ADHM instantons and the Gauss-Bonnet integral on non-compact hyper-Kahler manifolds.

Burgess, Andrew E. F

ADHM INSTANTONS AND THE GAUSS-BONNET INTEGRAL ON NON-COMPACT HYPER-KAHLER MANIFOLDS.

By
Andrew E. F. Burgess

SUBMITTED IN FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
AT
UNIVERSITY OF WALES SWANSEA
SINGLETON PARK, SWANSEA, SA2 8PP
JANUARY 2004

UNIVERSITY OF WALES SWANSEA

Date: **January 2004**

Author: **Andrew E. F. Burgess**

Title: **ADHM Instantons and the Gauss-Bonnet Integral on Non-Compact Hyper-Kahler Manifolds.**

Department: **Department of Physics**

Degree: **Ph.D.** Year: **2004**

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Date 6/4/'04

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by explicit references. A Bibliography is appended.

Date 6/4/'04

I hereby give consent for my thesis, if accepted, to be available for photocopying and inter-library loan, and for the title and summary to be made available to outside organisations.

Date 6/4/'04

Abstract

The main intention of this thesis is to calculate the Gauss-Bonnet integral on the moduli space of Yang-Mills instantons and in particular to test a conjecture of Dorey, Hollowood and Khoze which relates the D-instanton partition function (a quantity arising from string-theoretic considerations), and the Gauss-Bonnet integral on the resolved moduli space of instantons. We shall present two main results. Firstly we use the ADHM construction to determine the metric on the moduli space of a single $SU(3)$ instanton. The result obtained agrees with the previous result of [20]. From this metric we calculate the spin connection and the curvature. Ultimately we were able to evaluate the Gauss-Bonnet integral over this resolved moduli space. This involved a nontrivial integral over an eight dimensional hyper-kahler space. The result obtained confirms the prediction of [17].

Secondly, I have also been able to verify explicitly that the D-instanton partition function derived from string theory reduces to the Gauss-Bonnet integral on the resolved instanton moduli space for the case of a single instanton in an arbitrary gauge group.

In the introductory chapter we discuss in general terms the motivation for the calculations presented in this thesis. In chapter two we discuss zero modes and collective coordinates and introduce the notion of a moduli space. We also verify that the instanton moduli space is hyper-Kahler. Chapter three discusses the ADHM construction and we persue some of its consequences. Chapter 4 is devoted to obtaining the supersymmetric quantum mechanical sigma model on the moduli space of instantons and the elucidation of its geometrical significance. Chapter 5 is where

we illustrate the explicit implementation of the ADHM construction and calculate the Gauss-Bonnet integral in the single instanton $SU(3)$ case. The results of this calculation are compared with those obtained by [17]. Their method is reviewed in chapter 6. The results of both are in agreement.

Declaration

I declare that no material presented in this thesis has previously been submitted by myself for a degree at this or any other university.

The research described in this thesis has been conducted in collaboration with Dr. Nick Dorey.

The main contribution of the author is contained in chapters 5 and 6.

Acknowledgements

I would like to thank my supervisor, Nick Dorey, and Tim Hollowood for patiently explaining so much.

Thankyou to Wenji for never complaining, even when I did!

Most of all, I owe this to my parents for their continuing support.

Contents

Chapter 1

Introduction

1.1 Solitons and instantons

Non-linear classical field theories generically possess extended solutions. If these solutions are stable field configurations with a well defined energy and are nowhere singular then they are termed solitons. Generally, the stability of these solutions arises from (topological) classifications of the boundary conditions to the equations of motion. In fact such solutions are typically characterized by topologically distinct mappings between group space and coordinate space. A common feature of these circumstances is the existence of degenerate vacuum states. The stability of the soliton naturally implies the existence of a conservation law. However, the conserved topological charge and its associated current do not follow from the invariance of the Lagrangian under a symmetry transformation. That is to say a topological current is not a Noether current-its divergencelessness is not a consequence of the equations of motion. Thus topological currents differ from other conserved quantities such as energy, momentum or electric charge. Topological charges arise due to boundary conditions on the fields, conserved due to the requirement of finite energy.

There have been numerous attempts to understand the origin of charge itself in terms of geometry. For example, J. A. Wheeler has suggested a model in which electric charge may be regarded as an artifact of the topology of space-time [7]. Although suffering from numerous drawbacks, such models hinted at the possible importance of topological concepts when considering fundamental physical questions such as the nature of charge. More recently, the discovery of magnetic monopole solutions in Yang-Mills theories by t'Hooft and Polyakov has shown the essential part played by topological considerations in the construction of magnetically charged solutions. Furthermore, the existence of such solutions implies the existence of charge quantization. The topological considerations in this case arise not from the geometry of space-time but from the existence of topologically distinct classes of boundary conditions on Higgs fields with degenerate vacua.

More recently, the conjecture of Montonen and Olive [32] has led to the consideration of electromagnetic duality as a property of some field theories. This is an exciting development, for it offers the possibility of understanding the previously intractable strong coupling behavior of a theory by performing weak-coupling calculations in a dual formulation of the theory. In going from a field theory to its dual, topological and Noether charges are exchanged. One theory thought to possess an exact electromagnetic duality is $N = 4$ supersymmetric Yang-Mills theory. Thus there has been considerable interest in attempting to understand the dynamics of this theory. In this thesis we will consider the properties of finite action solutions or instantons in such theories. Instantons represent yet another species of topological object in quantum field theory. The object central to our considerations will be the moduli space of these solutions. Following [12] we shall review its general properties.

The approach we shall follow will be to take our instantons of the 4-dimensional

Euclidean gauge theory and embed them in a Minkowski space-time of $(4+1)$-dimensions. The instanton solution, which had finite action in 4-dimensions, will now become a soliton of finite energy for the 5-dimensional theory.

At very low energies the bosonic field theory in an instanton background reduces to quantum mechanics on the moduli space, similar to the analogous monopole case, [15]. The full supersymmetric Yang-Mills theory correspondingly yields a supersymmetric quantum mechanics on the moduli space.

In this thesis we shall be concerned with the calculation of the Gauss-Bonnet integral for non-compact hyper-Kahler spaces. Such spaces arise naturally as the moduli spaces of both instantons and monopoles in supersymmetric gauge field theory, (see chapter two and [33], [34]). The reason these quantities are of interest to physicists relates to currently fashionable notions of duality. For example, the electromagnetic duality conjecture of Montonen and Olive leads to predictions concerning the existence and properties of low energy monopole bound states. There exist similar duality conjectures involving instanton solutions. One test of such conjectures is to attempt to verify predictions for the number of bound states. Up to a sign, these are in fact counted by the Euler character on the moduli space. This contention will be elaborated later in this chapter.

The specific conjecture to which the calculation in this thesis is relevant is due to Seiberg, ([35], [36]). The claim is that 5-dimensional gauge theory is related to a theory in 6-dimensions called (2,0) super-conformal theory. The idea is to compactify the 6-dimensional theory on a circle whose radius is related to the 5-dimensional coupling constant, g_5, like so

$$R_6 = 8\pi^2 g_5 \tag{1.1}$$

Given such a 6-dimensional theory one may Fourier expand the fields in the periodic direction:

$$\phi(\mathbf{x}, x_6) = \sum_{k \in Z} e^{\frac{ikx_6}{R_6}} \phi_k(\mathbf{x}) \tag{1.2}$$

where $\mathbf{x}$ is the 5-vector of position and x_6 is the position coordinate in the compactified direction which has radius R_6. The free action then becomes

$$\partial_6 \phi_k \partial^6 \phi_k^* + \partial_\mu \phi_k \partial^\mu \phi_k^* = \frac{k^2}{R_6} \phi_k^2 + \partial_\mu \phi_k \partial^\mu \phi_k^* \tag{1.3}$$

This indicates that the compactification of a 6-dimensional theory yields a 5-dimensional theory in which the field ϕ_k has acquired a mass. In fact there is a tower of these massive states, one for each value of the integer k with mass

$$M_k = \frac{k}{R_6} = \frac{8\pi^2 k}{g_5^2} \tag{1.4}$$

This represents the classical mass of an instanton when regarded as a soliton in six dimensions.

In chapter 5 we shall explore gauge theory in 5-dimensions. We shall write the gauge field as $A_\mu = A_\mu(\mathbf{x}, X_i)$ where $\mathbf{x} \in R^4$ and X_i represent the collective coordinates of the instanton solution. Following [15] we shall develop an approximation in which we allow these collective coordinates to depend on time. We shall also examine supersymmetric gauge theory, so along-side the gauge field there will be scalar and fermion fields. We will find that the low energy effective action will take the form

$$L = -\frac{8\pi^2 k}{g_5^2} + \frac{1}{2} \int dx g_{ij} \dot{X}^i \dot{X}^j + \cdots \tag{1.5}$$

where g_{ij} is the metric on the moduli space of instantons, $\mathcal{M}_k$. We have expanded the Lagrangian around its instanton solutions. In the context of our discussion of the compactified 6-dimensional theory this is highly suggestive. The implication seems to be to identify the Kaluza-Klein modes of (1.4) with the Yang-Mills instanton (bound) states of the 5-dimensional theory.

The Hamiltonian corresponding to the above Lagrangian defines a supersymmetric quantum mechanics on the moduli space,

$$H = \frac{8\pi^2 k}{g_5^2} + H_{susy\ qm} \tag{1.6}$$

Thus we see that Sieberg's conjecture of the existence of a set of (unique) states of mass $\frac{8\pi^2 k}{g_5^2}$ requires that there exist a unique (normalizable) zero-energy eigenstate of the supersymmetric quantum mechanics on $\mathcal{M}_k$. Seiberg's conjecture requires that there exist a unique normalizable zero energy eigenstate of H_{susyqm} for each value of k.

1.2 Supersymmetric quantum mechanics

Here we shall adumbrate the concept of a supersymmetric quantum mechanics mentioned above. The basic object of our attention will be supersymmetric quantum mechanics with $2N$ supercharges Q^i and Q^{*i} and Hamiltonian H. (For details see [19], [25]). The supersymmetry algebra in this case is

$$\{Q^i, Q^{*j}\} = 2\delta^{ij} H, \qquad i,j = 1, \cdots N \tag{1.7}$$

$$\{Q^i, Q^j\} = \{Q^{*i}, Q^{*j}\} = 0 \tag{1.8}$$

The supersymmetry charges map bosons into fermions and fermions into bosons. The fermion number operator $(-1)^F$ is defined by

$$(-1)^F Q^i = -Q^i (-1)^F \tag{1.9}$$

This operator commutes with all the bosonic fields of the theory and anti-commutes with all the fermion fields. For a system with only one set of supersymmetric charges we drop the index i. The supersymmetry generators represent the square root of the Hamiltonian. In fact, if we define the combination of generators S as

$$S = \frac{1}{\sqrt{2}}(Q + Q^*) \tag{1.10}$$

Then we can write

$$S^2 = \frac{1}{2}\left(QQ + QQ^* + Q^*Q + Q^*Q^*\right) = H \tag{1.11}$$

We will label states by their energy eigenvalues E and a label f or b which will designate fermionic or bosonic states respectively. Since the supersymmetry charges commute with the Hamiltonian, the states $S|E, f>$ and $S|E, b>$ will each have energy E. However, $S|E, f>$ will have the opposite fermion number to $|E, f>$. Let $|E, b>$ be a normalized bosonic state of non-zero energy E. We may therefore define the normalized fermionic state $|E, f>= \frac{1}{\sqrt{E}}S|E, b>$, i.e.

$$S|E, b>= \sqrt{E}|E, f> \tag{1.12}$$

operating with S on both sides then yields

$$S|E, f>= \sqrt{E}|E, b> \tag{1.13}$$

This establishes that all states with non zero energy appear in pairs exhibiting the opposite spin statistics. However, for the zero energy states the number of bosonic states may not necessarily equal the number of fermionic states. In this case the supersymmetry generator annihilates the $|b>$ and $|f>$ states, so that they each separately form 1 dimensional representations and are therefore not paired as in the

$E \neq 0$ case. Let the number of bosonic zero-energy eigenstates be denoted by $n_b^{E=0}$ and the fermionic ones be $n_f^{E=0}$. Now cosider the effect of changing the parameters of the theory, such as the coupling constants or the masses. In general, for such transformations that also preserve supersymmetry, we would expect the energy levels to change, but of course the non zero energy states should still be arranged in bose fermi pairs. It may be possible, under such a variation of parameters, for a non-zero energy level to descend to a zero-energy eigenstate. Conversely, it may happen that a zero-energy state may become a state of non-zero energy. However, in this case, it is required that a supersymetric pair, one bose and one fermi state, must move up together. In each case, $n_b^{E=0}$ changes by the same amount as $n_f^{E=0}$, consequently the difference $n_b^{E=0} - n_f^{E=0}$ remains constant. It is straightforward to demonstrate that $n_b^{E=0} - n_f^{E=0}$ is a constant by evaluating $Tr(-1)^F e^{-\beta H}$, where the trace is taken over the Hilbert space of states, [25];

$$Tr(-1)^F e^{-\beta H} =< x|(-1)^F e^{-\beta H}|x > \tag{1.14}$$

(The opperator $Tr(-1)^F$ ensures that the fermionic fields satisfy periodic boundary conditions in the path integral representation). Expanding in eigenfunctions of the Hamiltonian we have

$$Tr(-1)^F e^{-\beta H} = \sum_n e^{-\beta E_n} < n|(-1)^F|n > \tag{1.15}$$

we can split this sum into a trace over zero energy states and one over non-zero energy eigenstates. Since the states of non-zero energy come in bose-fermi pairs, we can split these into their bose and fermi constituents

$$Tr(-1)^F e^{-\beta H} = \sum_{n(E=0)} < n|(-1)^F|n >$$

$$+ \sum_{n(E\neq 0)} e^{-\beta E_n} \left(< n, b|(-1)^F|n, b > + < n, f|(-1)^F|n, f > \right) \tag{1.16}$$

$$= \sum_{n(E=0)} < n|(-1)^F|n >= n_b^{E=0} - n_f^{E=0} \tag{1.17}$$

Note that this is independent of the value of β.

1.3 Supersymmetric quantum mechanics and differential geometry

Supersymmetric quantum mechanics has many points of contact with concepts of classical differential geometry. We now briefly elaborate these connections. Consider a compact Riemannian manifold M of dimension $2n$ with coordinates X^i. We denote the exterior derivative by d and its adjoint by d^*.

We expect bosonic wave-functions to resemble functions on the manifold.

$$f(X)|0> \tag{1.18}$$

We introduce the supersymmetric fermionic partners, ψ, to the bosonic coordinates. These will have the canonical commutation relations,

$$\{\psi^i, \psi^{j\dagger}\} = g^{ij} \tag{1.19}$$

The $\psi^\dagger$ therefore act as creation operators and we have the spectrum of states

$$f_{i\cdots j}\psi^{i\dagger}\cdots\psi^{j\dagger}|0> \tag{1.20}$$

Since the ψ's anti-commute, the $f_{i\cdots j}$ must be antisymmetric in its indices. So it would seem natural to identify fermion states with antisymmetric tensors on M. Now recall that the action of the exterior derivative d on forms maps r forms to $r+1$ forms. One can also define the adjoint exterior derivative d^* that maps r forms to $r-1$ forms.

This resembles the action of the supersymmetry charges, which change the fermion number of a state by ± 1. Pursuing this analogy, let us define the combination of derivatives Q_1 and Q_2 by

$$Q_1 = d + d^* \tag{1.21}$$

$$Q_2 = i(d - d^*) \tag{1.22}$$

Since the Q_i are to be identified as the supersymmetry charges, we know from their algebra that the Hamiltonian is given by

$$Q_1^2 = Q_2^2 = H \tag{1.23}$$

Substituting the forms for the Q_i's given above, we see that the Hamiltonian is actually the Laplacian on M. For details on differential geometry see [3] and [30].

$$Q_1^2 = Q_2^2 = dd^* + d^*d = H \tag{1.24}$$

This is entirely consistent, for using the nilpotent property of the exterior derivatives yields the remaining part of the supersymmetry algebra,

$$Q_1Q_2 - Q_2Q_1 = 0 \tag{1.25}$$

We are to regard the p-forms on M as bosonic or fermionic, depending upon whether p is even or odd.

A zero energy eigenfunction is therefore a solution of Laplace's equation on the manifold, otherwise known as a harmonic differential form. By Hodge's theorem, the number of such harmonic forms of degree r on a compact, smooth manifold is equal to the dimension of the r^{th} cohomology group [3], [11]. But these are the Betti numbers, so the number of linearly independent harmonic forms of degree r on a compact manifold is given by the r^{th} Betti number, b^r. The Euler character can be represented in terms of these quantities as

$$\chi(M) = \sum(-1)^r b^r \tag{1.26}$$

Thus in some sense, the Euler character can be said to count the number of zero energy states in the supersymmetric quantum mechanics defined on a manifold. In the supersymetric quantum mechanics r represents the fermion number of a wavefunction. The Witten index of a supersymmetric quantum mechanics is therefore a topological invariant of the moduli space and is in fact equal to the Euler character on M.

1.4 Evaluating the Witten index

Our expression for the regularized trace of $(-1)^F$ has a path integral representation, [26],

$$Tr(-1)^F e^{-\beta H} = \int_{PBC} d\phi(t) d\psi(t) e^{-S_E(\phi,\psi)} \tag{1.27}$$

where the fields are taken to satisfy periodic boundary conditions with period β and S_E is the euclidean action.

As discussed in section 1.1 the effective action of a gauge theory instanton reduces to a supersymmetric quantum mechanical sigma model on the instanton moduli space. A generic feature of such supersymmetric sigma models is the appearance of the curvature tensor in the fermionic part of the action. We now discus how we may manipulate such models to yield an expression for the Gauss-Bonnet integral, which as discussed above yields the Euler character of a compact space.

The archetypal bosonic σ model is defined by the Lagrangian

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j, \quad \dot{\phi}^i = \frac{d}{dt}\phi^i \tag{1.28}$$

where $\phi^i(t)$ is to be considered as a map from R or S^1 onto a manifold M and g is the metric on M which is of dimension $d = 2n$. The supersymmetric extension of this Lagrangian (corresponding to the Hamiltonian of (1.24) can be written as [26]

$$L = \frac{1}{2} g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + i\psi^{*i}\nabla\psi^j g_{ij} - \frac{1}{4}R_{ijkl}\psi^{*i}\psi^{*j}\psi^k\psi^l \tag{1.29}$$

where

$$\nabla\psi^i = \frac{d}{dt}\psi^i + \Gamma^i_{jk}\dot{\phi}^j\psi^k \tag{1.30}$$

In the path integral, (1.27), we impose periodic boundary conditions on both the bosons and fermions. As discussed previously, this is independent of β. Thus we are free to take the limit $\beta \to 0$. The first stage is to Fourier expand the fields

$$\phi(t) = \sum_k \phi^{(k)} e^{\frac{ikt}{\beta}} \tag{1.31}$$

Taking time derivatives yields exponential terms with coefficients proportional to $\frac{k}{\beta}$. In the limit $\beta \to 0$ these may be discarded so we need only retain the Fourier-zero mode quantities ϕ^0. Thus when we come to evaluate the path integral, we may drop all the time derivative terms in the exponential and we are left with just the fermion-field curvature term. Rescaling the fermions by a factor of $\beta^{-\frac{1}{4}}$ we get

$$Tr(-1)^F e^{-\beta H} = \int d(Vol) \prod_m d\psi^*_m d\psi_m exp\left(\int_0^{\beta/2\pi} dt \left(-\frac{1}{4}\right) R_{ijkl}\psi^{*i}\psi^{*j}\psi^k\psi^l\right) \tag{1.32}$$

Consider the functional integral above. This gives the Gauss-Bonnet formula for the (volume contribution) to the Euler character, as we now show. Expanding the exponential gives

$$Tr(-1)^F e^{-\beta H} = \frac{1}{(2\pi)^n} \int d(Vol) \prod_m d\psi^*_m d\psi_m \sum_{r=1}^{\infty} \left(\frac{-1}{4}\right)^r \left(\frac{1}{2\pi}\right)^r \frac{1}{r!} \left(R_{ijkl}\psi^{*i}\psi^{*j}\psi^k\psi^l\right)^r$$

Note that the terms in the exponential series bring down $2r$ powers of the fermionic Grassmann fields ψ. Thus to saturate the d Grassmann integrations we should only retain the $r = n$ term. This leaves us with

$$Tr(-1)^F e^{-\beta H} = \int d(Vol) \prod_m d\psi^*_m d\psi_m \left(\frac{-1}{4}\right)^n$$

$$\frac{1}{(n)!} R_{i_1 j_1 k_1 l_1} \ldots R_{i_n j_n k_n l_n} \psi^{*i_1} \psi^{*j_1} \psi^{k_1} \psi^{l_1} \ldots \psi^{*i_n} \psi^{*j_n} \psi^{k_n} \psi^{l_n}$$

$$= \frac{(-1)^n}{(8\pi)^n} \frac{1}{(n)!} \int d(Vol) \prod_m d\psi^*_m d\psi_m R_{i_1 j_1 k_1 l_1} \ldots R_{i_n j_n k_n l_n} \epsilon^{i_1 j_1 \ldots i_n j_n} \epsilon^{k_1 l_1 \ldots k_n l_n} \psi^{*1} \psi^{*2} \ldots \psi^{*n} \psi^1 \psi^2 \ldots \psi^n$$

$$= \frac{(-1)^n}{(8\pi)^n (n)!} \int d(Vol) \epsilon^{i_1 j_1 \ldots i_n j_n} \epsilon^{k_1 l_1 \ldots k_n l_n} R_{i_1 j_1 k_1 l_1} \ldots R_{i_n j_n k_n l_n} = \chi(M) \qquad (1.33)$$

Thus we have arrived at a statement of the Gauss-Bonnet theorem which gives an integral representation of the Euler character of a compact Reimannian manifold.

1.5 The ADHM construction

To calculate the Gauss-Bonnet integral it would seem that one requires knowledge of the metric on $\mathcal{M}_k$. Fortunately there exists a general procedure for determining the metric on the moduli space of k instantons in any gauge group. This is the ADHM procedure and will be outlined in chapter three. Starting with some initial information, called the ADHM data, and a Euclidean space called the mother space, one imposes certain constraints that restrict one to the moduli space of instantons realized as a subspace of the mother space. In principle this procedure will yield the metric on the moduli space. (However, note that it is not in general possible to solve the ADHM constraints). Given the metric it is straightforward, if tedious, to compute directly the curvature tensor and thence the Gauss-Bonnet integral, (1.33). This is precisely what we shall do for the case of a single $SU(3)$ instanton. In this case the moduli space is an eight dimensional hyper-Kahler manifold. In chapter five we shall use the ADHM construction to determine the metric on the resolved moduli space, (5.102). The result obtained agrees with the result of Gibbons et al, (5.39), derived in [20] using a different method.

From there on we use Cartan's structure equations (appendix B) to calculate the spin connection and then the curvature two-form. The calculation is cumbrous and in order to simplify it to a manageable extent we shall change to a more convenient coordinate system, (called the symplectic basis), in which we can then evaluate the Gauss-Bonnet Integral.

Our aim will then be to compare this result with one obtained by the indirect, conjectural but much more general method of [17].

1.6 Non-compact manifolds and singularities

The moduli space of instantons is non-compact. This reflects the fact that the collective coordinate describing the instanton's separation may take an arbitrarily large value. Likewise, the collective coordinate corresponding to the instanton's size may vary in a range unbounded from above. This imposes certain amendments upon the analysis above. Firstly note that the Witten index counts (up to a sign) the number of supersymmetric ground states. Thus it counts the number of normalizable (i.e. square integrable) solutions to Laplace's equation. This is termed the L^2 index of the Laplacian, $Ind_{L^2}(\Delta)$. In general this is not equal to the Euler character,

$$\chi \neq Ind_{L^2}(\Delta) \tag{1.34}$$

In fact, in this case one may regard the Euler character as being composed of a bulk term, I_{Bulk}, and a boundary term. The bulk term is given by the familiar Gauss-Bonnet integral and corresponds to the integral over the entire moduli space of an index density. The boundary term represents an integral over a surface at infinity, and therefore is not sensitive to the detailed interactions between instantons. This thesis will concentrate on evaluating the bulk term I_{Bulk}, for details on the boundary term see [36].

A further technical point should be discussed. The instanton moduli space actually possesses a singular point at which the instanton size shrinks to zero. This fact has the potential to render invalid the arguments constructed above. However, it proves possible to resolve this singularity by introducing a so-called non-commutativity parameter ζ. The practical difference that this will make to the calculations in this thesis is that we shall amend the ADHM constraint equations to

$$\tau^{c\dot{\alpha}}{}_{\dot{\beta}}(\bar{a}^{\dot{\beta}} a_{\dot{\alpha}}) = \tau^{c\dot{\alpha}}{}_{\dot{\beta}}(\bar{\omega}^{\dot{\beta}} \omega_{\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha} a'_{\alpha\dot{\alpha}}) = \zeta^c \tag{1.35}$$

(Strictly, the ADHM constraints, as outlined in chapter three, should be written with $\zeta = 0$). With ζ in place we shall calculate the metric on the 1-instanton $SU(3)$ moduli space and we will find that it depends on ζ. However, both the curvature and the Gauss-Bonnet integral will prove to be independent of ζ. Thus naively we may take the limit $\zeta \to 0$ to arrive at a quantity pertaining to the true ADHM moduli space.

1.7 The D-instanton partition function

Except for certain cases, the ADHM constraints are prohibitively difficult to solve so we cannot in general obtain the metric on $\mathcal{M}_k$. Despite this difficulty, Dorey, Hollowood and Khoze, [17], have proposed a means of calculating the Gauss-Bonnet integral for arbitrary instanton number in any gauge group. Their method does not require knowledge of the metric since the ADHM constraints need not be explicitly solved. Their contention is that the Gauss-Bonnet integral is equal to an entity derived from string theoretic considerations called the D-instanton partition function, defined thus;

$$Z_{I,N} = 2^{-2N-1}\pi^{-6N-9}\int d^{2N}\omega\, d^{2N}\bar{\omega}\, d^6\chi\, d^3D\, d^{4N}\mu\, d^{4N}\bar{\mu}\, d^8\lambda$$

$$exp\left[-\bar{\omega}_u^{\dot{\alpha}}\bar{\chi}^2\omega_{u\dot{\alpha}} - iD^c(\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}_u^{\dot{\beta}}\omega_{u\dot{\alpha}} - \zeta^c) + 2\sqrt{2}\pi i\bar{\mu}_u^A\chi_{AB}\mu_u^B + i\pi(\bar{\mu}_u^A\omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u}\mu_u^A)\lambda_A^{\dot{\alpha}}\right] \tag{1.36}$$

We shall evaluate this quantity in chapter six. These integrations may be evaluated in a certain order to give an explicit numerical result, which in the 1-instanton sector is

$$Z_{I,N} = \frac{2\Gamma\left(N+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(N)} = I_{Bulk} \tag{1.37}$$

The values obtained from this formula can be compared with known results for the cases $N=1$ and $N=2$. For $N=1$ the moduli space is simply a point, giving $\chi = 1$. For $N=2$ the resolved moduli space corresponds to the Eguchi-Hanson manifold. The Gauss-Bonnet integral in this case yields the value $\frac{3}{2}$, [30], again in agreement with (1.37). Our direct calculation of the Gauss-Bonnet integral on the resolved $SU(3)$ single instanton moduli space, (chapter five), provides a further test of (1.37) for a non-trivial case in which the moduli space is an eight-dimensional hyper-Kahler manifold. In this case (1.37) yields the value $\frac{15}{8}$. The values calculated by these two methods are in agreement, offering some support for the validity of the approach of [17].

We can test (1.37) in another way. We can evaluate the integrations in (1.36) in a different order. We integrate first over χ and then over the Lagrange multipliers

D and λ. We then obtain delta functions with the bosonic and fermionic ADHM constraints as their arguments.

$$\int d^3 D\, e^{-iD^c(\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}^{\dot{\beta}}_u \omega_{u\dot{\alpha}} - \zeta^c)} = (2\pi)^3 \delta\left(\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}^{\dot{\beta}}_u \omega_{u\dot{\alpha}} - \zeta^c\right) \tag{1.38}$$

$$\int d^8\lambda\, e^{i\pi(\bar{\mu}^A_u \omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u}\mu^A_u)\lambda^{\dot{\alpha}}_A} = (2)^8 \delta\left(\bar{\mu}^A_u \omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u}\mu^A_u\right) \tag{1.39}$$

In chapter six we show that the integration over χ provides the correct Jacobian, which along with the delta functions, restricts us to the instanton moduli space. Likewise, the fermionic integrations are restricted to the symplectic tangent space.

If we can solve the ADHM constraints then we may implement a change of variables to a coordinate system in which the delta function constraints are trivial. There will of course be another Jacobian associated with this transformation. The remaining integrations will then run over the coordinates on the moduli space and yield the Gauss-Bonnet formula (1.33) as we demonstrate in section 6.6.

Chapter 2

Zero modes and collective coordinates

In this chapter we introduce several related concepts which are key to understanding the instanton literature. These are the notions of a moduli space, collective coordinates and zero modes. The moduli space of a system is simply the space of inequivalent solutions of equal action to the equations of motion. The coordinates on this space parameterise the different solutions and are called the collective coordinates of a solution. When speaking of instantons, the most obvious classifcation of solutions is provided by the instanton number k. This is a discrete quantity and so may not be continuously deformed. Furthermore, k is a gauge invariant quantity. This suggests that there is a separate moduli space for each such topological charge, and these spaces are denoted by $\mathcal{M}_k$. Within each such space the collective coordinates will be denoted by X^μ and the gauge fields can be labeled by their collective coordinates: $A_n(x;X)$. To labour the point, the label X in the argument of the gauge field refers to the specification of a particular solution to the self-dual Yang-Mills equations. Since instantons are essentially localized objects, there will exist collective coordinates that designate the position of the instanton centre. Thus these solutions must necessarily break the translational symmetry of the gauge theory. Solutions with centre points at different locations can then be obtained by acting on a given solution with the group elements of the broken translational symmetry. More generally this example illustrates that one should normally expect to find a collective coordinate corresponding to each symmetry of the gauge theory that is broken by a particular solution to the equations of motion. However, there are also likely to be other collective coordinates that do not directly arise from the breaking of a symmetry. Varying the collective coordinates of a solution does not affect the value of the action, so such variations are termed the zero modes of our system. Zero modes are conceived to be small changes in the fields that leave the value of the action unaltered. Since an infinitesimal displacement on a manifold roughly corresponds to a tangent vector, one should regard the zero modes of our system as tangent vectors to the moduli space.

In this chapter we shall first explore the properties of fermionic fields propagating in an instanton background. Next we shall verify that the moduli space of instantons constitutes a hyper-Kahler manifold.

2.1 Deformations of the self-dual Yang-Mills system

Instantons in $SU(N)$ Yang-Mills theory are finite action configurations whose field strength tensors are self-dual, [40], [8],

$$F_{nm} = \frac{1}{2}\epsilon_{nmkl}F^{kl} = *F_{nm} \tag{2.1}$$

Let A_n be a solution of the self-dual Yang-Mills equations. We may consider small variations away from this solution, $A_n \to A'_n = A_n + \delta A_n$, such that the new field is also a solution of the self dual equations. To linear order the constraint upon the field fluctuations is derived by substituting A'_n into the self-dual condition

$$D_m \delta A_n - D_n \delta A_m = \epsilon_{mnkl} D_k \delta A_l \qquad (2.2)$$

(See appendix H for details on the conventions employed here). We will now show how (2.2) may be re-cast in quaternionic form. Multiplying throughout by $\bar{\sigma}_{mn}$ and using its anti-duality property gives

$$\bar{\sigma}_{mn}{}^{\dot{\alpha}}{}_{\dot{\beta}}(D_m \delta A_n - D_n \delta A_m) = \bar{\sigma}_{mn}{}^{\dot{\alpha}}{}_{\dot{\beta}} \epsilon_{mnkl} D_k \delta A_l = -2\bar{\sigma}_{kl}{}^{\dot{\alpha}}{}_{\dot{\beta}} D_k \delta A_l \qquad (2.3)$$

$$\Rightarrow (\bar{\sigma}_m^{\dot{\beta}\alpha} \sigma_{n\alpha\dot{\alpha}} - \delta_{mn} \delta^{\dot{\beta}}{}_{\dot{\alpha}}) D_m \delta A_n = 0 \qquad (2.4)$$

Multiplying throughout by the Pauli matrices,

$$\tau^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\sigma}_m^{\dot{\beta}\alpha} \sigma_{n\alpha\dot{\alpha}} D_m \delta A_n = \tau^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\not{D}}^{\dot{\beta}\alpha} \delta A_{\alpha\dot{\alpha}} = 0 \qquad (2.5)$$

where we have defined the usefull quaternionic quantities

$$\bar{\not{D}}^{\dot{\beta}\alpha} = \bar{\sigma}_n^{\dot{\beta}\alpha} D_n, \; \not{D}_{\alpha\dot{\alpha}} = \sigma_{n\alpha\dot{\alpha}} D_n, \; \delta A_{\alpha\dot{\alpha}} = \sigma_{n\alpha\dot{\alpha}} \delta A_n, \; \delta \bar{A}^{\dot{\beta}\alpha} = \bar{\sigma}_n^{\dot{\beta}\alpha} \delta A_n \qquad (2.6)$$

Now consider a variation in the gauge field due to an infinitessimal gauge trasformation

$$\delta A_n = D_n \Omega \qquad (2.7)$$

Substituting this into the above we see that (2.2) is trivially satisfied. Thus it would seem that any gauge transformation of our solution will yield a zero mode of the system. However, gauge transformations do not represent physically distinct field configurations and so are not to be regarded as true zero modes. We must therefore make a choice of gauge so that we may systematicaly elliminate non-physical variations of the instanton solution. In the instanton literature this is conventionally achieved by imposing the so-called covariant background gauge, which is defined thus

$$D_n^{cl} \delta A_n(x; X) = 0 \qquad (2.8)$$

With this choice of gauge the quantum fluctuations $\delta A_n(x;X)$ are functionally orthogonal to gauge transformations. To see this we convolute (2.8) with an arbitrary Lie algebra valued function $\Omega(x)$,

$$\int d^D x \, Tr \left[\Omega(x) D_n^{cl} \delta A_n(x;X)\right] = 0 \tag{2.9}$$

Integrating by parts gives

$$\int d^D x \, Tr \left[\delta A_n(x;X) D_n^{cl} \Omega(x)\right] = 0 \tag{2.10}$$

as required.

In quaternionic form the background gauge condition becomes

$$\bar{\not{D}}^{\dot{\alpha}\alpha} \delta A_{\alpha\dot{\alpha}} = 0 \tag{2.11}$$

Equations (2.5) and (2.11) may be combined into the single equation

$$\bar{\not{D}}^{\dot{\alpha}\alpha} \delta A_{\alpha\dot{\beta}} = 0 \tag{2.12}$$

The quaternionic notation affords more than an elegant concision. This is a useful point of view for we can now recognize that these conditions amount to a covariant Weyl equation for the spinor $\psi_\alpha = \delta A_{\alpha\dot{\beta}}$ in an instanton background. The free index $\dot{\beta}$ indicates that each gauge field zero mode corresponds to two independent solutions of the Weyl equation. Furthermore, the problem of counting the number of bosonic collective coordinates is now seen to be closely related to that of determining the index of the Dirac opperator. This is defined by

$$Ind(\bar{\not{D}}) = Dim\{ker \bar{\not{D}}\} - Dim\{ker \not{D}\} \tag{2.13}$$

From the Atiyah-Singer index theorem one can show that this index has the value $2kN$. Furthermore, it can be shown (see appendix E) that in the presence of an instanton there are no solutions to $\not{D}\lambda = 0$. Thus the Atiyah-Singer index theorem [3] gives the number of solutions to $\bar{\not{D}}\lambda = 0$. Therefore the number of solutions to (2.12) is $4kN$, (see for example [22]).

Zero modes are related to collective coordinates since the derivative of the gauge field with respect to a collective coordinate is certain to satisfy the zero mode equation (2.2). Any general solution of (2.2) can therfore be written as

$$\delta_\mu A_n(x) = \frac{\partial A_n(x;X)}{\partial X^\mu} \tag{2.14}$$

Since we are dealing with a gauge theory we should be free to gauge transform this solution to obtain another physically acceptable configuration. As discussed above, an infinitesimal gauge transformation can be effected by adding a covariant derivative term like so

$$\delta_\mu A_n(x) = \frac{\partial A_n(x;X)}{\partial X^\mu} - D_n \Omega_\mu \tag{2.15}$$

However, to ensure that this is a physical zero mode we must impose the gauge condition (2.8). Thus we require

$$D_n \delta_\mu A_n(x) = 0 \Rightarrow D_n \left(\frac{\partial A_n(x;X)}{\partial X^\mu} \right) = D^2 \Omega_\mu \tag{2.16}$$

2.2 The moduli space as a hyper-Kahler manifold

We begin this section by outlining some concepts of the differential geometry of complex manifolds, (for details see [3]). On a complex manifold of complex dimension m each coordinate neighbourhood is homeomorhic to complex Euclidean space C^m. The transition functions from one coordinate system to another are analytic. An almost complex structure, I, is a linear map of the tangent space onto itself such that $I^2 = -1$. Thus acting on a tangent vector with the complex structure corresponds roughly to multiplication by i. A complex manifold necessarily has real dimension $2m$.

We say that we have a complex structure if I is integrable. The condition for integrability is given by the Newlander-Nirenberg theorem which states that I is a complex structure if the Nijenhuis tensor N_I vanishes, [3], [16],

$$N_I(X,Y) = I[IX,IY] + [X,IY] + [IX,Y] - I[X,Y] = 0 \tag{2.17}$$

Where X, Y are tangent vectors.

A holomorphic coordinate basis is one that diagonalizes the complex structure. The complex structure partitions the tangent space into two disjoint vector spaces called the holomorphic and the anti-holomorphic vector spaces. The holomorphic vectors are eigenvectors of I with eigenvalue i. The anti-holomorphic vectors are eigenvectors of I with eigenvalue $-i$. Indices of the anti-holomorphic space are written with a bar (e.g. $\bar{\mu}$), to distinguish them from indices of the holomorphic variety (written μ).

A metric is said to be Hermitian if

$$g(IX,IY) = g(X,Y) \tag{2.18}$$

For a Hermitian metric, $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$ and so the metric takes the form

$$g = g_{\mu\bar{\nu}} dz^{\mu} \otimes d\bar{z}^{\bar{\nu}} + g_{\bar{\mu}\nu} d\bar{z}^{\bar{\mu}} \otimes dz^{\nu} \tag{2.19}$$

where $z, \bar{z}$ are the holomorphic and anti-holomorphic coordinate functions respectively.

A complex manifold is said to be a Kahler manifold if the complex structure is covariantly constant;

$$\nabla_{\mu} I = 0 \tag{2.20}$$

Given a complex structure I and a hermitian metric g we can define a 2-form ω as follows.

$$\omega(X,Y) = g(X,IY) \tag{2.21}$$

ω is called the Kahler form. For a Kahler manifold the Kahler form is closed,

$$d\omega = 0 \tag{2.22}$$

A further property of Kahler manifolds is that the metric may be obtained by differentiating a scalar function called the Kahler potential χ

$$g_{\mu\bar{\nu}} = \frac{\partial^2 \chi}{\partial z^\mu \partial \bar{z}^{\bar{\nu}}} \tag{2.23}$$

Conversely, if the metric on a manifold can be written in the above form, then the space is Kahler with a globally defined complex structure [11].

A hyper-Kahler space admits three independent complex structures, $I^{(c)}$, $c = 1,2,3$ that satisfy the quaternion algebra

$$I^c I^d = -\delta^{cd} + \epsilon^{cde} I^e \tag{2.24}$$

The metric is Hermitian with respect to all three complex structures.

In the rest of this chapter we aim to show that the moduli space of instantons is in fact a hyper-Kahler manifold. The approach we take is perhaps a little indirect, focussing on the construction of a hyper-Kahler potential function, [37]. This section is necessarily abstract and follows closely the treatment of [12].

$\mathcal{M}_k$ is a space of dimension $4kN$. It is a Riemannian manifold endowed with a natural metric defined as the functional inner product of the zero modes (in singular gauge).

$$g_{\mu\nu} = -2g^2 \int d^4x \; Tr_N \delta_\mu A_n(x;X) \delta_\nu A_n(x;X) \tag{2.25}$$

We shall show that $\mathcal{M}_k$ is a hyper-Kahler space. Euclidean space is itself hyper-Kahler. The three complex structures on R^4 can be chosen to be

$$I^c_{mn} = -\bar{\eta}^c_{mn}$$

where $\bar{\eta}^c_{mn}$ are the t'Hooft η-symbols. In the quaternionic basis we have

$$(I^c x)_{\alpha\dot{\alpha}} = i x_{\alpha\dot{\beta}} \tau^{c\dot{\beta}}{}_{\dot{\alpha}} \tag{2.26}$$

and

$$(I^c \bar{x})^{\dot{\alpha}\alpha} = -i\ \tau^{c\dot{\alpha}}{}_{\dot{\beta}} \bar{x}^{\dot{\beta}\alpha} \tag{2.27}$$

These complex structures on R^4 decend to give complex structures on $\mathcal{M}_k$. First note that in the zero mode equation, (2.12), $\dot{\beta}$ is a free index. Thus if $\delta_\mu A_{\alpha\dot{\alpha}}$ is a zero mode then so is $\delta_\mu A_{\alpha\dot{\beta}}\, G^{\dot{\beta}}{}_{\dot{\alpha}}$ for any constant matrix G,

$$\not{\mathcal{D}}^{\dot{\alpha}\alpha}(\delta_\mu A_{\alpha\dot{\beta}} G^{\dot{\beta}}_{\dot{\alpha}}) = (\not{\mathcal{D}}^{\dot{\alpha}\alpha}\delta_\mu A_{\alpha\dot{\beta}}) G^{\dot{\beta}}_{\dot{\alpha}} = 0$$

In particular we could have,

$$(I^c \delta_\mu A)_{\alpha\dot{\alpha}} = i\delta_\mu A_{\alpha\dot{\beta}} \tau^{c\dot{\beta}}{}_{\dot{\alpha}} \tag{2.28}$$

Since the zero modes form a complete set of vectors on the moduli space, the RHS must be some linear combination of zero modes, so there must exist a matrix $I^{(c)\nu}{}_\mu$ such that

$$(I^c \delta_\mu A)_{\alpha\dot{\alpha}} = \delta_\nu A_{\alpha\dot{\alpha}} I^{c\nu}{}_\mu \tag{2.29}$$

Comparing equations (2.28) and (2.28) gives

$$i\delta_\mu A_{\alpha\dot{\beta}} \tau^{c\dot{\beta}}{}_{\dot{\alpha}} = \delta_\nu A_{\alpha\dot{\alpha}} I^{c\nu}{}_\mu \tag{2.30}$$

This is an intertwining relation; the algebraic relations obeyed by the $I^{(c)}$ will also be satisfied by the $I^{(c)\nu}{}_{\mu}$. This includes equation (2.24). At this stage, the $I^{(c)\nu}{}_{\mu}$ remain almost complex structures. To show that they are indeed complex structures we shall examine an explicit construction for the hyper-kahler potential on $\mathcal{M}_k$. The expression for the potential given in [37], [12] is

$$\chi = -\frac{g^2}{4}\int d^4x\; x^2\, Tr_N(F_{mn}F^{mn}) \tag{2.31}$$

We will not derive this expression, but simply confirm that it is the correct hyper-Kahler potential for our manifold. To show this we first choose one of the complex structures $I^{(c)}$ of R^4, say $I^{(3)}$. We will also need to choose holomorphic coordinates. The holomorphic coordinates with respect to $I^{(3)}$ are

$$z^1 = ix^3 + x^4 \;\; ; \;\; z^2 = ix^1 - x^2$$

We can confirm that these are indeed the holomorphic coordinates for $I^{(3)}$. Recall the definition of the coordinates in quaternionic language,

$$\begin{pmatrix} ix^3 + x^4 & ix^1 + x^2 \\ ix^1 - x^2 & -ix^3 + x^4 \end{pmatrix} = \begin{pmatrix} z^1 & -\bar{z}^1 \\ z^2 & \bar{z}^2 \end{pmatrix}$$

The action of the complex structure on these coordinates is

$$(I^{(3)}x)_{\alpha\dot{\alpha}} = ix_{\alpha\dot{\beta}}\tau^{3\dot{\beta}}{}_{\dot{\alpha}} = \begin{pmatrix} iz^1 & i\bar{z}^1 \\ iz^2 & -i\bar{z}^2 \end{pmatrix}$$

thus we may write

$$I^{(3)}(z^1) = iz^1$$

$$I^{(3)}(z^2) = iz^2$$

$$I^{(3)}(-\bar{z}^1) = i\bar{z}^1 \Rightarrow I^3(\bar{z}^1) = -i\bar{z}^1$$

$$I^{(3)}(\bar{z}^2) = -i\bar{z}^2$$

as is required for a complex structure acting on its holomorphic coordinates. The complex structure $I^{(3)}$ can be associated to a complex structure on $\mathcal{M}_k$ with a matching set of holomorphic coordinates $(Z^i, \bar{Z}^i)$, with $i = 1, 2, ..., \frac{1}{2}dim(\mathcal{M}_k)$ for which the complex structure on the moduli space is

$$I^{(3)i}{}_j = \begin{pmatrix} i\delta^i_j & 0 \\ 0 & -i\delta^i_j \end{pmatrix}$$

From our previous discussion we know that the derivative of A_n with respect to a collective coordinate automatically provides a zero mode and so satisfies the zero mode equation, (although such derivatives do not in general satisfy the gauge condition). However, we will demonstrate that when we work in a holomorphic basis the derivatives of the gauge fields are in fact zero modes without the need for a compensating gauge transformation. With an eye on this result, we first concentrate on the zero modes, $\delta_i A_n$, generated by differentiating the gauge fields,

$$\delta_i A_n = \frac{\partial A_n}{\partial Z^i}\delta Z^i \tag{2.32}$$

and substitute these into the zero mode equation, (2.5)

$$\tau^{c\dot{\beta}}{}_{\dot{\alpha}}\bar{\not{D}}^{\alpha\dot{\alpha}}\left(\frac{\partial A_n}{\partial Z^i}\delta Z^i\right) = 0 \Rightarrow \tau^{c\dot{\beta}}{}_{\dot{\alpha}}\bar{\not{D}}^{\alpha\dot{\alpha}}\left(\frac{\partial A_n}{\partial Z^i}\right) = 0 \tag{2.33}$$

Likewise, differentiating with respect to the anti-holomorphic coordinates yields

$$\bar{\delta}_i A_n = \frac{\partial A_n}{\partial \bar{Z}^i}\delta\bar{Z}^i \; ; \; \tau^{c\dot{\beta}}{}_{\dot{\alpha}}\bar{\not{D}}^{\alpha\dot{\alpha}}\frac{\partial A_n}{\partial \bar{Z}^i}\delta\bar{Z}^i = 0 \tag{2.34}$$

Equation (2.29) then implies that

$$\left(I^{(3)}\frac{\partial A}{\partial Z^i}\right)_{\alpha\dot{\alpha}} = \frac{\partial A_{\alpha\dot{\alpha}}}{\partial Z^i}I^{3\,i}{}_j$$

$$\Rightarrow \frac{\partial A_{\alpha\dot{\alpha}}}{\partial Z^i} = \frac{\partial A_{\alpha\dot{\beta}}}{\partial Z^i}\tau^{3\,\dot{\beta}}{}_{\dot{\alpha}} = \begin{pmatrix} \frac{\partial A_{11}}{\partial Z^i} & -\frac{\partial A_{12}}{\partial Z^i} \\ \frac{\partial A_{21}}{\partial Z^i} & -\frac{\partial A_{22}}{\partial Z^i} \end{pmatrix}$$

This gives the following two relations

$$\frac{\partial A_{12}}{\partial Z^i} = -\frac{\partial A_{12}}{\partial Z^i} \Rightarrow \frac{\partial A_{12}}{\partial Z^i} = 0$$

$$\frac{\partial A_{22}}{\partial Z^i} = -\frac{\partial A_{22}}{\partial Z^i} \Rightarrow \frac{\partial A_{22}}{\partial Z^i} = 0$$

These can be combined into

$$\frac{\partial A_{\alpha 2}}{\partial Z^i} = 0 \tag{2.35}$$

Simmilarly, the anti-holomorphic coordinates yield

$$\frac{\partial A_{\alpha\dot{\beta}}}{\partial \bar{Z}^i}\tau^{3\,\dot{\beta}}{}_{\dot{\alpha}} = -\frac{\partial A_{\alpha\dot{\alpha}}}{\partial \bar{Z}^i} \Rightarrow \frac{\partial A_{\alpha 1}}{\partial \bar{Z}^i} = 0 \tag{2.36}$$

Using this information we will write out the zero mode equations in full

$$\tau^{1\,\dot{\alpha}}{}_{\dot{\beta}}\bar{\not{D}}^{\dot{\beta}\alpha}\frac{\partial A_{\alpha\dot{\alpha}}}{\partial Z^i} = Tr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{\not{D}}^{11} & \bar{\not{D}}^{12} \\ \bar{\not{D}}^{21} & \bar{\not{D}}^{22} \end{pmatrix}\begin{pmatrix} \frac{\partial A_{11}}{\partial Z^i} & 0 \\ \frac{\partial A_{21}}{\partial Z^i} & 0 \end{pmatrix}$$

$$\Rightarrow \bar{\not{D}}^{\dot{2}\alpha}\frac{\partial A_{\alpha\dot{1}}}{\partial Z^i} = 0 \tag{2.37}$$

The equation for $c = 2$ reproduces that above. For $c = 3$ we have

$$\bar{\not{D}}^{\dot{1}\alpha}\frac{\partial A_{\alpha\dot{1}}}{\partial Z^i} = 0 \tag{2.38}$$

The zero mode equations for the anti-holomorphic coordinates are

$$\bar{\not{D}}^{\dot{1}\alpha}\frac{\partial A_{\alpha\dot{2}}}{\partial \bar{Z}^i} = 0 \tag{2.39}$$

$$\bar{\not{D}}^{\dot{2}\alpha}\frac{\partial A_{\alpha\dot{2}}}{\partial \bar{Z}^i} = 0 \tag{2.40}$$

We now analyse the gauge conditions, (2.11), which under the above restrictions become

$$\bar{\not{D}}^{\dot{\alpha}\alpha}\frac{\partial A_{\alpha\dot{\alpha}}}{\partial Z^i} = \bar{\not{D}}^{\dot{1}\alpha}\frac{\partial A_{\alpha\dot{1}}}{\partial Z^i} = 0 \tag{2.41}$$

and

$$\bar{\not{D}}^{\dot{\alpha}\alpha}\frac{\partial A_{\alpha\dot{\alpha}}}{\partial\bar{Z}^i} = \bar{\not{D}}^{\dot{2}\alpha}\frac{\partial A_{\alpha\dot{2}}}{\partial\bar{Z}^i} = 0 \tag{2.42}$$

Here we see that the the gauge conditions are automatically satisfied by the zero modes for they correspond to a subset of the zero mode equations themselves. Consequently, the derivatives with respect to the holomorphic coordinates are zero modes directly without the need for compensating gauge transformations.

Equations (2.35) and (2.36) imply that the mixed derivative of the gauge field must vanish,

$$\frac{\partial^2 A_n}{\partial Z^i \partial \bar{Z}^j} = 0 \tag{2.43}$$

We will marshal this information to calculate the mixed derivative of the proposed hyper-Kahler potential χ. Since the derivative only acts on the moduli space variables contained in the specification of the A fields and has no effect on the space coordinates x, we get

$$\frac{\partial^2 \chi}{\partial Z^i \partial \bar{Z}^j} = -\frac{g^2}{4}\int d^4x\ x^2\ \frac{\partial^2}{\partial Z^i \partial \bar{Z}^j} Tr_N(F_{mn}F^{mn})$$

Taking into account the cyclic nature of the trace, the derivative in the integrand becomes

$$\frac{\partial^2}{\partial Z^i \partial \bar{Z}^j} Tr_N(F_{mn}F^{mn}) = 2Tr_N\left(\frac{\partial F_{mn}}{\partial \bar{Z}^j}\frac{\partial F_{mn}}{\partial Z^i} + F_{mn}\frac{\partial^2 F_{mn}}{\partial Z^i \partial \bar{Z}^j}\right)$$

Recalling the form of F_{mn} we may calculate the various derivatives required,

$$F_{mn} = \partial_m A_n - \partial_n A_m + g[A_m, A_n]$$

$$\begin{aligned}\frac{\partial F_{mn}}{\partial Z^i} &= \partial_m \delta_i A_n - \partial_n \delta_i A_m + g[\delta_i A_m, A_n] + g[A_m, \delta_i A_n] \\ &= D_m \delta_i A_n - D_n \delta_i A_m = \epsilon_{mnkl} D_k \delta_i A_l\end{aligned}$$

$$\frac{\partial F_{mn}}{\partial \bar{Z}^i} = \partial_m \bar{\delta}_j A_n - \partial_n \bar{\delta}_j A_m + g[\bar{\delta}_j A_m, A_n] + g[A_m, \bar{\delta}_j A_n]$$

$$\frac{\partial^2 F_{mn}}{\partial Z^i \partial \bar{Z}^j} = g[\bar{\delta}_j A_m, \delta_i A_n] + g[\delta_i A_m, \bar{\delta}_j A_n]$$

Using these equations we get,

$$\frac{1}{2}\frac{\partial^2}{\partial Z^i \partial \bar{Z}^j} Tr_N(F_{mn}F^{mn}) = Tr_n \Big[\partial_m \bar{\delta}_j A_n \partial_m \delta_i A_n - \partial_m \bar{\delta}_j A_n \partial_n \delta_i A_m + g\partial_m \bar{\delta}_j A_n [\delta_i A_m, A_n]$$

$$+g\partial_m \bar{\delta}_j A_n [A_m, \delta_i A_n] - \partial_n \bar{\delta}_j A_m \partial_m \delta_i A_n + \partial_n \bar{\delta}_j A_m \partial_n \delta_i A_m$$

$$-g\partial_n \bar{\delta}_j A_m [\delta_i A_m, A_n] - g\partial_n \bar{\delta}_j A_m [A_m, \delta_i A_n] + g[\bar{\delta}_j A_m, A_n]\partial_m \delta_i A_n$$

$$-g[\bar{\delta}_j A_m, A_n]\partial_n \delta_i A_m + g^2[\bar{\delta}_j A_m, A_n][A_m, \delta_i A_n] + g[A_m \bar{\delta}_j A_n]\partial_m \delta_i A_n$$

$$-g[A_m, \bar{\delta}_j A_n]\partial_n \delta_i A_m + g^2[A_m, \bar{\delta}_j A_n][\delta_i A_m, A_n] + g^2[A_m, \bar{\delta}_j A_n][A_m, \delta_i A_n]$$

$$+g\partial_m A_n[\bar{\delta}_j A_m, \delta_i A_n] + g\partial_m A_n[\delta_i A_m, \bar{\delta}_j A_n] - g\partial_n A_m[\bar{\delta}_j A_m, \delta_i A_n]$$

$$-g\partial_n A_m[\delta_i A_m, \bar{\delta}_j A_n] + g^2[A_m, A_n][\bar{\delta}_j A_m, \delta_i A_n] + g^2[A_m, A_n][\delta_i A_m, \bar{\delta}_j A_n]\Big]$$

The constraint that we must impose is

$$D_m \delta A_n = 0 \Rightarrow \partial_m \delta A_n = -g[A_m, \delta A_n]$$

Taking derivatives

$$\partial_m \partial_m \delta A_n = g[\delta A_n, \partial_m A_m] + g[\partial_m \delta A_n A_m]$$

$$\partial_n \partial_m \delta A_n = g[\delta A_n, \partial_n A_m] + g[\partial_n \delta A_n, A_m]$$

Using the above together with the trace property and re-labelling dummy indices gives

$$\frac{1}{2}\frac{\partial^2}{\partial Z^i \partial \bar{Z}^j} Tr_N F_{mn} F^{mn} = 2g Tr_N \left[(\partial_m A_n - \partial_n A_m)[\delta_i A_m, \bar{\delta}_j A_n]\right.$$

$$+g[\delta_i A_m, A_m][\bar{\delta}_j A_n, A_n] - g[\delta_i A_m, A_n][\bar{\delta}_j A_n, A_m]\Big]$$

$$= 2gTr_N\left[\partial_m\partial_m(\delta_i A_n\bar{\delta}_j A_n) - 2\partial_m\partial_n(\delta_i A_m\bar{\delta}_j A_n)\right]$$

Where the last line follows by expanding it out and using the same relations as previously. This result will allow us to calculate the mixed derivative of χ

$$\frac{\partial^2\chi}{\partial z^i\partial\bar{z}^j} = -\frac{g^2}{2}\int d^4x\, x^2 Tr_N\left[(\partial_m\partial_m(\delta_i A_n\bar{\delta}_j A_n) - 2\partial_m\partial_n(\delta_i A_m\bar{\delta}_j A_n)\right]$$

We wish to integrate by parts, so consider the following

$$Tr_N\partial_m\left(x^2\left[\partial_m(\delta_i A_n\bar{\delta}_j A_n) - 2\partial_n(\delta_i A_m\bar{\delta}_j A_n)\right]\right)$$

$$\Rightarrow x^2 Tr_N\left[\partial_m\partial_m(\delta_i A_n\bar{\delta}_j A_n) - 2\partial_m\partial_n(\delta_i A_m\bar{\delta}_j A_n)\right] =$$

$$Tr_N\partial_m\left(x^2\left[\partial_m(\delta_i A_n\bar{\delta}_j A_n) - 2\partial_n(\delta_i A_m\bar{\delta}_j A_n)\right]\right) - 2x_m Tr_N\left[\partial_m(\delta_i A_n\bar{\delta}_j A_n) - 2\partial_n(\delta_i A_m\bar{\delta}_j A_n)\right]$$

Upon integration over all space we may ignore the total divergence, for by Gauss' theorem this gives a surface integral at infinity and zero modes decay as $O(x^3)$. Likewise, we integrate by parts again, using the identity

$$x_m Tr_N\left[\partial_m(\delta_i A_n\bar{\delta}_j A_n)\right] - 2x_n Tr_N\left[\partial_m(\delta_i A_n\bar{\delta}_j A_m)\right] =$$

$$\partial_m Tr_N\left[x_m\delta_i A_n\bar{\delta}_j A_n - 2x_n\delta_i A_n\bar{\delta}_j A_m\right] - 2\delta_i A_m\bar{\delta}_j A_m$$

Again we dispose of the total divergence, leaving us with

$$\frac{\partial^2\chi}{\partial Z^i\partial\bar{Z}^j} = -2g^2\int d^4x\; Tr_N(\delta_i A_m\bar{\delta}_j A_m)$$

Observe that the above expression is a component of the metric on the space of zero modes;

$$g(X) = \frac{\partial^2\chi}{\partial Z^i\partial\bar{Z}^j}dZ^i d\bar{Z}^j$$

This proves that χ is the Kahler potential for the complex structure $I^{(3)}$. Evidently χ does not depend on the choice of index $c = 1, 2, 3$. Therefore it serves as a potential function for each of the three complex structures, so $\mathcal{M}_k$ is a hyper-Kahler space.

2.3 Summary

We have discussed the meaning of the terms zero mode and collective coordinate. The moduli space of solutions was also introduced. We have been able to relate the number of zero modes of the instanton solution to the index of the Dirac operator on the moduli space. Finally, we verified that the instanton moduli space is a hyper-Kahler manifold.

Chapter 3

The ADHM construction of instantons

The ADHM construction ([38], [12]), provides a general method by which one may construct the multi-instanton moduli space in terms of an over-complete set of variables and a set of constraints placed thereon. We shall briefly review this construction, obtaining the ADHM constraint equations and demonstrating that they do indeed yield field configurations with self-dual field strength tensors. We shall determine the asymptotic form of the ADHM gauge field in the so-called singular gauge and recover the well known expression for the gauge field of a single instanton and briefly discuss some simplifications that occur for the one instanton case. We will then briefly review the connection between the ADHM construction and the hyper-Kahler quotient. Finally we shall explore some properties of the ADHM moduli space that will be needed later, namely its Killing vector fields.

3.1 The ADHM field strength

The ADHM construction starts with the definition of the matrices Δ and $\bar{\Delta}$

$$\Delta_{\lambda i\dot{\alpha}}(x) = a_{\lambda i\dot{\alpha}} + b^{\alpha}_{\lambda i} x_n \sigma^n_{\alpha\dot{\alpha}} \tag{3.1}$$

$$\bar{\Delta}_i^{\dot{\alpha}\lambda} = \bar{a}_i^{\dot{\alpha}\lambda} + \bar{b}_{i\alpha}^{\lambda} x_n \bar{\sigma}_{\alpha\bar{\alpha}}^n \tag{3.2}$$

where $\lambda = 1, \ldots, N + 2k$, $i = 1, \ldots, k$ and as usual α is a spinor index covering the values 1 and 2. Note that Δ is linear in x_n. Conjugation is defined to raise both the spinor and ADHM indices, but does not change $\dot{\alpha}$ to α. Since $\bar{\Delta}$ has N fewer rows than columns, its null space is at least N dimensional. It is useful to arrange these $(N + 2k)$ dimensional N vectors into a matrix $U_{\lambda u}$, $u = 1, \ldots, N$. This matrix will then be annihilated by the $\bar{\Delta}$:

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}.U_{\lambda u} = 0 = \bar{U}_u^{\lambda}.\Delta_{\lambda i\dot{\alpha}} \tag{3.3}$$

We can orthonormalize these matrices:

$$\bar{U}_u^{\lambda}.U_{\lambda v} = \delta_{uv} \tag{3.4}$$

We now propose to write the gauge field of a multi-instanton solution as a generalization of the pure gauge form of the one instanton solution.

$$(A_n)_{uv} = \frac{1}{g}\bar{U}_u^{\lambda}.\partial_n U_{\lambda v} \tag{3.5}$$

Note, A_n is an $(N \times N)$ matrix as required. Note also that in the case $k = 0$, U is also an $(N \times N)$ matrix, and we have the usual pure gauge solution. We shall demonstrate that for the case of non-zero k the ADHM ansatz for the gauge field still gives a self-dual field tensor and therefore an instanton solution. However, to make this true we shall also require the following, sometimes called the ADHM condition,

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}.\Delta_{\lambda j\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}}(f^{-1})_{ij} \tag{3.6}$$

Where we can take f to be an arbitrary x-dependent $k \times k$ Hermitian matrix. This relation is required to ensure that the field-strength tensor $F_{\mu\nu}$ is self-dual and from this equation we shall obtain the ADHM constraint equations. One can show that the above considerations lead to the following completeness relation:

$$P_\lambda{}^\mu \equiv U_{\lambda u}.\bar{U}^\mu_u = \delta^\mu_\lambda - \Delta_{\lambda i\dot{\alpha}} f_{ij} \bar{\Delta}^{\dot{\alpha}\mu}_j \tag{3.7}$$

Using all this information we may now find an expression for the field strength tensor and confirm that it is self dual and therefore an instanton.

$$(F_{mn})_{uv} = \partial_m (A_n)_{uv} - \partial_n (A_m)_{uv} + g[A_m\,,\,A_n]_{uv} \tag{3.8}$$

$$= \frac{1}{g}\left(\partial_m(\bar{U}^\lambda_u \partial_n U_{\lambda v}) - \partial_n(\bar{U}^\lambda_u \partial_m U_{\lambda v}) + \bar{U}^\lambda_u \partial_m U_{\lambda w}.\bar{U}^\rho_w \partial_n U_{\rho v} - \bar{U}^\lambda_u \partial_n U_{\lambda w}.\bar{U}^\rho_w \partial_m U_{\rho v}\right)$$

$$= \frac{1}{g}\left(\delta^\rho_\lambda - U_{\lambda w}\bar{U}^\rho_w\right)\left(\partial_m \bar{U}^\lambda_u \partial_n U_{\rho v} - \partial_n \bar{U}^\lambda_u \partial_m U_{\rho v}\right)$$

$$= \frac{1}{g}(\Delta_{\lambda i\dot{\alpha}} f_{ij} \bar{\Delta}^{\dot{\alpha}\rho}_j).(\partial_m \bar{U}^\lambda_u.\partial_n U_{\rho v} - \partial_n \bar{U}^\lambda_u.\partial_m U_{\rho v})$$

Differentiating the null space conditions, (3.3), gives

$$\Rightarrow \partial_m \bar{U}^\lambda_u.\Delta_{\lambda i\alpha} = -\bar{U}^\lambda_u \partial_m \Delta_{\lambda i\alpha} \ \ \& \ \ \bar{\Delta}^{\dot{\alpha}\lambda}_i \partial_m U_{\lambda u} = -\partial_m \bar{\Delta}^{\dot{\alpha}\lambda}_i.U_{\lambda u}$$

Applying these relations to the F_{mn} (3.1) gives

$$(F_{mn})_{uv} = \frac{1}{g}(\bar{U}^\lambda_u \partial_m \Delta_{\lambda i\dot{\alpha}}.f_{ij}.\partial_n \bar{\Delta}^{\dot{\alpha}\rho}_j.U_{\rho v} - \bar{U}^\lambda_u \partial_n \Delta_{\lambda i\dot{\alpha}}.f_{ij}.\partial_m \bar{\Delta}^{\dot{\alpha}\rho}_j.U_{\rho v}) \tag{3.9}$$

Recalling the definitions of Δ and $\bar{\Delta}$ and differentiating we have,

$$\partial_m \Delta_{\lambda i\dot{\alpha}} = b^\alpha{}_{\lambda i} \sigma_{m\ \alpha\dot{\alpha}}$$

$$\partial_m \bar{\Delta}^{\dot{\alpha}\lambda}_i = \bar{\sigma}^{\dot{\alpha}\alpha}_m \bar{b}^\lambda_{i\alpha}$$

Substituting all these results into (3.9) we obtain our final result,

$$(F_{mn})_{uv} = 4g^{-1} \bar{U}^\lambda_u b^\alpha_{\lambda i} \sigma_{mn}{}^\beta{}_\alpha f_{ij} \bar{b}^\rho_{i\beta} U_{\rho v} \tag{3.10}$$

The self duality of the field strength is thus manifest due to the self duality of σ_{mn}.

3.2 The ADHM constraint equations

We will next analyze the restrictions that the ADHM condition (3.6) imposes on the Δ's. We can substitute in the given form of Δ (3.1, 3.2) into (3.6) to get

$$\delta^{\dot\alpha}{}_{\dot\beta}(f^{-1})_{ij} = \bar\Delta_i^{\dot\alpha\lambda}\Delta_{\lambda j\dot\beta} = (\bar a_i^{\dot\alpha\lambda} + \bar x^{\dot\alpha\alpha}\bar b_{i\alpha}^{\lambda}).(a_{\lambda j\dot\beta} + b_{\lambda j}^{\beta}x_{\beta\dot\beta})$$

$$= \bar a_i^{\dot\alpha\lambda}a_{\lambda j\dot\beta} + \bar a_i^{\dot\alpha\lambda}b_{\lambda j}^{\beta}x_{\beta\dot\beta} + \bar x^{\dot\alpha\alpha}\bar b_{i\alpha}^{\lambda}a_{\lambda j\dot\beta} + \bar x^{\dot\alpha\alpha}\bar b_{i\alpha}^{\lambda}b_{\lambda j}^{\beta}x_{\beta\dot\beta} \tag{3.11}$$

We can now Taylor expand $(f^{-1})_{ij}$ around $x^n = 0$:

$$(f^{-1})_{ij} = A_{ij} + B_{ij}^n x_n + C_{ij}^{nm}x_n x_m + \dots$$

Where A, B and C are constants. We can then equate coefficients on each side of (3.11). For the constant term we get

$$\bar a_i^{\dot\alpha\lambda}a_{\lambda j\dot\beta} \;=\; \delta_{\dot\beta}^{\dot\alpha}A_{ij} \tag{3.12}$$

Taking the trace over the spinor indices:

$$A_{ij} \;=\; \frac{1}{2}\bar a_i^{\dot\alpha\lambda}a_{\lambda j\dot\alpha}$$

Substituting this expression for A back into equation (4.97) yields

$$\bar a_i^{\dot\alpha\lambda}a_{\lambda j\dot\beta} \;=\; \frac{1}{2}(\bar a a)_{ij}\delta_{\dot\beta}^{\dot\alpha} \tag{3.13}$$

This is the first of the so-called ADHM equations. Equating coefficients for the linear and quadratic terms gives two further equations:

$$\bar a_i^{\dot\alpha\lambda}b_{\lambda j}^{\beta} = \bar b_i^{\beta\lambda}a_{\lambda j}^{\dot\alpha} \tag{3.14}$$

$$\bar b_{\alpha i}^{\lambda}b_{\lambda j}^{\beta} = \frac{1}{2}\left(\bar b b_{ij}\right)\delta_{\alpha}^{\beta} \tag{3.15}$$

These are the ADHM constraint equations.

3.3 Symmetries of the ADHM variables

We will now show that the ADHM field strength admits the following symmetries.

$$\Delta_{\lambda i\dot{\alpha}} \rightarrow \Lambda_\lambda{}^\rho \Delta_{\rho j\dot{\alpha}} \Upsilon^{-1}_{ji}, \;\; U_{\lambda u} \rightarrow \Lambda_\lambda{}^\rho U_{\rho u}, \;\; f_{ij} \rightarrow \Upsilon_{ik} f_{kl} \Upsilon^\dagger_{lj} \tag{3.16}$$

With $\Lambda \in U(N+2k)$ and $\Upsilon \in Gl(k, C)$. Suppressing indices, the field strength becomes

$$F_{mn} \rightarrow 4g^{-1} \bar{U} \Lambda^\dagger \Lambda b \Upsilon^{-1} \sigma_{mn} \Upsilon f \Upsilon^\dagger (\Upsilon^{-1})^\dagger \bar{b} \Lambda^\dagger \Lambda U$$

$$= 4g^{-1} b \sigma_{mn} f \bar{b} U = F_{mn}$$

as required. These symmetries are convenient for they allow us to write b in a simple canonical form. By splitting the index λ in the following fashion, $\lambda = u + i\alpha$, we may write

$$b^\beta_{\lambda j} = b^\beta_{(u+i\alpha)j} = \begin{pmatrix} 0 \\ \delta^\beta_\alpha \delta_{ij} \end{pmatrix} \tag{3.17}$$

$$\bar{b}^\lambda_{\beta j} = \bar{b}^{(u+i\alpha)}_{\beta j} = \begin{pmatrix} 0 & \delta^\alpha_\beta \delta_{ij} \end{pmatrix} \tag{3.18}$$

We then decompose the content of the a variables in the same way:

$$a_{\lambda j\dot{\alpha}} = a_{(u+i\alpha)j\dot{\alpha}} = \begin{pmatrix} \omega_{uj\dot{\alpha}} \\ (a'_{\alpha\dot{\alpha}})_{ij} \end{pmatrix} \tag{3.19}$$

$$\bar{a}^{\dot{\alpha}\lambda}_j = \bar{a}^{\dot{\alpha}(u+i\alpha)}_j = \begin{pmatrix} \bar{\omega}^{\dot{\alpha}}_{ju} & (\bar{a}'^{\alpha\dot{\alpha}})_{ji} \end{pmatrix} \tag{3.20}$$

Having written b in this form, the third ADHM constraint, (3.15) is immediately satisfied. We now show what the other two constraint equations become. Multiplying (3.13) by the Pauli matrices gives

$$\tau^{c\dot{\beta}}{}_{\dot{\alpha}} \bar{a}^{\dot{\alpha}\lambda}_i a_{\lambda j\dot{\beta}} = \frac{1}{2} (\bar{a}a)_{ij} \tau^{c\dot{\alpha}}{}_{\dot{\alpha}} = 0$$

Turning our attention to the second ADHM equation and using the above decomposition of the ADHM variables we have, (setting $\lambda = u + k\gamma$),

$$\bar{a}_i^{\dot{\alpha}\lambda} b_{\lambda j}^{\beta} = \begin{pmatrix} \bar{\omega}_{iu}^{\dot{\alpha}} & (\bar{a}'^{\dot{\alpha}\gamma})_{ik} \end{pmatrix} \begin{pmatrix} 0 \\ \delta_\gamma^\beta \delta_{kj} \end{pmatrix} = (\bar{a}'^{\dot{\alpha}\beta})_{ij}$$

Similarly we have

$$\bar{b}_i^{\beta\lambda} a_{\lambda j}^{\dot{\alpha}} = \epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}} \bar{b}_{\alpha i}^{\lambda} a_{\lambda j\dot{\beta}} = \epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}} (a'_{\alpha\dot{\beta}})_{ij}$$

Let us now define a quaternionic expansion for the $(a'_{\alpha\dot{\beta}})_{ij}$,

$$(a'_{\alpha\dot{\beta}})_{ij} = a'_{nij}\sigma_{\alpha\dot{\beta}} \tag{3.21}$$

$$\Rightarrow (a'_{\alpha\dot{\beta}})^\dagger = \bar{a}'^{\dot{\beta}\alpha} = a_n^\dagger \bar{\sigma}^{\dot{\beta}\alpha}$$

Inputting this into (3.14) gives

$$a_n^\dagger \bar{\sigma}^{\dot{\alpha}\beta} = \epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}} a_n \sigma_{n\alpha\dot{\beta}} = a_n \bar{\sigma}_n^{\dot{\alpha}\beta}$$

$$\Rightarrow a_n^\dagger = a_n \tag{3.22}$$

We can now find the form of the ADHM matrix f:

$$(f^{-1})_{ij}\delta^{\dot{\alpha}}{}_{\dot{\beta}} = \bar{\Delta}_i^{\dot{\alpha}\lambda}\Delta_{\lambda j\dot{\beta}} \tag{3.23}$$

$$\Rightarrow 2(f^{-1})_{ij} = \bar{\Delta}_i^{\dot{\alpha}\lambda}\Delta_{\lambda j\dot{\alpha}}$$

$$= (\bar{a}_i^{\dot{\alpha}\lambda} + \bar{x}^{\dot{\alpha}\alpha}\bar{b}_{i\alpha}^{\lambda})(a_{\lambda j\dot{\alpha}} + b_{\lambda j}^{\beta} x_{\beta\dot{\alpha}})$$

$$= \bar{\omega}_{iu}^{\dot{\alpha}}\omega_{uj\dot{\alpha}} + ((a'_n)_{ik}(a'_m)_{kj} + (a'_n)_{ij}x_m + x_n(a'_m)_{ij} + x_n x_m \delta^i{}_j)\bar{\sigma}_n^{\dot{\alpha}\alpha}.\sigma_{m\,\alpha\dot{\alpha}}$$

Now recall the identity $\bar{\sigma}_n^{\dot{\alpha}\alpha}\sigma_{m\alpha\dot{\alpha}} = \delta_{nm}$, using this gives

$$2(f^{-1})_{ij} = \bar{\omega}_{iu}^{\dot{\alpha}}\omega_{uj\dot{\alpha}} + (a'_n)_{ik}(a'_n)_{kj} + 2(a_n)_{ij}x_n + x_n^2\delta^i{}_j$$

$$\Rightarrow f = 2(\bar{\omega}^{\dot{\alpha}}\omega_{\dot{\alpha}} + (a'_n + x_n 1_{k\times k})^2)^{-1} \tag{3.24}$$

We now show that the canonical form for b is preserved by a $U(k)$ subgroup of the $U(N+2k) \times Gl(k,c)$ symmetry group. The specific group transformations involved are

$$\Lambda = \begin{pmatrix} 1_{N\times N} & 0 \\ 0 & \Xi\, 1_{2\times 2} \end{pmatrix}, \Upsilon = \Xi,\ \Xi \in U(k)$$

Recall the transformation law for b,

$$b^{\alpha}_{\lambda i} \to \Lambda_{\lambda}{}^{\rho} b^{\alpha}_{\rho j} \Upsilon^{-1}_{ji} \tag{3.25}$$

With b in its canonical form, the above quantity becomes (setting $\lambda = u + l\beta$ and $\rho = v + k\gamma$),

$$b^{\alpha}_{\lambda i} \to \begin{pmatrix} \delta_{uv} & 0 \\ 0 & \delta_{\beta}{}^{\gamma}\Xi_{lk} \end{pmatrix} \begin{pmatrix} 0 \\ \delta^{\alpha}_{\gamma}\delta_{kj} \end{pmatrix} (\Xi^{-1})_{ji} = \begin{pmatrix} 0 \\ \delta_{lj}\delta^{\alpha}_{\beta} \end{pmatrix} = b^{\alpha}_{\lambda i}$$

Thus this transformation leaves b in its canonical form as claimed. This $U(k)$ transformation acts on the remaining variables thus:

$$\omega_{ui\dot{\alpha}} \to \omega_{uj\dot{\alpha}}\Xi_{ji}$$

$$(a'_{\alpha\dot{\alpha}})_{ij} \to (\Xi^{\dagger})_{ik}(a'_{\alpha\dot{\alpha}})_{kl}\Xi_{lj} \tag{3.26}$$

As stated before, restricting b to take its canonical form renders the a_n Hermitian. We shall define the $k \times k$ matrices to be Hermitian from the outset. This leaves the first ADHM constraint still to be satisfied. Written in terms of the variables ω and a' this becomes

$$(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{a}^{\dot{\beta}}a_{\dot{\alpha}}) = (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{\omega}^{\dot{\beta}}\omega_{\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha}a'_{\alpha\dot{\alpha}}) = 0 \tag{3.27}$$

This is the equation that we will later refer to as the ADHM constraint.

3.4 The instanton number k

A Yang-Mills instanton is a solution to the euclidean equations of motion of the pure gauge theory with finite action. Such solutions should have zero action at infinity which requires that the field strength also tend to zero. However, this does not imply that the gauge field tend to zero. Rather the gauge field is required to approach a pure gauge configuration,

$$A_n \to U^{-1}\partial_n U$$

Fields which satisfy this boundary condition may be classified according to their Pontryagin class which assigns each sector an integer number as follows

$$k = -\frac{1}{16\pi^2}\int d^4x\, Tr_N F_{mn}F^{mn}$$

we would like to calculate this quantity for the generalized field strength arising from the ADHM construction. In this we are greatly aided by Osborn's identity, H.33,

$$Tr_N F_{mn}F^{mn} = -g^2(\partial^2)^2 Tr_k log(f) \tag{3.28}$$

Using this we can write

$$-\frac{1}{16\pi^2}\int d^4x\, Tr_N F_{mn}F^{mn} = \frac{1}{16\pi^2}\int d^4x\,(\partial^2)^2 Tr_k log(f) = \frac{1}{16\pi^2}\int d^4x\,(\partial^2)^2 log[det(f)] \tag{3.29}$$

To proceed we shall require an expression for the f matrix, (3.23). This equation involves the product of Δ and $\bar{\Delta}$. Asymptotically, as $x \to \infty$ this becomes,

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}\Delta_{\lambda j\dot{\beta}} \to x^n x^m \bar{\sigma}_n^{\dot{\alpha}\alpha}\sigma_{m\beta\dot{\alpha}}\bar{b}_{i\alpha}^{\lambda}b_{\lambda j}^{\beta}$$

With b in its canonical form we have

$$\bar{\Delta}_i^{\dot{\alpha}\lambda}\Delta_{\lambda j\dot{\beta}} \to x^n x^m \bar{\sigma}_n^{\dot{\alpha}\alpha}\sigma_{m\beta\dot{\alpha}}\delta_\alpha^\beta\delta_{ij} = 2x^n x^m\delta_{ij}$$

Therefore we can find an expression for f by substituting into (3.6),

$$f^{-1} \to x^n x_n \delta_{ij} \Rightarrow f \to \frac{1}{x^n x_n}\delta_{ij}$$

$$\Rightarrow Tr_k log(f) = log[det(f)] \to -klog(x^n x_n)$$

Returning to equation (3.29), applying Gauss' theorem, taking the surface integral over a sphere whose radius is large, and then substituting in the above asymptotic form yields

$$-\frac{k}{16\pi^2}\oint d\sigma^m \partial_m \partial^l \partial_l log(x^n x_n)$$

Where $d\sigma^m$ is the element of surface area. Carrying out the differentiations gives

$$\frac{k}{2\pi^2}\oint d\sigma^m x_m \frac{1}{(x^n x_n)^2}$$

The vector element of area is perpendicular to the surface over which we integrate, therefore we can write

$$d\sigma^m = \frac{x^m}{|x^m|}d\sigma = \frac{x^m}{\sqrt{x^l x_l}}d\sigma$$

Using this gives

$$\frac{k}{2\pi^2}\oint d\sigma (x^n x_n)^{-\frac{3}{2}} = \frac{k}{2\pi^2 r^3}\oint d\sigma$$

Where $r = \sqrt{x^n x_n}$ is the radius of the 3-sphere over whose surface we are integrating. The volume of a 3-sphere is a standard result and is given by $2\pi^2 r^3$. Thus our final result is

$$-\frac{1}{16\pi^2}\int d^4x\, Tr_N F_{mn} F^{mn} = k$$

So the ADHM field-strength reproduces the conventional instanton number, as required.

3.5 The shape of the moduli space

One may use single $SU(2)$ instantons to obtain instantons of $SU(N)$. This is achieved by simply embedding the $SU(2)$ solution in one of the $SU(2)$ subgroups of $SU(N)$. One such embedding we could choose is

$$A_\mu^{SU(N)} = \begin{pmatrix} A_\mu^{SU(2)} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.30}$$

Taubes has shown that all $SU(N)$ single instantons may be obtained in this way [41] One can act on this solution with a global gauge transformation like so:

$$A_\mu^{SU(N)} = U^\dagger . \begin{pmatrix} A_\mu^{SU(2)} & 0 \\ 0 & 0 \end{pmatrix} .U \tag{3.31}$$

In general, such an action could act non-trivially on the embedded $A_\mu^{SU(2)}$, yielding a different embedded solution. However, one can show that our embedded solution possesses a stability group, a subgroup of $SU(N)$ under which (3.30) is invariant.

In terms of the ADHM variables, the $SU(N)$ gauge group of our theory acts only on the $\omega_{ui\dot{\alpha}}$, since only these carry an $SU(N)$ index, u. One can envisage these quantities as constituting a set of $2k$ complex N-vectors. If we have $N > 2k$ then there will exist a subgroup of $SU(N)$ that will not affect the instanton solution. We shall embed the k instanton solution in an $SU(2k)$ subgroup of the gauge group. Essential to this argument is the fact that we may always choose to arrange the $N \times 2k$ matrix

ω in upper-triangular form;

$$\begin{pmatrix} \omega_{11} & \cdots & \omega_{12k} \\ \vdots & \ddots & \vdots \\ \omega_{1N} & \cdots & \omega_{1\,2k} \end{pmatrix} = U. \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1\,2k} \\ 0 & \xi_{22} & \cdots & \xi_{2\,2k} \\ & \ddots & \ddots & \\ \vdots & & & \xi_{2k\,2k} \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.32}$$

The ξ_{ab} are complex except for those on the diagonal. Note that all elements in the lower $(N-2k) \times (N-2k)$ corner of the $SU(N)$ matrix leave ξ unchanged. Thus at least schematically we have

$$U \in \frac{SU(N)}{SU(N-2k)} \tag{3.33}$$

In quaternionic language where the vector x_n is represented as a 2×2 matrix, the action of the conformal group can be written as

$$x \mapsto x' = (Ax + B)(Cx + D)^{-1} \tag{3.34}$$

where A, B, C and D are quaternions. There are 15 variables in this transformation, corresponding to the dimension of the conformal group. Acting on the ADHM variable $\Delta(x)$ with the conformal group we get

$$\Delta(x; a, b) \mapsto \Delta'(x; a, b) \tag{3.35}$$

$$= a + bx' = \Delta(x; aD + bB, aC + bA)(Cx + D)^{-1} \tag{3.36}$$

The gauge field depends on the matrices U and $\bar{U}$, defined in (3.3). Thus we may ignore the factor $(Cx + D)^{-1}$ and write the action of the conformal group on the ADHM variables as

$$a \mapsto aD + bB, \qquad b \mapsto aC + bA \tag{3.37}$$

However, it may be necessary in general to return the quantities so obtained to their canonical forms. This is achieved using the symmetries of (3.16). Specifically, for b we require the existence of transformations such that

$$\Lambda b'\Upsilon^{-1} = \Lambda(aC + bA)\Upsilon^{-1} = b \tag{3.38}$$

where b has been placed back in the canonical form (3.17). The corresponding transformation on a then takes the form

$$a \mapsto \Lambda(aD + bB)\Upsilon^{-1} \tag{3.39}$$

We shall examine the effect of space-time translations;

$$x_n \mapsto x_n + \xi_n \Rightarrow x_{\alpha\dot{\alpha}} \mapsto x'_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}} + \xi_{\alpha\dot{\alpha}} \tag{3.40}$$

Thus the effect on Δ is as follows,

$$\Delta(x; a, b) \mapsto a + b(x + \xi) = \Delta(x; a + b\xi, b) \tag{3.41}$$

i.e. $\Delta(x + \xi; a, b) \mapsto \Delta(x; a + b\xi, b)$. Including the index structure, we have

$$a'_{(u+i\alpha)j\dot{\alpha}} = a_{(u+i\alpha)j\dot{\alpha}} + b^{\beta}{}_{u+i\alpha)j}\xi_{\beta\dot{\alpha}} \tag{3.42}$$

Recalling that b is in its canonical form, we can write

$$a' = \begin{pmatrix} \omega_{uj\dot{\alpha}} \\ (a'_{\alpha\dot{\alpha}})_{ij} + \delta_{ij}\xi_{\alpha\dot{\alpha}} \end{pmatrix} \tag{3.43}$$

$$\Rightarrow \omega_{uj\dot{\alpha}} \mapsto \omega_{uj\dot{\alpha}}, \quad (a'_{\alpha\dot{\alpha}})_{ij} \mapsto (a'_{\alpha\dot{\alpha}})_{ij} + \delta_{ij}\xi_{\alpha\dot{\alpha}} \tag{3.44}$$

From equation (3.21) we can put

$$a'_n \mapsto a'_n + \xi 1_{k\times k} \tag{3.45}$$

This looks likes the transformation of the coordinates themselves and allows us to identify the components of the a_n's proportional to the identity matrix as representing the coordinates of the instanton centre.

3.6 k-instantons in singular gauge

In the usual 1-instanton calculations it often proves fruitful to define the so-called singular gauge. The main advantage to this viewpoint comes from the fact that the gauge fields decrease swiftly with distance from the instanton centre so that they asymptotically approach zero as $x \to \infty$.

It is possible to define the singular gauge for the generalized k-instanton gauge field of the ADHM construction. We begin by defining the following decompositions of the ADHM variables;

$$U_{\lambda v} = U_{(u+i\alpha)v} = \begin{pmatrix} V_{uv} \\ (U'_{\alpha})_{iv} \end{pmatrix} \tag{3.46}$$

$$\bar{U}_v^{\lambda} = \bar{U}_v{}^{u+i\alpha} = \begin{pmatrix} \bar{V}_{vu} & (\bar{U}'^{\alpha})_{vi} \end{pmatrix} \tag{3.47}$$

$$\Delta_{\lambda j\dot{\alpha}} = \Delta_{(u+i\alpha)j\dot{\alpha}} = \begin{pmatrix} \omega_{uj\dot{\alpha}} \\ (\Delta'_{\alpha\dot{\alpha}})_{ij} \end{pmatrix} \tag{3.48}$$

$$\bar{\Delta}_i^{\dot{\alpha}\lambda} = \bar{\Delta}_i^{\dot{\alpha}(u+i\alpha)} = \begin{pmatrix} \bar{\omega}_{ju}^{\dot{\alpha}} & (\Delta'^{\dot{\alpha}\alpha})_{ji} \end{pmatrix} \tag{3.49}$$

Thus in this basis we have, for the LHS of the completeness relation, (3.7);

$$U_{\lambda v}\bar{U}_V{}^{\mu} = U_{(u+l\alpha)v}\bar{U}_v^{(w+k\beta)} = \begin{pmatrix} V_{uv}\bar{V}_{vw} & V_{uv}(\bar{U}'^{\beta})_{vk} \\ (U'_{\alpha})_{lv}\bar{V}_{vw} & (U'_{\alpha})_{lv}(\bar{U}'^{\beta})_{vk} \end{pmatrix}$$

The RHS becomes

$$\delta_{\lambda}{}^{\mu} - \Delta_{\lambda i\dot{\alpha}} f_{ij} \bar{\Delta}_j^{\dot{\alpha}\mu} = \begin{pmatrix} \delta_{wu} - \omega_{ui\dot{\alpha}} f_{ij} \bar{\omega}_{jw}^{\dot{\alpha}} & -\omega_{ui\dot{\alpha}} f_{ij} (\Delta'^{\dot{\alpha}\beta})_{jk} \\ -(\Delta'_{\alpha\dot{\alpha}})_{li} f_{ij} \bar{\omega}_{jw}^{\dot{\alpha}} & \delta_{lk}\delta_{\alpha}^{\beta} - (\Delta'_{\alpha\dot{\alpha}})_{li} f_{ij} (\Delta'^{\dot{\alpha}\beta})_{jk} \end{pmatrix}$$

Comparing first entries in the above gives the equation

$$V_{uv}\bar{V}_{vw} = \delta_{wu} - \omega_{ui\dot{\alpha}} f_{ij} \bar{\omega}_{jw}^{\dot{\alpha}}$$

If V is real then we can write

$$V = (1_{N\times N} - \omega_{\dot{\alpha}} f \bar{\omega}^{\dot{\alpha}})^{\frac{1}{2}} \tag{3.50}$$

Having determined V we can then calculate U' using

$$(U'_\alpha)_{lv} \bar{V}_{vw} = -(\Delta'_{\alpha\dot{\alpha}})_{li} f_{ij} \bar{\omega}^{\dot{\alpha}}_{jw} \tag{3.51}$$

Having found U' we may proceed to the gauge field via (3.5). Using these results we may calculate the asymptotic form of the ADHM quantities in the singular gauge.

$$\Delta_{\lambda i\dot{\alpha}} \rightarrow b^{\alpha}_{\lambda i} x_{\alpha\dot{\alpha}} \ ; \ \bar{\Delta}^{\dot{\alpha}\lambda}_{i} \rightarrow \bar{x}^{\dot{\alpha}\alpha} \bar{b}^{\lambda}_{i\alpha} \tag{3.52}$$

$$f_{ij} \rightarrow \frac{1}{x^2}\delta_{ij} \tag{3.53}$$

$$(U'_\alpha)_{i\nu} \rightarrow -\frac{x_{\alpha\dot{\alpha}}}{x^2}\bar{\omega}^{\dot{\alpha}}_{i\nu} \ ; \ (\bar{U}'^{\alpha})_{\nu i} \rightarrow -\frac{\bar{x}^{\dot{\alpha}\alpha}}{x^2}\omega_{\nu i\dot{\alpha}} \ (since\ \sigma^\dagger = \bar{\sigma}) \tag{3.54}$$

$$A_{n\,uv} \rightarrow \frac{1}{g}\frac{x^m}{x^4}\omega_{ui\dot{\alpha}} \bar{\sigma}_{mn}{}^{\dot{\alpha}}{}_{\dot{\beta}} \tag{3.55}$$

Taylor expanding (3.50) and retaining the next-to-leading order term, we have

$$V_{uv} \rightarrow \delta_{uv} - \frac{1}{2}\omega_{ui\dot{\alpha}}\frac{1}{x^2}\delta_{ij}\bar{\omega}^{\dot{\alpha}}_{jv} = \delta_{uv} - \frac{1}{2x^2}\omega_{ui\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iv} \tag{3.56}$$

For completeness and ease of reference we also write down the asymptotic form of the ADHM matrices $U_{\lambda u}$ and $\bar{U}^{\lambda}_{u}$;

$$U_{\lambda v} = U_{(u+i\alpha)v} \rightarrow \begin{pmatrix} \delta_{uv} - \frac{1}{2x^2}\omega_{ui\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iv} \\ -\frac{1}{x^2}x_{\alpha\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{i\nu} \end{pmatrix} \tag{3.57}$$

$$\bar{U}^{\lambda}_{v} = \bar{U}^{(u+i\alpha)}_{v} \rightarrow \begin{pmatrix} \delta_{vu} - \frac{1}{2x^2}\omega_{iv\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iu} & -\frac{1}{x^2}\bar{x}^{\dot{\alpha}\alpha}\omega_{vi\dot{\alpha}} \end{pmatrix} \tag{3.58}$$

3.7 Recovery of the t'Hooft expression for $k = 1$

We use the canonical form of (3.17), (3.18), (3.19) and (3.20). Setting $k = 1$ we may omit the i, j indices. Thus the a'_n becomes a 4-vector. From the discussion of the moduli-space we know that we should identify this vector with the centre of the instanton.

$$a'_n = -X_n$$

We now examine the form of the ADHM constraint equation, (3.27), in this case. Multiplying (3.27) throughout by τ^c gives

$$\tau^{c\dot{\gamma}}{}_{\dot{\delta}}\tau^{c\dot{\alpha}}{}_{\dot{\beta}}(\bar{\omega}^{\dot{\beta}}_{iu}\omega_{ui\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha}a'_{\alpha\dot{\alpha}}) = 0$$

But

$$\tau^{c\dot{\gamma}}{}_{\dot{\delta}}\tau^{c\dot{\alpha}}{}_{\dot{\beta}} = 2\left(\delta^{\dot{\alpha}}{}_{\dot{\delta}}\delta^{\dot{\gamma}}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\gamma}}{}_{\dot{\delta}}\delta^{\dot{\alpha}}{}_{\dot{\beta}}\right)$$

Therefore

$$\bar{\omega}^{\dot{\gamma}}_u\omega_{ui\dot{\delta}} - \frac{1}{2}\bar{\omega}^{\dot{\alpha}}_u\omega_{u\dot{\alpha}}\delta^{\dot{\gamma}}{}_{\dot{\delta}} + \bar{a}'^{\dot{\gamma}\alpha}a'_{\alpha\dot{\delta}} - \frac{1}{2}\bar{a}'^{\dot{\alpha}\alpha}a'_{\alpha\dot{\alpha}}\delta^{\dot{\gamma}}{}_{\dot{\delta}} = 0 \tag{3.59}$$

Consider the third term above. Decomposing this in terms of the Pauli matrices gives

$$\bar{a}'^{\dot{\gamma}\alpha}a'_{\alpha\dot{\delta}} = a_n a_m \bar{\sigma}^{\dot{\gamma}\alpha}_n \sigma_{m\alpha\dot{\delta}}$$

Let $n, m \neq 4$, then,

$$a_b a_c \tau^{b\dot{\gamma}\alpha}\tau^c_{\alpha\dot{\delta}} = a_b a_c(\delta^{bc}\delta^{\dot{\gamma}}{}_{\dot{\delta}} + \epsilon^{bcd}\tau^{d\dot{\gamma}}{}_{\dot{\delta}}) = a_b a_b \delta^{\dot{\gamma}}{}_{\dot{\delta}}$$

If $n = m = 4$ then we get

$$a_4 a_4 \delta^{\dot{\gamma}}{}_{\dot{\delta}}$$

If $m = 4, n \neq 4$ then we have

$$i a_b a_4 \tau^{a\dot{\gamma}}{}_{\dot{\delta}}$$

Whilst $n = 4, m \neq 4$ gives

$$-ia_b a_4 \tau^{a\dot{\gamma}}{}_{\dot{\delta}}$$

Putting these results together,

$$\bar{a}'^{\dot{\gamma}\alpha} a'_{\alpha\dot{\delta}} = a_n a_n \delta^{\dot{\gamma}}{}_{\dot{\delta}}$$

Now we consider the fourth term in equation (3.59)

$$-\frac{1}{2}\bar{a}'^{\dot{\alpha}\alpha} a'_{\alpha\dot{\alpha}} \delta^{\dot{\gamma}}{}_{\dot{\delta}} = -\frac{1}{2} a_n a_m \bar{\sigma}_n^{\dot{\alpha}\alpha} \sigma_{m\alpha\dot{\alpha}} \delta^{\dot{\gamma}}{}_{\dot{\delta}}$$

If $n, m \neq 4$, then the above becomes,

$$-\frac{1}{2} a_b a_c Tr(\tau^b \tau^c) \delta^{\dot{\gamma}}{}_{\dot{\delta}} = -a_b a_b \delta^{\dot{\gamma}}{}_{\dot{\delta}}$$

If $m = n = 4$ then

$$-\frac{1}{2} a_4 a_4 Tr(1_{2\times 2}) \delta^{\dot{\gamma}}_{\dot{\delta}} = -a_4 a_4 \delta^{\dot{\gamma}}{}_{\dot{\delta}}$$

Whilst if $m = 4, n \neq 4$ or $n = 4, m \neq 4$ then

$$-\frac{1}{2} a_4 a_b Tr(\tau^b) \delta^{\dot{\gamma}}{}_{\dot{\delta}} = 0$$

Thus altogether the fourth term becomes

$$-a_n a_n \delta^{\dot{\gamma}}{}_{\dot{\delta}}$$

Using these results in the ADHM equations we notice that the a_n variables cancel out so the $k = 1$ ADHM constraint becomes

$$\bar{\omega}_u^{\dot{\alpha}} \omega_{u\dot{\beta}} = \frac{1}{2} \bar{\omega}_u^{\dot{\gamma}} \omega_{u\dot{\gamma}} \delta^{\dot{\alpha}}{}_{\dot{\beta}} = \rho^2 \delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{3.60}$$

Where we have defined the quantity $\rho^2 = \frac{1}{2}\bar{\omega}_u^{\dot{\alpha}} \omega_{u\dot{\alpha}}$.

For the 1-instanton case we solve this constraint by writing

$$\omega_{u\dot{\alpha}} = \rho U_{N\times N} \begin{pmatrix} 1_{2\times 2} \\ 0_{(N-2)\times 2} \end{pmatrix}$$

For $k = 1$, f is a scalar given by (3.24) as

$$f = \frac{1}{\rho^2 + (x_n - X_n)^2} \tag{3.61}$$

(where we have made a minor simplification to the standard form of f). We now follow the procedure outlined above, calculating first V via (3.50), which will allow us to find U using (3.51).

$$V_{uv} = \left(\delta_{uv} - \frac{1}{\rho^2 + (x_n - X_n)^2}\omega_{u\dot{\alpha}}\bar{\omega}_v^{\dot{\alpha}}\right)^{\frac{1}{2}} \tag{3.62}$$

Let us define V by an expansion in ω.

$$V_{uv} = A\delta_{uv} + B\omega_{\dot{\alpha}u}\bar{\omega}_v^{\dot{\alpha}}$$

Squaring and using (3.59) gives

$$\begin{aligned}(V^2)_{uv} &= A^2\delta_{uv} + 2AB\omega_{\dot{\alpha}u}\bar{\omega}_v^{\dot{\alpha}} + B^2\omega_{\dot{\alpha}u}\bar{\omega}_w^{\dot{\alpha}}\omega_{\dot{\beta}w}\bar{\omega}_v^{\dot{\beta}} \\ &= A^2\delta_{uv} + \left(2AB + B^2\rho^2\right)\omega_{\dot{\alpha}u}\bar{\omega}_v^{\dot{\alpha}}\end{aligned}$$

Comparing this with equation (3.62) we have

$$A^2 = 1$$

$$2AB + B^2\rho^2 = -\frac{1}{\rho^2 + (x_n - X_n)^2}$$

The solution to this quadratic equation is given by

$$B = -\frac{A}{\rho^2} \pm \frac{1}{\rho^2}\frac{|x_n - X_n|}{\sqrt{\rho^2 + (x - X)^2}}$$

So our result for V is

$$V_{uv} = A\delta_{uv} + \frac{1}{\rho^2}\left(-A \pm \frac{|x_n - X_n|}{\sqrt{\rho^2 + (x-X)^2}}\right)\omega_{\dot{\alpha}u}\bar{\omega}_v^{\dot{\alpha}} \tag{3.63}$$

If we want the 'positive square root' of the matrix V^2 then we can set A = +1 to give

$$V_{uv} = \delta_{uv} + \frac{1}{\rho^2}\left(\frac{|x_n - X_n|}{\sqrt{\rho^2 + (x-X)^2}} - 1\right)\omega_{\dot{\alpha}u}\bar{\omega}_v^{\dot{\alpha}} \tag{3.64}$$

Turning now to equation (3.51) for U, we have, (with a and b in thier canonical forms),

$$\Delta'_{\alpha\dot{\alpha}} = a'_{\alpha\dot{\alpha}} + x_{\alpha\dot{\alpha}} = (x_n - X_n)\sigma^n_{\alpha\dot{\alpha}}$$

Therefore we have

$$-\Delta'_{\alpha\dot{\alpha}} f \bar{\omega}_u^{\dot{\alpha}} = \frac{(X_n - x_n)}{\rho^2 + (x-X)^2}\sigma^n_{\alpha\dot{\alpha}}\bar{\omega}_u^{\dot{\alpha}}$$

To complete the expression for U we must find V^{-1}. Recall that $V^2 = F \Rightarrow V = FV^{-1}(= V^2V^{-1})$. Since we now know both V^2 and V we can find V^{-1}, which we shall write in the form

$$V^{-1}_{uv} = C\delta_{uv} + D\omega_{\dot{\alpha}u}\bar{\omega}_v^{\dot{\alpha}}$$

Therefore we have,

$$V_{uv} = C\delta_{uv} + \left(D - \frac{C}{\rho^2 + (x-X)^2} - \frac{D\rho^2}{\rho^2 + (x-X)^2}\right)\omega_{\dot{\alpha}u}\bar{\omega}_v^{\dot{\alpha}}$$

Comparing coefficients with equation (3.64) gives $C = 1$ and

$$D = \frac{1}{\rho^2}\left(\pm\frac{\sqrt{\rho^2 + (x-X)^2}}{|(x-X)|} - 1\right)$$

Substituting these results into equation (3.51) yields an expression for U',

$$U'_{u\alpha} = \frac{(X_n - x_n)}{2\rho^2 + (x-X)^2}\sigma_{n\alpha\dot{\alpha}}\bar{\omega}_v^{\dot{\alpha}}\left(\delta_{vu} + \frac{1}{\rho^2}\left(\pm\frac{\sqrt{\rho^2 + (x-X)^2}}{|(x-X)|} - 1\right)\omega_{\dot{\alpha}v}\bar{\omega}_u^{\dot{\alpha}}\right)$$

$$= \mp \frac{1}{|x - X|\sqrt{(x-X)^2 + \rho^2}} (X_n - x_n)\sigma_{n\alpha\dot{\alpha}}\bar{\omega}_u^{\dot{\alpha}}$$

Finally we wish to calculate the gauge field. Using 3.5

$$(A_n)_{uv} = \bar{V}_{wu}\partial_n V_{uv} + U'^{\alpha}_w \partial_n U'_{\alpha v}$$

$$= \frac{2\bar{\sigma}_{mn}(x_m - X_m)}{(x-X)^2((x-X)^2 + \rho^2)} \omega_{u\dot{\alpha}}\bar{\omega}_v^{\dot{\alpha}}$$

3.8 The ADHM construction for $k = 1$

Note from (3.60) that the a_n variables cancel out of the ADHM constraint equations. Since the ADHM equations thus have nothing to say about these quantities, we can drop them altogether and consider the moduli space metric to be

$$\tilde{g} = 2d\bar{\omega}_u^{\dot{\alpha}} d\omega_{u\dot{\alpha}} \tag{3.65}$$

In this case our coordinates on the moduli space become

$$z^{\bar{i}\dot{\alpha}} = \begin{pmatrix} \bar{\omega}_u^{\dot{\alpha}} \\ \epsilon^{\dot{\alpha}\dot{\beta}}\omega_{u\dot{\beta}} \end{pmatrix} \tag{3.66}$$

and so the moduli space has dimension $4(N-1)$.

3.9 The ADHM construction and the hyper-Kahler quotient

In the previous chapter we demonstrated that the instanton moduli space is an example of a hyper-Kahler manifold. So far in this chapter we have developed the ADHM construction. We now discuss how the ADHM construction is an example of a more general procedure called the hyper-Kahler quotient construction [39]. We start with some larger dimensional hyper-Kahler manifold called the mother space, $\tilde{\mathcal{M}}$, the metric and complex structure of which are preserved by a group of isometries G. The hyper-Kahler quotient procedure then yields another hyper-Kahler space called the daughter space $\mathcal{M}$. The fact that G preserves g and I is written as the vanishing of the Lie derivatives

$$L_{X_a}g = L_{X_a}I = 0 \tag{3.67}$$

This further implies that the Lie derivative of the Kahler form ω must also vanish. This can be written as

$$L_{X_a}\omega = (di_{X_a} + i_{X_a}d)\omega = 0$$

Because we have $d\omega = 0$ (closed), the condition $L_{X_a}\omega = 0$ is equivalent to the 1-form $i_{X_a}\omega$ being closed. We shall assume that this form is also exact, so that there exists a function μ_a such that

$$i_a\omega = d\mu_a$$

(This would be true if $\mathcal{M}$ were simply connected and had a trivial first cohomology group or if g were semi-simple). Such a function is called a Hamiltonian function and allows us to define the so-called moment mapping, $\mu : \mathcal{M} \to g^*$.

$$\mu = \sum_a \mu_a T^a \tag{3.68}$$

where the T^a are the generators of the group G. On a hyper-Kahler space there are three such moment maps, one for each of the complex structures. The quotient manifold is then

$$\mathcal{M} = \frac{\mu^{-1}(0)}{G} \tag{3.69}$$

Thus the daughter space is the subspace of $\tilde{\mathcal{M}}$ on which the moment map μ vanishes, quotiented by the group G.

3.9.1 Moment maps of the ADHM construction

We shall consider the one-instanton moduli space ($k = 1$). The metric in this case is given as

$$g = 2d\bar{\omega}_u^{\dot{\alpha}} d\omega_{u\dot{\alpha}}$$

Thus the mother space $\tilde{\mathcal{M}}$ is Euclidean. The three Kahler forms are

$$\omega^c = i(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} d\omega_{u\dot{\alpha}} \wedge d\bar{\omega}_u^{\dot{\beta}}$$

Now consider a $U(1)$ symmetry acting on the coordinates of the manifold. Such an action is evidently an isometry (symmetry of the metric)

$$\omega_{u\dot{\alpha}} \mapsto e^{i\theta} \omega_{u\dot{\alpha}}$$

$$\bar{\omega}_u^{\dot{\alpha}} \mapsto e^{-i\theta} \bar{\omega}_u^{\dot{\alpha}}$$

For infinitesimal transformations we then have

$$\delta\omega_{u\dot{\alpha}} = i\theta\omega_{u\dot{\alpha}}$$

$$\delta\bar{\omega}_u^{\dot{\alpha}} = -i\theta\bar{\omega}_u^{\dot{\alpha}}$$

Thus the required Killing vector is

$$\mathbf{X} = \omega_{u\dot{\alpha}} \frac{\partial}{\partial\omega_{u\dot{\alpha}}} - \bar{\omega}_u^{\dot{\alpha}} \frac{\partial}{\partial\bar{\omega}_u^{\dot{\alpha}}}$$

We can now find the contraction of the Kahler form with this vector,

$$i_{\mathbf{X}}\omega^c = \left(i(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} d\omega_{u\dot{\alpha}} \wedge d\bar{\omega}_u^{\dot{\beta}}\right) \left[\omega_{v\dot{\gamma}} \frac{\partial}{\partial\omega_{v\dot{\gamma}}} - \bar{\omega}_v^{\dot{\gamma}} \frac{\partial}{\partial\bar{\omega}_v^{\dot{\gamma}}}\right]$$

$$= i(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \left\langle d\omega_{u\dot{\alpha}} \, , \, \omega_{v\dot{\gamma}} \frac{\partial}{\partial\omega_{v\dot{\gamma}}} - \bar{\omega}_v^{\dot{\gamma}} \frac{\partial}{\partial\bar{\omega}_v^{\dot{\gamma}}} \right\rangle d\bar{\omega}_u^{\dot{\beta}} - i(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \left\langle d\bar{\omega}_u^{\dot{\beta}} \, , \, \omega_{v\dot{\gamma}} \frac{\partial}{\partial\omega_{v\dot{\gamma}}} - \bar{\omega}_v^{\dot{\gamma}} \frac{\partial}{\partial\bar{\omega}_v^{\dot{\gamma}}} \right\rangle d\bar{\omega}_u^{\dot{\beta}}$$

$$= i(\tau^c)^{\dot\alpha}{}_{\dot\beta}\delta_{uv}\delta^{\dot\alpha}{}_{\dot\beta}\omega_{v\dot\gamma}d\bar\omega^{\dot\beta}u + i(\tau^c)^{\dot\alpha}{}_{\dot\beta}\delta_{uv}\delta^{\dot\beta}{}_{\dot\gamma}\bar\omega^{\dot\gamma}_v d\omega_{u\dot\alpha}$$

$$= i(\tau^c)^{\dot\alpha}{}_{\dot\beta}\omega_{u\dot\alpha}d\bar\omega^{\dot\beta}_u + i(\tau^c)^{\dot\alpha}{}_{\dot\beta}\bar\omega^{\dot\beta}_u d\omega_{u\dot\alpha}$$

$$= d\left(i(\tau^c)^{\dot\alpha}{}_{\dot\beta}\omega_{u\dot\alpha}\bar\omega^{\dot\beta}_u\right)$$

This in turn yields the moment maps

$$(\tau^c)^{\dot\alpha}{}_{\dot\beta}\bar\omega^{\dot\beta}_u\omega_{u\dot\alpha} = \xi^c$$

Thus we have reproduced the ADHM constraint equation, (3.27). The group G by which we quotient the level set is then the $U(1)$ symmetry group of the canonical ADHM variables.

3.10 The ADHM construction and the metric on $\mathcal{M}_k$

The mother space $\tilde{\mathcal{M}}_k$ is the Euclidean space $R^{4k(k+N)}$. With coordinates $a_{\lambda j\dot\alpha}$ the metric $\tilde g$ on $\tilde{\mathcal{M}}$ can be written as

$$\tilde g = (2d\bar\omega^{\dot\alpha}_{iu}d\omega_{ui\dot\alpha} + d(\bar a'^{\dot\alpha\alpha})_{ij}d(a'_{\alpha\dot\alpha})_{ji}) = 2Tr_k\left(d\bar\omega^{\dot\alpha} + da'_n da'_n\right) \tag{3.70}$$

We can realize a symplectic structure on $\tilde{\mathcal{M}}_k$ by introducing the coordinates $z^{\tilde i\dot\alpha}, \tilde i = 1, ..., 2k(N+k)$. We choose the real set of coordinates;

$$z^{\tilde i\dot\alpha} = \begin{pmatrix} \bar\omega^{\dot\alpha}_{iu} \\ (\bar a'^{\dot\alpha 1})_{ij} \\ \epsilon^{\dot\alpha\dot\beta}\omega_{ui\dot\beta} \\ \epsilon^{\dot\alpha\dot\beta}(a'_{1\dot\beta})_{ij} \end{pmatrix} = \begin{pmatrix} \bar\omega^{\dot\alpha}_{iu} \\ (\bar a'^{\dot\alpha 1})_{ij} \\ W^{\dot\alpha}_{ui} \\ A^{\dot\alpha}_{ij} \end{pmatrix} \tag{3.71}$$

Where for convenience and clarity we sometimes use $\omega_{ui\dot\beta} = \epsilon_{\dot\beta\dot\alpha}W^{\dot\alpha}_{ui}$ and $A^{\dot\alpha}_{ij} = \epsilon^{\dot\alpha\dot\beta}(a'_{1\dot\beta})_{ij}$. Using this notation we can write the metric as

$$\tilde g = \tilde\Omega_{\tilde i\tilde j}\epsilon_{\dot\alpha\dot\beta}dz^{\tilde i\dot\alpha}dz^{\tilde j\dot\beta} \tag{3.72}$$

Where the symplectic matrix $\tilde{\Omega}$ is given by

$$\tilde{\Omega}_{\tilde{i}\tilde{j}} = \begin{pmatrix} 0 & 0 & \delta_{il}\delta_{uv} & 0 \\ 0 & 0 & 0 & \delta_{im}\delta_{jl} \\ -\delta_{il}\delta_{uv} & 0 & 0 & 0 \\ 0 & -\delta_{im}\delta_{jl} & 0 & 0 \end{pmatrix} \tag{3.73}$$

The indices $\tilde{i}$ and $\tilde{j}$ take the values $\{iu, ij, ui, ij\}$ and $\{lv, lm, vl, lm\}$ respectively.

3.10.1 Killing vector fields

Here we develop some results involving the Killing vector fields on $\mathcal{M}_k$ for use in the next chapter. A tensor field T is invariant under a vector field V if its Lie derivative vanishes, $L_V T = 0$. A Killing vector field is one under which the metric remains invariant. Let X be a vector field on a manifold $\mathcal{M}$. If an infinitesimal displacement, ξX generates an isometry of the metric then X is a Killing vector field. Under such a displacement the coordinates become $x^\mu \mapsto x^\mu + \xi X^\mu$. For an isometry we must then have

$$\frac{\partial(x^\kappa + \xi X^\kappa)}{\partial x^\mu}\frac{\partial(x^\lambda + \xi X^\lambda)}{\partial x^\nu} g_{\kappa\lambda}(x + \xi X) = g_{\mu\nu}(x) \tag{3.74}$$

If the group parameters do not depend on the coordinates (i.e. the symmetry is not gauged) then we get

$$\left(\delta^\kappa_\mu + \xi\frac{\partial X^\kappa}{\partial x^\mu}\right)\left(\delta^\lambda_\nu + \xi\frac{\partial X^\lambda}{\partial x^\nu}\right) g_{\kappa\lambda}(x + \xi X) = g_{\mu\nu}(x) \tag{3.75}$$

Expanding to first order in ξ,

$$X^\lambda\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial X^\lambda}{\partial x^\nu} g_{\mu\lambda} + \frac{\partial X^\kappa}{\partial x^\mu} g_{\kappa\nu} = 0 \tag{3.76}$$

All the Killing vectors of a manifold form a Lie algebra of the symmetric operations on the manifold. Consider now a group action on the coordinates of a manifold. The coordinates transform as,

$$x'^{\mu} = (e^{i\xi^r T^r})^{\mu}{}_{\nu} x^{\nu}$$

$$\approx (\delta^{\mu}_{\nu} + i\xi^r T^{r\mu}{}_{\nu}) x^{\nu}$$

Comparing with the above we see that we have r Killing vectors, X^r, one for each generator of the group, and their components are

$$X^{r\mu} = iT^{r\mu}{}_{\nu} x^{\nu} \tag{3.77}$$

Since the group generators are not functions of the coordinates we have the Killing equation

$$T^{r\kappa}{}_{\lambda} x^{\lambda} \partial_{\kappa} g_{\mu\nu} + T^{r\lambda}{}_{\nu} g_{\mu\lambda} + T^{r\kappa}{}_{\mu} g_{\kappa\nu} = 0$$

And the Killing vector fields are

$$X^r = iT^{r\mu}{}_{\nu} x^{\nu} \frac{\partial}{\partial x^{\mu}} \tag{3.78}$$

For a complex manifold there are two types of coordinate, z^{μ} and $\bar{z}^{\mu}$. For the real coordinates we proceed as before to get

$$X^r = iT^{r\mu}{}_{\nu} z^{\nu} \tag{3.79}$$

whilst the complex conjugate coordinate transforms as

$$\bar{z}'^{\mu} = \bar{z}^{\nu} [(e^{-i\xi T^r})^{\mu}_{\nu}]^* \tag{3.80}$$

$$\approx \bar{z}^{\nu} (\delta^{\mu}_{\nu} - i\xi (T^{r\mu}{}_{\nu})^*)$$

$$\Rightarrow \bar{X}^{r\mu} = -i\bar{z}^{\nu} (T^{r\mu}{}_{\nu})^* \tag{3.81}$$

Where we have assumed that the group parameters ξ are real. For Hermitian generators we get

$$\bar{X}^{r\mu} = -i\bar{z}^{\nu} T^r{}_{\nu}{}^{\mu} \tag{3.82}$$

In this case the Killing vector fields become

$$\begin{aligned} X^r &= X^{r\mu}\frac{\partial}{\partial z^\mu} + \bar{X}^{r\mu}\frac{\partial}{\partial \bar{z}^\mu} \\ &= iT^{r\mu}{}_\nu z^\nu \frac{\partial}{\partial z^\mu} - i\bar{z}^\nu T^r{}_\nu{}^\mu \frac{\partial}{\partial \bar{z}^\mu} \end{aligned} \tag{3.83}$$

3.10.2 $SU(N)$ Killing vector fields on $\mathcal{M}_k$

Since the mother space is Euclidean we need maintain no distinction between upper and lower indices. Thus we may re-write (3.83) as

$$X^r = iT^r_{\mu\nu}\left(z^\nu \frac{\partial}{\partial z^\mu} - \bar{z}^\mu \frac{\partial}{\partial \bar{z}^\nu}\right)$$

Using the normalization of the generators we can multiply throughout by the group generator, giving an object independent of the Lie algebra label r.

$$X^r T^r_{\mu\nu} = i\left(z^\mu \frac{\partial}{\partial z^\nu} - \bar{z}^\nu \frac{\partial}{\partial \bar{z}^\mu}\right)$$

The $SU(N)$ gauge symmetry of the theory acts as a global symmetry on (part of) the moduli space. The ω variables carry an $SU(N)$ index and transform under infinitesimal transformations as follows,

$$\omega'_{ui\dot{\alpha}} = (\delta_{uv} + i\xi T^r_{uv})\omega_{vi\dot{\alpha}} \;\Rightarrow\; X^r_u = iT^r_{uv}\omega_{vi\dot{\alpha}}$$

$$\bar{\omega}'^{\dot{\alpha}}_{iu} = \bar{\omega}^{\dot{\alpha}}_{vi}(\delta_{uv} - i\xi(T^r_{uv})^*) \;\Rightarrow\; \bar{X}^r_u = -i\bar{\omega}'^{\dot{\alpha}}_{iv}T^r_{vu}$$

The Killing vectors associated with these $SU(N)$ transformations are therefore

$$X^r = iT^r_{uv}\omega_{vi\dot{\alpha}}\frac{\partial}{\partial \omega_{ui\dot{\alpha}}} - i\bar{\omega}^{\dot{\alpha}}_{iv}T^r_{vu}\frac{\partial}{\partial \bar{\omega}^{\dot{\alpha}}_{iu}} \tag{3.84}$$

3.10.3 $U(k)$ Killing vector fields on $\mathcal{M}_k$

There are also Killing vector fields on $\mathcal{M}_k$ corresponding to the $U(k)$ subgroup of $Gl(k, C)$ that leaves invariant the canonical form of the ADHM quantity b. Up to an infinitesimal factor, the action of a Killing vector X_r is defined by

$$X_r = \delta_r(X^\mu)\frac{\partial}{\partial X^\mu}$$

In the case of the ADHM $U(k)$ group we will then have

$$X_k = \delta_r(\bar{\omega}^{\dot{\alpha}}_{iu})\frac{\partial}{\partial \bar{\omega}^{\dot{\alpha}}_{iu}} + \delta_r(\omega_{ui\dot{\alpha}})\frac{\partial}{\partial \omega_{ui\dot{\alpha}}} + \delta_r(a'_{n\,ij})\frac{\partial}{\partial a'_{n\,ij}}$$

The group action on these variables was given in equation (3.26). Writing this group element in terms of the exponentiated generators, we have

$$\Xi_{ij} = (e^{-i\theta^r T^r})_{ij}$$

For infinitesimal parameters θ^r we can approximate the exponential by the first term in its power series expansion.

$$\omega'_{ui\dot{\alpha}} = \omega_{uj\dot{\alpha}}(e^{-i\theta^r T^r})_{ji} \approx \omega_{uj\dot{\alpha}}(\delta_{ji} - i\theta^r T^r_{ji})$$

$$\Rightarrow \delta\omega_{ui\dot{\alpha}} = -i\theta^r \omega_{uj\dot{\alpha}} T^r_{ji} \tag{3.85}$$

Taking the Hermitian conjugate of the above equation gives

$$\delta\bar{\omega}^{\dot{\alpha}}_{iu} = i\theta^r T^r_{ij}\bar{\omega}^{\dot{\alpha}}_{ju} \tag{3.86}$$

Where we have used the fact that the generators are hermitian and we have assumed that the group parameters θ^r are real. Finally, we consider how the $a'_{n\,ij}$ transform under an infinitesimal group transformation.

$$a''_{n\,ij} = (e^{i\bar{\theta}^r T^r})_{ik} a'_{n\,kl}(e^{-i\theta^r T^r})_{lj}$$

$$\approx (\delta_{ik} + i\bar{\theta}^r T^r_{ik}) a'_{n\,kl} (\delta_{lj} - i\theta^r T^r_{lj})$$

$$\Rightarrow \delta a'_{n\,ij} = i\theta^r [T^r, a'_n]_{ij} \tag{3.87}$$

Removing the infinitesimal factor of θ^r, the Killing vector field associated to the r^{th} $U(k)$ generator is

$$X^r = iT^r_{ij}\bar{\omega}^{\dot{\alpha}}_{ju}\frac{\partial}{\partial\bar{\omega}^{\dot{\alpha}}_{iu}} - i\omega_{uj\dot{\alpha}}T^r_{ji}\frac{\partial}{\partial\omega_{ui\dot{\alpha}}} + i[T^r, a'_n]_{ij}\frac{\partial}{\partial a'_{n\,ij}} \tag{3.88}$$

Note that this expression is not written in terms of the coordinate basis of $\frac{\partial}{\partial z^{i\dot{\alpha}}}$. To do this we must express the derivatives in above in terms of derivatives with respect to the coordinate functions. Let

$$W^{\dot{\alpha}}_{ui} = \epsilon^{\dot{\alpha}\dot{\beta}}\omega_{ui\dot{\beta}} \;\Rightarrow\; \epsilon_{\dot{\gamma}\dot{\alpha}}W^{\dot{\alpha}}_{ui} = \omega_{ui\dot{\gamma}}$$

Therefore

$$\frac{\partial}{\partial W^{\dot{\alpha}}_{ui}} = \frac{\partial\omega_{vj\dot{\beta}}}{\partial W^{\dot{\alpha}}_{ui}}\frac{\partial}{\partial\omega_{vj\dot{\beta}}} = \epsilon_{\dot{\beta}\dot{\alpha}}\frac{\partial}{\partial\omega_{vi\dot{\beta}}}$$

$$\Rightarrow \frac{\partial}{\partial\omega_{ui\dot{\alpha}}} = \epsilon^{\dot{\beta}\dot{\alpha}}\frac{\partial}{\partial W^{\dot{\beta}}_{ui}} \tag{3.89}$$

Next we wish to express $\frac{\partial}{\partial a'_n}$ in terms of the coordinate basis for the tangent vectors, $\frac{\partial}{\partial\bar{a}'^{\dot{\alpha}1}}$ and $\frac{\partial}{\partial A^{\dot{\alpha}}_{ij}}$. To do this we introduce

$$A^{\dot{\alpha}}_{ij} = \epsilon^{\dot{\alpha}\dot{\beta}}(a'_{1\dot{\beta}})_{ij} = \epsilon^{\dot{\alpha}\dot{\beta}}a'_{mij}\sigma^m_{1\dot{\beta}} \tag{3.90}$$

Since $\bar{a}'^{\dot{\alpha}1}$ and $A^{\dot{\alpha}}_{ij}$ each have an index set equal to 1, they each account for half of the degrees of freedom in a'_n. Our particular choice of coordinates ensures that the coordinates and the associated tangent vector basis are real. Evaluating the derivative;

$$\frac{\partial}{\partial a'_{n\,ij}} = \frac{\partial\bar{a}'^{\dot{\alpha}1}_{kl}}{\partial a'_{n\,ij}}\frac{\partial}{\partial\bar{a}'^{\dot{\alpha}1}_{kl}} + \frac{\partial A^{\dot{\alpha}}_{kl}}{\partial a'_{n\,ij}}\frac{\partial}{\partial A^{\dot{\alpha}}_{kl}} \tag{3.91}$$

Where

$$\bar{a}'^{\dot{\alpha}1}_{kl} = a'_{m\,kl}\bar{\sigma}^{\dot{\alpha}1}_{m} \Rightarrow \frac{\partial \bar{a}'^{\dot{\alpha}1}_{kl}}{\partial a'_{n\,ij}} = \delta_{ki}\delta_{lj}\bar{\sigma}^{\dot{\alpha}1}_{n}$$

and from (3.90)

$$\frac{\partial A^{\dot{\alpha}}_{kl}}{\partial a'_{n\,ij}} = \epsilon^{\dot{\alpha}\dot{\beta}}\delta_{ki}\delta_{lj}\sigma_{n\,1\dot{\beta}}$$

Substituting this into (3.91) gives

$$\frac{\partial}{\partial a'_{n\,ij}} = \bar{\sigma}^{\dot{\alpha}1}_{n}\frac{\partial}{\partial \bar{a}'^{\dot{\alpha}1}_{ij}} + \epsilon^{\dot{\alpha}\dot{\beta}}\sigma_{n\,1\dot{\beta}}\frac{\partial}{\partial A^{\dot{\alpha}}_{ij}} \tag{3.92}$$

Using these results the Killing vector becomes

$$X^r = iT^r_{ij}\bar{\omega}^{\dot{\alpha}}_{ju}\frac{\partial}{\partial \bar{\omega}^{\dot{\alpha}}_{iu}} - i\epsilon^{\dot{\alpha}\dot{\beta}}\omega_{uj\dot{\beta}}T^r_{ji}\frac{\partial}{\partial W^{\dot{\alpha}}_{ui}} + i[T^r,\bar{a}'^{\dot{\alpha}1}]_{ij}\frac{\partial}{\partial \bar{a}'^{\dot{\alpha}1}_{ij}} + i\epsilon^{\dot{\alpha}\dot{\beta}}[T^r,a'_{1\dot{\beta}}]_{ij}\frac{\partial}{\partial A^{\dot{\alpha}}_{ij}} \tag{3.93}$$

For later use we shall also require the Killing vector written in the standard form

$$X^r = iT^r_{\tilde{i}\tilde{j}}\left(z^{\tilde{j}\dot{\alpha}}\frac{\partial}{\partial z^{\tilde{i}\dot{\alpha}}}\right) \tag{3.94}$$

It is a simple matter to re-write X^r in this form:

$$X^r = iT^r_{il}\delta_{uv}\left(\bar{\omega}^{\dot{\alpha}}_{lv}\frac{\partial}{\partial \bar{\omega}^{\dot{\alpha}}_{iu}}\right) - iT^r_{li}\delta_{uv}\left(W^{\dot{\alpha}}_{vl}\frac{\partial}{\partial W^{\dot{\alpha}}_{ui}}\right)$$

$$+i(T^r_{il}\delta_{jm} - T^r_{mj}\delta_{il})\left(\bar{a}'^{\dot{\alpha}1}_{lm}\frac{\partial}{\partial \bar{a}'^{\dot{\alpha}1}_{ij}}\right) + i(T^r_{il}\delta_{jm} - T^r_{mj}\delta_{il})\left(A^{\dot{\alpha}}_{lm}\frac{\partial}{\partial A^{\dot{\alpha}}_{ij}}\right) \tag{3.95}$$

From here we can read off the components of $T^r_{\tilde{i}\tilde{j}}$

$$T^r_{\tilde{i}\tilde{j}} = \begin{pmatrix} T^r_{il}\delta_{uv} & 0 & 0 & 0 \\ 0 & T^r_{il}\delta_{jm} - T^r_{mj}\delta_{il} & 0 & 0 \\ 0 & 0 & -T^r_{li}\delta_{uv} & 0 \\ 0 & 0 & 0 & T^r_{il}\delta_{jm} - T^r_{mj}\delta_{il} \end{pmatrix} \tag{3.96}$$

Where as before, the indices $\tilde{i}$ and $\tilde{j}$ take on the values $\{iu, ij, ui, ij\}$ and $\{lv, lm, vl, lm\}$ respectively.

Chapter 4

Supersymmetric instanton branes

4.1 Introduction

Classical Yang-Mils instantons are field configurations of finite action which solve the self-dual Yang-Mills equation. Here we develop the approximation discussed in the introduction whereby we allow the collective coordinates of our solution to depend on other variables, such as time. In fact a useful trick inspired by string theory is to embed these objects in a Minkowski space-time of higher dimension. The instanton can then be allowed to live on any four-dimensional Euclidean subspace. We intend to examine the status of such solutions in supersymmetric field theory. This will necessitate the introduction of additional terms into the action to include fermionic fields. Therefore the equations of motion will themselves be altered to accommodate these new fields. We will find that the pure gauge instanton solution will cease to be an exact solution of the fully coupled equations of motion. However, it will remain an approximate solution at lowest order in the coupling constant g. We then obtain a refinement of this solution by pursuing an expansion in powers of the coupling constant. The effective action we obtain will resemble (1.29). The terminology developed to describe these cases is part of the language of branes. For an instanton embedded in a D-dimensional Minkowski space, the solution will have no dependence on $D-5$ of the space-time coordinates, (one of them being the time coordinate). Thus it will represent an object which is extended in those $(D-5)$ directions. These are called $(D-5)$-branes.

This chapter follows the analysis of [12].

4.2 Pure gauge instanton branes

We now intend to embed four dimensional instanton solutions in gauge theories of higher dimensions. For simplicity our first consideration will be pure gauge theory.

Consider a gauge theory in a D dimensional Minkowski space. Let the space-time coordinates be denoted by $y^N, N = 0, 1, ..., D-1$. The pure gauge action is,

$$S = \frac{1}{2g_D^2} \int d^D y \; Tr_N(F_{MN}F^{MN})$$

To achieve the embedding we shall break up the D-dimensional Minkowski space into two parts. Firstly we will require a four-dimensional Euclidean space in which the conventional instanton solution must live. The remaining sub-space will be a $(D-4)$ dimensional Minkowski space. We denote this decomposition with an index deconstruction as follows, let $y^M = (\xi^a, x^m)$ where $m = 1, 2, 3, 4$, and $a = 0, ..., p$ with $p = D - 5$. We can then write the D-dimensional gauge field as

$$A_M(y) = (A_a, A_m(x; X)) \tag{4.1}$$

The Euler-Lagrange equations for this system are $D^M F_{MN} = 0$;

$$D^m F_{mn} + D^a F_{an} = 0 \;\Rightarrow\; D^m F_{mn} + D^a(\partial_a A_n - D_n A_a) = 0 \tag{4.2}$$

$$D^n F_{na} + D^b F_{ba} = 0. \;\Rightarrow\; D^n(D_n A_a - \partial_a A_n) + D^b F_{ba} = 0 \tag{4.3}$$

To embed the four-dimensional instanton in this space we set $A_a = 0$ and let A_m be given by the usual instanton solution in Euclidean space. In this case the field tensor becomes,

$$F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n].$$

$$F_{am} = F_{ab} = 0, \quad a, b = 0, 1, ..., D-5.$$

Thus the D-dimensional field tensor collapses to its 4-dimensional part. The equations of motion also reduce to the 4-dimensional equations $D^m F_{mn} = 0$. Since A_m is already assumed to satisfy these equations it is obvious that $(0, A_m)$ will satisfy the Euler-Lagrange equations on $R^{1,p}$. Since this solution contains no dependence on the ξ^a coordinates, it represents an object which is static and extended in the remaining p-dimensional (Minkowski) space-time.

4.3 The moduli space approximation

The pure gauge instantons are static solutions to the equations of motion. However, we may consider instantons that move extremely slowly. In this case, provided the motion starts out tangent to the moduli space we may construct an approximate description of slow moving instantons in terms of trajectories on the instanton moduli space. Any oscillations transverse to the moduli space will therefore be suppressed and motion is effectively constrained to the moduli space of static solutions. That is to say that at each instant we may envisage our slowly moving instanton to closely resemble a static solution, and time evolution simply picks out a series of static instantons. Furthermore, such motion along the moduli space can usually be shown to be geodesic [15]. The time evolution of slowly moving instantons is therefore represented by (geodesic) curves on the moduli space where time provides the parameterization along these curves. The points along these curves have (collective) coordinates on the moduli space, and as time progresses, we move along these curves and so the moduli space coordinates of our solution change. Thus we have effectively allowed the collective coordinates of the slow moving solution to depend on the time coordinate, ξ^0. We can generalize this approach further and investigate the effect of allowing the collective coordinates to depend upon several additional variables which we identify as coordinates in an enlarged space-time.

We then allow the collective coordinates of the instanton solution to have a dependence on these extra space-time coordinates ξ^a, i.e. we shall investigate the properties of a solution of the form

$$A_N(A_a, A_m(x; X(\xi))), \quad A_a \sim O\left(\frac{\partial}{\partial \xi^a}\right).$$

Of course this new object will not in general solve the Euler-Lagrange equations for the system. Our task at the moment is to determine the form of A_a that will compensate for the extra space-time dependence of the collective coordinates such that A_N is still an approximate solution to the full equations of motion. In fact, if the derivatives of $X(\xi)$ are sufficiently small then this expression will be an approximate solution to the equations of motion. We can regard this type of approximation as an expansion in powers of the derivatives of the extra space-time coordinates. We substitute this expression into the equations of motion, proceeding to linear order in derivatives with respect to ξ^a. Recalling that $D^m F_{mn} = 0$, the first equation of motion, (4.2) becomes,

$$D^a(\partial_a A_n - D_n A_a) = 0 \tag{4.4}$$

To linear order in the derivatives with respect to ξ we may ignore this expression.

For the same reason we may neglect F_{ab} in equation (4.3), which now becomes

$$D^n\left(\frac{\partial A_n}{\partial X^\mu}\frac{\partial X^\mu}{\partial \xi^a} - D_n A_a\right) = 0 \tag{4.5}$$

Recall the background gauge condition for an instanton zero mode given in (2.8). This comparison suggests that we should set

$$A_a = \Omega_\mu \partial_a X^\mu \tag{4.6}$$

where $D_n\Omega_\mu$ is the compensating gauge transformation associated to the collective coordinate X^μ. Making this substitution yields

$$D^n\left(\frac{\partial A_n}{\partial X^\mu}\partial_a X^\mu - D_n\left(\Omega_\mu \partial_a X^\mu\right)\right) = 0$$

$$\Rightarrow \left(D^n\left(\frac{\partial A_n}{\partial X^\mu}\right) - D^n D_n \Omega_\mu\right)\partial_a X^\mu = 0$$

where we have used $D_n(\partial_a X^\mu) = \partial_n(\partial_a X^\mu) = 0$. The case $\partial_a X^\mu = 0$ would correspond to $A_a = 0$, which we have already considered, so it is safe to ignore this solution and our solution to the equations of motion to first order in $\partial_a X^\mu$ is

$$A_N = (\Omega_\mu \partial_a X^\mu(\xi), A_n(x; X(\xi))) \tag{4.7}$$

We can now substitute these results into the action obtaining a result valid up to quadratic order in derivatives.

$$S = S^{(0)} + S^{(2)} = \frac{1}{2g_D^2}\int d^D y\ Tr_N(F_{mn}F^{mn}) + \frac{1}{g_D^2}\int d^D y\ Tr_N(F_{ma}F^{ma})$$

Using equation (4.6) we get

$$S^{(2)} = \frac{1}{g_D^2}\int d^{p+1}\xi\, d^4x\ \partial_a X^\mu \partial^a X^\nu Tr_N\left[\partial_m\Omega_\mu\partial^m\Omega_\nu - 2\partial_m\Omega_\mu\frac{\partial A^m}{\partial X^\nu}\right.$$

$$+\frac{\partial A_m}{\partial X^\mu}\frac{\partial A^m}{\partial X^\nu} + 2\partial_m\Omega_\mu[A^m,\Omega_\nu] - 2\frac{\partial A_m}{\partial X^\mu}[A^m,\Omega_\nu] + [A_m,\Omega_\mu][A^m,\Omega_\nu]\Bigg]$$

$$= \frac{1}{g_D^2}\int d^{p+1}\xi\, d^4x\; Tr_N[\delta_\mu A_m(x;X(\xi))\delta_\nu A_m(x;X(\xi))]\partial_a X^\mu \partial^a X^\nu$$

$$= \frac{1}{2g_D^2}\int d^{p+1}\xi g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu \tag{4.8}$$

where we have first used the cyclic property of the trace and then the equation defining the metric on the moduli space,

$$g_{\mu\nu} = -2\int d^4x Tr_N[\delta_\mu A_m(x;X(\xi))\delta_\nu A_m(x;X(\xi))]$$

Note the absence of the factor of g^2 at the front of this equation. This is due to our normalization of the A-fields in this section. $S^{(2)}$ has the form of an integral over a $(p+1)$-dimensional Minkowski space of a Lagrangian density $L = g_{\mu\nu}\partial_a X^\mu \partial^a X^\nu$. We can investigate the consequences of regarding the X^μ's as (dynamical) fields in $R^{1,p}$. The Euler-Lagrange equations for these fields are:

$$\frac{\partial L}{\partial X^\mu} - \partial_a\left(\frac{\partial L}{\partial(\partial_a X^\mu)}\right) = 0$$

Thus we have;

$$2g_{\mu\nu}\partial_a\partial^a X^\mu + \frac{\partial g_{\rho\sigma}}{\partial X^\mu}\partial^a X^\rho \partial_a X^\sigma = 0$$

$$\Rightarrow \partial_a\partial^a X^\mu + \frac{1}{2}g^{\mu\nu}\frac{\partial g_{\rho\sigma}}{\partial X^\mu}\partial^a X^\rho \partial_a X^\sigma = 0$$

For $p = 0$, these equations describe a $(D-5)$ brane and we recover the geodesic equation on $\mathcal{M}_k$.

4.4 Supersymmetric instanton branes

The action for theories with $N = 1$ supersymmetry in $D = 6$ and $D = 10$ dimensions may be written in a unified notation as [12]

$$S = \frac{1}{g_D^2}\int d^D y\; Tr_N\left(\frac{1}{2}F_{MN}F^{MN} - i\bar{\Psi}\Gamma^M D_M \Psi\right) \tag{4.9}$$

(For supersymmetry refferences see [9] and [16]). The supersymmetry transformations are (see appendix H for details about *Gamma* matrices),

$$\delta A_N = -\bar{\Xi}\Gamma_N\Psi \tag{4.10}$$

$$\delta\Psi = i\Gamma^{MN}\Xi F_{MN} \tag{4.11}$$

In both cases we can decompose the gamma matrices as

$$\Gamma_N = \{\Gamma_a \otimes \gamma_5, 1 \otimes \gamma_n\} = \{\Gamma'_a, \gamma'_n\} \tag{4.12}$$

and in both cases we can write the Γ_a's in terms of the Σ matrices;

$$\Gamma_a = \begin{pmatrix} 0 & \Sigma_a \\ \bar{\Sigma}_a & 0 \end{pmatrix} \tag{4.13}$$

We may also write the Weyl spinors as

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda^A_\alpha + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\lambda}^{\dot{\alpha}}_A \tag{4.14}$$

Adopting a 4×4 matrix notation in which each element is a square matrix of the same dimension as the Σ's the Weyl spinors become

$$\Psi = \begin{pmatrix} \lambda^A_\alpha \\ 0 \\ 0 \\ \bar{\lambda}^{\dot{\alpha}}_A \end{pmatrix} \tag{4.15}$$

And the Γ's are

$$\Gamma'_a = \Gamma_a \otimes \gamma_5 = \begin{pmatrix} 0 & \Sigma_a \\ \bar{\Sigma}_a & 0 \end{pmatrix} \otimes \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} = \begin{pmatrix} 0 & \Sigma_a & 0 & 0 \\ \bar{\Sigma}_a & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Sigma_a \\ 0 & 0 & -\bar{\Sigma}_a & 0 \end{pmatrix} \tag{4.16}$$

$$\Gamma'_n = \Gamma_n \otimes \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i\sigma_n \\ i\bar{\sigma}_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma_n & 0 & 0 \\ i\bar{\sigma}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_n \\ 0 & 0 & i\bar{\sigma}_n & 0 \end{pmatrix} \quad (4.17)$$

We now wish to assemble all the parts of the fermiomic action but to do this we need an expression for $\bar{\Psi}$:

$$\bar{\Psi} = \Psi^\dagger \Gamma_0 = \begin{pmatrix} (\lambda^A_\alpha)^\dagger & 0 & 0 & (\bar{\lambda}^{\dot{\alpha}}_A)^\dagger \end{pmatrix} \begin{pmatrix} 0 & \Sigma_{0\,AB} & 0 & 0 \\ \bar{\Sigma}_{0\,AB} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Sigma_{0\,AB} \\ 0 & 0 & -\bar{\Sigma}_{0\,AB} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (\lambda^A_\alpha)^\dagger \Sigma_{0\,AB} & -(\bar{\lambda}^{\dot{\alpha}}_A)^\dagger \bar{\Sigma}_{0\,AB} & 0 \end{pmatrix} \quad (4.18)$$

However, in both cases, the Weyl spinors satisfy the following,

$$(\lambda^A_\alpha)^\dagger = \bar{\Sigma}^0_{AB} \lambda^{\alpha B}, (\bar{\lambda}^{\dot{\alpha}}_A)^\dagger = \Sigma^{0AB} \bar{\lambda}_{\dot{\alpha}\,B} \quad (4.19)$$

(For $N = 2$ this equation amounts to a pseudo reality condition in $D = 6$. In the $D = 10$ case this equation represents the Majorana condition. For further details see [12]). Using this result gives

$$\bar{\Psi} = \begin{pmatrix} 0 & \bar{\Sigma}^0_{AC} \lambda^{\alpha C} \Sigma_{0AB} & -\Sigma^{0AC} \bar{\lambda}_{\dot{\alpha} C} \bar{\Sigma}_{0AB} & 0 \end{pmatrix}$$

Following the matrix multiplication through we get

$$-i\bar{\Psi}\Gamma'_a D_a \Psi = -i\left(\bar{\Sigma}^0_{AD} \lambda^{\alpha\, D} \Sigma_{0\,AB} \bar{\Sigma}_{0\,BC} D_a \lambda^C_\alpha + \Sigma^{0\,AD} \bar{\lambda}_{\dot{\alpha}\,D} \bar{\Sigma}_{0\,AB} \Sigma_{a\,BC} D_a \bar{\lambda}^{\dot{\alpha}}_C\right) \quad (4.20)$$

We would like to remove the awkward factors of $\Sigma_0 \bar{\Sigma}_0$. From appendix F we know that Σ_0 and $\bar{\Sigma}_0$ take the following forms,

$$\Sigma_0 = i\eta^3 \ , \ \bar{\Sigma}_0 = -i\eta^3$$

We can use this information to remove the factors of $\Sigma_0\bar{\Sigma}_0$ appearing in (4.20)

$$\bar{\Sigma}^0_{AD}\Sigma_{0\,AB} = (-i^2)\eta^3_{AD}\eta^3_{AB}$$
$$= \sum_{A=1}^{3} \eta^3_{AD}\eta^3_{AB} + \eta^3_{4D}\eta^3_{4B}$$
$$= \sum_{A=1}^{3} \epsilon_{3AD}\epsilon_{3AB} + \delta_{3D}\delta_{3B}$$
$$= \delta_{33}\delta_{DB} - \delta_{3B}\delta_{3D} + \delta_{3D}\delta_{3B}$$
$$= \delta_{DB}$$

So we can write $\bar{\psi}$ as

$$-i\bar{\Psi}\Gamma'_a D_a\Psi = -i\lambda^{\alpha A}\bar{\Sigma}_{0\;AB}D_a\lambda^C_\alpha - i\bar{\lambda}_{\dot{\alpha}\,A}\Sigma_{aAB}D_a\bar{\lambda}^{\dot{\alpha}}_B$$

Similarly, the other part of the fermionic action is

$$-i\bar{\Psi}\Gamma'_n D_n\Psi = 2D^n\bar{\lambda}_{\dot{\alpha}A}\bar{\sigma}_n\lambda^{\alpha A}$$

Assembling all these pieces, we may re-write the action, (4.9), as

$$S = \frac{1}{g_d^2}\int d^D y\, Tr_N\left[\frac{1}{2}F_{MN}F^{MN} + 2D^n\bar{\lambda}_{\dot{\alpha}A}\bar{\sigma}_n\lambda^{\alpha A} - i\lambda^{\alpha\,A}\bar{\Sigma}_{0AB}D_a\lambda^B_\alpha - i\bar{\lambda}_{\dot{\alpha}\,A}\Sigma_{aAB}D_a\bar{\lambda}^{\dot{\alpha}}_B\right] \tag{4.21}$$

The equations of motion are;

$$D^m F_{mn} + D^a F_{an} = 2\bar{\sigma}^{\dot{\alpha}\alpha}_n\{\lambda^A_\alpha, \bar{\lambda}^{\dot{\alpha}}_A\} \tag{4.22}$$

$$\bar{\sigma}^{\dot{\alpha}\alpha}_n D^n\lambda^A_\alpha = -i\Sigma^{a\,AB}D_a\bar{\lambda}^{\dot{\alpha}}_B \tag{4.23}$$

$$\sigma_{n\,\alpha\dot{\alpha}}D^n\bar{\lambda}^{\dot{\alpha}}_A = -i\bar{\Sigma}^a_{AB}D_a\lambda^B_\alpha \tag{4.24}$$

$$D^n F_{na} + D^b F_{ba} = i\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}\lambda^B_\alpha + i\Sigma^{AB}_a\bar{\lambda}_{\dot{\alpha}\,A}\bar{\lambda}^{\dot{\alpha}}_B \tag{4.25}$$

Previously for the pure gauge theory we described the moduli space dynamics with an expansion in ξ_a derivatives. However, in the supersymmetric version of the instanton action one is compelled to introduce fermion fields. To these will be associated Grassmann collective coordinates, which we shall also take to have an ξ_a dependence. The moduli space approximation then becomes an expansion in $n = n_\partial + \frac{1}{2}n_f$. Where n_f is the number of Grassmann collective coordinates. Such an expansion correctly takes into account the powers of the coupling constant that might otherwize appear in front of the fermion fields. We have already determined the lowest order solution, $n = 0$, during our discussion of the pure gauge case. We require the action to order $n = 2$, so we must solve the equations of motion to order $n = 1$. To this order, the fermionic equations of motion are

$$D^m F_{mn} = 0 \tag{4.26}$$

$$\bar{\sigma}_n^{\dot{\alpha}\alpha} D^n \lambda_\alpha^A = 0 \tag{4.27}$$

$$\sigma_{\alpha\dot{\alpha}} D^n \bar{\lambda}_A^{\dot{\alpha}} = 0 \tag{4.28}$$

$$D^n F_{na} = -i\bar{\Sigma}_{aAB}\lambda^{\alpha A}\lambda_\alpha^B + i\Sigma_a^{AB}\bar{\lambda}_{\dot{\alpha}A}\bar{\lambda}_B^{\dot{\alpha}} \tag{4.29}$$

Equations (4.27) and (4.28) are the covariant Weyl equations encountered previously. In chapter two we demonstrated that in the self-dual instanton background we have the solution $\bar{\lambda}^A = 0$, so we need only consider the λ field. To order $n = 1$ equation (4.29) becomes

$$D^n \left(\frac{\partial A_n}{\partial X^\mu} \partial_a X^\mu - D_n A_a \right) = -i\bar{\Sigma}_{aAB}\lambda^{\alpha A}\lambda_\alpha^B$$

We have already found the complementary function for this linear differential equation, (4.7). To proceed we note that this equation is linear in A_a. The particular integral will be denoted by $i\phi_a$ and is therefore a solution of

$$D^2\phi_a = \bar{\Sigma}_{aAB}\lambda^{\alpha A}\lambda^B_\alpha \tag{4.30}$$

This gauge-covariant Laplace equation has been solved elsewhere, [12]. Note that $i\phi_a$ must be a Hermitian field since multiplied by i it is part of the (anti-Hermitian) gauge field A_a, and we have

$$A_a\left(x; X(\xi), M^A(\xi)\right) = \Omega_\mu\left(x;\ X(\xi)\right)\partial_a X^\mu(\xi) + i\phi_a(x; X(\xi), M^A(\xi)\Big) \tag{4.31}$$

This is our solution to the fully coupled supersymmetric equations of motion for the gauge field up to order one in the derivative expansion.

4.5 Grassmann collective coordinates and symplectic tangent vectors

Here we consider how fermionic fields behave when a (bosonic) instanton is present. To the lowest non-trivial order in our approximation, the fermions satisfy the gauge-covariant Weyl equations, (4.27), (4.28), where the gauge field takes its classical instanton value. We will show that the fermionic collective coordinates may be assembled into a Grassmann-valued symplectic tangent vector to the moduli space $\mathcal{M}_k$.

The solution to the linear differential equation $\bar{\not{D}}\lambda = 0$ is given by, [12]

$$\lambda_{\alpha uv} = \Lambda_\alpha(M)_{uv} = \left(\bar{U}Mf\bar{b}_\alpha U - \bar{U}b_\alpha f\bar{M}U\right)_{uv} \tag{4.32}$$

Since λ represents a fermion field this is a Grassmann quantity. The quantities $M_{\lambda i}$ and $\bar{M}_i^\lambda$ are constant matrices of Grassmann collective coordinates, of dimensions $(N+2k)\times k$ and $k\times(N+2k)$ respectively. Note that the spinnor index α is not attached to the Grassmann collective coordinates. If the Grassmann matrices are to parameterize solutions of the Weyl equation then they must be constrained thus

$$\bar{\Delta}^{\dot{\alpha}} M + \bar{M} \Delta^{\dot{\alpha}} = 0 \tag{4.33}$$

Recalling the form for $\bar{\Delta}$ and Δ given earlier we may write the more explicit expression

$$\bar{M}_i^{\lambda} a_{\lambda j \dot{\alpha}} = -\bar{a}_{i\dot{\alpha}}^{\lambda} M_{\lambda j} \tag{4.34}$$

$$\bar{M}_i^{\lambda} b_{\lambda j}^{\alpha} = -\bar{b}_i^{\alpha\lambda} M_{\lambda j} \tag{4.35}$$

Writing the ADHM index as $\lambda = (u + i\alpha)$ we decompose the Grassmann matrices as

$$M_{\lambda j} = M_{(u+i\alpha)j} = \begin{pmatrix} \mu_{uj} \\ M'_{\alpha ij} \end{pmatrix} \tag{4.36}$$

$$\bar{M}_j^{\lambda} = \bar{M}_{j(u+i\alpha)} = \begin{pmatrix} \bar{\mu}_{ju} & (M'^{\alpha})_{ji} \end{pmatrix} \tag{4.37}$$

If we assume that b has been placed in its canonical form (3.17) then (4.35) becomes

$$\begin{pmatrix} \bar{\mu}_{iu} & (\bar{M}'^{\beta})_{ik} \end{pmatrix} \begin{pmatrix} 0 \\ \delta^{\alpha}{}_{\beta} \delta_{jk} \end{pmatrix} = \begin{pmatrix} 0 & \epsilon^{\alpha\beta} \delta_{ik} \end{pmatrix} \begin{pmatrix} \mu_{uj} \\ M'_{\beta kj} \end{pmatrix}$$

$$\Rightarrow \bar{M}'^{\alpha}_{ij} = \epsilon^{\alpha\beta} M'_{\beta ij} = M'^{\alpha}_{ij} \tag{4.38}$$

We use this information to re-write equation (4.34) as,

$$\begin{pmatrix} \bar{\mu}_{iu} & \bar{M}'^{\beta}_{ik} \end{pmatrix} \begin{pmatrix} \omega_{uj\dot{\alpha}} \\ (a'_{\beta\dot{\alpha}})_{kj} \end{pmatrix} = - \begin{pmatrix} \bar{\omega}_{i\dot{\alpha}u} & (a'_{\dot{\alpha}}{}^{\beta})_{ik} \end{pmatrix} \begin{pmatrix} \mu_{uj} \\ (M'_{\beta})_{kj} \end{pmatrix}$$

$$\Rightarrow \bar{\mu}_{iu} \omega_{uj\dot{\alpha}} + \bar{\omega}_{i\dot{\alpha}u} \mu_{uj} + [M'^{\alpha}, a'_{\alpha\dot{\alpha}}]_{ij} = 0 \tag{4.39}$$

Therefore the fermion fields are described by the collective coordinates $\{\mu, \bar{\mu}, M'_{\alpha}\}$. These number $k(N+2k)+kN = 2k(N+k)$. However, (4.39) contains the free indices α, i and j and so represents $2k^2$ equations. Thus the number of independent zero modes is $2kN$.

4.5.1 Relation to the hyper-Kahler quotient construction

Recall our definition of the symplectic variables $z^{\tilde{i}\dot{\alpha}}$, (3.71). We may write the Grassmann collective coordinates in an analogous form

$$M^{\tilde{i}} = \begin{pmatrix} \bar{\mu}_{iu} \\ (M'^{1})_{ij} \\ \mu_{ui} \\ (M'_1)_{ij} \end{pmatrix} \tag{4.40}$$

Where $\tilde{i} = \{iu, ij, ui, ij\}$. Unlike the (bosonic) coordinates on the moduli space, these objects only have a single index, $\tilde{i}$. In this form the Grassmann collective coordinates resemble symplectic tangent vectors to the mother space. We will now show that the fermionic ADHM constraints given above are equivalent to the condition that the $M^{\tilde{i}}$ indeed be symplectic tangent vectors to the quotient space. Since a basis of vectors orthogonal to the quotient space is given by the Killing vectors X_r, it will be sufficient to prove that the X_r are orthogonal to the $M^{\tilde{i}}$ with respect to the metric on the mother space, (3.70). Thus it will suffice to show that

$$M^{\tilde{i}}\tilde{\Omega}_{\tilde{i}\tilde{j}}X_r^{\tilde{j}\dot{\alpha}} = 0 \tag{4.41}$$

Setting $\tilde{i} = \{iu, ik, ui, ik\}$ and $\tilde{j} = \{jv, jl, vj, jl\}$ we have,

$$M^{\tilde{i}}\tilde{\Omega}_{\tilde{i}\tilde{j}}X_r^{\tilde{j}\dot{\alpha}} = \begin{pmatrix} \bar{\mu}_{iu} \\ (M'^{1})_{ik} \\ \mu_{ui} \\ (M'_1)_{ik} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & \delta_{uv}\delta_{ij} & 0 \\ 0 & 0 & 0 & \delta_{ij}\delta_{kl} \\ -\delta_{uv}\delta_{ij} & 0 & 0 & 0 \\ 0 & -\delta_{ij}\delta_{kl} & 0 & 0 \end{pmatrix} \begin{pmatrix} iT^r_{jm}\bar{\omega}^{\dot{\alpha}}_{mv} \\ i[T^r, a'_n]_{lj}\bar{\sigma}^{\dot{\alpha}1}_n \\ -iT^r_{mj}\omega_{vm\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} \\ i[T^r, a'_n]_{lj}\sigma_{n1\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$$

$$= i\left(-\mu_{vj}T^r_{jm}\bar{\omega}^{\dot{\alpha}}_{mv} - (M'_1)_{jl}[T^r, a'_n]_{lj}\bar{\sigma}^{\dot{\alpha}1}_n - \bar{\mu}_{jv}T^r_{mj}\omega_{vm\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}} + (M'^{1})_{jl}[T^r, a'_n]_{lj}\sigma_{n1\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}\right)$$

Now recall that

$$\bar{\sigma}^{m\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^m_{\beta\dot{\beta}} \Rightarrow \bar{\sigma}^{m\dot{\alpha}2} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{2\beta}\sigma^m_{\beta\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{21}\sigma^m_{1\dot{\beta}} = -\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^m_{1\dot{\beta}}$$

and

$$M_\alpha = \epsilon_{\dot\alpha\dot\beta} M^\beta \Rightarrow M_2 = M^1$$

Using these results gives

$$M^{\tilde i}\tilde\Omega X_r^{\tilde j\dot\alpha} = -i\left(T^r_{jm}\bar\omega^{\dot\alpha}_{mv}\mu_{vj} + T^r_{jm}\bar\mu_{mv}\omega^{\dot\alpha}_{vj} + (M_1')_{jl}[T^r, a_n']_{lj}\bar\sigma_n^{\dot\alpha 1} + (M_2')_{jl}[T^r, a_n']_{lj}\bar\sigma_n^{\dot\alpha 2}\right)$$

$$= -i\left(T^r_{jm}\bar\omega^{\dot\alpha}_{mv}\mu_{vj} + T^r_{jm}\bar\mu_{mv}\omega^{\dot\alpha}_{vj} + [T^r, \bar a'^{\dot\alpha\beta}]_{lj}(M_\beta')_{jl}\right)$$

Finally, we may use the cyclic property of the trace to alter the commutator term, producing

$$M^{\tilde i}\tilde\Omega X_r^{\tilde j\dot\alpha} = -iT^r_{jm}\left(\bar\omega^{\dot\alpha}_{mv}\mu_{vj} + \bar\mu_{mv}\omega^{\dot\alpha}_{vj} + [(M_\beta'), \bar a'^{\dot\alpha\beta}]_{mj}\right) \tag{4.42}$$

Thus the condition that the $M^{\tilde i}$ be symplectic tangent vectors to the quotient space (4.41) is equivalent to the fermionic ADHM constraints (4.39). In summary here we have shown that the Grassmann collective coordinates may be assembled into Grassmann-valued symplectic tangent vectors to the moduli space.

One can define a functional inner product of fermionic zero modes. The relevant formula is given in an appendix as (H.42)

$$\int d^4x\, Tr_N \Lambda(M)\lambda(N) = -\frac{\pi^2}{2} Tr_k\left[\bar M(P_\infty + 1)N + \bar N(P_\infty + 1)M\right] \tag{4.43}$$

Using equations (3.73) and (4.40) we can show that

$$\int d^4x\, Tr_N \Lambda(M)\lambda(N) = -\pi^2\tilde\Omega(M, N) \tag{4.44}$$

Thus the functional inner of fermionic zero modes coincides with the inner product of symplectic tangent vectors on $\tilde{\mathcal{M}}$.

We now define Grassmann valued symplectic tangent vectors to the moduli space $\mathcal{M}_k$. Firstly we must solve the fermionic ADHM constraints, (4.39), obtaining a solution $M = M(\psi, X)$. Just as the X's represent intrinsic coordinates on the moduli space, so are the ψ^i, $i = 1, \cdots, 2kN$ to represent intrinsic Grassmann coordinates. The symplectic tensor on the mother space, $\tilde{\Omega}_{\tilde{i}\tilde{j}}$ induces a symplectic tensor on the moduli space, denoted Ω_{ij}, in terms of which the symplectic inner product of fermionic zero modes can be written as

$$\tilde{\Omega}(M(\psi, X), N(\theta, X)) = \Omega_{ij}\psi^i\theta^j \tag{4.45}$$

4.6 Effective action

Having solved the equations of motion to order $n = 1$ we can write down the action to order $n = 2$,

$$S^{(2)} = \frac{1}{g_D^2}\int d^D y\, Tr_N \left[F_{na}F^{na} - i\lambda^{\alpha A}\bar{\Sigma}^a_{AB}D_a\lambda^B_\alpha\right] \tag{4.46}$$

Consider the first term in the above, it includes F_{na} where

$$F_{na} = D_n A_a - \partial_a A_n = D_n(\Omega_\mu \partial_a X^\mu + i\phi_a) - \frac{\partial A_n}{\partial X^\mu}\partial_a X^\mu$$

Recalling the gauge condition,

$$\delta_\mu A_n = \frac{\partial A_n}{\partial X^\mu} - D_n\Omega_\mu$$

we can write

$$F_{na} = iD_n\phi_a - \delta_\mu A_n \partial_a X^\mu \tag{4.47}$$

Therefore

$$F_{na}F^{na} = -D^n\phi^a D_n\phi_a + \delta_\mu A^n \delta_\nu A_n \partial_a X^\mu \partial^a X^\nu - 2iD^n\phi^a\delta_\mu A_n \partial_a X^\mu \tag{4.48}$$

We can integrate the last term by parts using the identity

$$\partial_n(\phi^a\delta_\mu A_n \partial_a X^\mu) = \partial_n\phi^a\delta_\mu A_n\partial_a X^\mu + \phi^a\partial_n\delta_\mu A_n\partial_a X^\mu \tag{4.49}$$

Therefore

$$\int d^D y \; D_n \phi^a \delta_\mu A_n \partial_a X^\mu = -\int d^D y \; \phi^a D_n \delta_\mu A_n \partial_a X^\mu = 0 \tag{4.50}$$

since $D^n \delta_\mu A_n = 0$, giving

$$\int d^D y \; Tr_N F_{na} F^{na} = \int d^D y \; Tr_N (-D^n \phi^a D_n \phi_a + \delta_\mu A^n \delta_\nu A_n \partial_a X^\mu \partial^a X^\nu) \tag{4.51}$$

Turning to the fermionic term,

$$\begin{aligned} D_a \lambda^B_\alpha &= \partial_a \lambda^B_\alpha + [A_a, \lambda^B_\alpha] \\ &= \Lambda_\alpha (\partial_a M^B) + \frac{\partial \lambda^B_\alpha}{\partial X^\mu} \frac{\partial X^\mu}{\partial \xi^a} - \Lambda_\alpha (\partial_a M^B) + [A_a, \lambda^B_\alpha] \\ &= \Lambda_\alpha (\partial_a M^B) + \left(\frac{\partial \lambda^B_\alpha}{\partial X^\mu} - \Lambda_\alpha \left(\frac{\partial M^B}{\partial X^\mu} \right) + [\Omega_\mu, \lambda^B_\alpha] \right) \partial_a X^\mu + i[\phi^a, \lambda^B_\alpha] \end{aligned} \tag{4.52}$$

Putting all this together gives

$$\begin{aligned} S^{(2)} = \frac{1}{g_D^2} \int d^D y \, Tr_N \Big[&-D^n \phi^a D_n \phi_a + \delta_\mu A^n \delta_\nu A_n \partial_a X^\mu \partial^a X^\nu - i \lambda^{\alpha \, A} \bar{\Sigma}^a_{AB} \Lambda_\alpha (\partial_a M^B) \\ &- i \lambda^{\alpha \, A} \bar{\Sigma}^a_{AB} \left(\frac{\partial \lambda^B_\alpha}{\partial X^\mu} - \Lambda_\alpha \left(\frac{\partial M^B}{\partial X^\mu} \right) + [\Omega_\mu, \lambda^B_\alpha] \right) \partial_a X^\mu + \lambda^{\alpha \, A} \bar{\Sigma}^a_{AB} [\phi^a, \lambda^B_\alpha] \Big] \end{aligned} \tag{4.53}$$

Breaking this up into parts, the second term we recognize as

$$-\frac{1}{2 g_D^2} \int d^{p+1} \xi \; g_{\mu\nu}(X) \partial_a X^\mu \partial^a X^\nu \tag{4.54}$$

We now concentrate on the third term, using the formula for the fermionic inner product, (H.42),

$$\begin{aligned} \frac{1}{g_D^2} \int d^D y \, Tr_N \left[-i \lambda^{\alpha \, A} \bar{\Sigma}^a_{AB} \Lambda_\alpha (\partial_a M^B) \right] &= \frac{1}{g_D^2} \int d^D y \, Tr_N \left[-i \Lambda^\alpha (M^A) \bar{\Sigma}^a_{AB} \Lambda_\alpha (\partial_a M^B) \right] \\ &= \frac{i \pi^2}{2 g_D^2} \int d^{p+1} \xi \; \bar{\Sigma}^a_{AB} \left[\bar{M}^{A\lambda}_i (P_\infty + 1)_\lambda{}^\mu \partial_a M^B_{\mu i} + \partial_a \bar{M}^B_{\mu i} (P_\infty + 1)_\lambda{}^\mu M^A_{\mu i} \right] \end{aligned} \tag{4.55}$$

with $\lambda = (u + k\alpha)$ and $\mu = (v + j\beta)$ we have

$$\frac{i\pi^2}{2g_D^2}\int d^{p+1}\xi\ \bar{\Sigma}^a_{AB}\left[\begin{pmatrix} \bar{\mu}^A_{iu} & \bar{M}'^{A\alpha}_{ik}\end{pmatrix}\begin{pmatrix} 2\delta_{uv} & 0 \\ 0 & \delta_{jk}\delta^\beta_\alpha\end{pmatrix}\begin{pmatrix}\partial_a\mu^B_{vi} \\ \partial_a M'^B_{\beta ji}\end{pmatrix}\right.$$
$$\left.+\begin{pmatrix}\partial_a\bar{\mu}^B_{iu} & \partial_a\bar{M}'^{B\alpha}_{ik}\end{pmatrix}\begin{pmatrix} 2\delta_{uv} & 0 \\ 0 & \delta_{jk}\delta^\beta_\alpha\end{pmatrix}\begin{pmatrix}\mu^A_{vi} \\ M'^A_{\beta ji}\end{pmatrix}\right]$$
$$= \frac{i\pi^2}{2g_D^2}\int d^{p+1}\xi\ \bar{\Sigma}^a_{AB}(2\bar{\mu}^A_{iv}\partial_a\mu^B_{vi} + 2\partial_a\bar{\mu}^B_{iv}\mu^A_{vi} + \bar{M}'^{A\alpha}_{ij}\partial_a M^B_{\alpha ji} + \partial_a\bar{M}'^{B\alpha}_{ij}M'^A_{\alpha ji}) \quad (4.56)$$

Using the fact that $\bar{M}' = M'$ and the fact that M' is Grassmann and changing the positions of the α's we can re-write the last term above to get;

$$\frac{i\pi^2}{g_D^2}\int d^{p+1}\xi\ \bar{\Sigma}^a_{AB}(\bar{\mu}^A_{iv}\partial_a\mu^B_{vi} + \partial_a\bar{\mu}^B_{iv}\mu^A_{vi} + M'^{A\alpha}_{ij}\partial_a M^B_{\alpha ji}) \quad (4.57)$$

By Gauss we have

$$\int d^{p+1}\xi\ \partial_a(\bar{\mu}^B_{iv}\mu^A_{vi}) = 0$$
$$\Rightarrow \int d^{p+1}\xi\ \partial_a\bar{\mu}^B_{iv}\mu^A_{vi} = -\int d^{p+1}\xi\ \partial_a\mu^A_{vi}\bar{\mu}^B_{iv} \quad (4.58)$$

Using this result with the antisymmetry of $\bar{\Sigma}^a_{AB}$ and the Grassmann property gives

$$\frac{1}{g_D^2}\int d^{p+1}\xi\ (2i\pi^2\bar{\Sigma}^a_{AB}\bar{\mu}^A_{iv}\partial_a\mu^B_{vi} + i\pi^2\bar{\Sigma}^a_{AB}M'^{A\alpha}_{ij}\partial_a M^B_{\alpha ji}) \quad (4.59)$$

The fourth term in the action can be re-written with the aid of an identity ((H.43) and H.44)) as

$$\int d^{p+1}\xi\ d^4x\ Tr_N\lambda^{A\alpha}\sigma_{n\alpha\dot{\alpha}}D^n\bar{\varrho}^B_\mu \quad (4.60)$$

Since $\sigma_{n\alpha\dot{\alpha}}D^n\lambda^{A\alpha} = 0$ w can write,

$$\lambda^{A\alpha}\sigma_{n\alpha\dot{\alpha}}D^n\bar{\varrho}^B_\mu = \sigma_{n\alpha\dot{\alpha}}D^n(\lambda^{A\alpha}\bar{\varrho}^B_\mu) \quad (4.61)$$

giving

$$\int d^4x\ Tr_N\sigma_{n\alpha\dot{\alpha}}D^n(\lambda^{A\alpha}\bar{\varrho}^B_\mu) = \int d^4x\ Tr_N\sigma_{n\alpha\dot{\alpha}}\partial_n(\lambda^{A\alpha}\bar{\varrho}^B_\mu) \quad (4.62)$$

since

$$\int d^4x \; Tr_N[A_n, \lambda^{A\alpha}\bar{\varrho}^B_\mu] = 0 \tag{4.63}$$

i.e. $Tr[A, B] = Tr(AB - BA) = Tr(AB) - Tr(BA) = 0$. Equation (4.62) can be written as a surface integral at infinity and therefore vanishes due to the asymptotic forms of λ and $\bar{\varrho}$.

In the interests of clarity we shall gather all our results so far. The second order contribution to the effective action is now given by

$$S^{(2)} = \tilde{S} + \frac{1}{g_D^2}\int d^{p+1}\xi \; \left(-\frac{1}{2}g_{\mu\nu}(X)\partial_a X^\mu \partial^a X^\nu + 2i\pi^2\bar{\Sigma}^a_{AB}\bar{\mu}^A_{iv}\partial_a\mu^B_{vi} + i\pi^2\bar{\Sigma}^a_{AB}M'^{A\alpha}_{ij}\partial_a M^B_{\alpha ji}\right) \tag{4.64}$$

where $\tilde{S}$ is given by

$$\tilde{S} = \frac{1}{g_D^2}\int d^D y \, Tr_N\left[-D^n\phi^a D_n\phi_a + \lambda^{\alpha\,A}\bar{\Sigma}^a_{AB}[\phi_a, \lambda^B_\alpha]\right] \tag{4.65}$$

We now wish to evaluate $\tilde{S}$. Recall that ϕ^a satisfies an inhomogeneous gauge covariant Laplace equation, (4.30). Since D_n is a linear derivation, we have

$$D^n(\phi^a D_n\phi_a) = D^n\phi^a D_n\phi_a + \phi^a D^2\phi_a = D^n\phi^a D_n\phi_a + \phi^a\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}\lambda^B_\alpha \tag{4.66}$$

We now investigate the behavior of this term under the trace.

$$Tr_N[D^n(\phi^a D_n\phi_a)] = Tr_N\{\partial^n\phi^a\partial_n\phi_a + \phi^a\partial^2\phi_a$$

$$+\partial^n\phi^a[A_n, \phi_a] + \phi^a[\partial_n A_n, \phi_a] + \phi^a[A_n, \partial_n\phi_a] + [A_n, \phi^a\partial_n\phi_a] + [A_n, \phi^a[A_n, \phi_a]]\}$$

$$= Tr_N\left(\partial^n\phi^a\partial_n\phi_a + \phi^a\partial^2\phi_a\right) = \partial^n Tr_N(\phi^a D_n\phi_a)$$

$$\Rightarrow -Tr_N(D^n\phi^a D_n\phi_a) = Tr_N\left[\phi^a\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}\lambda^B_\alpha - \partial^n(\phi^a D_n\phi_a)\right] \tag{4.67}$$

Next we treat the second term above in (4.66),

$$Tr_N(\phi^a D^2\phi_a) = Tr_N(\phi^a\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}\lambda^B_\alpha)$$

$$= Tr_N\left(\frac{1}{2}\phi^a\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}\lambda^B_\alpha + \frac{1}{2}\phi^a\bar{\Sigma}_{a\,BA}\lambda^{\alpha\,B}\lambda^A_\alpha\right)$$
$$= Tr_N\left(\frac{1}{2}\phi^a\bar{\Sigma}_{a\,AB}(\lambda^{\alpha\,A}\lambda^B_\alpha - \lambda^{\alpha\,B}\lambda^A_\alpha)\right)$$
$$= \frac{1}{2}Tr_N\left(\bar{\Sigma}_{a\,AB}(\lambda^{\alpha\,A}\lambda^B_\alpha\phi^a + \lambda^A_\alpha\phi^a\lambda^{\alpha\,B})\right)$$
$$= \frac{1}{2}Tr_N\left(\bar{\Sigma}_{a\,AB}(\lambda^{\alpha\,A}\lambda^B_\alpha\phi^a - \lambda^{\alpha\,A}\phi^a\lambda^B_\alpha)\right)$$
$$= \frac{1}{2}Tr_N\left(\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}[\lambda^B_\alpha, \phi^a]\right)$$

Where we have used the cyclic property of the trace together with the fact that the λ's are Grassmann quantities. (Recall the behavior of Grassmann quantities under a trace, $Tr(AB) = A_{ij}B_{ji} = -B_{ji}A_{ij} = -Tr(BA)$). Assembling these results gives

$$\tilde{S} = \frac{1}{g_D^2}\int d^D y\, Tr_N\left[-\partial^n(\phi^a D_n\phi_a) + \frac{1}{2}\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}[\phi^a, \lambda^B_\alpha]\right] \tag{4.68}$$

The first term is a total derivative. By Gauss' theorem, it may be written as a surface integral over the 4-dimensional sphere at infinity. Recall Gauss' theorem applied to a spherical surface,

$$\int_\Omega \partial^n M_n\, d^4x = \int_{\partial\Omega} M_n\, ds^n = \int_{\partial\Omega}\frac{x^n}{x}M_n\, dS$$

Where dS is the surface element on the sphere and $\frac{x^n}{x}$ is a unit vector normal to this surface.

We evaluate this gauge invariant quantity in the singular gauge. The asymptotic formulae for this gauge are given in chapter three, and ϕ is defined by, [12]

$$\phi_{a\,uv} = -\frac{1}{4}\bar{\Sigma}_{a\,AB}\bar{U}^\lambda_u M^A_{\lambda i} f_{ij}\bar{M}^{B\rho}_j U_{\rho v} + \bar{U}^\mu_u\begin{pmatrix}\phi^0_{a\,st} & 0\\ 0 & \varphi_{a\,ij}\delta^\beta_\alpha\end{pmatrix}U_{\nu v} \tag{4.69}$$

where the compound indices λ, ρ, , μ and ν are given by $(s+k\beta)$, $(t+h\gamma)$, $(s+i\alpha)$, and $(t+j\beta)$ respectively. We shall evaluate this to leading order in x. (Note that this will require that we consider the expansion of V in (3.56) to next to leading order). Substituting in the asymptotic forms for U and f gives

$$\phi_{a\,uv} \to -\frac{1}{4}\bar{\Sigma}_{a\,AB}\frac{1}{x^2}\mu^A_{ui}\bar{\mu}^B_{iv} + \phi^0_{a\,uv} - \frac{1}{2x^2}\omega_{iu\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{is}\phi^0_{a\,sv}$$
$$-\frac{1}{2x^2}\phi^0_{a\,us}\omega_{is\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iv} + \frac{1}{x^4}x_m x_p\bar{\sigma}^{\dot{\alpha}\alpha}_m\sigma_{p\,\alpha\dot{\beta}}\omega_{ui\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\beta}}_{jv} \tag{4.70}$$

Taking the derivative gives

$$\partial_n\phi_{a\,uv} \to \frac{1}{2}\frac{x^n}{x^4}\bar{\Sigma}_{a\,AB}\mu^A_{ui}\bar{\mu}^B_{iv} + \frac{x^n}{x^4}\omega_{iu\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{is}\phi^0_{a\,sv} + \frac{x^n}{x^4}\phi^0_{a\,us}\omega_{is\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iv}$$
$$-4\frac{x^n}{x^6}x_m x_p\bar{\sigma}^{\dot{\alpha}\alpha}_m\sigma_{p\,\alpha\dot{\beta}}\omega_{ui\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\beta}}_{jv} + \frac{1}{x^4}x_p\bar{\sigma}^{\dot{\alpha}\alpha}_n\sigma_{p\,\alpha\dot{\beta}}\omega_{ui\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\beta}}_{jv} + \frac{1}{x^4}x_m\bar{\sigma}^{\dot{\alpha}\alpha}_m\sigma_{n\,\alpha\dot{\beta}}\omega_{ui\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\beta}}_{jv}$$

Multiplying by $\frac{x^n}{x}$;

$$\frac{x^n}{x}\partial_n\phi_{a\,uv} \to \frac{1}{x^3}\left(\frac{1}{2}\bar{\Sigma}_{a\,AB}\mu^A_{ui}\bar{\mu}^B_{iv} + \omega_{iu\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{is}\phi^0_{a\,sv} + \phi^0_{a\,us}\omega_{is\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iv} - \frac{2}{x^2}x^m x^n\bar{\sigma}^{\dot{\alpha}\alpha}_m\sigma_{n\alpha\dot{\beta}}\omega_{iu\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\beta}}_{jv}\right)$$

Recalling the identity $\bar{\sigma}^{\dot{\alpha}\alpha}_m\sigma_{n\alpha\dot{\beta}} + \bar{\sigma}^{\dot{\alpha}\alpha}_n\sigma_{m\alpha\dot{\beta}} = 2\delta_{mn}\delta^{\dot{\alpha}}{}_{\dot{\beta}}$ we can transform the last term to give

$$\frac{x^n}{x}\partial_n\phi_{a\,uv} \to \frac{1}{x^3}\left(\frac{1}{2}\bar{\Sigma}_{a\,AB}\mu^A_{ui}\bar{\mu}^B_{iv} + \omega_{iu\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{is}\phi^0_{a\,sv} + \phi^0_{a\,us}\omega_{is\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iv} - 2\omega_{iu\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\alpha}}_{jv}\right) \tag{4.71}$$

Since the asymptotic form of the gauge field is $A_n \to x^m\bar{\sigma}_{mn}$ we must have

$$A_n x_n \to x_n x_m\bar{\sigma}_{mn} = 0 \tag{4.72}$$

due to the antisymmetry of the $\bar{\sigma}_{mn}$. Thus we must also have,

$$x^n D_n\phi_a \to x^n\partial_n\phi_a \tag{4.73}$$

Multiplying by $\phi^{0\,a}_{tu}$ and taking the trace over the $U(k)$ indices t and v gives

$$\frac{x^n}{x}Tr_N(\phi^{0\,a}D_n\phi_a) \to$$

$$\frac{1}{x^3}\left(\frac{1}{2}\bar{\Sigma}_{a\,AB}\phi^{0\,a}_{vu}\mu^{A}_{ui}\bar{\mu}^{B}_{iv} - 2\phi^{0\,a}_{vu}\omega_{iu\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\alpha}}_{jv} + \phi^{0\,a}_{vu}\omega_{iu\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{is}\phi^{0}_{a\,sv} + \phi^{0\,a}_{vu}\phi^{0}_{a\,us}\omega_{is\dot{\alpha}}\bar{\omega}^{\dot{\alpha}}_{iv}\right) \quad (4.74)$$

Using the antisymmetry of $\bar{\Sigma}_{a\,AB}$ and the fact that λ is a grassmann quantity yields,

$$\frac{x^n}{x}Tr_N(\phi^{0\,a}D_n\phi_a) \to \frac{1}{x^3}\left(\frac{1}{2}\bar{\Sigma}_{a\,AB}\bar{\mu}^{A}_{iv}\phi^{0\,a}_{vu}\mu^{B}_{ui} - 2\phi^{0\,a}_{vu}\omega_{iu\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\beta}}_{jv} + 2\bar{\omega}^{\dot{\alpha}}_{iv}\phi^{0\,a}_{vu}\phi^{0}_{a\,us}\omega_{is\dot{\alpha}}\right) \tag{4.75}$$

The volume element on the sphere S^3 of radius R in polar coordinates is

$$dS = R^3 dr\, d\phi\ sin\theta_1\, d\theta_1\ sin^2\theta_2\, d\theta_2,\ \ 0 \le R < \infty,\, 0 \le \phi < 2\pi,\, 0 \le \theta_i < \pi$$

Therefore

$$\oint dS^n\ \frac{x_n}{x}Tr_N(\phi^{0\,a}D_n\phi_a) \to 4\pi^2\left[\frac{1}{4}\bar{\Sigma}_{a\,AB}\bar{\mu}^{A}_{iv}\phi^{0\,a}_{vu}\mu^{B}_{ui} + \bar{\omega}^{\dot{\alpha}}_{iv}\phi^{0\,a}_{vu}\phi^{0}_{a\,us}\omega_{is\dot{\alpha}} - \phi^{0\,a}_{vu}\omega_{iu\dot{\alpha}}\varphi_{a\,ij}\bar{\omega}^{\dot{\beta}}_{jv}\right] \tag{4.76}$$

We now go back to the expression for $\tilde{S}$ and consider how we may manipulate the second term in (4.65).

$$\frac{1}{2}\int d^{p+1}\xi \int d^4x\, Tr_N\bar{\Sigma}_{a\,AB}\lambda^{\alpha\,A}[\phi^a, \lambda^B_\alpha] = \frac{1}{2}\int d^{p+1}\xi \int d^4x\, Tr_N\Lambda^\alpha(M^A)\left(\not{D}\bar{\psi}_{\alpha A} + \Lambda_\alpha(N_A)\right) \tag{4.77}$$

Where we have used an identity.

$$\bar{\Sigma}_{aAB}[\psi_a,\, \Lambda_\alpha(M^B)] = \not{D}\bar{\psi}_{\alpha A} + \Lambda_\alpha(N_A) \tag{4.78}$$

We know that the equations of motion to order $n = 1$ give $\bar{\not{D}}\lambda^A = 0$. Thus we have

$$\bar{\sigma}_{n\dot{\alpha}\alpha}D_n\lambda^{\alpha A} = 0 \Rightarrow \epsilon^{\dot{\beta}\dot{\alpha}}\bar{\sigma}_{n\dot{\alpha}\alpha}D_n\left(\epsilon^{\alpha\beta}\lambda^A_\beta\right) = 0 \tag{4.79}$$

Recall that $\epsilon^{\dot{\beta}\dot{\alpha}}\epsilon^{\beta\alpha}\bar{\sigma}_{n\dot{\alpha}\alpha} = \sigma_n^{\beta\dot{\beta}}$. Using this gives

$$-\sigma_n^{\beta\dot{\beta}}D_n\lambda^A_\beta = 0 \tag{4.80}$$

i.e $\bar{\not{\mathcal{D}}}\lambda^{\alpha A} = 0 \Rightarrow \not{\mathcal{D}}\lambda^A_\alpha = 0$. Using the cyclic property of the trace we may go on to show that

$$Tr_N \not{\mathcal{D}}_{\alpha\dot{\alpha}} \left(\lambda^{\alpha A}\bar{\psi}^{\dot{\alpha}A}\right) = \sigma_{n\alpha\dot{\alpha}}\partial_n Tr_N \left(\lambda^{\alpha A}\bar{\psi}^{\dot{\alpha}A}\right) \tag{4.81}$$

Thus the first term in (4.77) has reduced to a total divergence and may be evaluated on the sphere at infinity. Due to the asymptotic forms of $\bar{\psi}$ and λ this surface integral vanishes. The second term in (4.77) may be evaluated with recourse to the fermionic inner product formula, (H.42),

$$\frac{1}{2}\int d^4x\; Tr_N \Lambda^\alpha(M^A)\Lambda_\alpha(N_A) = \frac{\pi^2}{4} Tr_k \left[\bar{M}^{A\lambda}_j (P_\infty + 1)_\lambda{}^\mu N_{A\mu j} + \bar{N}^{A\lambda}_j (P_\infty + 1)_\lambda{}^\mu M_{A\mu j}\right] \tag{4.82}$$

$$= -\pi^2 \bar{\Sigma}_{a\,AB} Tr_k \left[\bar{\mu}^A \phi^0_a \mu^B - \bar{M}^A M^B \varphi_a\right] \tag{4.83}$$

Assembling both pieces, (4.83) and (4.76) gives:

$$\tilde{S} = -4\pi^2 \int d^{p+1}\xi\, Tr_k \left(\frac{1}{2}\bar{\Sigma}_{a\,AB}\bar{\mu}^A\phi^0_a\mu^B + \bar{\omega}^{\dot{\alpha}}\phi^0_a\phi^0_a\omega_{\dot{\alpha}} - \bar{\omega}^{\dot{\alpha}}\phi^0_a\omega_{\dot{\alpha}}\varphi_a - \frac{1}{4}\bar{\Sigma}_{a\,AB}\bar{M}^A M^B \varphi_a\right) \tag{4.84}$$

The definition of $L(\varphi_a)$ is, (H.39), [12]

$$L(\varphi_a) = \frac{1}{4}\bar{\Sigma}_{a\,AB}\bar{M}^A M^B + \bar{\omega}^{\dot{\alpha}}\phi^0_a\omega_{\dot{\alpha}}$$

$$\Rightarrow \varphi_a = L^{-1}\left[\frac{1}{4}\bar{\Sigma}_{a\,AB}\bar{M}^A M^B + \bar{\omega}^{\dot{\alpha}}\phi^0_a\omega_{\dot{\alpha}}\right] \tag{4.85}$$

we can recast the above as

$$\tilde{S} = -4\pi^2 \int d^{p+1}\xi\, Tr_k \left[\frac{1}{2}\bar{\Sigma}_{a\,AB}\bar{\mu}^A\phi^0_a\mu^B + \bar{\omega}^{\dot{\alpha}}\phi^0_a\phi^0_a\omega_{\dot{\alpha}} - (L(\varphi_a))\varphi_a\right]$$

$$= -4\pi^2 \int d^{p+1}\xi\, Tr_k \left[\frac{1}{2}\bar{\Sigma}_{a\,AB}\bar{\mu}^A\phi^0_a\mu^B + \bar{\omega}^{\dot{\alpha}}\phi^0_a\phi^0_a\omega_{\dot{\alpha}}\right.$$

$$\left. - \left(\frac{1}{4}\bar{\Sigma}_{a\,AB}\bar{M}^A M^B + \bar{\omega}^{\dot{\alpha}}\phi^0_a\omega_{\dot{\alpha}}\right) L^{-1}\left(\frac{1}{4}\bar{\Sigma}^a_{CD}\bar{M}^C M^D + \bar{\omega}^{\dot{\beta}}\phi^{0\,a}\omega_{\dot{\beta}}\right)\right] \tag{4.86}$$

For the $d = 6$ case, the Σ matrices are simply

$$\bar{\Sigma}_{1\,AB} = -i\epsilon_{AB}\ ;\ \bar{\Sigma}_{2\,AB} = \epsilon_{AB}$$

$$\Rightarrow \bar{\Sigma}^a_{AB}\bar{\Sigma}_{aCD} = \epsilon_{AB}\epsilon_{CD}((-i)^2 + 1^2) = 0$$

Thus for the $d = 6$ case, when the fields ϕ^0_a vanish then so does $\tilde{S}$. That is to say that in the $d = 10$ case there is a four fermion interaction whilst in the $d = 6$ case this interaction is absent. However, for $d = 10$ we have

$$\bar{\Sigma}^a_{AB}\bar{\Sigma}_{aCD} = 2\epsilon_{ABCD} \tag{4.87}$$

Thus even when $\phi^0_a = 0$ there is a non-vanishing $\tilde{S}$;

$$\tilde{S} = \frac{\pi^2}{2}\int d^{p+1}\xi\, \epsilon_{ABCD}Tr_k[(\bar{M}^A M^B)(L^{-1}(\bar{M}^C M^D))] \tag{4.88}$$

4.7 Geometric interpretation

From now on we shall only consider the case in which the vev's vanish. We will show that the expression for $\tilde{S}$, (4.88), is actually related to the symplectic curvature of $\mathcal{M}_k$, which is given by (see [12]).

$$R_{\bar{i}\bar{j}\bar{k}\bar{l}} = 2\sum_{rs}\left[(\tilde{\Omega}T^r)_{\bar{i}\bar{j}}L^{-1}_{rs}(\tilde{\Omega}T^s)_{\bar{k}\bar{l}} + (\tilde{\Omega}T^r)_{\bar{i}\bar{l}}L^{-1}_{rs}(\tilde{\Omega}T^s)_{\bar{j}\bar{k}} + (\tilde{\Omega}T^r)_{\bar{i}\bar{k}}L^{-1}_{rs}(\tilde{\Omega}T^s)_{\bar{j}\bar{l}}\right] \tag{4.89}$$

(For the definition of the symplectic curvature please see appendix B, specifically equation(B.12)). Therefore, using the fact that the $M^{\bar{i}}$'s are Grassman quantities,

$$R_{\bar{i}\bar{j}\bar{k}\bar{l}}M^{\bar{i}A}M^{\bar{j}B}M^{\bar{k}C}M^{\bar{l}D} = 2\sum_{rs}\left[M^{\bar{i}A}(\tilde{\Omega}T^r)_{\bar{i}\bar{j}}M^{\bar{j}B}L^{-1}_{rs}M^{\bar{k}C}(\tilde{\Omega}T^s)_{\bar{k}\bar{l}}M^{\bar{l}D}\right.$$

$$\left.+M^{\bar{i}A}(\tilde{\Omega}T^r)_{\bar{i}\bar{l}}M^{\bar{l}D}L^{-1}_{rs}M^{\bar{j}B}(\tilde{\Omega}T^s)_{\bar{j}\bar{k}}M^{\bar{k}C} - M^{\bar{i}A}(\tilde{\Omega}T^r)_{\bar{i}\bar{k}}M^{\bar{k}C}L^{-1}_{rs}M^{\bar{j}B}(\tilde{\Omega}T^s)_{\bar{j}\bar{l}}M^{\bar{l}D}\right]$$

$$\Rightarrow \epsilon_{ABCD}R_{\bar{i}\bar{j}\bar{k}\bar{l}}M^{\bar{i}A}M^{\bar{j}B}M^{\bar{k}C}M^{\bar{l}D} = 6\sum_{rs}\epsilon_{ABCD}M^{\bar{i}A}(\tilde{\Omega}T^r)_{\bar{i}\bar{j}}M^{\bar{j}B}L^{-1}_{rs}M^{\bar{k}C}(\tilde{\Omega}T^s)_{\bar{k}\bar{l}}M^{\bar{l}D} \tag{4.90}$$

Using some results developed in the appendices to this chapter, (4.99) we have;

$$M^A_{\bar{k}}\tilde{\Omega}_{\bar{k}\bar{i}}T^r_{\bar{i}\bar{j}}M^B_{\bar{j}} = -T_{kl}(\bar{M}^{A'\lambda}_l M^B_{\lambda k} - \bar{M}^{B\lambda}_l M^A_{\lambda k}) \tag{4.91}$$

Substituting (4.91) into (4.90);

$$\epsilon_{ABCD}R_{\bar{i}\bar{j}\bar{k}\bar{l}}M^{\bar{i}A}M^{\bar{j}B}M^{\bar{k}C}M^{\bar{l}D} =$$

$$\begin{aligned} 3\epsilon_{ABCD}&\left[T^r_{ij}(\bar{M}^{A\lambda}_j M^B_{\lambda i} - \bar{M}^{B\lambda}_j M^A_{\lambda i})\right]T^r_{mn}L^{-1}_{nm,pq}T^s_{qp}\left[T^s_{kl}(\bar{M}^{C\mu}_l M^D_{\mu k} - \bar{M}^{D\mu}_l M^C_{\mu k})\right] \\ &= 12\epsilon_{ABCD}\bar{M}^{A\lambda}_j M^B_{\lambda i}L^{-1}_{ij,kl}\bar{M}^{C\mu}_l M^D_{\mu k} \\ &= 12\epsilon_{ABCD}(\bar{M}^A M^B L^{-1}(\bar{M}^C M^D)) \end{aligned} \tag{4.92}$$

Where $L^{-1}_{rs} = \frac{1}{2}T^r_{mn}L^{-1}_{nm,pq}T^s_{qp}$. Thus our final result for $\tilde{S}$ for the case of $Nd = 6$ supersymmetric Yang-Mills with vanishing ϕ fields is,

$$\tilde{S} = \frac{\pi^2}{24}\int d^{p+1}\xi\,\epsilon_{ABCD}R_{\bar{i}\bar{j}\bar{k}\bar{l}}M^{\bar{i}A}M^{\bar{j}B}M^{\bar{k}C}M^{\bar{l}D} \tag{4.93}$$

Substituting this result into the expression for the effective action (4.64) yields

$$\begin{aligned} S^{(2)} = \frac{1}{g_D^2}\int d^{p+1}\xi\,\Big(&-\frac{1}{2}g_{\mu\nu}(X)\partial_a X^\mu\partial^a X^\nu + 2i\pi^2\bar{\Sigma}^a_{AB}\bar{\mu}^A_{iv}\partial_a\mu^B_{vi} + i\pi^2\bar{\Sigma}^a_{AB}M'^{A\alpha}_{ij}\partial_a M^B_{\alpha ji} \\ &-\frac{\pi^2}{24}\epsilon_{ABCD}R_{\bar{i}\bar{j}\bar{k}\bar{l}}M^{\bar{i}A}M^{\bar{j}B}M^{\bar{k}C}M^{\bar{l}D}\Big) \end{aligned} \tag{4.94}$$

We should like to write the entire effective action in terms of variables that are intrinsic to the moduli space. To this end we introduce the intrinsic Grassmann-valued symplectic tangent vectors to $\mathcal{M}_k$. These are denoted ψ^{iA} , $i - 1, 2, \cdots, 2kN$.

The quadratic fermionic term may be written in terms of quantities intrinsic to the moduli space. We shall not need this term in what follows so we shall not follow its development explicitly. This term may be re-written as (see [12]),

$$-\frac{i}{4g_D^2}\int d^{p+1}\xi\ \bar{\Sigma}^a_{AB}\Omega_{ij}(X)\psi^{iA}(\partial_a\delta^i{}_k+\omega_\mu{}^j{}_k\partial_a X^\mu)\psi^{kB} \tag{4.95}$$

Thus the second order contribution to the effective action becomes

$$S^{(2)}=-\frac{1}{g_D^2}\int d^{p+1}\xi\ \left(\frac{1}{2}g_{\mu\nu}(X)\partial_a X^\mu\partial^a X^\nu+\frac{i}{2}\bar{\Sigma}^a_{AB}\Omega_{ij}(X)\psi^{iA}(\partial_a\delta^i{}_k+\omega_\mu{}^j{}_k\partial_a X^\mu)\psi^{kB}\right.$$
$$\left.+\frac{\pi^2}{48}\epsilon_{ABCD}R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}M^{\tilde{i}A}M^{\tilde{j}B}M^{\tilde{k}C}M^{\tilde{l}D}\right) \tag{4.96}$$

This result represents the specialization of (1.29) to the hyper-Kahler case. Note the appearance of the metric tensor defined as the functional inner product of zero modes. We should sound a note of caution regarding the notation employed. In the equation above the tensor R_{ijkl} represents the symplectic curvature tensor on our hyper-Kahler manifold. However, the R_{ijkl} of (1.29) refers to the conventional curvature tensor on the moduli space.

4.8 Summary

A conventional instanton is a finite action configuration in four dimensional Euclidean space. We may embed these solutions in a five dimensional gauge theory in Minkowski space by introducing a time coordinate of which the instanton solution is independent. These instantons then represent static finite energy configurations of the five dimensional theory. They are therefore particle like solitons in the higher dimensional theory. Taking this hint we proceeded to embed the instanton solution in gauge theories of even higher dimensions. Thus we introduced the instanton as a $(D-5)$-brane.

Following the idea of Manton's moduli space approximation, [15], we allowed the instanton solution to depend on the extra space-time coordinates. We also introduce the supersymmetric extension of pure gauge theory. The field configurations so obtained are only approximate solutions to the full equations of motion and we proceed order by order in the coupling constant. After a lengthy analysis we arrive at a supersymmetric quantum mechanical sigma model on the moduli space. The terms in the action have natural interpretations in terms of intrinsic geometric quantities on the moduli space. When the vev's vanish, in the case of $N = 4$, there is a non-trivial contribution to the action involving the curvature on the moduli space. This is the most important term for us and we shall return to consider it in chapter six.

4.9 Appendices

4.9.1 Matrix maps

Consider a map of vectors onto vectors. These linear transformations may be effected by matrix multiplication;

$$M : V \rightarrow V' | V' = MV$$

Or in components

$$V'_i = M_{ij} V_j$$

Likewise we may define a map of matrices onto matrices. Consider such a map, L

$$L\Omega = \Omega'$$

i.e.

$$\Omega'_{ij} = (L(\Omega))_{ij}$$

We can write the action of L as follows,

$$(L(\Omega))_{ij} = L_{ij\,,kl}\Omega_{lk}$$

What is the inverse, L^{-1} ? Consider

$$(L^{-1}.L(\Omega))_{ij} = \Omega_{ij}$$

$$\Rightarrow L^{-1}_{ij\,,kl}L(\Omega)_{lk} = L^{-1}_{ij\,,kl}L_{lk\,,mn}\Omega_{nm} = \Omega_{ij}$$

$$\Rightarrow L^{-1}_{ij\,,kl}L_{lk\,,mn} = \delta_{in}\delta_{jm}$$

Likewise,

$$L_{ij\,,kl}L^{-1}_{lk\,,mn} = \delta_{in}\delta_{jm}$$

4.9.2 Lie algebras

The generators of U(k) form a complete set of $k \times k$ matrices. Thus any matrix, A say, can be written as

$$A_{ij} = \sum_a \lambda^a T^a_{ij}$$

Where the λ^a's are a set of numerical parameters and T^a are the U(k) generators. Using the orthogonality of the generators,

$$\lambda^a = A_{ij}T^a_{ji}$$

$$\Rightarrow A_{ij} = \sum_a (A_{lm}T^a_{ml})T^a_{ij} \tag{4.97}$$

If we choose A to be the matrix whose elements are all zero with the exception of one element. A_{pq} whose value is unity;

$$A = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

i.e.

$$A_{ij} = \delta_{ip}\delta_{jq}$$

Substituting the matrix into (4.97)

$$\delta_{ip}\delta_{jq} = \sum_a T^a_{qp} T^a_{ij}$$

We can use this result to determine the inverse of the operator L_{rs} in the following way. Recall the definition of L_{rs}

$$L_{rs} = 2Tr_k(T^r L T^s)$$

$$= 2T^r_{ij} L_{ji,lk} T^s_{kl}$$

We can use these relations to show that L^{-1}_{rs} is given by

$$L^{-1}_{rs} = \frac{1}{2} T^r_{ij} L^{-1}_{ji,lk} T^s_{kl} \tag{4.98}$$

4.9.3 Calculation of $M^A(\tilde{\Omega}T)M^B$

Recalling equations (3.73) and (3.96) we can calculate $\tilde{\Omega}T$

$$\tilde{\Omega}_{\tilde{k}\tilde{i}} T^r_{\tilde{i}\tilde{j}} = \begin{pmatrix} 0 & 0 & -T^r_{lk}\delta_{wv} & 0 \\ 0 & 0 & 0 & T^r_{nl}\delta_{km} - T^r_{mk}\delta_{nl} \\ -T^r_{kl}\delta_{vw} & 0 & 0 & 0 \\ 0 & T^r_{mk}\delta_{nl} - T^r_{nl}\delta_{km} & 0 & 0 \end{pmatrix}$$

Where the indices $\tilde{i}, \tilde{j}$ and $\tilde{k}$ are given by $\{iu, ij, ui, ij\}$, $\{lv, lm, vl, lm\}$ and $\{kw, kn, wk, kn\}$ respectively. Recall the forms for the Grassmann collective coordinates M;

$$M_{\tilde{k}}^{A} = \begin{pmatrix} \bar{\mu}_{kw}^{A} \\ M_{kn}^{\prime 1^{\prime}\,A} \\ \mu_{wk}^{A} \\ M_{1\,kn}^{\prime A} \end{pmatrix} \quad and \ M_{\tilde{j}}^{B} = \begin{pmatrix} \bar{\mu}_{lv}^{B} \\ M_{lm}^{\prime 1^{\prime}\,B} \\ \mu_{vl}^{B} \\ M_{1\,lm}^{\prime B} \end{pmatrix}$$

Using these we have

$$\begin{aligned} M_{\tilde{k}}^{A}\tilde{\Omega}_{\tilde{k}\tilde{i}}T_{\tilde{i}\tilde{j}}^{r}M_{\tilde{j}}^{B} &= T_{mk}^{r}(\bar{\mu}_{kw}^{A}\mu_{wm}^{B}+M_{1\,kl}^{\prime B}M_{lm}^{\prime 1^{\prime}\,A}+M_{kn}^{\prime 1^{\prime}\,A}M_{1\,nm}^{\prime B}-\bar{\mu}_{kw}^{B}\mu_{wm}^{A}-M_{1\,kn}^{\prime A}M_{nm}^{\prime 1^{\prime}\,B}-M_{kl}^{\prime 1^{\prime}\,B}M_{1\,lm}^{\prime A}) \\ &= T_{mk}(\bar{M}_{k}^{A^{\prime},\lambda}M_{\lambda m}^{B} - \bar{M}_{k}^{B\,\lambda}M_{\lambda m}^{A}) \qquad (4.99) \end{aligned}$$

Where we have made use of the Grassmann nature of the M's and we have used the property,

$$M_1 = M^2 \ and \ M_2 = -M^1$$

Chapter 5

The $SU(3)$-1 instanton moduli space

The specific case we shall pursue is that of 1-instanton in the gauge group $SU(3)$. We know from general considerations that the angular variables on the moduli space should correspond to the group manifold of $SU(3)/U(1)$. We first discuss the group space, defining its left invariant 1-forms and using the Maurer-Cartan equation to obtain their exterior derivatives. We calculate these for the general case of $SU(N+2)/SU(N)$ and then specialize to $N = 3$ using the method outlined by [20]. We will then have in place the necessary notation to quote the result of [20] for the metric on the $SU(3)$ single instanton moduli space. Our next step will be to derive the metric on the moduli space using the ADHM procedure. Again, this will be given in terms of the left-invariant 1-forms on the moduli space, facilitating comparison with the result of [20]. Having obtained the metric, we will then attempt to calculate the associated spin-connection. It should now be straight forward in principle to calculate the curvature two form going via the spin-connection and Cartan's equations of structure. Thence we should be able to proceed to the Gauss-Bonnet integral using (B.3). However, the scale of this computation proved prohibitively large, even for Mathematica. To negotiate this impasse we made use of the simplification afforded by the hyper-Kahler nature of the moduli space. We were able to identify the coordinate transformation to the so-called symplectic basis. In this basis our task is reduced to manipulating the symplectic curvature. Since this object is only four-dimensional, the problem simplifies to the extent that we are able to compute the Gauss-Bonnet integral by hand.

5.1 The left invariant 1-forms of SU(3)/U(1)

5.1.1 The metric on a Lie group

We define a metric for raising and lowering the indices on the generators of the lie-algebra of $SU(N)$. Consider a Lie algebra with generators T^a,

$$[T^a, T^b] = i f^{ab}{}_c T^c \tag{5.1}$$

The metric for raising indices is

$$g^{ab} = Tr(T^a T^b) \tag{5.2}$$

We wish to change to a basis in which the generators correspond to matrices with a single non-zero unit element in the (A, B) position. To achieve this it will be necessary to label the generators with two indices, so we have $T^a \rightarrow T_A{}^B$. Here we are changing the way in which we choose to label the generators. We exchange the label a which runs from 1 to $N^2 - 1$, to the index pair (A, B) each of which runs separately from 1 to N but upon which we place a tracelessness constraint so that the total number of generators is again $N^2 - 1$. The generators are traceless, so this unit element is prohibited from lying on the diagonal, which implies $T_A{}^A = 0$. We will also have $(T_A{}^B)^\dagger = T_B{}^A$. The metric for raising and lowering the generator indices is now.

$$g^{ab} \rightarrow g_A{}^B{}_{,C}{}^D = Tr(T_A{}^B T_C{}^D) \tag{5.3}$$

This metric actually has the effect of swapping the position of the two indices, i.e.

$$T_A{}^B g_A{}^B{}_{,C}{}^D = T^D{}_C \tag{5.4}$$

Likewise there will be a metric for the reverse swap,

$$T_A{}^B g^A{}_{B,}{}^C{}_D = T^C{}_D \tag{5.5}$$

Where

$$g^A{}_{B,}{}^C{}_D = Tr(T^A{}_B T^C{}_D) \tag{5.6}$$

In our new basis the components of the generators are

$$(T_A{}^B)_{ij} = \delta_{Ai}\delta^B{}_j \tag{5.7}$$

Using this we write the metric as

$$g_A{}^B{}_{,C}{}^D = Tr(T_A{}^B T_C{}^D) = \delta_{Ai}\delta^B{}_j\delta_{Cj}\delta^D{}_i = \delta_A{}^D\delta^B{}_C \tag{5.8}$$

We shall also require the inverse metric which we get as follows

$$g_{A'}{}^{B'}{}_{,C'}{}^{D'} = Tr(T_{A'}{}^{B'}\, T_{C'}{}^{D'}) \tag{5.9}$$

$$= Tr(T^A{}_B\, T^C{}_D)\; g_A{}^B{}_{,A'}{}^{B'}\; g_C{}^D{}_{,C'}{}^{D'} \tag{5.10}$$

$$= g^A{}_{B,}{}^C{}_D\; \delta^{B'}_A\; \delta^B_{A'}\; \delta^{D'}_C\; \delta^D_{C'} \;=\; g^{B'}{}_{A',}{}^{D'}{}_{C'} \tag{5.11}$$

i.e.

$$g_A{}^B{}_{,C}{}^D = g^B{}_{A,}{}^D{}_C = \delta^D_A\,\delta^B_C \tag{5.12}$$

Turning to the Lie algebra commutation relation;

$$[T_A{}^B\,,\, T_C{}^D] = T_A{}^B\, T_C{}^D \;-\; T_C{}^D\, T_A{}^B \tag{5.13}$$

$$\Rightarrow [T_A{}^B\,,\,T_C{}^D]_{ik} = (T_A{}^B)_{ij}(T_C{}^D)_{jk} - (T_C{}^D)_{ij}(T_A{}^B)_{jk} \tag{5.14}$$

$$= \delta_{Ai}\,\delta_j^B\,\delta_{Cj}\,\delta_k^D - \delta_{Ci}\,\delta_j^D\,\delta_{Aj}\,\delta_k^B \tag{5.15}$$

$$= \delta_C^B(T_A{}^D)_{ik} \;-\; \delta_A^D(T_C{}^B)_{ik}$$

$$\Rightarrow [T_A{}^B\,,\,T_C{}^D] = \delta_C^B\,T_A{}^D \;-\; \delta_A^D\,T_C{}^B \tag{5.16}$$

But we must also have

$$[T_A{}^B\,,\,T_C{}^D] = i f_A{}^B{}_{,C}{}^{D,E}{}_F\;T_E{}^F \tag{5.17}$$

Comparison of these two expressions yields

$$f_A{}^B{}_{,C}{}^{D,E}{}_F \;=\; -i(\delta_C^B\,\delta_{AE}\,\delta^{DF} \;-\; \delta_A^D\,\delta_{CE}\,\delta^{BF}) \tag{5.18}$$

We wish to know how the Maurer-Cartan structure equation looks in this basis. Recall that in terms of the left-invariant 1-forms $\mathcal{L}$ on a manifold the Maurer-Cartan structure equation is [3],

$$d\mathcal{L}^A = \frac{1}{2} f^A{}_{BC}\mathcal{L}^B \wedge \mathcal{L}^C \tag{5.19}$$

This now goes to

$$d\mathcal{L}_A{}^B = \frac{1}{2} f_A{}^{B,C}{}_{D,}{}^E{}_F\,\mathcal{L}_C{}^D \;\wedge\; \mathcal{L}_E{}^F \tag{5.20}$$

Comparison with the form of the structure constants given above shows that we must act on the middle pair of indices with the metric to get them in the required form, i.e.

$$f_A{}^{B,C}{}_{D,}{}^E{}_F \;=\; f_A{}^B{}_{,C'}{}^{D',E}{}_F\,g^{C'}{}_{D',}{}^C D \tag{5.21}$$

$$= -i(\delta_{C'}^B\,\delta_{AE}\,\delta^{D'F} \;-\; \delta_A^{D'}\,\delta_{C'E}\,\delta^{BF})\,\delta_{D'}^C\,\delta_D^{C'} \tag{5.22}$$

$$= -i(\delta_D^B\,\delta_{AE}\,\delta^{FC} \;-\; \delta_A^C\,\delta_{ED}\,\delta^{BF}) \tag{5.23}$$

Substituting this into (5.20) gives

$$d\mathcal{L}_A{}^B = -\frac{i}{2}(\delta_D^B\,\delta_{AE}\,\delta^{FC} - \delta_A^C\,\delta_{ED}\,\delta^{BF})\,\mathcal{L}_C{}^D \wedge \mathcal{L}_E{}^F \tag{5.24}$$

$$= -\frac{i}{2}(\mathcal{L}_C{}^B \wedge \mathcal{L}_A{}^C - \mathcal{L}_A{}^D \wedge \mathcal{L}_D{}^B)$$

$$= i\mathcal{L}_A{}^C \wedge \mathcal{L}_C{}^B \tag{5.25}$$

Like the generators, the left-invariant 1-forms are traceless and Hermitian.

5.1.2 Computation of the left-invariant 1-forms

We shall follow closely the discussion in [20]. We shall examine the case of $SU(n+2)/U(n)$. We divide the $SU(n+2)$ group indices A as follows, $A = \{1, 2, \alpha\}$. We require

$$\sum_A \mathcal{L}_A{}^A = 0\ ,\ Tr(\mathcal{L}_A{}^A) = 0 \tag{5.26}$$

This leads us to write the 1-forms as

$$(\mathcal{L}_A{}^A)_{ij} = \delta_{Ai}\delta_j^A - \frac{1}{n+2}\delta_{ij} \tag{5.27}$$

where there is no summation over the repeated index A. This satisfies both our conditions since

$$Tr(\mathcal{L}_A{}^A) = \delta_{Ai}\delta_i^A - \frac{1}{n+2}\delta_{ii} \tag{5.28}$$

$$= \delta_A^A - 1 = 0 \tag{5.29}$$

And

$$\sum_A \mathcal{L}_A{}^A = \mathcal{L}_1{}^1 + \mathcal{L}_2{}^2 + \ldots = \delta_{ij} - \frac{1}{n+2}(n+2)\delta_{ij} = 0 \tag{5.30}$$

We define the $SU(n)$ generators

$$\tilde{\mathcal{L}}_\alpha{}^\beta = \mathcal{L}_\alpha{}^\beta + \frac{1}{n}Q\delta^\beta{}_\alpha \tag{5.31}$$

Where Q is the $U(1)$ generator

$$Q = \mathcal{L}_1{}^1 + \mathcal{L}_2{}^2 \tag{5.32}$$

$$= \left[\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} - \frac{1}{n+2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\right]$$

$$+ \left[\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} - \frac{1}{n+2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} - \frac{2}{n+2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \tag{5.33}$$

Taking $\tilde{\mathcal{L}}_1{}^1$ as an example we have, using (5.27), (5.31) and (5.32),

$$\tilde{\mathcal{L}}_1{}^1 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} - \frac{1}{n+2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$+\frac{1}{n}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} - \frac{2}{n(n+2)}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} - \frac{1}{n} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \tag{5.34}$$

This is an $SU(n)$ generator with the trace subtracted. The orthogonal linear combination gives another $U(1)$ generator called λ.

$$\lambda = \mathcal{L}_1{}^1 - \mathcal{L}_2{}^2 = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \tag{5.35}$$

Note that λ is real. The generators of the coset will then be the complement of $\tilde{\mathcal{L}}_\alpha{}^\beta$ and Q, i.e. λ and the following

$$\sigma^\alpha = \mathcal{L}_1{}^\alpha \,,\; \Sigma^\alpha = \mathcal{L}_2{}^\alpha \,,\; \nu = \mathcal{L}_1{}^2 \tag{5.36}$$

Note that we have broken the left-invariant 1-form L into its components as follows

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1{}^1 & \mathcal{L}_1{}^2 & \mathcal{L}_1{}^3 & \mathcal{L}_1{}^4 & \dots & \dots & \mathcal{L}_1{}^{n+2} \\ \mathcal{L}_2{}^1 & \mathcal{L}_2{}^2 & \mathcal{L}_2{}^3 & \mathcal{L}_2{}^4 & \dots & \dots & \mathcal{L}_2{}^{n+2} \\ \mathcal{L}_3{}^1 & \mathcal{L}_3{}^2 & \mathcal{L}_3{}^3 & \mathcal{L}_3{}^4 & \dots & \dots & \mathcal{L}_3{}^{n+2} \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \mathcal{L}_{n+2}{}^1 & \mathcal{L}_{n+2}{}^2 & \mathcal{L}_{n+2}{}^3 & \mathcal{L}_{n+2}{}^4 & \dots & \dots & \mathcal{L}_{n+2}{}^{n+2} \end{pmatrix} \tag{5.37}$$

$$= \begin{pmatrix} \frac{1}{2}(Q+\lambda) & \nu & \sigma^1 & \sigma^2 & \dots & \dots & \sigma^n \\ \nu^* & \frac{1}{2}(Q-\lambda) & \Sigma^1 & \Sigma^2 & \dots & \dots & \Sigma^n \\ (\sigma^1)^* & (\Sigma^1)^* & \mathcal{L}_3{}^3 & \mathcal{L}_3{}^4 & \dots & \dots & \mathcal{L}_3{}^{n+2} \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ (\sigma^n)^* & (\Sigma^n)^* & \mathcal{L}_{n+2}{}^3 & \mathcal{L}_{n+2}{}^4 & \dots & \dots & \mathcal{L}_{n+2}{}^{n+2} \end{pmatrix} \tag{5.38}$$

In terms of these quantities the metric on the $SU(3)$ 1-instanton moduli space may be written as [20]

$$ds^2 = h^2 dr^2 + a^2(\sigma_1^2 + \sigma_2^2) + b^2(\Sigma_1^2 + \Sigma_2^2) + c^2(\nu_1^2 + \nu_2^2) + f^2\lambda^2 \tag{5.39}$$

where a, b, c, h and f are functions of r (the instanton scale size) only, given by

$$a^2 = \frac{1}{2}(r^2 - 1) \tag{5.40}$$

$$b^2 = \frac{1}{2}(r^2 + 1) \tag{5.41}$$

$$c^2 = r^2 \tag{5.42}$$

$$f^2 = \frac{1}{4}r^2(1 - r^{-4}) \tag{5.43}$$

$$h^2 = (1 - r^{-4})^{-1} \tag{5.44}$$

Later when we come to use Cartan's equations we will require the exterior derivatives of these 1-forms. These are calculated using the Maurer-Cartan equation (5.19).

$$d\mathcal{L}_1{}^{\alpha+2} = d\sigma^\alpha = i\mathcal{L}_1{}^c \wedge \mathcal{L}_c{}^{\alpha+2} \tag{5.45}$$

$$= i(\mathcal{L}_1{}^1 \wedge \mathcal{L}_1{}^{\alpha+2} + \mathcal{L}_1{}^2 \wedge \mathcal{L}_2{}^{\alpha+2} + \mathcal{L}_1{}^{\beta+2} \wedge \mathcal{L}_{\beta+2}{}^{\alpha+2}) \tag{5.46}$$

$$= i\left((Q + \lambda) \wedge \sigma^\alpha + \nu \wedge \Sigma^\alpha + \sigma^\beta \wedge (\tilde{\mathcal{L}}_\beta{}^\alpha - \frac{1}{n}Q\delta_\beta^\alpha)\right) \tag{5.47}$$

$$= \frac{i}{2}\lambda \wedge \sigma^\alpha + i\nu \wedge \Sigma^\alpha + i\sigma^\beta \wedge \tilde{\mathcal{L}}_\beta{}^\alpha + \frac{i}{2}\left(1 + \frac{2}{n}\right) Q \wedge \sigma^\alpha \tag{5.48}$$

Similarly we calculate the other exterior derivatives to be

$$d\Sigma^\alpha = -\frac{i}{2}\lambda \wedge \Sigma^\alpha + i\bar{\nu} \wedge \sigma^\alpha + \frac{i}{2}\left(1 + \frac{2}{n}\right) Q \wedge \Sigma^\alpha + i\Sigma^\beta \wedge \tilde{\mathcal{L}}_\beta{}^\alpha \tag{5.49}$$

$$d\nu = i\lambda \wedge \nu + i\sigma^\alpha \wedge \bar{\Sigma}_\alpha \tag{5.50}$$

$$d\lambda = 2i\nu \wedge \bar{\nu} + i\sigma^\alpha \wedge \bar{\sigma}_\alpha - i\Sigma^\alpha \wedge \bar{\Sigma}_\alpha \tag{5.51}$$

$$dQ = i\sigma^\alpha \wedge \bar{\sigma}_\alpha + i\Sigma^\alpha \wedge \bar{\Sigma}_\alpha \tag{5.52}$$

$$d\tilde{\mathcal{L}}_\alpha{}^\beta = i\bar{\sigma}_\beta + i\bar{\Sigma}_\alpha \wedge \Sigma^\beta - \frac{i}{n}\left(\bar{\sigma}_\gamma \wedge \sigma^\gamma + \bar{\Sigma}_\gamma \wedge \Sigma^\gamma\right)\delta^\beta{}_\alpha + i\tilde{\mathcal{L}}_\alpha{}^\gamma \wedge \tilde{\mathcal{L}}_\gamma{}^\beta \tag{5.53}$$

Specializing to the case $N = 3$ the coset space becomes the desired $SU(3)/U(1)$. Furthermore, the traceless $SU(1)$ generators $\tilde{\mathcal{L}}_\alpha{}^\beta$ vanish. We will also define real one-forms;

$$\sigma = \sigma^3 = \sigma_{13} + i\sigma_{23} \tag{5.54}$$

$$\Sigma = \Sigma^3 = \Sigma_1 + i\Sigma_2 \tag{5.55}$$

$$\nu = \nu_1 + i\nu_2 \tag{5.56}$$

Following these changes through, the exterior derivatives become;

$$d\sigma_1 = -\frac{1}{2}\lambda \wedge \sigma_2 - \nu_1 \wedge \Sigma_2 - \nu_2 \wedge \Sigma_1 - \frac{3}{2}Q \wedge \sigma_2 \tag{5.57}$$

$$d\sigma_2 = \frac{1}{2}\lambda \wedge \sigma_1 + \nu_1 \wedge \Sigma_1 - \nu_2 \wedge \Sigma_2 + \frac{3}{2}Q \wedge \sigma_1 \tag{5.58}$$

$$d\Sigma_1 = \frac{1}{2}\lambda \wedge \Sigma_2 - \nu_1 \wedge \sigma_2 + \nu_2 \wedge \sigma_1 - \frac{3}{2}Q \wedge \Sigma_2 \tag{5.59}$$

$$d\Sigma_2 = -\frac{1}{2}\lambda \wedge \Sigma_1 + \nu_1 \wedge \sigma_1 + \nu_2 \wedge \sigma_2 + \frac{3}{2}Q \wedge \Sigma_1 \tag{5.60}$$

$$d\nu_1 = -\lambda \wedge \nu_2 - \sigma_2 \wedge \Sigma_1 + \sigma_1 \wedge \Sigma_2 \tag{5.61}$$

$$d\nu_2 = \lambda \wedge \nu_1 + \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2 \tag{5.62}$$

$$d\lambda = 2\sigma_1 \wedge \sigma_2 - 2\Sigma_1 \wedge \Sigma_2 + 4\nu_1 \wedge \nu_2 \tag{5.63}$$

$$dQ = 2\sigma_1 \wedge \sigma_2 + 2\Sigma_1 \wedge \Sigma_2 \tag{5.64}$$

This brief summary contains the results that we shall need to implement the Cartan equations of structure, but first we must obtain the metric on $\mathcal{M}$.

5.2 Calculation of the metric using the ADHM procedure

In terms of the ω variables introduced in the ADHM construction, the metric on the 1-instanton moduli space is given by

$$g = 2d\bar{\omega}^{\dot{\alpha}}{}_{u}.d\omega_{u\dot{\alpha}} \tag{5.65}$$

Where

$$\bar{\omega}^{\dot{\alpha}}{}_{u} = (\omega_{u\dot{\alpha}})^{*} \tag{5.66}$$

These variables are to be subject to the ADHM constraints:

$$\tau^{c\dot{\alpha}}{}_{\dot{\beta}}.\bar{\omega}^{\dot{\beta}}{}_{u}\omega_{u\dot{\alpha}} = \zeta^{c} \tag{5.67}$$

(Note the introduction of the non-commutativity parameter as discussed in the introduction). This constraint can be satisfied for ζ along the 3-direction, (setting $\zeta = \zeta^3$ in what follows), by writing ω in the form,

$$\omega_{N\times N} = U.S = U_{N\times N}\begin{pmatrix} \sqrt{\rho^2+\frac{\zeta}{2}} & 0 \\ 0 & \sqrt{\rho^2-\frac{\zeta}{2}} \\ 0 & 0 \end{pmatrix} \tag{5.68}$$

Where U is an $N \times N$ unitary matrix. To shorten the notation we introduce

$$\rho_{+} = \sqrt{\rho^2+\frac{\zeta}{2}}\ ,\ \ \rho_{-} = \sqrt{\rho^2-\frac{\zeta}{2}} \tag{5.69}$$

Under a $U(1)$ transformation, the $\omega_{u\dot{\alpha}}$'s behave thus:

$$\omega'_{u\dot{\alpha}} = e^{i\phi}\omega_{u\dot{\alpha}} = e^{i\phi}U.S \tag{5.70}$$

$$\bar{\omega}^{\dot{\alpha}\prime}{}_{u} = e^{-i\phi}\bar{\omega}^{\dot{\alpha}}{}_{u} \tag{5.71}$$

We must also impose the so-called gauge condition. In terms of the quotient construction this amounts to the requirement that one take a section to the level set which is orthogonal to the integral curves generated by the Killing vector of the $U(1)$ group on $\mathcal{M}$. That is to say, the implementation of the gauge condition implements the $U(1)$ quotient part of the ADHM construction. The components of the Killing vector are given by

$$X_{Killing} = i\omega_{u\dot{\alpha}}\frac{\partial}{\partial\omega_{u\dot{\alpha}}} - i\bar{\omega}^{\dot{\alpha}}{}_{u}\frac{\partial}{\partial\bar{\omega}^{\dot{\alpha}}{}_{u}} \tag{5.72}$$

The requirement of orthogonality on tangent vectors is arrived at by using the metric:

$$g(X, X_{Killing}) = 0 \tag{5.73}$$

This can be converted to a condition on the one-forms by writing:

$$g(\_, X_{Killing}) = 0 \tag{5.74}$$

$$\Rightarrow i\, d\bar{\omega}^{\dot{\beta}}{}_{v}\left\langle d\omega_{v\dot{\beta}}\, , \omega_{u\dot{\alpha}}\frac{\partial}{\partial\omega_{u\dot{\alpha}}}\right\rangle - i\left\langle d\bar{\omega}^{\dot{\beta}}_{v}\, , \bar{\omega}^{\dot{\alpha}}{}_{u}\frac{\partial}{\partial\bar{\omega}^{\dot{\alpha}}{}_{u}}\right\rangle d\omega_{v\dot{\beta}} = 0 \tag{5.75}$$

Which yields the required gauge condition,

$$d\bar{\omega}^{\dot{\alpha}}{}_{u}\omega_{u\dot{\alpha}} - \bar{\omega}^{\dot{\alpha}}{}_{u}d\omega_{u\dot{\alpha}} = 0 \tag{5.76}$$

We now differentiate (5.70) to give

$$d\omega' = e^{i\phi}(dU.S + U.dS + id\phi\, U.S) \tag{5.77}$$

$$= e^{i\phi}\left(dU.\begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix} + U.d\rho\begin{pmatrix} \frac{\rho}{\rho_+} & 0 \\ 0 & \frac{\rho}{\rho_-} \\ 0 & 0 \end{pmatrix} + id\phi U.\begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix}\right) \tag{5.78}$$

Similarly we define the conjugate quantity $\bar{\omega}^{\dot{\alpha}}_{u}$:

$$\bar{\omega} = S^{\dagger}.U^{\dagger} \Rightarrow \bar{\omega}' = S^{\dagger}.U^{\dagger}e^{-i\phi} = \begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix} \tag{5.79}$$

Differentiating gives

$$d\bar{\omega}' = e^{-i\phi}(dS^\dagger.U^\dagger + S^\dagger.dU^\dagger - id\phi\, S^\dagger.U^\dagger) \tag{5.80}$$

$$= e^{-i\phi}\left(\begin{pmatrix} \frac{\rho}{\rho_+} & 0 & 0 \\ 0 & \frac{\rho}{\rho_-} & 0 \end{pmatrix}.U^\dagger d\rho + \begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix}.dU^\dagger - id\phi \begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix}.U^\dagger\right) \tag{5.81}$$

We can now calculate the parts of the expression for the gauge fixing condition. Firstly we have:

$$d\bar{\omega}^{\dot\alpha}{}_u\omega_{u\dot\alpha} = Tr(d\bar{\omega}.\omega) \tag{5.82}$$

$$= Tr\left\{\left[\begin{pmatrix} \frac{\rho}{\rho_+} & 0 & 0 \\ 0 & \frac{\rho}{\rho_-} & 0 \end{pmatrix} U^\dagger d\rho + \begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix} dU^\dagger \right.\right.$$

$$\left.\left. - id\phi \begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix}.U^\dagger\right] U \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix}\right\} \tag{5.83}$$

$$= 2\rho\, d\rho - 2i\rho^2\,, d\phi - Tr\left[\begin{pmatrix} \rho^2 + \frac{\zeta}{2} & 0 & 0 \\ 0 & \rho^2 - \frac{\zeta}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.U^\dagger.dU\right] \tag{5.84}$$

Where we have made use of the cyclic property of the trace and the identity $dU^\dagger.U = -U^\dagger.dU$, which derives from the unitary condition on U, $U^\dagger.U = 1$. Likewise, for the other part we get

$$\bar{\omega}^{\dot\alpha}_u d\omega_{u\dot\alpha} = Tr(\bar{\omega}.d\omega) \tag{5.85}$$

$$= Tr\left\{\begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix}.U^\dagger \left[dU.\begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix} + U.d\rho \begin{pmatrix} \frac{\rho}{\rho_+} & 0 \\ 0 & \frac{\rho}{\rho_-} \\ 0 & 0 \end{pmatrix} + id\phi U.\begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix}\right]\right\}$$

$$= 2\rho\, d\rho + 2id\phi\, \rho^2 + Tr\left[\begin{pmatrix} \rho^2 + \frac{\zeta}{2} & 0 & 0 \\ 0 & \rho^2 - \frac{\zeta}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^\dagger dU\right] \tag{5.86}$$

Inputing these results to (5.76) yields an equation for the parameter ϕ appearing in the $U(1)$ transformation.

$$2i\rho^2\, d\phi + Tr\left[\begin{pmatrix} \rho^2+\frac{\zeta}{2} & 0 & 0 \\ 0 & \rho^2-\frac{\zeta}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^{\dagger} dU\right] = 0 \tag{5.87}$$

$$\Rightarrow d\phi = \frac{i}{2\rho^2} Tr\left[\begin{pmatrix} \rho^2+\frac{\zeta}{2} & 0 & 0 \\ 0 & \rho^2-\frac{\zeta}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^{\dagger}.dU\right] \tag{5.88}$$

This expression for ϕ is not in its most useful form for our purposes. Ultimately we should like to compare the metric derived by this quotient construction to that deduced in [20]. To achieve this we shall need to give our expressions in terms of the left invariant one-forms on the coset space $SU(3)/U(1)$. Fortunately this is relatively straight forward to do. We shall make use of the expression

$$U^{\dagger}.dU = i\mathcal{L} \tag{5.89}$$

Where $\mathcal{L}$ is the matrix of left-invariant 1-forms given in the notation of [20] as

$$\mathcal{L} = \begin{pmatrix} \frac{1}{2}(\lambda+Q) & \nu & \sigma \\ \nu^* & \frac{1}{2}(Q-\lambda) & \Sigma \\ \sigma^* & \Sigma^* & ? \end{pmatrix} \tag{5.90}$$

As described in [20], Q is the $U(1)$ generator which lies outside the coset. The question mark denotes other 1-forms that lie outside the coset. We can now substitute this into our expression for $d\phi$:

$$d\phi = \frac{i^2}{2\rho^2} Tr\left[\begin{pmatrix} \rho^2+\frac{\zeta}{2} & 0 & 0 \\ 0 & \rho^2-\frac{\zeta}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} \frac{1}{2}(Q+\lambda) & \nu & \sigma \\ \nu^* & \frac{1}{2}(Q-\lambda) & \Sigma \\ \sigma^* & \Sigma^* & ? \end{pmatrix}\right] \tag{5.91}$$

$$= -\frac{1}{4\rho^2}(2\rho^2 Q + \lambda\zeta) \tag{5.92}$$

We can now substitute (5.92) back into (5.78) and (5.81). This represents the implementation of the $U(1)$ quotient.

$$d\omega' = dU \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix} + Ud\rho \begin{pmatrix} \frac{\rho}{\rho_+} & 0 \\ 0 & \frac{\rho}{\rho_-} \\ 0 & 0 \end{pmatrix} - \frac{i}{4\rho^2}(\lambda\zeta + 2\rho^2 Q)U \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix} \tag{5.93}$$

$$d\bar{\omega}' = \left(\begin{pmatrix} \frac{\rho}{\rho_+} & 0 & 0 \\ 0 & \frac{\rho}{\rho_-} & 0 \end{pmatrix} U^\dagger + \begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix} dU^\dagger + \frac{i}{4\rho^2}(\lambda\zeta + 2\rho^2 Q) \begin{pmatrix} \rho_+ & 0 & 0 \\ 0 & \rho_- & 0 \end{pmatrix} U^\dagger \right) \tag{5.94}$$

The above are still not in a convenient form to calculate the metric. Ideally we should like to pull out a pre-factor of U in $d\omega_{u\dot{\alpha}}$ and a post factor of $U^\dagger$ from $d\bar{\omega}_{u\dot{\alpha}}$. This would be convenient for then the unitary matrices will multiply to give the identity and so remove themselves from our consideration. Taking the first term in (5.93) and using the fact that U is unitary we may write

$$dU \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix} = UU^\dagger.dU. \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix} \tag{5.95}$$

$$= iU \begin{pmatrix} \frac{1}{2}(Q+\lambda) & \nu & \sigma \\ \nu^* & \frac{1}{2}(Q-\lambda) & \Sigma \\ \sigma^* & \Sigma^* & ? \end{pmatrix} . \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \\ 0 & 0 \end{pmatrix} \tag{5.96}$$

$$= iU \begin{pmatrix} \frac{1}{2}\rho_+(Q+\lambda) & \rho_-\nu \\ \rho_+\nu^* & \frac{1}{2}\rho_-(Q-\lambda) \\ \rho_+\sigma^* & \rho_-\Sigma^* \end{pmatrix} \tag{5.97}$$

Therefore $d\omega_{u\dot{\alpha}}$ becomes:

$$d\omega' = ie^{i\phi}U.\begin{pmatrix} \frac{1}{2\rho^2}\rho_+\rho_-^2\lambda - \frac{i}{\rho_+}\rho d\rho & \rho_-\nu \\ \rho_+\nu^* & -\frac{1}{2\rho^2}\rho_-\rho_+^2\lambda - \frac{i}{\rho_-}\rho d\rho \\ \rho_+\sigma^* & \rho_-\Sigma^* \end{pmatrix} \tag{5.98}$$

Performing similar manipulations upon $d\bar{\omega}_{u\dot{\alpha}}$ yields

$$d\bar{\omega}' = -ie^{-i\phi}\begin{pmatrix} \frac{1}{2\rho^2}\rho_+\rho_-^2 + \frac{i}{\rho_+}\rho d\rho & \rho_+\nu & \rho_+\sigma \\ \rho_-\nu^* & \frac{i}{\rho_-}\rho d\rho - \frac{1}{2\rho^2}\rho_-\rho_+^2\lambda & \rho_-\Sigma \end{pmatrix}.U^\dagger \tag{5.99}$$

Note that the 1-form outside the coset, Q, has obligingly canceled out of this expression and so it will not appear in the metric on the moduli space. Everything is now set up in a convenient form to allow calculation of the metric.

$$g = 2Tr(d\bar{\omega}'.d\omega') = 2Tr(d\bar{\omega}.d\omega) \tag{5.100}$$

$$= \frac{4\rho^4 d\rho^2}{\rho^4 - \left(\frac{\zeta}{2}\right)^2} + 4\rho^2\nu^2 + 2\left(\rho^2 + \frac{\zeta}{2}\right)\sigma^2 + 2\left(\rho^2 - \frac{\zeta}{2}\right)\Sigma^2 + \left(\rho^2 - \frac{\zeta^2}{4\rho^2}\right)\lambda^2 \tag{5.101}$$

Comparing the coefficients of the 1-forms with those given in [20] leads to;

$$ds_8^2 = h^2dr^2 + a^2(\sigma_1^2 + \sigma_2^2) + b^2(\Sigma_1^2 + \Sigma_2^2) + c^2(\nu_1^2 + \nu_2^2) + f^2\lambda^2 \tag{5.102}$$

$$a^2 = \frac{1}{2}(r^2 - 1) = 2\left(\rho^2 + \frac{\zeta}{2}\right) \tag{5.103}$$

$$b^2 = \frac{1}{2}(r^2 + 1) = 2\left(\rho^2 - \frac{\zeta}{2}\right) \tag{5.104}$$

$$c^2 = r^2 = 4\rho^2 \tag{5.105}$$

$$f^2 = \frac{1}{4}r^2(1 - r^{-4}) = \left(\rho^2 - \frac{\zeta^2}{4\rho^2}\right) \tag{5.106}$$

$$h^2 = (1 - r^{-4})^{-1} = \frac{\rho^4}{\rho^4 - (\frac{\zeta}{2})^2} \tag{5.107}$$

Which are in agreement if we set the noncomutivity parameter ζ as follows

$$\zeta = -\frac{1}{2} \tag{5.108}$$

and

$$r^2 = 4\rho^2 \tag{5.109}$$

Thus from here on we shall set $\zeta = -\frac{1}{2}$. However, note that there is nothing physically significant about the value of ζ. it merely depends upon the choice of scale for the radial variable, rescalinig ρ will also rescale the value of ζ. We can hereby write down the volume form for the 1-instanton moduli space,

$$d(Vol) = \frac{1}{2^3}r^3(r^2+1)(r^2-1)\,dr \wedge \sigma_1 \wedge \sigma_2 \wedge \Sigma_1 \wedge \Sigma_2 \wedge \nu_1 \wedge \nu_2 \wedge \lambda \tag{5.110}$$

$$= 2^5\rho^3\rho_+^2\rho_-^2\,d\rho \wedge \sigma_1 \wedge \sigma_2 \wedge \Sigma_1 \wedge \Sigma_2 \wedge \nu_1 \wedge \nu_2 \wedge \lambda \tag{5.111}$$

In summary, we have seen how to arrive at an explicit form for the metric on the moduli space of a single $SU(3)$ instanton by means of the A.D.H.M. construction.

5.3 Computation of the spin-connection and curvature 2-form

5.3.1 The spin-connection

We will use Cartan's first structure equation to compute the spin-connection on the moduli space. We will choose the obvious veilbien basis for the metric (5.102), namely

$$e^0 = dt,\ e^1 = a\sigma_1,\ e^2 = a\sigma_2,\ e^3 = b\Sigma_1,\ e^4 = b\Sigma_2,\ e^5 = c\nu_1,\ e^6 = c\nu_2,\ e^7 = f\lambda, \tag{5.112}$$

We will assume that the torsion tensor vanishes. One proceeds by writing out the spin connection with undetermined coefficients and using the Cartan equation to give a set of simultaneous equations which will fix these coefficients. The spin connection is a matrix-valued 1-form that we shall write in the following manner:

$$\omega^i{}_j = A^i{}_j dt + B^i{}_j\sigma_1 + C^i{}_j\sigma_2 + D^i{}_j\Sigma_1 + E^i{}_j\Sigma_2 + F^i{}_j\nu_1 + G^i{}_j\nu_2 + H^i{}_j\lambda + J^i{}_jQ \tag{5.113}$$

where i=0,1,...,8. Note the inclusion of the one-form Q in the spin connection. This is because the exterior derivatives of some of the e^i's include Q, so Q must necessarily appear in the spin connection. Due to the antisymmetry of the spin-connection we must have $\omega^i{}_i = 0$, (no summation over i). The detailed computation is presented in the appendix. The results of the analysis are as follows;

$$-\omega^0{}_1 = \omega^3{}_6 = \omega^4{}_5 = \omega^2{}_7 = \frac{b}{r}\sigma_1 \tag{5.114}$$

$$\omega^0{}_2 = \omega^1{}_7 = \omega^3{}_5 = -\omega^4{}_6 = -\frac{b}{r}\sigma_2 \tag{5.115}$$

$$\omega^0{}_3 = \omega^1{}_6 = -\omega^2{}_5 = \omega^4{}_7 = -\frac{a}{r}\Sigma_1 \tag{5.116}$$

$$\omega^0{}_4 = \omega^1{}_5 = \omega^2{}_6 = -\omega^3{}_7 = -\frac{a}{r}\Sigma_2 \tag{5.117}$$

$$-\omega^0{}_5 = \omega^6{}_7 = \sqrt{1-r^{-4}}\nu_1 \tag{5.118}$$

$$\omega^0{}_6 = \omega^5{}_7 = \sqrt{1-r^{-4}}\nu_2 \tag{5.119}$$

$$-\omega^0{}_7 = \omega^5{}_6 = \frac{1}{2}(1+r^{-4})\lambda \tag{5.120}$$

$$\omega^1{}_2 = -\frac{1}{2r^2}\lambda + \left(\frac{3}{2} - \frac{g^2}{a^2}\right) Q \tag{5.121}$$

$$\omega^3{}_4 = -\frac{1}{2r^2}\lambda + \left(\frac{3}{2} - \frac{g^2}{b^2}\right) Q \tag{5.122}$$

5.3.2 The curvature 2-form

We use the spin connection to calculate the curvature 2-form from Cartan's second structure equation;

$$R^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \tag{5.123}$$

In the appendix we will proceed to calculate explicitly a sample of the linearly independent components of the curvature. The results of this analysis are summarized below;

$$R^0_1 = -\frac{1}{r^4}(-e^0 \wedge e^1 + e^2 \wedge e^7 + e^3 \wedge e^6 + e^4 \wedge e^5) \tag{5.124}$$

$$R^0_2 = -\frac{1}{r^4}(-e^0 \wedge e^2 - e^1 \wedge e^7 - e^3 \wedge e^5 + e^4 \wedge e^6) \tag{5.125}$$

$$R^0_3 = -\frac{1}{r^4}(e^0 \wedge e^3 + e^1 \wedge e^6 - e^2 \wedge e^5 + e^4 \wedge e^7) \tag{5.126}$$

$$R^0_4 = -\frac{1}{r^4}(e^0 \wedge e^4 + e^1 \wedge e^5 + e^2 \wedge e^6 - e^3 \wedge e^7) \tag{5.127}$$

$$R^0_5 = -\frac{2}{r^6}(e^0 \wedge e^5 - e^6 \wedge e^7) \tag{5.128}$$

$$R^0_6 = -\frac{2}{r^6}(e^0 \wedge e^6 + e^5 \wedge e^7) \tag{5.129}$$

$$R^0_7 = \frac{4}{r^6}(e^0 \wedge e^7 - e^5 \wedge e^6) + \frac{2}{r^4}(e^1 \wedge e^2 + e^3 \wedge e^4) \tag{5.130}$$

$$R^1_2 = \frac{2}{r^4}(e^0 \wedge e^7 - e^5 \wedge e^6) + \frac{4}{r^2}(e^1 \wedge e^2 + e^3 \wedge e^4) \tag{5.131}$$

$$R^1_3 = -\frac{2}{r^2}(e^1 \wedge e^3 - e^2 \wedge e^4) \tag{5.132}$$

$$R^1_4 = -\frac{2}{r^2}(e^1 \wedge^4 + e^2 \wedge e^3) \tag{5.133}$$

$$R^1_5 = -\frac{1}{r^4}(e^0 \wedge e^4 + e^1 \wedge e^5 + e^2 \wedge e^6 - e^3 \wedge e^7) \tag{5.134}$$

$$R^1_6 = -\frac{1}{r^4}(e^0 \wedge e^3 + e^1 \wedge e^6 - e^2 \wedge e^5 + e^4 \wedge e^7) \tag{5.135}$$

$$R^1_7 = \frac{1}{r^4}(e^0 \wedge e^2 + e^1 \wedge e^7 + e^3 \wedge e^5 - e^4 \wedge e^6) \tag{5.136}$$

$$R^2_3 = -\frac{2}{r^2}(e^1 \wedge e^4 + e^2 \wedge e^3) \tag{5.137}$$

$$R^2_4 = -\frac{2}{r^2}(e^2 \wedge e^4 - e^1 \wedge e^3) \tag{5.138}$$

$$R^2_5 = -\frac{1}{r^4}(-e^0 \wedge e^3 - e^1 \wedge e^6 + e^2 \wedge e^5 - e^4 \wedge e^7) \tag{5.139}$$

$$R^2_6 = -\frac{1}{r^4}(e^0 \wedge e^4 + e^1 \wedge e^5 + e^2 \wedge e^6 - e^3 \wedge e^7) \tag{5.140}$$

$$R^2_7 = \frac{1}{r^4}(-e^0 \wedge e^1 + e^2 \wedge e^7 + e^3 \wedge e^6 + e^4 \wedge e^5) \tag{5.141}$$

$$R^3_4 = \frac{2}{r^4}(e^0 \wedge e^7 - e^5 \wedge e^6) + \frac{4}{r^2}(e^1 \wedge e^2 + e^3 \wedge e^4) \quad (5.142)$$

$$R^3_5 = \frac{1}{r^4}(e^0 \wedge e^2 + e^1 \wedge e^7 + e^3 \wedge e^5 - e^4 \wedge e^6) \quad (5.143)$$

$$R^3_6 = -\frac{1}{r^4}(e^0 \wedge e^1 - e^2 \wedge e^7 - e^3 \wedge e^6 - e^4 \wedge e^5) \quad (5.144)$$

$$R^3_7 = \frac{1}{r^4}(e^0 \wedge e^4 + e^1 \wedge e^5 + e^2 \wedge e^6 - e^3 \wedge e^7) \quad (5.145)$$

$$R^4_5 = \frac{1}{r^4}(-e^0 \wedge e^1 + e^2 \wedge e^7 + e^3 \wedge e^6 + e^4 \wedge e^5) \quad (5.146)$$

$$R^4_6 = -\frac{1}{r^4}(e^0 \wedge e^2 + e^1 \wedge e^7 + e^3 \wedge e^5 - e^4 \wedge e^6) \quad (5.147)$$

$$R^4_7 = -\frac{1}{r^4}(e^0 \wedge e^3 + e^1 \wedge e^6 - e^2 \wedge e^5 + e^4 \wedge e^7) \quad (5.148)$$

$$R^5_6 = -\frac{2}{r^4}(e^1 \wedge e^2 + e^3 \wedge e^4) - \frac{4}{r^6}(e^0 \wedge e^7 - e^5 \wedge e^6) \quad (5.149)$$

$$R^5_7 = -\frac{2}{r^6}(e^0 \wedge e^6 + e^5 \wedge e^7) \quad (5.150)$$

$$R^6_7 = \frac{2}{r^6}(e^0 \wedge e^5 - e^6 \wedge e^7) \quad (5.151)$$

Fortunately, unlike the spin connection, the curvature has no component proportional to the 1-form Q.

We are now in a position to calculate the Gauss-Bonnet integral according to (B.3) Unfortunately, the number of terms involved made the calculation of this integrand prohibitively large, even for Mathematica. To make further progress a profound simplification would have to be sought.

5.4 The symplectic curvature.

Having reached an impasse with the direct calculation of the Gauss-Bonnet integral using the above calculated curvature 2-forms, it was realized that an important simplification might result if we could obtain the coordinate change to a system in which the symplectic nature of the curvature tensor was explicitly realized. This should always be possible for a hyper-Kahler manifold. To start with, we note that on a hyper-Kahler manifold one can write the three complex structures as follows:

$$\omega^c = i\Omega_{ij}\tau^c_{\dot\beta\dot\gamma}h^{i\dot\beta}\wedge h^{j\dot\gamma} \tag{5.152}$$

Where $\tau^{(c)}_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\gamma}\tau^{(c)\dot\gamma}{}_{\dot\beta}$, and $\tau^{(c)\dot\gamma}{}_{\dot\beta}$ are the usual (components of) the Pauli matrices. And the curvature tensor in this basis may be expressed as

$$R_{(i_1\alpha_1)(i_2\alpha_2)(j_1\beta_1)(j_2\beta_2)} = R_{i_1i_2j_1j_2}\epsilon_{\alpha_1\alpha_2}\epsilon_{\beta_1\beta_2} \tag{5.153}$$

We will look for a basis in which Ω_{ij} has the simple form

$$[\Omega_{ij}] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{5.154}$$

We can then expand out the three complex structures as

$$\omega^1 = 2i\left(h^{1\dot2}\wedge h^{2\dot2} + h^{3\dot2}\wedge h^{4\dot2} - h^{1\dot1}\wedge h^{2\dot1} - h^{3\dot1}\wedge h^{4\dot1}\right) \tag{5.155}$$

$$\omega^2 = 2\left(h^{1\dot1}\wedge h^{2\dot1} + h^{1\dot2}\wedge h^{2\dot2} + h^{3\dot2}\wedge h^{4\dot2} + h^{3\dot1}\wedge h^{4\dot1}\right) \tag{5.156}$$

$$\omega^3 = 2i\left(h^{1\dot2}\wedge h^{2\dot1} + h^{1\dot1}\wedge h^{2\dot2} + h^{3\dot2}\wedge h^{4\dot1} + h^{3\dot1}\wedge h^{4\dot2}\right) \tag{5.157}$$

Fortunately for us, the work of finding the three complex structures on our space has already been done and can be found in [20]. Next we form the two linear combinations ω^+ and ω^-

$$\omega^+ = \omega^1 + i\omega^2 \tag{5.158}$$

$$= 4i(h^{1\dot2}\wedge h^{2\dot2} + h^{3\dot2}\wedge h^{4\dot2}) \tag{5.159}$$

$$\omega^- = \omega^1 - i\omega^2 \tag{5.160}$$

$$= -4i(h^{1\dot{1}} \wedge h^{2\dot{1}} + h^{3\dot{1}} \wedge h^{4\dot{1}}) \tag{5.161}$$

We shall then compare these with the combinations, K_+ and K_- given by [20] and attempt to identify terms.

$$K_+ = \epsilon^0 \wedge \epsilon^\# + i\epsilon^\alpha \wedge \epsilon^{\tilde{\alpha}} \tag{5.162}$$

$$K_- = \bar{\epsilon}^0 \wedge \bar{\epsilon}^\# - i\bar{\epsilon}^\alpha \wedge \bar{\epsilon}^{\tilde{\alpha}} \tag{5.163}$$

Where the $\epsilon's$ are given by

$$\epsilon^0 = e^0 + ie^7 \tag{5.164}$$

$$\epsilon^\# = e^5 + ie^6 \tag{5.165}$$

$$\epsilon^\alpha = e^1 + ie^2 \tag{5.166}$$

$$\epsilon^{\tilde{\alpha}} = e^3 - ie^4 \tag{5.167}$$

Setting ω^+ equal to twice K_+ gives

$$4i\left(h^{1\dot{2}} \wedge h^{2\dot{2}} + h^{3\dot{2}} \wedge h^{4\dot{2}}\right) = 2\left(\epsilon^0 \wedge \epsilon^\# + i\epsilon^\alpha \wedge \epsilon^{\tilde{\alpha}}\right) \tag{5.168}$$

We now make the following identifications,

$$4i\, h^{1\dot{2}} \wedge h^{2\dot{2}} = 2\epsilon^0 \wedge \epsilon^\# \tag{5.169}$$

$$4i\, h^{3\dot{2}} \wedge h^{4\dot{2}} = 2i\epsilon^\alpha \wedge \epsilon^{\tilde{\alpha}} \tag{5.170}$$

Comparison of the two sides of this equation suggests that we should make the following identifications:

$$h^{1\dot{2}} = \frac{1}{\sqrt{2}}\epsilon^0 = \frac{1}{\sqrt{2}}(e^0 + ie^7) \tag{5.171}$$

$$h^{2\dot{2}} = \frac{1}{i\sqrt{2}}\epsilon^{\#} = \frac{1}{\sqrt{2}}(-ie^5 + e^6) \tag{5.172}$$

Likewise, setting $\sqrt{2}h^{3\dot{2}} = \epsilon^{\alpha}$ gives

$$h^{3\dot{2}} = \frac{1}{\sqrt{2}}\epsilon^{\alpha} = \frac{1}{\sqrt{2}}(e^1 + ie^2) \tag{5.173}$$

$$h^{4\dot{2}} = \frac{1}{\sqrt{2}}\epsilon^{\tilde{\alpha}} = \frac{1}{\sqrt{2}}(e^3 - ie^4) \tag{5.174}$$

We proceed in a similar fashion for the other coordinates, giving:

$$h^{2\dot{1}} = \frac{1}{\sqrt{2}}\bar{\epsilon}^{0} = \frac{1}{\sqrt{2}}(e^0 - ie^7) \tag{5.175}$$

$$h^{1\dot{1}} = \frac{1}{\sqrt{2}}\bar{\epsilon}^{\#} = \frac{1}{\sqrt{2}}(-ie^5 - e^6) \tag{5.176}$$

$$h^{4\dot{1}} = \frac{1}{\sqrt{2}}\bar{\epsilon}^{\alpha} = \frac{1}{\sqrt{2}}(e^1 - ie^2) \tag{5.177}$$

$$h^{3\dot{1}} = -\frac{1}{\sqrt{2}}\bar{\epsilon}^{\tilde{\alpha}} = \frac{1}{\sqrt{2}}(-e^3 - ie^4) \tag{5.178}$$

This is our orthonormalised complex veilbien basis. We now wish to calculate the metric and the components of the curvature in this basis. It will therefore be helpful to study in general how one may express tensor components with respect to different basis'.

5.5 Rules for the change of basis

Consider a (dual) vector $\mathbf{V}$ expressed in both coordinate systems:

$$\mathbf{V} = V_{i\dot{\alpha}}h^{i\dot{\alpha}} = V_a e^a \tag{5.179}$$

Writing this out in components we have,

$$V_{1\dot{1}}h^{1\dot{1}} + V_{2\dot{1}}h^{2\dot{1}} + V_{3\dot{1}}h^{3\dot{1}} + V_{4\dot{1}}h^{4\dot{1}} + V_{1\dot{2}}h^{1\dot{2}} + V_{2\dot{2}}h^{2\dot{2}} + V_{3\dot{2}}h^{3\dot{2}} + V_{4\dot{2}}h^{4\dot{2}} \tag{5.180}$$

$$= V_0e^0 + V_1e^1 + V_2e^2 + V_3e^3 + V_4e^4 + V_5e^5 + V_6e^6 + V_7e^7 \quad (5.181)$$

Using equations (5.171)-(5.178) then gives

$$\mathbf{V} = \frac{1}{\sqrt{2}}(V_{2\dot{1}} + V_{1\dot{2}})e^0 + \frac{1}{\sqrt{2}}(V_{3\dot{2}} + V_{4\dot{1}})e^1 + \frac{i}{\sqrt{2}}(V_{3\dot{2}} - V_{4\dot{1}})e^2 + \frac{1}{\sqrt{2}}(V_{42} - V_{31})e^3 \quad (5.182)$$

$$-\frac{i}{\sqrt{2}}(V_{3\dot{1}} + V_{4\dot{2}})e^4 - \frac{i}{\sqrt{2}}(V_{1\dot{1}} + V_{2\dot{2}})e^5 + \frac{1}{\sqrt{2}}(V_{22} - V_{11})e^6 + \frac{i}{\sqrt{2}}(V_{12} - V_{21})e^7 \quad (5.183)$$

Comparing coefficients of the e^i's yields

$$V_0 = \frac{1}{\sqrt{2}}(V_{2\dot{1}} + V_{1\dot{2}}) \quad (5.184)$$

$$V_1 = \frac{1}{\sqrt{2}}(V_{3\dot{2}} + V_{4\dot{1}}) \quad (5.185)$$

$$V_2 = \frac{i}{\sqrt{2}}(V_{3\dot{2}} - V_{4\dot{1}}) \quad (5.186)$$

$$V_3 = \frac{1}{\sqrt{2}}(V_{4\dot{2}} - V_{3\dot{1}}) \quad (5.187)$$

$$V_4 = -\frac{i}{\sqrt{2}}(V_{3\dot{1}} + V_{4\dot{2}}) \quad (5.188)$$

$$V_5 = -\frac{i}{\sqrt{2}}(V_{1\dot{1}} + V_{2\dot{2}}) \quad (5.189)$$

$$V_6 = \frac{1}{\sqrt{2}}(V_{2\dot{2}} - V_{1\dot{1}}) \quad (5.190)$$

$$V_7 = \frac{i}{\sqrt{2}}(V_{1\dot{2}} - V_{2\dot{1}}) \quad (5.191)$$

Solving these equations for the $V_{i\alpha}$'s gives

$$V_{1\dot{1}} = \frac{1}{\sqrt{2}}(iV_5 - V_6) \quad (5.192)$$

$$V_{1\dot{2}} = \frac{1}{\sqrt{2}}(V_0 - iV_7) \quad (5.193)$$

$$V_{2\dot{1}} = \frac{1}{\sqrt{2}}(V_0 + iV_7) \quad (5.194)$$

$$V_{2\dot{2}} = \frac{1}{\sqrt{2}}(V_6 + iV_5) \tag{5.195}$$

$$V_{3\dot{1}} = \frac{1}{\sqrt{2}}(iV_4 - V_3) \tag{5.196}$$

$$V_{3\dot{2}} = \frac{1}{\sqrt{2}}(V_1 - iV_2) \tag{5.197}$$

$$V_{4\dot{1}} = \frac{1}{\sqrt{2}}(V_1 + iV_2) \tag{5.198}$$

$$V_{4\dot{2}} = \frac{1}{\sqrt{2}}(V_3 + iV4) \tag{5.199}$$

We can now give a set of rules for finding the components of a tensor in this new basis. For example, given a 2-form, $\mathbf{T}$, we can write

$$\mathbf{T} = T_{ab}e^a \wedge e^b = T_{(i\dot{\alpha})\,(j\dot{\beta})}\, h^{i\dot{\alpha}} \wedge h^{j\dot{\beta}} \tag{5.200}$$

where, for example, our particular coordinate change would provide the correspondence

$$T_{1\dot{1}\,1\dot{2}} = T_{\frac{1}{\sqrt{2}}(i5-6)\ \frac{1}{\sqrt{2}}(0-i7)} \tag{5.201}$$

$$= \frac{1}{2}(iT_{50} - i^2T_{57} + T_{06} + iT_{67}) \tag{5.202}$$

$$= \frac{1}{2}(-iT_{05} + T_{57} + T_{06} + iT_{67}) \tag{5.203}$$

We can check that we can recover our original metric using these rules. The non-zero components are as follows:

$$g_{(1\dot{1})\,(2\dot{2})} = g_{\frac{1}{\sqrt{2}}(i5-6)\ \frac{1}{\sqrt{2}}(6+i5)} = -1 \tag{5.204}$$

$$g_{(1\dot{2})\,(2\dot{1})} = g_{\frac{1}{\sqrt{2}}(0-i7)\ \frac{1}{\sqrt{2}}(0+i7)} = 1 \tag{5.205}$$

$$g_{(3\dot{1})\,(4\dot{2})} = g_{\frac{1}{\sqrt{2}}(i4-3)\ \frac{1}{\sqrt{2}}(3+i4)} = -1 \tag{5.206}$$

$$g_{(3\dot{2})\,(4\dot{1})} = g_{\frac{1}{\sqrt{2}}(1-i2)\ \frac{1}{\sqrt{2}}(1+i2)} = 1 \tag{5.207}$$

Using these gives:

$$g = 2\left(h^{1\dot{2}}h^{2\dot{1}} - h^{1\dot{1}}h^{2\dot{2}} + h^{3\dot{2}}h^{4\dot{1}} - h^{3\dot{1}}h^{4\dot{2}}\right) \tag{5.208}$$

$$= (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2 \tag{5.209}$$

as required. As an aside it will be convenient for later use to calculate the determinant of the metric in this symplectic basis. Written as a matrix, the metric looks like

$$g = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.210}$$

and so $det(g) = 1$. (Incidentally, one could transform the metric above to a diagonal form simply by changing the order in which the basis vectors are labeled).

Using the same coordinate change methods we may calculate the components of the curvature in this basis. The non-zero results are:

$$R_{(1\dot{1})\,(1\dot{2})} = -\frac{4}{r^6} h^{2\dot{1}} \wedge h^{2\dot{2}} \tag{5.211}$$

$$R_{(1\dot{1})\,(2\dot{2})} = \frac{2}{r^4}(h^{3\dot{2}} \wedge h^{4\dot{1}} - h^{3\dot{1}} \wedge h^{4\dot{2}}) + \frac{4}{r^6}(h^{1\dot{2}} \wedge h^{2\dot{1}} - h^{1\dot{1}} \wedge h^{2\dot{2}}) \tag{5.212}$$

$$R_{(1\dot{2})\,(2\dot{1})} = \frac{2}{r^4}(h^{3\dot{1}} \wedge h^{4\dot{2}} - h^{3\dot{2}} \wedge h^{4\dot{1}}) + \frac{4}{r^6}(h^{1\dot{1}} \wedge h^{2\dot{2}} - h^{1\dot{2}} \wedge h^{2\dot{1}}) \tag{5.213}$$

$$R_{(1\dot{1})\,(3\dot{2})} = \frac{2}{r^4}(h^{2\dot{2}} \wedge h^{4\dot{1}} - h^{2\dot{1}} \wedge h^{4\dot{2}}) \tag{5.214}$$

$$R_{(1\mathrm{i})\,(4\dot{2})} = \frac{2}{r^4}(h^{2\dot{2}} \wedge h^{3\mathrm{i}} - h^{2\mathrm{i}} \wedge h^{3\dot{2}}) \tag{5.215}$$

$$R_{(1\dot{2})\,(3\mathrm{i})} = \frac{2}{r^4}(h^{2\mathrm{i}} \wedge h^{4\dot{2}} - h^{2\dot{2}} \wedge h^{4\mathrm{i}}) \tag{5.216}$$

$$R_{(1\dot{2})\,(4\mathrm{i})} = \frac{2}{r^4}(h^{2\mathrm{i}} \wedge h^{3\dot{2}} - h^{22} \wedge h^{3\mathrm{i}}) \tag{5.217}$$

$$R_{(2\mathrm{i})\,(2\dot{2})} = -\frac{2}{r^6}h^{1\mathrm{i}} \wedge h^{1\dot{2}} \tag{5.218}$$

$$R_{(2\mathrm{i})\,(3\dot{2})} = \frac{2}{r^4}(h^{1\dot{2}} \wedge h^{4\mathrm{i}} - h^{1\mathrm{i}} \wedge h^{4\dot{2}}) \tag{5.219}$$

$$R_{(2\mathrm{i})\,(4\dot{2})} = \frac{2}{r^4}(h^{1\dot{2}} \wedge h^{3\mathrm{i}} - h^{1\mathrm{i}} \wedge h^{3\dot{2}}) \tag{5.220}$$

$$R_{(2\dot{2})\,(3\mathrm{i})} = \frac{2}{r^4}(h^{1\mathrm{i}} \wedge h^{4\dot{2}} - h^{1\dot{2}} \wedge h^{4\mathrm{i}}) \tag{5.221}$$

$$R_{(2\dot{2})\,(4\mathrm{i})} = \frac{2}{r^4}(h^{1\mathrm{i}} \wedge h^{3\dot{2}} - h^{1\dot{2}} \wedge h^{3\mathrm{i}}) \tag{5.222}$$

$$R_{(3\mathrm{i})\,(3\dot{2})} = -\frac{4}{r^2}h^{4\mathrm{i}} \wedge h^{4\dot{2}} \tag{5.223}$$

$$R_{(3\mathrm{i})\,(4\dot{2})} = \frac{2}{r^4}(h^{1\dot{2}} \wedge h^{2\mathrm{i}} - h^{1\mathrm{i}} \wedge h^{2\dot{2}}) + \frac{4}{r^2}(h^{3\dot{2}} \wedge h^{4\mathrm{i}} - h^{3\mathrm{i}} \wedge h^{4\dot{2}}) \tag{5.224}$$

$$R_{(3\dot{2})\,(4\mathrm{i})} = \frac{2}{r^4}(h^{1\mathrm{i}} \wedge h^{2\dot{2}} - h^{1\dot{2}} \wedge h^{2\mathrm{i}}) + \frac{4}{r^2}(h^{3\mathrm{i}} \wedge h^{4\dot{2}} - h^{3\dot{2}} \wedge h^{4\mathrm{i}}) \tag{5.225}$$

$$R_{(4\mathrm{i})\,(4\dot{2})} = -\frac{4}{r^2}h^{3\mathrm{i}} \wedge h^{3\dot{2}} \tag{5.226}$$

Where use has been made of the multiplication table for the $h^{i\dot{\alpha}}$'s which is given at the end of this chapter. We are now in a position to calculate the components of the symplectic curvature, (5.153). As an example of the method we calculate the three linearly independent non-zero components below, the rest being either zero or are related by symmetry.

$$\begin{aligned} R_{(1\mathrm{i})\,(1\dot{2})\,(2\mathrm{i})\,(2\dot{2})} &= R_{1122}\,\epsilon_{\mathrm{i}\,\dot{2}}\,\epsilon_{\mathrm{i}\,\dot{2}} = -\frac{4}{r^6} \\ \Rightarrow R_{1122} &= -\frac{4}{r^6} \end{aligned} \tag{5.227}$$

$$R_{(1\mathrm{i})\,(2\dot{2})\,(3\dot{2})\,(4\mathrm{i})} = R_{1234}\,\epsilon_{\mathrm{i}\,\dot{2}}\,\epsilon_{\dot{2}\,\mathrm{i}} = \frac{2}{r^4}$$

$$\Rightarrow R_{1\,2\,3\,4} = -\frac{2}{r^4} \tag{5.228}$$

$$R_{(3\dot{1})\,(3\dot{2})\,(4\dot{1})\,(4\dot{2})} = R_{3\,3\,4\,4}\,\epsilon_{\dot{1}\,\dot{2}}\,\epsilon_{\dot{1}\,\dot{2}} = -\frac{4}{r^2}$$

$$\Rightarrow R_{3\,3\,4\,4} = -\frac{4}{r^2} \tag{5.229}$$

This is our result for the symplectic curvature. We are now in a position to calculate the Gauss-Bonnet integral.

5.6 The calculation of the Gauss-Bonnet integral

Written in terms of an orthonormal non-coordinate vielbien basis, the volume-contribution to the Euler character of a $2n$-dimensional manifold is given by

$$\chi_{bulk} = \frac{(-1)^n}{(4\pi)^n n!}\int \epsilon^{i_1 i_2 \ldots i_{2n}}\, R_{i_1 i_2} \wedge R_{i_3 i_4} \wedge \ldots \wedge R_{i_{2n-1} i_{2n}} \tag{5.230}$$

$$= \frac{(-1)^n}{(8\pi)^n n!}\int \epsilon^{i_1 i_2 \ldots i_{2n}} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} \ldots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}}\, \theta^{j_1} \wedge \theta^{j_2} \wedge \ldots \wedge \theta^{j_{2n}} \tag{5.231}$$

$$= \frac{(-1)^n}{(8\pi)^n n!}\int \epsilon^{i_1 i_2 \ldots i_{2n}} \epsilon^{j_1 j_2 \ldots j_{2n}} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} \ldots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}}\, \theta^{1} \wedge \theta^{2} \wedge \ldots \wedge \theta^{2n} \tag{5.232}$$

where the θ^j are the basis of non-coordinate one-forms. In the case in hand these are given by

$$\theta^j \to h^{i\dot{\alpha}} \tag{5.233}$$

In a vielbein basis the volume element is given simply by

$$dV = h^{1\dot{1}} \wedge h^{1\dot{2}} \wedge h^{2\dot{1}} \wedge h^{2\dot{2}} \wedge h^{3\dot{1}} \wedge h^{3\dot{2}} \wedge h^{4\dot{1}} \wedge h^{4\dot{2}} \tag{5.234}$$

In terms of the left-invariant 1-forms of $SU(3)/U(1)$ this becomes;

$$dV = (e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \wedge e^7) \tag{5.235}$$

$$= \left(\frac{1}{2}\right)^3 r^3 (r^4 - 1)(dr \wedge \sigma_1 \wedge \sigma_2 \wedge \Sigma_1 \wedge \Sigma_2 \wedge \nu_1 \wedge \nu_2 \wedge \lambda) \tag{5.236}$$

Firstly we note that the curvature is a function of r only, so the integral over the coset elements separates out, giving the volume of the coset $SU(3)/U(1)$.

$$Vol(SU(3)/U(1)) = \int \sigma_1 \wedge \sigma_2 \wedge \Sigma_1 \wedge \Sigma_2 \wedge \nu_1 \wedge \nu_2 \wedge \lambda = \pi^4 \tag{5.237}$$

This result is arrived at in appendix G by comparing the ADHM measure with that obtained from the metric of [20].

The remaining integration is only over the radial coordinate.

$$\chi_{bulk} = \frac{1}{(8)^4 4!}\left(\frac{1}{2}\right)^3 \int dr r^3(r^4-1)\epsilon^{p_1\cdots p_8}\,\epsilon^{q_1\cdots q_8}\,R_{p_1p_2q_1q_2}\,R_{p_3p_4q_3q_4}\,R_{p_5p_6q_5q_6}\,R_{p_7p_8q_7q_8} \tag{5.238}$$

Now we make the change to the double index notation. In this notation the tensorial part of the integrand above becomes:

$$\epsilon^{(i_1\dot\alpha_1)\ldots(i_8\dot\alpha_8)}\,\epsilon^{(j_1\dot\beta_1)\ldots(j_8\dot\beta_8)}\,R_{(i_1\dot\alpha_1)\,(i_2\dot\alpha_2)\,(j_1\dot\beta_1)\,(j_2\dot\beta_2)}\cdots R_{(i_7\dot\alpha_7)\,(i_8\dot\alpha_8)\,(j_7\dot\beta_7)\,(j_8\dot\beta_8)} \tag{5.239}$$

Now recall that on a hyper-Kahler manifold we may write the curvature as in equation (5.153). Using this gives an expression in which all the spinor indices are contracted:

$$\epsilon^{i_1\dot\alpha_1\ldots i_8\dot\alpha_8}\,\epsilon^{j_1\dot\beta_1\ldots j_8\dot\beta_8}\,\epsilon_{\dot\alpha_1\dot\alpha_2}\,\epsilon_{\dot\alpha_3\dot\alpha_4}\,\epsilon_{\dot\alpha_5\dot\alpha_6}\,\epsilon_{\dot\alpha_7\dot\alpha_8}\,\epsilon_{\dot\beta_1\dot\beta_2}\,\epsilon_{\dot\beta_3\dot\beta_4}\,\epsilon_{\dot\beta_5\dot\beta_6}\,\epsilon_{\dot\beta_7\dot\beta_8}\,R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,R_{i_7i_8j_7j_8} \tag{5.240}$$

By laboriously expanding out the terms and using the symmetry of the symplectic curvature, one can show, (see appendix B).

$$\epsilon^{i_1\dot\alpha_1\ldots i_8\dot\alpha_8}\epsilon_{\dot\alpha_1\dot\alpha_2}\,\epsilon_{\dot\alpha_3\dot\alpha_4}\,\epsilon_{\dot\alpha_5\dot\alpha_6}\,\epsilon_{\dot\alpha_7\dot\alpha_8}\;R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,R_{i_7i_8j_7j_8}$$

$$= 16\epsilon^{i_1i_3i_5i_7}\epsilon^{i_2i_4i_6i_8}R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,R_{i_7i_8j_7j_8} \tag{5.241}$$

With a similar result holding for the terms involving the j_i's. We can now write an expression for χ in which all the spinor indices have been removed.

$$\chi_{bulk} = \frac{1}{(3)(2)^{10}} \int_1^\infty dr\, r^3(r^4-1)\, \epsilon^{i_1i_2i_3i_4}\, \epsilon^{j_1j_2j_3j_4}\, \epsilon^{k_1k_2k_3k_4}\, \epsilon^{l_1l_2l_3l_4}\, R_{i_1j_1k_1l_1}\, R_{i_2j_2k_2l_2}\, R_{i_3j_3k_3l_3}\, R_{i_4j_4\ldots} \quad (5.242)$$

Using the fact that there are only three linearly independent non-zero components to the symplectic curvature, together with the combinatoric arguments given in the appendix we may write the above as

$$\chi = \frac{1}{(3)(2)^{10}} \int_1^\infty dr\, r^3(r^4-1)$$

$$\left(216\,(R_{1122})^2\,(R_{3344})^2 + 1152\,R_{1122}\,R_{3344}\,(R_{1234})^2 + 576\,(R_{1234})^4\right) \quad (5.243)$$

$$= 45 \int_1^\infty dr \left(\frac{1}{r^9} - \frac{1}{r^{13}}\right) \quad (5.244)$$

$$= \frac{15}{8} \quad (5.245)$$

This is our final result for the volume contribution to the Euler character of the moduli space for a single $SU(3)$ instanton. This is the result that we shall compare with that given in [17].

5.7 Summary

We started this chapter with an explicit implementation of the ADHM construction for the $SU(3)$ 1-instanton case. We were able to obtain the metric on the moduli space and compare it with that previously obtained in [20]. Note that we have introduced the non-commutativity parameter, ζ, as discussed in the introduction. We have written the metric in terms of the left-invariant 1-forms of the quotient group $SU(3)/U(1)$. We then proceeded to use the Maurer-Cartan equation to obtain the exterior derivatives of these basis forms. This allowed us to use the Cartan structure equations to determine the spin connection and the curvature 2-form on the moduli space. Having obtained these results we transformed to a coordinate system called the symplectic basis that exploits the hyper-Kahler nature of our space. We were able to ascertain the components of the symplectic curvature tensor and use this simplification to calculate the Gauss-Bonnet integral over the moduli space. Since the Gauss-Bonnet integral represents a topological invariant, we expect its value to be independent of deformations of the moduli space induced by varying ζ. Thus naively we suspect that this value also pertains to the true instanton moduli space where we take the limit $\zeta \to 0$.

5.8 Multiplication table

$$h^{1\,\mathrm{i}} \wedge h^{1\,\dot{2}} = \frac{1}{\sqrt{2}}(-ie^5 - e^6) \wedge \frac{1}{\sqrt{2}}(e^0 + ie^7) \tag{5.246}$$

$$= \frac{1}{2}(ie^0 \wedge e^5 + e^5 \wedge e^7 + e^0 \wedge e^6 - ie^6 \wedge e^7) \tag{5.247}$$

$$h^{1\,\mathrm{i}} \wedge h^{2\,\mathrm{i}} = \frac{1}{2}(ie^0 \wedge e^5 - e^5 \wedge e^7 + e^0 \wedge e^6 + ie^6 \wedge e^7) \tag{5.248}$$

$$h^{1\,\mathrm{i}} \wedge h^{2\,\dot{2}} = -ie^5 \wedge e^6 \tag{5.249}$$

$$h^{1\,\mathrm{i}} \wedge h^{3\,\mathrm{i}} = \frac{1}{2}(-ie^3 \wedge e^5 + e^4 \wedge e^5 - e^3 \wedge e^6 - ie^4 \wedge e^6) \tag{5.250}$$

$$h^{1\,\mathrm{i}} \wedge h^{3\,\dot{2}} = \frac{1}{2}(ie^1 \wedge e^5 - e^2 \wedge e^5 + e^1 \wedge e^6 + ie^2 \wedge e^6) \tag{5.251}$$

$$h^{1\,\mathrm{i}} \wedge h^{4\,\mathrm{i}} = \frac{1}{2}(ie^1 \wedge e^5 + e^2 \wedge e^5 + e^1 \wedge e^6 - ie^2 \wedge e^6) \tag{5.252}$$

$$h^{1\,\mathrm{i}} \wedge h^{4\,\dot{2}} = \frac{1}{2}(ie^3 \wedge e^5 + e^4 \wedge e^5 + e^3 \wedge e^6 - ie^4 \wedge e^6) \tag{5.253}$$

$$h^{1\,\dot{2}} \wedge h^{2\,\mathrm{i}} = -ie^0 \wedge e^7 \tag{5.254}$$

$$h^{1\,\dot{2}} \wedge h^{2\,\dot{2}} = \frac{1}{2}(-ie^0 \wedge e^5 + e^0 \wedge e^6 - e^5 \wedge e^7 - ie^6 \wedge e^7) \tag{5.255}$$

$$h^{1\dot{2}} \wedge h^{3\dot{1}} = \frac{1}{2}(-e^0 \wedge e^3 - ie^0 \wedge e^4 + ie^3 \wedge e^7 - e^4 \wedge e^7) \tag{5.256}$$

$$h^{1\dot{2}} \wedge h^{3\dot{2}} = \frac{1}{2}(e^0 \wedge e^1 + ie^0 \wedge e^2 - ie^1 \wedge e^7 + e^2 \wedge e^7) \tag{5.257}$$

$$h^{1\dot{2}} \wedge h^{4\dot{1}} = \frac{1}{2}(e^0 \wedge e^1 - ie^0 \wedge e^2 - ie^1 \wedge e^7 - e^2 \wedge e^7) \tag{5.258}$$

$$h^{1\dot{2}} \wedge h^{4\dot{2}} = \frac{1}{2}(e^0 \wedge e^3 - ie^0 \wedge e^4 - ie^3 \wedge e^7 - e^4 \wedge e^7) \tag{5.259}$$

$$h^{2\dot{1}} \wedge h^{22} = \frac{1}{2}(-ie^0 \wedge e^5 + e^0 \wedge e^6 + e^5 \wedge e^7 + ie^6 \wedge e^7) \tag{5.260}$$

$$h^{21} \wedge h^{3\dot{1}} = \frac{1}{2}(-e^0 \wedge e^3 - ie^0 \wedge e^4 - ie^3 \wedge e^7 + e^4 \wedge e^7) \tag{5.261}$$

$$h^{2\dot{1}} \wedge h^{3\dot{2}} = \frac{1}{2}(e^0 \wedge e^1 + ie^0 \wedge e^2 + ie^1 \wedge e^7 - e^2 \wedge e^7) \tag{5.262}$$

$$h^{2\dot{1}} \wedge h^{4\dot{1}} = \frac{1}{2}(e^0 \wedge e^1 - ie^0 \wedge e^2 + ie^1 \wedge e^7 + e^2 \wedge e^7) \tag{5.263}$$

$$h^{2\dot{1}} \wedge h^{4\dot{2}} = \frac{1}{2}(e^0 \wedge e^3 - ie^0 \wedge e^4 + ie^3 \wedge e^7 + e^4 \wedge e^7) \tag{5.264}$$

$$h^{2\dot{2}} \wedge h^{3\dot{1}} = \frac{1}{2}(-ie^3 \wedge e^5 + e^4 \wedge e^5 + e^3 \wedge e^6 + ie^4 \wedge e^6) \tag{5.265}$$

$$h^{2\dot{2}} \wedge h^{3\dot{2}} = \frac{1}{2}(ie^1 \wedge e^5 - e^2 \wedge e^5 - e^1 \wedge e^6 - ie^2 \wedge e^6) \tag{5.266}$$

$$h^{2\dot{2}} \wedge h^{4\dot{1}} = \frac{1}{4}(ie^1 \wedge e^5 + e^2 \wedge e^5 - e^1 \wedge e^6 + ie^2 \wedge e^6) \tag{5.267}$$

$$h^{2\dot{2}} \wedge h^{4\dot{2}} = \frac{1}{2}(ie^3 \wedge e^5 + e^4 \wedge e^5 - e^3 \wedge e^6 + ie^4 \wedge e^6) \tag{5.268}$$

$$h^{3\dot{1}} \wedge h^{3\dot{2}} = \frac{1}{2}(e^1 \wedge e^3 + ie^2 \wedge e^3 + ie^1 \wedge e^4 - e^2 \wedge e^4) \tag{5.269}$$

$$h^{3\dot{1}} \wedge h^{4\dot{1}} = \frac{1}{2}(e^1 \wedge e^3 - ie^2 \wedge e^3 + ie^1 \wedge e^4 + e^2 \wedge e^4) \tag{5.270}$$

$$h^{3\dot{1}} \wedge h^{4\dot{2}} = ie^3 \wedge e^4 \tag{5.271}$$

$$h^{3\dot{2}} \wedge h^{4\dot{1}} = -ie^1 \wedge e^2 \tag{5.272}$$

$$h^{3\dot{2}} \wedge h^{4\dot{2}} = \frac{1}{2}(e^1 \wedge e^3 - ie^1 \wedge e^4 + ie^2 \wedge e^3 + e^2 \wedge e^4) \tag{5.273}$$

$$h^{4\dot{1}} \wedge h^{4\dot{2}} = \frac{1}{2}(e^1 \wedge e^3 - ie^1 \wedge e^4 - ie^2 \wedge^3 - e^2 \wedge e^4) \tag{5.274}$$

5.9 Combinatorial argument

We need to examine the possible combinations of index values appearing in (5.242). The ϵ symbols require that all the i's assume different values, and the same goes for all the j's, k's and l's. Thus there will only be contributions from the following terms $R_{1122}R_{2211}R_{3344}R_{4433}$, $R_{1122}R_{3344}R_{2431}R_{4213}$ and $R_{1234}R_{2341}R_{3412}R_{4123}$. Our task is to determine how these three terms contribute to the Gauss-Bonnet integral.

Let the index set (1122) be denoted by A, and let (1234) and (3344) be denoted by B and C respectively.

5.9.1 Case 1, $AACC$

If we fix the first index set to be $A = (1122)$, and if we then fix the second index set to be (2211) then there are 6 possibilities for the third configuration of indices, namely (3344), (4433), (3434), (3443), (4343), and (4334). This leaves the last set of indices completely determined. We can work out the sign of each such contribution using the ϵ symbols which appear in (5.242).

A	A	C	C	*Sign*
$i_1j_1k_1l_1$	$i_2j_2k_2l_2$	$i_3j_3k_3l_3$	$i_4j_4k_4l_4$	
1122	2211	3344	4433	+
		4433	3344	+
		3434	4343	+
		3443	4334	+
		4343	3434	+
		4334	3443	+

Note that there are 6 ways to arrange the numbers (1122) within the set A and there are a further six ways of arranging the index sets $AACC$. Thus the total number of contributions from the $AACC$ term is $6 \times 6 \times 6 = 216$.

5.9.2 Case 2, $ACBB$

Again we fix the first index set to be (1122) $\in A$. The next index set is fixed as C . We must then work through the possible configurations of these index sets.

A	C	B	B	*Sign*
$i_1j_1k_1l_1$	$i_2j_2k_2l_2$	$i_3j_3k_3l_3$	$i_4j_4k_4l_4$	
1122	3344	2431	4213	+
		2413	4231	+
		4213	2431	+
		4231	2413	+
	4433	2314	3241	+
		2341	3214	+
		3214	2341	+
		3241	2314	+
	3434	2341	4213	+
		4213	2341	+
	3443	2314	4213	+
		4213	2314	+
	4343	2431	3214	+
		3214	2431	+
	4334	2413	3241	+
		32241	2413	+

Above we have 16 index sets of allowed index values. But we fixed the first index set to be precisely (1122). However, there are 6 ways of arranging this configuration. There are a further 12 choices for the arrangement of the letters $ACBB$. Thus altogether there are $16 \times 12 \times 6 = 1152$ allowed terms whose indices are $ACBB$ or some permutation thereof.

5.9.3 Case 3, $BBBB$

We fix the first index set to be $(1234) \in B$ and proceed to work out the other allowed values of the indices,

B	B	B	B	*Sign*
$i_1j_1k_1l_1$	$i_2j_2k_2l_2$	$i_3j_3k_3l_3$	$i_4j_4k_4l_4$	
1234	2341	3412	4123	+
		4123	3412	+
	2413	3142	4321	+
		4321	3142	+
	2143	3412	4321	+
		3421	4312	+
		4312	3421	+
		4321	3412	+
	3412	2143	4321	+
		2341	4123	+
		4123	2341	+
		4321	2143	+
	3142	2413	4321	+
		4321	2413	+
	3421	2143	4312	+
		4312	2143	+
	4123	2341	3412	+
		3412	2341	+
	4321	2143	3412	+
		2413	3142	+
		3142	2413	+
		3412	2143	+
	4312	2143	3421	+
		3421	2143	+

There are 24 terms represented in this table. Furthermore, there are a possible 4! similar tables which may be drawn. This is because there are 4! ways of arranging our initial indices, i.e. there are 4! ways of arranging (1234). Thus the total contribution from terms of this type is $24 \times 4! = 576$.

Chapter 6

Integration over the mother space and the D-instanton partition function

In this chapter we shall discuss how we might calculate the Gauss-Bonnet integral of the single instanton moduli space $\mathcal{M}$ by integrating over the mother space $\tilde{\mathcal{M}}$ and imposing the ADHM constraints as delta functions on $\tilde{\mathcal{M}}$.

To restrict the domain of integration one cannot simply introduce these delta functions. They must be accompanied by a suitable Jacobian. In section 6.1 we explore the general form of this Jacobian J for a set of unspecified constraints. We demonstrate that by solving the constraints we may introduce a coordinate system in which the argument of the delta function corresponds to a subset of the coordinates themselves. Associated with this coordinate transformation will be another Jacobian, J'. The integrations over the delta functions are then trivial and one is left with an integration over the remaining coordinates, which correspond to the coordinates on the reduced space $\mathcal{M}$. In section 6.3 we follow this procedure explicitly for the $SU(3)$ 1-instanton moduli space. We solve the ADHM constraints thereby introducing a coordinate system that trivializes the ADHM delta functions. We also determine the appropriate Jacobian factors and thereby identify the volume form on the moduli space $\mathcal{M}$. We adopt the same procedure with the fermionic ADHM constraints. As detailed in chapter four, the fermionic collective coordinates of an instanton solution correspond to Grassmann-valued symplectic tangent vectors to the moduli space. Thus solving the fermionic ADHM constraints corresponds to identifying those symplectic tangent vectors in $\tilde{\mathcal{M}}$ that are also symplectic tangent vectors to $\mathcal{M}$. We find the coordinate change that trivializes the fermionic ADHM delta function constraints. Having identified such a symplectic basis we are then able to use equation (H.32) to obtain the components of the symplectic curvature on $\mathcal{M}$. This demonstrates how the fermionic ADHM constraints restrict the symplectic curvature of the mother space to that of the moduli space and allows us to confirm our previous expressions for these quantities.

In section 6.6 we introduce the D-instanton partition function. This is an integral over the mother space upon which Lagrange multipliers impose the ADHM constraints as delta functions. We can then assemble all the results of the previous sections to evaluate this integral. We show that in the 1-instanton sector the D-instanton partition function reproduces the conventional Gauss-Bonnet integral on the instanton moduli space. The remaining section is then devoted to developing the result of [17]. We perform the integrations in the D-instanton partition function in a different order and obtain a numerical result providing a general formula for the Gauss-Bonnet integral of a class of manifolds. We use this expression to calculate the Gauss-Bonnet integral for the $SU(3)$ case and compare this value with that obtained in chapter five.

6.1 Restricting the domain of integration to a subsurface of a manifold defined by constraints.

Let us start with a general situation in which one is given a manifold M (called the mother space) with $Dim(M) = m$, and coordinates ω^i. Consider n non-degenerate constraint equations, which are schematically of the form

$$f^c(\omega^i) = 0 \quad c = 0, 1, \cdots, n. \tag{6.1}$$

The imposition of these constraints will restrict us to a subsurface of M, called N, which will be of dimension $Dim(N) = m - n$. Analogous to the procedure for finding the metric, we first find the volume form on the mother space and then impose the constraints that will limit us to the embedded subspace. Now consider the embedded sub-surfaces of M defined by the n constraint equations $f^c = 0$, $c = 1, ..., n$. Since the f^c's have the constant value zero over the surface, variations in the value of f^c must take one out of these surfaces and thus be in a direction normal to them. Thus for each c, $\frac{\partial f^c}{\partial \omega^i}|_N$ must be a vector normal to N. Generically, given constraint equations, $f^c = 0$ then the volume form on the reduced space, Ω, will be related to that on the mother space, $\tilde{\Omega}$, in the following way:

$$\int_N \Omega = \int_M \tilde{\Omega} \prod_{c=1}^{n} \delta(f^c) J = \int d^m\omega \prod_{c=1}^{n} \delta(f^c) J \tag{6.2}$$

The delta function imposes the n constraints $f^c = 0$ that limit us to the space N, and J is a Jacobian factor, given by

$$J = \sqrt{det\left(\frac{\partial f^c}{\partial \omega^i}\frac{\partial f^d}{\partial \omega^i}\right)} \tag{6.3}$$

We can confirm that this is the correct choice for the Jacobian by investigating the consequences of a change of variables. Let f'^c be defined thus:

$$f'^c = M^{cd}.f^d \tag{6.4}$$

$$\Rightarrow \prod_{c=1}^{n} \delta(f^c) \mapsto \prod_{c=1}^{n} \delta(f'^c) = det(M^{-1}) \prod_{c=1}^{n} \delta(f^c) \tag{6.5}$$

And the Jacobian transforms as;

$$J \mapsto J' = \sqrt{det\left[\frac{\partial f'}{\partial \omega}\left(\frac{\partial f'}{\partial \omega}\right)^T\right]} = \sqrt{det\left[M\left(\frac{\partial f}{\partial \omega}.\left(\frac{\partial f}{\partial \omega}\right)^T\right)M^T\right]} = det(M).J \tag{6.6}$$

Thus the integral is invariant under the above reparameterisation, so the Jacobian serves its required function. Having restricted the domain of integration to the subspace, we will next like to make a change of variables that will simplify the integral by allowing us to integrate out the delta function constraints. The key idea is to change to a set of coordinates in which the f^c parameterize the directions in the mother space perpendicular to the embedded subspace, i.e. we aim to realize the the f^c as coordinates on the manifold M. The remaining coordinates will then naturally parameterize the sub-manifold N. We shall denote these coordinates by $\theta^\mu, \mu = 1, ..., m-n$. In short we wish to make the following coordinate change: $\{\omega\} \to \{f, \theta\}$. We therefore envisage that the θ^μ's be a set of parameters that satisfy the constraint equation. i.e.:

$$f^c(\omega) = 0, \quad \omega = \omega(f^c, \theta^\mu) \Rightarrow f^c(\omega(f^c, \theta^\mu))f^c(\theta) = 0 \tag{6.7}$$

There will be another Jacobian,J', associated to this change of variables, giving the volume element as:

$$d^m\omega = J'(f, \theta)d^n f.d^{m-n}\theta \tag{6.8}$$

And schematically, the overall integration becomes

$$\int_N d^n\omega = \int J'(f, \theta)d^n f.d^{m-n}\theta \prod_{c=0}^{n} \delta(f^c).J(\omega(f, \theta)) \tag{6.9}$$

$$= \int J'(0,\theta) d^{m-n}\theta J(\omega(0,\theta)) \tag{6.10}$$

Notice that the delta function constraints have allowed us to eliminate n of the integrations, leaving an integration over the coordinates of N and an appropriate Jacobian factor. We now wish to find the Jacobian J. Writing this out in full gives:

$$J^2 = \begin{vmatrix} \frac{\partial f^1}{\partial\omega^i}\frac{\partial f^1}{\partial\omega^i} & \frac{\partial f^1}{\partial\omega^i}\frac{\partial f^2}{\partial\omega^i} & \dots & \frac{\partial f^1}{\partial\omega^i}\frac{\partial f^n}{\partial\omega^i} \\ \frac{\partial f^2}{\partial\omega^i}\frac{\partial f^1}{\partial\omega^i} & \frac{\partial f^2}{\partial\omega^i}\frac{\partial f^2}{\partial\omega^i} & \dots & \frac{\partial f^2}{\partial\omega^i}\frac{\partial f^n}{\partial\omega^i} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f^n}{\partial\omega^i}\frac{\partial f^1}{\partial\omega^i} & \frac{\partial f^n}{\partial\omega^i}\frac{\partial f^2}{\partial\omega^i} & \dots & \frac{\partial f^n}{\partial\omega^i}\frac{\partial f^n}{\partial\omega^i} \end{vmatrix} \tag{6.11}$$

Note that this is just the determinant of the matrix of the inner products of vectors that are orthogonal to the constrained sub-surface.

6.2 Quotienting a Space by a Group Action.

Consider the case where a group G acts on a space $\tilde{\mathcal{M}}$. We wish to determine the volume form, Ω, on the quotient space $\tilde{\mathcal{M}}/G$. Somewhat schematically this may simply be written as

$$\int_{\mathcal{M}} \Omega = \int_{\tilde{\mathcal{M}}} \tilde{\Omega} \frac{1}{Vol_G(x)} \tag{6.12}$$

Where $Vol_G(x)$ is the volume of the G-orbit through the point $x \in \tilde{\mathcal{M}}$

6.3 Solution of the ADHM constraints

We have already found the form of ω which satisfies the case $f^c = 0$. (We did this when calculating the metric). For the case $f^c \neq 0$ we make the following ansatz:

$$\omega = U. \begin{pmatrix} \rho_+ & \alpha \\ 0 & \rho_- \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{6.13}$$

$$\bar{\omega} = \begin{pmatrix} \rho_+ & 0 & \cdots & 0 \\ \alpha^* & \rho_- & \cdots & 0 \end{pmatrix} .U^\dagger \tag{6.14}$$

Where $\rho_-, \rho_+ \in R; \alpha \in C$. Using this ansatz we get;

$$\bar{\omega}.\omega = \begin{pmatrix} \rho_+^2 & \alpha\rho_+ \\ \alpha^*\rho_+ & \alpha\alpha^* + \rho_-^2 \end{pmatrix} \tag{6.15}$$

Setting $c = 1, 2, 3$ and defining $f'^c = f^c + \zeta^c$ gives,

$$f'^1 = Tr. \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_+^2 & \alpha\rho_+ \\ \alpha^*\rho_+ & \alpha\alpha^* + \rho_-^2 \end{pmatrix} = \rho_+(\alpha + \alpha^*) \tag{6.16}$$

$$f'^2 = Tr. \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \rho_+^2 & \alpha\rho_+ \\ \alpha^*\rho_+ & \alpha\alpha^* + \rho_-^2 \end{pmatrix} = i\rho_+(\alpha - \alpha^*) \tag{6.17}$$

$$f'^3 = Tr. \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_+^2 & \alpha\rho_+ \\ \alpha^*\rho_+ & \alpha\alpha^* + \rho_-^2 \end{pmatrix} = \rho_+^2 - \rho_-^2 - \alpha\alpha^* \tag{6.18}$$

To clarify the situation we decompose α into its real and imaginary parts. Let $\alpha = a + ib$, $\alpha^* = a - ib$. This gives;

$$f'^1 = 2\rho_+ a \quad \Rightarrow \quad a = \frac{f'^1}{2\rho_+} \tag{6.19}$$

$$f'^2 = -2\rho_+ b \quad \Rightarrow \quad b = \frac{-f'^2}{2\rho_+} \tag{6.20}$$

$$f'^3 = \rho_+^2 - \rho_-^2 - a^2 - b^2 \tag{6.21}$$

To simplify the structure of these equations a little further we shall introduce the complex quantity F where $F = f'^1 - if'^2$ and so;

$$\alpha = \frac{F}{2\rho_+} = \frac{1}{2\rho_+}(f'^1 - if'^2) \tag{6.22}$$

The equation for f^3 becomes;

$$f'^3 = \rho_+^2 - \rho_-^2 - \frac{F.F^*}{4\rho_+^2} \tag{6.23}$$

This is an equation for ρ_+ and ρ_- which is to be solved in terms of f'^1, f'^2 and f'^3. To find a solution to this equation, let us set

$$\rho_-^2 = \rho^2 - \frac{f^3}{2} - \frac{F.F^*}{4\rho_+^2} \tag{6.24}$$

Substituting this into the equation for f'^3 gives,

$$f'^3 = \rho_+^2 + \frac{F.F^*}{4\rho_+^2} - \rho^2 + \frac{f'^3}{2} - \frac{F.F^*}{4\rho_+^2} = \rho_+^2 - \rho^2 + \frac{f'^3}{2} \tag{6.25}$$

$$\Rightarrow \quad \rho_+^2 = \rho^2 + \frac{f'^3}{2} \tag{6.26}$$

$$\Rightarrow \quad \alpha = \frac{F}{2\sqrt{\rho^2 + \frac{f'^3}{2}}} \quad , \quad \alpha^* = \frac{F^*}{2\sqrt{\rho^2 + \frac{f'^3}{2}}} \tag{6.27}$$

Thus ω is now;

$$\omega = U. \begin{pmatrix} \rho_+ & \frac{F}{2\rho_+} \\ 0 & \rho_- \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{6.28}$$

Our aim is to find the volume element on the quotient space. To do this we shall need to calculate $d\omega$ and wedge together all its constituents like so;

$$d\omega_{1\,\mathrm{i}} \wedge d\bar{\omega}_1{}^{\mathrm{i}} \wedge d\omega_{1\,\dot{2}} \wedge d\bar{\omega}_1{}^{\dot{2}} \wedge \cdots \wedge d\omega_{N\,\dot{2}} \wedge d\bar{\omega}_N{}^{\dot{2}} \tag{6.29}$$

Our strategy for making this somewhat easier is to change to a coordinate system that explicitly involves the f'^c's. The volume element will then include the factor $df'^1 \wedge df'^2 \wedge df'^3$ as well as wedge products of other one-forms which correspond to the $d\theta$'s mentioned above. The function which sits outside this wedge product will be the Jacobian $J'(f,\theta)$ of the transformation from $\{\omega\}$ to $\{f,\theta\}$. We take derivatives of the coordinates on the manifold to yield the basis one-forms;

$$d\omega = dU. \begin{pmatrix} \rho_+ & \frac{F}{2\rho_+} \\ 0 & \rho_- \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} + U. \begin{pmatrix} d\rho_+ & d\alpha \\ 0 & d\rho_- \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \tag{6.30}$$

We now calculate the exterior derivatives appearing above;

$$d\rho_+ = \frac{\rho}{\rho_+} d\rho + \frac{1}{4\rho_+} df'^3 \tag{6.31}$$

$$d\alpha = \frac{1}{2\rho_+} dF - \frac{F}{2\rho_+^2} d\rho_+ \tag{6.32}$$

$$= \frac{1}{2\rho_+} dF - \frac{F\rho}{2\rho_+^3} d\rho - \frac{F}{8\rho_+^3} df'^3 \tag{6.33}$$

$$d\rho_- = \frac{1}{\rho_-}\left[\left(1 + \frac{F.F^*}{4\rho_+^4}\right)\rho d\rho + \frac{1}{4}\left(\frac{F.F^*}{4\rho_+^4} - 1\right) df'^3 - \frac{F}{8\rho_+^2} dF^* - \frac{F^*}{8\rho_+^2} dF\right] \tag{6.34}$$

We must find the various component one-forms contained in the dU term in (6.30). We do this by writing dU in terms of the left invariant one forms arranged in the matrix $\mathcal{L}$, as discussed earlier.

$$dU = U.U^{\dagger}.dU = iU.\mathcal{L} = iU.\begin{pmatrix} \frac{1}{2}(Q+\lambda) & \nu & \sigma_1 & \sigma_2 & \cdots & \sigma_n \\ \nu^* & \frac{1}{2}(Q-\lambda) & \Sigma_1 & \Sigma_2 & \cdots & \Sigma_n \\ \sigma_1^* & \Sigma_1^* & \cdots & \cdots & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_n^* & \Sigma_n^* & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{6.35}$$

Thus we have

$$dU.\omega = U.\begin{pmatrix} i\frac{\rho_+}{2}(Q+\lambda) & i\frac{F}{4\rho_+}(Q+\lambda) + i\rho_-\nu \\ i\rho_+\nu^* & i\frac{iF}{2\rho_+}\nu^* + i\frac{\rho_-}{2}(Q-\lambda) \\ i\rho_+\sigma_1^* & i\frac{F}{2\rho_+}\sigma_1^* + i\rho_-\Sigma_1^* \\ \vdots & \vdots \\ i\rho_+\sigma_n^* & i\frac{F}{2\rho_+}\sigma_n^* + i\rho_-\Sigma_n^* \end{pmatrix} \tag{6.36}$$

We are now in a position to write out all the components of $d\omega$ in terms of our new coordinate system. The results are as follows;

$$d\omega_{1\,\dot{1}} = i\frac{\rho_+}{2}(Q+\lambda) + \frac{\rho}{\rho_+}d\rho + \frac{1}{4\rho_+}df^3 \tag{6.37}$$

$$d\omega_{1\,\dot{2}} = \frac{1}{2\rho_+}dF - \frac{F\rho}{2\rho_+^3}d\rho - \frac{F}{8\rho_+^3}df^3 + i\frac{F}{4\rho_+}(Q+\lambda) + i\rho_-\nu \tag{6.38}$$

$$d\omega_{2\,\dot{1}} = i\rho_+\nu \tag{6.39}$$

$$d\omega_{2\,\dot{2}} = i\frac{F}{2\rho_+}\nu^* + i\frac{\rho_-}{2}(Q-\lambda) + \frac{\rho}{\rho_-}\left(1 + \frac{F.F^*}{4\rho_+^4}\right)d\rho \tag{6.40}$$

$$+\frac{1}{4\rho_-}\left(\frac{F.F^*}{4\rho_+^4} - 1\right)df^3 - \frac{F}{8\rho_-\rho_+^2}dF^* - \frac{F^*}{8\rho_-\rho_+^2}dF \tag{6.41}$$

$$d\omega_{3\,\dot{1}} = i\rho_+\sigma^* \tag{6.42}$$

$$d\omega_{3\,\dot{2}} = i\frac{F}{2\rho_+}\sigma^* + i\rho_-\Sigma^* \tag{6.43}$$

Where we have used the fact that $df'^c = df^c$.

6.4 The volume form on the $SU(N)$ one instanton moduli space

In the case of a single instanton the mother space of the ADHM construction is R^{4N}. This space is trivially hyper-Kahler. The moduli space is obtained by applying the ADHM constraint equations and then performing a quotient by the group $U(1)$. In terms of an integration over the mother space this may be achieved by introducing the ADHM constraints as delta functions with an appropriate Jacobian, as discussed above. The Jacobian we need to introduce is the determinant of inner products of those vectors in $\tilde{\mathcal{M}}$ which are orthogonal to the surfaces defined by f^c = constant. Since we know the constraint equations we could directly calculate the determinant in (6.11). However, in the case of a single instanton the level set possesses a $U(1)$ isometry, (so the $U(1)$ Killing vector X must be tangent to the level set), so we have

$$\tilde{g}(\tilde{I}^c X, X) = \tilde{\omega}^c(X, X) = 0 \tag{6.44}$$

Where $\tilde{g}$ is the metric and $\tilde{\omega}$ the Kahler form on the manifold $\tilde{\mathcal{M}}$. Consequently a basis of vectors normal to the level set is provided by the vectors $I^c X$ which number $3dim(G) = 3$. Thus, using the Hermiticity of the metric, their inner products are

$$\tilde{g}(\tilde{I}^c X, \tilde{I}^d X) = \delta^{cd}\tilde{g}(X, X) - \epsilon^{cde}\tilde{g}(X, \tilde{I}^e X) = \delta^{cd}\tilde{g}(X, X) \tag{6.45}$$

This quantity is often denoted in the literature by L and we calculate it below for the general k instanton case and then set $k = 1$. Firstly we shall require the $U(k)$ Killing vector fields in the notation established in chapter three (see equation (3.94)),

$$X_r = iT^r_{\tilde{i}\tilde{j}} z^{\tilde{j}\dot{\alpha}} \frac{\partial}{\partial z^{\tilde{i}\dot{\alpha}}}$$

$$X_s = iT^s_{\tilde{k}\tilde{l}} z^{\tilde{l}\dot{\beta}} \frac{\partial}{\partial z^{\tilde{k}\dot{\beta}}}$$

$$\tilde{g} = \tilde{\Omega}_{\tilde{m}\tilde{n}}\epsilon_{\dot{\gamma}\dot{\delta}}dz^{\tilde{m}\dot{\gamma}}dz^{\tilde{n}\dot{\delta}}$$

Therefore we have

$$\begin{aligned}\tilde{g}(X^r, X^s) &= i^2\tilde{\Omega}_{\tilde{m}\tilde{n}}\epsilon_{\dot{\gamma}\dot{\delta}}T^r_{\tilde{i}\tilde{j}}z^{\tilde{j}\dot{\alpha}}T^s_{\tilde{k}\tilde{l}}z^{\tilde{l}\dot{\beta}}\left\langle dz^{\tilde{m}\dot{\gamma}}, \frac{\partial}{\partial z^{\tilde{i}\dot{\alpha}}}\right\rangle\left\langle dz^{\tilde{n}\dot{\delta}}, \frac{\partial}{\partial z^{\tilde{k}\dot{\beta}}}\right\rangle \\ &= -\tilde{\Omega}_{\tilde{m}\tilde{n}}\epsilon_{\dot{\gamma}\dot{\delta}}T^r_{\tilde{i}\tilde{j}}z^{\tilde{j}\dot{\alpha}}T^s_{\tilde{k}\tilde{l}}z^{\tilde{l}\dot{\beta}}\delta^m{}_i\delta^{\dot{\gamma}}{}_{\dot{\alpha}}\delta^{\tilde{n}}{}_{\tilde{k}}\delta^{\dot{\delta}}{}_{\dot{\beta}} \\ &= -\tilde{\Omega}_{\tilde{i}\tilde{k}}\epsilon_{\dot{\alpha}\dot{\beta}}T^r_{\tilde{i}\tilde{j}}z^{\tilde{j}\dot{\alpha}}T^s_{\tilde{k}\tilde{l}}z^{\tilde{l}\dot{\beta}}\end{aligned}$$

Setting $k = 1$ we have

$$\begin{aligned}g(X,X) &= -\epsilon_{\dot{\alpha}\dot{\beta}}z^{\dot{\alpha}}T^T\tilde{\Omega}Tz^{\dot{\beta}} \\ &= z^{\dot{2}}\begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix}z^{\dot{1}} - z^{\dot{1}}\begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix}z^{\dot{2}}\end{aligned}$$

Recall that for $k = 1$ the coordinates z are given by, (3.66),

$$z^{\dot{\alpha}} = \begin{pmatrix} \bar{\omega}^{\dot{\alpha}}{}_u \\ \epsilon^{\dot{\alpha}\dot{\beta}}\omega_{u\dot{\beta}} \end{pmatrix} \Rightarrow z^{\dot{1}} = \begin{pmatrix} \bar{\omega}^{\dot{1}}{}_u \\ \omega_{u\dot{2}} \end{pmatrix}, \, z^{\dot{2}} = \begin{pmatrix} \bar{\omega}^{\dot{2}}{}_u \\ -\omega_{u\dot{1}} \end{pmatrix}$$

Substituting this into the above gives (see equations (6.15),(6.24),(6.26) and (6.27))

$$\begin{aligned}g(X,X) &= \begin{pmatrix} \bar{\omega}^{\dot{2}}{}_u \\ -\omega_{u\dot{1}} \end{pmatrix}\begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix}\begin{pmatrix} \bar{\omega}^{\dot{1}}{}_{ui} \\ \omega_{u\dot{2}} \end{pmatrix} - \begin{pmatrix} \bar{\omega}^{\dot{1}}{}_u \\ \omega_{u\dot{2}} \end{pmatrix}\begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix}\begin{pmatrix} \bar{\omega}^{\dot{2}}{}_u \\ -\omega_{u\dot{1}} \end{pmatrix} \\ &= -2\bar{\omega}^{\dot{\alpha}}{}_u\omega_{u\dot{\alpha}} = -4\rho^2\end{aligned}$$

The Jacobian, being the square root of the determinant of inner products of basis vectors normal to the level set therefore becomes

$$J = |\sqrt{det(-4\rho^2 1_3)}| = (2\rho)^3 = 8\rho^3$$

In the case at hand, which we will treat explicitly, there are three A.D.H.M. constraints and one gauge fixing term. We shall attempt to define the coordinates f^c

which parameterize those directions that are orthogonal to the (embedded) $\mathcal{M}$, so that a point which satisfies the ADHM constraints has $f^c = 0$. Thus we define the f's as

$$f^c = \tau^{c\,\dot{\alpha}}{}_{\dot{\beta}} \bar{\omega}^{\dot{\beta}}{}_{u\cdot} \omega_{u\dot{\alpha}} - \zeta^c \tag{6.46}$$

To complete the ADHM construction we must also impose the further condition of gauge-fixing, which for us amounts to performing the $U(1)$ quotient. Although this condition may be written in differential form as

$$df^0 = d\bar{\omega}^{\dot{\alpha}}_u \omega_{u\dot{\alpha}} - \bar{\omega}^{\dot{\alpha}}_u d\omega_{u\dot{\alpha}} = 0 \tag{6.47}$$

we were not able to integrate this equation, so could not find a coordinate that could be used to parameterize this direction. In this respect we note that the action of G on $\mathcal{M}$ is free, so the $U(1)$ orbit through each point should be proportional to the volume of the $U(1)$ group space. The scale factor is $|det(L)|^{\frac{1}{2}}$, (see [12]), giving the volume of the orbit through a point on the level set is

$$Vol_G(x) = |det(L)|^{\frac{1}{2}} Vol_G$$

Which in our case gives the factor $2\rho Vol_G$. Thus, at least schematically, the overall result for the volume form is

$$\int_{\mathcal{M}} \Omega = \frac{4}{Vol_G} \int_{\tilde{\mathcal{M}}} \rho^2 \tilde{\Omega} \prod_{c=1}^{3} \delta(f'^c) \tag{6.48}$$

Where ω is the volume form on $\mathcal{M}$ and $\tilde{\omega}$ is that on $\tilde{\mathcal{M}}$. In our case this is just $U(1)$, so $Vol_G = Vol_{U(1)} = 2\pi$

$$\int_{\mathcal{M}} \Omega = \frac{2}{\pi} \int_{\tilde{\mathcal{M}}} \rho^2 \tilde{\Omega} \prod_{c=1}^{3} \delta(f'^c) \tag{6.49}$$

It is now straightforward to calculate the volume form on $\mathcal{M}$. We first calculate the wedge product of all the forms given in equations (6.37) to (6.43). As a preliminary we note that the one forms σ and Σ only occur in $d\omega_{2\dot{1}}$ and $d\omega_{2\dot{2}}$. Thus the only product involving σ and Σ is;

$$d\omega_{2\dot{1}} \wedge d\bar{\omega}_2{}^{\dot{1}} \wedge d\omega_{2\dot{2}} \wedge d\bar{\omega}_2{}^{\dot{2}} = \rho_+^2\rho_-^2\sigma \wedge \sigma^* \wedge \Sigma \wedge \Sigma^* \tag{6.50}$$

Wedging all these one-forms together gives

$$d\omega_{1\,\dot{1}} \wedge d\bar{\omega}_1^{\dot{1}} \wedge d\omega_{1\,\dot{2}} \wedge d\bar{\omega}_1^{\dot{2}} \wedge \cdots \wedge d\omega_{N\,\dot{2}} \wedge d\bar{\omega}_N^{\dot{2}} \tag{6.51}$$

$$= \frac{1}{4}(\rho_+\rho_-)^2\rho Q \wedge d\rho \wedge \sigma \wedge \sigma^* \wedge \Sigma \wedge \Sigma^* \wedge \nu \wedge \nu^* \wedge \lambda \wedge dF \wedge dF^* \wedge df^3$$

$$= 4\rho(\rho_+\rho_-)^2 Q \wedge d\rho \wedge \sigma_1 \wedge \sigma_2 \wedge \Sigma_1 \wedge \Sigma_2 \wedge \nu_1 \wedge \nu_2 \wedge \lambda \wedge df^1 \wedge df^2 \wedge df^3 \tag{6.52}$$

where we have used $\sigma \wedge \sigma* = -2i\sigma_1 \wedge \sigma_2$ etc. Now recall that from equation (5.88) we have,

$$Q = -2d\phi - \frac{\zeta}{2\rho^2}\lambda \tag{6.53}$$

Substituting this into the wedge product gives

$$= 8\rho(\rho_+\rho_-)^2 d\phi \wedge d\rho \wedge \sigma_1 \wedge \sigma_2 \wedge \Sigma_1 \wedge \Sigma_2 \wedge \nu_1 \wedge \nu_2 \wedge \lambda \wedge df^1 \wedge df^2 \wedge df^3 \tag{6.54}$$

Here we can read off the Jacobian factor, $J'(f,\theta)$ of (6.8).

$$J' = 8(\rho_+\rho_-)^2\rho \tag{6.55}$$

Thus have we changed coordinates to a form which will trivialize the action of the delta function constraints. In so doing we have determined the Jacobian factor J'. Note that J' is independent of the f coordinates, despite of the action of the delta function.

Thus, including the Jacobian factor from (6.49),replacing the f'^c's with the f^c's and integrating over the one form $d\phi$ (which yields a factor of 2π) gives the volume factor

$$= 2^5\rho^3(\rho_+\rho_-)^2\prod_{c=1}^{3}\delta(f^c-\zeta^c)d\rho\wedge\sigma_1\wedge\sigma_2\wedge\Sigma_1\wedge\Sigma_2\wedge\nu_1\wedge\nu_2\wedge\lambda\wedge df^1\wedge df^2\wedge df^3 \quad (6.56)$$

Fortunately, this agrees with that obtained from the explicit implementation of the hyper-Kahler quotient given previously, (5.111).

6.5 The Fermionic ADHM constraints

The components of the fermionic instanton solution can be arranged into a vector $\tilde{\psi} = (\bar{\mu},\, \mu)$ that can be shown to correspond to Grassmann valued symplectic tangent vectors to the mother space. For the 1-instanton case these fields are constrained to satisfy the following two equations, labeled by the index $\dot{\alpha}$,

$$\bar{\mu}_u\omega_{ui\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}iu}\mu_u = 0 \quad (6.57)$$

where $\bar{\omega}_{\dot{\alpha}iu} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\omega}^{\dot{\beta}}_{iu}$ and the ω's are already known from equation (6.28). The above equation is actually the requirement that the vector $\tilde{\psi} = (\bar{\mu},\, \mu)$ be a symplectic tangent vector to $\mathcal{M}_k$, as shown in chapter five. Now consider an integral over the symplectic tangent vectors to the mother space. If we wish to restrict this integration to tangent vectors to the reduced space, we shall need to introduce the two delta function constraints with the appropriate Jacobian factors ,

$$\int_{T_pM} d\psi = \int_{T_p\tilde{M}} d\tilde{\psi} J \prod_{\alpha}\delta(f^{\alpha}) \quad (6.58)$$

One should recall that for anti-commuting numbers the Jacobian is the inverse of what one would ordinarily expect.

Following the strategy of the previous section we shall attempt to change to a coordinate system which trivializes these constraints. The first stage is to again write the constraint equations as $f_{\dot\alpha}=0$ where

$$f_{\dot\alpha}=\bar\mu_u\omega_{ui\dot\alpha}+\bar\omega_{\dot\alpha iu}\mu_u \tag{6.59}$$

Taking the complex conjugate of this gives

$$\overline{(f_{\dot\alpha})}=\bar\omega^{\dot\alpha}_{iu}\mu_u+\bar\mu_u\omega^{\dot\alpha}_{ui}=f^{\dot\alpha}=\epsilon^{\dot\alpha\dot\beta}f_{\dot\beta} \tag{6.60}$$

$$\Rightarrow\overline{(f_{\dot 1})}=f^{\dot 1}=f_{\dot 2}\ ,\ \ \overline{(f_{\dot 2})}=f^{\dot 2}=-f_{\dot 1} \tag{6.61}$$

We set μ to be

$$\mu=\begin{pmatrix}\mu_1\\ \mu_2\\ \mu_n\end{pmatrix}\Rightarrow\bar\mu=\begin{pmatrix}\bar\mu_1 & \bar\mu_2 & \bar\mu_n\end{pmatrix} \tag{6.62}$$

Substituting this into (6.57) gives an explicit expression for the f's

$$f_{\dot 1}=\rho_+\bar\mu_1-\rho_-\mu_2\ ,\ \ f_{\dot 2}=\bar\mu_2\rho_-+\rho_+\mu_1 \tag{6.63}$$

The Jacobian J required to implement the coordinate change is calculated as previously prescribed,

$$J^{-2}=\begin{vmatrix}\frac{\partial f^1}{\partial\mu^a}\frac{\partial f^1}{\partial\mu^a}+\frac{\partial f^1}{\partial\bar\mu^a}\frac{\partial f^1}{\partial\bar\mu^a} & \frac{\partial f^1}{\partial\mu^a}\frac{\partial f^2}{\partial\mu^a}+\frac{\partial f^1}{\partial\bar\mu^a}\frac{\partial f^2}{\partial\bar\mu^a}\\ \frac{\partial f^2}{\partial\mu^a}\frac{\partial f^1}{\partial\mu^a}+\frac{\partial f^2}{\partial\bar\mu^a}\frac{\partial f^1}{\partial\bar\mu^a} & \frac{\partial f^1}{\partial\mu^a}\frac{\partial f^2}{\partial\mu^a}+\frac{\partial f^2}{\partial\bar\mu^a}\frac{\partial f^2}{\partial\bar\mu^a}\end{vmatrix} \tag{6.64}$$

$$=\begin{vmatrix}\rho_+^2+\rho_-^2 & 0\\ 0 & \rho_+^2+\rho_-^2\end{vmatrix}=4\rho^4 \tag{6.65}$$

Implementing the constraints $f_{\dot\alpha}=0$ gives the results

$$\bar\mu_1=\mu_2\frac{\rho_-}{\rho_+},\quad \bar\mu_2=-\mu_1\frac{\rho_+}{\rho_-} \tag{6.66}$$

As per the prescription in chapter four, we can assemble the fermionic fields that satisfy $f^{\dot\alpha}=0$ into a symplectic tangent vector on the reduced space, which we denote by ψ,

$$\psi = U\begin{pmatrix} \bar{\mu} \\ \mu \end{pmatrix} = U\begin{pmatrix} \mu_2\frac{\rho_-}{\rho_+} \\ -\mu_1\frac{\rho_+}{\rho_-} \\ \bar{\mu}_n \\ \mu_1 \\ \mu_2 \\ \mu_n \end{pmatrix} = U\begin{pmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \\ \bar{\mu}_n \\ -\bar{\mu}_2\frac{\rho_-}{\rho_+} \\ \bar{\mu}_1\frac{\rho_+}{\rho_-} \\ \mu_n \end{pmatrix} \tag{6.67}$$

This is a symplectic tangent vector to the moduli space, but is explicitly written in terms of the tangent basis to the mother space in which $\mathcal{M}$ is embedded. The vector ψ above leads to the following choice for the 4-dimensional symplectic tangent vectors to $\tilde{\mathcal{M}}$.

$$\mathbf{e}'^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ B \\ 0 \end{pmatrix}, \ \mathbf{e}'^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{B} \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}'^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}'^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.68}$$

Where for brevity we have set

$$B = \frac{\rho_+}{\rho_-} \tag{6.69}$$

In terms of the basis vectors on the mother space, $\mathbf{e}^i$ we can write the above as

$$\mathbf{e}'^1 = \mathbf{e}^1 + B\mathbf{e}^5 \tag{6.70}$$

$$\mathbf{e}'^2 = \mathbf{e}^2 - \frac{1}{B}\mathbf{e}^4 \tag{6.71}$$

$$\mathbf{e}'^3 = \mathbf{e}^3 \tag{6.72}$$

$$\mathbf{e}'^6 = \mathbf{e}^6 \tag{6.73}$$

However, these are not unit vectors, so we must divide these by their magnitude to yield an orthonormal basis on $\mathcal{M}$. We have the following unit vectors,

$$\hat{\mathbf{e}}'^1 = \frac{\rho_-}{\sqrt{2}\rho}\left(\mathbf{e}^1 + B\mathbf{e}^5\right) = \frac{1}{\sqrt{2}\rho}\left(\rho_-\mathbf{e}^1 + \rho_+\mathbf{e}^5\right) \tag{6.74}$$

$$\hat{\mathbf{e}}'^2 = \frac{\rho_+}{\sqrt{2}\rho}\left(\mathbf{e}^2 - \frac{1}{B}\mathbf{e}^4\right) = \frac{1}{\sqrt{2}\rho}\left(\rho_+\mathbf{e}^2 - \rho_-\mathbf{e}^4\right) \tag{6.75}$$

$$\hat{\mathbf{e}}'^3 = \mathbf{e}^3 \tag{6.76}$$

$$\hat{\mathbf{e}}'^6 = \mathbf{e}^6 \tag{6.77}$$

It is straightforward to determine the orthonormal complement to these vectors in $\mathcal{M}$,

$$\hat{\mathbf{e}}'^4 = -\frac{1}{\sqrt{B^2+1}}\left(\mathbf{e}^2 + B\mathbf{e}^4\right) = -\frac{1}{\rho\sqrt{2}}\left(\rho_-\mathbf{e}^2 + \rho_+\mathbf{e}^4\right) \tag{6.78}$$

$$\hat{\mathbf{e}}'^5 = \frac{1}{\sqrt{B^2+1}}\left(B\mathbf{e}^1 - \mathbf{e}^5\right) = \frac{1}{\rho\sqrt{2}}\left(\rho_+\mathbf{e}^1 - \rho_-\mathbf{e}^5\right) \tag{6.79}$$

Using the method of images we may write down the matrix which implements this transformation of the basis

$$M = \frac{1}{\rho\sqrt{2}}\begin{pmatrix} \rho_- & 0 & 0 & 0 & \rho_+ & 0 \\ 0 & \rho_+ & 0 & -\rho_- & 0 & 0 \\ 0 & 0 & \rho\sqrt{2} & 0 & 0 & 0 \\ 0 & -\rho_- & 0 & -\rho_+ & 0 & 0 \\ \rho_+ & 0 & 0 & 0 & -\rho_- & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho\sqrt{2} \end{pmatrix} \tag{6.80}$$

Applying this transformation to an arbitrary symplectic tangent vector of $\mathcal{M}$ gives

$$\tilde{\psi} \mapsto \tilde{\psi}' = M\tilde{\psi} = \frac{1}{\rho\sqrt{2}} \begin{pmatrix} \bar{\mu}_1\rho_- + \mu_2\rho_+ \\ \bar{\mu}_2\rho_+ - \mu_1\rho_- \\ \bar{\mu}_n\rho\sqrt{2} \\ -\mu_1\rho_+ - \bar{\mu}_2\rho_- \\ \bar{\mu}_1\rho_+ - \mu_2\rho_- \\ \mu_n\rho\sqrt{2} \end{pmatrix} = \frac{1}{\rho\sqrt{2}} \begin{pmatrix} \bar{\mu}_1\rho_- + \mu_2\rho_+ \\ \bar{\mu}_2\rho_+ - \mu_1\rho_- \\ \bar{\mu}_n\rho\sqrt{2} \\ -f_2 \\ f_1 \\ \mu_n\rho\sqrt{2} \end{pmatrix} = \begin{pmatrix} \bar{\mu}'_1 \\ \bar{\mu}'_2 \\ \bar{\mu}'_n \\ -\frac{f_2}{\rho\sqrt{2}} \\ \frac{f_1}{\rho\sqrt{2}} \\ \mu'_n \end{pmatrix} \tag{6.81}$$

Imposing the delta function constraints, $f_{\dot{\alpha}} = 0$ we see that a general vector under these constraints has no components in the 4 or 5 directions. An immediate consequence of this is that all tensor components which mention the 4 or 5 direction will also vanish (in this particular coordinate system). The Jacobian of the transformation from $(\mu\bar{\mu})$ to $(\mu'\bar{\mu}')$ is just the determinant of M^{-1}. Since M maps one orthonormal basis to another it is an orthogonal matrix, thus its determinant is unity. However, the Jacobian for the transformation from $(\bar{\mu}', \mu')$ to $(\bar{\mu}, f, \mu)$ will not be unity since we are changing to a basis to include f_1 and f_2 and these are not unit vectors. In fact we will have $J' = 2\rho^2$. (Where we have used the result $d\mu' = d\mu \left(\frac{d\mu'}{d\mu}\right)^{-1}$ which is valid for a change of variables in a Grassmann integration.) The remaining vectors may be assembled into a 4 component object representing a symplectic tangent vector to the reduced space $\mathcal{M}$.

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} \bar{\mu}'_1 \\ \bar{\mu}'_2 \\ \bar{\mu}'_n \\ \mu'_n \end{pmatrix} \tag{6.82}$$

The product of basis one-forms on this space becomes

$$d\bar{\mu}'_1 d\bar{\mu}'_2 d\bar{\mu}'_n d\mu'_1 d\mu'_2 d\mu'_n = d\bar{\mu}_1 d\bar{\mu}_2 d\bar{\mu}_n d\mu_1 d\mu_2 d\mu_n = 2\rho^2 d\bar{\mu}'_1 d\bar{\mu}'_2 d\bar{\mu}'_n df_1 df_2 d\mu_n \tag{6.83}$$

Since the Jacobian determinant factors cancel, the volume element on the tangent space becomes

$$\int_{T_p\mathcal{M}} = \int_{T_p\tilde{\mathcal{M}}} \prod_\alpha \delta(f^\alpha)\, d\bar{\mu}_1' d\bar{\mu}_2' d\bar{\mu}_n' df_1 df_2 d\mu_n = \int_{T_p\tilde{\mathcal{M}}} d\bar{\mu}_1' d\bar{\mu}_2' d\bar{\mu}_n' d\mu_n' = \int_{T_p\tilde{\mathcal{M}}} d\psi_1 d\psi_2 d\psi_3 d\psi_4 \tag{6.84}$$

We can use this information to compute the restriction of the symplectic curvature of $\tilde{\mathcal{M}}$ to the reduced space $\mathcal{M}$.

6.6 The symplectic curvature

Using the expression for the symplectic curvature given in the appendix, (H.32), [12], we may re-derive the components of the symplectic curvature for the $SU(3)$ 1-instanton moduli space. To start with, for $k = 1$, the ADHM gauge group is just $U(1)$ and there is only one group generator. Thus we may ignore the indices i, j and r, s in (4.89). The quantity L^{-1}_{rs} becomes an ordinary number and we have

$$R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}} = 2L^{-1}\left((\tilde{\Omega}T)_{\tilde{i}\tilde{j}}(\tilde{\Omega}T)_{\tilde{k}\tilde{l}} + (\tilde{\Omega}T)_{\tilde{i}\tilde{l}}(\tilde{\Omega}T)_{\tilde{j}\tilde{k}} + (\tilde{\Omega}T)_{\tilde{i}\tilde{k}}(\tilde{\Omega}T)_{\tilde{j}\tilde{l}}\right) \tag{6.85}$$

We have already calculated the general form of the matrices $\tilde{\Omega}$ and T appearing above, (3.73) and (3.96). Specializing to the $k = 1$ case we have

$$\tilde{\Omega} = \begin{pmatrix} 0_3 & 1_3 \\ -1_3 & 0_3 \end{pmatrix} \tag{6.86}$$

$$T = \begin{pmatrix} 1_3 & 0_3 \\ 0_3 & -1_3 \end{pmatrix} \tag{6.87}$$

We are therefore in a position to find the explicit form of $\tilde{\Omega}T$ and thus of the symplectic curvature. For brevity we shall denote $\tilde{\Omega}T$ by A, where

$$A = \tilde{\Omega}T = \begin{pmatrix} 0_3 & -1_3 \\ -1_3 & 0_3 \end{pmatrix} \tag{6.88}$$

A is a 6×6 matrix in terms of which the symplectic curvature becomes

$$R_{\bar{i}\bar{j}\bar{k}\bar{l}} = 2L^{-1}\left(A_{\bar{i}\bar{j}}A_{\bar{k}\bar{l}} + A_{\bar{i}\bar{l}}A_{\bar{j}\bar{k}} + A_{\bar{i}\bar{k}}A_{\bar{j}\bar{l}}\right) \tag{6.89}$$

$$= -\frac{1}{2\rho^2}\left((\tilde{\Omega}T)_{\bar{i}\bar{j}}(\tilde{\Omega}T)_{\bar{k}\bar{l}} + (\tilde{\Omega}T)_{\bar{i}\bar{l}}(\tilde{\Omega}T)_{\bar{j}\bar{k}} + (\tilde{\Omega}T)_{\bar{i}\bar{k}}(\tilde{\Omega}T)_{\bar{j}\bar{l}}\right) \tag{6.90}$$

We shall now write down the results for the components of the symplectic curvature on the mother space;

$$R_{\bar{1}\bar{1}\bar{4}\bar{4}} = -\frac{1}{\rho^2}(A_{\bar{1}\bar{1}}A_{\bar{4}\bar{4}} + A_{\bar{1}\bar{4}}A_{\bar{1}\bar{4}} + A_{\bar{1}\bar{4}}A_{\bar{1}\bar{4}}) = -\frac{1}{\rho^2} \tag{6.91}$$

$$R_{\bar{1}\bar{4}\bar{5}\bar{2}} = 2L^{-1} = -\frac{1}{2\rho^2} \tag{6.92}$$

$$R_{\bar{1}\bar{4}\bar{6}\bar{3}} = 2L^{-1} = -\frac{1}{2\rho^2} \tag{6.93}$$

$$R_{\bar{5}\bar{2}\bar{5}\bar{2}} = 4L^{-1} = -\frac{1}{\rho^2} \tag{6.94}$$

$$R_{\bar{6}\bar{3}\bar{6}\bar{3}} = 4L^{-1} = -\frac{1}{\rho^2} \tag{6.95}$$

$$R_{\bar{5}\bar{2}\bar{6}\bar{3}} = 2L^{-1} - \frac{1}{2\rho^2} \tag{6.96}$$

All other components are zero. We can use these results to calculate the components of the symplectic curvature of the reduced space

$$R_{1212} = \tilde{R}(\mathrm{e}'^1, \mathrm{e}'^2, \mathrm{e}'^1, \mathrm{e}'^2) = \frac{\rho_-^2\rho_+^2}{4\rho^4}R\left(\mathrm{e}^1 + B\mathrm{e}^5, \mathrm{e}^2 - \frac{1}{B}\mathrm{e}^4, \mathrm{e}^1 + B\mathrm{e}^5, \mathrm{e}^2 - \frac{1}{B}\mathrm{e}^4\right)$$

$$\begin{aligned}
= \frac{\rho_-^2\rho_+^2}{4\rho^4}\Big[&R(\mathrm{e}^1,\mathrm{e}^2,\mathrm{e}^1,\mathrm{e}^2) - \frac{1}{B}R(\mathrm{e}^1,\mathrm{e}^2,\mathrm{e}^1,\mathrm{e}^4) + BR(\mathrm{e}^1,\mathrm{e}^2,\mathrm{e}^5,\mathrm{e}^2) - R(\mathrm{e}^1,\mathrm{e}^2,\mathrm{e}^5,\mathrm{e}^4) \\
&-\frac{1}{B}R(\mathrm{e}^1,\mathrm{e}^4,\mathrm{e}^1,\mathrm{e}^2) + \frac{1}{B^2}R(\mathrm{e}^1,\mathrm{e}^4,\mathrm{e}^1,\mathrm{e}^4) - R(\mathrm{e}^1,\mathrm{e}^4,\mathrm{e}^5,\mathrm{e}^2) + \frac{1}{B}R(\mathrm{e}^1,\mathrm{e}^4,\mathrm{e}^5,\mathrm{e}^4) \\
&+BR(\mathrm{e}^5,\mathrm{e}^2,\mathrm{e}^1,\mathrm{e}^2) - R(\mathrm{e}^5,\mathrm{e}^2,\mathrm{e}^1,\mathrm{e}^4) + B^2R(\mathrm{e}^5,\mathrm{e}^2,\mathrm{e}^5,\mathrm{e}^2) - AR(\mathrm{e}^5,\mathrm{e}^2,\mathrm{e}^5,\mathrm{e}^4) \\
&-R(\mathrm{e}^5,\mathrm{e}^4,\mathrm{e}^1,\mathrm{e}^2) + \frac{1}{B}R(\mathrm{e}^5,\mathrm{e}^4,\mathrm{e}^1,\mathrm{e}^4) - BR(\mathrm{e}^5,\mathrm{e}^4,\mathrm{e}^5,\mathrm{e}^2) + R(\mathrm{e}^5,\mathrm{e}^4,\mathrm{e}^5,\mathrm{e}^4)\Big]
\end{aligned} \tag{6.97}$$

$$= -\frac{\rho_-^2\rho_+^2}{\rho^4}\left[R_{\tilde{1}\tilde{2}\tilde{5}\tilde{4}} + \frac{1}{B^2}R_{\tilde{1}\tilde{4}\tilde{1}\tilde{4}} + B^2 R_{\tilde{5}\tilde{2}\tilde{5}\tilde{2}}\right] \tag{6.98}$$

$$= \frac{\rho_-^2\rho_+^2}{\rho^4}L^{-1}(\frac{1}{B^2} + B^2 - 2) \tag{6.99}$$

$$= \frac{L^{-1}}{\rho^4}(\rho_+^2 - \rho_-^2)^2 \tag{6.100}$$

$$R_{1212} = \frac{L^{-1}\xi^2}{\rho^4} = -\frac{\xi^2}{4\rho^6} = -\frac{4}{r^6} \tag{6.101}$$

Next we have

$$\tilde{R}(\hat{e}'^3, \hat{e}'^4, \hat{e}'^3, \hat{e}'^4) = \tilde{R}(\hat{e}^3, \hat{e}^6, \hat{e}^3, \hat{e}^6) = \tilde{R}_{\tilde{3}\tilde{6}\tilde{3}\tilde{6}} \tag{6.102}$$

$$= 2L^{-1}(A_{\tilde{3}\tilde{6}} + A_{\tilde{3}\tilde{6}} + A_{\tilde{3}\tilde{3}}A_{\tilde{6}\tilde{6}}) \tag{6.103}$$

$$= 4L^{-1} = -\frac{1}{\rho^2} = -\frac{4}{r^2} \tag{6.104}$$

And

$$\tilde{R}(\hat{e}'^1, \hat{e}'^2, \hat{e}'^3, \hat{e}'^4) = \frac{\rho_-\rho_+}{2\rho^2}\tilde{R}\left[\hat{e}^1 + B\hat{e}^5, \hat{e}^2 - \frac{1}{B}\hat{e}^4, \hat{e}^3, \hat{e}^6\right] \tag{6.105}$$

$$= \frac{\rho_-\rho_+}{2\rho^2}\left[\tilde{R}(\hat{e}^1, \hat{e}^2, \hat{e}^3, \hat{e}^6) - \frac{1}{B}\tilde{R}(\hat{e}^1, \hat{e}^4, \hat{e}^3, \hat{e}^6) + B\tilde{R}(\hat{e}^5, \hat{e}^2, \hat{e}^3, \hat{e}^6) - \tilde{R}(\hat{e}^5, \hat{e}^4, \hat{e}^3, \hat{e}^6)\right] \tag{6.106}$$

$$= \frac{\rho_-\rho_+}{L\rho^2}\left(B - \frac{1}{B}\right) \tag{6.107}$$

$$= \frac{\xi}{L\rho^2} = -\frac{\xi}{4\rho^4} = +\frac{2}{r^4} \tag{6.108}$$

These results compare favorably with those obtained by direct methods, given in equations (5.227)-(5.229). Having obtained the correct symplectic curvature, one may proceed to calculate the Gauss-Bonnet integral directly, as in the previous chapter.

6.7 The D-instanton partition function

In [17] it was shown that the so called D-instanton partition function Z provides a means of determining the volume contribution to the Euler character of the instanton moduli space. The D-instanton partition function consists of a series of integrations. We will evaluate these integrations in two different orders. First we will choose an appropriate order of integration to show that Z does indeed equal the Gauss-Bonnet expression. In this case the ADHM constraints appear directly as delta functions, which along with the appropriate Jacobian factors calculated previously in this chapter, restrict the final integration to the moduli space $\mathcal{M}$. However, if one evaluates these integrals in a different order, then one can obtain a general formula for the Gauss-Bonnet integral for a single instanton in any $SU(N)$ gauge group. This calculation will be reproduced in full in the next section.

The D-instanton partition function is given by;

$$Z = 2^{-2N-1}\pi^{-6N-9}\int d^{2N}\omega\, d^{2N}\bar{\omega}\, d^6\chi\, d^3D\, d^{4N}\mu\, d^{4N}\bar{\mu}\, d^8\lambda$$

$$exp\left[-\bar{\omega}^{\dot{\alpha}}_u\bar{\chi}^2\omega_{u\dot{\alpha}} - iD^c(\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}^{\dot{\beta}}_u\omega_{u\dot{\alpha}} - \xi^c) + 2\sqrt{2}\pi i\bar{\mu}^A_u\chi_{AB}\mu^B_u + i\pi(\bar{\mu}^A_u\omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u}\mu^A_u)\lambda^{\dot{\alpha}}_A\right] \tag{6.109}$$

where, (see[23]),

$$\chi_{AB} = \frac{1}{\sqrt{8}}\bar{\Sigma}_{aAB}\chi_a\,, \quad a = 1, ..., 6;\;\; A, B = 1, ..., 4 \tag{6.110}$$

Before worrying about any of the other terms we propose to proceed with the integration over χ. This is a Gaussian, so we recall the general formula, given below;

$$\int_{-\infty}^{+\infty} dy_1...dy_n exp\left[-\frac{1}{2}Y^TAY + \rho^TY\right] = (2\pi)^{\frac{n}{2}}det(A)^{-\frac{1}{2}}exp\left[\frac{1}{2}\rho^TA^{-1}\rho\right] \tag{6.111}$$

Setting

$$A = 2\bar{\omega}_u^{\dot{\alpha}}\omega_{u\dot{\alpha}}1_{6\times 6} \tag{6.112}$$

$$\rho^a = \pi i \bar{\mu}_u^A \bar{\Sigma}_{AB}^a \mu_u^B \tag{6.113}$$

we have the result

$$\int d^6\chi \, exp\left[-\bar{\omega}_u^{\dot{\alpha}}\chi^2\omega_{u\dot{\alpha}} + 2\sqrt{2}\pi i \bar{\mu}_u^A \bar{\Sigma}_{AB}^a \chi^a \mu_u^B\right] = \frac{\pi^3}{(\bar{\omega}_u^{\dot{\alpha}}\omega_{u\dot{\alpha}})^3} exp\left[-\frac{\pi^2}{2\bar{\omega}_u^{\dot{\alpha}}\omega_{u\dot{\alpha}}}\epsilon_{ABCD}\bar{\mu}_u^A\mu_u^B\bar{\mu}_v^C\mu_v^D\right] \tag{6.114}$$

Substituting this into the expression for Z gives

$$Z = 2^{-2N-1}\pi^{-6N-6}\int d^{2N}\omega \, d^{2N}\bar{\omega} \, d^3D \, d^{4N}\mu \, d^{4N}\bar{\mu} \, d^8\lambda$$

$$\frac{1}{(\bar{\omega}_u^{\dot{\alpha}}\omega_{u\dot{\alpha}})^3} exp\left[-iD^c(\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}_u^{\dot{\beta}}\omega_{u\dot{\alpha}} - \xi^c) + i\pi(\bar{\mu}_u^A\omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u}\mu_u^A)\lambda_A^{\dot{\alpha}} - \frac{\pi^2}{2\bar{\omega}_u^{\dot{\alpha}}\omega_{u\dot{\alpha}}}\epsilon_{ABCD}\bar{\mu}_u^A\mu_u^B\bar{\mu}_v^C\mu_v^D\right] \tag{6.115}$$

The next stage is to recall the integral representation of the Dirac delta function,

$$\delta(x - x') = \frac{1}{(2\pi)^n}\int d^n p \, e^{ip.(x-x')} \tag{6.116}$$

Similarly the fermionic delta function may be represented as

$$\alpha \, \delta(\mu) = \int e^{\alpha\mu\theta} d\theta \tag{6.117}$$

With this in mind we recognize immediately that the integrations over D and λ yield delta functions. Furthermore, the argument of these delta functions are the ADHM constraints themselves, so these terms act to impose the ADHM constraints on the rest of the integrand. Our expression for Z now becomes

$$Z = 2^{-2N+2}\pi^{-6N+5}\int_{\tilde{\mathcal{M}}} d^{2N}\omega \, d^{2N}\bar{\omega}\int_{T(\tilde{\mathcal{M}})} d^{4N}\mu \, d^{4N}\bar{\mu}$$

$$\frac{1}{(\bar{\omega}_u^{\dot{\alpha}}\omega_{u\dot{\alpha}})^3}\prod_{c=1}^{3}\delta(\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}_u^{\dot{\beta}}\omega_{u\dot{\alpha}} - \xi^c)\prod_{A=1}^{4}\prod_{\dot{\alpha}=1}^{2}\delta(\bar{\mu}_u^A\omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u}\mu_u^A) \, exp\left[-\frac{\pi^2}{2\bar{\omega}_u^{\dot{\alpha}}\omega_{u\dot{\alpha}}}\epsilon_{ABCD}\bar{\mu}_u^A\mu_u^B\bar{\mu}_v^C\mu_v^D\right] \tag{6.118}$$

Also, it is convenient to note that for the 1-instanton case with which we are dealing, when the ADHM constraints are imposed, we may write

$$\bar{\omega}_u^{\dot{\alpha}} \omega_{u\dot{\alpha}} = 2\rho^2 \tag{6.119}$$

This gives Z to be

$$Z = 2^{-2N-1}\pi^{-6N+5} \int_{\tilde{\mathcal{M}}} d^{2N}\omega\, d^{2N}\bar{\omega} \int_{T(\tilde{\mathcal{M}})} d^{4N}\mu\, d^{4N}\bar{\mu}$$

$$\frac{1}{\rho^6} \prod_{c=1}^{3} \delta(\tau^{c\dot{\alpha}}{}_{\dot{\beta}} \bar{\omega}_u^{\dot{\beta}} \omega_{u\dot{\alpha}} - \xi^c) \prod_{A=1}^{4} \prod_{\alpha=1}^{2} \delta(\bar{\mu}_u^A \omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u} \mu_u^A)\, exp\left[-\frac{\pi^2}{4\rho^2} \epsilon_{ABCD} \bar{\mu}_u^A \mu_u^B \bar{\mu}_v^C \mu_v^D\right] \tag{6.120}$$

In equation (6.49) we gave the volume form on the moduli space in terms of variables defined on the mother space but constrained to lie on the moduli space by a delta function constraint as well as the condition of gauge fixing. We can use that result to write the integration above as an integration over the moduli space;

$$Z = 2^{-2N-2}\pi^{-6N+6} \int_{\mathcal{M}} d^{2(N-1)}\omega\, d^{2(N-1)}\bar{\omega} \int_{T(\tilde{\mathcal{M}})} d^{4N}\mu\, d^{4N}\bar{\mu}$$

$$\frac{1}{\rho^8} \prod_{A=1}^{4} \prod_{\alpha=1}^{2} \delta(\bar{\mu}_u^A \omega_{u\dot{\alpha}} + \bar{\omega}_{\dot{\alpha}u} \mu_u^A)\, e^{-\frac{\pi^2}{4\rho^2} \epsilon_{ABCD} \bar{\mu}_u^A \mu_u^B \bar{\mu}_v^C \mu_v^D} \tag{6.121}$$

where we have used the same symbol, ω, to denote the coordinates on the moduli-space and those on the mother space. From equation (6.65) we know that the fermionic Jacobian required to correctly implement the delta function constraint is $\frac{1}{2\rho^2}$. In our case we have four such fermions, each labeled by the index A. Thus we need four powers of this Jacobian, giving

$$Z = 2^{-2N+2}\pi^{-6N+6} \int_{\mathcal{M}} d^{2(N-1)}\omega d^{2(N-1)}\bar{\omega} \int_{T(\mathcal{M})} \prod_{A=1}^{4} d^{(N-1)}\mu^A d^{(N-1)}\bar{\mu}^A e^{-\frac{\pi^2}{4\rho^2} \epsilon_{ABCD} \bar{\mu}_u^A \mu_u^B \bar{\mu}_v^C \mu_v^D}$$

From our previous analysis we know that we may rewrite the quantity in the exponent, expressing it in terms of the symplectic curvature tensor. Recall equation

(6.26), which we reproduce below,

$$\frac{1}{12}\epsilon_{ABCD}R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}M^{\tilde{i}A}M^{\tilde{j}B}M^{\tilde{k}C}M^{\tilde{l}D} = \epsilon_{ABCD}(\bar{M}^A M^B L^{-1}(\bar{M}^C M^D)) \tag{6.122}$$

We can identify the M^A's with the μ^A's and $L = 2\rho^2$. As in the last chapter, we can assemble the μ^A's into a (real) symplectic tangent vector, $\psi^A = (\bar{\mu}^A\, \mu^A)$. These are what we called the $M^{\tilde{i}A}$ above. Using this information, Z now takes the form

$$Z = 2^{-2(N-1)}\pi^{-6(N-1)} \int_{\mathcal{M}} d^{4(N-1)}\omega \int_{T(\mathcal{M})} \prod_{A=1}^{4} d^{2(N-1)}\psi^A\, exp\left[\frac{1}{24}\pi^2\epsilon_{ABCD}R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}\psi^{\tilde{i}A}\psi^{\tilde{j}B}\psi^{\tilde{k}C}\psi^{\tilde{l}D}\right. \tag{6.123}$$

From equation (B.16) we know that this expression is going to be proportional to the Gauss-Bonnet integral. Looking at equation (B.17) we should set $A = 2\pi^3$. Expanding the exponential above, only the $2(N-1)^{th}$ term contributes (since $\mathcal{M}$ is of dimension $4(N-1)$). Thus the prefactors of 2 and π cancel exactly, yielding the result that Z_I is precisely equal to the Gauss-Bonnet integral over the moduli space of instantons.

$$Z = \int_{\mathcal{M}} d^{4(N-1)}\omega \int_{T(\mathcal{M})} \prod_{A=1}^{4} d^{2(N-1)}\psi^A\, exp\left[\frac{1}{48\pi}\epsilon_{ABCD}R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}\psi^{\tilde{i}A}\psi^{\tilde{j}B}\psi^{\tilde{k}C}\psi^{\tilde{l}D}\right] = \chi_{Bulk} \tag{6.124}$$

6.8 The D-instanton partition function integrated in a different order

Here we reiterate in some detail a calculation first performed in [17]. Although motivated by physical arguments, their result encapsulates the value for the Gauss-Bonnet integral for a class of manifolds in one simple expression. We begin with the so-called D-instanton partition function with vanishing vevs, equation (6.109).We shall perform these integrations in well defined stages and in the interests of clarity we shall summarize our progress at the end of each stage.

6.8.1 Integration over μ

After completing the square, the $\mu\bar{\mu}$ integrations become

$$I_\mu = \int d^{4N}\mu\, d^{4N}\bar{\mu}\, e^{2\sqrt{2}i\pi\left[\bar{\mu}^B_u+\frac{1}{2\sqrt{2}}\lambda_{\dot{\alpha}A}\bar{\omega}^{\dot{\alpha}}{}_u(\tilde{\chi}^{-1})^{AB}\right]\tilde{\chi}_{BC}\left[\mu^C_u+\frac{1}{2\sqrt{2}}(\tilde{\chi}^{-1})^{CD}\omega_{u\dot{\beta}}\lambda^{\dot{\beta}}_D\right]-\frac{1}{2\sqrt{2}}i\pi\lambda_{\dot{\alpha}A}\bar{\omega}^{\dot{\alpha}}{}_u(\tilde{\chi}^{-1})^{AB}\omega_{u\dot{\beta}}\lambda^{\dot{\beta}}_B} \tag{6.125}$$

Shifting the integration variable gives

$$I_\mu = e^{-\frac{1}{2\sqrt{2}}i\pi\lambda_{\dot{\alpha}A}\bar{\omega}^{\dot{\alpha}}{}_u(\tilde{\chi}^{-1})^{AB}\omega_{u\dot{\beta}}\lambda^{\dot{\beta}}_B}\int d^{4N}\mu'\, d^{4N}\bar{\mu}'\, e^{2\sqrt{2}\pi i\bar{\mu}'^B_u\tilde{\chi}_{BC}\mu'^C_u} \tag{6.126}$$

Considering just the integral above we have

$$\int d^{4N}\mu'\, d^{4N}\bar{\mu}'\, e^{2\sqrt{2}\pi i\bar{\mu}'^B_u\tilde{\chi}_{BC}\mu'^C_u} = \left[\int\prod_{A=1}^{4} d\mu'^A_1\, d\bar{\mu}'^A_1\, e^{2\sqrt{2}\pi i\bar{\mu}'^B_1\tilde{\chi}_{BC}\mu'^C_1}\right]$$

$$\left[\int\prod_{A=1}^{4} d\mu'^A_2\, d\bar{\mu}'^A_2\,, e^{2\sqrt{2}\pi i\bar{\mu}'^B_2\tilde{\chi}_{BC}\mu'^C_2}\right]\dots\left[\int\prod_{A=1}^{4} d\mu'^A_N\, d\bar{\mu}'^A_N\,, e^{2\sqrt{2}\pi i\bar{\mu}'^B_N\tilde{\chi}_{BC}\mu'^C_N}\right]$$

$$= det_{4\times4}(2\sqrt{2}\pi\tilde{\chi}).det_{4\times4}(2\sqrt{2}\pi\tilde{\chi})...det_{4\times4}(2\sqrt{2}\pi\tilde{\chi})$$

$$= \left(det_{4\times4}(2\sqrt{2}\pi\tilde{\chi})\right)^N = 2^{6N}\pi^{4N}(det_{4\times4}\tilde{\chi})^N$$

The quantity $det_{4\times4}\tilde{\chi}$ can be determined to be

$$det_{4\times4}\tilde{\chi} = det_{4\times4}\left(\frac{1}{\sqrt{8}}\bar{\Sigma}_{aAB}\chi_a\right) = \frac{1}{2^6}|\chi|^4$$

Thus the μ $\bar{\mu}$ integrations yield

$$I_\mu = \pi^{4N}|\chi|^{4N}e^{-\frac{1}{2\sqrt{2}}i\pi\lambda_{\dot{\alpha}A}\bar{\omega}^{\dot{\alpha}}{}_u(\tilde{\chi}^{-1})^{AB}\omega_{u\dot{\beta}}\lambda^{\dot{\beta}}_B} \tag{6.127}$$

And our partition function becomes;

$$Z = 2^{-2N-1}\pi^{-2N-9}\int d^{2N}\omega\, d^{2N}\bar{\omega}\, d^6\chi\, d^3D d^{4N}\mu\, d^{4N}\bar{\mu}\, d^8\lambda \tag{6.128}$$

$$|\chi|^{4N}exp\left[-\bar{\omega}^{\dot{\alpha}}_u\bar{\chi}^2_{uv}\omega_{v\dot{\alpha}} - iD^c(\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}^{\dot{\beta}}_u\omega_{u\dot{\alpha}} - \xi^c) - \frac{1}{2\sqrt{2}}i\pi\lambda_{\dot{\alpha}A}\bar{\omega}^{\dot{\alpha}}{}_u(\tilde{\chi}^{-1})^{AB}\omega_{u\dot{\beta}}\lambda^{\dot{\beta}}_B\right] \tag{6.129}$$

6.8.2 Changing to $U(N)$ invariant coordinates

To evaluate the remaining integrals we shall find it helpful to first effectuate a change of variables to $U(N)$-invariant coordinates defined by;

$$W^0_{ij} = \bar{\omega}^{\dot{\alpha}}{}_{iu}\omega_{uj\dot{\alpha}} \ , \ W^c_{ij} = (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}^{\dot{\beta}}{}_{iu}\omega_{uj\dot{\alpha}} \tag{6.130}$$

In terms of these quantities the ADHM constraints become linear in the W^c's,

$$W^c_{ij} = -(a_m)_{ik}(a_n)_{kj}(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\sigma}^{\dot{\beta}\alpha}_m\sigma_{n\alpha\dot{\alpha}} \tag{6.131}$$

We shall regard the ω's as $(N \times K)$-dimensional matrices where $K = 2k$, (k being the instanton number), and a is the composite index $a = i\dot{\alpha}$ so we write ω_{ua}. An appropriate $SU(N)$ transformation will yield an ω in upper triangular form. Explicitly the W variables are given in matrix form as

$$(W^{\dot{\alpha}}{}_{\dot{\beta}})_{ij} = \bar{\omega}^{\dot{\alpha}}{}_{iu}\omega_{uj\dot{\beta}} = \begin{pmatrix} \xi_{11} & 0 & \dots & 0 & 0 & \dots & 0 \\ \xi^*_{12} & \xi_{22} & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 & \dots & 0 \\ \xi^*_{1K} & \xi^*_{2K} & \dots & \xi_{KK} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \xi_{11} & \xi_{12} & \dots & \xi_{1K} \\ 0 & \xi_{22} & \dots & \xi_{2K} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \xi_{KK} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{6.132}$$

Note that the elements on the diagonal are real and that

$$det(W) = det(\bar{\omega}).det(\omega) = \left(\prod_{a=1}^{K} \xi_{aa}\right)^2 \tag{6.133}$$

We shall first effect a transitional change of variables to the ξ's, the Jacobian for which is given by

$$\int d^{K^2}W = 2^K \int d^{K^2}\xi \prod_{a=1}^{K} \xi_{aa}^{2K-2a+1} \tag{6.134}$$

To motivate this assertion we will consider explicitly the cases $K = 2$ and $K = 3$. For $K = 2$, W is

$$W = \begin{pmatrix} \xi_{11} & 0 \\ \xi_{12}^* & \xi_{22} \end{pmatrix} \begin{pmatrix} \xi_{11} & \xi_{12} \\ 0 & \xi_{22} \end{pmatrix} = \begin{pmatrix} \xi_{11}^2 & \xi_{11}\xi_{12} \\ \xi_{11}\xi_{12}^* & \xi_{12}\xi_{12}^* + \xi_{22}^2 \end{pmatrix} \tag{6.135}$$

Calculating the various partial derivatives;

$$\frac{1}{2}\frac{\partial W_{11}}{\partial \xi_{11}} = \frac{\partial W_{12}}{\partial \xi_{12}} = \frac{\partial W_{21}}{\partial \xi_{21}} = \frac{\partial W_{21}}{\partial \xi_{12}^*} = \xi_{11}$$

$$\frac{\partial W_{11}}{\partial \xi_{12}} = \frac{\partial W_{11}}{\partial \xi_{21}} = \frac{\partial W_{11}}{\partial \xi_{22}} = \frac{\partial W_{12}}{\partial \xi_{21}} = \frac{\partial W_{12}}{\partial \xi_{22}} = \frac{\partial W_{22}}{\partial \xi_{11}} = \frac{\partial W_{21}}{\partial \xi_{12}} = \frac{\partial W_{21}}{\partial \xi_{22}} = 0$$

$$\frac{\partial W_{12}}{\partial \xi_{11}} = \frac{\partial W_{22}}{\partial \xi_{21}} = \frac{\partial W_{22}}{\partial \xi_{12}^*} = \xi_{12}$$

$$\frac{\partial W_{21}}{\partial \xi_{11}} = \frac{\partial W_{22}}{\partial \xi_{12}} = \xi_{12}^*$$

$$\frac{\partial W_{22}}{\partial \xi_{22}} = 2\xi_{22}$$

The Jacobian for this change of variables is thus

$$J = \begin{vmatrix} \frac{\partial W_{11}}{\partial \xi_{11}} & \frac{\partial W_{11}}{\partial \xi_{12}} & \frac{\partial W_{11}}{\partial \xi_{21}} & \frac{\partial W_{11}}{\partial \xi_{22}} \\ \frac{\partial W_{12}}{\partial \xi_{11}} & \frac{\partial W_{12}}{\partial \xi_{12}} & \frac{\partial W_{12}}{\partial \xi_{21}} & \frac{\partial W_{12}}{\partial \xi_{22}} \\ \frac{\partial W_{21}}{\partial \xi_{11}} & \frac{\partial W_{21}}{\partial \xi_{12}} & \frac{\partial W_{21}}{\partial \xi_{21}} & \frac{\partial W_{21}}{\partial \xi_{22}} \\ \frac{\partial W_{22}}{\partial \xi_{11}} & \frac{\partial W_{22}}{\partial \xi_{12}} & \frac{\partial W_{22}}{\partial \xi_{21}} & \frac{\partial W_{22}}{\partial \xi_{22}} \end{vmatrix} = \begin{vmatrix} 2\xi_{11} & 0 & 0 & 0 \\ \xi_{12} & \xi_{11} & 0 & 0 \\ \xi_{12}^* & 0 & \xi_{11} & 0 \\ 0 & \xi_{12}^* & \xi_{12} & 2\xi_{22} \end{vmatrix} = 2\xi_{11} \begin{vmatrix} \xi_{11} & 0 & 0 \\ 0 & \xi_{11} & 0 \\ \xi_{12}^* & \xi_{12} & 2\xi_{22} \end{vmatrix} = 4\xi_{11}^3\xi_{22} \tag{6.136}$$

For the case $K = 3$ we have

$$W = \begin{pmatrix} \xi_{11} & 0 & 0 \\ \xi_{12}^* & \xi_{22} & 0 \\ \xi_{13}^* & \xi_{23}^* & \xi_{33} \end{pmatrix} \begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ 0 & \xi_{22} & \xi_{23} \\ 0 & 0 & \xi_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \xi_{11}^2 & \xi_{11}\xi_{12} & \xi_{11}\xi_{13} \\ \xi_{11}\xi_{12}^* & \xi_{12}\xi_{12}^* + \xi_{22}^2 & \xi_{12}^*\xi_{13} + \xi_{22}\xi_{23} \\ \xi_{11}\xi_{13}^* & \xi_{12}\xi_{13}^* + \xi_{22}\xi_{23}^* & \xi_{13}\xi_{13}^* + \xi_{23}\xi_{23}^* + \xi_{33}^2 \end{pmatrix}$$

We again calculate the various partial derivatives required for the Jacobian

$$\frac{1}{2}\frac{\partial W_{11}}{\partial \xi_{11}} = \frac{\partial W_{12}}{\partial \xi_{12}} = \frac{\partial W_{13}}{\partial \xi_{13}} = \frac{\partial W_{21}}{\partial \xi_{21}} = \frac{\partial W_{31}}{\partial \xi_{31}} = \xi_{11}$$

$$\frac{\partial W_{11}}{\partial \xi_{12}} = \frac{\partial W_{11}}{\partial \xi_{21}} = \frac{\partial W_{11}}{\partial \xi_{31}} = \frac{\partial W_{11}}{\partial \xi_{13}} = \frac{\partial W_{11}}{\partial \xi_{22}} = \frac{\partial W_{11}}{\partial \xi_{23}} = \frac{\partial W_{11}}{\partial \xi_{32}} = \frac{\partial W_{11}}{\partial \xi_{33}} = 0$$

$$\frac{\partial W_{12}}{\partial \xi_{11}} = \frac{\partial W_{22}}{\partial \xi_{21}} = \frac{\partial W_{32}}{\partial \xi_{31}} = \xi_{12}$$

$$\frac{\partial W_{13}}{\partial \xi_{11}} = \frac{\partial W_{23}}{\partial \xi_{21}} = \frac{\partial W_{33}}{\partial \xi_{31}} = \xi_{13}$$

$$\frac{\partial W_{21}}{\partial \xi_{11}} = \frac{\partial W_{22}}{\partial \xi_{12}} = \frac{\partial W_{23}}{\partial \xi_{13}} = \xi^*_{12}$$

$$\frac{1}{2}\frac{\partial W_{22}}{\partial \xi_{22}} = \frac{\partial W_{23}}{\partial \xi_{23}} = \frac{\partial W_{32}}{\partial \xi_{32}} = \xi_{22}$$

$$\frac{\partial W_{23}}{\partial \xi_{22}} = \frac{\partial W_{33}}{\partial \xi_{32}} = \xi_{23}$$

$$\frac{\partial W_{31}}{\partial \xi_{11}} = \frac{\partial W_{32}}{\partial \xi_{12}} = \frac{\partial W_{33}}{\partial \xi_{13}} = \xi^*_{13}$$

$$\frac{\partial W_{32}}{\partial \xi_{22}} = \xi^*_{23}$$

$$\frac{\partial W_{33}}{\partial \xi_{33}} = 2\xi_{33}$$

The Jacobian is therefore

$$J = \begin{vmatrix} 2\xi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_{12} & \xi_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_{13} & 0 & \xi_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_{21} & 0 & 0 & \xi_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^*_{12} & 0 & \xi_{12} & 2\xi_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^*_{12} & \xi_{13} & \xi_{23} & 2\xi_{22} & 0 & 0 & 0 \\ \xi^*_{13} & 0 & 0 & 0 & 0 & 0 & \xi_{11} & 0 & 0 \\ 0 & \xi^*_{13} & 0 & 0 & \xi^*_{23} & 0 & \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & \xi^*_{13} & 0 & 0 & \xi^*_{23} & \xi_{13} & \xi_{23} & 2\xi_{33} \end{vmatrix} = 2^3\xi_{11}^5\xi_{22}^3\xi_{33}$$

Notice that the Jacobian matrix is taking a lower diagonal form. Thus the determinant is the product of the diagonal entries. For $K = 2$ the diagonal entries were

$$\frac{\partial W_{11}}{\partial \xi_{11}}\frac{\partial W_{12}}{\partial \xi_{12}}\frac{\partial W_{21}}{\partial \xi_{21}}\frac{\partial W_{22}}{\partial \xi_{22}} = 4\xi_{11}^3\xi_{22}$$

and for $K = 3$ we had

$$\frac{\partial W_{11}}{\partial \xi_{11}}\frac{\partial W_{12}}{\partial \xi_{12}}\frac{\partial W_{13}}{\partial \xi_{13}}\frac{\partial W_{21}}{\partial \xi_{21}}\frac{\partial W_{22}}{\partial \xi_{22}}\frac{\partial W_{23}}{\partial \xi_{23}}\frac{\partial W_{31}}{\partial \xi_{31}}\frac{\partial W_{32}}{\partial \xi_{32}}\frac{\partial W_{33}}{\partial \xi_{33}}$$
$$= 2^3\xi_{11}^5\xi_{22}^3\xi_{33}$$

Thus we can speculate that the Jacobian for a general K will be

$$J = 2^K \prod_{a=1}^{K} \xi_{aa}^{2K-2a+1} \tag{6.137}$$

And in going from $K = 2$ to $K = 3$ we required the extra terms

$$\left(\frac{\partial W_{13}}{\partial \xi_{13}}\frac{\partial W_{31}}{\partial \xi_{31}}\right)\left(\frac{\partial W_{23}}{\partial \xi_{23}}\frac{\partial W_{32}}{\partial \xi_{32}}\right)\frac{\partial W_{33}}{\partial \xi_{33}}$$

In general, in going from $K = n - 1$ to $K = n$ the extra terms required will be

$$\left(\prod_{a=1}^{n-1}\frac{\partial W_{an}}{\partial \xi_{an}}\frac{\partial W_{na}}{\partial \xi_{na}}\right)\frac{\partial W_{nn}}{\partial \xi_{nn}}$$

And thus the Jacobian for the case $K = n$ is related to that for $K = n - 1$ by

$$J_{K=n} = \left(\prod_{a=1}^{n-1}\frac{\partial W_{an}}{\partial \xi_{an}}\frac{\partial W_{na}}{\partial \xi_{na}}\right)\frac{\partial W_{nn}}{\partial \xi_{nn}} J_{K=n-1}$$

Using (6.137) for $K = n - 1$ gives

$$J_{K=n} = \left(\prod_{a=1}^{n-1}\frac{\partial W_{an}}{\partial \xi_{an}}\frac{\partial W_{na}}{\partial \xi_{na}}\right)\frac{\partial W_{nn}}{\partial \xi_{nn}} 2^{n-1}\prod_{a=1}^{n-1}\xi_{aa}^{2n-2a-1}$$

However, we can calculate the partial derivatives;

$$\frac{\partial W_{nn}}{\partial \xi_{nn}} = 2\xi_{nn} \ \ and \ \ \frac{\partial W_{an}}{\partial \xi_{an}} = \frac{\partial W_{3n}}{\partial \xi_{3n}} = \xi_{aa}$$

which finally yields

$$J_{K=n} = 2\xi_{nn}\left(\prod_{a=1}^{n-1}\xi_{aa}^2\right)2^{n-1}\prod_{a=1}^{n-1}\xi_{aa}^{2n-2a-1} = 2^n\xi_{nn}\prod_{a=1}^{n-1}\xi_{aa}^{2n-2a+1} = 2^n\prod_{a=1}^{n}\xi_{aa}^{2n-2a+1} \tag{6.138}$$

as required.

We need to go further to calculate the change of variables from ω to $\{\xi, U\}$. Consider how ω is defined

$$\omega = \begin{pmatrix} U_{11} & U_{12} & U_{13} & \dots & U_{1N} \\ U_{21} & U_{22} & U_{23} & \dots & U_{2N} \\ \vdots & & & & \vdots \\ U_{2k1} & U_{2k2} & U_{2k3} & \dots & U_{2kN} \\ U_{2k+1\,1} & U_{2k+1\,2} & U_{2k+1\,3} & \dots & U_{2k+1\,N} \\ \vdots & & & & \vdots \\ U_{N1} & U_{N2} & U_{N3} & \dots & U_{NN} \end{pmatrix}\begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \dots & \xi_{1,2k} \\ 0 & \xi_{22} & \xi_{23} & \dots & \xi_{2,2k} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \xi_{2k,2k} \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Carrying out the matrix multiplication we can see that the only block in U to contribute to ω is the first $N \times 2k$ block. Note that ω is an $N \times 2k$ matrix whose first column is given by

$$\omega_{u1} = \begin{pmatrix} U_{11}\xi_{11} \\ U_{21}\xi_{11} \\ \vdots \\ U_{2k,1}\xi_{11} \\ U_{2k+1,1}\xi_{11} \\ \vdots \\ U_{N1}\xi_{11} \end{pmatrix}$$

Which we can write as

$$\omega_{u1} = U_{u1}\xi_{11} \tag{6.139}$$

Likewise the next column in ω is given by

$$\omega_{u2} = U_{u1}\xi_{12} + U_{u2}\xi_{22} \tag{6.140}$$

And in general we have

$$\omega_{ua} = \sum_{b=1}^{a} U_{ub}\xi_{ba} \tag{6.141}$$

It is useful to consider the columns of the matrix U as vectors, u, defined by $u^a = U_{ua}$. Since $U \in SU(N)$ we have

$$U^{\dagger}_{au}U_{ub} = (u^{\dagger})^a u^b = \delta^{ab}$$

Thus the vectors u are orthonormal. Selecting $2k$ such vectors, one can show that they furnish a representation of the coset in terms of a product of spheres. u^1 is a unit vector in an N-dimensional complex space and consequently parameterizes S^{2N-1}. The second vector u^2 is also a unit vector and is orthogonal to u^1. Thus u^2 parameterizes an S^{2N-3} which is orthogonal to the S^{2N-1}. That is to say, the vector u^1 parameterizes a $(2n-1)$-sphere. At every point p on this sphere, there will be a unit vector, namely u^2 that will be orthogonal to u^1 and will therefore parameterize a $(2N-3)$-sphere at p. There will then also be a unit vector u^3 orthogonal to both of these that will describe a $(2n-5)$-sphere, and so on. Thus we have

$$\frac{SU(N)}{SU(N-K)} \approx S^{2N-1} \times S^{2N-3} \times ... \times S^{2N-2K+1} \tag{6.142}$$

From (6.139) we see that ω_{u1} parameterizes a sphere of radius ξ_{11} in a complex N-dimensional space. We may therefore write the volume element associated with the ω_{u1}'s in polar coordinates as follows,

$$\int \prod_{u=1}^{N} d\omega_{u1}\, d\omega^*_{u1} = 2^N \int \xi_{11}^{2N-1} d\xi_{11}\, d^{2N-1}\Omega_1 \tag{6.143}$$

where $d^{2N-1}\Omega_1$ is the standard angular solid angle measure in $2N-1$ dimensions. The next vector in the series is ω_{u2}, (6.140). We can therefore write

$$(\omega_{u2} - U_{u1}\xi_{12})(\omega_{u2} - U_{u1}\xi_{12})^* = \xi_{22}^2 \tag{6.144}$$

Recalling the standard complex representation, $(z-a)(z-a)^* = r^2$ for a circle of radius r, center a, we conclude that the ω_{u2} coordinates parameterize a sphere of radius ξ_{22} in $(2N-3)$ dimensions, centre $U_{u1}\xi_{12}$. Thus in polar coordinates we may write

$$\int \prod_{u=1}^{N} d\omega_{u2}\, d\omega_{u2}^* = 2^{N-1} \int d\xi_{12}\, d\xi_{12}^*\, \xi_{22}^{2N-3} d\xi_{22}\, d^{2N-3}\Omega_1 \tag{6.145}$$

Continuing this iterative process, in general we have,

$$\int \prod_{u=1}^{N} d\omega_{ua}\, d\omega_{ua}^* = 2^{N-a+1} \int \left[\prod_{b=1}^{a-1} d\xi_{ba}\, d\xi_{ba}^* \right] \xi_{aa}^{2N-2a+1} d\xi_{22}\, d^{2N-2a+1}\Omega_a \tag{6.146}$$

where Ω_a is parameterized by u^a. Multiplying these terms together we have

$$\int \prod_{u=1}^{N} d\omega_{ua} d\omega_{ua}^* = 2^{2kN-k(2k-1)} \int \left[\prod_{a=1}^{2k} \xi_{aa}^{2N-2a+1} d\xi_{aa} d^{2N-2a+1}\Omega_a \right] \prod_{a<b} d\xi_{ab} d\xi_{ab}^* \tag{6.147}$$

Using equations (6.133) and (6.134) we may now finally write the integrations over the $SU(3)$invariant coordinates as

$$\int \prod_{u=1}^{N} d\omega_{ua} d\omega_{ua}^* = 2^{2kN-k(2k+1)} \int d^{4k^2}W\ |det(W)|^{N-2k} \prod_{a=1}^{2k} d^{2N-2a+1}\Omega_a \tag{6.148}$$

For the one-instanton case this becomes

$$\int \prod_{u=1}^{N} d\omega_{ua} d\omega_{ua}^* = 2^{2N-3} \int d^4W\ |det(W)|^{N-2} d^{2N-1}\Omega_1 d^{2N-3}\Omega_2 \tag{6.149}$$

From (6.130), we may write

$$W^{\dot\beta}{}_{\dot\alpha} = \frac{1}{2}\left(\tau^{c\dot\beta}{}_{\dot\alpha} W^c + \delta^{\dot\beta}{}_{\dot\alpha} W^0\right) \tag{6.150}$$

Taking the determinant gives

$$|det(W)| = \frac{1}{4}\left|(W^0)^2 - |W^c|^2\right| \tag{6.151}$$

We also wish to change variables in the integration from an integration over the components $W^{\dot{\alpha}}{}_{\dot{\beta}}$ to an integration over the W^c and W^0. The Jacobian for this transformation is

$$J = \left| \begin{array}{cc} \frac{\partial W^{\dot{\alpha}}{}_{\dot{\beta}}}{\partial W^0} & \frac{\partial W^{\dot{\alpha}}{}_{\dot{\beta}}}{\partial W^c} \end{array} \right| = \left| \begin{array}{cccc} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array} \right| = \left|\frac{-i}{4}\right| = \frac{1}{4} \tag{6.152}$$

Finally, we may calculate the angular factor with recourse to the general result for the measure on S^{N-1},

$$\int d\hat{\Omega}_N = 2\pi \prod_{k=1}^{N-2} \int_0^{\pi} sin^k\theta_k = 2\pi(\sqrt{\pi})^{N-2} \prod_{k=1}^{N-2} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} \tag{6.153}$$

Which gives

$$\int d^{2N-1}\Omega_1 = 2\pi^N \frac{\Gamma(1)}{\Gamma(N)} \tag{6.154}$$

$$\int d^{2N-3}\Omega_2 = 2\pi^{N-1} \frac{\Gamma(1)}{\Gamma(N-1)} \tag{6.155}$$

Substituting (6.151), (6.152), (6.154), (6.155) into (6.149) gives the result

$$\int d^{2N}\omega \int d^{2N}\bar{\omega} = \frac{2\pi^{2N-1}}{\Gamma(N)\Gamma(N-1)} \int dW^0 d^3W[(W^0)^2 - |W^c|^2]^{N-2} \tag{6.156}$$

Thus the partition function integration becomes

$$Z = \frac{2^{-2N}\pi^{-10}}{\Gamma(N)\Gamma(N-1)} \int dW^0\, d^3W\, d^6\chi\, d^3D\, d^8\lambda\, [(W^0)^2 - |W^c|^2]^{N-2}\chi^{4N} \tag{6.157}$$

$$e^{-W^0\chi^2 - iD^c(W^c - \xi^c) - \frac{i\pi}{2\sqrt{2}} W^c \tau^{c\dot{\alpha}}{}_{\dot{\beta}} (\chi^{-1})^{AB} \lambda_{\dot{\alpha}A} \lambda^{\dot{\beta}}_B} \tag{6.158}$$

6.8.3 Integration over λ

The λ integral is

$$I_\lambda = \int d^8\lambda e^{-\frac{i\pi}{2\sqrt{2}}W^c\tau^{c\dot{\alpha}}{}_{\dot{\beta}}(\chi^{-1})^{AB}\lambda_{\dot{\alpha}A}\lambda_B^{\dot{\beta}}} = \sqrt{det_{8\times 8}\left[\frac{\pi}{2\sqrt{2}}\tau^{c\dot{\alpha}}{}_{\dot{\beta}}W^c(\chi^{-1})^{AB}\right]} \tag{6.159}$$

$$= \frac{\pi^4}{2^6}\sqrt{det_{8\times 8}\left[W^c\tau^{c\dot{\alpha}}{}_{\dot{\gamma}}\delta^{CB}.(\chi^{-1})^{AC}\delta^{\dot{\gamma}}_{\dot{\beta}}\right]} \tag{6.160}$$

$$= \frac{\pi^4}{2^6}\sqrt{det\left(W^c\tau^c \otimes 1_{4\times 4}\right)\, det\left(\chi^{-1}\otimes 1_{2\times 2}\right)} \tag{6.161}$$

$$= \frac{\pi^4}{2^6}det(W^c\tau^c)^2 det(\chi^{-1}) \tag{6.162}$$

Where we have used

$$det\left(W\tau \otimes 1_{4\times 4}\right) = \begin{vmatrix} W^c\tau^c & 0 & 0 & 0 \\ 0 & W^c\tau^c & 0 & 0 \\ 0 & 0 & W^c\tau^c & 0 \\ 0 & 0 & 0 & W^c\tau^c \end{vmatrix} = \left(det(W^c\tau^c)\right)^4 \tag{6.163}$$

We can now determine $det(W^c\tau^c)$

$$det(W^c\tau^c) = \begin{vmatrix} W^3 & W^1 - iW^2 \\ W^1 + iW^2 & -W^3 \end{vmatrix} = -|W^c|^2 \tag{6.164}$$

Therefore

$$I_\lambda = \frac{\pi^4}{2^6}\frac{|W^c|^4}{det(\chi)} = \pi^4\frac{|W^c|^4}{|\chi|^4} \tag{6.165}$$

Thus Z is now

$$Z = \frac{2^{-2N}\pi^{-6}}{\Gamma(N)\Gamma(N-1)}\int dW^0\, d^3W\, d^6\chi\, d^3D\, d^8\lambda\, \chi^{4(N-1)}|W^c|^4[(W^0)^2 - |W^c|^2]^{N-2} \tag{6.166}$$

$$e^{-W^0\chi^2 - iD^c(W^c-\xi^c)} \tag{6.167}$$

6.8.4 Integration over χ

We proceed to the χ integral;

$$I_\chi = \int d^6\chi \chi^{4(N-1)} e^{-W^0\chi^2} \tag{6.168}$$

We will write this in terms of polar coordinates where

$$d^6\chi = |\chi|^5 d|\chi| d\hat{\Omega}_5 \tag{6.169}$$

Therefore

$$I_\chi = \int d\hat{\Omega}_5 \int |\chi|^5 d|\chi| \chi^{4(N-1)} e^{-W^0\chi^2} \tag{6.170}$$

$$= \int d\hat{\Omega}_5 \int d|\chi| \chi^{4N+1} e^{-W^0\chi^2} \tag{6.171}$$

Let

$$t = W^0\chi^2 \Rightarrow \chi = \sqrt{\frac{t}{W^0}} \tag{6.172}$$

$$\Rightarrow dt = 2W^0\chi d\chi \Rightarrow d\chi = \frac{1}{2\chi W^0} dt \tag{6.173}$$

Substituting this into the integral gives

$$I_\chi = \int d\hat{\Omega}_5 \int dt \frac{t^{2N}}{2W^{2N+1}} e^{-t} = \frac{\pi^3}{2(W^0)^{2N+1}} \Gamma(2N+1) \tag{6.174}$$

where we have used the standard results

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{6.175}$$

$$\int d\hat{\Omega}_5 = \pi^3 \tag{6.176}$$

This last equation follows from the general result for the angular measure given in equation (6.153), which for $N = 6$ gives

$$\int d\hat{\Omega}_5 = 2\pi(\sqrt{\pi})^4 \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{2})} \frac{\Gamma(\frac{4}{2})}{\Gamma(\frac{5}{2})} \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{6}{2})} = \pi^3 \tag{6.177}$$

Again, we substitute our progress so far into Z to get

$$Z = \frac{2^{-2N-1}\pi^{-3}\Gamma(2N+1)}{\Gamma(N)\Gamma(N-1)} \int dW^0\, d^3W\, d^6\chi\, d^3D\, d^8\lambda\, \chi^{4(N-1)} |W^c|^4 [(W^0)^2 - |W^c|^2]^{N-2} \tag{6.178}$$

$$e^{-W^0\chi^2 - iD^c(W^c - \xi^c) - \frac{i\pi}{2\sqrt{2}} W^c \tau^{c\dot{\alpha}}{}_{\dot{\beta}} (\chi^{-1})^{AB} \lambda_{\dot{\alpha}A} \lambda^{\dot{\beta}}_B} \tag{6.179}$$

6.8.5 Integration over W^0

We now proceed with the W^0 integral.

$$I_{W^0} = \int dW^0 [(W^0)^2 - |W^c|^2]^{N-2} (W^0)^{-2N-1} \tag{6.180}$$

This integral is of the general form

$$I_0 = \int_a^\infty (X^2 - a^2)^{N-2} X^{-2N-1} dX \tag{6.181}$$

To evaluate this let $A = N - 2$ and $B = 2N + 1$.

$$I_0 = \int_a^\infty (X^2 - a^2)^A X^{-B} dX \tag{6.182}$$

Now we integrate by parts,

$$u_1 = (X^2 - a^2)^{N-2} \Rightarrow du_1 = A(X^2 - a^2)^{A-1} 2X dX \tag{6.183}$$

$$dv_1 = X^{-B} dX \Rightarrow v_1 = \frac{X^{-B+1}}{1-B} \tag{6.184}$$

Using these results gives

$$I_0 = \int_a^\infty \frac{2A}{1-B} X^{2-B} (X^2 - a^2)^{A-1} dX = \frac{2A}{1-B} X^{2-B} I_1 \tag{6.185}$$

Next we integrate by parts again, setting

$$u_2 = (X^2 - a^2)^{A-1} \Rightarrow du_2 = (A-1)(X^2 - a^2)^{A-2} 2X dX \tag{6.186}$$

$$dv_2 = x^{2-B}dX \Rightarrow v_2 = \frac{X^{3-B}}{3-B} \tag{6.187}$$

Therefore

$$I_2 = 2\int_a^\infty \frac{A-1}{B-3}X^{4-B}(X^2-a^2)^{A-2}dX \tag{6.188}$$

Proceeding just one more step gives

$$I_3 = 2\int_a^\infty \frac{A-2}{B-5}X^{6-B}(X^2-a^2)^{A-3}dX \tag{6.189}$$

If we had $A = 3$ then our final result would be

$$I_0 = \frac{2^3 3!}{(B-1)(B-3)(B-5)}\int_a^\infty X^{6-B}dX \tag{6.190}$$

Generally after A iterations we would get

$$I_0 = \frac{2^A A!}{(B-1)(B-3)...(B+1-2A)}\int_a^\infty X^{2A-B}dX \tag{6.191}$$

Setting $A = N-2$ and $B = 2N+1$ gives

$$I_0 = \frac{2^{N-2}(N-2)!}{(2N)(2N-2)...(2N+2-2N+4)}\int_a^\infty X^{-5}dX = \frac{1}{2N(N-1)a^4} \tag{6.192}$$

Thus the W^0 integral becomes

$$I_{W^0} = \frac{1}{2N(N-1)|W^c|^4} \tag{6.193}$$

and Z is now

$$Z = \frac{2^{-2N-2}\pi^{-3}\Gamma(2N+1)}{\Gamma(N)\Gamma(N+1)}\int d^3W\, d^3D\, e^{-iD^c(W^c-\xi^c)} \tag{6.194}$$

6.8.6 Integration over D and W^c

There are now only two remaining integrals to be done, namely that over D^c and W^c. In fact, from equation (6.116) we should recognize the remaining integration as one over a delta function,

$$\int d^3De^{-iD^c(W^c-\xi^c)} = (2\pi)^3\delta(W^c-\xi^c) \tag{6.195}$$

Integrating this over all W then yields

$$2^3\pi^3 \int d^3W^c\, \delta^{(3)}(W^c - \xi^c) = 2^3\pi^3 \tag{6.196}$$

6.8.7 The final result

We may now assemble these results to give a remarkable result;

$$Z = \frac{2^{-2N+1}\,\Gamma(2N+1)}{\Gamma(N)\Gamma(N+1)} \tag{6.197}$$

There is just one final trick to perform to put this result in its final polished form. We expand the gamma functions so that we may perform some cancellations. This will also remove the ugly prefactors of two.

$$Z = \frac{2^{-2N+1}(2N) \times (2N-1) \times (2N-2) \times \ldots \times (2) \times (1)}{\Gamma(N) N \times (N-1) \times \ldots \times 2 \times 1} \tag{6.198}$$

$$= \frac{2^{-N+1}(2N-1) \times (2N-3) \times \ldots \times (3) \times (1)}{\Gamma(N)} \tag{6.199}$$

$$= \frac{2(N-\frac{1}{2}) \times (N-\frac{3}{2}) \times \ldots \times (\frac{3}{2}) \times (\frac{1}{2})}{\Gamma(N)} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})} \tag{6.200}$$

$$= \frac{2\Gamma\left(N+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(N)} \tag{6.201}$$

At the beginning of this chapter we demonstrated that this integral for the D-instanton partition function should yield the (volume contribution to) the Euler character on the 1-instanton moduli space.

We can now recover the result for the Euler character of the $SU(3)$ 1-instanton moduli space by setting $N = 3$ in the above. This gives

$$Z_{1,3} = \frac{2\Gamma\left(3+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(3)} \tag{6.202}$$

$$= \frac{2 \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}\right)} \tag{6.203}$$

$$= \frac{15}{8} \tag{6.204}$$

which fortunately is in agreement with the result obtained by brute force computation.

6.9 Summary

We started this chapter with a discussion of the machinery necessary to implement the construction of [17]. We explored how one might restrict the domain of integration to a sub-surface of a manifold by imposing constraints in the form of Dirac delta functions. We also reviewed the problem of quotienting a space by a group action as discussed in [12]. We were able to take the mother space of our $SU(3)$ 1-instanton moduli space and change to a coordinate system in which the ADHM delta functions were trivialized. In this way we were able to determine an explicit expression for the volume form on this moduli space, in agreement with the result obtained in chapter five. Furthermore, we were able to solve the fermionic ADHM constraints and so arrive at the Jacobian necessary to implement the fermionic ADHM constraints as delta functions. We then verified that such a method does actually yield the same symplectic curvature components as we found previously, (chapter five). With this apparatus in place we then showed that the D-instanton partition function of [17] reduces, in the 1-instanton sector, to the Gauss-Bonnet integral over the moduli space. The D-instanton partition function consists of a series of integrations and one is free to choose the order in which they are implemented. Following [17] we chose an order of integration that yielded a general numerical result covering all the $SU(N)$ 1-instanton moduli spaces. The result obtained for the Gauss-Bonnet integral for the case $N = 3$ agrees with that calculated earlier in chapter five. This offers some support for the validity of the result of [17].

Appendix A

Cartan's equations of structure

In a non-coordinate basis the tangent space is spanned by a linear combination of coordinate basis vectors e_μ

$$e_a = e_a{}^\mu e_\mu$$

With $e_a^\mu \in Gl(m.R)$ and $det e_a^\mu > 0$ The $\{e_a\}$ is the frame of basis vectors obtained by a Gl(m,R) rotation of the coordinate basis e_μ. If we require the e_a to be orthonormal then we have

$$g(e_a, e_b) = e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \delta_{ab}$$

We define the inverse of the matrixe_a^μ

$$e_a{}^\nu e^a{}_\mu = \delta^\nu_\mu$$

$$e_a{}^\mu e^b{}_\mu = \delta^a_b$$

Using these expressions we may now invert the equation involving the metric tensor to give

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \delta_{ab}$$

$$g_{\mu\nu} = e_{a\mu} e^a{}_\nu$$

Where δ_{ab} has been used to lower the tangent space index. Since a vector V is independent of the basis chosen, we may equate the expressions for V in these two basis.

$$V = V^{\mu} e_{\mu} = V^{a} e_{a} = V^{a} e_{a}^{\ \mu} e_{\mu}$$

$$\Rightarrow V^{\mu} = V^{a} e_{a}^{\ \mu} \ and \ V^{a} = e^{a}_{\ \mu} v^{\mu}$$

We now introduce the dual basis such that $\langle e^{a}, e_{b} \rangle = \delta^{a}_{b}$ Since the dual basis transforms oppositely to the tangent-space basis, so we must have:

$$e^{a} = e^{a}_{\ \mu} dx^{\mu}$$

In this non-coordinate basis the metric becomes

$$g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$$

$$= e^{a}_{\ \mu} e^{b}_{\ \nu} \delta_{ab} (e_{c}^{\ \mu} e^{c}) \otimes (e_{d}^{\ \nu} e^{d})$$

$$= \delta_{ab}\, e^{a} \otimes e^{b}$$

The coefficients $e_{a}^{\ \mu}$ are called vielbeins. The non-coordinate basis has a non-vanishing Lie bracket:

$$[e_{a}, e_{b}]|_{p} = [e_{a}^{\ \mu} e_{\mu}, e_{b}^{\ \nu} e_{\nu}]|_{p}$$

$$= e_{a}^{\ \mu} \partial_{\mu} (e_{b}^{\ \nu} e_{\nu}|_{p} - e_{b}^{\ \nu} \partial_{\nu} (e_{b}^{\ \mu} e_{\mu})|_{p}$$

$$= (e_{a}^{\ \nu} \partial_{\nu} e_{b}^{\ \mu} - e_{b}^{\ \nu} \partial_{\nu} e_{a}^{\ \mu}) e^{c}_{\ \mu} e^{c}|_{p}$$

i.e.

$$[e_{a}, e_{b}]|_{p} = c_{ab}^{\ \ c}(p) e_{c}|_{p}$$

Where

$$c_{ab}^{\ \ c}(p) = e^{c}_{\ \mu} (e_{a}^{\ \nu} \partial_{\nu} e_{b}^{\ \mu} - e_{b}^{\ \nu} \partial_{\nu} e_{a}^{\ \mu})(p)$$

Consider a non-coordinate basis of one forms e_a and vectors e^a where

$$[e_a, e_b] = c_{ab}{}^c e_c$$

We define the connection coefficients with respect to the basis e_a by

$$\nabla_a e_b = \nabla_{e_a} e_b = \omega^c{}_{ab} e_c$$

If we now write $e_a = e_a^{\ \mu} e_\mu$, then we have

$$\nabla_a e_b = e_a^{\ \mu} \nabla_\mu (e_b^{;\nu} e_\nu)$$

$$= e_a^{\ \mu}(\partial_\mu e_b^{\ \nu}.e_\nu + e_b^{\ \nu} \nabla_\mu e_\nu)$$

$$= e_a^{\ \mu}(\partial_\mu e_b^{\ \nu}.e_\nu + e_b^{\ \nu} \Gamma^\lambda_{\mu\nu} e_\lambda)$$

$$= e_a^{\ \mu}(\partial_\mu e_b^{\ \nu} + e_b^{\ \lambda} \Gamma^\nu_{\mu\lambda}) e_\nu$$

However, we also have the following:

$$\nabla_a e_b = \omega^c{}_{ab} e_c = \omega^c{}_{ab} e_c^{\ \nu} e_\nu$$

Equating these two expressions gives

$$\omega^c{}_{ab} = e^c{}_\nu e_a^{\ \mu}(\partial_\mu e_b^{\ \nu} + e_b^{\ \lambda} \Gamma^\lambda_{\mu\nu})$$

We will now calculate the components of the torsion T and the curvature R in this basis.

$$T^a_{bc} = \langle e^a, T(e_b, e_c) \rangle$$

$$= \langle e^a, \nabla_b e_c - \nabla_c e_b - [e_b, e_c] \rangle$$

$$= \langle e^a, \omega^d{}_{bc} e_d - \omega^d{}_{cb} e_d - c_{bc}{}^d e_d \rangle$$

$$\omega^a{}_{bc} - \omega^a{}_{cb} - c_{bc}{}^a$$

And

$$R^a_{bcd} = \langle e^a, \nabla_c \nabla_d e_b - \nabla_d \nabla_c e_b - \nabla_{[e_c, e_d]} e_b \rangle$$

$$= \langle e^a, \nabla_c(\omega^f{}_{db} e_f) - \nabla_d(\omega^f{}_{cb} e_f) - c_{cd}{}^f \nabla_f e_b \rangle$$

$$= \langle e^a, e_c[\omega^f{}_{db}] e_f + \omega^f{}_{db} \omega^g{}_{cf} e_g - e_d[\omega^f{}_{cb}] e_f - \omega^f{}_{cb} \omega^g{}_{df} e_g - c_{cd}{}^f \omega^g{}_{fb} e_g \rangle$$

$$= e_c[\omega^a{}_{db}] + \omega^f{}_{db} \omega^a{}_{cf} - e_d[\omega^a{}_{cb}] - \omega^f{}_{cb} \omega^a{}_{df} - c_{cd}{}^f \omega^a{}_{fb}$$

We now define a matrix-valued one-form ω^a_b called the connection one-form or spin connection,

$$\omega^a_b = \omega^a{}_{cb} e^c$$

The spin connection satisfies Cartan's equations of structure:

$$de^a + \omega^a_b \wedge e^b = T^a$$

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b$$

Where we have introduced the Torsion two-form

$$T^a = \frac{1}{2} T^a_{bc} e^b \wedge e^c$$

and the curvature two-form

$$R^a_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d$$

To verify Cartan's structure equations we let them act on the basis vectors e_c , the L.H.S. of the first Cartan equation then gives

$$de^a(e_c, e_d) + [\langle \omega^a_b, e_c \rangle \langle e^b, e_d \rangle - \langle e^b, e_c \rangle \langle \omega^a_b, e_d \rangle]$$

$$= de^a(e_c, e_d) + [\langle \omega^a_b, e_c \rangle \delta^b_d - \delta^b_c \langle \omega^a_b, e_d \rangle$$

$$= de^a(e_c, e_d) + \omega^a{}_{cb} - \omega^a{}_{dc}$$

To proceed we make use of the following identity

$$d\omega(X,Y) = X[\langle\omega,Y\rangle] - Y[\langle\omega,X\rangle] - \omega([X,Y])$$

Using this gives

$$de^a(e_c,e_d) = e_c[\langle e^a,e_d\rangle] - e_d[\langle e^a,e_c\rangle] - e^a([e_c,e_d])$$

$$= e_c[\delta^a_d] - e_d[\delta^a_c] - \langle e^a, c_{cd}{}^f e_f\rangle$$

$$= -c_{cd}{}^a$$

Substituting this into eqn? gives

$$-c_{cd}{}^a + \omega^a{}_{cb} - \omega^a{}_{dc} = T^a_{\ cd}$$

Simmilarly, we now consider the R.H.S. of the first Cartan equation acting on the basis vectors.

$$T^a(e_c,e_d) = \frac{1}{2}T^a_{\ bf}e^b \wedge e^f(e_c,e_d)$$

$$= \frac{1}{2}T^a_{\ bf}(\langle e^b,e_c\rangle\langle e^f,e_d\rangle - \langle e^b,e_d\rangle\langle e^f,e_c\rangle)$$

$$= \frac{1}{2}T^a_{\ bf}(\delta^b_c\delta^f_d - \delta^b_d\delta^f_c)$$

$$= T^a_{\ cd}$$

as required. Next we consider the L.H.S. of the second structure equation acting on the e^a

$$d\omega^a_{\ b}(e_c,e_d) + (\omega^a{}_f \wedge \omega^f{}_b)(e_c,e_d)$$

$$= d\omega^a_{\ b}(e_c,e_d) + \langle\omega^a{}_f,e_c\rangle\langle\omega^f{}_b,e_d\rangle - \langle\omega^a{}_f,e_d\rangle\langle\omega^f{}_b,e_c\rangle$$

$$= d\omega^a_{\ b}(e_c,e_d) + \omega^a{}_{db}\omega^f{}_{db} - \omega^a{}_{df}\omega^f{}_{cb}$$

As before, we use the coordinate independent expression for the exterior derivative of a form, which gives

$$d\omega^a_{\ b}(e_c, e_d) = e_c[\langle \omega^a_{\ b}, e_d\rangle] - e_d[\langle \omega^a_{\ b}, e_c\rangle] - \omega^a_{\ b}[e_c, e_d]$$

$$= e_c[\omega^a_{\ db}] - e_d[\omega^a_{\ cb}] - \langle \omega^a_{\ b}, c_{cd}^{\ \ f} e_f\rangle$$

$$e_c[\omega^a_{\ db}] - e_d[\omega^a_{\ cb}] - c_{ed}^{\ \ f}\,\omega^a_{\ fb}$$

Thus the L.H.S. of Cartan's second structure equation becomes

$$e_c[\omega^a_{\ db}] - e_d[\omega^a_{\ cb}] - c_{ed}^{\ \ f}\,\omega^a_{\ fb} + \omega^a_{\ db}\omega^f_{\ db} - \omega^a_{\ df}\omega^f_{\ cb}$$

$$= R^a_{\ bcd}$$

As required. The R.H.S. follows easily

$$R^a_{\ b}(e_c, e_d) = \frac{1}{2} R^a_{\ bgf}(e^f \wedge e^g)(e_c, e_d)$$

$$= \frac{1}{2} R^a_{\ bgf}[\langle e^f, e_c\rangle\langle e^g, e_d\rangle - \langle e^f, e_d\rangle\langle e^g, e_c\rangle]$$

$$= \frac{1}{2} R^a_{\ bgf}(\delta^f_c \delta^g_d - \delta^f_d \delta^g_c)$$

$$= R^a_{\ bcd}$$

Thus are the two equations verified.

Appendix B

The Gauss-Bonnet integral for a hyperkahler manifold

The purpose of this section is to demonstrate the different ways in which one may write down the Gauss-Bonnet formula for the (volume contribution) to the Euler character for a hyper-Kahler manifold. Usually we shall find it convenient to work in an orthonormal vielbein basis. The Gauss-Bonnet theorem states that the volume contribution to the Euler character of a $2n$-dimensional manifold $\mathcal{M}$ is given by

$$\chi = \frac{(-1)^n}{(4\pi)^n n!} \int \epsilon^{i_1 i_2 \ldots i_{2n}} R_{i_1 i_2} \wedge R_{i_3 i_4} \wedge \ldots \wedge R_{i_{2n-1} i_{2n}} \tag{B.1}$$

$$= \frac{(-1)^n}{(8\pi)^n n!} \int \epsilon^{i_1 i_2 \ldots i_{2n}} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} \ldots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}} \, \theta^{j_1} \wedge \theta^{j_2} \wedge \ldots \wedge \theta^{j_{2n}} \tag{B.2}$$

$$= \frac{(-1)^n}{(8\pi)^n n!} \int \epsilon^{i_1 i_2 \ldots i_{2n}} \epsilon^{j_1 j_2 \ldots j_{2n}} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} \ldots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}} \, \theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^{2n} \tag{B.3}$$

where the θ^j are the basis of non-coordinate one-forms. In a non-coordinate basis the invariant volume element is given simply as

$$dV = \theta^1 \wedge \theta^2 \wedge \ldots \wedge \theta^{2n} \tag{B.4}$$

We can express χ in terms of a coordinate basis by introducing the vielbien matrices $e_i{}^\mu$ which allow us to transform tensor components back and forth between the orthonormal non-coordinate basis and the coordinate basis with metric $g_{\mu\nu}$. The coordinate basis indices are μ, ν and those of the vielbein basis are i, j. The matrices $e_i{}^\mu$ have the following properties,

$$e_i{}^\mu e_j{}^\nu g_{\mu\nu} = \delta_{ij} \tag{B.5}$$

$$g_{\mu\nu} = e^i{}_\mu e^j{}_\nu \delta_{ij} \tag{B.6}$$

$$e^i{}_\mu e_j{}^\mu = \delta^i{}_j \tag{B.7}$$

We can find the components of an object in the coordinate basis from its components expressed in the non-coordinate basis as follows

$$V^\mu = V^i e_i{}^\mu \tag{B.8}$$

We can use this information to rewrite B.3 as follows

$$= \frac{(-1)^n}{(8\pi)^n n!} \int \epsilon^{i_1 i_2 \ldots i_{2n}} \epsilon^{j_1 j_2 \ldots j_{2n}} \delta_{i_1}{}^{k_1} \ldots \delta_{i_{2n}}{}^{k_{2n}} \delta_{j_1}{}^{l_1} \ldots \delta_{j_{2n}}{}^{l_{2n}} R_{k_1 k_2 l_1 l_2} R_{k_{2n-1} k_{2n} l_{2n-1} l_{2n}}\, dV$$

$$= \frac{(-1)^n}{(8\pi)^n n!} \int \epsilon^{i_1 i_2 \ldots i_{2n}} e_{i_1}{}^{\mu_1} \ldots e_{i_{2n}}{}^{\mu_{2n}} \epsilon^{j_1 j_2 \ldots j_{2n}} e_{j_1}{}^{\nu_1} \ldots e_{j_{2n}}{}^{\nu_{2n}}$$

$$R_{k_1 k_2 l_1 l_2} e^{k_1}{}_{\mu_1} e^{k_2}{}_{\mu_2} e^{l_1}{}_{\nu_1} e^{l_2}{}_{\nu_2} R_{k_{2n-1} k_{2n} l_{2n-1} l_{2n}} e^{k_{2n-1}}{}_{\mu_{2n-1}} e^{k_{2n}}{}_{\mu_{2n}} e^{l_{2n-1}}{}_{\nu_{2n-1}} e^{l_{2n}}{}_{\nu_{2n}}\, dV$$

$$= \frac{(-1)^n}{(8\pi)^n n!} \int \epsilon^{\mu_1 \mu_2 \ldots \mu_{2n}} (det(e^i{}_\mu))^2 \epsilon^{\nu_1 \nu_2 \ldots \nu_{2n}} R_{\mu_1 \mu_2 \nu_1 \nu_2} \ldots R_{\mu_{2n-1} \mu_{2n} \nu_{2n-1} \nu_{2n}}\, dV \tag{B.9}$$

We can evaluate the determinant of e in terms of the metric by taking the determinant of B.5

$$det(e^i{}_\mu)^2 = \frac{1}{det(g)} \tag{B.10}$$

Recalling that the invariant volume element expressed in terms of a coordinate basis is dV $= \prod_\mu \sqrt{det(g)} dX^\mu$, we have

$$\chi = \frac{(-1)^n}{(8\pi)^n n!} \int \frac{\prod_\mu dX^\mu}{\sqrt{det(g)}} \epsilon^{\mu_1 \mu_2 \ldots \mu_{2n}} \epsilon^{\nu_1 \nu_2 \ldots \nu_{2n}} R_{\mu_1 \mu_2 \nu_1 \nu_2} \ldots R_{\mu_{2n-1} \mu_{2n} \nu_{2n-1} \nu_{2n}} \tag{B.11}$$

On a hyper-Kahler manifold it is possible to split each index into a pair and express the curvature in terms of the so called symplectic curvature,

$$R_{abcd} \mapsto R_{(i\dot{\alpha})(j\dot{\beta})(k\dot{\gamma})(l\dot{\delta})} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}}R_{ijkl} \tag{B.12}$$

Naively we could just replace each index in the integrand above with an index pair. However, this would yield ϵ symbols with the unusual double index structure; $\epsilon^{(i_1\dot{\alpha}_1)(j_1\dot{\beta}_1)\dots(i_n\dot{\alpha}_n)(j_n\dot{\beta}_n)}$. How are we to interpret these objects? One method is to use a trick involving Grassmann quantities. Consider the integral

$$\int d\psi^n\dots d\psi^2 d\psi^1 \ \psi^{A_1}\psi^{A_2}\dots\psi^{A_n}$$

Clearly the result will yield a tensor like quantity involving the A_i's, call it $B^{A_1A_2\dots A_n}$. Furthermore, B should be totally antisymmetric in the A_i's since Grassmann quantities anti-commute. In fact, using the rules of Grassmann integration we can see that $B^{123\dots n} = 1$. These results suffice to fix B as it has the same properties as the Levi-Civitta tensor in n-dimensions, i.e. we have $B^{A_1A_2\dots A_n} = \epsilon^{A_1A_2\dots A_n}$. (We should note that there are at present no subtleties involving factors of the determinant of the metric when considering levi-civitta symbols with upper indices as we have chosen a vielbein basis in which the metric is Euclidean.) This result therefore allows us to express the totally antisymmetric tensor in terms of Grassmann integrations. It is this form that we will use to explore the meaning of the double index structure. Labeling our Grassmann quantities with two indices now gives (at least formally),

$$\epsilon^{i_1\alpha_1\dots i_n\alpha_n} = \int \prod_{i=1}^{n}\prod_{\alpha=1}^{2} d\psi^{i\alpha}\psi^{i_1\alpha_1}\dots\psi^{i_n\alpha_n}$$

For definiteness we will concentrate on the two cases, $n = 2$ and also that which is most relevant to us, namely $n = 4$, although the results we derive may be extended to cases involving ϵ's with arbitrary n. We illustrate the method first for $n = 2$. The entity we require is

$$\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon^{(i_1\alpha_1)(j_1\beta_1)(i_2\alpha_2)(j_2\beta_2)} = \int d\psi\, \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\psi^{i_1\alpha_1}\psi^{j_1\beta_1}\psi^{i_2\alpha_2}\psi^{j_2\beta_2}$$

$$= \int d\psi \,\left(\psi^{i_11}\psi^{j_12}\psi^{i_21}\psi^{j_22} + \psi^{i_11}\psi^{j_12}\psi^{j_21}\psi^{i_22} + \psi^{j_11}\psi^{i_12}\psi^{i_21}\psi^{j_22} + \psi^{j_11}\psi^{i_12}\psi^{j_21}\psi^{i_22}\right)$$

Our convention is to define the element $d\psi$ as $d\psi^{22}d\psi^{21}d\psi^{12}d\psi^{11}$. Notice that all ψ term with the same numerical index in the second place, must all differ in their first index entry. Thus we may write

$$\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\epsilon^{(i_1\alpha_1)(j_1\beta_1)(i_2\alpha_2)(j_2\beta_2)} = \int d\psi \,\left(\epsilon^{i_1i_2}\epsilon^{j_1j_2} + \epsilon^{i_1j_2}\epsilon^{j_1i_2} + \epsilon^{j_1i_2}\epsilon^{i_1j_2} + \epsilon^{j_1j_2}\epsilon^{i_1i_2}\right)\psi^{11}\psi^{12}\psi^{21}\psi^{2}$$

$$= 2\left(\epsilon^{i_1i_2}\epsilon^{j_1j_2} + \epsilon^{i_1j_2}\epsilon^{j_1i_2}\right)$$

Pursuing a similar method for the case $n = 4$ gives the result

$$\epsilon_{\alpha_1\beta_1}...\epsilon_{\alpha_4\beta_4}\epsilon^{(i_1\alpha_1)(j_1\beta_1)(i_2\alpha_2)(j_2\beta_2)(i_3\alpha_3)(j_3\beta_3)(i_4\alpha_4)(j_4\beta_4)}$$

$$= 2\left(\epsilon^{i_1i_2i_3i_4}\epsilon^{j_1j_2j_3j_4} + \epsilon^{i_1i_2i_3j_4}\epsilon^{j_1j_2j_3i_4} + \epsilon^{i_1i_2j_3i_4}\epsilon^{j_1j_2i_3j_4}\right.$$

$$\left.+\epsilon^{i_1i_2j_3j_4}\epsilon^{j_1j_2i_3i_4} + \epsilon^{i_1j_2i_3i_4}\epsilon^{j_1i_2j_3j_4} + \epsilon^{i_1j_2i_3j_4}\epsilon^{j_1i_2j_3i_4} + \epsilon^{i_1j_2j_3i_4}\epsilon^{j_1i_2j_3i_4} + \epsilon^{i_1j_2j_3j_4}\epsilon^{j_1i_2i_3i_4}\right)$$

Note that when we substitute this result into the expression for χ it will be contracted with the symplectic curvature, which is symmetric in i_a and j_a for $a = 1, ..., n$. Thus, upon this contraction, the above can be written simply as $16\epsilon^{i_1i_2i_3i_4}\epsilon^{j_1j_2j_3j_4}$. Our final result for $D = 4$ then becomes,

$$\chi(M) = \frac{1}{(2\pi)^4}\frac{1}{4!}\int dV\, \epsilon^{i_1i_2i_3i_4}\epsilon^{j_1j_2j_3j_4}\epsilon^{k_1k_2k_3k_4}\epsilon^{l_1l_2l_3l_4}R_{i_1j_1k_1l_1}...R_{i_4j_4k_4l_4} \qquad \text{(B.13)}$$

B.1 The Gauss-Bonnet integral re-written

The Gauss-Bonnet integral may be written in a convenient form involving integrations over Grassmann valued symplectic tangent vectors. We will require 4 types of these objects, labeled by the index A . We start with the integral

$$\int \prod_{i=1}^{d} \prod_{A=1}^{4} d\psi^{iA} \; e^{\frac{1}{48\pi}\epsilon_{ABCD} R_{ijkl} \psi^{iA} \psi^{jB} \psi^{kC} \psi^{lD}} \tag{B.14}$$

Where d is related to the dimension D of the manifold by $D = 2d$. Expanding the exponential in a power series yields

$$\int \prod_{i=1}^{d} \prod_{A=1}^{4} d\psi^{iA} \; \sum_n \frac{1}{n!} \left(\frac{1}{48\pi}\right)^n \left[\epsilon_{ABCD} R_{ijkl} \psi^{iA} \psi^{jB} \psi^{kC} \psi^{lD}\right]^n$$

To saturate the fermionic integrals we need only retain the term in the expansion which contributes exactly $4d$ Grassmann fields, i.e. we require $4n = 4d \Rightarrow n = d$, leaving us with

$$\int \prod_{i=1}^{d} \prod_{A=1}^{4} d\psi^{iA} \; \frac{1}{d!} \left(\frac{1}{48\pi}\right)^d \epsilon_{A_1B_1C_1D_1} \ldots \epsilon_{A_dB_dC_dD_d} R_{i_1j_1k_1l_1} \ldots R_{i_dj_dk_dl_d}$$
$$\psi^{i_1A_1} \psi^{j_1B_1} \psi^{k_1C_1} \psi^{l_1D_1} \ldots \psi^{i_dA_d} \psi^{j_dB_d} \psi^{k_dC_d} \psi^{l_dD_d} \tag{B.15}$$

Consider the quantity $\epsilon_{ABCD} \psi^{iA} \psi^{jB} \psi^{kC} \psi^{lD}$. There are 4! permutations of the indices ABCD. Since this is contracted with an entity which is symmetric in $ijkl$, a moments thought should convince the reader that, subject to such a symmetric contraction on these indices, we must have,

$$\epsilon_{ABCD} \psi^{iA} \psi^{jB} \psi^{kC} \psi^{lD} = 4! \psi^{i1} \psi^{j2} \psi^{k3} \psi^{l4}$$

Using this result we may write (subject to the symmetric contraction mentioned above)

$$\epsilon_{A_1B_1C_1D_1} \ldots \epsilon_{A_nB_nC_nD_n} \psi^{i_1A_1} \psi^{j_1B_1} \psi^{k_1C_1} \psi^{l_1D_1} \ldots \psi^{i_dA_d} \psi^{j_dB_d} \psi^{k_dC_d} \psi^{l_dD_d}$$

$$= (4!)^d \psi^{i_1 1}\psi^{j_1 2}\psi^{k_1 3}\psi^{l_1 4}...\psi^{i_d 1}\psi^{j_d 2}\psi^{k_d 3}\psi^{l_d 4}$$

$$= (4!)^d \left(\psi^{i_1 1}\psi^{i_2 1}...\psi^{i_d 1}\right)\left(\psi^{j_1 2}\psi^{j_2 2}...\psi^{j_d 2}\right)\left(\psi^{k_1 3}\psi^{k_2 3}...\psi^{k_d 3}\right)\left(\psi^{l_1 4}\psi^{l_2 4}...\psi^{l_d 4}\right)$$

$$= (4!)^d \epsilon^{i_1 i_2 ... i_d}\epsilon^{j_1 j_2 ... j_d}\epsilon^{k_1 k_2 ... k_d}\epsilon^{l_1 l_2 ... l_d}\psi^{11}\psi^{21}...\psi^{d1}\psi^{12}...\psi^{d2}...\psi^{d4}$$

Substituting this into B.15,

$$= \frac{1}{d!}\left(\frac{1}{2\pi}\right)^d \epsilon^{i_1 i_2 ... i_d}\epsilon^{j_1 j_2 ... j_d}\epsilon^{k_1 k_2 ... k_d}\epsilon^{l_1 l_2 ... l_d} R_{i_1 j_1 k_1 l_1}...R_{i_d j_d k_d l_d}$$

Which is precisely the Euler density, giving the Gauss-Bonnet expression for the Euler character when integrated over the manifold. Using this result we may write the Gauss-Bonnet integral for a hyper-Kahler manifold $\mathcal{M}$ as

$$\chi(M) = \int d(Vol_M) \int \prod_{i=1}^{d}\prod_{A=1}^{4} d\psi^{iA}\; e^{\frac{1}{48\pi}\epsilon_{ABCD}R_{ijkl}\psi^{iA}\psi^{jB}\psi^{kC}\psi^{lD}} \tag{B.16}$$

As an aside, note what would happen had we started with a different numerical coefficient in the exponent. Following through the analysis above, but with an arbitrary numerical coefficient A gives;

$$\int d(Vol_M) \int \prod_{i=1}^{d}\prod_{A=1}^{4} d\psi^{iA}\; e^{\frac{A}{48\pi}\epsilon_{ABCD}R_{ijkl}\psi^{iA}\psi^{jB}\psi^{kC}\psi^{lD}} = (A)^d \chi \tag{B.17}$$

Appendix C

Calculation of the spin connection

Recall the basis 1-forms of our metric,

$$e^0 = dt,\ e^1 = a\sigma_1,\ e^2 = a\sigma_2,\ e^3 = b\Sigma_1,\ e^4 = b\Sigma_2,\ e^5 = c\nu_1,\ e^6 = c\nu_2,\ e^7 = f\lambda,$$

One can use Cartan's first equation of structure to calculate the spin connection given that the torsion tensor vanishes. One proceeds by writing out the spin connection with undetermined coefficients and using Cartan's equation to give a set of simultaneous equations which will fix these coefficients. The spin connection is a matrix-valued 1-form which we shall write in the following manner:

$$\omega^i{}_j = A^i{}_j dt + B^i{}_j\sigma_1 + C^i{}_j\sigma_2 + D^i{}_j\Sigma_1 + E^i{}_j\Sigma_2 + F^i{}_j\nu_1 + G^i{}_j\nu_2 + H^i{}_j\lambda + J^i{}_jQ \quad \text{(C.1)}$$

where $i = 0, 1, ..., 8$. Note the inclusion of the one-form Q in the spin connection. This is because the exterior derivatives of some of the e^i's include Q, so Q must necessarily appear in the spin connection. Due to the antisymmetry of the spin-connection we must have $\omega^i{}_i = 0$, (no summation over i). Thus we have,

$$A^0{}_0 = A^1{}_1 = A^2{}_2 = A^3{}_3 = A^4{}_4 = A^5{}_5 = A^6{}_6 = A^7{}_7 = A^8{}_8 = 0 \quad \text{(C.2)}$$

$$B^0{}_0 = B^1{}_1 = B^2{}_2 = B^3{}_3 = B^4{}_4 = B^5{}_5 = B^6{}_6 = B^7{}_7 = B^8{}_8 = 0 \quad \text{(C.3)}$$

$$C^0{}_0 = C^1{}_1 = C^2{}_2 = C^3{}_3 = C^4{}_4 = C^5{}_5 = C^6{}_6 = C^7{}_7 = C^8{}_8 = 0 \qquad (C.4)$$

$$D^0{}_0 = D^1{}_1 = D^2{}_2 = D^3{}_3 = D^4{}_4 = D^5{}_5 = D^6{}_6 = D^7{}_7 = D^8{}_8 = 0 \qquad (C.5)$$

$$E^0{}_0 = E^1{}_1 = E^2{}_2 = E^3{}_3 = E^4{}_4 = E^5{}_5 = E^6{}_6 = E^7{}_7 = E^8{}_8 = 0 \qquad (C.6)$$

$$F^0{}_0 = F^1{}_1 = F^2{}_2 = F^3{}_3 = F^4{}_4 = F^5{}_5 = F^6{}_6 = F^7{}_7 = F^8{}_8 = 0 \qquad (C.7)$$

$$G^0{}_0 = G^1{}_1 = G^2{}_2 = G^3{}_3 = G^4{}_4 = G^5{}_5 = G^6{}_6 = G^7{}_7 = G^8{}_8 = 0 \qquad (C.8)$$

$$H^0{}_0 = H^1{}_1 = H^2{}_2 = H^3{}_3 = H^4{}_4 = H^5{}_5 = H^6{}_6 = H^7{}_7 = H^8{}_8 = 0 \qquad (C.9)$$

$$J^0{}_0 = J^1{}_1 = J^2{}_2 = J^3{}_3 = J^4{}_4 = J^5{}_5 = J^6{}_6 = J^7{}_7 = J^8{}_8 = 0 \qquad (C.10)$$

C.1 1^{st} Cartan equation, $(i = 0)$

Since $e^0 = d^2t = 0$ we have, for the first of these expressions,

$$\omega^0{}_j \wedge e^j = 0 \qquad (C.11)$$

$$\Rightarrow \omega^0{}_1 \wedge a\sigma_1 + \omega^0{}_2 \wedge a\sigma_2 + \omega^0{}_3 \wedge d\Sigma_1 + \omega^0{}_4 \wedge b\Sigma_2 + \omega^0{}_5 \wedge c\nu_1 + \omega^0{}_6 \wedge c\nu_2 + \omega^0{}_7 \wedge f\lambda + \omega^0{}_8 \wedge Q = 0 \qquad (C.12)$$

Substituting in from equation (C.1) gives

$$0 = a(A^0{}_1 dt + B^0{}_1\sigma_1 + C^0{}_1\sigma_2 + D^0{}_1\Sigma_1 + E^0{}_1\Sigma_2 + F^0{}_1\nu_1 + G^0{}_1\nu_2 + H^0{}_1\lambda + J^0{}_1 Q) \wedge \sigma_1$$

$$+a(A^0{}_2 dt + B^0{}_2\sigma_1 + C^0{}_2\sigma_2 + D^0{}_2\Sigma_1 + E^0{}_2\Sigma_2 + F^0{}_2\nu_1 + G^0{}_2\nu_2 + H^0{}_2\lambda + J^0{}_2 Q) \wedge \sigma_2$$

$$+b(A^0{}_3 dt + +B^0{}_3\sigma_1 + C^0{}_3\sigma_2 + D^0{}_3\Sigma_1 + E^0{}_3\Sigma_2 + F^0{}_3\nu_1 + G^0{}_3\nu_2 + H^0{}_3\lambda + J^0{}_3 Q) \wedge \Sigma_1$$

$$+b(A^0{}_4 dt + +B^0{}_4\sigma_1 + C^0{}_4\sigma_2 + D^0{}_4\Sigma_1 + E^0{}_4\Sigma_2 + F^0{}_4\nu_1 + G^0{}_4\nu_2 + H^0{}_4\lambda + J^0{}_4 Q) \wedge \Sigma_2$$

$$+c(A^0{}_5 dt + +B^0{}_5\sigma_1 + C^0{}_5\sigma_2 + D^0{}_5\Sigma_1 + E^0{}_5\Sigma_2 + F^0{}_5\nu_1 + G^0{}_5\nu_2 + H^0{}_5\lambda + J^0{}_5 Q) \wedge \nu_1$$

$$+c(A^0{}_6 dt + +B^0{}_6\sigma_1 + C^0{}_6\sigma_2 + D^0{}_6\Sigma_1 + E^0{}_6\Sigma_2 + F^0{}_6\nu_1 + G^0{}_6\nu_2 + H^0{}_6\lambda + J^0{}_6 Q) \wedge \nu_2$$

$$+f(A^0{}_7dt + +B^0{}_7\sigma_1 + C^0{}_7\sigma_2 + D^0{}_7\Sigma_1 + E^0{}_7\Sigma_2 + F^0{}_7\nu_1 + G^0{}_7\nu_2 + H^0{}_7\lambda + J^0{}_7Q) \wedge \lambda$$

$$+(A^0{}_8dt + +B^0{}_8\sigma_1 + C^0{}_8\sigma_2 + D^0{}_8\Sigma_1 + E^0{}_8\Sigma_2 + F^0{}_8\nu_1 + G^0{}_8\nu_2 + H^0{}_8\lambda + J^0{}_8Q) \wedge gQ$$

Setting the coefficients of each of the 2-forms to zero yields the following set of equations.

$$A^0{}_i = 0 \ i = 1, .2, ..., 7 \quad \text{(C.13)}$$

$$A^0{}_8 = 0 \quad \text{(C.14)}$$

$$B^0{}_2 = C^0{}_1 \quad \text{(C.15)}$$

$$bB^0{}_3 = aD^0{}_1 \quad \text{(C.16)}$$

$$bB^0{}_4 = aE^0{}_1 \quad \text{(C.17)}$$

$$cB^0{}_5 = aF^0{}_1 \quad \text{(C.18)}$$

$$cB^0{}_6 = aG^0{}_1 \quad \text{(C.19)}$$

$$fB^0{}_7 = aH^0{}_1 \quad \text{(C.20)}$$

$$bC^0{}_3 = aD^0{}_2 \quad \text{(C.21)}$$

$$bC^0{}_4 = aE^0{}_2 \quad \text{(C.22)}$$

$$cC^0{}_5 = aF^0{}_2 \quad \text{(C.23)}$$

$$cC^0{}_6 = aG^0{}_2 \quad \text{(C.24)}$$

$$fC^0{}_7 = aH^0{}_2 \quad \text{(C.25)}$$

$$D^0{}_4 = E^0{}_3 \quad \text{(C.26)}$$

$$cD^0{}_5 = bF^0{}_3 \quad \text{(C.27)}$$

$$cD^0{}_6 = bG^0{}_3 \quad \text{(C.28)}$$

$$fD^0{}_7 = bH^0{}_3 \tag{C.29}$$

$$cE^0{}_5 = bF^0{}_4 \tag{C.30}$$

$$cE^0{}_6 = bG^0{}_4 \tag{C.31}$$

$$fE^0{}_7 = bH^0{}_4 \tag{C.32}$$

$$F^0{}_6 = G^0{}_5 \tag{C.33}$$

$$fF^0{}_7 = cH^0{}_5 \tag{C.34}$$

$$fG^0{}_7 = cH^0{}_6 \tag{C.35}$$

C.2 2^{nd} Cartan equation ($i = 1$)

Passing now to the second torsion-free equation and recalling previous expressions for the exterior derivatives, we have

$$de^1 + \omega^1{}_i \wedge e^i + \omega^1{}_8 \wedge Q = 0 \tag{C.36}$$

where

$$de^1 = da \wedge \sigma_1 + ad\sigma_1 = \frac{\partial a}{\partial t} dt \wedge \sigma_1 + a\left(-\frac{1}{2}\lambda \wedge \sigma_2 - \nu_1 \wedge \Sigma_2 - \nu_2 \wedge \Sigma_1 - \frac{3}{2}Q \wedge \sigma_2\right) \tag{C.37}$$

Therefore

$$0 = \frac{\partial a}{\partial t} dt \wedge \sigma_1 + a\left(-\frac{1}{2}\lambda \wedge \sigma_2 - \nu_1 \wedge \Sigma_2 - \nu_2 \wedge \Sigma_1 - \frac{3}{2}Q \wedge \sigma_2\right)$$

$$+(A^1{}_0 dt + B^1{}_0 \sigma_1 C^1{}_0 \sigma_2 + D^1{}_0 \Sigma_1 + E^1{}_0 \Sigma_2 + F^1{}_0 \nu_1 + G^1{}_0 \nu_2 + H^1{}_0 \lambda + J^1{}_0 Q) \wedge dt$$

$$+a(A^1{}_2 dt + B^1{}_2 \sigma_1 + C^1{}_2 \sigma_2 + D^1{}_2 \Sigma_1 + E^1{}_2 \Sigma_2 + F^1{}_2 \nu_1 + G^1{}_2 \nu_2 + H^1{}_2 \lambda + J^1{}_2 Q) \wedge \sigma_2$$

$$+b(A^1{}_3 dt + B^1{}_3 \sigma_1 + C^1{}_3 \sigma_2 + D^1{}_3 \Sigma_1 + E^1{}_3 \Sigma_2 + F^1{}_3 \nu_1 + G^1{}_3 \nu_2 + H^1{}_3 \lambda + J^1{}_3 Q) \wedge \Sigma_1$$

$$+b({A^1}_4 dt + {B^1}_4\sigma_1 + {C^1}_4\sigma_2 + {D^1}_4\Sigma_1 + {E^1}_4\Sigma_2 + {F^1}_4\nu_1 + {G^1}_4\nu_2 + {H^1}_4\lambda + {J^1}_4 Q) \wedge \Sigma_2$$

$$+c({A^1}_5 dt + {B^1}_5\sigma_1 + {C^1}_5\sigma_2 + {D^1}_5\Sigma_1 + {E^1}_5\Sigma_2 + {F^1}_5\nu_1 + {G^1}_5\nu_2 + {H^1}_5\lambda + {J^1}_5 Q) \wedge \nu_1$$

$$+c({A^1}_6 dt + {B^1}_6\sigma_1 + {C^1}_6\sigma_2 + {D^1}_6\Sigma_1 + {E^1}_6\Sigma_2 + {F^1}_6\nu_1 + {G^1}_6\nu_2 + {H^1}_6\lambda + {J^1}_6 Q) \wedge \nu_2$$

$$+f({A^1}_7 dt + {B^1}_7\sigma_1 + {C^1}_7\sigma_2 + {D^1}_7\Sigma_1 + {E^1}_7\Sigma_2 + {F^1}_7\nu_1 + {G^1}_7\nu_2 + {H^1}_7\lambda + {J^1}_7 Q) \wedge \lambda$$

$$+({A^1}_8 dt + {B^1}_8\sigma_1 + {C^1}_8\sigma_2 + {D^1}_8\Sigma_1 + {E^1}_8\Sigma_2 + {F^1}_8\nu_1 + {G^1}_8\nu_2 + {H^1}_8\lambda + {J^1}_8 Q) \wedge gQ$$

Proceeding as before, the exhaustive list of conditions following from the above is as follows,

$$a{A^1}_2 = {C^0}_1 \tag{C.38}$$

$$b{A^1}_3 = D1_0 \tag{C.39}$$

$$b{A^1}_4 = {E^1}_0 \tag{C.40}$$

$$c{A^1}_5 = {F^1}_0 \tag{C.41}$$

$$c{A^1}_6 = {G^1}_0 \tag{C.42}$$

$$f{A^1}_7 = {H^1}_0 \tag{C.43}$$

$${B^1}_0 = \frac{\partial a}{\partial t} \tag{C.44}$$

$${B^1}_i = 0 \quad i = 1, ..., 7 \tag{C.45}$$

$$b{C^1}_3 = a{D^1}_2 \tag{C.46}$$

$$b{C^1}_4 = a{E^1}_2 \tag{C.47}$$

$$c{C^1}_5 = a{F^1}_2 \tag{C.48}$$

$$c{C^1}_6 = a{G^1}_2 \tag{C.49}$$

$$f{C^1}_7 = a{H^1}_2 - \frac{a}{2} \tag{C.50}$$

$$D^1{}_4 = E^1{}_3 \tag{C.51}$$

$$cD^1{}_5 = bF^1{}_3 \tag{C.52}$$

$$cD^1{}_6 = bG^1{}_3 - a \tag{C.53}$$

$$fD^1{}_7 = bH^1{}_3 \tag{C.54}$$

$$cE^1{}_5 = bF^1{}_4 - a \tag{C.55}$$

$$cE^1{}_6 = bG^1{}_4 \tag{C.56}$$

$$fE^1{}_7 = bH^1{}_4 \tag{C.57}$$

$$F^1{}_6 = G^1{}_5 \tag{C.58}$$

$$fF^1{}_7 = cH^1{}_5 \tag{C.59}$$

$$fG^1{}_7 = cH^1{}_5 \tag{C.60}$$

C.3 3^{rd} Cartan equation ($i = 2$)

$$de^2 = \omega^2{}_a e^a = 0 \tag{C.61}$$

Where

$$de^2 = d(a\sigma_2) = \frac{\partial a}{\partial t} dt \wedge \sigma_2 + a\left(\frac{1}{2}\lambda \wedge \sigma_1 + \nu_1 \wedge \Sigma_1 - \nu_2 \wedge \Sigma_2\right) \tag{C.62}$$

Therefore

$$0 = \frac{\partial a}{\partial t} dt \wedge \sigma_2 + a\left(\frac{1}{2}\lambda \wedge \sigma_1 + \nu_1 \wedge \Sigma_1 - \nu_2\right) \wedge \Sigma_2 \tag{C.63}$$

$$+(A^2{}_0 dt + B^2{}_0\sigma_1 + C^2{}_0\sigma_2 + D^2{}_0\Sigma_1 + E^2{}_0\Sigma_2 + F^2{}_0\nu_1 + G^2{}_0\nu_2 + H^2{}_0\lambda + J^2{}_0 Q) \wedge dt$$

$$+a(A^2{}_1 dt + B^2{}_1\sigma_1 + C^2{}_1\sigma_2 + D^2{}_1\Sigma_1 + E^2{}_1\Sigma_2 + F^2{}_1\nu_1 + G^2{}_1\nu_2 + H^2{}_1\lambda + J^2{}_1 Q) \wedge \sigma_1$$

$$+b(A^2{}_3 dt + B^2{}_3\sigma_1 + C^2{}_3\sigma_2 + D^2{}_3\Sigma_1 + E^2{}_3\Sigma_2 + F^2{}_3\nu_1 + G^2{}_3\nu_2 + H^2{}_3\lambda + J^2{}_3 Q) \wedge \Sigma_1$$

$$+b(A^2{}_4dt + B^2{}_4\sigma_1 + C^2{}_4\sigma_2 + D^2{}_4\Sigma_1 + E^2{}_4\Sigma_2 + F^2{}_4\nu_1 + G^2{}_4\nu_2 + H^2{}_4\lambda + J^2{}_4Q) \wedge \Sigma_2$$

$$+c(A^2{}_5dt + B^2{}_5\sigma_1 + C^2{}_5\sigma_2 + D^2{}_5\Sigma_1 + E^2{}_5\Sigma_2 + F^2{}_5\nu_1 + G^2{}_5\nu_2 + H^2{}_5\lambda + J^2{}_5Q) \wedge \nu_1$$

$$+c(A^2{}_6dt + B^2{}_6\sigma_1 + C^2{}_6\sigma_2 + D^2{}_6\Sigma_1 + E^2{}_6\Sigma_2 + F^2{}_6\nu_1 + G^2{}_6\nu_2 + H^2{}_6\lambda + J^2{}_6Q) \wedge \nu_2$$

$$+f(A^2{}_7dt + B^2{}_7\sigma_1 + C^2{}_7\sigma_2 + D^2{}_7\Sigma_1 + E^2{}_7\Sigma_2 + F^2{}_7\nu_1 + G^2{}_7\nu_2 + H^2{}_7\lambda + J^2{}_7Q) \wedge \lambda$$

$$+(A^2{}_8dt + B^2{}_8\sigma_1 + C^2{}_8\sigma_2 + D^2{}_8\Sigma_1 + E^2{}_8\Sigma_2 + F^2{}_8\nu_1 + G^2{}_8\nu_2 + H^2{}_8\lambda + J^2{}_8Q) \wedge gQ$$

Comparing coefficients,

$$aA^2{}_1 = B^2{}_0 \tag{C.64}$$

$$bA^2{}_3 = D^2{}_0 \tag{C.65}$$

$$bA^2{}_4 = E^2{}_0 \tag{C.66}$$

$$cA^2{}_5 = F^2{}_0 \tag{C.67}$$

$$cA^2{}_6 = G^2{}_0 \tag{C.68}$$

$$fA^2{}_7 = H^2{}_0 \tag{C.69}$$

$$bB^2{}_3 = aD^2{}_1 \tag{C.70}$$

$$bB^2{}_4 = aE^2{}_1 \tag{C.71}$$

$$cB^2{}_5 = aF^2{}_1 \tag{C.72}$$

$$cB^2{}_6 = aG^2{}_1 \tag{C.73}$$

$$fB^2{}_7 = aH^2{}_1 + \frac{a}{2} \tag{C.74}$$

$$C^2{}_0 = \frac{\partial a}{\partial t} \tag{C.75}$$

$$C^2{}_i = 0 \quad i = 1, ..., 8 \tag{C.76}$$

$$D^2{}_4 = E^2{}_3 \tag{C.77}$$

$$cD^2{}_5 = bF^2{}_3 + a \tag{C.78}$$

$$cD^2{}_6 = bG^2{}_3 \tag{C.79}$$

$$fD^2{}_7 = bH^2{}_3 \tag{C.80}$$

$$cE^2{}_5 = bF^2{}_4 \tag{C.81}$$

$$cE^2{}_6 = bG^2{}_4 - a \tag{C.82}$$

$$fE^2{}_7 = bH^2{}_4 \tag{C.83}$$

$$F^2{}_6 = G^2{}_5 \tag{C.84}$$

$$fF^2{}_7 = cH^2{}_5 \tag{C.85}$$

$$fG^2{}_7 = cH^2{}_6 \tag{C.86}$$

C.4 4^{th} Cartan equation ($i = 3$)

$$de^3 + \omega^3{}_a e^a = 0 \tag{C.87}$$

Where

$$de^3 = d(b\Sigma_1) = \frac{\partial b}{\partial t} dt \wedge \Sigma_1 + bd\Sigma_1$$
$$= \frac{\partial b}{\partial t} dt \wedge \Sigma_1 + b\left(\frac{1}{2}\lambda \wedge \Sigma_2 - \nu_1 \wedge \sigma_2 + \nu_2 \wedge \sigma_1\right)$$

Therefore

$$0 = \frac{\partial b}{\partial t} dt \wedge \Sigma_1 + b\left(\frac{1}{2}\lambda \wedge \Sigma_2 - \nu_1 \wedge \sigma_2 + \nu_2 \wedge \sigma_1\right)$$
$$+(A^3{}_0 dt + B^3{}_0\sigma_1 + C^3{}_0\sigma_2 + D^3{}_0\Sigma_1 + E^3{}_0\Sigma_2 + F^1{}_0\nu_1 + G^1{}_0\nu_2 + H^1{}_0\lambda + J^1{}_0 Q) \wedge dt$$
$$+b(A^3{}_1 dt + B^3{}_1\sigma_1 + C^3{}_1\sigma_2 + D^3{}_1\Sigma_1 + E^3{}_1\Sigma_2 + F^3{}_1\nu_1 + G^3{}_1\nu_2 + H^3{}_1\lambda + J^3{}_1 Q) \wedge \Sigma_1$$
$$+a(A^3{}_2 dt + B^3{}_2\sigma_1 + C^3{}_2\sigma_2 + D^3{}_2\Sigma_1 + E^3{}_2\Sigma_2 + F^3{}_2\nu_1 + G^3{}_2\nu_2 + H^3{}_2\lambda + J^3{}_2 Q) \wedge \sigma_2$$
$$+b(A^3{}_4 dt + B^3{}_4\sigma_1 + C^3{}_4\sigma_2 + D^3{}_4\Sigma_1 + E^3{}_4\Sigma_2 + F^3{}_4\nu_1 + G^3{}_4\nu_2 + H^3{}_4\lambda + J^3{}_4 Q) \wedge \Sigma_2$$

$$+c({A^3}_5 dt + {B^3}_5\sigma_1 + {C^3}_5\sigma_2 + {D^3}_5\Sigma_1 + {E^3}_5\Sigma_2 + {F^3}_5\nu_1 + {G^3}_5\nu_2 + {H^3}_5\lambda + {J^3}_5 Q) \wedge \nu_1$$

$$+c({A^3}_6 dt + {B^3}_6\sigma_1 + {C^3}_6\sigma_2 + {D^3}_6\Sigma_1 + {E^3}_6\Sigma_2 + {F^3}_6\nu_1 + {G^3}_6\nu_2 + {H^3}_6\lambda + {J^3}_6 Q) \wedge \nu_2$$

$$+f({A^3}_7 dt + {B^3}_7\sigma_1 + {C^3}_7\sigma_2 + {D^3}_7\Sigma_1 + {E^3}_7\Sigma_2 + {F^3}_7\nu_1 + {G^3}_7\nu_2 + {H^3}_7\lambda + {J^3}_7 Q) \wedge \lambda$$

$$+({A^3}_8 dt + {B^3}_8\sigma_1 + {C^3}_8\sigma_2 + {D^3}_8\Sigma_1 + {E^3}_8\Sigma_2 + {F^3}_8\nu_1 + {G^3}_8\nu_2 + {H^3}_8\lambda + {J^3}_8 Q) \wedge gQ$$

Therefore

$$a{A^3}_1 = {B^3}_0 \tag{C.88}$$

$$a{A^3}_2 = {C^3}_0 \tag{C.89}$$

$$b{A^3}_4 = {E^3}_0 \tag{C.90}$$

$$c{A^3}_5 = {F^3}_0 \tag{C.91}$$

$$c{A^3}_6 = {G^3}_0 \tag{C.92}$$

$$f{A^3}_7 = {H^3}_0 \tag{C.93}$$

$${B^3}_2 = {C^3}_1 \tag{C.94}$$

$$b{B^3}_4 = a{E^3}_1 \tag{C.95}$$

$$c{B^3}_5 = a{F^3}_1 \tag{C.96}$$

$$c{B^3}_6 = a{G^3}_1 + b \tag{C.97}$$

$$f{B^3}_7 = a{H^3}_1 \tag{C.98}$$

$$b{C^3}_4 = a{E^3}_2 \tag{C.99}$$

$$c{C^3}_5 = a{F^3}_2 - b \tag{C.100}$$

$$c{C^3}_6 = a{G^3}_2 \tag{C.101}$$

$$f{C^3}_7 = a{H^3}_2 \tag{C.102}$$

$$D^3{}_i = 0 \quad i = 1,2,3,4,5,6,7,8 \tag{C.103}$$

$$D^3{}_0 = \frac{\partial b}{\partial t} \tag{C.104}$$

$$cE^3{}_5 = bF^3{}_4 \tag{C.105}$$

$$cE^3{}_6 = bG^3{}_4 \tag{C.106}$$

$$fE^3{}_7 = bH^3{}_4 + \frac{b}{2} \tag{C.107}$$

$$F^3{}_6 = G^3{}_5 \tag{C.108}$$

$$fF^3{}_7 = cH^3{}_5 \tag{C.109}$$

$$fG^3{}_7 = cH^3{}_6 \tag{C.110}$$

C.5 5^{th} Cartan equation $(i = 4)$

$$de^4 + \omega^4{}_a e^a = 0 \tag{C.111}$$

Where

$$de^4 = d(b\Sigma_2) = \frac{\partial b}{\partial t} dt \wedge \Sigma_2 + bd\Sigma_2 \tag{C.112}$$

$$= \frac{\partial b}{\partial t} dt \wedge \Sigma_2 + b\left(-\frac{1}{2}\lambda \wedge \Sigma_1 + \nu_1 \wedge \sigma_1 + \nu_2 \wedge \sigma_2\right) \tag{C.113}$$

Therefore,

$$0 = \frac{\partial b}{\partial t} dt \wedge \Sigma_2 + b\left(-\frac{1}{2}\lambda \wedge \Sigma_1 + \nu_1 \wedge \sigma_1 + \nu_2 \wedge \sigma_2\right)$$

$$+(A^4{}_0 dt + B^4{}_0\sigma_1 + C^4{}_0\sigma_2 + D^4{}_0\Sigma_1 + E^4{}_0\Sigma_2 + F^4{}_0\nu_1 + G^4{}_0\nu_2 + H^4{}_0\lambda + J^4{}_0 Q) \wedge dt$$

$$+a(A^4{}_1 dt + B^4{}_1\sigma_1 + C^4{}_1\sigma_2 + D^4{}_1\Sigma_1 + E^4{}_1\Sigma_2 + F^4{}_1\nu_1 + G^4{}_1\nu_2 + H^4{}_1\lambda + J^3{}_1 Q) \wedge \sigma_1$$

$$+a(A^4{}_2 dt + B^4{}_2\sigma_1 + C^4{}_2\sigma_2 + D^4{}_2\Sigma_1 + E^4{}_2\Sigma_2 + F^4{}_2\nu_1 + G^4{}_2\nu_2 + H^4{}_2\lambda + J^3{}_2 Q) \wedge \sigma_2$$

$$+b(A^4{}_3 dt + B^4{}_3\sigma_1 + C^4{}_3\sigma_2 + D^4{}_3\Sigma_1 + E^4{}_3\Sigma_2 + F^4{}_3\nu_1 + G^4{}_3\nu_2 + H^4{}_3\lambda + J^4{}_3 Q) \wedge \Sigma_1$$

$$+c(A^4{}_5 dt + B^4{}_5\sigma_1 + C^4{}_5\sigma_2 + D^4{}_5\Sigma_1 + E^4{}_5\Sigma_2 + F^4{}_5\nu_1 + G^4{}_5\nu_2 + H^4{}_5\lambda + J^4{}_5 Q) \wedge \nu_1$$

$$+c(A^4{}_6dt + B^4{}_6\sigma_1 + C^4{}_6\sigma_2 + D^4{}_6\Sigma_1 + E^4{}_6\Sigma_2 + F^4{}_6\nu_1 + G^4{}_6\nu_2 + H^4{}_6\lambda + J^4{}_6Q) \wedge \nu_2$$

$$+f(A^4{}_7dt + B^4{}_7\sigma_1 + C^4{}_7\sigma_2 + D^4{}_7\Sigma_1 + E^4{}_7\Sigma_2 + F^4{}_7\nu_1 + G^4{}_7\nu_2 + H^4{}_7\lambda + J^4{}_7Q) \wedge \lambda$$

$$+(A^4{}_8dt + B^4{}_8\sigma_1 + C^4{}_8\sigma_2 + D^4{}_8\Sigma_1 + E^4{}_8\Sigma_2 + F^4{}_8\nu_1 + G^4{}_8\nu_2 + H^4{}_8\lambda + J^4{}_8Q) \wedge gQ$$

Therefore, comparing coefficients gives,

$$aA^4{}_1 = B^4{}_0 \tag{C.114}$$

$$aA^4{}_2 = C^4{}_0 \tag{C.115}$$

$$bA^4{}_3 = D^4{}_0 \tag{C.116}$$

$$cA^4{}_5 = F^4{}_0 \tag{C.117}$$

$$cA^4{}_6 = G^4{}_0 \tag{C.118}$$

$$fA^4{}_7 = H^4{}_0 \tag{C.119}$$

$$B^4{}_2 = C^4{}_1 \tag{C.120}$$

$$bB^4{}_3 = aD^4{}_1 \tag{C.121}$$

$$cB^4{}_5 = aF^4{}_1 + b \tag{C.122}$$

$$cB^4{}_6 = aG^4{}_1 \tag{C.123}$$

$$fB^4{}_7 = aH^4{}_1 \tag{C.124}$$

$$bC^4{}_3 = aD^4{}_2 \tag{C.125}$$

$$cC^4{}_5 = aF^4{}_2 \tag{C.126}$$

$$cC^4{}_6 = aG^4{}_2 + b \tag{C.127}$$

$$fC^4{}_7 = aH^4{}_2 \tag{C.128}$$

$$cD^4{}_5 = bF^4{}_3 \tag{C.129}$$

$$cD^4{}_6 = bG^4{}_3 \tag{C.130}$$

$$fD^4{}_7 = bH^4{}_3 - \frac{b}{2} \tag{C.131}$$

$$E^4{}_0 = \frac{\partial b}{\partial t} \tag{C.132}$$

$$E^4{}_i = 0 \quad i = 1, 2, ..., 8 \tag{C.133}$$

$$F^4{}_6 = G^4{}_5 \tag{C.134}$$

$$fF^4{}_7 = cH^4{}_5 \tag{C.135}$$

$$fG^4{}_7 = cH^4{}_6 \tag{C.136}$$

C.6 6^{th} Cartan equation ($i = 5$)

$$de^5 + \omega^5{}_a e^a = 0 \tag{C.137}$$

Where

$$de^5 = d(c\nu_1) = \frac{\partial c}{\partial t} dt \wedge \nu_1 + c d\nu_1 \tag{C.138}$$

$$= \frac{\partial c}{\partial t} dt \wedge \nu_1 + c\left(-\lambda \wedge \nu_2 - \sigma_2 \wedge \Sigma_1 + \sigma_1 \wedge \Sigma_2\right) \tag{C.139}$$

Therefore

$$0 = \frac{\partial c}{\partial t} dt \wedge \nu_1 + c\left(-\lambda \wedge \nu_2 - \sigma_2 \wedge \Sigma_1 + \sigma_1 \wedge \Sigma_2\right) \tag{C.140}$$

$$+(A^5{}_0 dt + B^5{}_0 \sigma_1 + C^5{}_0 \sigma_2 + D^5{}_0 \Sigma_1 + E^5{}_0 \Sigma_2 + F^5{}_0 \nu_1 + G^5{}_0 \nu_2 + H^5{}_0 \lambda + J^5{}_0 Q) \wedge dt$$

$$+a(A^5{}_1 dt + B^5{}_1 \sigma_1 + C^5{}_1 \sigma_2 + D^5{}_1 \Sigma_1 + E^5{}_1 \Sigma_2 + F^5{}_1 \nu_1 + G^5{}_1 \nu_2 + H^5{}_1 \lambda + J^5{}_1 Q) \wedge \sigma_1$$

$$+a(A^5{}_2 dt + B^5{}_2 \sigma_1 + C^5{}_2 \sigma_2 + D^5{}_2 \Sigma_1 + E^5{}_2 \Sigma_2 + F^5{}_2 \nu_1 + G^5{}_2 \nu_2 + H^5{}_2 \lambda + J^5{}_2 Q) \wedge \sigma_2$$

$$+b(A^5{}_3 dt + B^5{}_3 \sigma_1 + C^5{}_3 \sigma_2 + D^5{}_3 \Sigma_1 + E^5{}_3 \Sigma_2 + F^5{}_3 \nu_1 + G^5{}_3 \nu_2 + H^5{}_3 \lambda + J^5{}_3 Q) \wedge \Sigma_1$$

$$+b(A^5{}_4dt + B^5{}_4\sigma_1 + C^5{}_4\sigma_2 + D^5{}_4\Sigma_1 + E^5{}_4\Sigma_2 + F^5{}_4\nu_1 + G^5{}_4\nu_2 + H^5{}_4\lambda + J^5{}_4Q) \wedge \Sigma_2$$

$$+c(A^5{}_6dt + B^5{}_6\sigma_1 + C^5{}_6\sigma_2 + D^5{}_6\Sigma_1 + E^5{}_6\Sigma_2 + F^5{}_6\nu_1 + G^5{}_6\nu_2 + H^5{}_6\lambda + J^5{}_6Q) \wedge \nu_2$$

$$+f(A^5{}_7dt + B^5{}_7\sigma_1 + C^5{}_7\sigma_2 + D^5{}_7\Sigma_1 + E^5{}_7\Sigma_2 + F^5{}_7\nu_1 + G^5{}_7\nu_2 + H^5{}_7\lambda + J^5{}_7Q) \wedge \lambda$$

$$+(A^5{}_8dt + B^5{}_8\sigma_1 + C^5{}_8\sigma_2 + D^5{}_8\Sigma_1 + E^5{}_8\Sigma_2 + F^5{}_8\nu_1 + G^5{}_8\nu_2 + H^5{}_8\lambda + J^5{}_8Q) \wedge gQ$$

Therefore, comparing coefficients gives,

$$aA^5{}_1 = B^5{}_0 \tag{C.141}$$

$$aA^5{}_2 = C^5{}_0 \tag{C.142}$$

$$bA^5{}_3 = D^5{}_0 \tag{C.143}$$

$$bA^5{}_4 = E^5{}_0 \tag{C.144}$$

$$cA^5{}_6 = G^5{}_0 \tag{C.145}$$

$$fA^5{}_7 = H^5{}_0 \tag{C.146}$$

$$B^5{}_2 = C^5{}_1 \tag{C.147}$$

$$bB^5{}_3 = aD^5{}_1 \tag{C.148}$$

$$bB^5{}_4 = aE^5{}_1 - c \tag{C.149}$$

$$cB^5{}_6 = aG^5{}_1 \tag{C.150}$$

$$fB^5{}_7 = aH^5{}_1 \tag{C.151}$$

$$bC^5{}_3 = aD^5{}_2 + c \tag{C.152}$$

$$bC^5{}_4 = aE^5{}_2 \tag{C.153}$$

$$cC^5{}_6 = aG^5{}_2 \tag{C.154}$$

$$fC^5{}_7 = aH^5{}_2 \tag{C.155}$$

$$D^5{}_4 = E^5{}_3 \tag{C.156}$$

$$cD^5{}_6 = bG^5{}_3 \tag{C.157}$$

$$fD^5{}_7 = bH^5{}_3 \tag{C.158}$$

$$cE^5{}_6 = bG^5{}_4 \tag{C.159}$$

$$fE^5{}_7 = bH^5{}_4 \tag{C.160}$$

$$F^5{}_0 = \frac{\partial c}{\partial t} \tag{C.161}$$

$$F^5{}_i = 0 \quad i = 1, ..., 8 \tag{C.162}$$

$$fG^5{}_7 = cH^5{}_6 - c \tag{C.163}$$

C.7 7^{th} Cartan equation ($i = 6$)

$$de^6 + \omega^6{}_a e^a = 0 \tag{C.164}$$

Where

$$de^6 = d(c\nu_2) = \frac{\partial c}{\partial t} dt \wedge \nu_2 + c\,(\lambda \wedge \nu_1 + \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2) \tag{C.165}$$

Therefore

$$0 = d(c\nu_2) = \frac{\partial c}{\partial t} dt \wedge \nu_2 + c\,(\lambda \wedge \nu_1 + \sigma_1 \wedge \Sigma_1 + \sigma_2 \wedge \Sigma_2)$$

$$+(A^6{}_0 dt + B^6{}_0 \sigma_1 + C^6{}_0 \sigma_2 + D^6{}_0 \Sigma_1 + E^6{}_0 \Sigma_2 + F^6{}_0 \nu_1 + G^6{}_0 \nu_2 + H^6{}_0 \lambda + J^6{}_0 Q) \wedge dt$$

$$+a(A^6{}_1 dt + B^6{}_1 \sigma_1 + C^6{}_1 \sigma_2 + D^6{}_1 \Sigma_1 + E^6{}_1 \Sigma_2 + F^6{}_1 \nu_1 + G^6{}_1 \nu_2 + H^6{}_1 \lambda + J^6{}_1 Q) \wedge \sigma_1$$

$$+a(A^6{}_2 dt + B^6{}_2 \sigma_1 + C^6{}_2 \sigma_2 + D^6{}_2 \Sigma_1 + E^6{}_2 \Sigma_2 + F^6{}_2 \nu_1 + G^6{}_2 \nu_2 + H^6{}_2 \lambda + J^6{}_2 Q) \wedge \sigma_2$$

$$+b(A^6{}_3 dt + B^6{}_3 \sigma_1 + C^6{}_3 \sigma_2 + D^6{}_3 \Sigma_1 + E^6{}_3 \Sigma_2 + F^6{}_3 \nu_1 + G^6{}_3 \nu_2 + H^6{}_3 \lambda + J^6{}_3 Q) \wedge \Sigma_1$$

$$+b(A^6{}_4dt + B^6{}_4\sigma_1 + C^6{}_4\sigma_2 + D^6{}_4\Sigma_1 + E^6{}_4\Sigma_2 + F^6{}_4\nu_1 + G^6{}_4\nu_2 + H^6{}_4\lambda + J^6{}_4Q) \wedge \Sigma_2$$

$$+c(A^6{}_5dt + B^6{}_5\sigma_1 + C^6{}_5\sigma_2 + D^6{}_5\Sigma_1 + E^6{}_5\Sigma_2 + F^6{}_5\nu_1 + G^6{}_5\nu_2 + H^6{}_5\lambda + J^6{}_5Q) \wedge \nu_1$$

$$+f(A^6{}_7dt + B^6{}_7\sigma_1 + C^6{}_7\sigma_2 + D^6{}_7\Sigma_1 + E^6{}_7\Sigma_2 + F^6{}_7\nu_1 + G^6{}_7\nu_2 + H^6{}_7\lambda + J^6{}_7Q) \wedge \lambda$$

$$+(A^6{}_8dt + B^6{}_8\sigma_1 + C^6{}_8\sigma_2 + D^6{}_8\Sigma_1 + E^6{}_8\Sigma_2 + F^6{}_8\nu_1 + G^6{}_8\nu_2 + H^6{}_8\lambda + J^6{}_8Q) \wedge gQ$$

Therefore, comparing coefficients gives,

$$aA^6{}_1 = B^6{}_0 \tag{C.166}$$

$$aA^6{}_2 = C^6{}_0 \tag{C.167}$$

$$bA^6{}_3 = D^6{}_0 \tag{C.168}$$

$$bA^6{}_4 = E^6{}_0 \tag{C.169}$$

$$cA^6{}_5 = F^6{}_0 \tag{C.170}$$

$$fA^6{}_7 = H^6{}_0 \tag{C.171}$$

$$B^6{}_2 = C^6{}_1 \tag{C.172}$$

$$bB^6{}_3 = aD^6{}_1 - c \tag{C.173}$$

$$bB^6{}_4 = aE^6{}_1 \tag{C.174}$$

$$cB^6{}_5 = aF^6{}_1 \tag{C.175}$$

$$fB^6{}_7 = aH^6{}_1 \tag{C.176}$$

$$bC^6{}_3 = aD^6{}_2 \tag{C.177}$$

$$bC^6{}_4 = aE^6{}_2 - c \tag{C.178}$$

$$cC^6{}_5 = aF^6{}_2 \tag{C.179}$$

$$fC^6{}_7 = aH^6{}_2 \tag{C.180}$$

$$D^6{}_4 = E^6{}_3 \tag{C.181}$$

$$cD^6{}_5 = bF^6{}_3 \tag{C.182}$$

$$fD^6{}_7 = bH^6{}_3 \tag{C.183}$$

$$cE^6{}_5 = bF^6{}_4 \tag{C.184}$$

$$fE^6{}_7 = bH^6{}_4 \tag{C.185}$$

$$fF^6{}_7 = cH^6{}_5 + c \tag{C.186}$$

$$G^6{}_0 = \frac{\partial c}{\partial t} \tag{C.187}$$

$$G^6{}_i = 0 \quad i = 1, ..., 8 \tag{C.188}$$

C.8 8^{th} Cartan equation $(i = 7)$

$$de^7 + \omega^7{}_a e^a = 0 \tag{C.189}$$

Where

$$de^7 = d(f\lambda) = \frac{\partial f}{\partial t} dt \wedge \lambda + f d\lambda \tag{C.190}$$

$$= \frac{\partial f}{\partial t} dt \wedge \lambda + f\left(2\sigma_1 \wedge \sigma_2 - 2\Sigma_1 \wedge \Sigma_2 + 4\nu_1 \wedge \nu_2\right) \tag{C.191}$$

Therefore,

$$0 = \frac{\partial f}{\partial t} dt \wedge \lambda + f\left(2\sigma_1 \wedge \sigma_2 - 2\Sigma_1 \wedge \Sigma_2 + 4\nu_1 \wedge \nu_2\right)$$

$$+(A^7{}_0 dt + B^7{}_0\sigma_1 + C^7{}_0\sigma_2 + D^7{}_0\Sigma_1 + E^7{}_0\Sigma_2 + F^7{}_0\nu_1 + G^7{}_0\nu_2 + H^7{}_0\lambda + J^7{}_0 Q) \wedge dt$$

$$+a(A^7{}_1 dt + B^7{}_1\sigma_1 + C^7{}_1\sigma_2 + D^7{}_1\Sigma_1 + E^7{}_1\Sigma_2 + F^7{}_1\nu_1 + G^7{}_1\nu_2 + H^7{}_1\lambda + J^7{}_1 Q) \wedge \sigma_1$$

$$+a(A^7{}_2 dt + B^7{}_2\sigma_1 + C^7{}_2\sigma_2 + D^7{}_2\Sigma_1 + E^7{}_2\Sigma_2 + F^7{}_2\nu_1 + G^7{}_2\nu_2 + H^7{}_2\lambda + J^7{}_2 Q) \wedge \sigma_2$$

$$+b(A^7{}_3 dt + B^7{}_3\sigma_1 + C^7{}_3\sigma_2 + D^7{}_3\Sigma_1 + E^7{}_3\Sigma_2 + F^7{}_3\nu_1 + G^7{}_3\nu_2 + H^7{}_3\lambda + J^7{}_3 Q) \wedge \Sigma_1$$

$$+b(A^7{}_4dt + B^7{}_4\sigma_1 + C^7{}_4\sigma_2 + D^7{}_4\Sigma_1 + E^7{}_4\Sigma_2 + F^7{}_4\nu_1 + G^7{}_4\nu_2 + H^7{}_4\lambda + J^7{}_4Q) \wedge \Sigma_2$$

$$+c(A^7{}_5dt + B^7{}_5\sigma_1 + C^7{}_5\sigma_2 + D^7{}_5\Sigma_1 + E^7{}_5\Sigma_2 + F^7{}_5\nu_1 + G^7{}_5\nu_2 + H^7{}_5\lambda + J^7{}_5Q) \wedge \nu_1$$

$$+c(A^7{}_6dt + B^7{}_6\sigma_1 + C^7{}_6\sigma_2 + D^7{}_6\Sigma_1 + E^7{}_6\Sigma_2 + F^7{}_6\nu_1 + G^7{}_6\nu_2 + H^7{}_6\lambda + J^7{}_6Q) \wedge \nu_2$$

$$+(A^7{}_8dt + B^7{}_8\sigma_1 + C^7{}_8\sigma_2 + D^7{}_8\Sigma_1 + E^7{}_8\Sigma_2 + F^7{}_8\nu_1 + G^7{}_8\nu_2 + H^7{}_8\lambda + J^7{}_8Q) \wedge gQ$$

Therefore, comparing coefficients gives,

$$aA^7{}_1 = B^7{}_0 \tag{C.192}$$

$$aA^7{}_2 = C^7{}_0 \tag{C.193}$$

$$bA^7{}_3 = D^7{}_0 \tag{C.194}$$

$$bA^7{}_4 = E^7{}_0 \tag{C.195}$$

$$cA^7{}_5 = F^7{}_0 \tag{C.196}$$

$$cA^7{}_6 = G^7{}_0 \tag{C.197}$$

$$aB^7{}_2 = aC^7{}_1 - 2f \tag{C.198}$$

$$bB^7{}_3 = aD^7{}_1 \tag{C.199}$$

$$bB^7{}_4 = aE^7{}_1 \tag{C.200}$$

$$cB^7{}_5 = aF^7{}_1 \tag{C.201}$$

$$cB^7{}_6 = aG^7{}_1 \tag{C.202}$$

$$bC^7{}_3 = aD^7{}_2 \tag{C.203}$$

$$bC^7{}_4 = aE^7{}_2 \tag{C.204}$$

$$cC^7{}_5 = aF^7{}_2 \tag{C.205}$$

$$cC^7{}_6 = aG^7{}_2 \tag{C.206}$$

$$bD^7{}_4 = bE^7{}_3 + 2f \tag{C.207}$$

$$cD^7{}_5 = bF^7{}_3 \tag{C.208}$$

$$cD^7{}_6 = bG^7{}_3 \tag{C.209}$$

$$cE^7{}_5 = bF^7{}_4 \tag{C.210}$$

$$cE^7{}_6 = bG^7{}_4 \tag{C.211}$$

$$cF^7{}_6 = cG^7{}_5 - 4f \tag{C.212}$$

$$H^7{}_0 = \frac{\partial f}{\partial t} \tag{C.213}$$

$$H^7{}_i = 0 \quad i = 1, ..., 8 \tag{C.214}$$

C.9 solutions

This set of simultaneous equations is straightforward to solve. We will also make use of the antisymmetry of the spin connection ω. This follows because in our vielbein basis the metric is just the Kronecker delta

$$\omega_{ij} = -\omega_{ji} \Rightarrow \omega_{ij}\delta^{jk} = -\omega_{ji}\delta^{jk} \tag{C.215}$$

$$\Rightarrow \omega_i{}^k = -\omega^k{}_i \tag{C.216}$$

The non trivial simultaneous equations are presented below Adding C.50 to C.74 gives

$$C^1{}_7 = -B^2{}_7 \tag{C.217}$$

Substituting into C.198;

$$B^2{}_7 = \frac{f}{a} = \frac{b}{r} \Rightarrow C^1{}_7 = -\frac{b}{r} \tag{C.218}$$

Substituting these results into C.50 gives

$$H^1{}_2 = -\frac{1}{2r^2} \tag{C.219}$$

Adding a times C.53 to b times C.97 gives

$$D^1{}_6 = \frac{1-cb}{ac} B^3{}_6 \tag{C.220}$$

Substituting into C.173;

$$B^3{}_6 = \frac{c^2+1}{2cb} = \frac{b}{r} \Rightarrow D^1{}_6 = -\frac{a}{r} \tag{C.221}$$

Substituting these results into C.53 gives

$$G^1{}_3 = 0 \tag{C.222}$$

Adding a times C.55 to b times C.122 gives

$$E^1{}_5 = \frac{1-cb}{ac} B^4{}_5 \tag{C.223}$$

Substituting into C.149;

$$B^4{}_5 = \frac{c^2+1}{2cb} = \frac{b}{r} \Rightarrow E^1{}_5 = -\frac{a}{r} \tag{C.224}$$

Substituting these results into C.55 gives

$$F^1{}_4 = 0 \tag{C.225}$$

Adding a times C.78 to b times C.100 gives

$$D^2{}_5 = -\frac{1+bcC^3{}_5}{ac} \tag{C.226}$$

Substituting into C.152;

$$C^3{}_5 = -\frac{b}{r} \Rightarrow D^2{}_5 = \frac{a}{r} \tag{C.227}$$

Substituting these results into C.78 gives

$$F^2{}_3 = 0 \tag{C.228}$$

Adding a times C.82 to b times C.127 gives

$$E^2{}_6 = \frac{1 - bcC^4{}_6}{ac} \tag{C.229}$$

Substituting into C.178;

$$C^4{}_6 = \frac{c^2+1}{2bc} = \frac{b}{r} \Rightarrow E^2{}_6 = -\frac{a}{r} \tag{C.230}$$

Substituting these results into C.82 gives

$$G^2{}_4 = 0 \tag{C.231}$$

Adding C.107 to C.131 gives

$$E^3{}_7 = -D^4{}_7 \tag{C.232}$$

Substituting into C.207;

$$D^4{}_7 = \frac{f}{b} = -\frac{a}{r} \Rightarrow E^3{}_7 = \frac{a}{r} \tag{C.233}$$

Substituting these results into C.107 gives

$$H^3{}_4 = -\frac{1}{2r^2} \tag{C.234}$$

Adding C.163 to C.186 gives

$$F^6{}_7 = -G^5{}_7 \tag{C.235}$$

Substituting into C.212;

$$F^6{}_7 = \sqrt{1 - r^{-4}} \Rightarrow G^5{}_7 = -\sqrt{1 - r^{-4}} \tag{C.236}$$

Substituting these results into C.163 gives

$$H^5{}_6 = -\frac{2f^2}{c} = \frac{1}{2}(1 + r^{-4}) \tag{C.237}$$

Using ?? and ?? gives

$$J^1{}_2 = \frac{3}{2} \tag{C.238}$$

Adding ?? to ?? gives

$$B^2{}_8 = -C^1{}_8 = 0 \tag{C.239}$$

Adding ?? to ?? gives

$$D^4{}_8 = -E^3{}_8 = 0 \tag{C.240}$$

Substituting these results into ?? gives

$$J^3{}_4 = \frac{3}{2} \tag{C.241}$$

C.10 Spin connection results

$$-\omega^0{}_1 = \omega^3{}_6 = \omega^4{}_5 = \omega^2{}_7 = \frac{b}{r}\sigma_1 \tag{C.242}$$

$$\omega^0{}_2 = \omega^1{}_7 = \omega^3{}_5 = -\omega^4{}_6 = -\frac{b}{r}\sigma_2 \tag{C.243}$$

$$\omega^0{}_3 = \omega^1{}_6 = -\omega^2{}_5 = \omega^4{}_7 = -\frac{a}{r}\Sigma_1 \tag{C.244}$$

$$\omega^0{}_4 = \omega^1{}_5 = \omega^2{}_6 = -\omega^3{}_7 = -\frac{a}{r}\Sigma_2 \tag{C.245}$$

$$-\omega^0{}_5 = \omega^6{}_7 = \sqrt{1 - r^{-4}}\nu_1 \tag{C.246}$$

$$\omega^0{}_6 = \omega^5{}_7 = \sqrt{1 - r^{-4}}\nu_2 \tag{C.247}$$

$$-\omega^0{}_7 = \omega^5{}_6 = \frac{1}{2}(1 + r^{-4})\lambda \tag{C.248}$$

$$\omega^1{}_2 = -\frac{1}{2r^2}\lambda + \left(\frac{3}{2}\right) Q \tag{C.249}$$

$$\omega^3{}_4 = -\frac{1}{2r^2}\lambda + \left(\frac{3}{2}\right) Q \tag{C.250}$$

Appendix D

Computation of the curvature 2-form

We use the spin connection to calculate the curvature 2-form from Cartan's second structure equation.

$$R^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$

Although I have calculated explicitly all the linearly independent components of the curvature 2-form, necessary considerations of concision compel me to include only a representative sample of this work as an illustration of the methods employed.

D.1 $R^0{}_1$

$$R^0{}_1 = d\omega^0{}_1 + \omega^0{}_k \wedge \omega^k{}_1$$

Where

$$d\omega^0{}_1 = -\frac{\partial^2 a}{\partial t^2} dt \wedge \sigma_1 - \frac{\partial a}{\partial t}\left(-\frac{1}{2}\lambda \wedge \sigma_2 - \nu_1 \wedge \Sigma_2 - \nu_2 \wedge \Sigma_1 - \frac{3}{2}Q \wedge \sigma_2\right)$$

And

$$\omega^0{}_2 \wedge \omega^2{}_1 = -\frac{\partial a}{\partial t}\sigma_2 \wedge \left(\frac{1}{2r^2}\lambda - \frac{3}{2}Q\right)$$

$$-\frac{b}{2r^3}\sigma_2 \wedge \lambda + \frac{3b}{2r}\sigma_2 \wedge Q$$

$$\omega^0{}_3 \wedge \omega^3{}_1 = \omega^0{}_4 \wedge \omega^4{}_1 = 0$$

$$\omega^0{}_5 \wedge \omega^5{}_1 = -\frac{a}{r}\sqrt{1-r^{-4}}\nu_1 \wedge \Sigma_2$$

$$\omega^0{}_6 \wedge \omega^6{}_1 = -\frac{a}{r}\sqrt{1-r^{-4}}\nu_2 \wedge \Sigma_1$$

$$\omega^0{}_7 \wedge \omega^7{}_1 = -\frac{1}{2}\frac{b}{r}(1+r^{-4})\lambda \wedge \sigma_2$$

We now collect terms. The $dt \wedge \sigma_1$ term is

$$\frac{a}{r^4}dt \wedge \sigma_1 = \frac{1}{r^4}e^0 \wedge e^1$$

The coefficient of the $\lambda \wedge \sigma_2$ term is

$$\frac{1}{2}\frac{b}{r} + \frac{1}{2r^2}\frac{b}{r} - \frac{1}{2}\frac{b}{r}(1+r^{-4}) = \frac{af}{r4}$$

Therefore

$$\frac{af}{r^4}\lambda \wedge \sigma_2 = -\frac{1}{r^4}e^2 \wedge e^7$$

The coefficient of $\nu_1 \wedge \Sigma_2$ is

$$\frac{b}{r} - \frac{a}{r}\sqrt{1-r^{-4}} = \frac{bc}{r^4}$$

Therefore

$$\frac{bc}{r^4}\nu_1 \wedge \Sigma_2 = -\frac{1}{r^4}e^4 \wedge e^5$$

The $\nu_2 \wedge \Sigma_1$ term is

$$\frac{bc}{r^4}\nu_2 \wedge \Sigma_1 = -\frac{1}{r^4}e^3 \wedge e^6$$

The coefficient of $Q \wedge \sigma_2$ is zero.

Collecting our results we have

$$R^0{}_1 = -\frac{1}{r^4}\left(-e^0 \wedge e^1 + e^2 \wedge e^7 + e^3 \wedge e^6 + e^4 \wedge e^5\right)$$

D.2 $R^0{}_2$

$$R^0{}_2 = d\omega^0{}_2 + \omega^0{}_k \wedge \omega^k{}_2$$

Where

$$d\omega^0{}_1 = -\frac{\partial^2 a}{\partial t^2} dt \wedge \sigma_2 - \frac{\partial a}{\partial t}\left(\frac{1}{2}\lambda \wedge \sigma_2 + \nu_1 \wedge \Sigma_1 - \nu_2 \wedge \Sigma_2 + \frac{3}{2} Q \wedge \sigma_1\right)$$

And

$$\omega^0{}_1 \wedge \omega^1{}_2 = \frac{b}{2r^3}\sigma_1 \wedge \lambda - frac32\frac{b}{r}\sigma_1 \wedge Q$$

$$\omega^0{}_3 \wedge \omega^3{}_2 = \omega^0{}_4 \wedge \omega^4{}_2 = 0$$

$$\omega^0{}_5 \wedge \omega^5{}_2 = \frac{a}{r}\sqrt{1 - r^{-4}}\nu_1 \wedge \Sigma_1$$

$$\omega^0{}_6 \wedge \omega^6{}_2 = -\frac{a}{r}\sqrt{1 - r^{-4}}\nu_2 \wedge \Sigma_2$$

$$\omega^0{}_7 \wedge \omega^7{}_2 = \frac{1}{2}\frac{b}{r}(1 + r^{-4})\lambda \wedge \sigma_1$$

We now collect terms. The $dt \wedge \sigma_2$ term is

$$\frac{a}{r^4} dt \wedge \sigma_2 = \frac{1}{r^4} e^0 \wedge e^2$$

The $\lambda \wedge \sigma_1$ term is

$$\frac{1}{2}\frac{b}{r} - \frac{1}{2r^2}\frac{b}{r} + \frac{1}{2}\frac{b}{r}(1 + r^{-4}) = -\frac{af}{r^4}$$

Therefore

$$-\frac{af}{r^4}\lambda \wedge \sigma_1 = \frac{1}{r^4} e^1 \wedge e^7$$

The coefficient of the $\nu_1 \wedge \Sigma_1$ term is

$$-\frac{b}{r} + \frac{a}{r}\sqrt{1 - r^{-4}} = -\frac{bc}{r^4}$$

Therefore

$$-\frac{bc}{r^4}\nu_1 \wedge \Sigma_1 = \frac{1}{r^4}e^3 \wedge e^5$$

The coefficient of the $\nu_2 \wedge \Sigma_2$ term is

$$\frac{b}{r} - \frac{a}{r}\sqrt{1-r^{-4}} = \frac{bc}{r^4}$$

Therefore

$$\frac{bc}{r^4}\nu_2 \wedge \Sigma_2 = -\frac{bc}{r^4}e^4 \wedge e^6$$

The coefficient of $Q \wedge \sigma_1$ is zero.

Collecting all the results we have

$$R^0{}_2 = -\frac{1}{r^4}\left(-e^0 \wedge e^2 - e^1 \wedge e^7 + e^3 \wedge e^5 + e^4 \wedge e^6\right)$$

D.3 $R^0{}_3$

$$R^0{}_3 = d\omega^0{}_3 + \omega^0{}_k \wedge \omega^k{}_3$$

Where

$$d\omega^0{}_3 = -\frac{\partial^2 b}{\partial t^2}dt \wedge \Sigma_1 - \frac{\partial b}{\partial t}\left(\frac{1}{2}\lambda \wedge \Sigma_2 - \nu_1 \wedge \sigma_2 + \nu_2 \wedge \sigma_1 - \frac{3}{2}Q \wedge \Sigma_2\right)$$

And

$$\omega^0{}_1 \wedge \omega^1{}_3 = \omega^0{}_2 \wedge \omega^2{}_3 = 0$$

$$\omega^0{}_4 \wedge \omega^4{}_3 = -\frac{1}{2r^2}\frac{\partial b}{\partial t}\Sigma_2 \wedge \lambda + \frac{3}{2}\frac{\partial b}{\partial t}\Sigma_2 \wedge Q$$

$$\omega^0{}_5 \wedge \omega^5{}_3 = -\frac{b}{r}\sqrt{1-r^{-4}}\nu_1 \wedge \sigma_2$$

$$\omega^0{}_6 \wedge \omega^6{}_3 = \frac{b}{r}\sqrt{1-r^{-4}}\nu_2 \wedge \sigma_1$$

$$\omega^0{}_7 \wedge \omega^7{}_3 = \frac{1}{2}\frac{a}{r}(1 + r^{-4})\lambda \wedge \Sigma_2$$

The $dt \wedge \Sigma_1$ term is

$$-\frac{\partial^2 b}{\partial t^2} dt \wedge \Sigma_1 = -\frac{b}{r^4} dt \wedge \Sigma_1 = -\frac{1}{r^4} e^0 \wedge e^3$$

The coefficient of the $\lambda \wedge \Sigma_2$ term is

$$-\frac{1}{2}\frac{a}{r} + \frac{1}{2r^2}\frac{a}{r} + \frac{1}{2}\frac{a}{r}(1 + r^{-4}) = \frac{bf}{r^4}$$

Therefore

$$\frac{bf}{r^4} = \lambda \wedge \Sigma_2 = -\frac{1}{r^4} e^4 \wedge e^7$$

The $\nu_1 \wedge \sigma_2$ coefficient is

$$\frac{a}{r} - \frac{b}{r}\sqrt{1 - r^{-4}} = -\frac{ac}{r^4}$$

Therefore

$$-\frac{ac}{r^4}\nu_1 \wedge \sigma_2 = \frac{1}{r^4} e^2 \wedge e^5$$

The $\nu_2 \wedge \sigma_1$ term is

$$\frac{ac}{r^4}\nu_2 \wedge \sigma_1 = -\frac{1}{r^4} e^1 \wedge e^6$$

The coefficient of the $Q \wedge \Sigma_2$ is zero. Assembling these results

$$R^0{}_3 = -\frac{1}{r^4}(e^0 \wedge e^3 + e^1 \wedge e^6 - e^2 \wedge e^5 + e^4 \wedge e^7)$$

D.4 $R^0{}_4$

$$R^0{}_4 = d\omega^0{}_4 + \omega^0{}_k \wedge \omega^k{}_4$$

Where

$$d\omega^0{}_4 = -\frac{\partial^2 b}{\partial t^2} dt \wedge \Sigma_2 - \frac{\partial b}{\partial t}\left(-\frac{1}{2}\lambda \wedge \Sigma_1 + \nu_1 \wedge \sigma_1 + \nu_2 \wedge \sigma_2 + \frac{3}{2} Q \wedge \Sigma_1\right)$$

And

$$\omega^0{}_1 \wedge \omega^1{}_4 = \omega^0{}_2 \wedge \omega^2{}_4 = 0$$

$$\omega^0{}_3 \wedge \omega^3{}_4 = \frac{1}{2r^2}\frac{a}{r}\Sigma_1 \wedge \lambda - \frac{3}{2}\frac{a}{r}\Sigma_1 \wedge Q$$

$$\omega^0{}_5 \wedge \omega^5{}_4 = \frac{b}{r}\sqrt{1 - r^{-4}}\nu_1 \wedge \sigma_1$$

$$\omega^0{}_6 \wedge \omega^6{}_4 = \frac{b}{r}\sqrt{1 - r^{-4}}\nu_2 \wedge \sigma_2$$

$$\omega^0{}_7 \wedge \omega^7{}_4 = -\frac{1}{2}\frac{a}{r}(1 + r^{-4})\lambda \wedge \Sigma_1$$

The $dt \wedge \Sigma_2$ term is

$$-\frac{\partial^2 b}{\partial t^2}dt \wedge \Sigma_2 = -\frac{b}{r^4}dt \wedge \Sigma_2 = -\frac{1}{r^4}e^0 \wedge e^4$$

The $\lambda \wedge \Sigma_1$ coefficient is

$$\frac{1}{2}\frac{a}{r} - \frac{1}{2r^2}\frac{a}{r} - \frac{a}{r}\frac{1}{2}(1 + r^{-4}) = -\frac{bf}{r^4}$$

Therefore

$$-\frac{bf}{r^4}\lambda \wedge \Sigma_1 = \frac{1}{r^4}e^3 \wedge e^7$$

The $\nu_1 \wedge \sigma_1$ coefficient is

$$-\frac{a}{r} + \frac{b}{r}\sqrt{1 - r^{-4}} = \frac{ac}{r^4}$$

Therefore

$$\frac{ac}{r^4}\nu_1 \wedge \sigma_1 = -\frac{1}{r^4}e^1 \wedge e^5$$

The $\nu_2 \wedge \sigma_2$ coefficient is

$$-\frac{a}{r} + \frac{b}{r}\sqrt{1 - r^{-4}} = \frac{ac}{r^4}$$

Therefore

$$\frac{ac}{r^4}\nu_2 \wedge \sigma_2 = -\frac{1}{r^4}e^2 \wedge e^6$$

The coefficient of $Q \wedge \Sigma_1$ is zero.

Gathering results,

$$R^0{}_4 = -\frac{1}{r^4}(e^0 \wedge e^4 + e^1 \wedge e^5 + e^2 \wedge e^6 - e^3 \wedge e^7)$$

Appendix E

Fermionic zero modes in an instanton background

We show that in an instanton background the operator $\bar{\not{D}}$ has no zero modes. Thus there can be no left-handed Weyl fermions in an instanton background. Here we shall consider a Dirac fermion, in an arbitrary representation of the gauge group. In the presence of an instanton background the Dirac equation becomes

$$\gamma^\mu D^{cl}_\mu \psi = 0$$

where D^{cl}_μ involves the instanton gauge field A_μ. We decompose ψ into its chiral and anti-chiral parts,

$$\lambda = \frac{1}{2}(1+\gamma^5)\psi, \quad \bar{\chi} = \frac{1}{2}(1-\gamma^5)\psi$$

A Euclidean representation for the Clifford algebra is given by

$$\gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu_{\alpha\dot{\beta}} \\ i\bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Dirac equation then splits into two independent equations for λ and $\bar{\chi}$.

$$\sigma^{\mu\alpha\dot{\beta}} D^{cl}_\mu \bar{\chi}_{\dot{\beta}} = \not{D}^{cl}\bar{\chi} = 0 \qquad \& \qquad \bar{\sigma}^\mu_{\dot{\alpha}\beta} D^{cl}_\mu \lambda^\beta = \bar{\not{D}}^{cl}\lambda = 0$$

Where $\not{D}$ is a 2×2 matrix of derivative operators. We aim to show that, in the background of an instanton, the above has solutions for $\bar{\chi}$ but not for λ. (Obversely, in the vicinity of an anti-instanton there exist solutions for λ but not $\bar{\chi}$). Let us suppose that we are given a solution of $\not{D}\bar{\chi} = 0$. Given any such $\bar{\chi}$ we must also have

$$\bar{\not{D}}\not{D}\bar{\chi} = 0$$

(Technically, we could write that $ker\not{D} \subset \{\bar{\not{D}}\not{D}\}$). We proceed by evaluating $\bar{\not{D}}\not{D}$,

$$\bar{\not{D}}\not{D} = \bar{\sigma}^\mu \sigma^\nu D_\mu D_\nu = (2\delta^{\mu\nu} - \bar{\sigma}^\nu \sigma^\mu) D_\mu D_\nu = 2D^\mu D_\mu - \bar{\sigma}^\nu \sigma^\mu D_\mu D_\nu$$

Now recall $\bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)$. Keeping this in mind we may write out expression for $\bar{\not{D}}\not{D}$ as

$$\begin{aligned}
\bar{\not{D}}\not{D} &= \frac{1}{2}\bar{\sigma}^\mu \sigma^\nu D_\mu D_\nu + \frac{1}{2}\bar{\sigma}^\mu \sigma^\nu D_\mu D_\nu \\
&= \frac{1}{2}(2\delta^{\mu\nu} - \bar{\sigma}^\nu \sigma^\mu) D_\mu D_\nu + \frac{1}{2}\bar{\sigma}^\mu \sigma^\nu D_\mu D_\nu \\
&= D^\mu D_\mu + 2\bar{\sigma}^{\mu\nu} D_\mu D_\nu \\
&= D^\mu D_\mu + \bar{\sigma}^{\mu\nu}(D_\mu D_\nu - D_\nu D_\mu) \\
&= D^\mu D_\mu + \bar{\sigma}^{\mu\nu} F_{\mu\nu}
\end{aligned}$$

The tensor $\bar{\sigma}^{\mu\nu}$ is anti-self dual whereas the instanton field-strength is self-dual. Therefore the second term above vanishes leaving us with

$$D^\mu D_\mu \bar{\chi} = \bar{\not{D}}\not{D}\bar{\chi} = 0$$

We can use this result to show that $\bar{\chi} = 0$; consider the following manipulations,

$$D^\mu(\bar{\chi}^* D_\mu \bar{\chi}) = \partial^\mu(\bar{\chi}^* D_\mu \bar{\chi}) = \bar{\chi}^* D^\mu D_\mu \bar{\chi} + D^\mu \bar{\chi}^* D_\mu \bar{\chi}$$

Integrating over all space removes the divergence by Gauss' theorem, giving

$$\int d^4x\, D^\mu \bar{\chi}^* D_\mu \bar{\chi} = 0 \Rightarrow \int d^4x\, |D_\mu \bar{\chi}|^2 = 0$$

This implies that $\bar{\chi}$ is covariantly constant, since

$$D_\mu \bar{\chi}^a = 0$$

$$\Rightarrow \bar{\chi}^a D_\mu \bar{\chi}^a = \bar{\chi}^a(\partial_\mu \bar{\chi}^a - f^{abc} A^b_\mu \bar{\chi}^c) = 0$$

$$\Rightarrow \bar{\chi}^a \partial_\mu \bar{\chi}^a = 0$$

$$\Rightarrow \partial_\mu(\bar{\chi}^a \bar{\chi}^a) = \partial_\mu |\bar{\chi}|^2 = 0$$

i.e. $|\bar{\chi}|$ is a constant. The only square-integrable solution which vanishes on the boundary is therefore $\bar{\chi} = 0$.

Similarly we may show that

$$\not{D}\bar{\not{D}} = D^\mu D_\mu + \frac{1}{2}\sigma^{\mu\nu} F_{\mu\nu}$$

This time the second term does not vanish in the presence of an instanton, so zero modes are possible.

Appendix F

The ADHM constraint equations

We will now show that the ADHM constraint equations for one instanton, (5.67) and (5.76) may be condensed into a single equation. The constraints that must be satisfied by the ω's are

$$\tau^{c\dot{\alpha}}{}_{\dot{\beta}}\bar{\omega}^{\dot{\beta}}_{iu}\omega_{ui\dot{\alpha}} = \zeta^c \tag{F.1}$$

$$d\bar{\omega}^{\dot{\alpha}}{}_{iu}\omega_{ui\dot{\alpha}} - \bar{\omega}^{\dot{\alpha}}{}_{iu}d\omega_{ui\dot{\alpha}} = 0 \tag{F.2}$$

Now recall that the Pauli matrices satisfy

$$\tau^{c\dot{\gamma}}{}_{\dot{\delta}}\tau^{c\dot{\alpha}}{}_{\dot{\beta}} = 2(\delta^{\dot{\alpha}}{}_{\dot{\delta}}\delta^{\dot{\gamma}}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\gamma}}{}_{\dot{\delta}}\delta^{\dot{\alpha}}{}_{\dot{\beta}})$$

Multiplying F.1 by τ^c and substituting in the above,

$$2(\delta^{\dot{\alpha}}{}_{\dot{\delta}}\delta^{\dot{\gamma}}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\gamma}}{}_{\dot{\delta}}\delta^{\dot{\alpha}}{}_{\dot{\beta}})\bar{\omega}^{\dot{\beta}}_{iu}\omega_{ui\dot{\alpha}} = \tau^{c\dot{\gamma}}{}_{\dot{\delta}}\zeta^c$$

$$\Rightarrow 2(\bar{\omega}^{\dot{\gamma}}{}_{iu}\omega_{ui\dot{\delta}} - \frac{1}{2}\bar{\omega}^{\dot{\alpha}}{}_{iu}\omega_{ui\dot{\alpha}}\delta^{\dot{\gamma}}{}_{\dot{\delta}}) = \tau^{c\dot{\gamma}}{}_{\dot{\delta}}\zeta^c \tag{F.3}$$

Taking the derivative gives

$$d\bar{\omega}^{\dot{\gamma}}{}_{iu}\omega_{ui\dot{\delta}} + \bar{\omega}^{\dot{\gamma}}{}_{iu}d\omega_{ui\dot{\delta}} - \frac{1}{2}d\bar{\omega}^{\dot{\alpha}}{}_{iu}\omega_{ui\dot{\alpha}}\delta^{\dot{\gamma}}{}_{\dot{\delta}} - \frac{1}{2}\bar{\omega}^{\dot{\alpha}}{}_{iu}d\omega_{ui\dot{\alpha}}\delta^{\dot{\gamma}}{}_{\dot{\delta}} = 0 \tag{F.4}$$

Using F.2 gives

$$d\bar{\omega}^{\dot{\gamma}}{}_{iu}\omega_{ui\dot{\delta}} + \bar{\omega}^{\dot{\gamma}}{}_{iu}d\omega_{ui\dot{\delta}} - \bar{\omega}^{\dot{\alpha}}{}_{iu}d\omega_{ui\dot{\alpha}}\delta^{\dot{\gamma}}{}_{\dot{\delta}} = 0$$

$$\Rightarrow \epsilon^{\dot\gamma\dot\beta} d\bar\omega_{\dot\beta iu}\omega_{ui\dot\delta} + \epsilon^{\dot\gamma\dot\beta}\bar\omega_{\dot\beta iu} d\omega_{ui\dot\delta} - \bar\omega_{\dot\beta iu}\epsilon^{\dot\alpha\dot\beta} d\omega_{ui\dot\alpha}\delta^{\dot\gamma}{}_{\dot\delta} = 0 \quad \text{(F.5)}$$

Multiplying throughout by $\epsilon_{\dot\rho\dot\gamma}$,

$$\delta^{\dot\beta}{}_{\dot\rho}(d\bar\omega_{\dot\beta iu}\omega_{ui\dot\delta} + \bar\omega_{\dot\beta iu} d\omega_{ui\dot\delta}) - \epsilon_{\dot\rho\dot\gamma}\epsilon^{\dot\alpha\dot\beta}\delta^{\dot\gamma}{}_{\dot\delta}\bar\omega_{\dot\beta iu} d\omega_{ui\dot\alpha} = 0$$

$$d\bar\omega_{\dot\rho iu}\omega_{ui\dot\delta} + \bar\omega_{\dot\rho iu} d\omega_{ui\dot\delta} - \epsilon_{\dot\rho\dot\delta}\epsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\beta iu} d\omega_{ui\dot\alpha} = 0$$

Now recall that the product of ϵ tensors is given by

$$\epsilon_{\dot\rho\dot\delta}\epsilon^{\dot\alpha\dot\beta} = \begin{vmatrix} \delta^{\dot\alpha}{}_{\dot\rho} & \delta^{\dot\alpha}{}_{\dot\delta} \\ \delta^{\dot\beta}{}_{\dot\rho} & \delta^{\dot\beta}{}_{\dot\delta} \end{vmatrix} = \delta^{\dot\alpha}{}_{\dot\delta}\delta^{\dot\beta}{}_{\dot\rho} - \delta^{\dot\alpha}{}_{\dot\rho}\delta^{\dot\beta}{}_{\dot\delta}$$

This leads to the expression

$$d\bar\omega_{\dot\rho iu}\omega_{ui\dot\delta} + \bar\omega_{\dot\rho iu} d\omega_{ui\dot\delta} - (\delta^{\dot\alpha}{}_{\dot\delta}\delta^{\dot\beta}{}_{\dot\rho} - \delta^{\dot\alpha}{}_{\dot\rho}\delta^{\dot\beta}{}_{\dot\delta})\epsilon_{\dot\rho\dot\delta}\epsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\beta iu} d\omega_{ui\dot\alpha} = 0$$

$$\Rightarrow d\bar\omega_{\dot\rho iu}\omega_{ui\dot\delta} + \bar\omega_{\dot\delta iu} d\omega_{ui\dot\rho} = 0 \quad \text{(F.6)}$$

This is our final result. It unites the entire ADHM procedure, including the quotient, into one equation for the case $k = 1$.

We have used the convention

$$\epsilon_{\dot\rho\dot\delta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Appendix G

The volume of $SU(3)/U(1)$

In chapter six we effected a change of variables from the ADHM coordinates to $U(N)$ invariant coordinates. In so doing we were able to integrate over the angular variables in the measure. We shall exploit this result to determine the the contribution of the left-invariant one forms of equation (5.39), which we interpret as the volume of the coset $SU(3)/U(1)$.

To start, recall equation (6.149), which we reproduce here of the case $N = 3$..

$$\int \prod_{u=1}^{N} d\omega_{ua} d\omega_{ua}^{*} = 2^3 \int d^4W \ |det(W)| d^5\Omega_1 d^3\Omega_2 \tag{G.1}$$

The W's are defined by equation (6.135), which in the notation of chapter five becomes

$$W = \begin{pmatrix} \rho_+^2 & \rho_+\alpha \\ \rho_+\alpha^* & \alpha\alpha^* + \rho_-^2 \end{pmatrix} \tag{G.2}$$

The determinant of this matrix is $\rho_+^2\rho_-^2$. To complete the integral in (G.1) we will also need to change variables as detailed in (6.134). Using the fact that $\xi_{11} = \rho_+$, $\xi_{22} = \rho_-$ and $\xi_{12} = \alpha$, we have

$$d^4\xi = d\rho_+ \wedge d\rho_- \wedge d\alpha \wedge d\alpha^* \tag{G.3}$$

Using equations (6.31)-(6.34) gives the result

$$d^4\xi = \frac{\rho}{4\rho_+^3\rho_-} d\rho \wedge df^1 \wedge df^2 \wedge df^3 \tag{G.4}$$

Substituting this and (6.134) into (G.1) gives the result

$$\int d\omega d\omega^* = 2^4\pi^5 \int \rho\rho_+^2\rho_-^2 d\rho df^1 df^2 df^3 \tag{G.5}$$

where we have used the results $Vol.(S^3) = 2\pi^2$ and $Vol.(S^5) = \pi^3$. However, we have yet to perform the $U(1)$ quotient. The relevant result is given in equation (6.49). Using this result and imposing the delta function constraints $\delta(f^c)$ gives our result for the volume form on the moduli space:

$$\int_{\mathcal{M}} \Omega = 2^5\pi^4 \int \rho^3\rho_+^2\rho_-^2 d\rho \tag{G.6}$$

Comparing this result with the volume element of (5.111) leads to the identification

$$\int \sigma_1 \wedge \sigma_2 \wedge \Sigma_1 \wedge \Sigma_2 \wedge \nu_1 \wedge \nu_2 \wedge \lambda = \pi^4 \tag{G.7}$$

Appendix H

Conventions and formulae

We follow [12] and normalize Lie group generators as

$$Tr(T_a T_b) = \delta_{ab}$$

H.1 Index conventions

i, j, k : Multi-instanton indices

u, v, w : $SU(N)$ gauge group indices

μ, ν : Moduli space coordinate indices

m, n, p : Minkowski or Euclidean space-time indices

$\tilde{i}, \tilde{j}, \tilde{k}$: ADHM composite index notation

$\alpha, \beta, \dot{\alpha}, \dot{\beta}$: Spinor indices

H.2 Symbols

$\mathcal{M}$: 1-instanton moduli space

$\tilde{\mathcal{M}}$: 1-instanton mother space

H.3 Formulae and results

Below is a list of formulae and results used in the main body of the text. For further details on any of these points the reader is referred to the review of [12].

H.3.1 Pauli matrix stuff

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{H.1}$$

$$\tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{H.2}$$

$$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{H.3}$$

$$\epsilon_{21} = \epsilon^{12} = 1 \tag{H.4}$$

$$\epsilon^{21} = \epsilon_{12} = -1 \tag{H.5}$$

$$\epsilon^{11} = \epsilon^{22} = 0 \tag{H.6}$$

$$[\tau^i, \tau^j] = 2i\epsilon^{ijk}\tau^k \tag{H.7}$$

$$\sum_i (\tau^i)^\alpha{}_\beta (\tau^i)^\gamma{}_\delta = 2\left(\delta^\gamma{}_\beta \delta^\alpha{}_\delta - \frac{1}{2}\delta^\alpha{}_\beta \delta^\gamma{}_\delta\right) \tag{H.8}$$

$$\sigma_n = (i\tau^i, 1_{2\times 2}) \tag{H.9}$$

$$\bar{\sigma}_n = \sigma_n^\dagger = (-i\tau^i, 1_{2\times 2}) \tag{H.10}$$

$$\sigma_{mn} = \frac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m) \tag{H.11}$$

$$\bar{\sigma}_{mn} = \frac{1}{4}(\bar{\sigma}_m\sigma_n - \bar{\sigma}_n\sigma_m) \tag{H.12}$$

$$\sigma_{mn} = \frac{1}{2}\epsilon_{mnkl}\sigma_{kl} \tag{H.13}$$

$$\bar{\sigma}_{mn} = -\frac{1}{2}\epsilon_{mnkl}\bar{\sigma}_{kl} \tag{H.14}$$

H.3.2 The t'Hooft η symbols

These form a basis for the self-dual and the anti-self-dual antisymmetric matrices in 4-dimensions.

$$\eta^c_{AB} = \frac{1}{2}\epsilon_{ABCD}\eta^c_{CD} \tag{H.15}$$

$$\bar{\eta}^c_{AB} = \epsilon_{ABCD}\bar{\eta}^c_{CD} \tag{H.16}$$

$$\eta^c_{AB} = \bar{\eta}^c_{AB} = \epsilon_{cAB}\,, \quad A, B \in \{1,2,3\} \tag{H.17}$$

$$\bar{\eta}^c_{4A}\eta^c_{A4} = \delta_{cA} \tag{H.18}$$

$$\eta^c_{AB} = \eta^c_{BA}\,, \qquad \bar{\eta}^c_{AB} = -\bar{\eta}^c_{BA} \tag{H.19}$$

H.3.3 The Σ matrices

In six-dimensional Euclidean space we define the 4×4 matrices

$$\Sigma_a = (\eta^3, i\bar{\eta}^3, \eta^2, i\bar{\eta}^2, \eta^1, i\bar{\eta}^1) \tag{H.20}$$

$$\bar{\Sigma}_a = (-\eta^3, i\bar{\eta}^3, -\eta^2, i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1) \tag{H.21}$$

Whilst in Minkowski space we define;

$$\Sigma_a = (i\eta^3, i\bar{\eta}^3, \eta^2, i\bar{\eta}^2, \eta^1, i\bar{\eta}^1) \tag{H.22}$$

$$\bar{\Sigma}_a = (-i\eta^3, i\bar{\eta}^3, -\eta^2, i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1) \tag{H.23}$$

The next two formulae are valid in both Euclidean and Minkowski space;

$$\Sigma_{aAB}\Sigma_{aCD} = \bar{\Sigma}_{aAB}\bar{\Sigma}_{aCD} = 2\epsilon_{ABCD} \tag{H.24}$$

$$\Sigma_{aAB}\bar{\Sigma}_{aCD} = -2\delta_{AC}\delta_{BD} - 2\delta_{AD}\delta_{BC} \tag{H.25}$$

H.4 A useful result

We shall prove a result used to derive (2.10) in the text above.

$$\int d^D x \; Tr_N(D_n\Omega_\mu\delta_\nu A_n) = \int d^D x \; Tr_N\left\{(\partial_n\Omega_\mu + [A_n, \Omega_\mu])\,\delta_\nu A_n\right\} \tag{H.26}$$

$$= \int d^D x \; Tr_N\left\{\partial_n(\Omega_\mu\delta_\nu A_n) - \Omega_\mu\partial_n\delta_\nu A_n\right\} + \int d^D x \; Tr_N\left\{[A_n, \Omega_\mu]\,\delta_\nu A_n\right\} \tag{H.27}$$

The total derivative gives rise to a surface term which vanishes due to the asymptotic form of Ω_μ and $\delta_\nu A_n$, so we have

$$\int d^D x \; Tr_N\left\{-\Omega_\mu\partial_n\delta_\nu A_n + [A_n, \Omega_\mu]\,\delta_\nu A_n\right\} \tag{H.28}$$

Using the cyclic property of the trace gives

$$\int d^D x \; Tr_N\left\{-\Omega_\mu\partial_n\delta_\nu A_n + \Omega_\mu[\delta_\nu A_n A_n]\right\} \tag{H.29}$$

$$= -\int d^D x \; Tr_N\left\{\Omega_\mu D_n\delta_\nu A_n\right\} = 0 \tag{H.30}$$

H.4.1 The symplectic curvature

The components of the curvature tensor on the moduli space may be written in the $z^{\tilde{i}\dot{\alpha}}$ coordinate basis of the mother space as

$$R_{(\tilde{i}\dot{\alpha})(\tilde{j}\dot{\beta})(\tilde{k}\dot{\gamma})(\tilde{l}\dot{\delta})} 2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}} \sum_{rs}\left[(\tilde{\Omega}T^r)_{\tilde{i}\tilde{j}}L^{-1}_{rs}(\tilde{\Omega}T^r)_{\tilde{k}\tilde{l}} + (\tilde{\Omega}T^r)_{\tilde{i}\tilde{l}}L^{-1}_{rs}(\tilde{\Omega}T^r)_{\tilde{j}\tilde{k}} + (\tilde{\Omega}T^r)_{\tilde{i}\tilde{k}}L^{-1}_{rs}(\tilde{\Omega}T^r)_{\tilde{j}\tilde{l}}\right] \tag{H.31}$$

We can extract from this the symplectic curvature of the quotient

$$R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}} = 2\sum_{rs}\left[(\tilde{\Omega}T^r)_{\tilde{i}\tilde{j}}L^{-1}_{rs}(\tilde{\Omega}T^r)_{\tilde{k}\tilde{l}} + (\tilde{\Omega}T^r)_{\tilde{i}\tilde{l}}L^{-1}_{rs}(\tilde{\Omega}T^r)_{\tilde{j}\tilde{k}} + (\tilde{\Omega}T^r)_{\tilde{i}\tilde{k}}L^{-1}_{rs}(\tilde{\Omega}T^r)_{\tilde{j}\tilde{l}}\right] \tag{H.32}$$

H.5 Results that employ the ADHM algebra

H.5.1 Osborn's formula

$$Tr_N(F_{mn}F^{mn}) = -\frac{1}{g^2}(\partial^2)^2 Tr_k ln(f) \tag{H.33}$$

H.5.2 Zero modes of the Weyl equation

The quantity

$$\Lambda_\alpha = \bar{U}Cf\bar{b}_\alpha U - \bar{U}b_\alpha f\bar{C}U \tag{H.34}$$

satisfies the covariant Weyl equation

$$\bar{D}^{\dot{\alpha}\alpha}\Lambda_\alpha \tag{H.35}$$

subject to the condition

$$\bar{\Delta}^{\dot{\alpha}}C + \bar{C}\Delta^{\dot{\alpha}} = 0 \tag{H.36}$$

i.e.

H.5.3 The covariant Laplace equation with bi-fermion source

The equation to be solved is

$$D^2\phi = \Lambda(C)\Lambda(C') \tag{H.37}$$

subject to the boundary condition $lim_{x\to\infty}\phi(x) = \phi^0$ the solution is

$$\phi = -\frac{1}{4}\bar{\Sigma}_{aAB}\bar{U}M^A f\bar{M}^B U + \bar{U}\begin{pmatrix} \phi^0 & 0 \\ 0 & \varphi 1_{2\times 2} \end{pmatrix} \tag{H.38}$$

where

$$\varphi = L^{-1}\left(\bar{\omega}^{\dot{\alpha}}\phi^0\omega_{\dot{\alpha}} + \frac{1}{4}\bar{\Sigma}_{aAB}\bar{M}^A M^B\right) \tag{H.39}$$

H.5.4 Anti-fermion source

$$\bar{\Sigma}_{aAB}[\phi_a, \Lambda(M^B)] = D_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_A + \Lambda(N_A) \tag{H.40}$$

H.5.5 The inner product formula

$$\int d^4x\, Tr_N \Lambda(C)\lambda(C') = -\frac{\pi^2}{2} Tr_k \left[\bar{C}(P_\infty + 1)C' - \bar{C}'(P_\infty + 1)C\right] \tag{H.41}$$

H.5.6 The fermionic inner product formula

$$\int d^4x\, Tr_N \Lambda(M)\lambda(N) = -\frac{\pi^2}{2} Tr_k \left[\bar{M}(P_\infty + 1)N + \bar{N}(P_\infty + 1)M\right] \tag{H.42}$$

H.5.7 Miscellaneous identities and definitions

$$\frac{\partial\Lambda(M)}{\partial X^\mu} + [\Omega_\mu, \Lambda(M)] = \not{D}\bar{\varrho}_\mu + \Lambda\left(\frac{\partial M}{\partial X^\mu}\right) \tag{H.43}$$

where

$$\bar{\varrho}^{\dot{\alpha}}_\mu = \frac{1}{4}\bar{U}\frac{\partial a^{\dot{\alpha}}}{\partial X^\mu} f\bar{M}U \tag{H.44}$$

Bibliography

[1] S. Coleman, "Aspects of Symmetry", 1988, Cambridge University Press; ISBN: 0521318270

[2] R. Rajaraman, "Solitons and Instantons", 1987, * North-Holland; ISBN: 0444870474

[3] M. Nakahara, Geometry, "Topology and Physics", 1990, The Institute of Physics; ISBN: 0852740956

[4] M. Peskin and D. Schroeder, "An Introduction to Quantum Field Theory", 1995, Perseus Publishing; ISBN: 0201503972

[5] L. H. Ryder, "Quantum Field Theory", 1996, Cambridge University Press; ISBN: 0521478146

[6] V. I. Arnold, "Mathematical Methods of Classical Mechanics", 1995, Springer-Verlag Berlin ISBN: 3540968903

[7] B. Schultz, "Geometrical Methods of Mathematical Physics", 1980, Cambridge University Press; ISBN: 0521298873

[8] Ta-Pei Cheng and Ling-Fong Li, "Gauge Theory of Elementary Particle Physics", 1984, Oxford University Press; ISBN: 0198519613

[9] J. Wess and J. Bagger, "Supersymmetry and Supergravity", 1992, Princeton University Press; ISBN: 0691025304 .

[10] L. Schulman, "Techniques and Applications of Path Integration", 1996, John Wiley; ISBN: 0471166103 .

[11] M. Green, J. Schwartz, E. Witten, "Superstring Theory", 1987, Cambridge University Press.

[12] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, "The calculus of many instantons," Phys. Rept. **371** (2002) 231 [arXiv:hep-th/0206063].

[13] N. Michael Davies, Semiclassical Monopole Calculations in Supersymmetric Gauge Theories, Ph.D thesis submitted to University of Durham.

[14] J. A. Harvey, "Magnetic monopoles, duality, and supersymmetry," arXiv:hep-th/9603086.

[15] N. S. Manton, "A Remark On The Scattering Of Bps Monopoles," Phys. Lett. B **110** (1982) 54.

[16] J. M. Figueroa-O'Farrill, Electromagnetic Duality For Children.

[17] N. Dorey, T. J. Hollowood and V. V. Khoze, "The D-instanton partition function," JHEP **0103** (2001) 040 [arXiv:hep-th/0011247].

[18] N. Dorey, V. V. Khoze and M. P. Mattis, "Multi-instantons, three-dimensional gauge theory, and the Gauss-Bonnet-Chern theorem," Nucl. Phys. B **502** (1997) 94 [arXiv:hep-th/9704197].

[19] E. Witten, "Supersymmetry And Morse Theory," J. Diff. Geom. **17** (1982) 661.

[20] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, "Hyper-Kaehler Calabi metrics, L**2 harmonic forms, resolved M2-branes, and AdS(4)/CFT(3) correspondence," Nucl. Phys. B **617** (2001) 151 [arXiv:hep-th/0102185].

[21] C. W. Bernard, "Gauge Zero Modes, Instanton Determinants, And Quantum-Chromodynamic Calculations," Phys. Rev. D **19** (1979) 3013.

[22] A. V. Belitsky, S. Vandoren and P. van Nieuwenhuizen, "Yang-Mills and D-instantons," Class. Quant. Grav. **17** (2000) 3521 [arXiv:hep-th/0004186].

[23] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis and S. Vandoren, "Multi-instanton calculus and the AdS/CFT correspondence in N = 4 superconformal field theory," Nucl. Phys. B **552**, 88 (1999) [arXiv:hep-th/9901128].

[24] R. Gilmore, "Lie Groups, Lie Algebras and some of their Applications", 1974, John Wiley; ISBN: 0471301795

[25] E. Witten, "Constraints On Supersymmetry Breaking," Nucl. Phys. B **202** (1982) 253.

[26] L. Alvarez-Gaume, "Supersymmetry And The Atiyah-Singer Index Theorem," Commun. Math. Phys. **90** (1983) 161.

[27] G. W. Gibbons and N. S. Manton, "Classical And Quantum Dynamics Of Bps Monopoles," Nucl. Phys. B **274** (1986) 183.

[28] H. Osborn, "Semiclassical Functional Integrals For Selfdual Gauge Fields," Annals Phys. **135** (1981) 373.

[29] J. M. Figueroa-O'Farrill, "BUSSTEPP lectures on supersymmetry," arXiv:hep-th/0109172.

[30] T. Eguchi, P. B. Gilkey and A. J. Hanson, "Gravitation, Gauge Theories And Differential Geometry," Phys. Rept. **66** (1980) 213.

[31] J. P. Gauntlett, "Low-energy dynamics of N=2 supersymmetric monopoles," Nucl. Phys. B **411** (1994) 443 [arXiv:hep-th/9305068].

[32] C. Montonen and D. I. Olive, "Magnetic Monopoles As Gauge Particles?," Phys. Lett. B **72** (1977) 117.

[33] L. Alvarez-Gaume and D. Z. Freedman, "Potentials For The Supersymmetric Nonlinear Sigma Model," Commun. Math. Phys. **91** (1983) 87.

[34] L. Alvarez-Gaume and D. Z. Freedman, "Geometrical Structure And Ultraviolet Finiteness In The Supersymmetric Sigma Model," Commun. Math. Phys. **80** (1981) 443.

[35] M. Berkooz, M. Rozali and N. Seiberg, "On transverse fivebranes in M(atrix) theory on T**5," Phys. Lett. B **408** (1997) 105 [arXiv:hep-th/9704089].

[36] N. Dorey, T. J. Hollowood and V. V. Khoze, "Notes on soliton bound-state problems in gauge theory and string theory," arXiv:hep-th/0105090.

[37] A. Maciocia, "Metrics On The Moduli Spaces Of Instantons Over Euclidean Four Space," Commun. Math. Phys. **135** (1991) 467.

[38] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, "Construction Of Instantons," Phys. Lett. A **65** (1978) 185.

[39] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, "Hyperkahler Metrics And Supersymmetry," Commun. Math. Phys. **108** (1987) 535.

[40] G. 't Hooft, "Computation Of The Quantum Effects Due To A Four-Dimensional Pseudoparticle," Phys. Rev. D **14** (1976) 3432 [Erratum-ibid. D **18** (1978) 2199].

[41] C. Callias and C. H. Taubes, "Functional Determinants In Euclidean Yang-Mills Theory," Commun. Math. Phys. **77** (1980) 229.

[42] A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Y. S. Tyupkin, "Pseudoparticle Solutions Of The Yang-Mills Equations," Phys. Lett. B **59** (1975) 85.