r - i\Phi = A_{\bar{z}} = 0$, and again translations do not have a simple action on solutions to this equation. Recall that we dubbed a solution to the Jarvis equation $\mathcal{H}(\mathbf{x})$ satisfying $\mathcal{H}(\mathbf{0}) = \mathbb{I}$ a *metric*, following Jarvis, and we refer to the gauge in which $A_r + i\Phi = \mathcal{H}^{-1}\partial_r\mathcal{H}$ and $A_z = \mathcal{H}^{-1}\partial_z\mathcal{H}$ as the *Jarvis gauge.*

By introducing an extra parameter it is possible to write the Bogomol'nyi equations as a pair of commuting operators. That these two operators commute is equivalent to the statement that they have a simultaneous solution. These two equations and their solution form the linear system for the Bogomol'nyi equations.

We will show that the linear system is covariant under Galiliean transformations in the sense that, given a solution, we can define translated and rotated solutions corresponding to a linear system in which the Higgs and gauge fields are translated or rotated. The linear system also has the property that the gauge transformation which takes us to a gauge in which $A_r - i\Phi = A_{\bar{z}} = 0$ is obtained by evaluating the solution when the parameter λ takes the value $-1/z$. Therefore, once we have a solution in any gauge, it is a simple matter to translate it and then gauge transform to the Jarvis gauge, thus obtaining a solution to the Jarvis equation.

Using the explicit solution to the Jarvis equation for the BPS monopole at the origin, we can find the corresponding solution to the linear system, which therefore enables us to find the translated one monopole solution in the Jarvis gauge and the corresponding Jarvis rational map.

4.1 The Linear System for the Bogomol'nyi Equations

Here we will present the linear system for the Bogomol'nyi equations. The starting point is the observation that the Bogomol'nyi equations

$$B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk} = D_i\Phi \qquad i = 1\dots3, \tag{4.1}$$

are equivalent to the equations

$$\begin{aligned}[D_3 - i\Phi, D_1 - iD_2] &= 0\\ [D_3 + i\Phi, D_1 + iD_2] &= 0\\ [D_3 - i\Phi, D_3 + i\Phi] + [D_1 - iD_2, D_1 + iD_2] &= 0.\end{aligned} \tag{4.2}$$

Introducing the spectral parameter λ we can write these equations as a zero curvature condition or "Lax pair"

$$[\lambda(D_3 + i\Phi) + (D_1 - iD_2), -\lambda(D_1 + iD_2) + (D_3 - i\Phi)] = 0. \tag{4.3}$$

Thus the Bogomolny equations can be thought of as the condition that the following linear system has a solution

$$\begin{aligned}(\lambda(D_3 + i\Phi) + (D_1 - iD_2))\,\Psi(\lambda, \mathbf{x}) &= 0\\ (-\lambda(D_1 + iD_2) + (D_3 - i\Phi))\,\Psi(\lambda, \mathbf{x}) &= 0.\end{aligned} \tag{4.4}$$

We will take $\Psi(\lambda, \mathbf{x})$ to be a 2×2 matrix so that the gauge and Higgs field are in the fundamental representation of $su(2)$ and act on Ψ by left multiplication.

This is the linear system used by Forgács, Horváth and Palla [44–46] in their inverse scattering approach, and related to that originally found by Belavin and Zakharov in the context of the self-duality equations on $\mathbb{R}^4$ [43]. The Bogomol'nyi equations on $\mathbb{R}^3$ are equivalent to the Euclidean self-duality equations on $\mathbb{R}^4$ if the fourth component of the gauge field is identified with the Higgs field, and all the fields are taken to be independent of the fourth Euclidean direction [39, 40]. In this case, the linear system of Belavin and Zakharov reduces to the linear system (4.4).

We will rewrite the linear system in terms of the spherical polar coordinates r, z and $\bar{z}$ introduced in Chapter 2. For reference, we list the following covariant derivatives in these coordinates

$$
\begin{aligned}
D_3 - i\Phi &= \frac{z\bar{z}}{1+z\bar{z}}(D_r - i\Phi) - \frac{1}{1+z\bar{z}}(D_r + i\Phi) + \frac{z}{r}D_z + \frac{\bar{z}}{r}D_{\bar{z}} \\
D_3 + i\Phi &= -\frac{1}{1+z\bar{z}}(D_r - i\Phi) + \frac{z\bar{z}}{1+z\bar{z}}(D_r + i\Phi) + \frac{z}{r}D_z + \frac{\bar{z}}{r}D_{\bar{z}} \\
D_1 + iD_2 &= \frac{z}{1+z\bar{z}}(D_r - i\Phi) + \frac{z}{1+z\bar{z}}(D_r + i\Phi) - \frac{z^2}{r}D_z + \frac{1}{r}D_{\bar{z}} \\
D_1 - iD_2 &= \frac{\bar{z}}{1+z\bar{z}}(D_r - i\Phi) + \frac{\bar{z}}{1+z\bar{z}}(D_r + i\Phi) + \frac{1}{r}D_z - \frac{\bar{z}^2}{r}D_{\bar{z}}.
\end{aligned} \tag{4.5}
$$

The zero curvature condition (4.3) is

$$
\begin{aligned}
&\left[(\bar{z}-\lambda)\left(\frac{1}{1+z\bar{z}}(D_r - i\Phi) - \frac{\bar{z}}{r}D_{\bar{z}}\right) + (\lambda z + 1)\left(\frac{\bar{z}}{1+z\bar{z}}(D_r + i\Phi) + \frac{1}{r}D_z\right),\right. \\
&\left.\quad (\bar{z}-\lambda)\left(\frac{z}{1+z\bar{z}}(D_r - i\Phi) + \frac{1}{r}D_{\bar{z}}\right) + (\lambda z + 1)\left(-\frac{1}{1+z\bar{z}}(D_r + i\Phi) + \frac{z}{r}D_z\right)\right] = 0,
\end{aligned} \tag{4.6}
$$

and expanding this out we obtain

$$
\begin{aligned}
&\frac{(\bar{z}-\lambda)^2}{r}[D_r - i\Phi, D_{\bar{z}}] \quad + \quad \frac{(\lambda z+1)^2}{r}[D_r + i\Phi, D_z] \quad + \\
&\frac{(\bar{z}-\lambda)(\lambda z+1)}{1+z\bar{z}}\left([D_r - i\Phi, D_r + i\Phi] + \frac{(1+z\bar{z})^2}{r^2}[D_{\bar{z}}, D_z]\right) = 0,
\end{aligned} \tag{4.7}
$$

which is equivalent to the Jarvis equations

$$
\begin{aligned}
[D_r - i\Phi, D_{\bar{z}}] &= 0 \\
[D_r + i\Phi, D_z] &= 0 \\
[D_r - i\Phi, D_r + i\Phi] + \frac{(1+z\bar{z})^2}{r^2}[D_{\bar{z}}, D_z] &= 0.
\end{aligned} \tag{4.8}
$$

Taking linear combinations of the operators in (4.6) we obtain the linear system

$$
\begin{aligned}
\left((\bar{z}-\lambda)(D_r - i\Phi) + (\lambda z + 1)\frac{1+z\bar{z}}{r}D_z\right)\Psi(\lambda, \mathbf{x}) &= 0 \\
\left((\bar{z}-\lambda)\frac{1+z\bar{z}}{r}D_{\bar{z}} - (\lambda z + 1)(D_r + i\Phi)\right)\Psi(\lambda, \mathbf{x}) &= 0
\end{aligned} \tag{4.9}
$$

Note that, since we have taken linear combinations depending on z and $\bar{z}$, these operators no longer commute. However, the algebra they define closes and this is sufficient to be able to find a simultaneous solution to the linear system.

There is a freedom in the solution $\Psi(\lambda, \mathbf{x})$. The function

$$\gamma(\lambda, \mathbf{x}) = \frac{r(1+\lambda z)(\lambda - \bar{z})}{(1+z\bar{z})} = \frac{1}{2}\left(\lambda^2(x_1 + ix_2) - 2\lambda x_3 - (x_1 - ix_2)\right) \tag{4.10}$$

satisfies

$$\begin{aligned} \left((\bar{z}-\lambda)\partial_r + (\lambda z + 1)\frac{1+z\bar{z}}{r}\partial_z\right)\gamma(\lambda, \mathbf{x}) &= 0 \\ \left((\bar{z}-\lambda)\frac{1+z\bar{z}}{r}\partial_{\bar{z}} - (\lambda z + 1)\partial_r\right)\gamma(\lambda, \mathbf{x}) &= 0. \end{aligned} \tag{4.11}$$

It follows that multiplying $\Psi(\lambda, \mathbf{x})$ on the right by a matrix depending on λ and $\gamma(\lambda, \mathbf{x})$ leaves the fields appearing in the linear system unchanged. It will be important to us later to note that

$$\gamma(\lambda, \mathbf{0}) = \gamma(\bar{z}, \mathbf{x}) = \gamma(-1/z, \mathbf{x}) = 0, \tag{4.12}$$

and that γ is linear in $\mathbf{x}$ so that a translation $\mathbf{x} \to \mathbf{x} - \mathbf{A}$, where $\mathbf{A}$ is a constant vector in $\mathbb{R}^3$, results in the addition of a function of λ to γ

$$\gamma(\lambda, \mathbf{x}) \to \gamma(\lambda, \mathbf{x}) + \gamma(\lambda, \mathbf{A}). \tag{4.13}$$

4.2 Galilean Covariance of the Linear System

The Galilean group has a well-defined action on a solution of the linear system, and it is in this sense that we will talk about the covariance of the linear system under Galilean transformations. To show how translations act on the linear system we use the linear system in the form (4.4)

$$\begin{aligned} (\lambda(D_3 + i\Phi) + (D_1 - iD_2))\,\Psi(\lambda, \mathbf{x}) &= 0 \\ (-\lambda(D_1 + iD_2) + (D_3 - i\Phi))\,\Psi(\lambda, \mathbf{x}) &= 0. \end{aligned} \tag{4.14}$$

Clearly $\Psi'(\lambda, \mathbf{x}) = \Psi(\lambda, \mathbf{x} - \mathbf{A})$ is a solution to the linear system corresponding to the translated fields $A_i(\mathbf{x} - \mathbf{A})$ and $\Phi(\mathbf{x} - \mathbf{A})$.

Writing the linear system in the form (4.9)

$$\left(\left(\frac{\lambda - \bar{z}}{1+\lambda z}\right)\frac{1+z\bar{z}}{r}D_{\bar{z}} + (D_r + i\Phi)\right)\Psi(\lambda, \mathbf{x}) = 0 \tag{4.15}$$

$$\left((D_r - i\Phi) - \left(\frac{1+\lambda z}{\lambda - \bar{z}}\right)\frac{1+z\bar{z}}{r}D_z\right)\Psi(\lambda, \mathbf{x}) = 0, \tag{4.16}$$

allows us to define the action of a rotation about the origin.

The operators

$$\Delta_z^\lambda = \frac{1+\lambda z}{\lambda - \bar{z}}(1+z\bar{z})\partial_z \qquad \Delta_{\bar{z}}^\lambda = \frac{\lambda - \bar{z}}{1+\lambda z}(1+z\bar{z})\partial_{\bar{z}}, \tag{4.17}$$

are invariant under the simultaneous rotation

$$z \to z' = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}} \qquad \lambda \to \lambda' = \frac{\bar{\alpha}\lambda + \bar{\beta}}{-\beta\lambda + \alpha}, \tag{4.18}$$

therefore $\Psi'(\lambda, r, z, \bar{z}) = \Psi(\lambda', r, z', \bar{z}')$ is a solution to the linear system corresponding to the rotated fields $A_i(r, z', \bar{z}')$ and $\Phi(r, z', \bar{z}')$.

These transformations generate the Galilean group and therefore define the full action on solutions of the linear system.

4.3 The Conjugate Solution

There is a sense in which the operators appearing in the linear system are conjugates of each other. Suppose we are in a unitary gauge in which the fields A_i and Φ are anti-Hermitian, and we have a solution $\psi(\lambda, \mathbf{x})$ to the linear system (4.4). For the time being we will concentrate on the first equation in the linear system and for clarity write it as

$$(\lambda\partial_3 + (\partial_1 - i\partial_2))\,\psi(\lambda,\mathbf{x}) = -\left(\lambda(A_3 + i\Phi) + (A_1 - iA_2)\right)\psi(\lambda,\mathbf{x}). \tag{4.19}$$

Multiplying on the left and right by $\psi^{-1}(\lambda, \mathbf{x})$ we obtain

$$-\left(\lambda\partial_3 + (\partial_1 - i\partial_2)\right)\psi^{-1}(\lambda,\mathbf{x}) = -\psi^{-1}(\lambda,\mathbf{x})\left(\lambda(A_3 + i\Phi) + (A_1 - iA_2)\right) \tag{4.20}$$

Now defining $\lambda' = -1/\bar{\lambda}$ and taking the Hermitian conjugate we have

$$-\left(-\frac{1}{\lambda'}\partial_3 + (\partial_1 + i\partial_2)\right)\psi^{-1}(-1/\bar{\lambda}',\mathbf{x})^\dagger = \left(-\frac{1}{\lambda'}(A_3 - i\Phi) + (A_1 + iA_2)\right)\psi^{-1}(-1/\bar{\lambda}',\mathbf{x})^\dagger, \tag{4.21}$$

which implies that

$$(-\lambda'(D_1 + iD_2) + (D_3 - i\Phi))\,\psi^{-1}(-1/\bar{\lambda}',\mathbf{x})^\dagger = 0. \tag{4.22}$$

A similar manipulation of the second equation implies that $\psi^{-1}(-1/\bar{\lambda},\mathbf{x})^\dagger$ is a solution of the first. Therefore, given a solution $\psi(\lambda, \mathbf{x})$ in a unitary gauge, we can define a conjugate solution $\psi^{-1}(-1/\bar{\lambda},\mathbf{x})^\dagger$ which is a solution of the same linear system.

Given two solutions to the same linear system $\psi_1(\lambda, \mathbf{x})$ and $\psi_2(\lambda, \mathbf{x})$, it is straightforward to show that $G = \psi_2^{-1}(\lambda, \mathbf{x})\psi_1(\lambda, \mathbf{x})$ must satisfy

$$\begin{aligned}(\lambda\partial_3 + (\partial_1 - i\partial_2))\, G &= 0\\ (-\lambda(\partial_1 + i\partial_2) + \partial_3)\, G &= 0,\end{aligned} \tag{4.23}$$

implying $G = G(\gamma(\lambda, \mathbf{x}), \lambda)$, where $\gamma(\lambda, \mathbf{x})$ was introduced in (4.10) above. Thus the solution $\psi(\lambda, \mathbf{x})$ and its conjugate are related by

$$\psi(\lambda, \mathbf{x}) = \psi^{-1}(-1/\bar{\lambda}, \mathbf{x})^{\dagger} G(\gamma(\lambda, \mathbf{x})\, \lambda). \tag{4.24}$$

Now suppose that $a(\mathbf{x})$ is a gauge transformation that takes us to a non-unitary gauge, so that $\Psi(\lambda, \mathbf{x}) = a^{-1}(\mathbf{x})\psi(\lambda, \mathbf{x})$ is the solution to the linear system in this gauge. Then (4.24) gives

$$\Psi(\lambda, \mathbf{x}) = H^{-1}(\mathbf{x})\Psi^{-1}(-1/\bar{\lambda}, \mathbf{x})^{\dagger} G(\gamma(\lambda, \mathbf{x})\, \lambda), \tag{4.25}$$

where $H(\mathbf{x}) = a(\mathbf{x})a(\mathbf{x})^{\dagger}$. Therefore the conjugate solution in a non-unitary gauge is of the form $H^{-1}(\mathbf{x})\Psi^{-1}(-1/\bar{\lambda}, \mathbf{x})^{\dagger}$ where $H(\mathbf{x})$ is a Hermitian matrix.

4.4 Solutions in the Jarvis Gauge

In this section we will show how any solution to the linear system satisfying a conjugate relation of the form (4.25) gives rise to a to a unimodular Hermitian matrix $\mathcal{H}$ with $\mathcal{H}(\mathbf{0}) = \mathbb{I}$ satisfying the Jarvis equation

$$\partial_r(\mathcal{H}^{-1}\partial_r\mathcal{H}) + \frac{(1+z\bar{z})^2}{r^2}\partial_{\bar{z}}(\mathcal{H}^{-1}\partial_z\mathcal{H}) = 0. \tag{4.26}$$

Suppose we have a solution to the linear system

$$\begin{aligned}\left((\bar{z} - \lambda)(D_r - i\Phi) + (\lambda z + 1)\frac{1+z\bar{z}}{r}D_z\right)\psi(\lambda, \mathbf{x}) &= 0\\ \left((\bar{z} - \lambda)\frac{1+z\bar{z}}{r}D_{\bar{z}} - (\lambda z + 1)(D_r + i\Phi)\right)\psi(\lambda, \mathbf{x}) &= 0,\end{aligned} \tag{4.27}$$

satisfying the conjugate relation

$$\psi(\lambda, \mathbf{x}) = H^{-1}(\mathbf{x})\psi^{-1}(-1/\bar{\lambda}, \mathbf{x})^{\dagger} G(\gamma(\lambda, \mathbf{x})\, \lambda), \tag{4.28}$$

where $H(\mathbf{x})$ is some Hermitian matrix. Any such solution corresponds to fields Φ, A_r, A_z, $A_{\bar{z}}$ which are independent of λ but may be in any (possibly non-unitary) gauge.

Assuming that $\psi(\lambda, \mathbf{0})$ is well-defined, we can define a new solution

$$\hat{\psi}(\lambda, \mathbf{x}) = \psi(\lambda, \mathbf{x})\psi^{-1}(\lambda, \mathbf{0})N, \tag{4.29}$$

where N is a constant 2×2 matrix. Since this corresponds to multiplying on the right by a matrix depending on λ, it is a solution to the same linear system but now satisfying

$$\hat{\psi}(\lambda, \mathbf{0}) = N, \tag{4.30}$$

and the conjugate relation

$$\hat{\psi}(\lambda, \mathbf{x}) = H^{-1}(\mathbf{x})\hat{\psi}^{-1}(-1/\bar{\lambda}, \mathbf{x})^{\dagger}\hat{G}(\gamma(\lambda, \mathbf{x}), \lambda), \tag{4.31}$$

where H^{-1} is the same as in (4.28). Evaluating this at the origin implies that $\hat{G}(0, \lambda) = NH(\mathbf{0})N^{\dagger}$. Writing $H^{-1}(0) = B^{\dagger}B$, where B is unique up to left multiplication by an element of $U(2)$, we can take $N = B$ so that

$$\hat{G}(0, \lambda) = \mathbb{I}. \tag{4.32}$$

The determinant of $\hat{\psi}(\lambda, \mathbf{x})$ is the Wronskian of two vector solutions to the linear system and satisfies

$$\begin{aligned} \left((\bar{z} - \lambda)\partial_r + (\lambda z + 1)\frac{1 + z\bar{z}}{r}\partial_z\right) \det \hat{\psi}(\lambda, \mathbf{x}) &= 0 \\ \left((\bar{z} - \lambda)\frac{1 + z\bar{z}}{r}\partial_{\bar{z}} - (\lambda z + 1)\partial_r\right) \det \hat{\psi}(\lambda, \mathbf{x}) &= 0, \end{aligned} \tag{4.33}$$

which implies that

$$\det \hat{\psi}(\lambda, \mathbf{x}) = f(\gamma(\lambda, \mathbf{x}), \lambda). \tag{4.34}$$

Equation (4.30) then tells us that $f(0, \lambda) = \det N$.

If we evaluate the linear system when $\lambda = -1/z$, we find that $\hat{\psi}(-1/z, \mathbf{x})$ satisfies

$$(D_r - i\Phi)\hat{\psi}(-1/z, \mathbf{x}) = 0 \qquad D_{\bar{z}}\hat{\psi}(-1/z, \mathbf{x}) = 0, \tag{4.35}$$

which means that $\hat{\psi}(-1/z, \mathbf{x})$ is a gauge transformation taking us to a gauge in which $A_r - i\Phi = A_{\bar{z}} = 0$. Therefore $\Psi(\lambda, \mathbf{x}) = \hat{\psi}^{-1}(-1/z, \mathbf{x})\hat{\psi}(\lambda, \mathbf{x})$ is a solution of the linear system

$$\begin{aligned} \left((\bar{z} - \lambda)\partial_r + (\lambda z + 1)\frac{1 + z\bar{z}}{r}D_z\right) \Psi(\lambda, \mathbf{x}) &= 0 \\ \left((\bar{z} - \lambda)\frac{1 + z\bar{z}}{r}\partial_{\bar{z}} - (\lambda z + 1)(D_r + i\Phi)\right) \Psi(\lambda, \mathbf{x}) &= 0. \end{aligned} \tag{4.36}$$

We will refer to this as the Jarvis linear system. (We are being sloppy in that the gauge fields A_z and $A_r + i\Phi$ are gauge transforms of those appearing in the linear system (4.27)).

Now evaluating $\Psi(\lambda, \mathbf{x})$ when $\lambda = \bar{z}$ we find

$$A_r + i\Phi = -\partial_r \Psi(\bar{z}, \mathbf{x}) \cdot \Psi^{-1}(\bar{z}, \mathbf{x}) \qquad A_z = -\partial_z \Psi(\bar{z}, \mathbf{x}) \cdot \Psi^{-1}(\bar{z}, \mathbf{x}), \tag{4.37}$$

so $\mathcal{H}(\mathbf{x}) = \Psi^{-1}(\bar{z}, \mathbf{x}) = \hat{\psi}^{-1}(\bar{z}, \mathbf{x})\hat{\psi}(-1/z, \mathbf{x})$ is a solution of the Jarvis equation (4.26). Using the properties of $\hat{\psi}(\lambda, \mathbf{x})$, (4.30) and (4.34), and the properties (4.12) of $\gamma(\lambda, \mathbf{x})$, we find

$$\begin{aligned} \tilde{H}(\mathbf{0}) &= \hat{\psi}^{-1}(\bar{z}, \mathbf{0})\hat{\psi}(-1/z, \mathbf{0}) \\ &= N^{-1}N \\ &= \mathbb{I} \end{aligned} \tag{4.38}$$

$$\begin{aligned} \det \tilde{H}(\mathbf{x}) &= \det\left(\hat{\psi}^{-1}(\bar{z}, \mathbf{x})\right)^{-1} \det \hat{\psi}(-1/z, \mathbf{x}) \\ &= f(\gamma(\bar{z}, \mathbf{x}), \bar{z})^{-1} f(\gamma(-1/z, \mathbf{x}), -1/z) \\ &= f(0, \bar{z})^{-1} f(0, -1/z) \\ &= (\det N)^{-1} \det N \\ &= 1. \end{aligned} \tag{4.39}$$

We can calculate $\mathcal{H}^\dagger$ using (4.32) and the relation (4.31)

$$\begin{aligned} \mathcal{H}(\mathbf{x})^\dagger &= \hat{\psi}(-1/z, \mathbf{x})^\dagger \hat{\psi}^{-1}(\bar{z}, \mathbf{x})^\dagger \\ &= \hat{G}(0, -1/z)^\dagger \hat{\psi}^{-1}(\bar{z}, \mathbf{x})\hat{\psi}(-1/z, \mathbf{x})\hat{G}^{-1}(0, -1/z) \\ &= \mathcal{H}(\mathbf{x}). \end{aligned} \tag{4.40}$$

Thus $\mathcal{H}(\mathbf{x})$ is a solution to the Jarvis equation in the Jarvis gauge, unique up to the freedom in choosing B which amounts to a constant $SU(2)$ transformation $\mathcal{H} \to U^{-1}\mathcal{H}U$.

4.5 The Translated Solution in the Jarvis Gauge

We now have all the tools necessary to find solutions to the Jarvis equation corresponding to monopoles which have undergone a translation. Given a solution to the linear system $\psi(\lambda, \mathbf{x})$, covariance of the linear system under a translation means that $\psi(\lambda, \mathbf{x} - \mathbf{A})$ is also a solution to the linear system corresponding to translating the monopole by $\mathbf{A}$. The conjugate relation is just obtained by translating (4.28)

$$\psi(\lambda, \mathbf{x} - \mathbf{A}) = H^{-1}(\mathbf{x} - \mathbf{A})\psi^{-1}(-1/\bar{\lambda}, \mathbf{x} - \mathbf{A})^\dagger G(\gamma(\lambda, \mathbf{x}) - \gamma(\lambda, \mathbf{A}), \lambda). \tag{4.41}$$

Defining $\hat{\psi}(\lambda, \mathbf{x}) = \psi(\lambda, \mathbf{x} - \mathbf{A})\psi^{-1}(\lambda, -\mathbf{A})N$ we simply follow the procedure detailed above to obtain the solution in the Jarvis gauge:

$$\Psi(\lambda, \mathbf{x}) = N^{-1}\psi(-1/z, -\mathbf{A})\psi^{-1}(-1/z, \mathbf{x} - \mathbf{A})\psi(\lambda, \mathbf{x} - \mathbf{A})\psi^{-1}(\lambda, -\mathbf{A})N, \tag{4.42}$$

where $N^\dagger N = H^{-1}(-\mathbf{A})$.

4.6 The Vacuum Solution

We begin by finding the solution to the Jarvis linear system corresponding to the vacuum solution of the Jarvis equation. In the process we will find variables which will be of use to us later.

The simplest vacuum solution is

$$\mathcal{H}_0 = \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}. \tag{4.43}$$

The Jarvis linear system in this case is

$$\left(\frac{\lambda - \bar{z}}{1 + \lambda z} \frac{1 + z\bar{z}}{r} \partial_{\bar{z}} + \partial_r \right) \Psi_0(\lambda, \mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Psi_0(\lambda, \mathbf{x}) \tag{4.44}$$

$$\left(\partial_r - \frac{1 + \lambda z}{\lambda - \bar{z}} \frac{1 + z\bar{z}}{r} \partial_z \right) \Psi_0(\lambda, \mathbf{x}) = 0. \tag{4.45}$$

We solve this by setting

$$\Psi_0 = \begin{pmatrix} e^{2rf} & 0 \\ 0 & e^{-2rf} \end{pmatrix}, \tag{4.46}$$

where f is taken to be a function of z, $\bar{z}$, λ and $\bar{\lambda}$. This ansatz is motivated by the desire to have no worse than e^{2r} behaviour asymptotically and to have $\Psi_0(\lambda, \mathbf{0}) = \mathbb{I}$. The equations (4.44) and (4.45) reduce to

$$\partial_{\bar{z}} f + \left(\frac{1 + \lambda z}{\lambda - \bar{z}} \right) \frac{f}{1 + z\bar{z}} = \left(\frac{1 + \lambda z}{\lambda - \bar{z}} \right) \frac{1}{1 + z\bar{z}} \tag{4.47}$$

$$\partial_z f - \left(\frac{\lambda - \bar{z}}{1 + \lambda z} \right) \frac{f}{1 + z\bar{z}} = 0 \tag{4.48}$$

The solution is now straightforward. (4.48) implies

$$f = g(\lambda, \bar{\lambda}, \bar{z}) \frac{1 + \lambda z}{1 + z\bar{z}}, \tag{4.49}$$

and plugging this in to (4.47) gives

$$\partial_{\bar{z}} g + \frac{1}{\lambda - \bar{z}} g = \frac{1}{\lambda - \bar{z}}. \tag{4.50}$$

A particular solution for this is clearly given by $g = 1$ while the general solution to the homogeneous part is $g = h(\lambda, \bar{\lambda})(\lambda - \bar{z})$, so we have

$$f = \frac{1 + \lambda z}{1 + z\bar{z}} + h(\lambda, \bar{\lambda}) \frac{(1 + \lambda z)(\lambda - \bar{z})}{1 + z\bar{z}}. \tag{4.51}$$

This general solution satisfies $\Psi_0(-1/z, \mathbf{x}) = \mathbb{I}$ and $\Psi_0(\bar{z}, \mathbf{x}) = \mathcal{H}_0^{-1}$ as required.

Two choices of the function $h(\lambda, \bar{\lambda})$ will be of use to us in the following work. The first is the simplest choice $h(\lambda, \bar{\lambda}) = 0$, which gives us the variable

$$s = \frac{1 + \lambda z}{1 + z\bar{z}}. \tag{4.52}$$

The second is given by $h(\lambda, \bar{\lambda}) = -\bar{\lambda}/(1 + \lambda\bar{\lambda})$, giving the variable

$$p = \frac{(1 + \lambda z)(1 + \bar{\lambda}\bar{z})}{(1 + z\bar{z})(1 + \lambda\bar{\lambda})}. \tag{4.53}$$

This is the cross ratio of the points z, $-1/\lambda$, $\bar{\lambda}$ and $-1/\bar{z}$ which is invariant under the rotation (4.18) and is well-defined for z and λ taking values in the whole of the Riemann sphere. We will also defined the variables $t = 1 - s$ and $q = 1 - p$. The relationship between Ψ and its conjugate (4.25) is especially simple for $f = p$ since $\overline{p(-1/\bar{\lambda})} = q(\lambda)$, namely

$$\Psi_0(\lambda, \mathbf{x}) = \begin{pmatrix} e^{2rp} & 0 \\ 0 & e^{-2rp} \end{pmatrix} = \mathcal{H}_0^{-1} \Psi_0^{-1}(-1/\bar{\lambda})^{\dagger}. \tag{4.54}$$

4.7 The Spherically Symmetric One-Monopole

In this section we will construct solutions to the linear system corresponding to the spherically symmetric BPS monopole. We will do this in two ways: firstly, by exploiting the spherical symmetry of the Jarvis linear system to find a solution which also has "hedgehog" symmetry, and secondly, following similar reasoning, a solution which is meromorphic in λ (and independent of $\bar{\lambda}$). We then show how these solutions are related by a matrix function of γ and λ.

4.7.1 The Hedgehog Solution

We write the spherically symmetric one-monopole solution in the Jarvis gauge in the form given by Ioannidou and Sutcliffe [47]

$$\mathcal{H} = e^{\frac{g}{2}}\mathbb{P} + e^{-\frac{g}{2}}(\mathbb{I} - \mathbb{P}), \tag{4.55}$$

where g is a function of r and $\mathbb{P}$ is a Hermitian projector

$$g = 2\log\left(\frac{2r}{\sinh 2r}\right) \qquad \mathbb{P} = \frac{1}{1+z\bar{z}}\begin{pmatrix}1\\ z\end{pmatrix}\begin{pmatrix}1 & \bar{z}\end{pmatrix}. \tag{4.56}$$

This is a gauge transformation of the solution found at the end of Chapter 2.

The corresponding non-zero fields are

$$\begin{aligned} A_r + i\Phi &= \mathcal{H}^{-1}\partial_r\mathcal{H} = \frac{\partial_r g}{2}(2\mathbb{P} - \mathbb{I}) \\ A_z &= \mathcal{H}^{-1}\partial_z\mathcal{H} = (e^g - 1)\partial_z\mathbb{P}, \end{aligned} \tag{4.57}$$

and the linear system, in terms of the operators introduced in (4.17), is

$$D^\lambda_{\bar{z}r}\Psi(\lambda,\mathbf{x}) = \left(\Delta^\lambda_{\bar{z}} + r\partial_r + r\frac{\partial_r g}{2}(2\mathbb{P} - \mathbb{I})\right)\Psi(\lambda,\mathbf{x}) = 0 \tag{4.58}$$

$$D^\lambda_{rz}\Psi(\lambda,\mathbf{x}) = \left(r\partial_r - \Delta^\lambda_z - (e^g - 1)\mathbb{P}^\lambda_-\right)\Psi(\lambda,\mathbf{x}) = 0. \tag{4.59}$$

Here

$$\mathbb{P}^\lambda_- = \Delta^\lambda_z\mathbb{P} = \frac{1+\lambda z}{\lambda - \bar{z}}\frac{1}{1+z\bar{z}}\begin{pmatrix}-\bar{z}\\ 1\end{pmatrix}\begin{pmatrix}1 & \bar{z}\end{pmatrix}. \tag{4.60}$$

The operators $D^\lambda_{\bar{z}r}$ and D^λ_{rz} are invariant under the simultaneous rotation of z and λ (4.18) together with the gauge rotation

$$\begin{aligned} D^\lambda_{\bar{z}'r} &= \begin{pmatrix}\bar{\alpha} & -\bar{\beta}\\ \beta & \alpha\end{pmatrix} D^\lambda_{\bar{z}r}\begin{pmatrix}\alpha & \bar{\beta}\\ -\beta & \bar{\alpha}\end{pmatrix} \\ D^\lambda_{rz'} &= \begin{pmatrix}\bar{\alpha} & -\bar{\beta}\\ \beta & \alpha\end{pmatrix} D^\lambda_{rz}\begin{pmatrix}\alpha & \bar{\beta}\\ -\beta & \bar{\alpha}\end{pmatrix}, \end{aligned} \tag{4.61}$$

therefore we can look for a solution Ψ with the same symmetry.

A basis for Ψ which is invariant under these simultaneous rotations is given by

$$\begin{aligned} B_1 &= \frac{1+\bar{\lambda}\bar{z}}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}1\\ z\end{pmatrix}\begin{pmatrix}1 & \lambda\end{pmatrix} \\ B_2 &= \frac{\bar{z}-\lambda}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}1\\ z\end{pmatrix}\begin{pmatrix}-\bar{\lambda} & 1\end{pmatrix} \\ B_3 &= \frac{\bar{\lambda}-z}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}-\bar{z}\\ 1\end{pmatrix}\begin{pmatrix}1 & \lambda\end{pmatrix} \\ B_4 &= \frac{1+\lambda z}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}-\bar{z}\\ 1\end{pmatrix}\begin{pmatrix}-\bar{\lambda} & 1\end{pmatrix}. \end{aligned} \tag{4.62}$$

Writing

$$\Psi = aB_1 + bB_2 + cB_3 + dB_4, \tag{4.63}$$

the coefficient functions must be functions of the invariant quantities r, p and $q = 1 - p$ where

$$p = \frac{(1+\lambda z)(1+\bar{\lambda}\bar{z})}{(1+z\bar{z})(1+\lambda\bar{\lambda})} \qquad q = \frac{(\lambda - \bar{z})(\bar{\lambda} - z)}{(1+z\bar{z})(1+\lambda\bar{\lambda})}. \tag{4.64}$$

We list the following derivatives with respect to the operators (4.17)

$$\begin{aligned}
\Delta_z^\lambda p &= p & \Delta_{\bar{z}}^\lambda p &= q \\
\Delta_z^\lambda q &= -p & \Delta_{\bar{z}}^\lambda q &= -q \\
\Delta_z^\lambda B_1 &= \frac{p}{q} B_3 & \Delta_{\bar{z}}^\lambda B_1 &= \frac{q}{p} B_1 \\
\Delta_z^\lambda B_2 &= -B_4 & \Delta_{\bar{z}}^\lambda B_2 &= -B_2 \\
\Delta_z^\lambda B_3 &= -\frac{p}{q} B_3 & \Delta_{\bar{z}}^\lambda B_3 &= -\frac{q}{p} B_1 \\
\Delta_z^\lambda B_4 &= B_4 & \Delta_{\bar{z}}^\lambda B_4 &= B_2.
\end{aligned} \tag{4.65}$$

We also need

$$\begin{aligned}
\mathbb{P} B_1 &= B_1 & \mathbb{P}_-^\lambda B_1 &= \frac{p}{q} B_3 \\
\mathbb{P} B_2 &= B_2 & \mathbb{P}_-^\lambda B_2 &= -B_4 \\
\mathbb{P} B_3 &= 0 & \mathbb{P}_-^\lambda B_3 &= 0 \\
\mathbb{P} B_4 &= 0 & \mathbb{P}_-^\lambda B_4 &= 0.
\end{aligned} \tag{4.66}$$

Substituting (4.63) into the first equation of the linear system (4.58) and expanding out the result in the basis (4.62) gives us the equations

$$\Delta_{\bar{z}}^\lambda a + \frac{q}{p}(a - c) + r\partial_r a + r\frac{\partial_r g}{2} a = 0 \tag{4.67}$$

$$\Delta_{\bar{z}}^\lambda b - b + d + r\partial_r b + r\frac{\partial_r g}{2} b = 0 \tag{4.68}$$

$$\Delta_{\bar{z}}^\lambda c + r\partial_r c - r\frac{\partial_r g}{2} c = 0 \tag{4.69}$$

$$\Delta_{\bar{z}}^\lambda d + r\partial_r d - r\frac{\partial_r g}{2} d = 0. \tag{4.70}$$

Similarly, the second equation (4.59) becomes

$$r\partial_r a - \Delta_z^\lambda a = 0 \tag{4.71}$$
$$r\partial_r b - \Delta_z^\lambda b = 0 \tag{4.72}$$
$$r\partial_r c - \Delta_z^\lambda c + \frac{p}{q}(a-c) - \frac{p}{q}(e^g - 1)a = 0 \tag{4.73}$$
$$r\partial_r d - \Delta_z^\lambda d - b + d + (e^g - 1)b = 0. \tag{4.74}$$

Equations (4.71) and (4.72) immediately imply that $a = \alpha(rp)$ and $b = \beta(rp)$, while (4.69) and (4.70) imply that $c = e^{\frac{g}{2}}\gamma(rq)$ and $d = e^{\frac{g}{2}}\delta(rq)$.

Using (4.67) we have

$$rp\alpha'(rp) + rp\frac{\partial_r g}{2}\alpha(rp) + q(\alpha(rp) - e^{\frac{g}{2}}\gamma(rq)). \tag{4.75}$$

When $p = 1$, $q = 0$ this is

$$r\alpha'(r) + r\frac{\partial_r g}{2}\alpha(r) = 0, \tag{4.76}$$

which implies that $\alpha(r) = ke^{-\frac{g}{2}}$. Using the explicit form of g (4.56) we have

$$\alpha(rp) = k\frac{\sinh 2rp}{2rp}. \tag{4.77}$$

Now when $p = 0$, $q = 1$ we get

$$\begin{aligned} k - e^{\frac{g}{2}}\gamma(r) &= 0 \\ \Rightarrow \gamma(rq) &= k\,\frac{\sinh 2rq}{2rq}. \end{aligned} \tag{4.78}$$

To find β and γ we use (4.68):

$$r\beta'(rp) - 2r\beta(rp)\coth 2r + 2r\delta(rq)\,\mathrm{cosech}\,2r = 0. \tag{4.79}$$

Evaluating this when $p = 1$, $q = 0$ implies that

$$\beta(rp) = \delta(0)\cosh 2rp + k_1 \sinh 2rp, \tag{4.80}$$

and then evaluating when $p = 0$, $q = 1$ gives us

$$\delta(rq) = \delta(0)\cosh 2rq - k_1 \sinh 2rq. \tag{4.81}$$

We determine the values of the constants $\delta(0)$, k and k_1 by requiring that $\Psi(-1/z, \mathbf{x}) = \mathbb{I}$, and we should then have $\Psi(\bar{z}, \mathbf{x}) = \mathcal{H}^{-1}$. Recall that $\Psi = aB_1 + bB_2 + cB_3 + dB_4$. When $\lambda = -1/z$ we have

$$\Psi = bP + c(1-P) \qquad p = 0 \qquad q = 1, \tag{4.82}$$

implying that $b = \delta(0) = 1$ and $c = k = 1$. When $\lambda = \bar{z}$ we have

$$\Psi = aP + d(1-P) \qquad p = 1 \qquad q = 0, \tag{4.83}$$

which gives $a = \frac{\sinh 2r}{2r}$ and $d = \frac{2r}{\sinh 2r}$ as required.

We are free to set $k_1 = 0$ and so we have, finally

$$\begin{aligned} a(r,p) &= \frac{\sinh 2rp}{2rp} \\ b(r,p) &= \cosh 2rp \\ c(r,q) &= \frac{2r}{\sinh 2r}\frac{\sinh 2rq}{2rq} \\ d(r,q) &= \frac{2r}{\sinh 2r}\cosh 2rq. \end{aligned} \tag{4.84}$$

4.7.2 The Conjugate Relation

We can now discuss the relationship with the conjugate solution as per (4.25). To do this we will have to calculate the inverse of Ψ which will also allow us to confirm that the solution found above has determinant 1.

It is useful to define another basis as follows

$$\begin{aligned} C_1(\lambda) = B_1(\lambda)^\dagger &= \frac{1+\lambda z}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}1\\ \bar{\lambda}\end{pmatrix}\begin{pmatrix}1 & \bar{z}\end{pmatrix} \\ C_2(\lambda) = B_2(\lambda)^\dagger &= \frac{z-\bar{\lambda}}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}-\lambda\\ 1\end{pmatrix}\begin{pmatrix}1 & \bar{z}\end{pmatrix} \\ C_3(\lambda) = B_3(\lambda)^\dagger &= \frac{\lambda-\bar{z}}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}1\\ \bar{\lambda}\end{pmatrix}\begin{pmatrix}-z & 1\end{pmatrix} \\ C_4(\lambda) = B_4(\lambda)^\dagger &= \frac{1+\bar{\lambda}\bar{z}}{(1+z\bar{z})(1+\lambda\bar{\lambda})}\begin{pmatrix}-\lambda\\ 1\end{pmatrix}\begin{pmatrix}-z & 1\end{pmatrix}. \end{aligned} \tag{4.85}$$

These are related under $\lambda \to -1/\bar{\lambda}$ by $C_1(-1/\bar{\lambda}) = C_2(\lambda)$ and $C_3(-1/\bar{\lambda}) = C_4(\lambda)$. In terms of this basis it can be shown that the inverse of $\Psi = aB_1 + bB_2 + cB_3 + dB_4$ is

$$\Psi^{-1} = \frac{1}{adp + bcq}(dC_1 + cC_2 + bC_3 + aC_4). \tag{4.86}$$

For the solution above we have

$$adp + bcq = \frac{\sinh 2rp \cosh 2rq + \cosh 2rp \sinh 2rq}{\sinh 2r} = 1. \tag{4.87}$$

To find the conjugate relationship we first calculate $\Psi^{-1}(-1/\bar{\lambda})^{\dagger}$

$$\begin{aligned}
\Psi(\lambda) &= a(r,p)B_1 + b(r,p)B_2 + c(r,q)B_3 + d(r,q)B_4 \\
\Psi^{-1}(\lambda) &= d(r,q)C_1 + c(r,q)C_2 + b(r,p)C_3 + a(r,p)C_4 \\
\Psi^{-1}(-1/\bar{\lambda}) &= d(r,p)C_2 + c(r,p)C_1 + b(r,q)C_4 + a(r,q)C_3 \\
\Psi^{-1}(-1/\bar{\lambda})^{\dagger} &= \overline{d(r,p)}B_2 + \overline{c(r,p)}B_1 + \overline{b(r,q)}B_4 + \overline{a(r,q)}B_3.
\end{aligned} \tag{4.88}$$

Now

$$\begin{aligned}
\mathcal{H}^{-1}\Psi^{-1}(-1/\bar{\lambda})^{\dagger} &= \left(e^{-\frac{g}{2}}\mathbb{P} + e^{\frac{g}{2}}(\mathbb{I} - \mathbb{P})\right)\Psi^{-1}(-1/\bar{\lambda})^{\dagger} \\
&= e^{-\frac{g}{2}}\overline{c(r,p)}B_1 + e^{-\frac{g}{2}}\overline{d(r,p)}B_2 + e^{\frac{g}{2}}\overline{a(r,q)}B_3 + e^{\frac{g}{2}}\overline{b(r,q)}B_4.
\end{aligned} \tag{4.89}$$

For the solution (4.84) we have

$$\begin{aligned}
a(r,p) &= e^{-\frac{g}{2}}\overline{c(r,p)} \\
b(r,p) &= e^{-\frac{g}{2}}\overline{d(r,p)} \\
c(r,p) &= e^{\frac{g}{2}}\overline{a(r,q)} \\
d(r,q) &= e^{\frac{g}{2}}\overline{b(r,q)},
\end{aligned} \tag{4.90}$$

and so the conjugate relation is

$$\Psi(\lambda, \mathbf{x}) = \mathcal{H}^{-1}\Psi^{-1}(-1/\bar{\lambda}, \mathbf{x})^{\dagger}. \tag{4.91}$$

4.7.3 The Meromorphic Solution

The solution found above has explicit hedgehog behaviour which required that the solution depended on $\bar{\lambda}$ as well as λ. In general we want to deal with solutions depending only on λ, so in this section we will present a solution which is meromorphic in λ. To do this, it will be necessary to relax the rotational invariance of the hedgehog solution by using coefficient functions which are no longer rotationally invariant.

We can use a basis for Ψ which is spherically symmetric, but now independent of $\bar{\lambda}$

$$\begin{aligned}
\mathbb{P} &= \frac{1}{1+z\bar{z}}\begin{pmatrix}1\\ z\end{pmatrix}\begin{pmatrix}1 & \bar{z}\end{pmatrix} \\
\mathbb{I}-\mathbb{P} &= \frac{1}{1+z\bar{z}}\begin{pmatrix}-\bar{z}\\ 1\end{pmatrix}\begin{pmatrix}-z & 1\end{pmatrix} \\
\mathbb{P}^\lambda_+ &= \frac{\lambda-\bar{z}}{1+\lambda z}\frac{1}{1+z\bar{z}}\begin{pmatrix}1\\ z\end{pmatrix}\begin{pmatrix}-z & 1\end{pmatrix} \\
\mathbb{P}^\lambda_- &= \frac{1+\lambda z}{\lambda-\bar{z}}\frac{1}{1+z\bar{z}}\begin{pmatrix}-\bar{z}\\ 1\end{pmatrix}\begin{pmatrix}1 & \bar{z}\end{pmatrix}.
\end{aligned} \tag{4.92}$$

The price we pay is that $\mathbb{P}^\lambda_+$ and $\mathbb{P}^\lambda_-$ are singular when $\lambda = -1/z$ and $\lambda = \bar{z}$ respectively, so we will have to check that the solution we obtain is defined at these points.

The variables s and t

$$s = \frac{1+\lambda z}{1+z\bar{z}} \qquad t = 1-s, \tag{4.93}$$

are analogues of p and q and similarly satisfy

$$\begin{aligned}
\Delta^\lambda_z s &= s & \Delta^\lambda_{\bar{z}} s &= t \\
\Delta^\lambda_z t &= -s & \Delta^\lambda_{\bar{z}} t &= -t.
\end{aligned} \tag{4.94}$$

We therefore anticipate that we can find a solution

$$\Psi = a\mathbb{P} + b\mathbb{P}^\lambda_+ + c\mathbb{P}^\lambda_- + d(\mathbb{I}-\mathbb{P}), \tag{4.95}$$

where the coefficient functions are functions of r, s and t.

We list the derivatives of the basis (4.92)

$$\begin{aligned}
\Delta^\lambda_z \mathbb{P} &= \mathbb{P}^\lambda_- & \Delta^\lambda_{\bar{z}} \mathbb{P} &= \mathbb{P}^\lambda_+ \\
\Delta^\lambda_z \mathbb{P}^\lambda_+ &= -\mathbb{P}^\lambda_+ - (2\mathbb{P}-\mathbb{I}) & \Delta^\lambda_{\bar{z}} \mathbb{P}^\lambda_+ &= -\mathbb{P}^\lambda_+ \\
\Delta^\lambda_z \mathbb{P}^\lambda_- &= \mathbb{P}^\lambda_- & \Delta^\lambda_{\bar{z}} \mathbb{P}^\lambda_- &= \mathbb{P}^\lambda_- - (2\mathbb{P}-\mathbb{I}).
\end{aligned} \tag{4.96}$$

Plugging (4.95) into (4.59) gives

$$r\partial_r a - \Delta^\lambda_z a + b = 0 \tag{4.97}$$

$$r\partial_r b - \Delta^\lambda_z b + b = 0 \tag{4.98}$$

$$r\partial_r c - \Delta^\lambda_z c - c + d - e^g a = 0 \tag{4.99}$$

$$r\partial_r d - \Delta^\lambda_z d - e^g b = 0. \tag{4.100}$$

Multiplying (4.98) by r we obtain the equation

$$(r\partial_r - \Delta_z^\lambda)(rb) = 0, \tag{4.101}$$

which implies that $b = \beta(rs)/2r$. Subtracting (4.98) from (4.97) implies

$$(r\partial_r - \Delta_z^\lambda)(a - b) = 0, \tag{4.102}$$

so $a - b = \alpha(rs)$ and therefore

$$a = \alpha(rs) + \frac{\beta(rs)}{2r}. \tag{4.103}$$

Now if we substitute Ψ into (4.58) we obtain

$$\Delta_{\bar{z}}^\lambda a - c + r\partial_r a + r\frac{\partial_r g}{2}a = 0 \tag{4.104}$$

$$\Delta_{\bar{z}}^\lambda b - b + a - d + r\partial_r b + r\frac{\partial_r g}{2}b = 0 \tag{4.105}$$

$$\Delta_{\bar{z}}^\lambda c + c + r\partial_r c - r\frac{\partial_r g}{2}c = 0 \tag{4.106}$$

$$\Delta_{\bar{z}}^\lambda d + c + r\partial_r d - r\frac{\partial_r g}{2}d = 0 \tag{4.107}$$

Using $r\frac{\partial_r g}{2} = 1 - 2r\coth 2r$ and multiplying (4.106) by $\sinh 2r$ we obtain

$$(\Delta_z^\lambda + r\partial_r)(\sinh 2rc) = 0, \tag{4.108}$$

implying that $c = \gamma(rt)/\sinh 2r$. Now subtracting (4.106) from (4.107) and multiplying by $e^{-\frac{g}{2}}$ gives the equation

$$(\Delta_z^\lambda + r\partial_r)(e^{-\frac{g}{2}}(d - c)) = 0, \tag{4.109}$$

which implies that

$$d - c = e^{\frac{g}{2}}\delta(rt) = \frac{2r}{\sinh 2r}\delta(rt). \tag{4.110}$$

Now subtracting (4.99) from (4.100), we obtain the equation

$$r\partial_r(d - c) - \Delta_z^\lambda(d - c) - (d - c) + e^g(a - b) = 0, \tag{4.111}$$

and substituting the formulae for $a - b$ and $d - c$ into this implies

$$\delta'(rt)\sinh 2r - 2\delta(rt)\cosh 2r + 2\alpha(rs) = 0, \tag{4.112}$$

with solution

$$\begin{aligned}\delta(rt) &= k_1 \cosh 2rt + k_2 \sinh 2rt \\ \alpha(rs) &= k_1 \cosh 2rs - k_2 \sinh 2rs.\end{aligned} \tag{4.113}$$

We now turn to (4.100) with

$$d = \frac{2r}{\sinh 2r}(k_1 \cosh 2rt + k_2 \sinh 2rt) + \frac{\gamma(rt)}{\sinh 2r} \qquad b = \frac{\beta(rs)}{2r}. \tag{4.114}$$

Plugging this in and multiplying by $\frac{\sinh^2 2r}{r}$ we obtain

$$\begin{aligned}2k_1 \sinh 2r \cosh 2rt - 4rk_1 \cosh 2r \cosh 2rt + 4rk_1 \sinh 2r \sinh 2rt - \\ 2\gamma(rt) \cosh 2r + \gamma'(rt) \sinh 2r - 2\beta(rs) = 0.\end{aligned} \tag{4.115}$$

Evaluating this when $t = 1$ and $s = 0$ gives

$$\gamma'(r) \sinh 2r - 2\gamma(r) \cosh 2r = 2\beta(0) + 2k_1(2r - \cosh 2r \sinh 2r). \tag{4.116}$$

A particular solution is seen to be given by $\gamma(r) = -(\beta(0)+2rk_1) \cosh 2r$ while the general solution to the homogeneous equation is $\gamma(r) = k_3 \sinh 2r$. Hence

$$\gamma(rt) = k_3 \sinh 2rt - (\beta(0) + 2rtk_1) \cosh 2rt. \tag{4.117}$$

Substituting this back into (4.115) gives us

$$\beta(rs) = k_3 \sinh 2rs + (\beta(0) - 2rsk_1) \cosh 2rs. \tag{4.118}$$

We will set $\beta(0)$ to 0. The conditions that $\det(\Psi) = 1$ and $\Psi(-1/z) = 1$ fix $k_1 = k_3 = 1$ so finally we have

$$\begin{aligned}a &= t \cosh 2rs + \frac{\sinh 2rs}{2r} \\ b &= -s \cosh 2rs + \frac{\sinh 2rs}{2r} \\ c &= \frac{2r}{\sinh 2r}\left(-t \cosh 2rt + \frac{\sinh 2rt}{2r}\right) \\ d &= \frac{2r}{\sinh 2r}\left(s \cosh 2rt + \frac{\sinh 2rt}{2r}\right).\end{aligned} \tag{4.119}$$

It can be checked that $b\mathbb{P}^\lambda_+$ and $c\mathbb{P}^\lambda_-$ are nonsingular.

4.7.4 The Relationship Between the Solutions

Since both the spherically symmetric and the meromorphic solution give rise to the same gauge fields, they must differ by multiplication on the right by a matrix depending on $\gamma(\lambda, \mathbf{x})$ and λ. We will compare the solutions and, in the process, rewrite them in a form which will be convenient for finding the translated solution.

We rewrite the hedgehog solution as follows:

$$\begin{aligned}\Psi^S(\lambda,\mathbf{x}) = &\begin{pmatrix}1\\ z\end{pmatrix}\frac{1}{2r(1+\lambda z)}\left[\sinh 2rp\,(1\ \lambda) - \cosh 2rp\,\frac{2\gamma}{1+\lambda\bar{\lambda}}\,(-\bar{\lambda}\ 1)\right]\\ &+\begin{pmatrix}-\bar{z}\\ 1\end{pmatrix}\frac{1}{(\lambda-\bar{z})\sinh 2r}\left[\sinh 2rq\,(1\ \lambda) + \cosh 2rq\,\frac{2\gamma}{1+\lambda\bar{\lambda}}\,(-\bar{\lambda}\ 1)\right]\\ =&\begin{pmatrix}1 & -\bar{z}\\ z & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2r(1+\lambda z)} & 0\\ 0 & \frac{1}{(\lambda-\bar{z})\sinh 2r}\end{pmatrix}\begin{pmatrix}\sinh 2rp & -\cosh 2rp\\ \sinh 2rq & \cosh 2rq\end{pmatrix}\begin{pmatrix}1 & \lambda\\ -\frac{2\gamma\bar{\lambda}}{1+\lambda\bar{\lambda}} & \frac{2\gamma}{1+\lambda\bar{\lambda}}\end{pmatrix}.\end{aligned} \tag{4.120}$$

Similarly the meromorphic solution can be written as

$$\Psi^M(\lambda,\mathbf{x}) = \begin{pmatrix}1 & -\bar{z}\\ z & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2r(1+\lambda z)} & 0\\ 0 & \frac{1}{(\lambda-\bar{z})\sinh 2r}\end{pmatrix}\begin{pmatrix}\sinh 2rs & -\cosh 2rs\\ \sinh 2rt & \cosh 2rt\end{pmatrix}\begin{pmatrix}1 & \lambda\\ 0 & 2\gamma\end{pmatrix}. \tag{4.121}$$

Noting that $p = s - \frac{2\gamma\bar{\lambda}}{1+\lambda\bar{\lambda}}$ we find

$$\begin{pmatrix}\sinh 2rp & -\cosh 2rp\\ \sinh 2rq & \cosh 2rq\end{pmatrix} = \begin{pmatrix}\sinh 2rs & -\cosh 2rs\\ \sinh 2rt & \cosh 2rt\end{pmatrix}\begin{pmatrix}\cosh\frac{2\gamma\bar{\lambda}}{1+\lambda\bar{\lambda}} & \sinh\frac{2\gamma\bar{\lambda}}{1+\lambda\bar{\lambda}}\\ \sinh\frac{2\gamma\bar{\lambda}}{1+\lambda\bar{\lambda}} & \cosh\frac{2\gamma\bar{\lambda}}{1+\lambda\bar{\lambda}}\end{pmatrix}. \tag{4.122}$$

Since the hedgehog solution satisfies $\Psi^S(\lambda,\mathbf{x}) = \mathcal{H}^{-1}(\mathbf{x})\Psi^{-1}(-1/\bar{\lambda},\mathbf{x})^\dagger$, the meromorphic solution satisfies a conjugate relation of the form

$$\Psi^M(\lambda,\mathbf{x}) = \mathcal{H}^{-1}(\mathbf{x})\Psi^{M-1}(-1/\bar{\lambda},\mathbf{x})^\dagger G^M(\gamma(\lambda,\mathbf{x}),\lambda), \tag{4.123}$$

where $G^M(0,\lambda) = \mathbb{I}$.

4.8 The Translated Solution

In this section, we will apply the method for finding a translated solution of the Jarvis equation to the one-monopole solution of the linear system. We begin by rewriting the expression for the meromorphic solution (4.121) given in the previous section as

$$\Psi^M(\lambda,r,z,\bar{z}) = \mathcal{A}(r,z,\bar{z})\mathcal{D}(\lambda,z,\bar{z})\mathcal{B}(\lambda,r,z,\bar{z})\mathcal{C}(\gamma(\lambda,\mathbf{x}),\lambda), \tag{4.124}$$

where

$$
\begin{aligned}
\mathcal{A}(r,z,\bar{z}) &= \frac{1}{\sqrt{1+z\bar{z}}}\begin{pmatrix}1 & -\bar{z}\\ z & 1\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\sinh 2r}{2r}} & 0\\ 0 & \sqrt{\frac{2r}{\sinh 2r}}\end{pmatrix}\\
\mathcal{D}(\lambda,z,\bar{z}) &= \begin{pmatrix}\frac{1}{1+\lambda z} & 0\\ 0 & \frac{1}{\lambda-\bar{z}}\end{pmatrix}\\
\mathcal{B}(\lambda,r,z,\bar{z}) &= \sqrt{\frac{1+z\bar{z}}{2r\sinh 2r}}\begin{pmatrix}\sinh 2rs & -\cosh 2rs\\ \sinh 2rt & \cosh 2rt\end{pmatrix}\\
\mathcal{C}(\gamma,\lambda) &= \begin{pmatrix}1 & \lambda\\ 0 & 2\gamma\end{pmatrix}.
\end{aligned} \tag{4.125}
$$

Here, again, $s = (1+\lambda z)/(1+z\bar{z}) = 1-t$. As already discussed, $\mathcal{C}(\gamma,\lambda)$ does not contribute to the fields in the linear system, so we will actually begin with the solution

$$\Psi^M(\lambda,r,z,\bar{z}) = \mathcal{A}(r,z,\bar{z})\mathcal{D}(\lambda,z,\bar{z})\mathcal{B}(\lambda,r,z,\bar{z}). \tag{4.126}$$

Recall that the coordinates r, z and $\bar{z}$ are polar coordinates from the origin. When we translate the solution it will be useful to use two systems of polar coordinates, one defined from the origin and the other defined from the position of the monopole. Let $\mathbf{x}$ be the coordinate vector from the origin and $\mathbf{X}$ be the coordinate vector from the monopole, so that $\mathbf{X} = \mathbf{x} + \mathbf{A}$ defines the position of the monopole for some constant vector $\mathbf{A}$. The corresponding polar coordinates are defined by writing

$$
\begin{aligned}
\mathbf{x} &= r\left(\frac{z+\bar{z}}{1+z\bar{z}},\ -i\frac{z-\bar{z}}{1+z\bar{z}},\ \frac{z\bar{z}-1}{1+z\bar{z}}\right)\\
\mathbf{X} &= R\left(\frac{Z+\bar{Z}}{1+Z\bar{Z}},\ -i\frac{Z-\bar{Z}}{1+Z\bar{Z}},\ \frac{Z\bar{Z}-1}{1+Z\bar{Z}}\right)\\
\mathbf{A} &= A\left(\frac{W+\overline{W}}{1+W\overline{W}},\ -i\frac{W-\overline{W}}{1+W\overline{W}},\ \frac{W\overline{W}-1}{1+W\overline{W}}\right).
\end{aligned} \tag{4.127}
$$

To obtain the translated solution in the Jarvis gauge we follow the procedure described in section 4.4. Translating the solution (4.126), ie. taking $\Psi^M(\lambda,R,Z,\bar{Z})$, gives us a solution to the linear system corresponding to a monopole at $-\mathbf{A}$, although not now in a gauge satisfying the Jarvis equation. The conjugate relation is

$$\Psi^M(\lambda,R,Z,\bar{Z}) = \mathcal{H}^{-1}(R,Z,\bar{Z})\Psi^{M-1}(-1/\bar{\lambda},R,Z,\bar{Z})^\dagger\mathcal{G}(\gamma(\lambda,\mathbf{x}),\lambda), \tag{4.128}$$

for some matrix function $\mathcal{G}$. At the origin $r=0$, $R=A$, $Z=W$ and $\bar{Z}=\overline{W}$ are constants and

$$
\begin{aligned}
\mathcal{H}^{-1}(A,W,\overline{W}) &= \frac{1}{1+z\bar{z}}\begin{pmatrix}1 & -\bar{z}\\ z & 1\end{pmatrix}\begin{pmatrix}\frac{\sinh 2r}{2r} & 0\\ 0 & \frac{2r}{\sinh 2r}\end{pmatrix}\begin{pmatrix}1 & \bar{z}\\ -z & 1\end{pmatrix}\\
&= \mathcal{A}(A,W,\overline{W})^\dagger\mathcal{A}(A,W,\overline{W}).
\end{aligned} \tag{4.129}
$$

Therefore we take $N = \mathcal{A}(A, W, \overline{W})$ and define

$$\begin{aligned}\hat{\psi}(\lambda, r, z, \bar{z}) &= \psi(\lambda, R, Z, \bar{Z})\psi^{-1}(\lambda, A, W, \overline{W})N \\ &= \mathcal{A}(R, Z, \bar{Z})\mathcal{D}(\lambda, Z, \bar{Z})\mathcal{B}(\lambda, R, Z, \bar{Z})\mathcal{B}^{-1}(\lambda, A, W, \overline{W})\mathcal{D}^{-1}(\lambda, W, \overline{W}). \quad (4.130)\end{aligned}$$

The solution to the linear system in the Jarvis gauge is therefore

$$\begin{aligned}\Psi(\lambda, r, z, \bar{z}) &= \hat{\psi}^{-1}(-1/z, r, z, \bar{z})\hat{\psi}(\lambda, r, z, \bar{z}) \\ &= \tilde{\mathcal{B}}^{-1}(-1/z)\tilde{\mathcal{D}}(\lambda)\tilde{\mathcal{B}}(\lambda), \quad (4.131)\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathcal{B}}(\lambda) &= \mathcal{B}(\lambda, R, Z, \bar{Z})\mathcal{B}^{-1}(\lambda, A, W, \overline{W})\mathcal{D}^{-1}(\lambda, W, \overline{W}) \\ \tilde{\mathcal{D}}(\lambda) &= \mathcal{D}^{-1}(-1/z, Z, \bar{Z})\mathcal{D}(\lambda, Z, \bar{Z}), \quad (4.132)\end{aligned}$$

and the solution to the Jarvis equation $\mathcal{H}$ is given by

$$\mathcal{H}^{-1} = \Psi(\bar{z}) = \tilde{\mathcal{B}}^{-1}(-1/z)\tilde{\mathcal{D}}(\bar{z})\tilde{\mathcal{B}}(\bar{z}). \quad (4.133)$$

Explicity,

$$\begin{aligned}\tilde{\mathcal{B}}(\lambda) &= \frac{1}{\sqrt{\sinh 2A}}\begin{pmatrix} (1+\lambda W)\sinh(A+R+r(2s-1)) & (\overline{W}-\lambda)\sinh(A-R-r(2s-1)) \\ -(1+\lambda W)\frac{\sinh(A-R+r(2s-1))}{\sinh 2R} & (\lambda-\overline{W})\frac{\sinh(A+R-r(2s-1))}{\sinh 2R}\end{pmatrix} \\ \tilde{\mathcal{D}}(\lambda) &= \frac{1}{z}\begin{pmatrix}\frac{z-Z}{1+\lambda Z} & 0 \\ 0 & -\frac{1+z\bar{Z}}{\lambda-\bar{Z}}\end{pmatrix} \\ \tilde{\mathcal{B}}^{-1}(-1/z) &= \frac{z}{\sqrt{\sinh 2A}}\begin{pmatrix}\frac{1}{z-W}\frac{\sinh(A+R+r)}{\sinh 2R} & \frac{1}{z-W}\sinh(A-R+r) \\ -\frac{1}{1+z\overline{W}}\frac{\sinh(A-R-r)}{\sinh 2R} & -\frac{1}{1+z\overline{W}}\sinh(A+R-r)\end{pmatrix}. \quad (4.134)\end{aligned}$$

Rewriting $z\tilde{\mathcal{D}}(\bar{z})$ as follows

$$\begin{pmatrix}\frac{z-Z}{1+\bar{z}Z} & 0 \\ 0 & -\frac{1+z\bar{Z}}{\bar{z}-\bar{Z}}\end{pmatrix} = \frac{R(1+W\overline{W})}{A(1+Z\bar{Z})}\begin{pmatrix}\frac{(z-Z)(\bar{z}-\bar{Z})}{(\bar{z}-\overline{W})(1+\bar{z}W)} & 0 \\ 0 & -\frac{(1+z\bar{Z})(1+\bar{z}Z)}{(\bar{z}-\overline{W})(1+\bar{z}W)}\end{pmatrix}, \quad (4.135)$$

we obtain the following explicit form of the solution

$$\mathcal{H}^{-1} = \frac{R(1+W\overline{W})}{A(1+Z\bar{Z})\sinh(2A)\sinh(2R)}\left(\mathbf{v}_1\mathbf{v}_1^\dagger + \mathbf{v}_2\mathbf{v}_2^\dagger\right), \quad (4.136)$$

where

$$\mathbf{v}_1 = \begin{pmatrix} \frac{z-Z}{z-W}\sinh(A+R+r) \\ -\frac{z-Z}{1+z\overline{W}}\sinh(A-R-r) \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} -\frac{1+z\bar{Z}}{z-W}\sinh(A-R+r) \\ \frac{1+z\bar{Z}}{1+z\overline{W}}\sinh(A+R-r) \end{pmatrix}. \tag{4.137}$$

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are related in an interesting way. Suppose we send r to $-r$ and z to its antipodal point $-1/\bar{z}$. Then this describes the same position vector $\mathbf{x} \in \mathbb{R}^3$. The coordinates R, Z and $\bar{Z}$ depend only on the Cartesian coordinates and so remain unchanged under this transformation. We find that

$$\mathbf{v}_1(r,z,\bar{z}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\mathbf{v}_2(-r,-1/\bar{z},-1/z)}. \tag{4.138}$$

This corresponds to the condition that

$$\mathcal{H}(r,z,\bar{z}) = \mathcal{H}^{-1}(-r,-1/\bar{z},-1/z). \tag{4.139}$$

Interestingly, this relates exponentially growing terms in $\mathcal{H}^{-1}$ to exponentially decaying ones.

The metric is required to be smooth away from the origin and continuous at the origin. Examining the solution, we see that there is one particular line on which smoothness of the solution (4.136) is not obvious. This is the line $z = W, -1/\overline{W}$ shown in figure 4.1. To check that the solution is well-defined around this line, we investigated the solution for the following five cases to leading order in ϵ using MATHEMATICA

1. $z \approx W + \epsilon\, v$, $R \approx A + r$
2. $z \approx -1/\overline{W} + \epsilon\, v$, $R \approx A - r$
3. $z \approx -1/\overline{W} + \epsilon\, v$, $R \approx r - A$
4. $r \approx \epsilon\, u$
5. $R \approx \epsilon\, u$

The result is that on this line

$$\mathcal{H}^{-1} = \begin{cases} \begin{pmatrix} \frac{2A}{\sinh 2A}\frac{\sinh 2R}{2R} & 0 \\ 0 & \frac{\sinh 2A}{2A}\frac{2R}{\sinh 2R} \end{pmatrix} & z = W \\ \begin{pmatrix} \frac{\sinh 2A}{2A}\frac{2R}{\sinh 2R} & 0 \\ 0 & \frac{2A}{\sinh 2A}\frac{\sinh 2R}{2R} \end{pmatrix} & z = -\frac{1}{\overline{W}} \end{cases}. \tag{4.140}$$

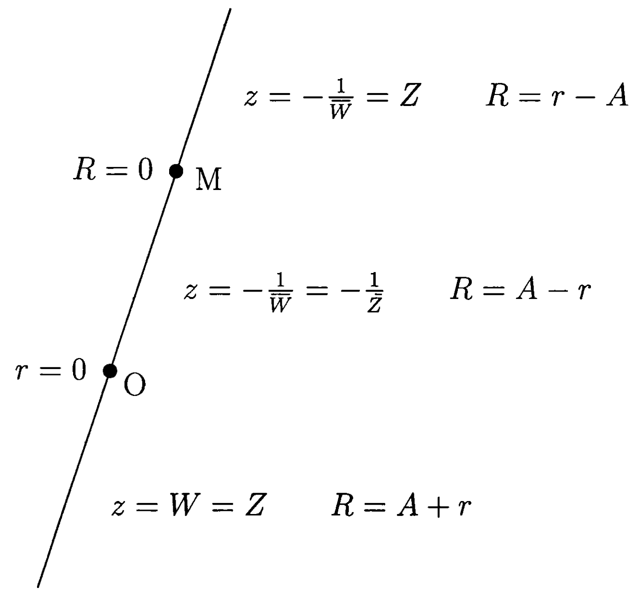


Figure 4.1: Coordinates on the line through the monopole and the origin.

This is smooth away from the origin and continuous at the origin as required.

Although $\mathcal{H}^{-1}$ no longer has explicit spherical symmetry in the Jarvis gauge, it retains a manifest axially symmetry about the line through the origin and the centre of the monopole. By using the freedom to rotate the solution to set $W = \overline{W} = 0$, we can exhibit this axial symmetry

$$\tilde{\mathcal{H}}^{-1} = \frac{2}{(1+z\bar{z})\sinh 2A \sinh 2R}$$
$$\left[\frac{1}{R+r-Ax_3} \begin{pmatrix} \sinh(A+R+r) \\ -z\sinh(A-R-r) \end{pmatrix} \left(\sinh(A+R+r),\ -\bar{z}\sinh(A-R-r)\right) \right.$$
$$\left. + \frac{1}{R-r+Ax_3} \begin{pmatrix} \sinh(A-R+r) \\ -z\sinh(A+R-r) \end{pmatrix} \left(\sinh(A-R+r),\ -\bar{z}\sinh(A+R-r)\right) \right] \tag{4.141}$$

where $R = \sqrt{x_1^2 + x_2^2 + (x_3 - A)^2}$. For $z' = e^{i\phi}z$ we have

$$\mathcal{H}^{-1}(r, z, \bar{z}, R) = \begin{pmatrix} e^{-\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix} \mathcal{H}^{-1}(r, z', \bar{z}', R) \begin{pmatrix} e^{\frac{i\phi}{2}} & 0 \\ 0 & e^{-\frac{i\phi}{2}} \end{pmatrix}. \tag{4.142}$$

We plot $\mathcal{H}^{-1}_{12}$ and $\mathcal{H}^{-1}_{22}$ around the origin for a monopole at $x_3 = 1$ in figure 4.2. The results appear to be consistent with the required properties of $\mathcal{H}$.

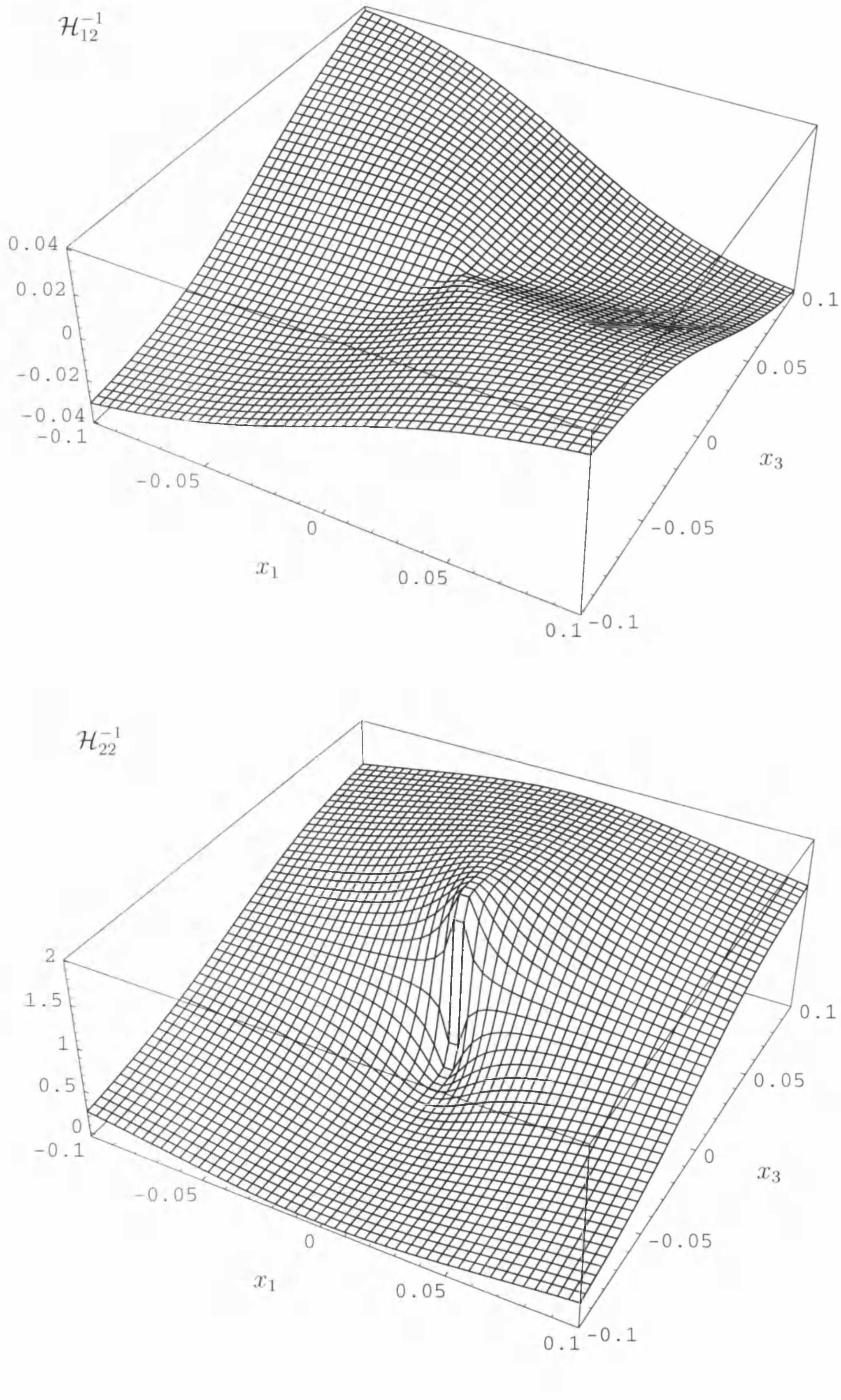


Figure 4.2: The components $\mathcal{H}_{12}^{-1}$ and $\mathcal{H}_{22}^{-1}$ close to the origin for a monopole at $x_3 = 1$. The coordinate x_2 here takes the value zero.

4.9 The Higgs Field and Jarvis Rational Map

In this section we will compute the Higgs field in the Jarvis gauge corresponding to the solution to the Jarvis equation found in the previous section. The Jarvis rational map is then found straightforwardly in terms of the eigenvector of $-i\Phi^\infty$ with eigenvalue $+1$ as explained in Chapter 2.

Recall that the Higgs field is given by $-i\Phi = \frac{1}{2}\mathcal{H}^{-1}\partial_r\mathcal{H}$ where

$$\mathcal{H}^{-1} = \tilde{\mathcal{B}}^{-1}(-1/z)\tilde{\mathcal{D}}(\bar{z})\tilde{\mathcal{B}}(\bar{z}), \tag{4.143}$$

in terms of the functions defined in (4.134). To perform the computation it is convenient to write the matrix functions $\tilde{\mathcal{B}}(-1/z)$, $\tilde{\mathcal{B}}(\bar{z})$ and $\tilde{\mathcal{D}}(\bar{z})$ in a slightly more restrictive way as functions of r, z, $\bar{z}$ and R, where $R = \sqrt{r^2 + 2r\,\hat{\mathbf{x}}\cdot\mathbf{A} + A^2}$. The vectors $\mathbf{A}$ and $\hat{\mathbf{x}} = \mathbf{x}/r$ are defined by (4.127). The form of the functions we take is

$$\begin{aligned}
\tilde{\mathcal{B}}(-1/z) &= \frac{1}{\sqrt{\sinh 2A}}\begin{pmatrix} \sinh A{+}R{-}r & \sinh A{-}R{+}r \\ -\frac{\sinh A-R-r}{\sinh 2R} & -\frac{\sinh A+R+r}{\sinh 2R}\end{pmatrix}\begin{pmatrix} z-W & 0 \\ 0 & 1+z\overline{W}\end{pmatrix}\\
\tilde{\mathcal{B}}(\bar{z}) &= \frac{1}{\sqrt{\sinh 2A}}\begin{pmatrix} \sinh A{+}R{+}r & \sinh A{-}R{-}r \\ -\frac{\sinh A-R+r}{\sinh 2R} & -\frac{\sinh A+R-r}{\sinh 2R}\end{pmatrix}\begin{pmatrix} 1+\bar{z}W & 0 \\ 0 & \overline{W}-\bar{z}\end{pmatrix}\\
\tilde{\mathcal{D}}(\bar{z}) &= \frac{1+W\overline{W}}{A(1+\bar{z}W)(\bar{z}-W)}\frac{1+z\bar{z}}{2}\begin{pmatrix} R-r-\hat{\mathbf{x}}\cdot\mathbf{A} & 0 \\ 0 & -(R+r+\hat{\mathbf{x}}\cdot\mathbf{A})\end{pmatrix}.
\end{aligned}\tag{4.144}$$

We compute

$$\begin{aligned}
&-i\tilde{\mathcal{B}}(-1/z)\Phi\tilde{\mathcal{B}}^{-1}(-1/z) = \\
&\frac{1}{2}\left(\tilde{\mathcal{D}}(\bar{z})\partial_r\tilde{\mathcal{B}}(\bar{z})\cdot\tilde{\mathcal{B}}^{-1}(\bar{z})\tilde{\mathcal{D}}^{-1}(\bar{z}) + \partial_r\tilde{\mathcal{D}}(\bar{z})\cdot\tilde{\mathcal{D}}^{-1}(\bar{z}) - \partial_r\tilde{\mathcal{B}}(-1/z)\cdot\tilde{\mathcal{B}}^{-1}(-1/z)\right),
\end{aligned}\tag{4.145}$$

the terms being explicitly

$$\begin{aligned}
\tilde{\mathcal{D}}(\bar{z})\partial_r\tilde{\mathcal{B}}(\bar{z})\cdot\tilde{\mathcal{B}}^{-1}(\bar{z})\tilde{\mathcal{D}}^{-1}(\bar{z}) &= \begin{pmatrix} \frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}+R}{R}\coth 2R & \frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}-R}{R} \\ \frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}+R}{R}\frac{1}{\sinh^2 2R} & -\frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}+R}{R}\coth 2R\end{pmatrix}\\
\partial_r\tilde{\mathcal{B}}(-1/z)\cdot\tilde{\mathcal{B}}^{-1}(-1/z) &= \begin{pmatrix} \frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}-R}{R}\coth 2R & \frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}-R}{R} \\ \frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}+R}{R}\frac{1}{\sinh^2 2R} & -\frac{r+\hat{\mathbf{x}}\cdot\mathbf{A}-R}{R}\coth 2R\end{pmatrix}\\
\partial_r\tilde{\mathcal{D}}(\bar{z})\cdot\tilde{\mathcal{D}}^{-1}(\bar{z}) &= -\frac{1}{R}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}.
\end{aligned}\tag{4.146}$$

Thus we obtain

$$\begin{aligned}
-i\Phi &= \left(\coth(2R) - \frac{1}{2R}\right)\tilde{\mathcal{B}}^{-1}(-1/z)\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\tilde{\mathcal{B}}(-1/z)\\
&= \left(\coth(2R) - \frac{1}{2R}\right)(2\tilde{P} - \mathbb{I}),
\end{aligned}\tag{4.147}$$

where

$$\tilde{P} = \tilde{\mathcal{B}}^{-1}(-1/z)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\tilde{\mathcal{B}}(-1/z)$$
$$= \frac{1}{\sinh 2A \sinh 2R}\begin{pmatrix}(1+z\overline{W})\sinh(A+R+r)\\ (W-z)\sinh(A-R-r)\end{pmatrix}\begin{pmatrix}\frac{\sinh(A+R-r)}{1+z\overline{W}} & \frac{\sinh(A-R+r)}{z-W}\end{pmatrix}. \quad (4.148)$$

Clearly the Higgs field satisfies $||\Phi|| = \coth 2R - 1/(2R)$ which corresponds to that of a translated one-monopole.

As $r \to \infty$ we have $R \to r + \hat{\mathbf{x}} \cdot \mathbf{A} + O(r^{-1})$ and

$$\tilde{P}^{\infty} = \frac{1}{\sinh 2A}\begin{pmatrix}(1+z\overline{W})e^{A-\hat{\mathbf{x}}\cdot\mathbf{A}}\\ (z-W)e^{-A-\hat{\mathbf{x}}\cdot\mathbf{A}}\end{pmatrix}\begin{pmatrix}\frac{\sinh(A+\hat{\mathbf{x}}\cdot\mathbf{A})}{1+z\overline{W}}, & \frac{\sinh(A-\hat{\mathbf{x}}\cdot\mathbf{A})}{z-W}\end{pmatrix} \quad (4.149)$$

Therefore the eigenvector of $-i\Phi^{\infty} = 2\tilde{P}^{\infty} - \mathbb{I}$ with eigenvalue $+1$ is

$$\begin{pmatrix}(1+z\overline{W})e^{A}\\ (z-W)e^{-A}\end{pmatrix}, \quad (4.150)$$

and the Jarvis rational map of the translated monopole is

$$\frac{z-W}{1+z\overline{W}}e^{-2A}. \quad (4.151)$$

4.10 Comments

We have succeeded in finding the action of translations on the Jarvis rational map of the one-monopole by explicitly constructing a metric corresponding to a monopole with arbitrary position. This has turned out to be an extremely natural procedure using the linear system for the Bogomol'nyi equations, which tallies with the fact that the solution to the linear system is a more fundamental object than a solution to the Bogomol'nyi equations. We will continue the study of solutions to the linear system in Chapter 6 when we discuss the inverse scattering method.

Chapter 5

Results on the Metric and Jarvis Rational Map

In this chapter, we will present a menagerie of results relating to the Jarvis rational map and metric $\mathcal{H}$ valid, we believe, for a general multimonopole. The first of these concerns the relationship between the Jarvis rational map and spectral lines which pass through the origin, so we begin by introducing the concepts of spectral lines and the spectral curve associated to a monopole. By defining a dual rational map, we find the spectral lines through the origin in terms of the Jarvis map and its dual.

In the previous chapter, we noted that the one-monopole solution in the Jarvis gauge has an interesting property under $(r, z, \bar{z}) \to (-r, -1/\bar{z}, -1/z)$. In this chapter we will give an argument to show that this is a general property of the metric in the Jarvis gauge, although some care seems necessary to interpret this result.

We will also discuss the asymptotic behaviour of the Higgs field in the Jarvis gauge and the boundary conditions on the metric. We find disagreement with the analysis of Ioannidou and Sutcliffe [47]. We will suggest relating the boundary conditions to the functional property of the metric found and the dual rational map.

Also included is some work which chronologically precedes the work of the previous chapter. The first idea is to consider the effect of an *infinitesimal* translation on a solution to the Jarvis equation. We find that we can find a new solution involving an integral of the fields of the initial solution, and this allows us to perform this translation on the BPS monopole at the origin and calculate the change in the rational map. The other work concerns the relation between the centre of a monopole and the Jarvis rational map. This is an idea which is close to fruition.

5.1 The Spectral Curve Associated to a Monopole

Underlying the linear system is the mini-twistor space $\mathbb{T}$ introduced by Hitchin [25] and the fact that the Galilean group acts on this space in a nice way. Mini-twistor space is the space of oriented straight lines in $\mathbb{R}^3$, which is isomorphic to $T\mathbb{C}P^1$, the tangent bundle to $\mathbb{C}P^1$. We can specify a given straight line by a point in the base space λ and a holomorphic tangent vector $\gamma\frac{\partial}{\partial\lambda}$, where λ specifies the direction of the line and γ is a complex coordinate specifying the intersection of the line with the plane orthogonal to this direction. We will represent this line as the point (γ, λ). This can be understood in terms of simple geometry of $\mathbb{R}^3$ if we relate λ to a unit normal vector $\mathbf{n} \in \mathbb{R}^3$ as follows

$$\mathbf{n} = \left(\frac{\lambda+\bar{\lambda}}{1+\lambda\bar{\lambda}}, \frac{i(\lambda-\bar{\lambda})}{1+\lambda\bar{\lambda}}, \frac{\lambda\bar{\lambda}-1}{1+\lambda\bar{\lambda}}\right). \tag{5.1}$$

The vectors

$$\begin{aligned} \frac{\partial\mathbf{n}}{\partial\lambda} &= \frac{1}{(1+\lambda\bar{\lambda})^2}\left(1-\bar{\lambda}^2, i(1+\bar{\lambda}^2), 2\bar{\lambda}\right) \\ \frac{\partial\mathbf{n}}{\partial\bar{\lambda}} &= \frac{1}{(1+\lambda\bar{\lambda})^2}\left(1-\lambda^2, -i(1+\lambda^2), 2\lambda\right), \end{aligned} \tag{5.2}$$

are orthogonal to $\mathbf{n}$, so we can specify a straight line in $\mathbb{R}^3$ as

$$\mathbf{x} = c\frac{\partial\mathbf{n}}{\partial\lambda} + \bar{c}\frac{\partial\mathbf{n}}{\partial\bar{\lambda}} + u\mathbf{n}, \tag{5.3}$$

where u is the parameter along the line. In mini-twistor space, this corresponds to the point (c, λ). We can find c from the equation of the straight line by taking the dot product with $\frac{\partial\mathbf{n}}{\partial\bar{\lambda}}$:

$$c = \frac{(1+\lambda\bar{\lambda})^2}{2}\frac{\partial\mathbf{n}}{\partial\bar{\lambda}}\cdot\mathbf{x} = -\frac{1}{2}\gamma(\lambda, \mathbf{x}), \tag{5.4}$$

where $\gamma(\lambda, \mathbf{x})$ is the function introduced in Chapter 4. This is the *twistor transform* which relates a point (c, λ) in $T\mathbb{C}P^1$ to a straight line in $\mathbb{R}^3$. (Note that our conventions concerning the definition of λ and $\gamma(\lambda, \mathbf{x})$ are not those normally used.)

Following Hitchin [25], we can consider the spectrum of the scattering operator $D_u - i\Phi$ along a given straight line in $\mathbb{T}$. As in the case of the scattering operator $D_r - i\Phi$ used to define the Jarvis rational map along half-lines from the origin, there are solutions $\mathbf{s}_1$ and $\mathbf{s}_2$ satisfying

$$e^u u^{-\frac{n}{2}}\mathbf{s}_1 \to \begin{pmatrix}1\\0\end{pmatrix} \qquad e^{-u}u^{\frac{n}{2}}\mathbf{s}_2 \to \begin{pmatrix}0\\1\end{pmatrix} \qquad \text{as } u \to \infty, \tag{5.5}$$

in some unitary gauge. Similarly, there are solutions $\mathbf{s}'_1$ and $\mathbf{s}'_2$ satisfying

$$e^{-u}u^{\frac{n}{2}}\mathbf{s}'_1 \to \begin{pmatrix}1\\0\end{pmatrix} \qquad e^{u}u^{-\frac{n}{2}}\mathbf{s}'_2 \to \begin{pmatrix}0\\1\end{pmatrix} \qquad \text{as } u \to -\infty. \tag{5.6}$$

Lines on which there is a solution which decays as both $u \to \pm\infty$ are called spectral lines. The set of spectral lines for a given monopole is described by a curve in $\mathbb{T}$ called the spectral curve.

Hitchin shows that the spectral curve of an n-monopole is a curve of genus $(n-1)^2$ in $\mathbb{T}$ of the form

$$S = \gamma^n + \gamma^{n-1}a_1(\lambda) + \cdots + \gamma^i a_{n-i}(\lambda) + \cdots + \gamma a_{n-1}(\lambda) + a_n(\lambda) = 0, \tag{5.7}$$

where each $a_i(\lambda)$ is a polynomial in λ with maximum degree $2i$. If a line in the λ direction is spectral, then so is the same line considered in the opposite direction $-1/\bar{\lambda}$ which amounts to the following reality condition on the coefficient functions $a_r(\lambda)$

$$a_r(\lambda) = (-1)^r\lambda^{2r}\overline{a_r(-1/\bar{\lambda})}. \tag{5.8}$$

There are also some non-singularity conditions which will not have cause to discuss here.

The spectral curve of a monopole centred at the origin is $\gamma = 0$. By considering

$$\gamma(\lambda, \mathbf{x}+\mathbf{A}) = \gamma(\lambda, \mathbf{x}) + \gamma(\lambda, \mathbf{A}) = 0, \tag{5.9}$$

one obtains the spectral curve of a monopole with position $-\mathbf{A}$

$$\gamma + \frac{A(\lambda - \overline{W})(1+\lambda W)}{1+W\overline{W}} = 0. \tag{5.10}$$

This describes the *star* of lines through the point $-\mathbf{A}$.

5.1.1 Spectral Lines Through the Origin

We can ask for all the spectral lines which pass through the origin. Since a line through the origin corresponds to $\gamma = 0$, such lines are solutions to the equation

$$a_n(\lambda) = 0. \tag{5.11}$$

We can look at this in terms of the solutions to the scattering operator $D_r - i\Phi$ along half-lines from the origin studied in the definition of the Jarvis rational map. We will work in a unitary gauge in which the Higgs and Euclidean gauge fields are anti-Hermitian. The variable u along a line in the direction λ through the origin corresponds to

$$\begin{aligned} r &= u \qquad & z &= \lambda \qquad & u &\geq 0 \\ r &= -u \qquad & z &= -1/\bar{\lambda} \qquad & u &\leq 0. \end{aligned} \tag{5.12}$$

The solution which decays as $u \to \infty$ is

$$(D_r - i\Phi)\mathbf{s}_1(r,\lambda) = (D_u - i\Phi)\mathbf{s}_1(u,\lambda) = 0, \tag{5.13}$$

while the solution decaying as $u \to -\infty$ is

$$(D_r - i\Phi)\mathbf{s}_1(r,-1/\bar{\lambda}) = (-D_u - i\Phi)\mathbf{s}_1(-u,-1/\bar{\lambda}) = 0. \tag{5.14}$$

Now using the fact that a traceless, anti-Hermitian matrix M satisfies

$$J\overline{M}J^{-1} = M \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.15}$$

we obtain the solution

$$(D_u - i\Phi)J\,\overline{\mathbf{s}_1(-u,-1/\bar{\lambda})}. \tag{5.16}$$

If the line $z = \lambda$ is a spectral line then the solutions $\mathbf{s}_1(u,\lambda)$, which decays as $u \to +\infty$, and $J\,\overline{\mathbf{s}_1(-u,-1/\bar{\lambda})}$, which decays as $u \to -\infty$, are linearly dependent and their Wronskian vanishes. Since the Wronskian is independent of u, we can evaluate it at the origin where $\mathbf{s}_1(u,\lambda)$ is of the form

$$\mathbf{s}_1(0,\lambda) = \alpha(\lambda,\bar{\lambda})\begin{pmatrix} p(\lambda) \\ q(\lambda) \end{pmatrix}, \tag{5.17}$$

where $f(z) = q(z)/p(z)$ is the rational map of the monopole. Similarly

$$J\,\overline{\mathbf{s}_1(0,-1/\bar{\lambda})} = \overline{\alpha(\lambda,\bar{\lambda})}\begin{pmatrix} \overline{q(-1/\bar{\lambda})} \\ -\overline{p(-1/\bar{\lambda})} \end{pmatrix}. \tag{5.18}$$

This corresponds to a rational map

$$\tilde{f}(z) = -\frac{1}{\overline{f(-1/\bar{z})}} = -\frac{\overline{p(-1/\bar{z})}}{\overline{q(-1/\bar{z})}}, \tag{5.19}$$

which we will term the *dual* rational map.

Thus spectral lines through the origin are those lines $z = \lambda$ for which

$$p(\lambda)\overline{p(-1/\bar{\lambda})} + q(\lambda)\overline{q(-1/\bar{\lambda})} = 0. \tag{5.20}$$

Note that this condition is independent of the constant $SU(2)$ Möbius gauge action on p and q, as we would hope since a spectral line is a gauge invariant object.

This relates the Jarvis rational map to the function $a_n(\lambda)$ in the spectral curve which is a degree $2n$ polynomial. They must be related by

$$a_n(\lambda) = c\lambda^n \left(p(\lambda)\overline{p(-1/\bar{\lambda})} + q(\lambda)\overline{q(-1/\bar{\lambda})}\right), \tag{5.21}$$

for some constant c.

We can test this for the Jarvis rational map of the one-monopole with position $-\mathbf{A}$ which was found to be $f(z) = e^{-2A}(z-W)/(1+z\overline{W})$. The dual rational map in this case is $\tilde{f}(z) = e^{2A}(z-W)/(1+z\overline{W})$ which corresponds to the rational map of a monopole at the point $\mathbf{A}$. In particular the one-monopole at the origin is seen to have a self-dual Jarvis map.

According to the analysis given above, the spectral lines through the origin are solutions to the equation $f(\lambda) = \tilde{f}(\lambda)$ giving

$$(\lambda - W)(1+\lambda\overline{W}) = 0. \tag{5.22}$$

This describes the line $\lambda = W, -1/\overline{W}$, which is the line on which the smoothness of the one-monopole metric (4.136) was unclear, and coincides with the line through the origin desribed by the one-monopole spectral curve (5.10).

5.2 A Functional Condition on the Metric

We saw in the previous chapter that the one-monopole metric $\mathcal{H}$ had the interesting property that $\mathcal{H}(r,z,\bar{z}) = \mathcal{H}^{-1}(-r,-1/\bar{z},-1/z)$. In this section we will show that this is true in general. Normally we would only consider non-negative values of r since this is enough to cover $\mathbb{R}^3$, but, as a function of r, z and $\bar{z}$, $\mathcal{H}$ is equally well-defined when r is negative.

Recall from the definition of the metric in Chapter 2 that there is a solution $a(r,z,\bar{z})$ to the equations

$$(D_r - i\Phi)a(r,z,\bar{z}) = D_{\bar{z}}a(r,z,\bar{z}) = 0, \tag{5.23}$$

in a smooth unitary gauge, which satisfies $a(0,z,\bar{z}) = \mathbb{I}$. The metric is defined in terms of this by $\mathcal{H}(r,z,\bar{z}) = a(r,z,\bar{z})^\dagger a(r,z,\bar{z})$. In this gauge the Cartesian gauge fields A_i and the Higgs field Φ are well-defined functions of the Cartesian coordinates, so that, as functions of the spherical polar coordinates, $A_i(r,z,\bar{z}) = A_i(-r,-1/\bar{z},-1/z)$ and $\Phi(r,z,\bar{z}) = \Phi(-r,-1/\bar{z},-1/z)$. Under $r \to -r$, $z \to -1/\bar{z}$

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial r} &= \left(\frac{z+\bar{z}}{1+z\bar{z}}, -i\frac{z-\bar{z}}{1+z\bar{z}}, \frac{z\bar{z}-1}{1+z\bar{z}}\right) \to -\frac{\partial \mathbf{x}}{\partial r} \\ \frac{\partial \mathbf{x}}{\partial \bar{z}} &= r\left(\frac{1-z^2}{(1+z\bar{z})^2}, i\frac{1+z^2}{(1+z\bar{z})^2}, \frac{2z}{(1+z\bar{z})^2}\right) \to z^2\frac{\partial \mathbf{x}}{\partial z}, \end{aligned} \tag{5.24}$$

therefore, as functions of r, z and $\bar{z}$, $A_r = \frac{\partial x^i}{\partial r} A_i$ and $A_{\bar{z}} = \frac{\partial x^i}{\partial \bar{z}} A_i$ satisfy

$$\begin{aligned} A_r(-r, -1/\bar{z}, -1/z) &= -A_r(r, z, \bar{z}) \\ A_{\bar{z}}(-r, -1/\bar{z}, -1/z) &= z^2 A_z(r, z, \bar{z}). \end{aligned} \tag{5.25}$$

Furthermore, in a unitary gauge A_r, A_z, $A_{\bar{z}}$ and Φ are traceless matrices which obey

$$A_r{}^\dagger = -A_r \qquad A_{\bar{z}}{}^\dagger = -A_z \qquad \Phi^\dagger = -\Phi. \tag{5.26}$$

Therefore we deduce

$$A_r(r, z, \bar{z}) - i\Phi(r, z, \bar{z}) = (A_r(-r, -1/\bar{z}, -1/z) - i\Phi(-r, -1/\bar{z}, -1/z))^\dagger \tag{5.27}$$

$$A_{\bar{z}}(r, z, \bar{z}) = -\frac{1}{\bar{z}^2} A_{\bar{z}}(-r, -1/\bar{z}, -1/z)^\dagger. \tag{5.28}$$

Now, using (5.23), we have $A_r(r, z, \bar{z}) - i\Phi(r, z, \bar{z}) = -(\partial_r a(r, z, \bar{z}))a^{-1}(r, z, \bar{z})$ and therefore (5.27) implies

$$\begin{aligned} (\partial_r a(r, z, \bar{z}))a^{-1}(r, z, \bar{z}) &= a^{-1}(-r, -1/\bar{z}, -1/z)^\dagger \partial_{(-r)} a(-r, -1/\bar{z}, -1/z)^\dagger \\ &= (\partial_r a^{-1}(-r, -1/\bar{z}, -1/z)^\dagger) a(-r, -1/\bar{z}, -1/z)^\dagger. \end{aligned} \tag{5.29}$$

Similarly, $A_{\bar{z}}(r, z, \bar{z}) = -(\partial_{\bar{z}} a(r, z, \bar{z}))a^{-1}(r, z, \bar{z})$ and (5.28) imply that

$$\begin{aligned} (\partial_{\bar{z}} a(r, z, \bar{z}))a^{-1}(r, z, \bar{z}) &= -\frac{1}{\bar{z}^2} a^{-1}(-r, -1/\bar{z}, -1/z)^\dagger \partial_{(-1/\bar{z})} a(-r, -1/\bar{z}, -1/z)^\dagger \\ &= (\partial_{\bar{z}} a^{-1}(-r, -1/\bar{z}, -1/z)^\dagger) a(-r, -1/\bar{z}, -1/z)^\dagger. \end{aligned} \tag{5.30}$$

Therefore $a(r, z, \bar{z})$ and $a^{-1}(-r, -1/\bar{z}, -1/z)^\dagger$ must differ by multiplication on the right by a matrix depending solely on z. Since $a(0, z, \bar{z}) = \mathbb{I}$, it follows that

$$a(r, z, \bar{z}) = a^{-1}(-r, -1/\bar{z}, -1/z)^\dagger, \tag{5.31}$$

which implies that the metric $\mathcal{H}$ satisfies

$$\begin{aligned} \mathcal{H}^{-1}(-r, -1/\bar{z}, -1/z) &= a^{-1}(-r, -1/\bar{z}, -1/z) a^{-1}(-r, -1/\bar{z}, -1/z)^\dagger \\ &= a(r, z, \bar{z})^\dagger a(r, z, \bar{z}) \\ &= \mathcal{H}(r, z, \bar{z}). \end{aligned} \tag{5.32}$$

We believe that this condition may find a use, for instance, in specifying asymptotic conditions on $\mathcal{H}$ in which the Jarvis map and its dual appear explicitly. More work will have to be done to see if this is the case.

5.3 The Asymptotics of the Metric and Higgs Field

In this section, we will discuss the asymptotic boundary conditions on the metric and Higgs field in the Jarvis gauge. The argument is based on the definition of the metric presented in Chapter 2. We shall support our argument with the example of the one-monopole solution in the Jarvis gauge found in the previous chapter.

In [47], Ioannidou and Sutcliffe present an argument based on the assumption that the Higgs field has the following asymptotic expansion valid in the Jarvis gauge

$$\Phi = \Phi^{\infty}(z,\bar{z})\left(1-\frac{n}{2r}\right)+O(\frac{1}{r^2}), \tag{5.33}$$

where $||\Phi^{\infty}||^2 = 1$.

The asymptotic Higgs field of the solution that we found in Chapter 4 is of the form

$$\Phi \sim i\left(1-\frac{1}{2r}\right)(2\tilde{P}-\mathbb{I}), \tag{5.34}$$

where the precise form of the projector is

$$\tilde{P} = \frac{1}{\sinh 2A \sinh 2R}\begin{pmatrix}(1+z\overline{W})\sinh(A+R+r)\\(W-z)\sinh(A-R-r)\end{pmatrix}\begin{pmatrix}\frac{\sinh(A+R-r)}{1+z\overline{W}} & \frac{\sinh(A-R+r)}{z-W}\end{pmatrix}. \tag{5.35}$$

This does not satisfy the boundary condition (5.33) since $\tilde{P}$ contains an $O(r^{-1})$ term which necessarily points in a different direction in the algebra to the $O(1)$ term. That this does not contradict the analysis presented in Chapter 2 which defines the metric will now be shown.

According to this analysis, $\mathcal{H}$ is of the form

$$\mathcal{H} = K(z,\bar{z})^{\dagger}S^{\dagger}SK(z,\bar{z}), \tag{5.36}$$

where the matrix S satisfies

$$S\begin{pmatrix}e^{r}r^{-\frac{n}{2}} & 0\\ 0 & e^{-r}r^{\frac{n}{2}}\end{pmatrix} \to \begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \text{as } r\to\infty, \tag{5.37}$$

and K is of the form

$$K(z,\bar{z}) = \begin{pmatrix}(1+z\bar{z})^{\frac{n}{2}} & 0\\ 0 & (1+z\bar{z})^{-\frac{n}{2}}\end{pmatrix}\begin{pmatrix}1 & v(z,\bar{z})\\ 0 & 1\end{pmatrix}F(z)^{-1}. \tag{5.38}$$

The matrix $F(z)$ is a holomorphic representative of the rational map viewed as an element of the coset space $SL(2,\mathbb{C})/B$, where B is the group of complex unimodular upper triangular matrices. Given the Jarvis map $f(z) = q(z)/p(z)$, this takes the form

$$F(z) = \begin{pmatrix}p(z) & a(z)\\ q(z) & b(z)\end{pmatrix}, \tag{5.39}$$

where $a(z)$ and $b(z)$ are polynomials satisfying $p(z)b(z) - q(z)a(z) = 1$. That such polynomials can be found is a consequence of Euclid's algorithm since p and q have no non-constant common factor.

The Higgs field is $-\frac{i}{2}\mathcal{H}^{-1}\partial_r\mathcal{H}$ where

$$\mathcal{H}^{-1}\partial_r\mathcal{H} = K^{-1}(z,\bar{z})(S^\dagger S)^{-1}\partial_r(S^\dagger S)K(z,\bar{z}). \tag{5.40}$$

A unimodular matrix S of the form

$$S = \begin{pmatrix} e^{-r}r^{\frac{n}{2}} & e^{-r}r^{\frac{n}{2}}b(r) \\ 0 & e^{r}r^{-\frac{n}{2}} \end{pmatrix}, \tag{5.41}$$

satisfies the asymptotic condition (5.37) and we find that

$$(S^\dagger S)^{-1}\partial_r(S^\dagger S) = \left(-2+\frac{n}{r}\right)\begin{pmatrix} 1 & 2b(r) \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & b'(r) \\ 0 & 0 \end{pmatrix} + e^{-4r}r^{2n}b'(r)\begin{pmatrix} -b(r) & -b(r)^2 \\ 1 & b(r) \end{pmatrix}. \tag{5.42}$$

Therefore, if $b(r) = b_0 + b_1 r^{-1} + O(r^{-2})$, then the asymptotic Higgs field is not of the form in (5.33) since the $O(1)$ and $O(r^{-1})$ terms do not point in the same direction. Note that the modulus of the asymptotic Higgs field is unaffected by the function b. It is easily verified that the eigenvector of $-i\Phi^\infty$ with eigenvalue $+1$ is $(p(z), q(z))^t$ which reproduces the required Jarvis map.

The boundary condition on S (5.37) implies that the leading order term in the metric is

$$\frac{e^{2r}}{(r(1+z\bar{z}))^n}\begin{pmatrix} -\overline{q(z)} \\ \overline{p(z)} \end{pmatrix}\begin{pmatrix} -q(z) & p(z) \end{pmatrix}. \tag{5.43}$$

However, it does not allow us to say much about the decaying part of the metric.

We can again illustrate the problems in determining the decaying terms in the metric using the parametrisation given in [47]. This consists of expressing a general unimodular Hermitian matrix $\mathcal{H}$ in the form

$$\mathcal{H} = e^{\frac{g}{2}}P + e^{-\frac{g}{2}}(\mathbb{I} - P), \tag{5.44}$$

where P is a Hermitian projector. In this case the Higgs field is found to be

$$\Phi = -\frac{i}{2}\left(\frac{\partial_r g}{2}(2P - \mathbb{I}) + (1-e^{-g})P\partial_r P + (e^g - 1)(\mathbb{I} - P)\partial_r P\right). \tag{5.45}$$

The claimed leading order behaviour of g and P is

$$g = -4r + 2n\log r + O(1) \qquad P \sim \mathbb{P}(z,\bar{z}), \tag{5.46}$$

where $\mathbb{P}$ is defined in terms of the rational map $q(z)/p(z)$ as follows:

$$\mathbb{P} = \frac{1}{p(z)\overline{p(z)} + q(z)\overline{q(z)}} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \begin{pmatrix} \overline{p(z)} & \overline{q(z)} \end{pmatrix}. \tag{5.47}$$

By comparing the exponentially growing term with (5.43), we see that this is consistent with our analysis so far. However, we will show that the conclusions drawn in [47] about the asymptotic Higgs field are too narrow.

We will introduce the traceless matrices $\mathbb{P}_+$ and $\mathbb{P}_- = \mathbb{P}_+^\dagger$ given by

$$\begin{aligned} \mathbb{P}_+ &= \frac{1}{p(z)\overline{p(z)} + q(z)\overline{q(z)}} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \begin{pmatrix} -q(z) & p(z) \end{pmatrix} \\ \mathbb{P}_- &= \frac{1}{p(z)\overline{p(z)} + q(z)\overline{q(z)}} \begin{pmatrix} -\overline{q(z)} \\ \overline{p(z)} \end{pmatrix} \begin{pmatrix} \overline{p(z)} & \overline{q(z)} \end{pmatrix}, \end{aligned} \tag{5.48}$$

and obeying

$$\begin{array}{lll} \mathbb{P}\mathbb{P}_+ = \mathbb{P}_+ & \mathbb{P}_+\mathbb{P} = 0 & \mathbb{P}_+\mathbb{P}_- = \mathbb{P} \\ \mathbb{P}\mathbb{P}_- = 0 & \mathbb{P}_-\mathbb{P} = \mathbb{P}_- & \mathbb{P}_-\mathbb{P}_+ = \mathbb{I} - \mathbb{P}. \end{array} \tag{5.49}$$

Now suppose that the projector P has an expansion

$$P = \mathbb{P} + e^g(b\mathbb{P}_+ + \bar{b}\mathbb{P}_-) + \cdots, \tag{5.50}$$

where b is a complex function and $\bar{b}$ its conjugate. It is easily verified that $P^2 = P$ to first order in e^g. This gives rise to a Higgs field

$$\Phi = -\frac{i}{2}\left(\frac{\partial_r g}{2}(2\mathbb{P} - \mathbb{I}) + (e^g - 1)((b\partial_r g + \partial_r b)\mathbb{P}_+ + e^g(\bar{b}\partial_r g + \partial_r \bar{b})\mathbb{P}_-)\right). \tag{5.51}$$

The analysis of [47] corresponds to taking $b = 0$, which leads to an asymptotic Higgs field of the form

$$\Phi \sim i\left(1 + \frac{n}{2r}\right)(2\mathbb{P} - \mathbb{I}). \tag{5.52}$$

It seems unlikely that the asymptotic Higgs field of a generic multimonopole should be of this form since, in the non-unitary Jarvis gauge, there is no reason to expect that $-i\Phi^\infty$ should be Hermitian. The spherically symmetric one monopole centred at the origin is a special case in which it is, but, as we have seen for the translated solution in the Jarvis gauge, this ceases to be the case as soon as we move the monopole. It would be interesting to see if there are any other cases where the asymptotic Higgs field is Hermitian. One possibility would be that this occurs for centred monopoles.

We believe that the question of how to specify decaying terms in a boundary condition on the metric remains unanswered, and a better understanding of what it means for the metric to be close to some asymptotic form involving exponentially growing and decaying terms is necessary. To what extent the numerical analysis of [48] is dependent on the decaying term in the boundary condition on $\mathcal{H}$ is unclear.

5.4 Infinitesimal Translations of the Metric

It is possible to consider the effect of infinitesimal translations on solutions to the Jarvis equation without recourse to the linear system. Such translations are examples of *zero modes* of the Bogomol'nyi equations.

5.4.1 Zero Modes of the Bogomol'nyi Equations

Consider a solution Φ, A_i of the Bogomol'nyi equations

$$D_i\Phi = -\frac{1}{2}\epsilon_{ijk}F_{jk}. \tag{5.53}$$

A zero mode is an infinitesimal perturbation of a solution $\Phi \to \Phi + \varepsilon\delta\Phi$, $A_i \to A_i + \varepsilon\delta A_i$, which satisfies the linearised equation

$$D_i\delta\Phi + [\delta A_i, \Phi] + \epsilon_{ijk}D_j\delta A_k = 0. \tag{5.54}$$

We can define a positive definite scalar product on the space of zero modes. For two infinitesimal perturbations δ and δ' the scalar product is

$$(\delta, \delta') = -\int d^3x\, \mathrm{tr}(\delta A_i\, \delta' A_i + \delta\Phi\, \delta'\Phi). \tag{5.55}$$

Zero modes corresponding to gauge transformations are of the form

$$\delta^{\mathrm{gauge}} A_i = [D_i, \Lambda] \qquad \delta^{\mathrm{gauge}}\Phi = [\Phi, \Lambda]. \tag{5.56}$$

Since we are interested in zero modes which correspond to physical perturbations of a solution, rather than infinitesimal gauge transformations, we can look for zero modes that are orthogonal to (5.56). From

$$\begin{aligned} 0 &= (\delta, \delta^{\mathrm{gauge}}) \\ &= -\int d^3x\, \mathrm{tr}(\delta A_i[D_i, \Lambda] + \delta\Phi[\Phi, \Lambda]) \\ &= \int d^3x\, \mathrm{tr}((D_i\delta A_i + [\Phi, \delta\Phi])\Lambda), \end{aligned} \tag{5.57}$$

we see that such zero modes satisfy the background gauge condition

$$D_i \delta A_i + [\Phi, \delta\Phi] = 0. \tag{5.58}$$

We can write the linearised Bogomol'nyi equations (5.54) and the background gauge condition (5.58) in the form of a Dirac equation [50, 51]

$$(\sigma_i D_i - i\Phi)\Psi = 0, \tag{5.59}$$

where

$$\Psi = \delta\Phi - i\sigma_j \delta A_j. \tag{5.60}$$

Given a zero mode Ψ, we can obtain other ones by multiplying on the right by the quaternions $i\sigma_1$, $i\sigma_2$, $i\sigma_3$. We will be dealing with zero modes Ψ which are normalisable with respect to the scalar product (5.55).

It is easily checked that $\Psi = i\sigma_j D_j \Phi$ is a solution of (5.59) for which

$$\begin{aligned} \delta A_1 &= -D_1\Phi \\ \delta A_2 &= -D_2\Phi \\ \delta A_3 &= -D_3\Phi \\ \delta\Phi &= 0. \end{aligned} \tag{5.61}$$

Right-multiplying by $i\sigma_i$ gives us solutions corresponding to translation of the monopole in the i-direction. We will consider a translation in the x_3 direction for which

$$\begin{aligned} \delta A_1 &= -D_2\Phi \\ \delta A_2 &= D_1\Phi \\ \delta A_3 &= 0 \\ \delta\Phi &= D_3\Phi. \end{aligned} \tag{5.62}$$

In terms of the spherical polar coordinates r, z and $\bar{z}$, where z is the complex coordinate on the Riemann sphere, the Bogomol'nyi equations are

$$[D_r - i\Phi, D_{\bar{z}}] = 0 \tag{5.63}$$

$$[D_r + i\Phi, D_z] = 0 \tag{5.64}$$

$$F_{z\bar{z}} = \frac{2ir^2}{(1+z\bar{z})^2} D_r\Phi. \tag{5.65}$$

and zero modes of these equations satisfy

$$[D_r - i\Phi, \delta A_{\bar{z}}] = D_{\bar{z}}(\delta A_r - i\delta\Phi) \tag{5.66}$$
$$[D_r + i\Phi, \delta A_z] = D_z(\delta A_r + i\delta\Phi) \tag{5.67}$$
$$D_z\delta A_{\bar{z}} - D_{\bar{z}}\delta A_z = \frac{2ir^2}{(1+z\bar{z})^2}(D_r\delta\Phi - [\Phi, \delta A_r]). \tag{5.68}$$

We can write the translational zero mode (5.62) in these coordinates, obtaining

$$\delta A_r - i\delta\Phi = i\frac{1-z\bar{z}}{1+z\bar{z}}D_r\Phi - 2i\frac{z}{r}D_z\Phi \tag{5.69}$$
$$\delta A_{\bar{z}} = 2ir\frac{z}{(1+z\bar{z})^2}D_r\Phi + i\frac{1-z\bar{z}}{1+z\bar{z}}D_{\bar{z}}\Phi \tag{5.70}$$
$$\delta A_r + i\delta\Phi = -i\frac{1-z\bar{z}}{1+z\bar{z}}D_r\Phi + 2i\frac{\bar{z}}{r}D_{\bar{z}}\Phi \tag{5.71}$$
$$\delta A_z = -2ir\frac{\bar{z}}{(1+z\bar{z})^2}D_r\Phi - i\frac{1-z\bar{z}}{1+z\bar{z}}D_z\Phi. \tag{5.72}$$

We will work in the Jarvis gauge in which $A_r - i\Phi = A_{\bar{z}} = 0$, $A_r + i\Phi = \mathcal{H}^{-1}\partial_r\mathcal{H}$, and $A_z = \mathcal{H}^{-1}\partial_z\mathcal{H}$. The first thing to note is that equation (5.66) in this gauge becomes

$$\partial_r\delta A_{\bar{z}} = \partial_{\bar{z}}(\delta A_r - i\delta\Phi), \tag{5.73}$$

which implies that we can find a matrix function $\delta Q(r, z, \bar{z})$ such that

$$\begin{aligned}\delta A_r - i\delta\Phi &= \partial_r\delta Q\\ \delta A_{\bar{z}} &= \partial_{\bar{z}}\delta Q.\end{aligned} \tag{5.74}$$

The infinitesimal gauge transformation which trivialises these gauge fields is $(1 - \delta Q)$

$$\begin{aligned}\delta A_r - i\delta\Phi &\to (1+\delta Q)(\delta A_r - i\delta\Phi)(1-\delta Q) + (1+\delta Q)\partial_r(1-\delta Q)\\ &\sim \delta A_r - i\delta\Phi - \partial_r\delta Q = 0.\end{aligned} \tag{5.75}$$

We require $\delta Q(r=0) = \mathbf{0}$ if we are to end up in the Jarvis gauge.

Using the fact that $A_r = i\Phi$, (5.64) gives

$$2iD_z\Phi = \partial_r A_z, \tag{5.76}$$

so we have the following covariant derivatives of the Higgs field in this gauge:

$$D_r\Phi = \partial_r\Phi \qquad 2iD_z\Phi = \partial_r A_z \qquad D_{\bar{z}}\Phi = \partial_{\bar{z}}\Phi. \tag{5.77}$$

Equation (5.65) also implies that

$$\frac{2ir^2}{(1+z\bar{z})^2}\partial_r\Phi = -\partial_{\bar{z}}A_z. \tag{5.78}$$

Using these expressions we find that, in the Jarvis gauge,

$$\begin{aligned} \delta A_r - i\delta\Phi &= i\frac{1-z\bar{z}}{1+z\bar{z}}\partial_r\Phi - \frac{z}{r}\partial_r A_z \\ \delta A_{\bar{z}} &= i\frac{1-z\bar{z}}{1+z\bar{z}}\partial_{\bar{z}}\Phi - \frac{z}{r}\partial_{\bar{z}}A_z. \end{aligned} \tag{5.79}$$

Now we can find the function Q defined in (5.74). Integrating the first equation with the boundary condition $\delta Q(r=0) = \mathbf{0}$ we obtain

$$\delta Q = \left[i\frac{1-z\bar{z}}{1+z\bar{z}}\Phi - \frac{z}{r}A_z\right]_0^r - \int_0^r dr\,\frac{z}{r^2}A_z. \tag{5.80}$$

Differentiating with respect to $\bar{z}$ and using (5.78) gives

$$\begin{aligned} \partial_{\bar{z}}\delta Q &= \left[i\frac{1-z\bar{z}}{1+z\bar{z}}\partial_{\bar{z}}\Phi - \frac{2iz}{(1+z\bar{z})^2}\Phi - \frac{z}{r}\partial_{\bar{z}}A_z\right]_0^r + \int_0^r dr\,\frac{2iz}{(1+z\bar{z})^2}\partial_r\Phi \\ &= \left[i\frac{1-z\bar{z}}{1+z\bar{z}}\partial_{\bar{z}}\Phi - \frac{z}{r}\partial_{\bar{z}}A_z\right]_{0.}^r \end{aligned} \tag{5.81}$$

The fact that $\mathcal{H}(\mathbf{0}) = \mathbb{I}$ means that $\partial_z\Phi(\mathbf{0}) = 0$ and $A_z(\mathbf{0}) = 0$. In other words, Φ and A_z are well-defined at the origin. Thus $\partial_{\bar{z}}\delta Q = \delta A_{\bar{z}}$ as required.

The metric corresponding to the translated monopole is

$$\tilde{\mathcal{H}} = (1-\delta Q^\dagger)\mathcal{H}(1-\delta Q), \tag{5.82}$$

from which we obtain the Higgs field

$$\begin{aligned} \tilde{\Phi} &= -\frac{i}{2}\tilde{\mathcal{H}}^{-1}\partial_r\tilde{\mathcal{H}} \\ &= -\frac{i}{2}\left(\mathcal{H}^{-1}\partial_r\mathcal{H} - \mathcal{H}^{-1}\partial_r\delta Q^\dagger\mathcal{H} - [\mathcal{H}^{-1}\partial_r\mathcal{H}, \delta Q] - \partial_r\delta Q\right). \end{aligned} \tag{5.83}$$

In principle, this can be used to find the Jarvis rational map of the translated monopole.

The square of the modulus of the Higgs field is

$$\parallel \tilde{\Phi} \parallel^2 = -\frac{1}{2}\mathrm{tr}(\tilde{\Phi}^2) \sim \parallel \Phi \parallel^2 + \frac{1}{4}\mathrm{tr}(\Phi\,\partial_r\delta Q + (\Phi\,\partial_r\delta Q)^\dagger). \tag{5.84}$$

5.4.2 An Example

We will apply the above technique to the case of the one monopole centred at the origin and find the infinitesimal change in the rational map corresponding to translation in the x_3 direction. We find that the answer is in agreement with the rational map found in Chapter 4.

We work in the Jarvis gauge with the metric in the form

$$\mathcal{H} = \frac{2r}{\sinh 2r}\mathbb{P} + \frac{\sinh 2r}{2r}(\mathbb{I} - \mathbb{P}) \qquad \text{where} \qquad \mathbb{P} = \frac{1}{1+z\bar{z}}\begin{pmatrix}1\\ z\end{pmatrix}\begin{pmatrix}1 & \bar{z}\end{pmatrix}. \tag{5.85}$$

The Higgs field and gauge field A_z are

$$\begin{aligned}\Phi &= -\frac{i}{2}\mathcal{H}^{-1}\partial_r\mathcal{H} = -\frac{i}{2}\left(\frac{1}{r} - 2\coth 2r\right)(2\mathbb{P} - \mathbb{I})\\ A_z &= \mathcal{H}^{-1}\partial_z\mathcal{H} = \left(\frac{4r^2}{\sinh^2 2r} - 1\right)\partial_z\mathbb{P}.\end{aligned} \tag{5.86}$$

The infinitesimal gauge transformation which trivialises $\delta A_r - i\delta\Phi$ and $\delta A_{\bar{z}}$ is found to be

$$\delta Q = \frac{1}{2}\left(\frac{1}{r} - 2\coth 2r\right)\frac{z\bar{z}-1}{1+z\bar{z}}(2\mathbb{P} - \mathbb{I}) + \left(\frac{4r}{\sinh^2 2r} - 2\coth 2r\right) z\,\partial_z\mathbb{P}. \tag{5.87}$$

Note that $\delta Q(\mathbf{0}) = 0$ so $\tilde{\mathcal{H}} = \mathcal{H} - \varepsilon(\delta Q^\dagger\mathcal{H} - \mathcal{H}\delta Q)$ satisfies $\tilde{\mathcal{H}}(\mathbf{0}) = \mathbb{I}$ and is in the Jarvis gauge.

As $r \to \infty$ we have

$$\delta Q \to \frac{1-z\bar{z}}{1+z\bar{z}}(2\mathbb{P} - \mathbb{I}) - 2z\,\partial_z\mathbb{P} \qquad \partial_r\delta Q = \delta A_r - i\delta\Phi \to 0. \tag{5.88}$$

Therefore

$$\delta\Phi^\infty = -[\Phi^\infty, \delta Q^\infty] = 2iz[2\mathbb{P} - \mathbb{I}, \partial_z\mathbb{P}] = -4iz\,\partial_z\mathbb{P}. \tag{5.89}$$

Recall that the rational map is determined by the eigenvector of $-i\Phi^\infty$ with eigenvalue $+1$. Now $(-\bar{z}, 1)^t$ is an eigenvector with eigenvalue -1 and the eigenvector with eigenvalue $+1$ is

$$\begin{pmatrix}1\\ z\end{pmatrix} + \frac{2\varepsilon z}{1+z\bar{z}}\begin{pmatrix}-\bar{z}\\ 1\end{pmatrix}. \tag{5.90}$$

Thus the rational map is

$$\left(z + \frac{2\varepsilon z}{1+z\bar{z}}\right)\left(1 + \frac{2\varepsilon z\bar{z}}{1+z\bar{z}}\right) \sim z(1+2\varepsilon). \tag{5.91}$$

The Jarvis rational map of a monopole with position $(0, 0, A)$ is in the $SU(2)$ orbit of $e^{2A}z$. Therefore $(1 + 2\varepsilon)z$ corresponds to the rational map of a monopole with position $(0, 0, \varepsilon)$ as hoped.

5.4.3 Comments

We have shown that it is possible to perform an infinitesimal translation on a solution to the Jarvis equation without recourse to the linear system. Interestingly, Belavin and Zahkarov's derivation of the linear system [43] makes use of the Dirac equation (5.59), so presumably there is a direct way to relate zero modes to solutions of the linear system.

5.5 The Centre of a Monopole

Every multi-monopole has a well-defined centre, which is a vector in $\mathbb{R}^3$ [32]. The centre of a monopole is defined from the asymptotic expansion of the length of the Higgs field which was shown by Hurtubise to be harmonic [42] and is therefore of the form

$$\parallel \Phi \parallel = 1 - \frac{n}{2r} + \frac{a^i x^i}{r^3} + O(r^{-3}). \tag{5.92}$$

By comparing with the harmonic function

$$-\frac{n}{2|\mathbf{x}-\mathbf{A}|} = -\frac{n}{2r} - \frac{nA^i x^i}{2r^3} + O(r^{-3}), \tag{5.93}$$

we are led to define the centre as

$$A^i = -\frac{2a^i}{n}. \tag{5.94}$$

Rotations have a well-defined action on the Jarvis rational map and so it is possible to look for quantities defined in terms of this map which are real 3-vectors under rotations of the coordinate z. Furthermore, this vector should be a scalar under the residual Möbius gauge action, since this corresponds to a gauge transformation of the fields of the monopole. A real 3-vector satisfying this will be a candidate for the centre of the monopole. There will also be real scalar quantities which we can associate to the Jarvis map, and the length of the 3-vector will be a function of these.

Recall that a rational map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ is a map of the form

$$f(z) = \frac{q(z)}{p(z)} \tag{5.95}$$

where p and q are polynomials and z is an inhomogeneous complex coordinate on $\mathbb{C}P^1$. If p and q have no common (non-constant) factor then the degree of the map is the maximum of the degrees of p and q. What we mean by the degree of a polynomial $p(z)$ is rather ambiguous since the polynomial 1 can be thought of as a root at infinity, therefore to avoid this ambiguity we will work with homogeneous coordinates x and y such that $z = x/y$. Multiplying the top and bottom by y^n where n is the degree gives us the rational map in terms of homogeneous polynomials

$$f(x,y) = \frac{q(x,y)}{p(x,y)}, \tag{5.96}$$

where the degrees of p and q are now explicitly both n.

Suppose we have a homogeneous polynomial $p(x,y)$ of degree n. Then we can define an action of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \tag{5.97}$$

as follows

$$p(x,y) \to p(x',y') = p(ax+by, cx+dy) \tag{5.98}$$

In terms of the coefficients of the polynomial this gives us an irreducible $\underline{n+1}$-dimensional representation of $SL(2,\mathbb{C})$ [52].

The generators of this action are

$$\begin{aligned} T^3 &= \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) \\ T^+ &= x\frac{\partial}{\partial y} \\ T^- &= y\frac{\partial}{\partial x}, \end{aligned} \tag{5.99}$$

obeying the usual algebra

$$\begin{aligned} [\,T^3, T^+\,] &= T^+ \\ [\,T^3, T^-\,] &= -T^- \\ [\,T^+, T^-\,] &= 2T^3. \end{aligned} \tag{5.100}$$

The eigenfunctions of T^3 are the polynomials $p^n_m = x^m y^{n-m}$ which satisfy

$$\begin{aligned} T^+ p^n_m &= (n-m)p^n_{m+1} \\ T^- p^n_m &= m p^n_{m-1} \\ T^3 p^n_m &= \left(m - \frac{n}{2}\right) p^n_m. \end{aligned} \tag{5.101}$$

The degree is given by the eigenvalue of the operator $x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ which commutes with the T^i

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)p^n_m = np^n_m. \tag{5.102}$$

Given two homogeneous polynomials $p(x,y)$ and $q(x,y)$, we can construct the $SL(2,\mathbb{C})$ tensor $\tilde{f}(x_1,y_1,x_2,y_2)=p(x_1,y_1)\,q(x_2,y_2)$. To decompose this into vector representations, we use the operator

$$D_{12}=\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2}-\frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}, \tag{5.103}$$

which is invariant under simultaneous $SL(2,\mathbb{C})$ transformations of (x_1,y_1) and (x_2,y_2). Setting

$$\tilde{f}_n(x_1,y_1,x_2,y_2)=(D_{12})^n p(x_1,y_1)q(x_2,y_2), \tag{5.104}$$

we obtain vectors by taking $f_n(x,y)=\tilde{f}_n(x,y,x,y)$. Notice that each application of D_{12} reduces the degree of the resulting polynomial by 2 so we see that we obtain the standard decomposition in this way.

In addition to considering holomorphic polynomials $p(x,y)$, we can also consider antiholomorphic polynomials $\bar{q}(\bar{x},\bar{y})$. These correspond to vectors transforming in the conjugate representation. In general we can consider polynomials which are functions of both x,y and $\bar{x},\bar{y}$ and we will denote the degree by (m,n) where m and n are the holomorphic and antiholomorphic degrees respectively. The basic tool for decomposing products of holomorphic and antiholomorphic polynomials is the $SU(2)$ invariant operator

$$D_{1\bar{2}}=\frac{\partial}{\partial x_1}\frac{\partial}{\partial \bar{x}_2}+\frac{\partial}{\partial y_1}\frac{\partial}{\partial \bar{y}_2}. \tag{5.105}$$

The $SU(2)$ gauge action on the polynomials $p(x,y)$ and $q(x,y)$ is

$$\begin{pmatrix} p(x,y)\\ q(x,y)\end{pmatrix}\to\begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\begin{pmatrix} p(x,y)\\ q(x,y)\end{pmatrix}. \tag{5.106}$$

There are two quadratic quantities we can consider that are invariant under this transformation

$$Q_{12}=p(x_1,y_1)q(x_2,y_2)-q(x_1,y_1)p(x_2,y_2) \tag{5.107}$$
$$Q_{1\bar{2}}=p(x_1,y_1)\bar{p}(\bar{x}_2,\bar{y}_2)+q(x_1,y_1)\bar{q}(\bar{x}_2,\bar{y}_2) \tag{5.108}$$

The first is actually invariant under an $SL(2,\mathbb{C})$ transformation and the second is the Hermitian scalar product preserved by $SU(2)$. We obtain vector quantities by taking

$(D_{12})^m Q_{12}$ and $(D_{1\bar{2}})^m Q_{1\bar{2}}$ and then setting $x_1 = x_2 = x$ etc. Note that the antisymmetric nature of Q_{12} and D_{12} means that only odd powers of D_{12} give us non-zero vectors.

A real $SU(2)$ 3-vector is related to a vector in $\mathbb{R}^3$ by

$$(x_1, x_2, x_3) = \left(\frac{x\bar{y} + y\bar{x}}{x\bar{x} + y\bar{y}}, -i\frac{x\bar{y} - y\bar{x}}{x\bar{x} + y\bar{y}}, \frac{x\bar{x} - y\bar{y}}{x\bar{x} + y\bar{y}} \right). \tag{5.109}$$

We shall look at rational maps of degree 1 and 2 as examples. This will also tie in with the work on two-lumps in Chapter 3.

5.5.1 Degree 1 Maps

A degree 1 rational map is of the form

$$\frac{p(x,y)}{q(x,y)} = \frac{ax + by}{cx + dy}. \tag{5.110}$$

The two gauge invariant quantities (5.107) and (5.108) are

$$\begin{gathered}(ad - bc)(x_1 y_2 - y_1 x_2) \\ (a\bar{a} + c\bar{c})x_1\bar{x}_2 + (a\bar{b} + c\bar{d})x_1\bar{y}_2 + (b\bar{a} + d\bar{c})y_1\bar{x}_2 + (b\bar{b} + d\bar{d})y_1\bar{y}_2 \,. \end{gathered} \tag{5.111}$$

Applying D_{12} once to the first quantity, we obtain the $SL(2, \mathbb{C})$ invariant $2(ad - bc)$. The second quantity is reducible and we can decompose it into a vector

$$\frac{a\bar{a} - b\bar{b} + c\bar{c} - d\bar{d}}{2}(x\bar{x} - y\bar{y}) + \frac{a\bar{b} + b\bar{a} + c\bar{d} + d\bar{c}}{2}(x\bar{y} + y\bar{x}) + \frac{a\bar{b} - b\bar{a} + c\bar{d} - d\bar{c}}{2}(x\bar{y} - y\bar{x}), \tag{5.112}$$

and a scalar $\frac{1}{2}(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d})(x\bar{x} + y\bar{y})$. Dividing through by this scalar, we obtain the $SO(3)$ vector

$$\left(\frac{a\bar{b} + b\bar{a} + c\bar{d} + d\bar{c}}{a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}}, -\frac{i(a\bar{b} - b\bar{a} + c\bar{d} - d\bar{c})}{a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}}, \frac{a\bar{a} - b\bar{b} + c\bar{c} - d\bar{d}}{a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}} \right). \tag{5.113}$$

This vector is invariant under a rescaling of the polynomials p and q which leaves the rational map unchanged.

Using this, the rational map of a monopole at $\mathbf{A}$, $f(z) = e^{-2A}(z - W)/(1 + z\overline{W})$ gives rise to the vector

$$\frac{\tanh 2A}{2} \left(\frac{W + \overline{W}}{1 + W\overline{W}}, -i\frac{W - \overline{W}}{1 + W\overline{W}}, \frac{W\overline{W} - 1}{1 + W\overline{W}} \right), \tag{5.114}$$

which correctly describes the line on which the centre of the monopole lies.

We can also define the scalar

$$\frac{(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d})}{(ad - bc)(\bar{a}\bar{d} - \bar{b}\bar{c})}, \tag{5.115}$$

which is invariant under a rescaling of the numerator and denominator of the rational map.

If we write a degree 1 map as an $SL(2, \mathbb{C})$ matrix which we decompose as the product of a unitary and a Hermitian unimodular matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \Lambda + \lambda^3 & \lambda^1 - i\lambda^2 \\ \lambda^1 + i\lambda^2 & \Lambda - \lambda^3 \end{pmatrix}, \tag{5.116}$$

where $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ and $\Lambda = \sqrt{1 + \lambda^i\lambda^i}$, then we obtain the scalar

$$s = 1 + 2\lambda^i\lambda^i, \tag{5.117}$$

and the vector

$$v = \frac{2\Lambda}{s}\left(\lambda^1, \lambda^2, \lambda^3\right). \tag{5.118}$$

5.5.2 Degree 2 Maps

We can take the general form of a degree two map given in Chapter 3.

$$A = UH\frac{1}{\sqrt{2}}\begin{pmatrix} -i & -\sec\zeta & -i\tan\zeta \\ i & -\sec\zeta & -i\tan\zeta \end{pmatrix} R^t, \tag{5.119}$$

where $U \in SU(2)$, H is a unimodular Hermitian matrix of the same form as above, ζ is a real parameter, and $R \in SO(3)$.

This describes the map with respect to the basis of degree two homogeneous polynomials

$$l_1 = \frac{i}{2}(x^2 - y^2) \qquad l_2 = -\frac{1}{2}(x^2 + y^2) \qquad l_3 = -ixy. \tag{5.120}$$

Writing

$$p = A_{11}l_1 + A_{12}l_2 + A_{13}l_3 \qquad q = A_{21}l_1 + A_{22}l_2 + A_{23}l_3, \tag{5.121}$$

we decompose $p\bar{p} + q\bar{q}$ using $D_{1\bar{2}}$ to obtain a vector

$$v^t = \frac{i}{s}\begin{pmatrix} A_{13}\bar{A}_{12} - A_{12}\bar{A}_{13} + A_{23}\bar{A}_{22} - A_{22}\bar{A}_{23} \\ A_{11}\bar{A}_{13} - A_{13}\bar{A}_{11} + A_{21}\bar{A}_{23} - A_{23}\bar{A}_{21} \\ A_{12}\bar{A}_{11} - A_{11}\bar{A}_{12} + A_{22}\bar{A}_{21} - A_{21}\bar{A}_{22} \end{pmatrix}, \tag{5.122}$$

where we divide through by the scalar quantity

$$A_{11}\bar{A}_{11} + A_{12}\bar{A}_{12} + A_{13}\bar{A}_{13} + A_{21}\bar{A}_{21} + A_{22}\bar{A}_{22} + A_{23}\bar{A}_{23}, \tag{5.123}$$

in order to obtain a vector which is invariant under a rescaling of p and q.

Substituting in A given by (5.119) gives the vector

$$v = \frac{4\sec^2\zeta}{s}\left((1 + 2\lambda^i\lambda^i + 2\Lambda\lambda^1)\sin\zeta,\ -2\Lambda\lambda^2\sin\zeta\cos\zeta,\ 2\Lambda\lambda^3\cos\zeta\right)R^t, \tag{5.124}$$

where

$$s = 4\sec^2\zeta(1 + 2\lambda^i\lambda^i + 2\Lambda\lambda^1\sin^2\zeta). \tag{5.125}$$

The parameter space of two-lumps describing configurations fixed by an antipodal map studied in Chapter 3 had $\zeta = 0$ and

$$H = \begin{pmatrix} \cosh\alpha/2 & \sinh\alpha/2 \\ \sinh\alpha/2 & \cosh\alpha/2 \end{pmatrix}, \tag{5.126}$$

which corresponds to $(\lambda_1,\ \lambda_2,\ \lambda_3) = (\sinh\alpha/2, 0, 0)$. In this case, $v = \mathbf{0}$, which suggests that such Jarvis rational maps describe monopoles whose centre is at the origin. These are known as *centred* monopoles.

We can also obtain the complex $SL(2,\mathbb{C})$ vector

$$(0, 2\tan\zeta, 2i\sec\zeta)R^t, \tag{5.127}$$

from this map. By taking i times the vector product with its complex conjugate we obtain the real $SO(3)$ vector

$$(4\sec\zeta\tan\zeta, 0, 0)R^t, \tag{5.128}$$

which is a scalar under a "gauge" $SL(2,\mathbb{C})$ Möbius action. This vector cannot describe the centre of the monopole since it would imply a space of centred monopoles with too many dimensions.

5.5.3 Comments

In general, there will be many real $SO(3)$ 3-vectors which we can associate to a given rational map. However, we can impose the restriction that the space of centred monopoles has the correct dimension. Whether this will uniquely specify a 3-vector is not yet clear. In any case, without an argument fixing the length of the vector, the best we can do is to determine the line through the origin on which the centre of the monopole lies.

Chapter 6

One Monopole Inverse Scattering

In Chapter 4 we succeeded in finding the solution to the linear system in the Jarvis gauge corresponding to a single monopole with arbitrary position in $\mathbb{R}^3$. To do this, we required the solution to the linear system corresponding to a monopole at some position, which we obtained from the explicit BPS monopole solution in the Jarvis gauge. It is difficult to see how this method could be generalised to multi-monopoles, since explicit descriptions of the fields of multi-monopoles are few and far between. However, there is a method for generating solutions of the linear system which has been successfully used to find multi-monopole solutions, the *inverse scattering method* [44–46].

This method consists of generating new solutions starting from a "seed" solution of the linear system which corresponds to a vacuum solution of the Bogomol'nyi equations. Viewing a solution as a meromorphic function of the parameter λ, solutions with higher topological charge are obtained by multiplying the seed solution by factors involving poles which are functions of the spatial variables, and projectors which may be related to the seed solution. The difficulty in using the method lies in the fact that there is considerable arbitrariness in choosing the seed solution, the poles, and the projectors, before imposing the restriction that the solution be regular.

What we would like to be able to do is to find a solution to the linear system, and hence a solution to the Jarvis equation, given the Jarvis rational map as initial data. The aim of this chapter, then, is to make a start on this programme by comparing the inverse scattering method for a single monopole with the solution to the linear system obtained in Chapter 4. We can "derive" the one-monopole inverse scattering ansatz from this solution and, in this way, find the data needed to obtain a smooth solution in terms of the rational map. Interestingly, it turns out that the Jarvis rational map and the dual rational map introduced in the previous chapter will make an explicit appearance in the seed solution.

6.1 The One-Monopole Inverse Scattering Ansatz

We shall begin by describing the one monopole inverse scattering ansatz in some detail, expanding on the argument presented by Forgács, Horváth and Palla in [46].

Let

$$\rho = \frac{1}{2}(x_1 + ix_2) \qquad t = x_3. \tag{6.1}$$

The linear system (4.4) is then

$$\begin{aligned} (\lambda(D_t + i\Phi) + D_\rho)\,\Psi(\lambda, \mathbf{x}) &= 0 \\ (-\lambda D_{\bar\rho} + (D_t - i\Phi))\,\Psi(\lambda, \mathbf{x}) &= 0. \end{aligned} \tag{6.2}$$

We can transform to a gauge in which the commutator $[D_t + i\Phi, D_{\bar\rho}]$ is trivialised, and rewrite the linear system in this gauge as follows

$$\begin{aligned} (\lambda\partial_t + \partial_\rho)\Psi \cdot \Psi^{-1} &= -A_\rho \\ (-\lambda\partial_{\bar\rho} + \partial_t)\Psi \cdot \Psi^{-1} &= -A_t + i\Phi. \end{aligned} \tag{6.3}$$

This is what we might call the Donaldson linear system. We will present the argument in this gauge and then use it to obtain the ansatz for the Jarvis linear system.

If we define

$$g = \Psi(\lambda = 0), \tag{6.4}$$

then evaluating (6.3) when $\lambda = 0$ gives

$$A_r = -\partial_\rho g \cdot g^{-1} \tag{6.5}$$

$$A_t - i\Phi = -\partial_t g \cdot g^{-1} \tag{6.6}$$

The method of Forgács et al. requires a known "seed" solution to the linear system, which is a solution Ψ_0 satisfying

$$\begin{aligned} (\lambda\partial_t + \partial_\rho)\Psi_0 \cdot \Psi_0^{-1} &= -A_\rho^0 \\ (-\lambda\partial_{\bar\rho} + \partial_t)\Psi_0 \cdot \Psi_0^{-1} &= -A_t^0 + i\Phi^0. \end{aligned} \tag{6.7}$$

The seed solution normally corresponds to a vacuum solution of the Bogomol'nyi equations. We then look for a new solution of the form

$$\Psi(\lambda) = \chi(\lambda)\Psi_0(\lambda). \tag{6.8}$$

Substituting this into (6.3) gives

$$\begin{aligned}(\lambda\partial_t + \partial_\rho)\chi \cdot \chi^{-1} - \chi A^0_\rho \chi^{-1} &= -A_\rho \\ (-\lambda\partial_{\bar\rho} + \partial_t)\chi \cdot \chi^{-1} - \chi(A^0_t + i\Phi^0)\chi^{-1} &= -A_t + i\Phi. \end{aligned} \tag{6.9}$$

A priori the left hand side of this system is meromorphic in the complex variable λ while the right hand side is required to be independent of λ. Therefore the method consists of choosing $\chi(\lambda)$ in such a way that the left hand side is independent of λ, giving rise to a new solution for A_ρ and $A_t + i\Phi$.

For a charge one monopole we make the following choice for $\chi(\lambda)$, the so-called one pole ansatz

$$\chi(\lambda) = \mathbb{I} + \frac{R}{\lambda - \mu}, \tag{6.10}$$

where R is a matrix depending on ρ, $\bar\rho$ and t but independent of λ. In addition we assume that χ^{-1} has a similar form

$$\chi(\lambda)^{-1} = \mathbb{I} + \frac{S}{\lambda - \nu}, \tag{6.11}$$

where, again, S is a matrix function independent of λ. This implies that R and S are proportional to a projector as we will now see.

By Liouville's theorem it is sufficient to check that $\chi(\lambda)\chi(\lambda)^{-1} = \mathbb{I}$ has no poles. Taking residues of $\chi(\lambda)\chi(\lambda)^{-1} = \mathbb{I}$ at $\lambda = \mu, \nu$ gives

$$\begin{aligned} R &= \frac{RS}{\nu - \mu} \\ S &= \frac{RS}{\mu - \nu}, \end{aligned} \tag{6.12}$$

from which we see that $S = -R$ and

$$R = -\frac{R^2}{\nu - \mu}. \tag{6.13}$$

Defining

$$P = \frac{R}{\mu - \nu}, \tag{6.14}$$

we see that P is a projector and therefore χ and χ^{-1} are of the form

$$\begin{aligned} \chi(\lambda) &= 1 + \frac{\mu - \nu}{\lambda - \mu}P = (\mathbb{I} - P) + \frac{\lambda - \nu}{\lambda - \mu}P \\ \chi(\lambda)^{-1} &= 1 + \frac{\nu - \mu}{\lambda - \nu}P = (\mathbb{I} - P) + \frac{\lambda - \mu}{\lambda - \nu}P. \end{aligned} \tag{6.15}$$

In the $SU(2)$ case, P is a rank 1 projector so we can write it in the form

$$P = nm^\dagger \qquad \text{where} \qquad m^\dagger n = 1. \tag{6.16}$$

Defining $m_\perp$ and $n_\perp$ as orthogonal vectors satisfying $m^\dagger m_\perp = n_\perp^\dagger n = 0$ and normalised so that $n_\perp^\dagger m_\perp = 1$, we have the following basis for 2×2 matrices

$$\begin{aligned} P &= nm^\dagger & \mathbb{I} - P &= m_\perp n_\perp^\dagger \\ P_+ &= nn_\perp^\dagger & P_- &= m_\perp m^\dagger. \end{aligned} \tag{6.17}$$

They satisfy the following relations

$$\begin{aligned} PP_+ &= P_+ & P_+P &= 0 & P_+P_- &= P \\ PP_- &= 0 & P_-P &= P_- & P_-P_+ &= \mathbb{I} - P. \end{aligned} \tag{6.18}$$

If we think of $(2P - \mathbb{I})$ as being the Cartan subalgebra generator then P_+ and P_- are step operators.

Using these relations and the fact that P and $(\mathbb{I} - P)$ have unit trace, whilst P_+ and P_- are traceless, we can expand out an arbitary matrix M in this basis as follows

$$\begin{aligned} M &= \operatorname{tr}(PM)P + \operatorname{tr}((\mathbb{I} - P)M)(\mathbb{I} - P) + \operatorname{tr}(P_-M)P_+ + \operatorname{tr}(P_+M)P_- \\ &= (m^\dagger Mn)P + (n_\perp^\dagger Mm_\perp)(\mathbb{I} - P) + (m^\dagger Mm_\perp)P_+ + (n_\perp^\dagger Mn)P_-. \end{aligned} \tag{6.19}$$

We can use this to find an expression for a derivative of P in this basis. Using the fact that $\partial(m^\dagger n) = 0$ we find

$$\partial P = (\partial m^\dagger)m_\perp P_+ + n_\perp^\dagger(\partial n)P_-. \tag{6.20}$$

We are now in a position to evaluate the left hand side of (6.9) for χ given by equation (6.15). Firstly

$$\partial^\lambda \chi(\lambda) = \frac{(\lambda - \nu)\partial^\lambda \mu}{(\lambda - \mu)^2} P - \frac{\partial^\lambda \nu}{\lambda - \mu} P + \frac{\mu - \nu}{\lambda - \mu}\left((\partial m^\dagger)m_\perp P_+ + n_\perp^\dagger(\partial n)P_-\right), \tag{6.21}$$

where ∂^λ denotes $\lambda\partial_t + \partial_\rho$ or $-\lambda\partial_{\bar\rho} + \partial_t$. Now using (6.19) we expand $A^0 = A^0_\rho$ or $A^0_t + i\Phi^0$ as

$$A^0 = (m^\dagger A^0 n)P + (n_\perp^\dagger A^0 m_\perp)(\mathbb{I} - P) + (m^\dagger A^0 m_\perp)P_+ + (n_\perp^\dagger A^0 n)P_-. \tag{6.22}$$

The left hand side of equation (6.9) becomes

$$\begin{aligned} &\frac{\partial^\lambda \mu}{\lambda - \mu} P - \frac{\partial^\lambda \nu}{\lambda - \nu} P + (\mu - \nu)\left(\frac{(\partial^\lambda m^\dagger)m_\perp}{\lambda - \mu} P_+ + \frac{n_\perp^\dagger(\partial^\lambda n)}{\lambda - \nu} P_-\right) \\ &- (m^\dagger A^0 n)P - (n_\perp^\dagger A^0 m_\perp)(\mathbb{I} - P) - \frac{\lambda - \nu}{\lambda - \mu}(m^\dagger A^0 m_\perp)P_+ - \frac{\lambda - \mu}{\lambda - \nu}(n_\perp^\dagger A^0 n)P_-. \end{aligned} \tag{6.23}$$

The requirement that this has no poles at $\lambda = \mu$ and $\lambda = \nu$, or equivalently that the residues vanish at these points, for the different coefficients of P, P_+, P_- gives us 4 equations

$$\partial^\mu \mu = 0 \tag{6.24}$$
$$\partial^\nu \nu = 0 \tag{6.25}$$
$$\left(\partial^\mu m^\dagger - m^\dagger A^0\right) m_\perp = 0 \tag{6.26}$$
$$n_\perp^\dagger \left(\partial^\nu n + A^0 n\right) = 0. \tag{6.27}$$

We begin with the first equation (6.24) which, written out in full, is

$$\begin{aligned} \mu\partial_t\mu + \partial_\rho\mu &= 0 \\ -\mu\partial_{\bar\rho}\mu + \partial_t\mu &= 0. \end{aligned} \tag{6.28}$$

We solve these equations using the idea of characteristic curves. Taking a linear combination of these equations

$$\left((a\mu + b)\partial_t + a\partial_\rho - b\mu\partial_{\bar\rho}\right)\mu = 0 \tag{6.29}$$

where a and b are arbitrary functions of $t, \rho, \bar\rho$ we define a curve with parameter s as follows

$$\frac{dt}{ds} = a\mu + b \qquad \frac{d\rho}{ds} = a \qquad \frac{d\bar\rho}{ds} = -b\mu. \tag{6.30}$$

Equation (6.29) then implies that

$$\frac{d\mu}{ds} = 0, \tag{6.31}$$

so μ is constant along the curve. By taking a linear combination of the equations in (6.30) we can also show that

$$\frac{d}{ds}\gamma(\mu, \mathbf{x}) = 0, \tag{6.32}$$

where $\gamma(\mu, \mathbf{x}) = 2(\mu^2\rho - \mu t - \bar\rho)$ is again the function introduced in Chapter 4. Hence $\gamma(\mu, \mathbf{x})$ is also constant along the curve. Surfaces on which $\gamma(\mu, \mathbf{x})$ and μ are constant are simply straight lines in $\mathbb{R}^3$, corresponding to points $(\gamma(\mu), \mu)$ in the mini-twistor space $\mathbb{TT}$. The general solution to the equations (6.28) is

$$h(\gamma(\mu, \mathbf{x}), \mu) = 0, \tag{6.33}$$

where h is any sufficiently nice function. Similarly, the general solution to the second equation (6.25) is $\tilde{h}(\gamma(\nu),\nu)=0$ for some function $\tilde{h}$.

To solve (6.26) and (6.27) for the vectors $m^\dagger$ and n we substitute

$$\begin{aligned} m^\dagger &= M^\dagger\Psi_0^{-1}(\mu) \qquad & n &= \Psi_0(\nu)N \\ m_\perp &= \Psi_0(\mu)M_\perp \qquad & n_\perp^\dagger &= N_\perp^\dagger\Psi_0^{-1}(\nu), \end{aligned} \tag{6.34}$$

obtaining the equations

$$\partial^\mu M^\dagger \cdot M_\perp = 0 \qquad N_\perp^\dagger \cdot \partial^\nu N = 0. \tag{6.35}$$

These imply that

$$\partial^\mu M^\dagger = aM^\dagger \qquad \partial^\nu N = bN, \tag{6.36}$$

for some functions a and b. Given a solution to (6.36), we can rescale $M^\dagger$ which will have the effect of changing the function a. Therefore we can take $M^\dagger = \begin{pmatrix}1 & f\end{pmatrix}$, which must then satisfy

$$\partial^\mu M^\dagger = 0, \tag{6.37}$$

and this implies that $f = f(\gamma(\mu),\mu)$.

Similarly, we can write

$$N = c\begin{pmatrix}1\\ g\end{pmatrix}, \tag{6.38}$$

and equation (6.27) then implies that $g = g(\gamma(\nu),\nu)$. Requiring $m^\dagger n$ to equal 1 then gives

$$c = \frac{1}{M^\dagger\Psi_0^{-1}(\mu)\Psi_0(\nu)N}. \tag{6.39}$$

So far the solution has not been required to have unit determinant, and therefore the gauge fields obtained are not necessarily traceless. Here we will show that we can scale $\chi(\lambda)$ so that it has unit determinant. From (6.15) we see that

$$\det\chi = \frac{\lambda-\nu}{\lambda-\mu}. \tag{6.40}$$

Now we can show that

$$\begin{aligned} (\lambda\partial_t + \partial_\rho)\left(\frac{\lambda-\nu}{\lambda-\mu}\right) &= \left(\frac{\lambda-\nu}{\lambda-\mu}\right)\left(\frac{\partial_\rho\nu}{\nu} - \frac{\partial_\rho\mu}{\mu}\right) \\ (-\lambda\partial_{\bar\rho} + \partial_t)\left(\frac{\lambda-\nu}{\lambda-\mu}\right) &= \left(\frac{\lambda-\nu}{\lambda-\mu}\right)\left(\frac{\partial_t\nu}{\nu} - \frac{\partial_t\mu}{\mu}\right), \end{aligned} \tag{6.41}$$

and therefore

$$\left(\frac{\lambda-\mu}{\lambda-\nu}\right)^{\frac{1}{2}}\Psi(\lambda), \tag{6.42}$$

is a solution of the linear system with determinant equal to that of $\Psi_0(\lambda)$.

6.1.1 The One-Monopole Ansatz for the Jarvis Linear System

We can easily obtain an ansatz for the Jarvis linear system from that for the Donaldson linear system using the technique of Chapter 4. The ansatz for the Donaldson system (6.3) is of the form

$$\Psi(\lambda) = \chi(\lambda)\Psi_0(\lambda) = \left(\frac{\lambda-\mu}{\lambda-\nu}\right)^{\frac{1}{2}} \left[(\mathbb{I}-P) + \frac{\lambda-\nu}{\lambda-\mu}P\right] \Psi_0(\lambda). \tag{6.43}$$

To obtain an ansatz for the Jarvis linear system (4.36) we simply compute $\tilde{\Psi}(\lambda) = \Psi^{-1}(-1/z)\Psi(\lambda)$, obtaining

$$\tilde{\Psi}(\lambda) = \tilde{\chi}(\lambda)\tilde{\Psi}_0(\lambda) = \left(\frac{(1+\nu z)(\lambda-\mu)}{(1+\mu z)(\lambda-\nu)}\right)^{\frac{1}{2}} \left[(\mathbb{I}-\tilde{P}) + \frac{(1+\mu z)(\lambda-\nu)}{(1+\nu z)(\lambda-\mu)}\tilde{P}\right] \tilde{\Psi}_0(\lambda), \tag{6.44}$$

where

$$\tilde{P} = \Psi_0^{-1}(-1/z)P\Psi_0(-1/z) \qquad \tilde{\Psi}_0(\lambda) = \Psi_0^{-1}(-1/z)\Psi_0(\lambda). \tag{6.45}$$

Here $\tilde{P}$ is again a projector and $\tilde{\Psi}_0(\lambda)$ is a seed solution for the Jarvis linear system.

6.2 Comparison with the One-Monopole Solution

We will compare the solution we found in Chapter 4 with the ansatz given above and use it to determine the seed solution, the functions μ and ν and the vectors M and N which give rise to the general one-monopole.

Recall that the solution (4.131) was of the form $\Psi(\lambda) = \tilde{\mathcal{B}}^{-1}(-1/z)\tilde{\mathcal{D}}(\lambda)\tilde{\mathcal{B}}(\lambda)$ where

$$\begin{aligned}
\tilde{\mathcal{B}}(\lambda) &= \frac{1}{\sqrt{\sinh 2A}}\begin{pmatrix} (1+\lambda W)\sinh(A+R+r(2s-1)) & (\overline{W}-\lambda)\sinh(A-R-r(2s-1)) \\ -(1+\lambda W)\frac{\sinh(A-R+r(2s-1))}{\sinh 2R} & (\lambda-\overline{W})\frac{\sinh(A+R-r(2s-1))}{\sinh 2R} \end{pmatrix} \\
\tilde{\mathcal{D}}(\lambda) &= \frac{1}{z}\begin{pmatrix} \frac{z-Z}{1+\lambda Z} & 0 \\ 0 & -\frac{1+z\bar{Z}}{\lambda-\bar{Z}} \end{pmatrix} \\
\tilde{\mathcal{B}}^{-1}(-1/z) &= \frac{z}{\sqrt{\sinh 2A}}\begin{pmatrix} \frac{1}{z-W}\frac{\sinh(A+R+r)}{\sinh 2R} & \frac{1}{z-\overline{W}}\sinh(A-R+r) \\ -\frac{1}{1+zW}\frac{\sinh(A-R-r)}{\sinh 2R} & -\frac{1}{1+z\overline{W}}\sinh(A+R-r) \end{pmatrix}.
\end{aligned} \tag{6.46}$$

To compare this with the one-monopole ansatz, we rewrite the solution as follows

$$\tilde{\Psi}(\lambda) = \tilde{\mathcal{B}}^{-1}(-1/z)\frac{1}{z}\begin{pmatrix} \frac{z-Z}{1+\lambda Z} & 0 \\ 0 & -\frac{1+z\bar{Z}}{\lambda-\bar{Z}} \end{pmatrix}\tilde{\mathcal{B}}(-1/z)\cdot\tilde{\mathcal{B}}^{-1}(-1/z)\tilde{\mathcal{B}}(\lambda) \tag{6.47}$$

$$= \left[\frac{z-Z}{z(1+\lambda Z)}\tilde{P} - \frac{1+z\bar{Z}}{z(\lambda-\bar{Z})}(\mathbb{I}-\tilde{P})\right]\tilde{\Psi}_0(\lambda), \tag{6.48}$$

obtaining it in terms of the projector

$$\tilde{P} = \tilde{\mathcal{B}}^{-1}(-1/z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{\mathcal{B}}(-1/z), \tag{6.49}$$

and the seed solution

$$\tilde{\Psi}_0(\lambda) = \tilde{\mathcal{B}}^{-1}(-1/z)\tilde{\mathcal{B}}(\lambda). \tag{6.50}$$

6.2.1 The Seed Solution

Substituting the explicit expression for $\tilde{\mathcal{B}}(-1/z)$, we find that $\tilde{\Psi}_0$ can be written in the form

$$\tilde{\Psi}_0(\lambda) = \frac{z}{2\sinh 2A} \begin{pmatrix} \frac{e^A}{z-W} & \frac{e^{-A}}{z-W} \\ \frac{e^{-A}}{1+\overline{W}z} & \frac{e^A}{1+\overline{W}z} \end{pmatrix} \begin{pmatrix} e^{2rs} & 0 \\ 0 & -e^{-2rs} \end{pmatrix} \begin{pmatrix} e^A(1+\lambda W) & e^{-A}(\lambda - \overline{W}) \\ e^{-A}(1+\lambda W) & e^A(\lambda - \overline{W}) \end{pmatrix}, \tag{6.51}$$

where $s = (1+\lambda z)/(1+z\bar{z})$. Defining $H_0 = \tilde{\Psi}_0^{-1}(\bar{z})$, the corresponding Higgs field is

$$\Phi_0 = \frac{i}{2}\partial_r H_0 \cdot H_0^{-1} = \frac{i}{\sinh 2A} \begin{pmatrix} \cosh 2A & \frac{1+\overline{W}z}{W-z} \\ \frac{z-W}{1+\overline{W}z} & -\cosh 2A \end{pmatrix}, \tag{6.52}$$

satisfying $\| \Phi_0 \| = 1$. Note that H_0 is not hermitian. Now $-i\Phi_0$ has eigenvectors

$$\begin{pmatrix} e^{2A}(1+\overline{W}z) \\ z - W \end{pmatrix} \qquad \begin{pmatrix} e^{-2A}(1+\overline{W}z) \\ z - W \end{pmatrix}, \tag{6.53}$$

with eigenvalues $+1$ and -1 respectively, which correspond to the Jarvis rational map of the one-monopole $f(z) = e^{-2A}(z-W)/(1+\overline{W}z)$ and its dual.

Thus the seed solution for the one-monopole depends explicitly on the Jarvis rational map and its dual. Furthermore, since on the spectral line through the origin, the Jarvis map and the dual map are proportional, it degenerates on the lines $z = W, -1/\overline{W}$ and $\lambda = \overline{W}, -1/W$. The functions μ and ν and the projector $\tilde{P}$ must be chosen in such a way as to ensure that Ψ is well-defined when z and λ take these values.

6.2.2 The Poles

Comparing (6.48) with (6.44) we see that the functions μ and ν are

$$\mu = -\frac{1}{Z} \qquad \nu = \bar{Z}. \tag{6.54}$$

Now $\mu = -1/z$ and $\mu = \bar{z}$ are solutions to the equation

$$\gamma(\mu) = \frac{r(1+\mu z)(\mu - \bar{z})}{1 + z\bar{z}} = 0, \tag{6.55}$$

and similarly, $-1/Z$ and $\bar{Z}$ are the solutions to

$$h(\gamma(\mu), \mu) = \gamma(\mu) + \frac{A(1+\mu W)(\mu - \overline{W})}{1 + W\overline{W}} = 0, \tag{6.56}$$

which we recognise as the spectral curve of the one-monopole at $-\mathbf{A}$ described in the previous chapter.

6.2.3 The Projector

Above, we described the projector in terms of vectors $m^\dagger$ and n. Our solution satisfies $\tilde{\Psi}(\lambda, \mathbf{0}) = \mathbb{I}$, independent of λ. At the origin we find

$$\tilde{\Psi}_0(\lambda, \mathbf{0}) = \begin{pmatrix} \frac{z(1+\lambda W)}{z-W} & 0 \\ 0 & \frac{z(\lambda - \overline{W})}{1+\overline{W}z} \end{pmatrix}, \tag{6.57}$$

so comparing with $\tilde{\chi}$ we see that the projector must satisfy

$$\tilde{P}(\mathbf{0}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.58}$$

We can now calculate the vectors $M^\dagger$ and N defined in (6.34). Using

$$\tilde{P} = \tilde{\Psi}_0(-1/Z) N M^\dagger \tilde{\Psi}_0^{-1}(\bar{Z}) = \tilde{\mathcal{B}}^{-1}(-1/z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{\mathcal{B}}(-1/z), \tag{6.59}$$

and comparing with (6.49) we find

$$M^\dagger = \begin{pmatrix} 1 & 0 \end{pmatrix} \tilde{\mathcal{B}}(-1/Z) \qquad N = \tilde{\mathcal{B}}^{-1}(\bar{Z}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6.60}$$

We use

$$\begin{aligned} r(2s(\lambda) - 1) &= \frac{r}{1 + z\bar{z}}(1 - z\bar{z} + 2\lambda z) \\ &= \frac{R}{1 + Z\bar{Z}}(1 - Z\bar{Z} + 2\lambda Z) - \frac{A}{1 + W\overline{W}}(1 - W\overline{W} + 2\lambda W), \end{aligned} \tag{6.61}$$

giving

$$r(2s(-1/Z)-1) = -R - \frac{A}{1+W\overline{W}}\left(1 - W\overline{W} - 2\frac{W}{Z}\right)$$
$$r(2s(\bar{Z})-1) = R - \frac{A}{1+W\overline{W}}(1 - W\overline{W} + 2\bar{Z}W). \tag{6.62}$$

Thus

$$M^{\dagger} = \frac{1}{Z\sqrt{\sinh 2A}}\left((Z-W)\sinh\frac{2A\overline{W}(1+WZ)}{Z(1+W\overline{W})}\ (1+Z\overline{W})\sinh\frac{2A(Z-W)}{Z(1+W\overline{W})}\right). \tag{6.63}$$

6.3 Comments

The standard way of finding solutions with higher topological charge is to multiply the ansatz on the left by terms of the form $\chi(\lambda)$, each involving a projector and a simple pole in λ. The explicit solution to the linear system has allowed us to begin to put together the necessary ingredients for doing this, although clearly more work is necessary. The crucial starting point seems to be the fact that the rational map and its dual appear in the seed solution, so that it degenerates on spectral lines through the origin. It ought to be possible to determine the poles and the projector by demanding that the solution is well-defined on these lines, although we have not yet been able to do so.

Chapter 7

Conclusions

In this thesis we have studied two different models whose topological solitons can be parametrised in terms of a rational map. Our study of two $\mathbb{C}P^1$ lumps on the sphere found some superficially similar behavour to that of monopoles. It would be interesting if there were a deeper relationship between these two models.

We succeeded in making the Jarvis correspondence between BPS monopoles and rational maps explicit in the simplest case of a single BPS monopole, achieving this by means of the linear system for the Bogomol'nyi equations. Solutions to the linear system could equally be used to find the Donaldson rational map of a monopole, and therefore to see how the Donaldson and Jarvis rational maps are related.

The most promising method we have for finding explicit multi-monopole solutions of the linear system is the inverse scattering method. We have seen, at least for the one-monopole, that the seed solution of the inverse scattering ansatz depends explicitly on the Jarvis map and its dual. Once the choice of the poles and the projectors has been clarified, it should be possible to find monopoles with higher topological charge starting from a seed solution involving a rational map of higher degree.

We would like to understand better the relationship between the solution to the linear system, the spectral curve and the rational map. Hidden in the solution to the linear system must be the spectral curve of the monopole. That a monopole can be specified in terms of a rational map means that, in principle, the spectral curve can be determined from the Jarvis map. This seems a rather contradictory statement, since the rational map is directly related to one term in the spectral curve.

An extension of our methods using the linear system to monopoles in higher gauge groups would be possible. In this case the Jarvis map is a map from $\mathbb{C}P^1$ into a flag manifold isomorphic to the orbit of the asymptotic Higgs field. Perhaps the study of the analogous sigma model where the static field is a holomorphic map from the two-sphere to this flag manifold would also be interesting.

We would like to find some use for the functional condition on the metric found in Chapter 5. We suspect that the solution to the problem of determining the asymptotic boundary condition on the decaying part of the metric will involve this condition and the dual rational map.

It is hoped that our work will find applications in high energy physics, where monopole moduli spaces in particular have come to play an important rôle.

Appendix A

Mathematica Code

A.1 The Code for Figures 3.1, 3.2 and 3.3

```
<< Graphics`ParametricPlot3D`
p1 = I/2(z^2 - 1);
p2 = -1/2(z^2 + 1);
p3 = -I z;
p1d = -I/2(zd^2 - 1);
p2d = -1/2(zd^2 + 1);
p3d = I zd;
Clear[z, zd]
Simplify[Solve[Tan[A]p2 + I Sec[A]p3 == 0, z]]
(* H1 *)
H = {{Cosh[L/2], Sinh[L/2]}, {Sinh[L/2], Cosh[-L/2]}};
HD = {{Cosh[L/2], Sinh[L/2]}, {Sinh[L/2], Cosh[-L/2]}};
M = H.{{-I, -Sec[A], -I Tan[A]}, {I, -Sec[A], -I Tan[A]}};
MD = HD.{{I, -Sec[A], I Tan[A]}, {-I, -Sec[A], I Tan[A]}};
Clear[z, zd]
kahler = (M[[1, 1]]p1 + M[[1, 2]]p2 + M[[1, 3]]p3)(MD[[1, 1]]p1d +
            MD[[1, 2]]p2d + MD[[1, 3]]p3d) + (M[[2, 1]]p1 + M[[2, 2]]p2 +
            M[[2, 3]]p3)(MD[[2, 1]]p1d + MD[[2, 2]]p2d + MD[[2, 3]]p3d);
Clear[x, y]
twopot = Together[(1 + z zd)^2D[D[Log[kahler], z], zd]];
z = x + I y;
zd = x - I y;
twoplot :=
    Expand[Numerator[Together[twopot]]]/
```

```
      Expand[Denominator[Together[twopot]]];
x = Cos[\[Phi]]Tan[\[Theta]/2];
y = Sin[\[Phi]]Tan[\[Theta]/2];
H10 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. {A -> 0})/3], {\[Theta], 0, Pi}, {\[Phi], 0,
        2Pi}, Compiled -> True, PlotPoints -> 70], {L, -1.2, 1.2, 0.6}]
H18 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. {A -> Pi/8})/3], {\[Theta], 0, Pi}, {\[Phi], 0,
         2Pi}, Compiled -> True, PlotPoints -> 70], {L, -1.2, 1.2, 0.6}]
(* H2 *)
H = {{Cosh[L/2], -I Sinh[L/2]}, {I Sinh[L/2], Cosh[-L/2]}};
HD = {{Cosh[L/2], I Sinh[L/2]}, {-I Sinh[L/2], Cosh[-L/2]}};
M = H.{{-I, -Sec[A], -I Tan[A]}, {I, -Sec[A], -I Tan[A]}};
MD = HD.{{I, -Sec[A], I Tan[A]}, {-I, -Sec[A], I Tan[A]}};
Clear[z, zd]
kahler = (M[[1, 1]]p1 + M[[1, 2]]p2 + M[[1, 3]]p3)(MD[[1, 1]]p1d +
            MD[[1, 2]]p2d + MD[[1, 3]]p3d) + (M[[2, 1]]p1 + M[[2, 2]]p2 +
            M[[2, 3]]p3)(MD[[2, 1]]p1d + MD[[2, 2]]p2d + MD[[2, 3]]p3d);
Clear[x, y]
twopot = Together[(1 + z zd)^2D[D[Log[kahler], z], zd]];
z = x + I y;
zd = x - I y;
twoplot :=
    Expand[Numerator[Together[twopot]]]/
      Expand[Denominator[Together[twopot]]];
x = Cos[\[Phi]]Tan[\[Theta]/2];
y = Sin[\[Phi]]Tan[\[Theta]/2];
H20 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. {A -> 0})/3], {\[Theta], 0, Pi}, {\[Phi], 0,
        2Pi}, Compiled -> True, PlotPoints -> 70], {L, -1.2, 1.2, 0.6}]
H28 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. {A -> Pi/8})/3], {\[Theta], 0, Pi}, {\[Phi], 0,
         2Pi}, Compiled -> True, PlotPoints -> 70], {L, -1.2, 1.2, 0.6}]
```

```
(* H3 *)
H = {{Exp[L/2], 0}, {0, Exp[-L/2]}};
HD = {{Exp[L/2], 0}, {0, Exp[-L/2]}};
M = H.{{-I, -Sec[A], -I Tan[A]}, {I, -Sec[A], -I Tan[A]}};
MD = HD.{{I, -Sec[A], I Tan[A]}, {-I, -Sec[A], I Tan[A]}};
Clear[z, zd]
kahler = (M[[1, 1]]p1 + M[[1, 2]]p2 + M[[1, 3]]p3)(MD[[1, 1]]p1d +
           MD[[1, 2]]p2d + MD[[1, 3]]p3d) + (M[[2, 1]]p1 + M[[2, 2]]p2 +
           M[[2, 3]]p3)(MD[[2, 1]]p1d + MD[[2, 2]]p2d + MD[[2, 3]]p3d);
Clear[x, y]
twopot = Together[(1 + z zd)^2D[D[Log[kahler], z], zd]];
z = x + I y;
zd = x - I y;
twoplot :=
    Expand[Numerator[Together[twopot]]]/
      Expand[Denominator[Together[twopot]]];
x = Cos[\[Phi]]Tan[\[Theta]/2];
y = Sin[\[Phi]]Tan[\[Theta]/2];
H30 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. {A -> 0})/3], {\[Theta], 0, Pi}, {\[Phi], 0,
        2Pi}, Compiled -> True, PlotPoints -> 70], {L, -2, 2, 1}]
H38 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. {A -> Pi/8})/3], {\[Theta], 0, Pi}, {\[Phi], 0,
         2Pi}, Compiled -> True, PlotPoints -> 70], {L, -2, 2, 1}]
(* Generate plots *)
Table[Show[
    Graphics3D[
      Prepend[Part[H10, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-10, 10}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Graphics3D[{Line[{{-10, 0, 0}, {-5, 0, 0}}],
        Line[{{5, 0, 0}, {10, 0, 0}}], Line[{{0, -10, 0}, {0, -5, 0}}],
        Line[{{0, 5, 0}, {0, 10, 0}}], Line[{{0, 0, -10}, {0, 0, -5}}],
        Line[{{0, 0, 5}, {0, 0, 10}}]}]], {i, 1, 5}]
Table[Show[
```

```
    Graphics3D[
      Prepend[Part[H18, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-10, 10}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Graphics3D[{Line[{{-10, 0, 0}, {-5, 0, 0}}],
        Line[{{5, 0, 0}, {10, 0, 0}}], Line[{{0, -10, 0}, {0, -5, 0}}],
        Line[{{0, 5, 0}, {0, 10, 0}}], Line[{{0, 0, -10}, {0, 0, -5}}],
        Line[{{0, 0, 5}, {0, 0, 10}}]}]], {i, 1, 5}]
Table[Show[
    Graphics3D[
      Prepend[Part[H20, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-10, 10}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Graphics3D[{Line[{{-10, 0, 0}, {-5, 0, 0}}],
        Line[{{5, 0, 0}, {10, 0, 0}}], Line[{{0, -10, 0}, {0, -5, 0}}],
        Line[{{0, 5, 0}, {0, 10, 0}}], Line[{{0, 0, -10}, {0, 0, -5}}],
        Line[{{0, 0, 5}, {0, 0, 10}}]}]], {i, 1, 5}]
Table[Show[
    Graphics3D[
      Prepend[Part[H28, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-10, 10}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Graphics3D[{Line[{{-10, 0, 0}, {-5, 0, 0}}],
        Line[{{5, 0, 0}, {10, 0, 0}}], Line[{{0, -10, 0}, {0, -5, 0}}],
        Line[{{0, 5, 0}, {0, 10, 0}}], Line[{{0, 0, -10}, {0, 0, -5}}],
        Line[{{0, 0, 5}, {0, 0, 10}}]}]], {i, 1, 5}]
Table[Show[
    Graphics3D[
      Prepend[Part[H30, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-10, 10}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Graphics3D[{Line[{{-10, 0, 0}, {-5, 0, 0}}],
        Line[{{5, 0, 0}, {10, 0, 0}}], Line[{{0, -10, 0}, {0, -5, 0}}],
        Line[{{0, 5, 0}, {0, 10, 0}}], Line[{{0, 0, -10}, {0, 0, -5}}],
        Line[{{0, 0, 5}, {0, 0, 10}}]}]], {i, 1, 5}]
Table[Show[
    Graphics3D[
```

```
      Prepend[Part[H38, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-10, 10}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Graphics3D[{Line[{{-10, 0, 0}, {-5, 0, 0}}],
        Line[{{5, 0, 0}, {10, 0, 0}}], Line[{{0, -10, 0}, {0, -5, 0}}],
        Line[{{0, 5, 0}, {0, 10, 0}}], Line[{{0, 0, -10}, {0, 0, -5}}],
        Line[{{0, 0, 5}, {0, 0, 10}}]}]], {i, 1, 5}]
```

A.2 The Code for Figures 3.8, 3.12 and 3.16

```
<< Graphics‘ParametricPlot3D‘
CA = Cos[\[Psi][t]];
SA = Sin[\[Psi][t]];
EE = EllipticE[SA^2];
EB = (EllipticE[SA^2] - CA^2EllipticK[SA^2])/SA^2;
EE1 = EE - 1;
EB1 = EB - 1;
f = Pi (EE1 - EB1)/(4 CA^2);
a = Pi(1/2(EE1 + SA^2 EB1)/CA^2 + SA/2(EB1 + EE1)/CA^2);
b = Pi(1/2(EE1 + SA^2 EB1)/CA^2 - SA/2(EB1 + EE1)/CA^2);
c = Pi(1 - (EE1 + SA^2 EB1)/CA^2);
d = Pi CA/2((EE1 + SA^2 EB1)/CA^2 - 1);
e = Pi(1 - EE1)/4;
g = Simplify[c e - d^2]/e;
fp = Simplify[D[f, \[Psi][t]]];
ap = Simplify[D[a, \[Psi][t]]];
bp = Simplify[D[b, \[Psi][t]]];
gp = Simplify[D[g, \[Psi][t]]];
(* Momentum about x - axis *)
eqn1 = 2f \[Psi]''[t] + fp(\[Psi]'[t])^2 == ap M1^2/a^2;
eqn2 = \[Gamma]'[t] == M1/a;
M1 = 0.04;
sol = First[
    NDSolve[{eqn1,
        eqn2, \[Psi]'[0] == 0.08, \[Psi][0] == -Pi/2 + 0.4, \[Gamma][0] ==
          0}, {\[Psi], \[Gamma]}, {t, 0, 20}]]
Plot[Evaluate[\[Psi][t] /. sol], {t, 0, 20}, AxesLabel -> {t, psit}]
```

```
Plot[Evaluate[\[Gamma][t] /. sol], {t, 0, 20}, AxesLabel -> {t, git}]
Clear[z, zd, Z, ZD, x, y]
twopot = Together[(1 + z zd)^2D[
          D[Log[1 + (z zd)^2 + (z^2 + zd^2)Sin[\[Psi][t]]], z], zd]];
z = (Z - I Tan[\[Gamma][t]/2])/(-I Tan[\[Gamma][t]/2]Z + 1);
zd = (ZD + I Tan[\[Gamma][t]/2])/(I Tan[\[Gamma][t]/2]ZD + 1);
Z = x + I y;
ZD = x - I y;
twoplot :=
    Expand[Numerator[Together[twopot]]]/
      Expand[Denominator[Together[twopot]]];
x = Cos[\[Phi]]Tan[\[Theta]/2];
y = Sin[\[Phi]]Tan[\[Theta]/2];
M1 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. sol)/3], {\[Theta], 0, Pi}, {\[Phi], 0, 2Pi},
      Compiled -> True, PlotPoints -> 70], {t, 0, 17.5, 2.5}]
z1 = Simplify[(W - I Tan[\[Gamma][t]/2])/(-I Tan[\[Gamma][t]/2]W + 1) /.
             {W -> 1}];
z2 = Simplify[(W - I Tan[\[Gamma][t]/2])/(-I Tan[\[Gamma][t]/2]W + 1) /.
             {W -> I}];
z3 = Simplify[(W - I Tan[\[Gamma][t]/2])/(-I Tan[\[Gamma][t]/2]W + 1) /.
             {W -> 0}];
zd1 = Simplify[(WD + I Tan[\[Gamma][t]/2])/(I Tan[\[Gamma][t]/2]WD +
              1) /. {WD -> 1}];
zd2 = Simplify[(WD + I Tan[\[Gamma][t]/2])/(I Tan[\[Gamma][t]/2]WD +
              1) /. {WD -> -I}];
zd3 = Simplify[(WD + I Tan[\[Gamma][t]/2])/(I Tan[\[Gamma][t]/2]WD +
              1) /. {WD -> 0}];
v1 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z1, wd -> zd1}]
v2 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z2, wd -> zd2}]
v3 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z3, wd -> zd3}]
M1axes = Table[
    Graphics3D[{Line[{5(v1 /. sol), 10(v1 /. sol)}],
```

```
        Line[{-5(v1 /. sol), -10(v1 /. sol)}],
        Line[{5(v2 /. sol), 10(v2 /. sol)}],
        Line[{-5(v2 /. sol), -10(v2 /. sol)}],
        Line[{5(v3 /. sol), 10(v3 /. sol)}],
        Line[{-5(v3 /. sol), -10(v3 /. sol)}]}], {t, 0, 17.5, 2.5}]
Table[Show[
    Graphics3D[
      Prepend[Part[M1, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-40, 40}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Part[M1axes, i]], {i, 1, 8}]
(* Momentum about y - axis *)
eqn1 = 2f \[Psi]''[t] + fp(\[Psi]'[t])^2 == bp M2^2/b^2;
eqn2 = \[Gamma]'[t] == M2/b;
M2 = 0.19;
sol = First[
    NDSolve[{eqn1,
        eqn2, \[Psi]'[0] == 0.2, \[Psi][0] == -Pi/2 + 0.4, \[Gamma][0] ==
          0}, {\[Psi], \[Gamma]}, {t, 0, 20}]]
Plot[Evaluate[\[Psi][t] /. sol], {t, 0, 20}, AxesLabel -> {t, psit}]
Plot[Evaluate[\[Gamma][t] /. sol], {t, 0, 20}, AxesLabel -> {t, git}]
Clear[z, zd, Z, ZD, x, y]
twopot = Together[(1 + z zd)^2D[
          D[Log[1 + (z zd)^2 + (z^2 + zd^2)Sin[\[Psi][t]]], z], zd]];
z = (Z - Tan[\[Gamma][t]/2])/( Tan[\[Gamma][t]/2]Z + 1);
zd = (ZD - Tan[\[Gamma][t]/2])/(Tan[\[Gamma][t]/2]ZD + 1);
Z = x + I y;
ZD = x - I y;
twoplot :=
    Expand[Numerator[Together[twopot]]]/
      Expand[Denominator[Together[twopot]]];
x = Cos[\[Phi]]Tan[\[Theta]/2];
y = Sin[\[Phi]]Tan[\[Theta]/2];
M2 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. sol)/3], {\[Theta], 0, Pi}, {\[Phi], 0, 2Pi},
      Compiled -> True, PlotPoints -> 70], {t, 0, 20, 10/3}]
```

```
z1 = Simplify[(W - Tan[\[Gamma][t]/2])/(Tan[\[Gamma][t]/2]W + 1) /.
            {W -> 1}];
z2 = Simplify[(W - Tan[\[Gamma][t]/2])/(Tan[\[Gamma][t]/2]W + 1) /.
            {W -> I}];
z3 = Simplify[(W - Tan[\[Gamma][t]/2])/(Tan[\[Gamma][t]/2]W + 1) /.
            {W -> 0}];
zd1 = Simplify[(WD - Tan[\[Gamma][t]/2])/(Tan[\[Gamma][t]/2]WD + 1) /.
            {WD -> 1}];
zd2 = Simplify[(WD - Tan[\[Gamma][t]/2])/(Tan[\[Gamma][t]/2]WD + 1) /.
            {WD -> -I}];
zd3 = Simplify[(WD - Tan[\[Gamma][t]/2])/(Tan[\[Gamma][t]/2]WD + 1) /.
            {WD -> 0}];
v1 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z1, wd -> zd1}]
v2 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z2, wd -> zd2}]
v3 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z3, wd -> zd3}]
M2axes = Table[
    Graphics3D[{Line[{5(v1 /. sol), 10(v1 /. sol)}],
        Line[{-5(v1 /. sol), -10(v1 /. sol)}],
        Line[{5(v2 /. sol), 10(v2 /. sol)}],
        Line[{-5(v2 /. sol), -10(v2 /. sol)}],
        Line[{5(v3 /. sol), 10(v3 /. sol)}],
        Line[{-5(v3 /. sol), -10(v3 /. sol)}]}], {t, 0, 20, 10/3}]
Table[Show[
    Graphics3D[
      Prepend[Part[M2, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-40, 40}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Part[M2axes, i]], {i, 1, 7}]
(* Momentum about z - axis *)
eqn1 = 2f \[Psi]''[t] + fp(\[Psi]'[t])^2 == gp M3^2/g^2;
eqn2 = \[Gamma]'[t] == M3/g;
M3 = 0.02;
sol = First[
    NDSolve[{eqn1,
```

```
        eqn2, \[Psi]'[0] == 0.1, \[Psi][0] == -Pi/2 + 0.4, \[Gamma][0] ==
          0}, {\[Psi], \[Gamma]}, {t, 0, 20}]]
Plot[Evaluate[\[Psi][t] /. sol], {t, 0, 20}, AxesLabel -> {t, psit}]
Plot[Evaluate[\[Gamma][t] /. sol], {t, 0, 20}, AxesLabel -> {t, git}]
Clear[z, zd, Z, ZD, x, y]
twopot = Together[(1 + z zd)^2D[
          D[Log[1 + (z zd)^2 + (z^2 + zd^2)Sin[\[Psi][t]]], z], zd]];
z = (Cos[\[Gamma][t]] - I Sin[\[Gamma][t]])Z;
zd = (Cos[\[Gamma][t]] + I Sin[\[Gamma][t]])ZD;;
Z = x + I y;
ZD = x - I y;
twoplot :=
    Expand[Numerator[Together[twopot]]]/
      Expand[Denominator[Together[twopot]]];
x = Cos[\[Phi]]Tan[\[Theta]/2];
y = Sin[\[Phi]]Tan[\[Theta]/2];
M3 = Table[
    SphericalPlot3D[
      Evaluate[5 + (twoplot /. sol)/3], {\[Theta], 0, Pi}, {\[Phi], 0, 2Pi},
      Compiled -> True, PlotPoints -> 70], {t, 0, 17.5, 2.5}]
z1 = Simplify[(Cos[\[Gamma][t]] - I Sin[\[Gamma][t]])W /. {W -> 1}];
z2 = Simplify[(Cos[\[Gamma][t]] - I Sin[\[Gamma][t]])W /. {W -> I}];
z3 = Simplify[(Cos[\[Gamma][t]] - I Sin[\[Gamma][t]])W /. {W -> 0}];
zd1 = Simplify[(Cos[\[Gamma][t]] + I Sin[\[Gamma][t]])WD /. {WD -> 1}];
zd2 = Simplify[(Cos[\[Gamma][t]] + I Sin[\[Gamma][t]])WD /. {WD -> -I}];
zd3 = Simplify[(Cos[\[Gamma][t]] + I Sin[\[Gamma][t]])WD /. {WD -> 0}];
v1 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z1, wd -> zd1}]
v2 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z2, wd -> zd2}]
v3 = Simplify[{(w + wd)/(1 + w wd), -I(w - wd)/(1 + w wd), (w wd - 1)/
              (1 + w wd)} /. {w -> z3, wd -> zd3}]
M3axes = Table[
    Graphics3D[{Line[{5(v1 /. sol), 10(v1 /. sol)}],
        Line[{-5(v1 /. sol), -10(v1 /. sol)}],
        Line[{5(v2 /. sol), 10(v2 /. sol)}],
        Line[{-5(v2 /. sol), -10(v2 /. sol)}],
```

```
        Line[{5(v3 /. sol), 10(v3 /. sol)}],
        Line[{-5(v3 /. sol), -10(v3 /. sol)}]}], {t, 0, 17.5, 2.5}]
Table[Show[
    Graphics3D[
      Prepend[Part[M3, i, 1], EdgeForm[]], {SphericalRegion -> True,
        PlotRange -> {{-40, 40}, {-40, 40}, {-40, 40}}, Axes -> False,
        Boxed -> False, RenderAll -> False, ImageSize -> 500}],
    Part[M3axes, i]], {i, 1, 8}]
```

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