Drift parameter estimates for stochastic differential equations of mean-reversion type arising from financial modelings.

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Drift Parameter Estimates For Stochastic Differential Equations Of Mean-Reversion Type Arising From Financial Modelings

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Abstract

In this thesis, we aim to study parametric and nonparametric estimation with stochastic differential equations of mean-reversion type and their applications. we have done two lines of research works in this thesis. First, we apply mean reversion feature to the problem of parameter estimation for stochastic differential equation with small noise on discrete observations. Girsanov Transformation and least squares method are applied to get an estimator of the drift parameter. Then we begin to study the consistency and the rate of convergence of the least squares estimator and establish the asymptotic distribution of the least squares estimator. Moreover, an illustrative example with linear case and an application to a credit risk modeling is given. After that, we focus our study on nonlinear mean functional parameter estimation. With similar method, we give proof on the consistency and the asymptotic distribution. For the second line of this thesis, we focus on the nonparametric drift estimation for inhomogeneous stochastic differential equation driven by Brownian motion with sampling data. Kernel density function is applied to get the estimator of the drift parameter. Meanwhile the consistency and rate of convergence of the drift estimator are proved. Then the asymptotic distribution of the least squares estimator is established. Finally, we study an example case.

Keywords: Girsanov transformation; least squares method; discrete observation; mean-reverting processes; Brownian motion; consistency of least squares estimator; asymptotic distribution of LSE; nonparametric; inhomogeneous; rating process; zero coupon bond; nonparametric

Chapter 1

Introduction

This thesis is mainly concerned with parametric and nonparametric estimation for mean reversion feature stochastic differential equations driven by Brownian motion with sampling data and their applications.

In the past, for the parametric estimation method, when the driving noise is Brownian motion, with small white noise based on continuous-time observations, Prakasa Rao [41], Liptser and Shiryaev [29], Kutoyants [26] use maximum likelihood estimator method based on the Girsanov density with the continuous observation. Meanwhile, Sϕrensen [48] gave a survey of existing estimation techniques for stationary and ergodic diffusion processes observed at discrete points in time. For the least squares estimate method, Dorogoveev [6] and Le Breton [27] proved the convergence in probability and in Kasonga [21], they defined the least squares estimator and show the strong consistency under some regularity conditions. Moreover, Prakasa Rao [39] gave a study on the asymptotic distribution. Further, Shimizu and Yoshida [45] considered a multidimensional diffusion process with jumps whose jump term is driven by a compound poisson process. They let $\alpha(x, \theta)$ be the drift part and $b(x, \sigma)$ be the diffusion coefficient and study estimation of the parameter $\alpha(x, \theta) = (\theta, \sigma)$. Under certain assumptions, the consistency and asymptotic normality of an estimator were shown. Shimizu [46] considered a similar case and proposed an estimating function under complicated situation.

The asymptotic theory of parametric estimation for diffusion processes with small white noise, based on continuous-time observations is well developed (see, e.g., Kutoyants [25], Kutoyants [24], Uchida and Yoshida [56], Yoshida [60] and Yoshida [59]). For instance, Yoshida [56] considered the evaluation problem of statistical models for diffusion processes with small noise. There were many applications of small noise asymptotic to mathematical finance,(see, e.g.,Kunitomo and Takahashi [23], Long [31], Takahashi [49], Takahashi and Yoshida [50], Uchida and Yoshida [55], and Yoshida [58]). For example, Kunitomo and Takahashi [23] proposed a new methodology for the valuation problem of financial contingent claims when the underlying asset prices follow a general class of continuous Itô processes. And then they gave two examples on the valuation problems of average options for interest rates. However, estimation for diffusion processes with small noise based on discretely observations is used more frequently, since the actual data may be obtained discretely. So, we begin with our study on this direction. Long [30] gave a study on the parameter estimation for discretely observed one dimensional Ornstein-Uhlenbeck processes with small Lévy noises. It assumed that the drift function $b(x, \theta) = -\theta x$ was linear for both x and θ. Meanwhile the driving Lévy process was $L_t = aB_t + bZ_t$, where a and b were known constants, $\{B_t, t \geq 0\}$ was a standard Brownian motion and Z_t was a $\alpha-$stable Lévy motion independent of $\{B_t, t \geq 0\}$. Under this framework, he established the consistency and asymptotic normality for the proposed estimators. In Long [31], he investigated the parameter estimation problem for discrete observations with small Levy noises. In Long [31], he gave a discussion on a case of the drift function $b(x, \theta) = \theta b(x)$. Under some regularity conditions, he obtained the consistency and rate of convergence of the least squares estimator when a small dispersion parameter $\varepsilon \to 0$ and $n \to \infty$ simultaneously. Meanwhile, they gave the result of asymptotic distribution which was shown to be the convolution of a stable distribution. In a similar framework, Ma [33] extended the results of Long [31] to the case where the driving noise was a general Lévy process. After that, in Hu and Long [16], least squares estimator for Ornstein-Uhlenbeck processes driven by α-stable motions was studied. The main focus of Hu and Long [16] was to study the strong consistency and asymptotic distributions of the least squares estimator for generalized O-U processes. After using least squares method to get a least squares estimator, they gave a proof on the strong consistency and the rate of convergence of the estimator. Finally they got an asymptotic distribution. In Hu and Long [17], results in [16] was extended for the case that the drift function $b(x, \theta) = \alpha_0 - \theta_0 x$. When $\alpha_0 = 0$, the mean-reverting α-stable motion becomes O-U process. Under certain conditions, by using least squares method, they proved the consistency and asymptotic distribution.

On the other side, when the drift function is unknown, nonparametric estimation method was used. The problem of nonparametric estimation of a density function has received extensive attention since Nadaraya [35] and Watson [57] introduced the Nadaraya-Watson estimator of the regression function. In these papers they extended the methods for estimation of probability densities to regression functions. After that consistency and rate of convergence were established by Härdle [14]. A growing body of literature exists on the related problem of nonparametric estimation of unknown regression functions(see Collomb [5]). Most of the literature on nonparametric regression function estimation deals with the kernel method and its variants. When the stochastic process is stationary, Robinson [42], Roussas [44], Tran [51], Kim and Cox [22], Nze and Rios [36], Liebscher [28] derived the strong convergence and central limit theorem of the kernel density estimator. For instance, in Roussas [44], they considered the nonparametric estimation in mixing sequences of random variables. Under some conditions, they showed the strongly consistent estimates. Moreover, Hall, Peng and Yao [13] gave the nonparametric regression estimation for time series with heavy tail. To the extension of the discrete time series with heavy tail, a regression type of estimation for stochastic processes driven by Lévy motion was discussed by Long and Qian [32].

Meanwhile, in the financial field, stochastic differential equations with mean reversion type play an important role. The phenomenon of mean reversion is a tendency generally for a stochastic process to remain near, or return over time to a long-run average. Specifically, mean reversion in credit risk means that good credits today tend to become somewhat worse credits over time and bad credits tend to become better credits over time. Hodges and Carverhill [15] characterized the behavior of the drift function in an equilibrium economy.

They assumed the stock price follows a mean reversion type stochastic differential equation

$$\frac{dX_t}{X_t} = (r + \sigma\alpha(X_t, t))dt + \sigma dW_t$$

where the risk free interest rate r and volatility σ are constants, W_t is a Brownian motion, the correction term $\alpha(X_t, t)$ with $\alpha : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a $\mathcal{C}^{2,1}$−fucntion. They used a binomial tree argument in the discrete time setting and Taylor expansion to prove that the the instantaneous reward per unit of risk α follows the Burgers Equation

$$\frac{\partial}{\partial t}\alpha(x,t) = -\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\alpha(x,t) - \sigma\alpha(x,t)\frac{\partial}{\partial x}\alpha(x,t).$$

In addition, Stein and Stein [47] proved a similar result for their stochastically varying volatility, which was driven by an arithmetic Ornstein-Uhlenbeck process.

Motivated by the above research, we have done two lines of research works in this thesis. First, we apply mean reversion SDEs to the problem of parameter estimation for stochastic differential equations with small noise on discrete observations. Then we study the consistency and the rate of convergence of the estimator and establish the asymptotic distribution of the least squares estimator with nonlinear mean parameter. Second, we focus on the nonparametric drift estimation for inhomogeneous stochastic differential equations driven by Brownian motion with sampling data. Kernel density function is applied to get the estimator of the drift parameter. Then the consistency and rate of convergence of the drift estimator are proved. After that the asymptotic distribution of the estimator is established.

The rest of this thesis is organized as follow:

Chapter 2 prepares some preliminaries, which will be used in later derivations and proofs. First, we give a brief introduction on SDEs, especially on the existence and uniqueness of the solutions to SDEs. Then we discuss the Girsanov theorem. Finally, we show some useful estimates, limits and inequalities.

In Chapter 3, we consider the mean-reversion type SDE, for which the drift function is $[r+\alpha(X_t, t, \varepsilon)]b(X_t, t)$, and diffusion coefficient is $\varepsilon\sigma(X_t, t)$, where $b(x,t) : \mathbb{R}\times[0,1] \to \mathbb{R}\backslash\{0\}$ and $\sigma(x,t) : \mathbb{R} \times [0,1] \to \mathbb{R}$ are continuous with respect to t; $\varepsilon \in (0,1]$ is a parameter; $\alpha(x,t,\varepsilon) : \mathbb{R} \times [0,1] \times (0,1] \to \mathbb{R}$ is twice differentiable with respect to x and differentiable with respect to t. Due to the complexity of the drift item $\alpha(X_t, t, \varepsilon)$, we utilize the Girsanov Transformation to get rid of this part, which changes measure P to measure Q_ε. Then we provide an explicit least squares estimator from which we can make a proof for the convergence from least squares estimator to the true value with certain conditions. Moreover, we also give a proof on the asymptotic of the least squares estimator. Then, the asymptotic distribution is proved under probability measure Q. After that, we give an illustrative example. Finally, as an application, we apply our mean-reversion approach to a rating process in the range of zero coupon bond.

In Chapter 4, we extend the mean reversion stochastic differential equation to the situation where the mean parameter is nonlinear. We change the drift part to $[r(\theta)+\alpha(X_t, t, \varepsilon)]X_t$. With the similar method, after using Girsanov Transformation, we have a new SDE under measure Q_ε. Then we prove the consistency of least square estimator and illustrate the asymptotic distribution.

In Chapter 5, we extend the situation to nonparametric approach. We give a study on the problem of nonparametric estimation for inhomogeneous stochastic differential equations driven by Brownian motion as following

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t \quad 0 \leq t \leq T$$

where $\mu(x, t) : \mathbb{R} \times [0, T] \to \mathbb{R}$ is a measurable function which is continuous with respect to t and $\sigma(x, t) : \mathbb{R} \times [0, T] \to \mathbb{R}_+$ is a positive function which is continuous with respect to t. Then we prove the the consistency and the asymptotic distribution of least squares estimator. Finally, an example is given for $\mu(X_t, t) = [a(X_t, t) + br(X_t, t)]$ and $\sigma(X_t, t) = b$.

In summary, in this thesis, we address three topics from mathematical modeling for finance. At the very beginning of our topics, we will outline some preliminaries in Chapter 2.

Chapter 2

Preliminaries

In this chapter, we introduce some notations and preliminaries. In the whole thesis, we use notation "$\to_Q$" to denote "convergence in probability Q"; notation "$\to_P$" to denote "convergence in probability P" and notation "$\Rightarrow$" to denote "convergence in distribution". Moreover, $o_P(1)$ denotes a sequence of random variables converging to zero in probability; $O_P(1)$ means a sequence of random variables converging to a finite constant in probability. All the contents in this chapter is mainly based on the books by Økesenda [37], Ikeda and Watanabe [19] and Mao [34].

2.1 Introduction to Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with right-continuous increasing family $\{\mathcal{F}_t\}_{t\geq 0}$ of sub σ-fields of $\mathcal{F}$. Let $W = (W_t)_{t\geq 0}$ be a one-dimensional complete $\{\mathcal{F}_t\}$-Brownian motion, i.e., a stochastic process starting at 0 with independent and stationary increments and normally distributed with mean 0 and variance t.

Let $\mathcal{A}$ denote the collection of all $\mathcal{B}(\mathbb{R} \times [0,\infty))/\mathcal{B}(\mathbb{R})$-measurable functions $a : \mathbb{R} \times [0,\infty) \to \mathbb{R}$. Given $a, b \in \mathcal{A}$, we consider a stochastic differential equation(SDE) of the form

$$dX_t = a(X_t,t)dt + b(X_t,t)dW_t, \qquad t \geq 0. \tag{2.1}$$

Here, $a(X_t, t)$ is called the drift coefficient and $b(X_t, t)$ the diffusion coefficient.

A solution of the equation (2.1) is a continuous stochastic process $X = (X_t)_{t\geq 0}$ on the given probability setting $(\Omega, \mathcal{F}, P)$ with a $\{\mathcal{F}_t\}$-Brownian motion $W = (W_t)_{t\geq 0}$ such that $X = (X_t)_{t\geq 0}$ is $\{\mathcal{F}_t\}$-adapted and, with probability one

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s, \quad t \geq 0 \tag{2.2}$$

where $X_0 \in \mathbb{R}$ is a given initial data. This solution $X = (X_t)_{t\geq 0}$ is called an Itô diffusion. If the coefficients a and b are only functions of variable x, that is

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s,$$

we call $(X_t)_{t\geq 0}$ a time-homogeneous Itô diffusion process.

The existence and uniqueness of a solution of (2.1) can be verified by the local Lipshcitz condition and the linear growth condition. Let $a : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $b : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ continue with respect to t. For the local Lipschitz condition, we assume that there exists a constant $D_n > 0$ and $x, y \in \mathbb{R}$ with $|x| \vee |y| \leq n$ for every integer $n \geq 1$ such that

$$|a(x,t) - a(y,t)| \vee |b(x,t) - b(y,t)| \leq D_n|x - y| \quad t \geq 0.$$

For the linear growth condition, there exists a constant $C > 0$ such that,

$$|a(x,t)| + |b(x,t)| \leq C(1 + |x|) \ \ x \in \mathbb{R}, \ t \geq 0.$$

One can refer to a solution in a strong sense or in a weak sense. A strong solution means that the solution is constructed on a given probability space, e.g. $(\Omega, \mathcal{F}, P)$ with respect to a given filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and a given Brownian motion W_t on it. In contrast, a weak solution is to say that, given the two functions $a(x,t)$ and $b(x,t)$, we can find a pair (X_t, W_t) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ which satisfy equation (2.2).

Throughout the thesis, we mean a solution to a SDE in the unique strong sense.

2.2 Girsanov Theorem

Apart from the general theory on existence and uniqueness for SDEs, there is a powerful probabilistic tool called the Girsanov Transformation. It solves SDEs by changing the underlying probability measure. We state the result according to Øksendal [37].

The Girsanov Theorem 1

Let $Y(t) \in \mathbb{R}$ solve the Itô SDE

$$dY(t) = a(\omega, t)dt + dB(t); \qquad t \leq T, \;\; Y_0 = 0,$$

where $T \leq \infty$ is a given constant and $B(t)$ is a one dimensional Brownian motion. Let

$$M_t = \exp\Big(- \int_0^t a(\omega, s)dB_s - \frac{1}{2}\int_0^t a^2(\omega, s)ds \Big); \qquad 0 \leq t \leq T.$$

Assume that M_t is a martingale with respect to $\mathcal{F}_t$ and P. Define the measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) = M_T(\omega)dP(\omega). \tag{2.3}$$

Q is a probability measure on $\mathcal{F}_T$ and $(Y(t))_{t\in[0,T]}$ is a Brownian motion w.r.t. Q.

Remark 2.2.1.

(1) The transformation $P \to Q$ given by (2.3) is called the Girsanov transformation of measures.

(2) The following Novikov condition is sufficient to guarantee that $\{M_t\}_{t\leq T}$ is a martingale (w.r.t. $\mathcal{F}_t$ and P):

$$E\Big[\exp\Big(\frac{1}{2}\int_0^T a^2(\omega, s)ds\Big)\Big] < \infty$$

where $E = E_P$ is the expectation w.r.t. P.

(3) Since M_t is a martingale, we have

$$M_T dP|\mathcal{F}_t = M_t dP. \tag{2.4}$$

The Girsanov Theorem 2

Let $Y(t) \in \mathbb{R}$ be an Itô process of the form

$$dY(t) = \beta(\omega, t)dt + \theta(\omega, t)dB(t); \qquad t \leq T$$

where $B(t) \in \mathbb{R}$, $\beta(\omega, t) \in \mathbb{R}$ and $\theta(\omega, t) \in \mathbb{R}$. Suppose there exist processes $u(\omega, t) \in \mathcal{A}$ and $\alpha(\omega, t) \in \mathcal{A}$, such that

$$\theta(\omega, t)u(\omega, t) = \beta(\omega, t) - \alpha(\omega, t).$$

Let

$$M_t = \exp\Big(- \int_0^t u(\omega, s)dB_s - \frac{1}{2}u^2(\omega, s)ds\Big); \quad t \leq T \tag{2.5}$$

and

$$dQ(\omega) = M_T(\omega)dP(\omega) \quad on\ \mathcal{F}_T. \tag{2.6}$$

Assume that M_t is a martingale (w.r.t. $\mathcal{F}_t$ and P). Then Q is a probability measure on $\mathcal{F}_T$, the process

$$\hat{B}(t) := \int_0^t u(\omega, s)ds + B(t); \quad t \leq T \tag{2.7}$$

is a Brownian motion w.r.t. Q and in terms of $\hat{B}(t)$ the process $Y(t)$ solves the equation

$$dY(t) = \alpha(\omega, t)dt + \theta d\hat{B}(t).$$

Remark 2.2.2.

1. We note that the following Novikov condition is sufficient to guarantee that M_T is a martingale:

$$E\Big[\exp\Big(\frac{1}{2}\int_0^T u^2(\omega, s)ds\Big)\Big] < \infty.$$

2. In most applications, the process $\alpha(\omega, t)$ is chosen to be 0. Then the process $Y(t)$ becomes

$$dY(t) = \theta(\omega, t)d\tilde{B}(t),$$

which implies that $Y(t)$ is a local martingale w.r.t. Q. In this case Q is called an equivalent local martingale measure.

The Girsanov Theorem 3

Let $X(t) \in \mathbb{R}$ and $Y(t) \in \mathbb{R}$ be an Itô diffusion and an Itô process, respectively,

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t); \quad t \leq T, \quad X(0) = x$$

$$dY(t) = [\gamma(\omega, t) + b(Y(t))]dt + \sigma(Y(t))dB(t); \quad t \leq T, \quad Y(0) = x$$

where the functions $b : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$ satisfy the lipschiz condition and linear growth condition and $\gamma \in \mathcal{A}$, $x \in \mathbb{R}$. Suppose there exists a process $u(\omega, t) \in \mathcal{A}$ such that

$$\sigma(Y(t))u(\omega, t) = \gamma(\omega, t).$$

Then, define M_t, Q and $\hat{B}(t)$ as in (2.5), (2.6) and (2.7). Assume that M_t is a martingale w.r.t. $\mathcal{F}_t$ and P. Then Q is a probability measure on $\mathcal{F}_T$ and

$$dY(t) = b(Y(t))dt + \sigma(Y(t))d\hat{B}(t).$$

Therefore, the Q-law of $Y(t)$ is the same as the P-law of $X(t)$.

2.3 Some Useful Estimates, Limits and Inequalities

Lemma 2.3.1. *(Burkholder-Davis-Gundy Inequality) Let $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R})$. Define, for $0 \leq t \leq T$,*

$$x(t) = \int_0^t g(s)dB(s)$$

and

$$A(t) = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, there exist universal positive constants c_p, C_p(depending only on p), such that

$$c_p E|A(t)|^{\frac{p}{2}} \leq E\left(\sup_{0 \leq s \leq t} |x(s)|^p\right) \leq C_p E|A(t)|^{\frac{p}{2}}$$

for all $t \geq 0$. In particular, one may take $c_p = (p/2)^p, C_p = (32/p)^{p/2}$, if $0 < p < 2$; $c_p = 1, C_p = 4$, if $p = 2$; $c_p = (2p)^{-p/2}, C_p = [p^{p+1}/2(p-1)^{p-1}]^{p/2}$, if $p > 2$.

Lemma 2.3.2. *(Gronwall's Inequality) Let $f(t)$, $g(t)$ and $h(t)$ be continuous function on some interval $[a, b]$ and $h(t) \geq 0$. If*

$$f(t) \leq g(t) + \int_a^t h(s)f(s)ds, \qquad for \quad t \in [a, b],$$

then

$$f(t) \leq g(t) + \int_a^t g(s)h(s)e^{\int_a^t h(\alpha)d\alpha}ds \qquad \alpha \in [a, b].$$

Lemma 2.3.3. *(Markov Inequality) For each constant $c > 0$, any non-negative integrable random variable Y satisfies the inequalities*

$$P[Y \geq c] \leq E\Big(\frac{Y}{c}\Big).$$

Lemma 2.3.4. *(Cauchy-Schwarz Inequality) Let $(a_1,\ a_2, \ldots\ a_n)$ and $(b_1,\ b_2, \ldots,\ b_n)$ be two sequences of real numbers, then*

$$\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \geq \Big(\sum_{i=1}^{n} a_i b_i\Big)^2.$$

Lemma 2.3.5. *(Lebesgue Dominated Convergence Theorem) Suppose $f_n : \mathbb{R} \rightarrow [-\infty, \infty]$ are measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Assume there is an integrable $g : \mathbb{R} \rightarrow [0, \infty]$ with $|f_n(x)| \leq g(x)$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu.$$

Lemma 2.3.6. *(Hölder's Inequality) Let $a_1,\ a_2, \ldots,\ a_n,\ b_1,\ b_2, \ldots, b_n$ be nonnegative numbers. Let $p > 1,\ q > 1$ be real number with*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$\sum_{j=1}^{n} a_j b_j \leq \Big(\sum_{j=1}^{n} a_j^p\Big)^{\frac{1}{p}} \cdot \Big(\sum_{j=1}^{n} b_j^q\Big)^{\frac{1}{q}}.$$

Lemma 2.3.7. *(Slutsky's Theorem) Suppose that $X_n \rightarrow_d X$ and $Y_n \rightarrow_p \theta$ (a constant). Then*

$$(a) X_n + Y_n \rightarrow_d X + \theta.$$

$$(b) X_n Y_n \rightarrow_d \theta X.$$

Lemma 2.3.8. *(Taylor's Expansion) If a function $f(x)$ has continuous derivatives up to $(n+1)^{th}$ order, then this function can be expanded in the following fashion:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n$$

where R_n *is called the remainder after* $n+1$ *terms and* R_n *is given by:*

$$R_n = \int_a^x f^{(n+1)}(u)\frac{(x-u)^n}{n!}du = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!} \qquad a < \xi < x$$

when this expansion converges over a certain range of x*, that is,* $\lim_{n\to\infty} R_n = 0$*, then the expansion is called the Taylor Series of* $f(x)$ *expanded about* a*.*

Remark 2.3.1. *Formally, let* $\{X_n\}$ *be a sequence of random variables defined on the probability space* $(\Omega, \mathcal{F}, Q)$*. We say that* $\{X_n\}$ *is convergent in probability to a random variable* X *defined on* $(\Omega, \mathcal{F}, Q)$ *if and only if*

$$\lim_{n\to\infty} Q(|X_n - X| > \varepsilon) = 0$$

where $\varepsilon > 0$*.*

Remark 2.3.2. *A sequence of random variables,* X_1*,* X_2*,..., converge in distribution to a random variable* X *if*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

at all points x *where* $F_X(x)$ *is continuous.*

Chapter 3

Drift Parameter Estimates For SDEs with Discrete Observations

3.1 Introduction

First, we let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ i.e.$\mathcal{F}_t \subset \mathcal{F}_1$ for $0 \leq t \leq 1$. Then the stochastic process $X = (X_t, 0 \leq t \leq 1)$ with a given initial value $X_0 = x \in \mathbb{R}$, is determined by the following mean reversion stochastic differential equation (SDE)

$$dX_t = [r + \alpha(X_t, t, \varepsilon)]b(X_t, t)dt + \varepsilon\sigma(X_t, t)dB_t, \qquad 0 \leq t \leq 1. \tag{3.1}$$

Where $b : \mathbb{R} \times [0, 1] \to \mathbb{R} \setminus \{0\}$ and $\sigma : \mathbb{R} \times [0, 1] \to \mathbb{R}$ are continuous with respect to t; $\varepsilon \in (0, 1]$ is a parameter; $\alpha : \mathbb{R} \times [0, 1] \times (0, 1] \to \mathbb{R}$ is twice differentiable with respect to x and differentiable with respect to t; B_t is a one dimensional $\{\mathcal{F}_t\}$−Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{0\leq t\leq 1}\}$. We assume that it meets the following condition:

(1) $|b(x, t) - b(y, t)| \leq L|x - y|$, where $L > 0$ is a constant, $x, y \in \mathbb{R}$.

(2) $|\alpha(x, t, \varepsilon)b(x, t) - \alpha(y, t, \varepsilon)b(y, t)| \leq L''|x - y|$, where $L'' > 0$ is a constant, $x, y \in \mathbb{R}$.

(3) $|\sigma(x, t) - \sigma(y, t)| \leq L'|x - y|$, where $L' > 0$ is a constant, $x, y \in \mathbb{R}$;

(4) $\sigma^{-2}(x, t) \leq K'(1 + |x|^m)$ where $K' > 0, m > 0$ and $x \in \mathbb{R}$.

The purpose of this chapter is to investigate the least squares estimator for the true value of r based on the sampling data $(X_{t_i})_{i=1}^n$.

In this chapter, we consider a general class of stochastic process with a mean reverting feature satisfying (3.1). The main difficulty in such a case is the stochastic item $\alpha(X_t, t)$ on (3.1). We use Girsanov Transformation to get rid of the item $\alpha(X_t, t)$.

By (3.1), we consider the discrete-time system

$$X_{t_k} = x + \sum_{i=1}^{n}[r + \alpha(X_{t_{i-1}}, t_{i-1}, \varepsilon)]b(X_{t_{i-1}}, t_{i-1})\Delta t_i + \varepsilon \sum_{i=1}^{i} \sigma(X_{t_{i-1}}, t_{i-1})(B_{t_i} - B_{t_{i-1}})$$

where $\Delta t_i = t_i - t_{i-1}$. We want to obtain the true value of r based on the sampling data $(X_{t_i})_{i=1}^n$. We define

$$u_\varepsilon(X_t, t) := \frac{\alpha(X_t, t, \varepsilon)b(X_t, t)}{\varepsilon\sigma(X_t, t)}, \tag{3.2}$$

which satisfies the condition

$$E\left[\exp\left(\frac{1}{2}\int_0^t |u_\varepsilon(X_s, s)|^2 ds\right)\right] < \infty, \quad t \geq 0. \tag{3.3}$$

Then, we define

$$M_t^\varepsilon = \exp\left(-\int_0^t u_\varepsilon(X_s, s)dB_s - \frac{1}{2}\int_0^t u_\varepsilon^2(X_s, s)ds\right), \quad t \geq 0 \tag{3.4}$$

where M_t^ε is an $\{\mathcal{F}_t\}$−martingale. Let Q_ε be a probability measure on $\mathcal{F}_1$, satisfying

$$dQ_\varepsilon := M_1^\varepsilon dP. \tag{3.5}$$

Then define

$$\hat{B}_t^\varepsilon := \int_0^t u_\varepsilon(X_s, s)ds + B_t \tag{3.6}$$

where $\hat{B}_t^\varepsilon$ is an $\mathcal{F}_t$− Brownian motion with respect to Q_ε. Then we have

$$dX_t = rb(X_t, t)dt + \varepsilon\sigma(X_t, t)d\hat{B}_t^\varepsilon. \tag{3.7}$$

Assume that the process X_t is observed at regularly spaced time points $\{t_i = \frac{i}{n}, i = 1, 2, ..., n\}$. We represent the true value of the parameter r by r_0 and least square estimator of r by $\hat{r}$. As mentioned before we focus on investigation of the least squares estimator for the true value r_0 based on the sampling data $(X_{t_i})_{i=1}^n$ determined by

$$X_{t_i} = x + \sum_{i=1}^{n} rb(X_{t_{i-1}}, t_{i-1})\Delta t_i + \varepsilon \sum_{i=1}^{n} \sigma(X_{t_{i-1}}, t_{i-1})(\hat{B}^\varepsilon_{t_i} - \hat{B}^\varepsilon_{t_{i-1}}).$$

Let us start with the use of the least squares method to get a consistent estimator. First of all, we discretize (4.1)

$$X_{t_i} - X_{t_{i-1}} = rb(X_{t_{i-1}}, t_{i-1})\Delta t_i + \varepsilon\sigma(X_{t_{i-1}}, t_{i-1})\Delta\hat{B}^\varepsilon_{t_i}$$

where $\Delta t_i = t_i - t_{i-1} = \frac{1}{n}$; $\Delta\hat{B}^\varepsilon_{t_i} = \hat{B}^\varepsilon_{t_i} - \hat{B}^\varepsilon_{t_{i-1}}$ is the increment of Brownian motion. Then

$$\frac{X_{t_i} - X_{t_{i-1}} - rb(X_{t_{i-1}}, t_{i-1})\Delta t_i}{\varepsilon\sigma(X_{t_{i-1}}, t_{i-1})} = \Delta\hat{B}^\varepsilon_{t_i}$$

Since $\Delta\hat{B}^\varepsilon_{t_i}$ is a normal distribution with zero mean on $\{\Omega, \mathcal{F}, Q_\varepsilon\}$, we obtain the variance of $\Delta\hat{B}^\varepsilon_{t_i}$ and denote it by the following contrast function

$$\rho_{n,\varepsilon}(r) = \sum_{i=1}^{n} \left| \frac{X_{t_i} - X_{t_{i-1}} - rb(X_{t_{i-1}}, t_{i-1})\Delta t_i}{\varepsilon\sigma(X_{t_{i-1}}, t_{i-1})} \right|^2 .$$

In order to get the least square estimator $\hat{r}_{n,\varepsilon}$, let

$$\frac{\partial\rho_{n,\varepsilon}(r)}{\partial r} = 0.$$

Then we get the solution, denoted by $\hat{r}_{n,\varepsilon}$ which is given as

$$\hat{r}_{n,\varepsilon} = \frac{\sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1})(X_{t_i} - X_{t_{i-1}})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})}{n^{-1}\sum_{i=1}^{n} b^2(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})}. \tag{3.8}$$

In the following sections, we focus on the asymptotic of the least square estimator $\hat{r}_{n,\varepsilon}$ with high frequency $n \to \infty$ and small dispersion $\varepsilon \to 0$. In Section 3.2, we aim to prove that $\hat{r}_{n,\varepsilon} \to_{Q_\varepsilon} r_0$ in probability. In Section 3.3 we establish the rate of convergence and the asymptotic distribution, after that we give an illustrative example in Section 3.4. Finally a further application will be discussed in Section 3.5.

3.2 Consistency of The Least Squares Estimator

At the beginning of this part, we give two lemmas as follows.

Let X_t^0 be the solution of the following ordinary differential equation under the true value of the drift parameter,

$$dX_t^0 = r_0 b(X_t^0, t)dt, \quad X_0^0 = x_0. \tag{3.9}$$

where r_0 be the real value of r.

Lemma 3.2.1. *Under conditions (1),(2),(3), we have*

$$|X_t - X_t^0| \leq \varepsilon e^{L|r_0|t} \sup_{\delta \in [0,t]} \Big| \int_0^\delta \sigma(X_s, s) d\hat{B}_t^\varepsilon \Big|. \tag{3.10}$$

Proof. We have

$$X_t^0 = x_0 + r_0 \int_0^t b(X_s^0, s)ds. \tag{3.11}$$

From (3.7) we have

$$X_{t_i} - X_{t_{i-1}} = r_0 \int_{t_{i-1}}^{t_i} b(X_s, s)ds + \varepsilon \int_{t_{i-1}}^{t_i} \sigma(X_s, s) d\hat{B}_s^\varepsilon. \tag{3.12}$$

Together with (3.11) and (3.12), we obtain

$$X_t - X_t^0 = r_0 \int_0^t (b(X_s, s) - b(X_s^0, s))ds + \varepsilon \int_0^t \sigma(X_s, s) d\hat{B}_s^\varepsilon. \tag{3.13}$$

By the condition (1), we get

$$|X_t - X_t^0| \leq L|r_0| \int_0^t |X_s - X_s^0| ds + \varepsilon \Big| \int_0^t \sigma(X_s, s) d\hat{B}_s^\varepsilon \Big|. \tag{3.14}$$

By the Gronwall's Inequality, we get

$$|X_t - X_t^0| \leq \varepsilon e^{L|r_0|t} \sup_{\delta \in [0,t]} \Big| \int_0^\delta \sigma(X_s, s) d\hat{B}_s^\varepsilon \Big|.$$

□

Lemma 3.2.2. *Under conditions (1),(2),(3), we have*

$$\sup_{0 \leq t \leq 1} |X_t - X_t^0| \rightarrow_{Q_\varepsilon} 0 \text{ as } \varepsilon \rightarrow 0. \tag{3.15}$$

Proof. By Lemma 3.2.1, it yields that

$$\sup_{t\in[0,1]} |X_t - X_t^0| \leq \varepsilon e^{L|r_0|} \sup_{\delta\in[0,t]} \left| \int_0^\delta \sigma(X_s, s) d\hat{B}_t^\varepsilon \right|. \tag{3.16}$$

Let $\eta > 0$, by Lemma 2.3.1, Markov Inequality and condition (4), we have

$$\begin{aligned} Q_\varepsilon(\varepsilon e^{L|r_0|} \sup_{\delta\in[0,t]} \left| \int_0^\delta \sigma(X_s, s) d\hat{B}_s^\varepsilon \right| > \eta) &\leq \eta^{-1} e^{L|r_0|} \varepsilon E_{Q_\varepsilon} \Big[\sup_{\delta\in[0,t]} \left| \int_0^\delta \sigma(X_s, s) d\hat{B}_s^\varepsilon \right| \Big] \\ &\leq 4\sqrt{2}\eta^{-1} e^{L|r_0|} \varepsilon E_{Q_\varepsilon} \Big[\Big(\int_0^\delta |\sigma(X_s, s)|^2 ds \Big)^{\frac{1}{2}} \Big] \end{aligned} \tag{3.17}$$

Since $\sigma(x, t)$ satisfies the Lipshcitz condition and continues with respect to t, so that $\sigma(x, t)$ meets following linear growth condition.

$$|\sigma(x, t)|^2 \leq K(1 + |x|^2)$$

where $K > 0$ is a constant. Then we have

$$Q_\varepsilon(\varepsilon e^{L|r_0|} \sup_{\delta\in[0,t]} \left| \int_0^\delta \sigma(X_s, s) d\hat{B}_s^\varepsilon \right| > \eta) \leq 4\sqrt{2}\eta^{-1} e^{L|r_0|} \varepsilon \Big(\int_0^\delta K(1 + E_{Q_\varepsilon}|X_s|^2) ds \Big)^{\frac{1}{2}}. \tag{3.18}$$

By Hölder's Inequality, Itô isometry and Gronwall's Inequality, we can obtain $E_{Q_\varepsilon}|X_s|^2 \leq C$, where $C > 0$ is a constant. Then we have

$$Q_\varepsilon(\varepsilon e^{L|r_0|} \sup_{\delta\in[0,t]} \left| \int_0^\delta \sigma(X_s, s) d\hat{B}_s^\varepsilon \right| > \eta) \leq 4\sqrt{2}\eta^{-1} e^{L|r_0|} \varepsilon [K(1 + C)\delta]^{\frac{1}{2}}. \tag{3.19}$$

The above equation implies that

$$\sup_{0\leq t\leq 1} |X_t - X_t^0| \to_{Q_\varepsilon} 0 \text{ as } \varepsilon \to 0.$$

□

At the beginning, we set r_0 be the true value of r, note from (3.7) that

$$X_{t_i} - X_{t_{i-1}} = r_0 \int_{t_{i-1}}^{t_i} b(X_s, s) ds + \varepsilon \int_{t_{i-1}}^{t_i} \sigma(X_s, s) d\hat{B}_s^\varepsilon. \tag{3.20}$$

This, together with (3.8), (3.12), yields that

$$
\begin{aligned}
\hat{r}_{n,\varepsilon} &= \frac{\sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1})(X_{t_i} - X_{t_{i-1}})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})}{n^{-1}\sum_{i=1}^{n} b^2(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})} \\
&= \frac{r_0 \sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} b(X_s, s)ds}{n^{-1}\sum_{i=1}^{n} b^2(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})} + \\
&\quad \frac{\varepsilon \sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} \sigma(X_s, s)d\hat{B}_s}{n^{-1}\sum_{i=1}^{n} b^2(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})} \\
&= r_0 + \frac{r_0 \sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} (b(X_s, s) - b(X_{t_{i-1}}, t_{i-1})ds}{n^{-1}\sum_{i=1}^{n} b^2(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})} \\
&\quad + \frac{\varepsilon \sum_{i=1}^{n} b(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} \sigma(X_s, s)d\hat{B}_s}{n^{-1}\sum_{i=1}^{n} b^2(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1})} \\
&:= r_0 + \frac{\phi_2(n, r)}{\phi_1(n, r)} + \frac{\phi_3(n, r)}{\phi_1(n, r)}.
\end{aligned}
$$

Theorem 3.2.1. *We have* $\hat{r}_{n,\varepsilon} \to_{Q_\varepsilon} r_0$*, as* $n \to \infty$ *and* $\varepsilon \to 0$ *with* $\varepsilon n^{\frac{1}{2}} \to 0$.

Lemma 3.2.3. *Let* $n \to \infty$ *and* $\varepsilon \to 0$ *with* $\varepsilon n^{\frac{1}{2}} \to 0$*, under conditions (1)-(4), we have* $\phi_1(n, \varepsilon) \to_{Q_\varepsilon} \int_0^t \sigma^{-2}(X_t^0, t) b^2(X_t^0, t)dt$.

Proof.

$$
\begin{aligned}
\phi_1(n, \varepsilon) &= n^{-1} \sum_{i=1}^{n} b^2(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) \\
&= n^{-1} \sum_{i=1}^{n} b^2(X_{t_{i-1}}^0, t_{i-1})\sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) \\
&\quad + n^{-1} \sum_{i=1}^{n} (b^2(X_{t_{i-1}}, t_{i-1}) - b^2(X_{t_{i-1}}^0, t_{i-1}))\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) \\
&\quad + n^{-1} \sum_{i=1}^{n} b^2(X_{t_{i-1}}^0, t_{i-1})(\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) - \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) \\
&:= \phi_{1,1}(n, \varepsilon) + \phi_{1,2}(n, \varepsilon) + \phi_{1,3}(n, \varepsilon).
\end{aligned}
$$

For $\phi_{1,1}(n, \varepsilon)$, according to the definition of Riemann Integral, we obtain $\phi_{1,1}(n, \varepsilon) \to_{Q_\varepsilon} \int_0^1 b^2(X_s^0, s)\sigma^{-2}(X_s^0, s)ds$ as $n \to \infty$.

For $\phi_{1,2}(n,\varepsilon)$, we have

$$
\begin{aligned}
|\phi_{1,2}(n,\varepsilon)| &\le n^{-1}\sum_{i=1}^{n} |b^2(X_{t_{i-1}},t_{i-1}) - b^2(X^0_{t_{i-1}},t_{i-1})|\sigma^{-2}(X_{t_{i-1}},t_{i-1}) \\
&\le n^{-1}\sum_{i=1}^{n} K'(1+|X_{t_{i-1}}|^m)|b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})||b(X_{t_{i-1}},t_{i-1}) + b(X^0_{t_{i-1}},t_{i-1})| \\
&\le n^{-1}\sum_{i=1}^{n} K'(1+|X_{t_{i-1}}|^m)\Big(|b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})|^2 \\
&\quad + 2|b(X^0_{t_{i-1}},t_{i-1})||b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})|\Big) \\
&\le n^{-1}K'\sum_{i=1}^{n} |b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})|^2 \\
&\quad + 2n^{-1}K'\sum_{i=1}^{n} |b(X^0_{t_{i-1}},t_{i-1})||b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})| \\
&\quad + n^{-1}K'\sum_{i=1}^{n} |X_{t_{i-1}}|^m|b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})|^2 \\
&\quad + 2n^{-1}K'\sum_{i=1}^{n} |b(X^0_{t_{i-1}},t_{i-1})||X_{t_{i-1}}|^m|b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})| \\
&:= \phi^1_{1,2}(n,\varepsilon) + \phi^2_{1,2}(n,\varepsilon) + \phi^3_{1,2}(n,\varepsilon) + \phi^4_{1,2}(n,\varepsilon).
\end{aligned}
$$

By condition (1), we have

$$
\begin{aligned}
\phi^1_{1,2}(n,\varepsilon) &\le n^{-1}L^2K'\sum_{i=1}^{n}|X_{t_{i-1}} - X^0_{t_{i-1}}|^2 \\
&\le L^2K'\Big(\sup_{0\le t\le 1}|X_t - X^0_t|\Big)^2.
\end{aligned}
$$

By Lemma 3.2.2, we get $\phi^1_{1,2}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $\varepsilon \to 0$.

$$
\begin{aligned}
\phi^2_{1,2}(n,\varepsilon) &\le 2n^{-1}LK'\sum_{i=1}^{n}|b(X^0_{t_{i-1}},t_{i-1})||X_{t_{i-1}} - X^0_{t_{i-1}}| \\
&\le 2LK'\sup_{0\le t\le 1}|X_t - X^0_t|\int_0^1 |b(X^0_t,t)|dt.
\end{aligned}
$$

By Lemma 3.2.2, we have $\phi^2_{1,2}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n\to\infty$ and $\varepsilon\to 0$.

For $\phi^3_{1,2}(n,\varepsilon)$, since

$$
\begin{aligned}
|X_{t_{i-1}}|^m &= (|X_{t_{i-1}} - X^0_{t_{i-1}}| + |X^0_{t_{i-1}}|)^m \\
&\le 2^m(|X_{t_{i-1}} - X^0_{t_{i-1}}|^m + |X^0_{t_{i-1}}|^m)
\end{aligned}
\tag{3.21}
$$

where $m \geq 1$. By condition (1), we have

$$\phi_{1,2}^3(n,\varepsilon) \leq n^{-1}K'L^2 2^m \sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|^{m+2} + n^{-1}K'L^2 2^m \sum_{i=1}^n |X_{t_{i-1}}^0|^m |X_{t_{i-1}} - X_{t_{i-1}}^0|^2.$$

By Lemma 3.2.2, we obtain $\phi_{1,2}^3(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. And by the same way, we can get $\phi_{1,2}^4(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. So that, we get $\phi_{1,2}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$. For $\phi_{1,3}(n,\varepsilon)$, by condition (1), (4) and (3.21), we have

$$\begin{aligned}
|\phi_{1,3}(n,\varepsilon)| &\leq n^{-1} \sum_{i=1}^n |\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) - \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})| b^2(X_{t_{i-1}}^0, t_{i-1}) \\
&\leq n^{-1} \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}, t_{i-1}) \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) |\sigma^2(X_{t_{i-1}}, t_{i-1}) - \sigma^2(X_{t_{i-1}}^0, t_{i-1})| b^2(X_{t_{i-1}}^0, t_{i-1}) \\
&\leq n^{-1} 2K'LK \sum_{i=1}^n (1 + |X_{t_{i-1}}|^m) |X_{t_{i-1}} - X_{t_{i-1}}^0| \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) b^2(X_{t_{i-1}}^0, t_{i-1}) \\
&\leq n^{-1} 2K'LK \sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0| \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) b^2(X_{t_{i-1}}^0, t_{i-1}) \\
&\quad + n^{-1} 2K'LK 2^m \sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0|^{m+1} \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) b^2(X_{t_{i-1}}^0, t_{i-1}) \\
&\quad + n^{-1} 2K'LK 2^m \sum_{i=1}^n |X_{t_{i-1}} - X_{t_{i-1}}^0| |X_{t_{i-1}}^0|^m \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) b^2(X_{t_{i-1}}^0, t_{i-1}) \\
&\leq 2K'LK \Big(\sup_{0 \leq t \leq 1} |X_t - X_t^0| + 2^m \sup_{0 \leq t \leq 1} |X_t - X_t^0|^{m+1} \Big) n^{-1} \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) b^2(X_{t_{i-1}}^0, t_{i-1}) \\
&\quad + 2K'LK 2^m \sup_{0 \leq t \leq 1} |X_t - X_t^0| n^{-1} \sum_{i=1}^n |X_{t_{i-1}}^0|^m \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1}) b^2(X_{t_{i-1}}^0, t_{i-1}).
\end{aligned}$$

We get $\phi_{1,3}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. □

Lemma 3.2.4. *We have $\phi_2(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$.*

Proof. From (3.12), we have

$$\begin{aligned}
|X_t - X_{t_{i-1}}| &\leq |r_0| \int_{t_{i-1}}^t (|b(X_s, s) - b(X_{t_{i-1}}, t_{i-1})| + |b(X_{t_{i-1}}, t_{i-1})|) ds + \varepsilon \Big| \int_{t_{i-1}}^t \sigma(X_s, s) d\hat{B}_s^\varepsilon \Big|. \\
&\leq |r_0| L \int_{t_{i-1}}^t |X_s - X_{t_{i-1}}| ds + n^{-1} |r_0| |b(X_{t_{i-1}}, t_{i-1})| + \varepsilon \sup_{t_{i-1} \leq t \leq t_i} \Big| \int_{t_{i-1}}^t \sigma(X_s, s) d\hat{B}_s^\varepsilon \Big|.
\end{aligned}$$

By Gronwall's Inequality, we obtain

$$|X_t - X_{t_{i-1}}| \le e^{r_0 L(t-t_{i-1})}\Big[n^{-1}|r_0||b(X_{t_{i-1}}, t_{i-1})| + \varepsilon \sup_{t_{i-1}\le t\le t_i}\Big|\int_{t_{i-1}}^{t} \sigma(X_s, s)d\hat{B}_s^{\varepsilon}\Big|\Big].$$

It yields that

$$\sup_{t_{i-1}\le t\le t_i}|X_t - X_{t_{i-1}}| \le e^{r_0 L n^{-1}}\Big[n^{-1}|r_0||b(X_{t_{i-1}}, t_{i-1})| + \varepsilon \sup_{t_{i-1}\le t\le t_i}\Big|\int_{t_{i-1}}^{t} \sigma(X_s, s)d\hat{B}_s^{\varepsilon}\Big|\Big].$$

By conditions (1), (4), we have

$$\begin{aligned}
\phi_2(n,\varepsilon) &\le |r_0|\sum_{i=1}^{n} K'(1+|X_{t_{i-1}}|^m)|b(X_{t_{i-1}}, t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}(b(X_s,s)-b(X_{t_{i-1}}, t_{i-1}))ds\Big| \\
&\le K'L|r_0|\sum_{i=1}^{n}(1+|X_{t_{i-1}}|^m)|b(X_{t_{i-1}}, t_{i-1})|\int_{t_{i-1}}^{t_i}|X_s - X_{t_{i-1}}|ds \\
&\le K'L|r_0|\sum_{i=1}^{n}(1+|X_{t_{i-1}}|^m)|b(X_{t_{i-1}}, t_{i-1})|n^{-1}\sup_{t_{i-1}\le t\le t_i}|X_t - X_{t_{i-1}}| \\
&\le K'L|r_0|^2 e^{\frac{|r_0|L}{n}} n^{-2}\sum_{i=1}^{n}(1+|X_{t_{i-1}}|^m)|b(X_{t_{i-1}}, t_{i-1})|^2 \\
&\quad + K'L|r_0|e^{\frac{|r_0|L}{n}} n^{-1}\varepsilon\sum_{i=1}^{n}(1+|X_{t_{i-1}}|^m)|b(X_{t_{i-1}}, t_{i-1})|\sup_{t_{i-1}\le t\le t_i}\Big|\int_{t_{i-1}}^{t}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&:= \phi_{2,1}(n,\varepsilon) + \phi_{2,2}(n,\varepsilon).
\end{aligned} \tag{3.22}$$

For $\phi_{2,1}(n,\varepsilon)$, by condition (1) and (3.21), we have

$$\begin{aligned}
\phi_{2,1}(n,\varepsilon) \le\ & K'L|r_0|^2 e^{\frac{|r_0|L}{n}} n^{-2}\sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m + 2^m|X_{t_{i-1}} - X^0_{t_{i-1}}|^m) \\
& (|b(X^0_{t_{i-1}}, t_{i-1})| + L|X_{t_{i-1}} - X^0_{t_{i-1}}|)^2 \\
\le\ & K'L|r_0|^2 e^{\frac{|r_0|L}{n}} n^{-2}\sum_{i=1}^{n}|b(X^0_{t_{i-1}}, t_{i-1})|^2(1+2^m|X^0_{t_{i-1}}|^m) \\
& + K'L|r_0|^2 e^{\frac{|r_0|L}{n}} n^{-2}\sum_{i=1}^{n}|b(X^0_{t_{i-1}}, t_{i-1})|^2|X_{t_{i-1}} - X^0_{t_{i-1}}|^m \\
& + K'L|r_0|^2 e^{\frac{|r_0|L}{n}} n^{-2}\sum_{i=1}^{n}|X_{t_{i-1}} - X^0_{t_{i-1}}|^2(1+2^m|X^0_{t_{i-1}}|^m) \\
& + K'L|r_0|^2 e^{\frac{|r_0|L}{n}} n^{-2}\sum_{i=1}^{n}|X_{t_{i-1}} - X^0_{t_{i-1}}|^{m+2} \\
\le\ & K'L|r_0|^2 e^{\frac{|r_0|L}{n}} n^{-2}\sum_{i=1}^{n}|b(X^0_{t_{i-1}}, t_{i-1})|^2(1+2^m|X^0_{t_{i-1}}|^m) \\
& + K'L|r_0|^2 e^{\frac{|r_0|L}{n}} \sup_{0\le t\le 1}|X_{t_{i-1}} - X^0_{t_{i-1}}|^m n^{-2}\sum_{i=1}^{n}|b(X^0_{t_{i-1}}, t_{i-1})|^2 \\
& + K'L|r_0|^2 e^{\frac{|r_0|L}{n}} \sup_{0\le t\le 1}|X_{t_{i-1}} - X^0_{t_{i-1}}|^2 n^{-2}\sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m) \\
& K'L|r_0|^2 e^{\frac{|r_0|L}{n}} \sup_{0\le t\le 1}|X_{t_{i-1}} - X^0_{t_{i-1}}|^{m+2} n^{-1}.
\end{aligned} \tag{3.23}$$

By Lemma 3.2.2, it is easy to see that $\phi_{2,1}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n\to\infty$ and $\varepsilon\to 0$.

For $\phi_{2,2}(n,\varepsilon)$, we have

$$\phi_{2,2} \leq K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m+2^m|X_{t_{i-1}}-X^0_{t_{i-1}}|^m)$$

$$(|b(X^0_{t_{i-1}},t_{i-1})|+L|X_{t_{i-1}}-X^0_{t_{i-1}}|)\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$\leq K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X^0_{t_{i-1}},t_{i-1})|(1+2^m|X^0_{t_{i-1}}|^m)\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$+K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X^0_{t_{i-1}},t_{i-1})||X_{t_{i-1}}-X^0_{t_{i-1}}|^m\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$+K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|X_{t_{i-1}}-X^0_{t_{i-1}}|(1+2^m|X^0_{t_{i-1}}|^m\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$+K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|X_{t_{i-1}}-X^0_{t_{i-1}}|^{m+1}\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$\leq K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X^0_{t_{i-1}},t_{i-1})|(1+2^m|X^0_{t_{i-1}}|^m)\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$+K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sup_{0\leq t\leq 1}|X_{t_{i-1}}-X^0_{t_{i-1}}|^m\sum_{i=1}^{n}|b(X^0_{t_{i-1}},t_{i-1})|\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$+K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sup_{0\leq t\leq 1}|X_{t_{i-1}}-X^0_{t_{i-1}}|^m\sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$+K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sup_{0\leq t\leq 1}|X_{t_{i-1}}-X^0_{t_{i-1}}|^{m+1}\sum_{i=1}^{n}\sup_{t_{i-1}\leq t\leq t_i}\left|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}^\varepsilon_s\right|$$

$$:=\phi^1_{2,2}(n,\varepsilon)+\phi^2_{2,2}(n,\varepsilon)+\phi^3_{2,2}(n,\varepsilon)+\phi^4_{2,2}(n,\varepsilon). \tag{3.24}$$

For $\phi^1_{2,2}(n,\varepsilon)$, by Markov Inequality, Hölder's Inequality, Gronwall's Inequality and Lemma

2.3.1, for any given $\gamma > 0$, we have

$$Q_\varepsilon(|\phi_{2,2}^1(n,\varepsilon)| > \gamma) \leq \gamma^{-1}K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X_{t_{i-1}}^0, t_{i-1})|(1+2^m|X_{t_{i-1}}^0|^m)$$

$$E_{Q_\varepsilon}\Big[\sup_{t_{i-1}\leq t\leq t_i}\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^\varepsilon\Big|\Big]$$

$$\leq \gamma^{-1}K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X_{t_{i-1}}^0, t_{i-1})|(1+2^m|X_{t_{i-1}}^0|^m)$$

$$4\sqrt{2}E_{Q_\varepsilon}\Big[\Big(\int_{t_{i-1}}^{t_i}|\sigma(X_s,s)|^2ds\Big)^{\frac{1}{2}}\Big]$$

$$\leq \gamma^{-1}K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X_{t_{i-1}}^0, t_{i-1})|(1+2^m|X_{t_{i-1}}^0|^m)$$

$$4\sqrt{2}\Big(\int_{t_{i-1}}^{t_i}E_{Q_\varepsilon}|\sigma(X_s,s)|^2ds\Big)^{\frac{1}{2}}$$

$$\leq \gamma^{-1}K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X_{t_{i-1}}^0, t_{i-1})|(1+2^m|X_{t_{i-1}}^0|^m)$$

$$4\sqrt{2}\Big(\int_{t_{i-1}}^{t_i}K(1+E_{Q_\varepsilon}|X_s|^2)ds\Big)^{\frac{1}{2}}$$

$$\leq \gamma^{-1}K'L|r_0|e^{\frac{|r_0|L}{n}}\varepsilon n^{-1}\sum_{i=1}^{n}|b(X_{t_{i-1}}^0, t_{i-1})|(1+2^m|X_{t_{i-1}}^0|^m)$$

$$4\sqrt{2}K^{\frac{1}{2}}(1+C)^{\frac{1}{2}}n^{-\frac{1}{2}}$$

It implies that $\phi_{2,2}^1(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. By the same way, through using Lemma 3.2.2, we can get $\phi_{2,2}^2(n,\varepsilon)$, $\phi_{2,2}^3(n,\varepsilon)$, $\phi_{2,2}^4(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. □

Lemma 3.2.5. *Under conditions (1)-(4), We have* $\phi_3(n,\varepsilon) \to_{Q_\varepsilon} 0$ *as* $n \to \infty$ *and* $\varepsilon \to 0$.

Proof. By condition (1), (4), and (3.21), we have

$$
\begin{aligned}
|\phi_3(n,\varepsilon)| &\le \varepsilon \sum_{i=1}^{n} K'(1+|X_{t_{i-1}}|^m)|b(X_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\le \varepsilon \sum_{i=1}^{n} K'(1+2^m|X^0_{t_{i-1}}|^m+2^m|X_{t_{i-1}}-X^0_{t_{i-1}}|^m) \\
&\quad (|b(X^0_{t_{i-1}},t_{i-1})|+L|X_{t_{i-1}}-X^0_{t_{i-1}}|)\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\le \varepsilon K' \sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)|b(X^0_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\quad + \varepsilon K' 2^m \sum_{i=1}^{n}|X_{t_{i-1}}-X^0_{t_{i-1}}|^m|b(X^0_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\quad + \varepsilon K' L \sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)|X_{t_{i-1}}-X^0_{t_{i-1}}|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\quad + \varepsilon K' 2^m L \sum_{i=1}^{n}|X_{t_{i-1}}-X^0_{t_{i-1}}|^{m+1}\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\le \varepsilon K' \sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)|b(X^0_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\quad + \varepsilon K' 2^m \sup_{0\le t\le 1}|X_{t_{i-1}}-X^0_{t_{i-1}}|^m \sum_{i=1}^{n}|b(X^0_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\quad + \varepsilon K' L \sup_{0\le t\le 1}|X_{t_{i-1}}-X^0_{t_{i-1}}| \sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&\quad + \varepsilon K' 2^m L \sup_{0\le t\le 1}|X_{t_{i-1}}-X^0_{t_{i-1}}|^{m+1} \sum_{i=1}^{n}\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big| \\
&:= \phi_{3,1}(n,\varepsilon)+\phi_{3,2}(n,\varepsilon)+\phi_{3,3}(n,\varepsilon)+\phi_{3,4}(n,\varepsilon).
\end{aligned}
$$

For $\phi_{3,1}(n,\varepsilon)$, Markov Inequality and Lemma 2.3.1, for any given $\gamma > 0$, we have

$$
\begin{aligned}
Q_\varepsilon(|\phi_{3,1}(n,\varepsilon)|>\gamma) &\le \gamma^{-1}K'\varepsilon \sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)|b(X^0_{t_{i-1}},t_{i-1})|E_{Q_\varepsilon}\Big[\sup_{t_{i-1}\le t\le t_i}\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s^{\varepsilon}\Big|\Big] \\
&\le \gamma^{-1}K'\varepsilon \sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)|b(X^0_{t_{i-1}},t_{i-1})|4\sqrt{2}E_{Q_\varepsilon}\Big[\Big(\int_{t_{i-1}}^{t_i}|\sigma(X_s,s)|^2ds\Big)^{\frac{1}{2}}\Big] \\
&\le \gamma^{-1}K'\varepsilon \sum_{i=1}^{n}(1+2^m|X^0_{t_{i-1}}|^m)|b(X^0_{t_{i-1}},t_{i-1})| \quad 4\sqrt{2}K^{\frac{1}{2}}(1+C)^{\frac{1}{2}}n^{-\frac{1}{2}}
\end{aligned}
$$

It implies that $\phi_{3,1}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n\to\infty$ and $\varepsilon n^{\frac{1}{2}} \to 0$. In the same way, we can obtain

$\phi_{3,2}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon n^{\frac{1}{2}} \to 0$; $\phi_{3,3}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon n^{\frac{1}{2}} \to 0$; $\phi_{3,1}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon n^{\frac{1}{2}} \to 0$. □

Proof. Proof of Theorem 3.2.1, let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$. By using Lemma 3.2.4, Lemma 3.2.5 and Lemma 3.2.6, we have

$$\hat{r}_{n,\varepsilon} = r_0 + \frac{\phi_2(n,r)}{\phi_1(n,r)} + \frac{\phi_3(n,r)}{\phi_1(n,r)} \to_{Q_\varepsilon} r_0.$$

□

3.3 Asymptotic of the Least Squares Estimator

In this section, we assume that $\alpha(x,t,\varepsilon) = \varepsilon\alpha(x,t)$, so that $Q_\varepsilon = Q$ is independent of ε.

Theorem 3.3.1. *There exist two independent Q-random variables U_1 and U_2 with distribution $N(0,1)$ such that*

$$\varepsilon^{-1}(\hat{r}_{n,\varepsilon} - r_0) \rightarrow_Q \frac{\left(\int_0^1 |\sigma(X_s^0,s)|^{-4}[(b(X_s^0,s)\sigma(X_s^0,s))_+]^2 ds\right)^{\frac{1}{2}} U_1}{\int_0^1 \sigma^{-2}(X_s^0,s)b^2(X_s^0,s)ds} - \frac{\left(\int_0^1 |\sigma(X_s^0,s)|^{-4}[(b(X_s^0,s)\sigma(X_s^0,s))_-]^2 ds\right)^{\frac{1}{2}} U_2}{\int_0^1 \sigma^{-2}(X_s^0,s)b^2(X_s^0,s)ds}$$

as $n \to \infty$, $\varepsilon \to 0$, $n\varepsilon \to \infty$, and $\varepsilon n^{\frac{1}{2}} \to 0$. By theorem 3.2.1, we have

$$\begin{aligned}\varepsilon^{-1}(\hat{r}_{n,\varepsilon} - r_0) &= \frac{\varepsilon^{-1}\phi_2(n,\varepsilon)}{\phi_1(n,\varepsilon)} + \frac{\varepsilon^{-1}\phi_3(n,\varepsilon)}{\phi_1(n,\varepsilon)} \\ &:= \frac{\Phi_2(n,\varepsilon)}{\phi_1(n,\varepsilon)} + \frac{\Phi_3(n,\varepsilon)}{\phi_1(n,\varepsilon)}.\end{aligned}$$

Lemma 3.3.1. *Under condition (1)-(4), we have $\Phi_2(n,\varepsilon) \rightarrow_Q 0$ as $n \to \infty$, $\varepsilon \to 0$ and $n\varepsilon \to \infty$.*

Proof. From Lemma 3.2.5, by (3.22), we have

$$\begin{aligned}|\Phi_2(n,\varepsilon)| &= \varepsilon^{-1}|\phi_2(n,\varepsilon)| \\ &\le \varepsilon^{-1}\phi_{2,1}(n,\varepsilon) + \varepsilon^{-1}\phi_{2,2}(n,\varepsilon) \\ &:= \Phi_{2,1}(n,\varepsilon) + \Phi_{2,2}(n,\varepsilon).\end{aligned}$$

By (3.23), it is easy to see that $\Phi_{2,1}(n,\varepsilon) \rightarrow_Q 0$ as $n \to \infty$, $\varepsilon \to 0$ and $n\varepsilon \to \infty$. Similarly, $\Phi_{2,2}(n,\varepsilon) \rightarrow_Q 0$ as $n \to \infty$ and $\varepsilon \to 0$. ☐

Lemma 3.3.2. *Under condition(1)-(4), we have*

$$\Phi_3(n,\varepsilon) \rightarrow_Q \left(\int_0^1 |\sigma(X_s^0,s)|^{-4}[(b(X_s^0,s)\sigma(X_s^0,s))_+]^2 ds\right)^{\frac{1}{2}} U_1 - \left(\int_0^1 |\sigma(X_s^0,s)|^{-4}[(b(X_s^0,s)\sigma(X_s^0,s))_-]^2 ds\right)^{\frac{1}{2}} U_2$$

as $n \to \infty$ and $\varepsilon \to 0$, $\varepsilon n^{\frac{1}{2}} \to 0$.

Proof.

$$\Phi_3(n,\varepsilon) = \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}, t_{i-1}) b(X_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} \sigma(X_s, s) d\hat{B}_s$$

$$= \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1}) b(X^0_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} \sigma(X_s, s) d\hat{B}_s$$

$$+ \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1}) b(X^0_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} (\sigma(X_s, s) - \sigma(X^0_s, s)) d\hat{B}_s$$

$$+ \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1}) [b(X_{t_{i-1}}, t_{i-1}) - b(X^0_{t_{i-1}}, t_{i-1})] \int_{t_{i-1}}^{t_i} \sigma(X_s, s) d\hat{B}_s$$

$$+ \sum_{i=1}^{n} [\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) - \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})] b(X^0_{t_{i-1}}, t_{i-1}) \int_{t_{i-1}}^{t_i} \sigma(X_s, s) d\hat{B}_s$$

$$+ \sum_{i=1}^{n} [\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) - \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})][b(X_{t_{i-1}}, t_{i-1}) - b(X^0_{t_{i-1}}, t_{i-1})] \int_{t_{i-1}}^{t_i} \sigma(X_s, s) d\hat{B}_s$$

$$:= \Phi_{3,1}(n,\varepsilon) + \Phi_{3,2}(n,\varepsilon) + \Phi_{3,3}(n,\varepsilon) + \Phi_{3,4}(n,\varepsilon) + \Phi_{3,5}(n,\varepsilon).$$

Define a deterministic process $V(s)$ by $V(s) = \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1}) b(X^0_{t_{i-1}}, t_{i-1}) \sigma(X^0_s, s) \mathbb{1}_{(t_{i-1}, t_i]}(s)$. Let $V_+(s)$ and $V_-(s)$ denote the positive and negative part of $V(s)$. By Theorem 4.1 of Kallenberg [20], there exist two independent Q-Brownian motions $\hat{B}'$, $\hat{B}''$, which have the same distribution of $\hat{B}$, such that

$$\Phi_{3,1}(n,\varepsilon) = \int_0^1 V(s) d\hat{B}_s = \hat{B}' \circ \int_0^1 V_+^2(s) ds - \hat{B}'' \circ \int_0^1 V_-^2(s) ds.$$

Note that

$$V_+^2 = \sum_{i=1}^{n} |\sigma(X^0_{t_{i-1}}, t_{i-1})|^{-4} ((b(X^0_{t_{i-1}}, t_{i-1})) \sigma(X^0_s, s))_+^2 \mathbb{1}_{(t_{i-1}, t_i]}(s)$$

and

$$V_-^2 = \sum_{i=1}^{n} |\sigma(X^0_{t_{i-1}}, t_{i-1})|^{-4} ((b(X^0_{t_{i-1}}, t_{i-1})) \sigma(X^0_s, s))_-^2 \mathbb{1}_{(t_{i-1}, t_i]}(s).$$

Then we have

$$\int_0^1 V_+^2(s) ds \to \int_0^1 |\sigma(X^0_s, s)|^{-4} (b(X^0_s, s) \sigma(X^0_s, s))_+^2 ds$$

and

$$\int_0^1 V_-^2(s) ds \to \int_0^1 |\sigma(X^0_s, s)|^{-4} (b(X^0_s, s) \sigma(X^0_s, s))_-^2 ds$$

as $n \to \infty$. Then,

$$\hat{B}' \circ \int_0^1 V_+^2(s)ds \to \hat{B}' \circ \int_0^1 |\sigma(X_s^0, s)|^{-4}[(b(X_s^0, s)\sigma(X_s^0, s))_+]^2 ds$$

and

$$\hat{B}'' \circ \int_0^1 V_-^2(s)ds \to \hat{B}'' \circ \int_0^1 |\sigma(X_s^0, s)|^{-4}[(b(X_s^0, s)\sigma(X_s^0, s))_-]^2 ds.$$

So we get

$$\Phi_{3,1}(n,\varepsilon) \to_Q \left(\int_0^1 |\sigma(X_s^0, s)|^{-4}[(b(X_s^0, s)\sigma(X_s^0, s))_+]^2 ds\right)^{\frac{1}{2}} U_1 - \left(\int_0^1 |\sigma(X_s^0, s)|^{-4}[(b(X_s^0, s)\sigma(X_s^0, s))_-]^2 ds\right)^{\frac{1}{2}} U_2$$

as $n \to \infty$.

For $\Phi_{3,2}(n,\varepsilon)$, by condition (3), Markov Inequality and Lemma 2.3.1, Lemma 3.2.1, for any given $\gamma > 0$, we have

$$\begin{aligned}
Q(|\Phi_{3,2}(n,\varepsilon)| > \gamma) &\le \gamma^{-1} E_Q\Big[\sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})|b(X_{t_{i-1}}^0, t_{i-1})|\Big|\int_{t_{i-1}}^{t_i} (\sigma(X_s, s) - \sigma(X_s^0, s))d\hat{B}_s\Big|\Big] \\
&\le \gamma^{-1} \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})|b(X_{t_{i-1}}^0, t_{i-1})| E_Q\Big[\Big|\int_{t_{i-1}}^{t_i} (\sigma(X_s, s) - \sigma(X_s^0, s))d\hat{B}_s\Big|\Big] \\
&\le \gamma^{-1} \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})|b(X_{t_{i-1}}^0, t_{i-1})| 4\sqrt{2} E_Q\Big[\Big(\int_{t_{i-1}}^{t_i} |\sigma(X_s, s) - \sigma(X_s^0, s)|^2 ds\Big)^{\frac{1}{2}}\Big] \\
&\le \gamma^{-1} L 4\sqrt{2} \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})|b(X_{t_{i-1}}^0, t_{i-1})| E_Q\Big[\Big(\int_{t_{i-1}}^{t_i} |X_s - X_s^0|^2 ds\Big)^{\frac{1}{2}}\Big] \\
&\le \gamma^{-1} L 4\sqrt{2} n^{-\frac{1}{2}} \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})|b(X_{t_{i-1}}^0, t_{i-1})| E_Q\Big[\sup_{t_{i-1} \le t \le t_i} |X_s - X_s^0|\Big] \\
&\le \gamma^{-1} L 4\sqrt{2} n^{-\frac{1}{2}} \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})|b(X_{t_{i-1}}^0, t_{i-1}) \\
&\qquad E_Q\Big[\varepsilon e^{L|r_0|t_i} \sup_{t_{i-1} \le t \le t_i} \Big|\int_0^t \sigma(X_s, s)d\hat{B}_s\Big|\Big] \\
&\le \gamma^{-1} L 32 e^{L|r_0|} n^{-\frac{1}{2}} \varepsilon \sum_{i=1}^n \sigma^{-2}(X_{t_{i-1}}^0, t_{i-1})|b(X_{t_{i-1}}^0, t_{i-1}) E_Q\Big[\Big(\int_0^{t_i} |\sigma(X_s, s)|^2 ds\Big)^{\frac{1}{2}}\Big]
\end{aligned} \tag{3.25}$$

which tend to zero as $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$.

For $\Phi_{3,3}(n,\varepsilon)$, we have

$$\begin{aligned}\Phi_{3,3}(n,\varepsilon) &= \sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})[b(X_{t_{i-1}},t_{i-1})-b(X^0_{t_{i-1}},t_{i-1})]\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s\\ &= \sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})[b(X_{t_{i-1}},t_{i-1})-b(X^0_{t_{i-1}},t_{i-1})]\int_{t_{i-1}}^{t_i}\sigma(X^0_s,s)d\hat{B}_s\\ &+\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})[b(X_{t_{i-1}},t_{i-1})-b(X^0_{t_{i-1}},t_{i-1})]\\ &\int_{t_{i-1}}^{t_i}(\sigma(X_s,s)-\sigma(X^0_s,s))d\hat{B}_s\\ &:= \Phi^1_{3,3}(n,\varepsilon)+\Phi^2_{3,3}(n,\varepsilon).\end{aligned}$$

For $\Phi^1_{3,3}(n,\varepsilon)$, by condition (1), we have

$$\begin{aligned}\Phi^1_{3,3}(n,\varepsilon) &\le \Big|\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})[b(X_{t_{i-1}},t_{i-1})-b(X^0_{t_{i-1}},t_{i-1})]\int_{t_{i-1}}^{t_i}\sigma(X^0_s,s)d\hat{B}_s\Big|\\ &\le L\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})|X_{t_{i-1}}-X^0_{t_{i-1}}|\Big|\int_{t_{i-1}}^{t_i}\sigma(X^0_s,s)d\hat{B}_s\Big|\\ &\le L\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})\varepsilon e^{L|r_0|t_{i-1}}\Big|\int_0^{t_{i-1}}\sigma(X_s,s)d\hat{B}_s\Big|\Big|\int_{t_{i-1}}^{t_i}\sigma(X^0_s,s)d\hat{B}_s\Big|.\end{aligned}$$

By Markov Inequality and Lemma 2.3.1, for any given $\gamma>0$ we get

$$\begin{aligned}Q(|\Phi_{3,2}(n,\varepsilon)|>\gamma) &\le \gamma^{-1}L\varepsilon\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})e^{L|r_0|t_{i-1}}E_Q\Big|\int_0^{t_{i-1}}\sigma(X_s,s)d\hat{B}_s\Big|E_Q\Big|\int_{t_{i-1}}^{t_i}\sigma(X^0_s,s)d\hat{B}_s\Big|\\ &\le 32\gamma^{-1}L\varepsilon\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})e^{L|r_0|t_{i-1}}E_Q\Big[\Big(\int_0^{t_{i-1}}|\sigma(X_s,s)|^2ds\Big)^{\frac{1}{2}}\Big]\\ &\Big[\Big(\int_{t_{i-1}}^{t_i}|\sigma(X^0_s,s)|^2ds\Big)^{\frac{1}{2}}\Big]\\ &\le 32e^{L|r_0|}n^{-\frac{1}{2}}K(1+C)\gamma^{-1}L\varepsilon\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})t^{\frac{1}{2}}_{i-1}\\ &= 32e^{L|r_0|}\varepsilon n^{\frac{1}{2}}K(1+C)\gamma^{-1}Ln^{-1}\sum_{i=1}^{n}\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})t^{\frac{1}{2}}_{i-1}\end{aligned}$$

which tends to zero as $n\to\infty$ and $\varepsilon n^{\frac{1}{2}}\to 0$.

For $\Phi^2_{3,3}(n,\varepsilon)$, by condition (1), Lemma 3.2.2 and the same arguments used in (3.25), we find

$$\Phi^2_{3,3}(n,\varepsilon) \le \Big|\sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}},t_{i-1})[b(X_{t_{i-1}},t_{i-1}) - b(X^0_{t_{i-1}},t_{i-1})] \int_{t_{i-1}}^{t_i} (\sigma(X_s,s) - \sigma(X^0_s,s))d\hat{B}_s\Big|$$

$$\le L\sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}},t_{i-1})|X_{t_{i-1}} - X^0_{t_{i-1}}|\Big|\int_{t_{i-1}}^{t_i} (\sigma(X_s,s) - \sigma(X^0_s,s))d\hat{B}_s\Big|$$

$$\le L\sup_{0\le t\le 1} |X_t - X^0_t|\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})\Big|\int_{t_{i-1}}^{t_i} (\sigma(X_s,s) - \sigma(X^0_s,s))d\hat{B}_s\Big|$$

which converges to zero as $n\to\infty$ $\varepsilon\to 0$ and $\varepsilon n^{\frac{1}{2}}\to 0$. So that it is easy to see $\Phi_{3,3}(n,\varepsilon)\to_Q 0$ as $n\to\infty$, $\varepsilon\to 0$ and $\varepsilon n^{\frac{1}{2}}\to 0$.

For $\Phi_{3,4}(n,\varepsilon)$, by condition (1)-(4), and (3.21), we have

$$|\Phi_{3,4}(n,\varepsilon)| \le \sum_{i=1}^{n} |\sigma^{-2}(X_{t_{i-1}},t_{i-1}) - \sigma^{-2}(X^0_{t_{i-1}},t_{i-1})||b(X^0_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s\Big|$$

$$\le \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}},t_{i-1})\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})|\sigma^2(X_{t_{i-1}},t_{i-1}) - \sigma^2(X^0_{t_{i-1}},t_{i-1})| |b(X^0_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s\Big|$$

$$\le \sum_{i=1}^{n} K'(1+|X_{t_{i-1}}|^m)\sigma^{-2}(X^0_{t_{i-1}},t_{i-1})2K|\sigma(X_{t_{i-1}},t_{i-1}) - \sigma(X^0_{t_{i-1}},t_{i-1})| |b(X^0_{t_{i-1}},t_{i-1})|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s\Big|$$

$$\le 2KK'L\sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}},t_{i-1})|b(X^0_{t_{i-1}},t_{i-1})||X_{t_{i-1}} - X^0_{t_{i-1}}|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s\Big|$$

$$+ 2KK'L2^m\sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}},t_{i-1})|b(X^0_{t_{i-1}},t_{i-1})||X_{t_{i-1}} - X^0_{t_{i-1}}|^{m+1}\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s\Big|$$

$$+ 2KK'L2^m\sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}},t_{i-1})|b(X^0_{t_{i-1}},t_{i-1})||X^0_{t_{i-1}}|^m|X_{t_{i-1}} - X^0_{t_{i-1}}|\Big|\int_{t_{i-1}}^{t_i}\sigma(X_s,s)d\hat{B}_s\Big|$$

$$:= \Phi^1_{3,4}(n,\varepsilon) + \Phi^2_{3,4}(n,\varepsilon) + \Phi^3_{3,4}(n,\varepsilon).$$

From the method of the convergence of $\Phi_{3,3}(n,\varepsilon)$, we get $\Phi^1_{3,4}(n,\varepsilon)\to_Q 0$ and $\Phi^3_{3,4}(n,\varepsilon)\to_Q 0$ as $n\to\infty$ $\varepsilon\to 0$ and $\varepsilon n^{\frac{1}{2}}\to 0$.

For $\Phi^2_{3,4}(n,\varepsilon)$, we have

$$\Phi^2_{3,4}(n,\varepsilon) \le \sup_{0\le t\le 1} |X_{t_{i-1}} - X^0_{t_{i-1}}|^m 2KK'L2^m \sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})$$
$$|b(X^0_{t_{i-1}}, t_{i-1})||X_{t_{i-1}} - X^0_{t_{i-1}}|\left|\int_{t_{i-1}}^{t_i} \sigma(X_s, s)d\hat{B}_s\right|$$

which converges to zero in probability, since $m \ge 1$, $\sup_{0\le t\le 1} |X_{t_{i-1}} - X^0_{t_{i-1}}|^m$ converges to zero in probability as $\varepsilon \to 0$. Hence, $\Phi_{3,4}(n,\varepsilon) \to_Q 0$ as $n \to \infty$ $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$.

For $\Phi_{3,5}(n,\varepsilon)$, to some $G > 0$ we have

$$\begin{aligned}
|\Phi_{3,5}(n,\varepsilon)| &\le \sum_{i=1}^{n} |\sigma^{-2}(X_{t_{i-1}}, t_{i-1}) - \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})||b(X_{t_{i-1}}, t_{i-1}) - b(X^0_{t_{i-1}}, t_{i-1})|\left|\int_{t_{i-1}}^{t_i} \sigma(X_s, s)d\hat{B}_s\right| \\
&\le \sum_{i=1}^{n} \sigma^{-2}(X_{t_{i-1}}, t_{i-1})\sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})|\sigma^2(X_{t_{i-1}}, t_{i-1}) - \sigma^2(X^0_{t_{i-1}}, t_{i-1})| \\
&\quad |b(X_{t_{i-1}}, t_{i-1}) - b(X^0_{t_{i-1}}, t_{i-1})|\left|\int_{t_{i-1}}^{t_i} \sigma(X_s, s)d\hat{B}_s\right| \\
&\le G\sum_{i=1}^{n} \sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})(1 + |X_{t_{i-1}}|^m)|X_{t_{i-1}} - X^0_{t_{i-1}}|^2\left|\int_{t_{i-1}}^{t_i} \sigma(X_s, s)d\hat{B}_s\right| \\
&\le \sup_{0\le t\le 1} |X_{t_{i-1}} - X^0_{t_{i-1}}| G\sum_{i=1}^{n} |\sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})(1 + |X_{t_{i-1}}|^m)|X_{t_{i-1}} - X^0_{t_{i-1}}|\left|\int_{t_{i-1}}^{t_i} \sigma(X_s, s)d\hat{B}_s\right|
\end{aligned}$$

which converges to zero in probability Q, since $\sup_{0\le t\le 1} |X_{t_{i-1}} - X^0_{t_{i-1}}| \to_Q 0$ as $\varepsilon \to 0$ by Lemma 3.2.2, and

$$G\sum_{i=1}^{n} |\sigma^{-2}(X^0_{t_{i-1}}, t_{i-1})(1 + |X_{t_{i-1}}|^m)|X_{t_{i-1}} - X^0_{t_{i-1}}|\left|\int_{t_{i-1}}^{t_i} \sigma(X_s, s)dB_s\right| \to_Q 0$$

as $n \to \infty$ $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$ by the same arguments for the convergence of $\Phi_{3,4}(n,\varepsilon)$. □

Proof. Proof of Theorem 3.3.1, by using Lemma 3.2.3, Lemma 3.3.1 and Lemma 3.3.2, we have

$$\begin{aligned}
\varepsilon^{-1}(\hat{r}_{n,\varepsilon} - r_0) &= \frac{\Phi_2(n,\varepsilon)}{\phi_1(n,\varepsilon)} + \frac{\Phi_3(n,\varepsilon)}{\phi_3(n,\varepsilon)} \\
&\to_Q \frac{\left(\int_0^1 |\sigma(X^0_s, s)|^{-4}[(b(X^0_s, s)\sigma(X^0_s, s))_+]^2 ds\right)^{\frac{1}{2}} U_1}{\int_0^1 \sigma^{-2}(X^0_s, s)b^2(X^0_s, s)ds} \\
&\quad - \frac{\left(\int_0^1 |\sigma(X^0_s, s)|^{-4}[(b(X^0_s, s)\sigma(X^0_s, s))_-]^2 ds\right)^{\frac{1}{2}} U_2}{\int_0^1 \sigma^{-2}(X^0_s, s)b^2(X^0_s, s)ds}
\end{aligned}$$

as $n \to \infty$, $n\varepsilon \to \infty$, $\varepsilon n^{\frac{1}{2}} \to 0$ and $\varepsilon \to 0$. $\square$

3.4 An Illustrative Example

In this section, we are aiming to give a linear example for (3.1). We consider the following equation with initial value $X_0 \in (0,1]$

$$dX_t = [r + \varepsilon^k \alpha(X_t, t)]X_t dt + \varepsilon X_t dB_t, \qquad 0 \leq t \leq 1, \tag{3.26}$$

Where $\varepsilon \in (0,1]$ and $k \geq 0$ are constants; $\alpha(x,t) : \mathbb{R} \times [0,1] \to \mathbb{R}$ is a $\mathcal{C}^{2,1}$−function satisfying $E\Big[\exp\Big(\frac{1}{2}\int_0^t |\varepsilon^{k-1}\alpha(X_s,s)|^2 ds\Big)\Big] < \infty$; B_t is a one dimensional $\{\mathcal{F}_t\}$−Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{0\leq t\leq 1}\}$.

We assume equation (3.26) satisfies the following condition

$$|\alpha(x,t)x - \alpha(y,t)y| \leq D|x-y| \tag{3.27}$$

where $D > 0$ is a constant, $x, y \in \mathbb{R}$. By (3.26), we can get the system

$$X_{t_i} = x + \sum_{i=1}^{n} [r + \varepsilon^k \alpha(X_{t_{i-1}}, t_{i-1})] X_{t_{i-1}} \Delta t_i + \varepsilon \sum_{i=1}^{n} X_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})$$

where $\Delta t_i = t_i - t_{i-1}$.

Under condition (3.3), (3.4), (3.5) and (3.6), we set

$$u_\varepsilon(X_t, t) = \alpha(X_t, t)\varepsilon^{k-1} \tag{3.28}$$

by (3.1),(3.6), (3.2), under measure Q_ε,

$$dX_t = rX_t dt + \varepsilon X_t d\hat{B}_t^\varepsilon. \tag{3.29}$$

Assume that the process X_t is observed at regularly spaced time intervals $\{t_i = \frac{i}{n}, i = 1, 2, ..., n\}$. We represent the true value of the parameter r by r_0 and least square estimator of r by $\hat{r}$. As mentioned before we focus on investigation of the least squares estimator for the true value r_0 based on the sampling data $(X_{t_i})_{i=1}^n$ determined by

$$X_{t_i} = x + \sum_{i=1}^{n} rX_{t_{i-1}} \Delta t_i + \varepsilon \sum_{i=1}^{i} X_{t_{i-1}} (\hat{B}_{t_i}^\varepsilon - \hat{B}_{t_{i-1}}^\varepsilon).$$

Where $\Delta t_i = t_i - t_{i-1}$. We start with the use of the least squares method to obtain an asymptotically consistent estimator. Then we get the following contrast function

$$\rho_{n,\varepsilon}(r) = \sum_{i=1}^{n} \left| \frac{X_{t_i} - X_{t_{i-1}} - rX_{t_{i-1}}\Delta t_i}{\varepsilon X_{t_{i-1}}} \right|^2 .$$

In order to get the least square estimator $\hat{r}_{n,\varepsilon}$, let

$$\frac{\partial \rho_{n,\varepsilon}(r)}{\partial r} = 0.$$

Then we have

$$\hat{r}_{n,\varepsilon} = \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}. \tag{3.30}$$

Theorem 3.4.1. *Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$. Then, we have $\hat{r}_{n,\varepsilon} \to_{Q_\varepsilon} r_0$.*

Theorem 3.4.2. *Let $k = 1$, so that Q is independent of ε. Then let U be a Q-random variables with standard normal distribution N(0,1), we have*

$$\varepsilon^{-1}(\hat{r}_{n,\varepsilon} - r_0) \to_Q U$$

as $n \to \infty$ and $\varepsilon \to 0$.

3.5 Application to A Credit Risk Modeling

In this section, we consider the zero coupon bonds as an example to apply our mean-reversion approach, and discuss a defaultable zero-coupon bond in an rating based model.

A zero coupon bond is a special type bond of which can be bought at a price lower than its face value, with the face value repaid at the time of maturity T. A credit rating estimates the credit worthiness of an individual, corporation, or even a country. It is an evaluation made by credit bureaus of a borrower's overall credit history. Although credit rating is not to be a measure of a firm's default probability over some time horizon, but, to some extent, a measure of relative credit quality among firms. Usually, the credit rating is assigned by credit rating agencies such as A.M. Best, Dun & Bredstreet, Standard & Poor's, Moody's or Fitch Ratings and have letter designations such as A, B, C. The standard & Poor's rating scale is as follows, from excellent to poor: AAA, AA+, AA, AA-, A+, A, A-, BBB+, BBB, BBB-, BB+, BB, BB-, B+, B, B-, CCC+, CCC, CCC-, CC, C, D. Anything lower than a BBB- rating is considered a speculative or junk bond.

If we put the rating provided by agencies into a rating process, together with unexpected changes of the credit quality, it is useful to give each issuer a continuous rating that follows a diffusion process[see Douady and Jeanblanc [7]]. In Douady and Jeanblanc [7], they assign each issuer with a continuous rating process $R = (R_t)_{t\geq 0}$. Then a given agency rating corresponds to some sub-interval$(n_i, n_{i+1}) \subset [0, 1]$. Rating migrations correspond to crossing one threshold $n_i \in (0, 1)$. After that they let the continuous rating process $R = (R_t)_{0\leq t\leq 1} \in [0, 1]$ determined by the following SDE

$$dR_t = h_t dt + \sigma(R_t, t) dW_t, \qquad R_0 \in [0, 1], \qquad 0 \leq t \leq 1$$

where W_t is a Brownian motion, h_t is an integrable function of t and $\sigma(R_t, t)$ is a deterministic function of R_t and t. With h_t and $\sigma(R_t, t)$, it can ensure that for each R_0 and all $0 \leq t \leq 1$, we have $R_t < 1$; if $R_0 = 1$, it implies $R_t \equiv 1$ for $0 \leq t \leq 1$, and it is a nondefaultable bond; if $R_t = 0$, then default happens a.s..

In this part, we would like to investigate a rating process with a mean-reversion approach.

Let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{0\le t\le 1}$ i.e.$\mathcal{F}_t \subset \mathcal{F}_1$ for $0 \le t \le 1$.First we define

$$\tau_1 := inf\{0 \le t \le 1;\ X_t = 1\}, \tag{3.31}$$

Then we define a killed diffusion process $\tilde{X} = (\tilde{X}_t)_{0\le t\le 1}$ by

$$\tilde{X}_t := X_{t\wedge\tau_1},\ \forall 0 \le t \le 1 \tag{3.32}$$

where $X = (X_t)$ is the solution of (3.26).Then, we define

$$R_t := 1 - \tilde{X}_t. \tag{3.33}$$

where R_t is the rating process with initial value $R_0 = y \in [0,1]$ satisfies the following SDE:

$$dR_t = -[r + \varepsilon^k\alpha(1 - R_t, t)](1 - R_t)dt - \varepsilon(1 - R_t)dB_t,\ 0 \le t \le 1. \tag{3.34}$$

Where $\varepsilon \in (0,1]$; $r \in \mathbb{R}$; $k \ge 0$; $\alpha(X_t, t)$ is the mean correction with the function $\alpha(1-R_t, t)$: $\mathbb{R} \times [0,1] \to \mathbb{R}$ being twice differentiable with respect to x and differentiable with respect to t; B_t is a one dimensional $\mathcal{F}_t-$Brownian motion on $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{0\le t\le 1}\}$.
When $\tau_1 \le t$ and $R_0 \ne 1$, we have

$$\tilde{X}_t = 1 - R_t = X_{\tau_1} = 1$$

and $R_t \equiv 0$ which implies default's happen case.
When $t < \tau_1$ and $R_0 \ne 1$, we have

$$\tilde{X}_t = 1 - R_t = X_t$$

then by (3.33) and (3.34), for $0 \le t \le 1$ we obtain

$$dX_t = [r + \varepsilon^k\alpha(X_t, t)]X_t dt + \varepsilon X_t dB_t. \tag{3.35}$$

We find that it is the same as our stochastic differential equation (3.26). According to our main results in Section 3.4, if we set r_0 and $\hat{r}_{n,\varepsilon}$ be the true value and least square estimator of r. Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, we have $\hat{r}_{n,\varepsilon} \to_{Q_\varepsilon} r_0$. Moreover, when we replace $\varepsilon^k\alpha$ by $\varepsilon\alpha$, the asymptotic distribution is

$$\varepsilon^{-1}(\hat{r}_{n,\varepsilon} - r_0) \to_Q U.$$

As $n \to \infty$ and $\varepsilon \to 0$, where U is a random variables with standard normal distribution $N(0,1)$. In Hodges and Carverhill [15], Hodges and Carverhill suggested a link of Burgers Equation for the mean-reversion type stochastic differential equation. By the proof, we can give the Burgers Equation of (3.35):

$$\frac{\partial}{\partial t}\alpha(x,t) = -\frac{1}{2}\varepsilon^2\frac{\partial^2}{\partial x^2}\alpha(x,t) - \alpha(x,t)\frac{\partial}{\partial x}\alpha(x,t).$$

The mathematical justification with multi-dimensional extension can be found in Truman, Wang, Wu and Yang [53].

Chapter 4

Least Squares Estimators For SDEs With Nonlinear Mean Functional Parameter

4.1 Introduction

Let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{0\le t\le 1}$. The stochastic process $X = (X_t, 0 \le t \le 1)$ with a given initial value $X_0 = x \in \mathbb{R}$, is determined by the following mean reversion stochastic differential equation (SDE)

$$dX_t = [r(\theta) + \alpha(X_t, t, \varepsilon)]X_t dt + \varepsilon X_t dB_t, \qquad 0 \le t \le 1. \tag{4.1}$$

Where $\varepsilon \in (0,1]$ is a parameter; $\alpha : \mathbb{R} \times [0,1] \times (0,1] \to \mathbb{R}$ is twice differentiable with respect to x and differentiable with respect to t; B_t is a one dimensional $\{\mathcal{F}_t\}$−Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{0\le t\le 1}\}$; $r(\theta)$ is a $\mathcal{C}^2$-function of parameter θ with $r'(\theta) \neq 0$, $r''(\theta) \neq 0$ and $\inf_{\theta\in\Theta} |r'(\theta)| > 0$ for all $\theta \in \Theta = \bar{\Theta}_0$(the closure of Θ_0) with Θ_0 being an open bounded subset of $\mathbb{R}$. We assume equation (4.1) satisfies the following condition:

(1') $\alpha(x,t,\varepsilon)x - \alpha(y,t,\varepsilon)y \le H|x-y|$, where $H > 0$ is a constant, $x, y \in \mathbb{R}$.

It is well-known that the condition is enough to ensure the existence and uniqueness of solutions to (4.1).

So the only unknown quantity in (4.1) is θ. The purpose of this paper is to investigate the least squares estimator for the true value of θ based on the sampling data $(X_{t_i})_{i=1}^n$.

By (4.1), we can get the system

$$X_{t_i} = x + \sum_{i=1}^{i}[r(\theta) + \alpha(X_{t_{i-1}}, t_{i-1}, \varepsilon)]X_{t_{i-1}}\Delta t_i + \varepsilon \sum_{i=1}^{n} X_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$$

where $\Delta t_i = t_i - t_{i-1}$. We want to obtain the true value of θ based on the sampling data $(X_{t_i})_{i=1}^n$. Since (1') implies that α is bounded, we have

$$u_\varepsilon(X_t, t) = \frac{\alpha(X_t, t, \varepsilon)}{\varepsilon}. \tag{4.2}$$

Let u satisfy

$$E\left[\exp\left(\frac{1}{2}\int_0^t |u_\varepsilon(X_s, s)|^2 ds\right)\right] < \infty, \qquad t \geq 0. \tag{4.3}$$

Then, we define

$$M_t^\varepsilon = \exp\left(-\int_0^t u_\varepsilon(X_s, s)dB_s - \frac{1}{2}\int_0^t u_\varepsilon^2(X_s, s)ds\right), \qquad t \geq 0 \tag{4.4}$$

where M_t^ε is an $\{\mathcal{F}_t\}-$martingale. Let Q_ε be a probability measure on $\mathcal{F}_1$, satisfying

$$dQ_\varepsilon := M_1^\varepsilon dP. \tag{4.5}$$

Then, we say Q_ε is absolutely continuous with respect to $\mathcal{F}_t$ and P. Moreover, we have

$$\hat{B}_t^\varepsilon := \int_0^t u_\varepsilon(X_s, s)ds + B_t \tag{4.6}$$

where $\hat{B}_t^\varepsilon$ is an $\mathcal{F}_t-$ Brownian motion with respect to Q_ε. So, X_t solves the equation

$$dX_t = r(\theta)X_t dt + \varepsilon X_t d\hat{B}_t^\varepsilon. \tag{4.7}$$

Assume that the process X_t is observed at regularly spaced time intervals $\{t_i = \frac{i}{n}, i = 1, 2, ..., n\}$. We represent the true value of the parameter $r(\theta)$ by $r(\theta_0)$ and least square estimator of $r(\theta)$ by $r(\hat{\theta})$. We focus on investigation of the least squares estimator for the true value $r(\theta_0)$ based on the sampling data $(X_{t_i})_{i=1}^n$ determined by

$$X_{t_i} = x + r(\theta)\sum_{i=1}^{n} X_{t_{i-1}}\Delta t_i + \varepsilon \sum_{i=1}^{n} X_{t_{i-1}}(\hat{B}_{t_i}^\varepsilon - \hat{B}_{t_{i-1}}^\varepsilon)$$

where $\Delta t_i = t_i - t_{i-1}$. Let us start with the use of the least squares method to get a consistent estimator. First of all, we consider the following contrast function

$$\rho_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - r(\theta)X_{t_{i-1}}\Delta t_i|^2}{\varepsilon^2 X_{t_{i-1}}^2 \Delta t_i}.$$

Then the least square estimator $\hat{\theta}_{n,\varepsilon}$ is defined as

$$\hat{\theta}_{n,\varepsilon} := arg \min \rho_{n,\varepsilon}(\theta).$$

Let θ_0 denote the true value of the parameter θ. The purpose of this chapter is to study the least squares estimator for the true value θ_0 based on the sampling data $(X_{t_i})_{i=1}^n$ with small dispersion ε and large sample size n. This chapter is organized as follows. In Section 4.2 we aim to establish the consistency of the LSE $\hat{\theta}_{n,\varepsilon}$. In Section 4.3 the rate of convergence and the asymptotic distribution are established.

4.2 Consistency of The Least Squares Estimator

Since minimizing $\rho_{n,\varepsilon}(\theta)$ is equivalent to minimizing

$$\phi_{n,\varepsilon}(\theta) := \varepsilon^2(\rho_{n,\varepsilon}(\theta) - \rho_{n,\varepsilon}(\theta_0)). \tag{4.8}$$

We have

$$\begin{aligned}
\phi_{n,\varepsilon}(\theta) &= \varepsilon^2(\rho_{n,\varepsilon}r(\theta) - \rho_{n,\varepsilon}(\theta_0)) \\
&= \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - X_{t_{i-1}} r(\theta)\Delta t_{i-1}|^2 - |X_{t_i} - X_{t_{i-1}} - X_{t_{i-1}} r(\theta_0)\Delta t_{i-1}|^2}{X_{t_{i-1}}^2 \Delta t_i} \\
&= \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}}|^2 - 2X_{t_{i-1}} r(\theta)\Delta t_{i-1}(X_{t_i} - X_{t_{i-1}}) + |X_{t_{i-1}} r(\theta)\Delta t_{i-1}|^2}{X_{t_{i-1}}^2 \Delta t_i} \\
&\quad - \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}}|^2 - 2X_{t_{i-1}} r(\theta_0)\Delta t_{i-1}(X_{t_i} - X_{t_{i-1}}) + |X_{t_{i-1}} r(\theta_0)\Delta t_{i-1}|^2}{X_{t_{i-1}}^2 \Delta t_i} \qquad (4.9) \\
&= \sum_{i=1}^{n} \frac{2(X_{t_i} - X_{t_{i-1}})X_{t_{i-1}}\Delta t_{i-1}(r(\theta_0) - r(\theta)) + X_{t_{i-1}}^2 \Delta t_{i-1}^2(|r(\theta)|^2 - |r(\theta_0)|^2)}{X_{t_{i-1}}^2 \Delta t_i} \\
&= 2(r(\theta_0) - r(\theta)) \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} + (r^2(\theta) - r^2(\theta_0)).
\end{aligned}$$

Theorem 4.2.1. *Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, we have $\hat{\theta}_{n,\varepsilon} \to_{Q_\varepsilon} \theta_0$.*

Proof. Let

$$\Phi_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}. \tag{4.10}$$

Since

$$X_{t_i} - X_{t_{i-1}} = r(\theta_0) \int_{t_{i-1}}^{t_i} X_s ds + \varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^\varepsilon. \tag{4.11}$$

We have

$$\begin{aligned}
\Phi_{n,\varepsilon}(\theta) &= \sum_{i=1}^{n} \frac{r(\theta_0)\int_{t_{i-1}}^{t_i} X_s ds + \varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^\varepsilon}{X_{t_{i-1}}} \\
&= \sum_{i=1}^{n} \frac{r(\theta_0)\int_{t_{i-1}}^{t_i} (X_s - X_{t_{i-1}})ds + r(\theta_0)\int_{t_{i-1}}^{t_i} X_{t_{i-1}} ds + \varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^\varepsilon}{X_{t_{i-1}}} \\
&= r(\theta_0) + \sum_{i=1}^{n} \frac{r(\theta_0)\int_{t_{i-1}}^{t_i} (X_s - X_{t_{i-1}})ds}{X_{t_{i-1}}} + \sum_{i=1}^{n} \frac{\varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^\varepsilon}{X_{t_{i-1}}} \\
&:= r(\theta_0) + \Phi_1(n,\varepsilon) + \Phi_2(n,\varepsilon).
\end{aligned}$$

Proposition 4.2.1. *Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, we have $\Phi_{n,\varepsilon}(\theta) \to_{Q_\varepsilon} r(\theta_0)$.*

Proof. This result follows from the following lemmas.

Lemma 4.2.1. *We have $\Phi_1(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$.*

Proof. From (4.11), we get

$$\begin{aligned}|X_t - X_{t_{i-1}}| &\le \int_{t_{i-1}}^{t} |r(\theta_0)||X_s|ds + |\varepsilon \int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon| \\ &\le |r(\theta_0)| \int_{t_{i-1}}^{t} |X_s - X_{t_{i-1}}| + |X_{t_{i-1}}|ds + |\varepsilon \int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|.\end{aligned}$$

By Gronwall's Inequality

$$|X_t - X_{t_{i-1}}| \le e^{|r(\theta_0)|(t-t_{i-1})}\Big(n^{-1}|r(\theta_0)||X_{t_{i-1}}| + \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|\Big).$$

It yields

$$\sup_{t_{i-1}\le t\le t_i} |X_t - X_{t_{i-1}}| \le e^{n^{-1}|r(\theta_0)|}\Big(n^{-1}|r(\theta_0)||X_{t_{i-1}}| + \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|\Big). \tag{4.12}$$

On the other side, from $\Phi_1(n,\varepsilon)$, it is seen that

$$\begin{aligned}|\Phi_1(n,\varepsilon)| &\le |r(\theta_0)| \sum_{i=1}^{n} \frac{\int_{t_{i-1}}^{t_i} |X_s - X_{t_{i-1}}|ds}{|X_{t_{i-1}}|} \\ &\le |r(\theta_0)| \sum_{i=1}^{n} \frac{n^{-1} \sup_{t_{i-1}\le t\le t_i} |X_t - X_{t_{i-1}}|}{|X_{t_{i-1}}|}.\end{aligned}$$

From (4.12), note that

$$\begin{aligned}|\phi_1(n,\varepsilon)| &\le |r(\theta_0)| \sum_{i=1}^{n} \frac{n^{-1} e^{\frac{|r(\theta_0)|}{n}}\Big(n^{-1}|r(\theta_0)||X_{t_{i-1}}| + \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|\Big)}{|X_{t_{i-1}}|} \\ &= \frac{|r(\theta_0)|^2 e^{\frac{|r(\theta_0)|}{n}}}{n} + \sum_{i=1}^{n} \frac{|r(\theta_0)| n^{-1} e^{\frac{|r(\theta_0)|}{n}} \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|}{|X_{t_{i-1}}|}.\end{aligned}$$

We set

$\Phi_1^{(1)}(n,\varepsilon) := \frac{|r(\theta_0)|^2 e^{\frac{|r(\theta_0)|}{n}}}{n}$;

$\Phi_1^{(2)}(n,\varepsilon) := \sum_{i=1}^{n} \frac{|r(\theta_0)| n^{-1} e^{\frac{|r(\theta_0)|}{n}} \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|}{|X_{t_{i-1}}|}$.

It is clear that $\Phi_1^{(1)}(n,\varepsilon) \to 0$, as $n \to \infty$. Then we consider $\Phi_1^{(2)}(n,\varepsilon)$, by Holder's Inequality, Markov Inequality and Lemma 2.3.1, for $\delta > 0$, we have

$$\begin{aligned}
Q_\varepsilon(|\Phi_1^{(2)}(n,\varepsilon)| > \delta) &\le \frac{E_{Q_\varepsilon}|\Phi_1^{(2)}(n,\varepsilon)|}{\delta} \\
&\le \frac{1}{\delta} E_{Q_\varepsilon} \sum_{i=1}^{n} \frac{|r(\theta_0)| n^{-1} e^{\frac{|r(\theta_0)|}{n}} \varepsilon \sup\limits_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|}{|X_{t_{i-1}}|} \\
&= \frac{1}{\delta} \sum_{i=1}^{n} E_{Q_\varepsilon} X_{t_{i-1}}^{-1} \Big(|r(\theta_0)| n^{-1} e^{\frac{|r(\theta_0)|}{n}} \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon| \Big) \\
&\le \frac{1}{\delta} \sum_{i=1}^{n} (E_{Q_\varepsilon} X_{t_{i-1}}^{-2})^{\frac{1}{2}} \Big[E_{Q_\varepsilon} \Big(|r(\theta_0)| n^{-1} e^{\frac{|r(\theta_0)|}{n}} \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon| \Big)^2 \Big]^{\frac{1}{2}} \\
&= \frac{1}{\delta} \sum_{i=1}^{n} (E_{Q_\varepsilon} X_{t_{i-1}}^{-2})^{\frac{1}{2}} \Big[|r(\theta_0)|^2 n^{-2} e^{\frac{|2r(\theta_0)|}{n}} \varepsilon^2 E_{Q_\varepsilon} \Big(\sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon| \Big)^2 \Big]^{\frac{1}{2}} \\
&= \frac{1}{\delta} \sum_{i=1}^{n} X_0^{-1} e^{\frac{3\varepsilon^2}{2} t_{i-1} - r t_{i-1}} \Big[|r(\theta_0)|^2 n^{-2} e^{\frac{|2r(\theta_0)|}{n}} \varepsilon^2 E_{Q_\varepsilon} \Big(\sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon| \Big)^2 \Big]^{\frac{1}{2}} \\
&:= A.
\end{aligned}$$

We set $E_{Q_\varepsilon}(\sup\limits_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s^\varepsilon|^2) = \vartheta$.

It yields

$$\begin{aligned}
\vartheta &\le 4E_{Q_\varepsilon}\Big(\int_{t_{i-1}}^{t_i} |X_s|^2 ds \Big) \\
&\le 4 \int_{t_{i-1}}^{t_i} E_{Q_\varepsilon} |X_s|^2 ds \qquad (4.13) \\
&= 4n^{-1} X_0^2 exp(\varepsilon^2 s + 2r(\theta)s).
\end{aligned}$$

So that, we have

$$\begin{aligned}
A &\le \frac{1}{\delta} \sum_{i=1}^{n} X_0^{-1} \exp\Big(\frac{3\varepsilon^2}{2} t_{i-1} - r(\theta) t_{i-1} \Big) |r_0| n^{-1} e^{\frac{|r_0|}{n}} \varepsilon 2 n^{-\frac{1}{2}} X_0 \exp\Big(\frac{1}{2}\varepsilon^2 s + r(\theta)s \Big) \\
&= |r(\theta_0)| \frac{2}{\delta} \exp\Big(\frac{3\varepsilon^2}{2} t_{i-1} - r(\theta)t_{i-1} + \frac{|r(\theta_0)|}{n} + \frac{1}{2}\varepsilon^2 s + r(\theta)s \Big) \varepsilon n^{-\frac{1}{2}}
\end{aligned}$$

which imply $A \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. Then we have $\phi_1^{(2)}(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. Finally, we get $\Phi_1(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$ and $\varepsilon \to 0$. □

Lemma 4.2.2. *Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, we have $\Phi_2(n,\varepsilon) \to_{Q_\varepsilon} 0$.*

Proof. Since $\Phi_2(n,\varepsilon) = \sum_{i=1}^{n} \frac{\varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^\varepsilon}{X_{t_{i-1}}}$.

Together with Holder's Inequality, Markov Inequality and Lemma 2.3.1, we obtain, for δ

$$\begin{aligned} Q_\varepsilon(|\Phi_2(n,\varepsilon)| > \delta) &\le \frac{E_{Q_\varepsilon}|(\Phi_2(n,\varepsilon))|}{\delta} \\ &= \frac{1}{\delta}\sum_{i=1}^{n} E_{Q_\varepsilon} X_{t_{i-1}}^{-1} \varepsilon \Big| \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^\varepsilon \Big| \\ &\le \frac{1}{\delta}\sum_{i=1}^{n} \left(E_{Q_\varepsilon} X_{t_{i-1}}^{-2}\right)^{\frac{1}{2}} \left[E_{Q_\varepsilon}(\varepsilon \sup_{t_{i-1}\le t\le t_i} | \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s^\varepsilon)|^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{\delta}\sum_{i=1}^{n} X_0^{-1} exp\Big(\frac{3\varepsilon^2}{2} t_{i-1} - r(\theta)t_{i-1}\Big) 2\varepsilon n^{-\frac{1}{2}} X_0 exp\Big(\frac{1}{2}\varepsilon^2 s + r(\theta)s\Big) \\ &= \frac{2}{\delta}\exp\Big(\frac{3\varepsilon^2}{2} t_{i-1} - r(\theta)t_{i-1} + \frac{1}{2}\varepsilon^2 s + r(\theta)s\Big)\varepsilon n^{\frac{1}{2}} \end{aligned}$$

which implies that $\Phi_2(n,\varepsilon) \to_{Q_\varepsilon} 0$ as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$. □

Then by using Lemma 4.2.2 and 4.2.3, we have

$$\Phi(n,\varepsilon) := r(\theta_0) + \phi_1(n,\varepsilon) + \phi_2(n,\varepsilon) \to_{Q_\varepsilon} r(\theta_0)$$

as $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$. □

Proof of Theorem 4.2.1.

Recall that $\phi_{n,\varepsilon}(\theta) = 2(r(\theta_0) - r(\theta)) \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} + (r^2(\theta) - r^2(\theta_0))$, by Proposition 4.2.1, as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$, we have

$$\phi_{n,\varepsilon}(\theta) \to_{Q_\varepsilon} (r(\theta) - r(\theta_0))^2. \tag{4.14}$$

Recall that our contrast function is

$$\rho_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \frac{|X_{t_i} - X_{t_{i-1}} - r(\theta)X_{t_{i-1}}\Delta t_i|^2}{\varepsilon^2 X_{t_{i-1}}^2 \Delta t_i}.$$

In order to obtain the least square estimator $r(\hat{\theta}_{n,\varepsilon})$, we let

$$\frac{\partial \rho_{n,\varepsilon}(r(\theta))}{\partial \theta} = r'(\theta).$$

Since $r'(\theta) \neq 0$, we get

$$\frac{\partial \rho_{n,\varepsilon}(r(\hat{\theta}))}{\partial r(\hat{\theta})} = 0$$

So that

$$r(\hat{\theta}_{n,\varepsilon}) = \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}}. \tag{4.15}$$

From Proposition 4.2.1, we know $\Phi_{n,\varepsilon}(\theta) \to_{Q_\varepsilon} r(\theta_0)$, as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$. Together with (4.15) and (4.10) we get $r(\hat{\theta}_{n,\varepsilon}) \to_{Q_\varepsilon} r(\theta_0)$ as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$. Since $r(\theta)$ is a $\mathcal{C}^2$-function of θ with $r'(\theta) \neq 0$, we have

$$\{|\hat{\theta}_{n,\varepsilon} - \theta_0| > \eta\} \subset \left\{|r(\hat{\theta}_{n,\varepsilon})| > \frac{\eta}{\inf_{\theta \in \Theta} |r'(\theta)|}\right\}$$

for $\eta > 0$. This implies $\hat{\theta}_{n,\varepsilon} \to_{Q_\varepsilon} \theta_0$ as $n \to \infty$, $\varepsilon \to 0$ and $\varepsilon n^{\frac{1}{2}} \to 0$. □

4.3 Asymptotic of The Least Squares Estimator

In this section we assume that $\alpha(x,t,\varepsilon) = \varepsilon\alpha(x,t)$ such that $Q = Q_\varepsilon$ is independent of ε.

Theorem 4.3.1. *Let $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$, we have*

$$\varepsilon^{-1}(\hat{\theta}_{n,\varepsilon} - \theta_0) \to_Q \left(r'(\theta_0)\right)^{-1} U$$

where U is a Q-random variable with standard normal distribution $N(0,1)$.

Proof. Before we give a proof, we introduce

$$I(\theta) = \left(r'(\theta)\right)^2$$

and

$$D(\theta) = -\left(r'(\theta)\right)^2. \tag{4.16}$$

Since

$$\phi_{n,\varepsilon}(\theta) = 2(r(\theta_0) - r(\theta))\sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} + (r^2(\theta) - r^2(\theta_0)).$$

We have

$$\phi'_{n,\varepsilon}(\theta) = -2\sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} r'(\theta) + 2r(\theta)r'(\theta).$$

Set

$$f_{n,\varepsilon}(\theta) = \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} r'(\theta) - r(\theta)r'(\theta)$$

and

$$\begin{aligned} D_{n,\varepsilon}(\theta) &= f'_{n,\varepsilon}(\theta) \\ &= \sum_{i=1}^{n} \frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} r''(\theta) - \left(r'(\theta)\right)^2 - r(\theta)r''(\theta). \end{aligned}$$

Let $B(\theta_0;\rho) = \{\theta : |\theta - \theta_0| \le \rho\}$ for $\rho > 0$. Then, by the consistency of $\hat{\theta}_{n,\varepsilon}$, there exists a sequence $\eta_{n,\varepsilon} \to 0$ as $n \to \infty$ and $\varepsilon \to 0$ such that $B(\theta_0;\eta_{n,\varepsilon}) \subset \Theta_0$, and $P_{\theta_0}[\hat{\theta}_{n,\varepsilon} \in B(\theta_0;\eta_{n,\varepsilon})] \to 1$. When $\hat{\theta}_{n,\varepsilon} \in B(\theta_0;\eta_{n,\varepsilon})$, we have

$$\begin{aligned} &\varepsilon^{-1}\{f_{n,\varepsilon}(\hat{\theta}_{n,\varepsilon}) - f_{n,\varepsilon}(\theta_0)\} \\ =&\varepsilon^{-1}(\hat{\theta}_{n,\varepsilon} - \theta_0)\int_0^1 D_{n,\varepsilon}(\theta_0 + u(\hat{\theta}_{n,\varepsilon} - \theta_0))du. \end{aligned}$$

But

$$\Big|\int_0^1 D_{n,\varepsilon}(\theta_0+u(\hat{\theta}_{n,\varepsilon}-\theta_0))du - D_{n,\varepsilon}(\theta_0)\Big|\mathbb{1}_{\{\theta_{n,\varepsilon}\in B(\hat{\theta}_0;\eta_{n,\varepsilon})\}}$$

$$\leq \sup_{\theta\in B(\theta_0;\eta_{n,\varepsilon})}|D_{n,\varepsilon}(\theta)-D_{n,\varepsilon}(\theta_0)|$$

$$\leq \sup_{\theta\in B(\theta_0;\eta_{n,\varepsilon})}|D_{n,\varepsilon}(\theta)-D(\theta)| + \sup_{\theta\in B(\theta_0;\eta_{n,\varepsilon})}|D(\theta)-D(\theta_0)| + \sup_{\theta\in B(\theta_0;\eta_{n,\varepsilon})}|D_{n,\varepsilon}(\theta_0)-D(\theta_0)|$$

$$:= A_1 + A_2 + A_3.$$

Since

$$D_{n,\varepsilon}(\theta) = \sum_{i=1}^{n}\frac{X_{t_k}-X_{t_{k-1}}}{X_{t_{k-1}}}r''(\theta) - \Big(r'(\theta)\Big)^2 - r(\theta)r''(\theta).$$

According to Proposition 4.2.1 and (4.16), let $n\to\infty$ and $\varepsilon\to 0$ with $\varepsilon n^{\frac{1}{2}}\to 0$, then, we obtain

$$D_{n,\varepsilon}(\theta)\to_Q D(\theta).$$

Consequently, we have $A_1\to 0$, $A_2\to 0$ and $A_3\to 0$ as $n\to\infty$ and $\varepsilon\to 0$ with $\varepsilon n^{\frac{1}{2}}\to 0$. So that, we get

$$\int_0^1 D_{n,\varepsilon}(\theta_0+u(\hat{\theta}_{n,\varepsilon}-\theta_0))du \to_Q D(\theta_0)$$

It is easy to see

$$\varepsilon^{-1}f_{n,\varepsilon}(\hat{\theta}_{n,\varepsilon}) = 0$$

as

$$\begin{aligned}\varepsilon^{-1}f_{n,\varepsilon}(\hat{\theta}_{n,\varepsilon}) &= \varepsilon^{-1}\sum_{k=1}^{n}\frac{X_{t_i}-X_{t_{i-1}}}{X_{t_{i-1}}}r'(\hat{\theta}_{n,\varepsilon}) - \varepsilon^{-1}r(\hat{\theta}_{n,\varepsilon})r'(\hat{\theta}_{n,\varepsilon})\\ &= \varepsilon^{-1}r(\hat{\theta}_{n,\varepsilon}))r'(\hat{\theta}_{n,\varepsilon}) - \varepsilon^{-1}r(\hat{\theta}_{n,\varepsilon})r'(\hat{\theta}_{n,\varepsilon})\\ &= 0\end{aligned}$$

by (4.15).

Proposition 4.3.1. *We have*

$$\varepsilon^{-1}f_{n,\varepsilon}(\theta_0)\to_Q r'(\theta_0)U$$

as $n\to\infty$ and $\varepsilon\to 0$.

Proof. Note that from (4.10)

$$\begin{aligned}\varepsilon^{-1} f_{n,\varepsilon}(\theta_0) &= \varepsilon^{-1}\Big(\sum_{k=1}^{n} \frac{X_{t_k} - X_{t_{k-1}}}{X_{t_{k-1}}} r'(\theta_0) - r(\theta_0) r'(\theta_0)\Big) \\ &= \varepsilon^{-1}\Big(\sum_{i=1}^{n} \frac{r(\theta_0)\int_{t_{i-1}}^{t_i} (X_s - X_{t_{i-1}})ds}{X_{t_{i-1}}} + \sum_{i=1}^{n} \frac{\varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s}{X_{t_{i-1}}}\Big) r'(\theta_0).\end{aligned} \tag{4.17}$$

We set

$$\begin{aligned}C(n,\varepsilon) &= \varepsilon^{-1}\Big(\sum_{i=1}^{n} \frac{r(\theta_0)\int_{t_{i-1}}^{t_i} (X_s - X_{t_{i-1}})ds}{X_{t_{i-1}}} + \sum_{i=1}^{n} \frac{\varepsilon \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s}{X_{t_{i-1}}}\Big) \\ &= \varepsilon^{-1} \sum_{i=1}^{n} \frac{r(\theta_0)\int_{t_{i-1}}^{t_i} (X_s - X_{t_{i-1}})ds}{X_{t_{i-1}}} + \sum_{i=1}^{n} \frac{\int_{t_{i-1}}^{t_i} X_s d\hat{B}_s}{X_{t_{i-1}}} \\ &:= C_1(n,\varepsilon) + C_2(n,\varepsilon).\end{aligned}$$

Lemma 4.3.1. *We have $C_1(n,\varepsilon) \to_Q 0$ as $n \to \infty$ and $\varepsilon \to 0$.*

Proof. From $C_1(n,\varepsilon)$, we have

$$\begin{aligned}|C_1(n,\varepsilon)| &\le |\varepsilon^{-1}||r(\theta_0)| \sum_{i=1}^{n} \frac{\int_{t_{i-1}}^{t_i} |X_s - X_{t_{i-1}}| ds}{|X_{t_{i-1}}|} \\ &\le |\varepsilon^{-1}||r(\theta_0)| \sum_{i=1}^{n} \frac{n^{-1} \sup_{t_{i-1} \le t \le t_i} |X_t - X_{ti-1}|}{|X_{t_{i-1}}|}.\end{aligned}$$

From (4.12), we have

$$\begin{aligned}|C_1(n,\varepsilon)| &\le |\varepsilon^{-1}||r(\theta_0)| \sum_{i=1}^{n} \frac{n^{-1} e^{\frac{|r(\theta_0)|}{n}} (n^{-1}|r(\theta_0)||X_{t_{i-1}}| + \varepsilon \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s|)}{|X_{t_{i-1}}|} \\ &= \frac{|\varepsilon^{-1}||r(\theta_0)|^2 e^{\frac{|r(\theta_0)|}{n}}}{n} + |r(\theta_0)| \sum_{i=1}^{n} \frac{n^{-1} e^{\frac{|r(\theta_0)|}{n}} \sup_{t_{i-1}\le t\le t_i} |\int_{t_{i-1}}^{t} X_s d\hat{B}_s|}{|X_{t_{i-1}}|} \\ &:= C_1^1(n,\varepsilon) + C_1^2(n,\varepsilon).\end{aligned}$$

It is easy to see that $C_1^1(n,\varepsilon) \to_Q 0$, as $n \to \infty$.

Then we consider $C_1^2(n,\varepsilon)$, by Holder's Inequality, Markov Inequality, Lemma 2.3.1, we have

$$\begin{aligned}
Q(|C_1^2(n,\varepsilon)| > \delta) &\le \frac{E_Q|C_1^2(n,\varepsilon)|}{\delta} \\
&\le \frac{1}{\delta}|r(\theta_0)|E_Q\sum_{i=1}^{n}\frac{n^{-1}e^{\frac{|r(\theta_0)|}{n}}\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|}{|X_{t_{i-1}}|} \\
&= \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}E_Q X_{t_{i-1}}^{-1}(n^{-1}e^{\frac{|r(\theta_0)|}{n}}\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|) \\
&\le \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}(E_Q X_{t_{i-1}}^{-2})^{\frac{1}{2}}[E_Q(n^{-1}e^{\frac{|r(\theta_0)|}{n}}\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)^2]^{\frac{1}{2}} \\
&= \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}(E_Q X_{t_{i-1}}^{-2})^{\frac{1}{2}}[n^{-2}e^{\frac{|2r(\theta_0)|}{n}}E_Q(\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)^2]^{\frac{1}{2}} \\
&= \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}X_0^{-1}e^{\frac{3\varepsilon^2}{2}t_{i-1}-r(\theta_0)t_{i-1}}[n^{-2}e^{\frac{|2r(\theta_0)|}{n}}E_Q(\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|)^2]^{\frac{1}{2}} \\
&= \iota.
\end{aligned}$$

We recall that $E_Q(\sup_{t_{i-1}\le t\le t_i}|\int_{t_{i-1}}^{t}X_s d\hat{B}_s|^2) = \vartheta$.

By equation (4.13), we have

$$\begin{aligned}
\iota &\le \frac{1}{\delta}|r(\theta_0)|\sum_{i=1}^{n}X_0^{-1}\exp(\frac{3\varepsilon^2}{2}t_{i-1} - r(\theta)t_{i-1})n^{-1}e^{\frac{|r(\theta_0)|}{n}}2n^{-\frac{1}{2}}X_0\exp(\frac{1}{2}\varepsilon^2 s + r(\theta_0)s) \\
&= \frac{2}{\delta}|r(\theta_0)|\exp(\frac{3\varepsilon^2}{2}t_{i-1} - r(\theta)t_{i-1} + \frac{|r(\theta_0)|}{n} + \frac{1}{2}\varepsilon^2 s + r(\theta)s)n^{-\frac{1}{2}}
\end{aligned}$$

which implies $\iota \to_Q 0$ as $n \to \infty$. Then we have $C_1^2(n,\varepsilon) \to_Q 0$ as $n \to \infty$ and $\varepsilon \to 0$. Finally, we get $C_1(n,\varepsilon) \to_Q 0$ as $n \to \infty$ and $\varepsilon \to 0$. □

Let X_t^0 be the solution of the underlying ordinary differential equation under the true value of the drift parameter:

$$dX_t^0 = r(\theta_0)X_t^0 dt, \qquad X_0^0 = x_0. \tag{4.18}$$

Lemma 4.3.2. *We have $C_2(n,\varepsilon) \to_Q U$ as $n \to \infty$ and $\varepsilon \to 0$, where U is a Q-random variable with standard normal distribution $N(0,1)$.*

Proof. Since

$$
\begin{aligned}
C_2(n,\varepsilon) &= \sum_{i=1}^{n} \frac{\int_{t_{i-1}}^{t_i} X_s d\hat{B}_s}{X_{t_{i-1}}}. \\
&= \sum_{i=1}^{n} X_{t_{i-1}}^{-1} \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\
&= \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-1} \int_{t_{i-1}}^{t_i} X_s^0 d\hat{B}_s + \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-1} \int_{t_{i-1}}^{t_i} (X_s - X_s^0) d\hat{B}_s \\
&+ \sum_{i=1}^{n} (X_{t_{i-1}}^{-1} - (X_{t_{i-1}}^0)^{-1}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\
&:= C_2^1(n,\varepsilon) + C_2^2(n,\varepsilon) + C_2^3(n,\varepsilon).
\end{aligned}
$$

Define a deterministic process $V(s)$ by $V(s) = \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-1} X_s^0 \mathbb{1}_{(t_{i-1},t_i]}(s)$. Let $V_+(s)$ and $V_-(s)$ denote the positive and negative part of $V(s)$. By Theorem 4.1 of Kallenberg [20], there exist two independent Q-Brownian motions $\hat{B}'$, $\hat{B}''$, which have the same distribution of $\hat{B}$, such that

$$
C_2^1(n,\varepsilon) = \int_0^1 V(s) d\hat{B}_s = \hat{B}' \circ \int_0^1 V_+^2(s) ds - \hat{B}'' \circ \int_0^1 V_-^2(s) ds.
$$

Note that

$$
V_+^2 = \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-2} (X_s^0)_+^2 \mathbb{1}_{(t_{i-1},t_i]}(s)
$$

and

$$
V_-^2 = \sum_{i=1}^{n} (X_{t_{i-1}}^0)^{-2} (X_s^0)_-^2 \mathbb{1}_{(t_{i-1},t_i]}(s).
$$

Then we have

$$
\int_0^1 V_+^2(s) ds \to \int_0^1 (X_s^0)^{-2} (X_s^0)_+^2 ds
$$

and

$$
\int_0^1 V_-^2(s) ds \to \int_0^1 (X_s^0)^{-2} (X_s^0)_-^2 ds
$$

as $n \to \infty$. Then,

$$
\hat{B}' \circ \int_0^1 V_+^2(s) ds \to \hat{B}' \circ \int_0^1 (X_s^0)^{-2} (X_s^0)_+^2 ds
$$

and

$$
\hat{B}'' \circ \int_0^1 V_-^2(s) ds \to \hat{B}'' \circ \int_0^1 (X_s^0)^{-2} (X_s^0)_-^2 ds.
$$

We get

$$C_2^1(n,\varepsilon) \to_Q U_1\Big(\int_0^1 (X_s^0)^{(-2)}(X_s^0)_+^2 ds\Big)^{\frac{1}{2}} - U_2\Big(\int_0^1 (X_s^0)^{-2}(X_s^0)_-^2 ds\Big)^{\frac{1}{2}}$$

where U_1 and U_2 are two random variables with standard normal distribution $N(0,1)$ as $n \to \infty$. Since

$$X_{s+}^0 = \max(X_s^0, 0)$$

and

$$X_{s-}^0 = \max(-X_s^0, 0).$$

$$C_2^1(n,\varepsilon) \to_Q \begin{cases} U_1, & X_s^0 \geq 0 \\ U_2, & X_s^0 \leq 0 \end{cases}$$

as $n \to \infty$.

So it can be summarized by

$$C_2^1(n,\varepsilon) \to_Q U$$

as $n \to \infty$.

Now, let us consider $C_2^2(n,\varepsilon)$. By using Holder's Inequality, Markov Inequality, Lemma 2.3.1, we get

$$\begin{aligned} Q(|C_2^2(n,\varepsilon)| > \delta) &\leq \delta^{-1} E_Q[\sum_{i=1}^n (X_{t_{i-1}}^0)^{-1} \sup_{t_{i-1}\leq t\leq t_i} | \int_{t_{i-1}}^{t_i} (X_s - X_s^0) d\hat{B}_s|| \\ &\leq \frac{1}{\delta}\sum_{i=1}^n (E_Q[(X_{t_{i-1}}^0)^{-2}]^{\frac{1}{2}} [E_Q(\sup_{t_{i-1}\leq t\leq t_i} | \int_{t_{i-1}}^{t_i} (X_s - X_s^0) d\hat{B}_s)|^2]^{\frac{1}{2}} \\ &\leq \frac{1}{\delta}\sum_{i=1}^n (E_Q[(X_{t_{i-1}}^0)^{-2}]^{\frac{1}{2}} [E_Q \int_{t_{i-1}}^{t_i} |X_s - X_s^0|^2 ds)]^{\frac{1}{2}} \\ &\leq \frac{1}{\delta}\sum_{i=1}^n (E_Q[(X_{t_{i-1}}^0)^{-2}]^{\frac{1}{2}} [2t^{\frac{1}{2}} E_Q \sup_{t_{i-1}\leq t\leq t_i} |X_s - X_s^0| ds] \end{aligned}$$

which tends to zero as $n \to \infty$ and $\varepsilon \to 0$. For $C_2^3(n,\varepsilon)$, we have

$$\begin{aligned} C_2^3(n,\varepsilon) &= \sum_{i=1}^n (X_{t_{i-1}}^{-1} - (X_{t_{i-1}}^0)^{-1}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\ &= \sum_{i=1}^n (-\frac{X_{t_{i-1}} - X_{t_{i-1}}^0}{X_{t_{i-1}}^0 X_{t_{i-1}}}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s \\ &\leq \sum_{i=1}^n (-\frac{\sup_{t_{i-1}\leq t\leq t_i} |X_{t_{i-1}} - X_{t_{i-1}}^0|}{X_{t_{i-1}}^0 X_{t_{i-1}}}) \int_{t_{i-1}}^{t_i} X_s d\hat{B}_s. \end{aligned}$$

By Lemma 3.2.2, we have $C_2^3(n,\varepsilon) \to_Q 0$ as $n \to \infty$. □

Proof of Proposition 4.3.1, Combining Lemma 4.3.1 and Lemma 4.3.2, we have

$$\begin{aligned} C(n,\varepsilon) &= C_1(n,\varepsilon) + C_2(n,\varepsilon) \\ &\to_Q U \end{aligned}$$

as $n \to \infty$ and $\varepsilon \to 0$.

By (4.17),

$$\varepsilon^{-1} f_{n,\varepsilon}(\theta_0) \to_Q r'(\theta_0) U$$

as $n \to \infty$ and $\varepsilon \to 0$. □

Proof of Theorem 4.3.1. With previous proof, we have

$$\begin{aligned} \varepsilon^{-1}(\hat{\theta}_{n,\varepsilon} - \theta_0) &= -\Big(\int_0^1 D_{n,\varepsilon}(\theta_0 + u(\hat{\theta}_{n,\varepsilon} - \theta_0)) d\theta \Big)^{-1} \varepsilon^{-1} f_{n,\varepsilon}(\theta_0) \\ &\to_Q \Big(r'(\theta_0) \Big)^{-1} U \end{aligned}$$

as $n \to \infty$ and $\varepsilon \to 0$ with $\varepsilon n^{\frac{1}{2}} \to 0$. □

Chapter 5

Nonparametric Drift Estimation For Inhomogeneous Stochastic Differential Equations

5.1 Introduction

Let $\{\Omega, \mathcal{F}, \mathcal{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The stochastic process $X = (X_t, 0 \leq t \leq T)$ with a given initial value $X_0 = x \in \mathbb{R}$, is determined by the following Itô stochastic differential equation (SDE)

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t, \qquad 0 \leq t \leq T. \tag{5.1}$$

where $\mu : \mathbb{R} \times [0, T] \to \mathbb{R}$ is a measurable function which is continuous with respect to t; $\sigma : \mathbb{R} \times [0, T] \to \mathbb{R}_+$ is a positive function which is continuous with respect to t; $B_t(0 \leq t \leq T)$ is a one dimensional $\mathcal{F}_t$-Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{0\leq t\leq T}\}$. We assume the following conditions $\forall t \in [0, T]$:

(1*) $|\mu(x_1, t) - \mu(x_2, t)| + |\sigma(x_1, t) - \sigma(x_2, t)| \leq L|x_1 - x_2|$, $\forall\, x_1, x_2 \in \mathbb{R}$, where $L > 0$ is a constant.

(2*) There exist positive constants σ_0 and σ_1 such that $0 < \sigma_0 \leq \sigma(x, t) \leq \sigma_1$.

(3*) Dissipative condition: $\frac{\mu(x_1,t)-\mu(x_2,t)}{x_1-x_2} \leq G$, $\forall\ x_1, x_2 \in \mathbb{R}$, where $G < 0$ is a constant.

Remark 5.1.1. *Condition (1*) guarantees that the equation* (5.1) *has an unique solution and the solution does not explode; condition (2*) means that $\sigma(t,x)$ is uniformly elliptic; under condition (3*), the solution of* (5.1) *is stationary.*

We assume that the process X_t is observed at discrete time intervals $\{t_i = i\Delta, i = 0, 1, 2, ..., n\}$, where Δ is the time frequency for observation and n is the sample size. By (5.1), we can get the system

$$X_{t_i} = x + \sum_{i=0}^{n-1} \mu(X_{t_i}, t_i)\Delta t_i + \sum_{i=0}^{n-1} \sigma(X_{t_i}, t_i)(B_{t_{i+1}} - B_{t_i}),$$

where $\Delta t_i = t_{i+1} - t_i$. We represent the estimator of $\mu_n(x,t)$ by $\hat{\mu}_n(x,t)$. To get the expression of $\hat{\mu}_n(x,t)$, first we minimize an object function given below with certain weights:

$$\sum_{i=0}^{n-1} W_{n,i}(x, t_i)(Y_i - \Delta)^2$$

where $Y_i := X_{t_{i+1}} - X_{t_i}$ and $\Delta := t_{i+1} - t_i, i = 0, 1, 2, ..., n-1$. The weight function is given by

$$W_{n,i}(x,t) = \frac{\sum_{i=0}^{n-1} K_h(X_{t_i} - x)}{\sum_{i=0}^{n-1} K_h(X_{t_i} - x)}, i = 0, 1, 2, ..., n-1$$

where $K_h(.) = K(./h)/h$, K is a kernel density function with mean zero and finite variance, and h is the bandwidth for the kernel. Then, we get the expression of $\hat{\mu}_n(x,t)$ by

$$\hat{\mu}_n(x, t_i) = \frac{\sum_{i=0}^{n-1} Y_i K_h(X_{t_i} - x)}{\Delta \sum_{i=0}^{n-1} K_h(X_{t_i} - x)}. \tag{5.2}$$

In this chapter, we focus on the asymptotic of the estimator $\hat{\mu}_n(x,t)$ with high frequency $n \to \infty$. In Section 5.2, we aim to prove that $\hat{\mu}_n(x,t) \to_P \mu(x,t)$ in probability as $n \to 0$. And Section 5.3 we establish the rate of convergence and the asymptotic distribution, after that we give an example in Section 5.4.

5.2 Consistency of The Nonparametric Drift Estimator

Before this section, we give two propositions. From Theorem 5.3 of Ikeda and Watanabe, [19], we have the following propositions:

Proposition 5.2.1. *Let $F : \Omega \times [0, \infty) \to \mathbb{R}$ be such that*

$$E \exp\left\{\alpha \int_0^t |F_s|^2 ds\right\} < \infty \tag{5.3}$$

for every $\alpha > 0$ and $t > 0$. Let B_s be a Brownian motion. Then for every $\lambda \in \mathbb{R}$

$$Z_F(t) = \exp\left\{i\lambda \int_0^t F_s dB_s + |\lambda|^2 \int_0^t |F_s|^2 ds\right\}$$

is a complex-valued $\{\mathcal{F}_t\}$-martingale.

Proposition 5.2.2. *Let $F : \Omega \times [0, \infty) \to \mathbb{R} \setminus \{0\}$ be such that $\tau(u) = \int_0^u |F_t|^2 dt \to \infty$ as $u \to \infty$. Let*

$$\tau^{-1}(t) = \inf\{u : \tau(u) > t\} \quad \text{and} \quad \mathcal{A}_t = \mathcal{F}_{\tau^{-1}(t)},$$

Then the time-changed stochastic integral

$$\tilde{B}(t) = \int_0^{\tau^{-1}(t)} F_s dB_s$$

is an $\{\mathcal{A}_t\}$-Brownian motion. Consequently, for each $t > 0$

$$\int_0^t F_s dB_s = \tilde{B}(\tau(t)).$$

Lemma 5.2.1. *we assume that there is a non-negative adapted process $\varphi(t)$ satisfying $\int_0^T |\varphi|^2 dt < \infty$ for $T < \infty$. For any given $\xi > 0$ and $\rho > 0$, there is some constant $b > 0$ such that*

$$P\Big(\sup_{0\le t\le T}\Big|\int_0^t \varphi(s)dB_s\Big| > \xi\Big) \le \frac{b\rho}{\xi^2} + P\Big(\int_0^T |\varphi(t)|^2 dt > \rho\Big)$$

$$P\Big(\sup_{0\le t\le T}\Big|\int_0^t \varphi(s)dB_s\Big| > \xi\Big) \le \frac{b\rho}{\xi^2} + P\Big(\int_0^t |\varphi(t)|^2 dt > \rho\Big).$$

Proof. Let $S_t = \int_0^t |\varphi(s)|^2 ds$. Then by Proposition 5.2.2, there exists a Brownian motion B' with the same distribution as B such that $\int_0^t \varphi(s)dB_s = B'(S_t)$. By the classical maximal inequality(see Proposition 10.2 of Fristedt [11]), we find that

$$\begin{aligned}
P\Big(\sup_{0\le t\le T}\Big|\int_0^t \varphi(s)dB_s\Big| > \xi\Big) &\le P\Big(\sup_{0\le t\le T}\Big|\int_0^t \varphi(s)dB_s\Big| > \xi,\ S_T \le \rho\Big) + P(S_T > \rho) \\
&\le P\Big(\sup_{0\le t\le T}|B'(S_t)| > \xi,\ S_T \le \rho\Big) + P(S_T > \rho) \\
&\le P\Big(\sup_{0\le s\le \rho}|B'(s)| > \xi\Big) + P(S_T > \rho) \\
&\le \frac{b\rho}{\xi^2} + P\Big(\int_0^T |\varphi(t)|^2 dt > \rho\Big).
\end{aligned}$$

□

Lemma 5.2.2. *Suppose that there is a deterministic and nonnegative function χ. And we assume that there is a adapted process $\varphi(t)$ satisfying $\int_0^{T'} |\varphi|^2 dt < \infty$ for $T' < \infty$ and $\varphi(T') = \chi^{-1}(T')$ on the interval $(T', T'+1]$. If*

$$\chi^2(T') \int_0^{T'} |\varphi(t)|^2 dt \to_P 1 \quad \text{as} \quad T' \to \infty.$$

Then, we have

$$\chi(T') \int_0^{T'} \varphi(t)dB_t \Rightarrow N(0,1) \tag{5.4}$$

where B_t is a one dimensional Brownian Motion.

Proof. We define

$$R_t = \chi^2(T') \int_0^t |\varphi(s)|^2 ds$$

and

$$\tau_{T'} = \inf\{t \ge 0, R_t > 1\}.$$

Then we have $\tau_{T'} \in [0, T'+1]$. Then by proposition 5.2.2, there is a Brownian motion B' with the same distribution as B such that $\chi(T') \int_0^t |\varphi(s)| dB_s = B'_{R_t}$. We have

$$\chi(T') \int_0^{\tau_{T'}} \varphi(t)dB_t = B'_1 \sim N(0,1).$$

Then by using Lemma 5.2.1 and following the same arguments as in the proof of Theorem 1.19 in [26], we can see that the characteristic function of $\chi(T') \int_0^{T'} \varphi(t)dB_t$ converges to the characteristic function of $\chi(T') \int_0^{T'} \varphi(t)dB_t$ as $T' \to \infty$. By the continuity theorem (see Theorem 26.3 of Billingsley [3]), we have

$$\chi(T') \int_0^{T'} \varphi(t)dB_t \Rightarrow N(0,1).$$

□

Under Condition (1*), (2*) and (3*), there exists a unique invariant distribution ψ of the solution X_t. Let $f(x,t)$ be the density function of ψ. Then we define the kernel estimator $\hat{f}_n(x,t)$ of $f(x,t)$ by

$$\hat{f}_n(x,t) = \frac{1}{n}\sum_{i=0}^{n-1} K_h(X_{t_i} - x), \qquad 0 \le t \le T \tag{5.5}$$

and $\hat{g}_n(x,t_i)$ by

$$\hat{g}_n(x,t) = \frac{1}{n\Delta}\sum_{i=0}^{n-1} Y_i K_h(X_{t_i} - x), \qquad 0 \le t \le T. \tag{5.6}$$

From (5.2), we obtain

$$\hat{\mu}_n(x,t) = \frac{\hat{g}_n(x,t)}{\hat{f}_n(x,t)}, \qquad 0 \le t \le T. \tag{5.7}$$

Define the strong mixing coefficient of X by

$$\alpha_X(t) = \sup_{s\in\mathbb{R}_+} \sup_{A,B\in\mathcal{B}(\mathbb{R})} E[\lambda_A(X_s)\lambda_B(X_{s+t})] - E(\lambda_A(X_s))E(\lambda_B(X_{s+t})), \qquad 0 \le t \le T \tag{5.8}$$

where A, B are measurable sets in the σ-algebras.

We need specify some new conditions as follows:

(4*) The kernel function $K(.)$ satisfies

$$\int_{-\infty}^{\infty} u^2 K(u)du < \infty$$

and

$$\int_{-\infty}^{\infty} K^2(u)du < \infty.$$

(5*) As $n \to \infty$, $h \to 0$, $\Delta \to 0$ and $n\Delta h \to \infty$.

(6*) The solution X_t admits a unique invariant distribution ψ and is geometrically strong mixing(GSM), i.e. there exist $c_0 > 0$ and $\rho \in (0,1)$ such that $\alpha_X(t) \le c_0\rho^t$, $t \ge 0$.

(7*) The density function $f(x)$ of the stationary distribution ψ is continuous.

Theorem 5.2.1. *Assume that condition (1*)-(3*) hold, and $f(x,t) > 0$, then $\hat{\mu}_n(x,t) \rightarrow_P \mu(x,t)$ as $n \rightarrow \infty$.*

Lemma 5.2.3. *Under conditions (1*)-(3*), we have*

$$\hat{f}_n(x,t) \rightarrow_P f(x,t) \quad as \quad n \rightarrow \infty. \tag{5.9}$$

Proof. Recall that $\hat{f}_n(x,t) \rightarrow_P f(x,t)$ as $n \rightarrow \infty$ means that $\forall v > 0$, $\lim_{n\rightarrow\infty} P(|\hat{f}_n(x,t) - f(x,t)| \geq v) = 0$.
And note that

$$\hat{f}_n(x,t) - f(x,t) = \hat{f}_n(x,t) - E[\hat{f}_n(x,t)] + E[\hat{f}_n(x,t)] - f(x,t).$$

For $E[\hat{f}_n(x,t)] - f(x,t)$, by the stationarity of the process X_t, we have

$$\begin{aligned} E[\hat{f}_n(x,t)] &= E[K_h(X_0 - x)] \\ &= \int_{-\infty}^{\infty} K_h(y-x) f(y,t) dy \\ &= \int_{-\infty}^{\infty} K(u) f(x+uh,t) du \end{aligned}$$

which converges to $f(x,t)$ for each x as $n \rightarrow \infty$ by Lebesgue Dominated Convergence Theorem.
For $\hat{f}_n(x,t) - E[\hat{f}_n(x,t)]$, we have

$$\begin{aligned} \hat{f}_n(x,t) - E[\hat{f}_n(x,t)] &= \frac{1}{n}\sum_{i=0}^{n-1} K_h(X_{t_i} - x) - \frac{1}{n}\sum_{i=0}^{n-1} E[K_h(X_{t_i} - x)] \\ &= \frac{1}{n}\sum_{i=0}^{n-1} [K_h(X_{t_i} - x) - E[K_h(X_{t_i} - x)]]. \end{aligned}$$

Let $\gamma_{n,i}(x,t) = K_h(X_{t_{i-1}} - x) - EK_h(X_{t_{i-1}} - x)$, i=1,2,...,n. Note that $\sup_{1\leq i\leq n} |\gamma_{n,i}(x,t)| \leq Dh^{-1}$ for some positive constant $D < \infty$. By applying Theorem 1.3 of Bosq [4], we have for each integer $q \in [1, \frac{n}{2}]$ and any $\delta > 0$

$$\begin{aligned} P\Big(\frac{1}{n}\Big|\sum_{i=1}^{n} \gamma_{n,i}(x,t)\Big| > \delta\Big) \leq 4exp\Big(-\frac{\delta^2 q}{8v^2(q)}\Big) \\ + 22\Big(1 + \frac{4Dh^{-1}}{\delta}\Big)^{\frac{1}{2}} q\alpha_X([p]\Delta t_i), \end{aligned} \tag{5.10}$$

where

$$v^2(q) = \frac{2}{p^2} s(q) + \frac{Dh^{-1}\delta}{2}$$

with $p = \frac{n}{2q}$ and

$$s(q) = \max_{0 \le j \le 2q-1} E[([jp] + 1 - jp)\gamma_{n,[jp]+1}(x,t) + \gamma_{n,[jp]+2}(x,t)$$
$$+ \ldots + \gamma_{n,[(j+1)p]}(x,t) + ((j+1)p - [(j+1)p])\gamma_{n,[(j+1)p]+1}(x,t)]^2.$$

We set $\gamma_{n,n+1}(x,t) = 0$ for the $s(q)$. By using Cauchy-Schwarz Inequality and stationarity of $\gamma_{n,i}(x,t)$, it is easy to find that $s(q) = O(p^2h^{-1})$. Then let $q = [\frac{\sqrt{n\Delta}}{\sqrt{h}}]$ and $p = \frac{n}{2q} = O(\frac{\sqrt{nh}}{\sqrt{\Delta}})$, we have

$$\frac{\delta^2 q}{8v^2(q)} = \delta^2 O(qh) = O(\delta^2\sqrt{n\Delta h}). \tag{5.11}$$

By the GSM property of X_t and some basic calculations, we find

$$22\Big(1 + \frac{4Dh^{-1}}{\delta}\Big)^{\frac{1}{2}} q\alpha_X([p]\Delta) \le C(\delta)exp(-O(\sqrt{n\Delta h})). \tag{5.12}$$

This, together with (5.10), and(5.11), we have

$$P\Big(\frac{1}{n}\Big|\sum_{i=1}^{n}\gamma_{n,i}(x,t)\Big|\Big) \le C(\delta)exp(-O(\delta^2\sqrt{n\Delta h})). \tag{5.13}$$

By the above proof, we have

$$\lim_{n\to\infty} P(|\hat{f}_n(x,t) - f(x,t)| \ge v) = 0.$$

Therefore $\hat{f}_n(x,t) \to_P f(x,t) \;\; as \; n \to \infty$. □

Lemma 5.2.4.

$$\hat{g}_n(x,t) \to_P f(x,t)\mu(x,t) \quad as \quad n \to \infty. \tag{5.14}$$

Proof. Since

$$Y_i = X_{t_{i+1}} - X_{t_i} = \int_{t_i}^{t_{i+1}} \mu(X_s,s)ds + \int_{t_i}^{t_{i+1}} \sigma(X_s,s)dB_s$$
$$= \mu(X_{t_i},t_i)\Delta + \int_{t_i}^{t_{i+1}} (\mu(X_s,s) - \mu(X_{t_i},t_i))ds + \int_{t_i}^{t_{i+1}} \sigma(X_s,s)dB_s.$$

This together with (5.6), we have

$$\begin{aligned}\hat{g}_n(x,t) &= \frac{1}{n}\sum_{i=0}^{n-1}\mu(X_{t_i},t_i)K_h(X_{t_i}-x)\\ &+ \frac{1}{n\Delta}\sum_{i=0}^{n-1}K_h(X_{t_i}-x)\int_{t_i}^{t_{i+1}}(\mu(X_s,s)-\mu(X_{t_i},t_i))ds\\ &+ \frac{1}{n\Delta}\sum_{i=0}^{n-1}K_h(X_{t_i}-x)\int_{t_i}^{t_{i+1}}\sigma(X_s,s)dB_s\\ &:= g_n^{(1)}(x,t)+g_n^{(2)}(x,t)+g_n^{(3)}(x,t).\end{aligned} \tag{5.15}$$

For $g_n^{(1)}(x,t)$, we have

$$\begin{aligned}g_n^{(1)}(x,t) &= \mu(x,t)\frac{1}{n}\sum_{i=0}^{n-1}K_h(X_{t_i}-x)+\frac{1}{n}\sum_{i=0}^{n-1}(\mu(X_{t_i},t_i)-\mu(x,t))K_h(X_{t_i}-x)\\ &=: g_n^{(1)}(1)(x,t)+g_n^{(1)}(2)(x,t).\end{aligned} \tag{5.16}$$

For $g_n^{(1)}(1)(x,t)$, by Lemma 5.2.3, it is clear that $g_n^{(1)}(1)(x,t) \to_P f(x,t)\mu(x,t)$ as $n \to \infty$. For $g_n^{(1)}(2)(x,t)$, by condition (1*), we have

$$\begin{aligned}|g_n^1(2)(x,t)| &\le \frac{1}{n}\sum_{i=0}^{n-1}L|X_{t_i}-x|K_h(X_{t_i}-x)\\ &\le \frac{1}{n}\sum_{i=0}^{n-1}L(|X_{t_i}-x|K_h(X_{t_i}-x)-E[|X_{t_i}-x|K_h(X_{t_i}-x)])\\ &+ LE[|X_0-x|K_h(X_0-x)]\end{aligned} \tag{5.17}$$

By the proof of Lemma 5.2.3, we can prove that

$$\frac{1}{n}\sum_{i=0}^{n-1}L(|X_{t_i}-x|K_h(X_{t_i}-x)-E[|X_{t_i}-x|K_h(X_{t_i}-x)]) \to_P 0 \quad as \quad n\to\infty. \tag{5.18}$$

□

By the continuity of $f(x,t)$ and Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned}\lim_{h\to 0}\frac{E[|X_0-x|K_h(X_0-x)]}{h} &= \lim_{h\to 0}\frac{1}{h}\int_{-\infty}^{\infty}|y-x|K_h(y-x)f(y)dy\\ &= \lim_{h\to 0}\int_{-\infty}^{\infty}|u|K(u)f(x+uh)du\\ &= f(x)\int_{-\infty}^{\infty}|u|K(u)du.\end{aligned} \tag{5.19}$$

By (5.17), (5.18) and (5.19), we gain $g_n^{(1)}(2)(x,t) \to_P 0$ as $n \to 0$. Then we obtain $g_n^{(1)}(x,t) \to_P f(x,t)\mu(x,t)$, as $n \to 0$.

For $g_n^{(2)}(x,t)$, by condition (1*), we obtain

$$\begin{aligned} |g_n^{(2)}(x,t)| &\leq \frac{1}{n\Delta}\sum_{i=0}^{n-1} K_h(X_{t_i}-x)\int_{t_i}^{t_{i+1}} |\mu(X_s,s)-\mu(X_{t_i},t_i)|ds \\ &\leq \frac{1}{n\Delta}\sum_{i=0}^{n-1} LK_h(X_{t_i}-x)\int_{t_i}^{t_{i+1}} |X_s - X_{t_i}|ds \\ &\leq \frac{1}{n}\sum_{i=0}^{n-1} LK_h(X_{t_i}-x)\sup_{t_i\leq t\leq t_{i+1}} |X_t - X_{t_i}| \end{aligned} \tag{5.20}$$

Since

$$X_t - X_{t_i} = \int_{t_i}^{t} \mu(X_s,s)ds + \int_{t_i}^{t} \sigma(X_s,s)dB_s.$$

By condition (1*), we get

$$\begin{aligned} |X_t - X_{t_i}| &= \int_{t_i}^{t} |\mu(X_s,s)|ds + \Big|\int_{t_i}^{t} \sigma(X_s,s)dB_s\Big| \\ &\leq \int_{t_i}^{t} (|\mu(X_s,s)-\mu(X_{t_i},t_i)| + |\mu(X_{t_i},t_i)|)ds + \Big|\int_{t_i}^{t} \sigma(X_s,s)dB_s\Big| \\ &\leq |\mu(X_{t_i},t_i)|\Delta + \sup_{t_i\leq t\leq t_{i+1}}\Big|\int_{t_i}^{t} \sigma(X_s,s)dB_s\Big| + L\int_{t_i}^{t} |X_s - X_{t_i}|ds \end{aligned}$$

According to Gronwall's Inequality, we have

$$|X_t - X_{t_i}| \leq \Big(|\mu(X_{t_i},t_i)|\Delta + \sup_{t_i\leq t\leq t_{i+1}}\Big|\int_{t_i}^{t} \sigma(X_s,s)dB_s\Big)e^{L(t-t_i)}.$$

Then we get

$$\sup_{t_i\leq t\leq t_{i+1}} |X_t - X_{t_i}| \leq e^{L\Delta}\Big(|\mu(X_{t_i},t_i)|\Delta + \sup_{t_i\leq t\leq t_{i+1}}\Big|\int_{t_i}^{t} \sigma(X_s,s)dB_s\Big|\Big). \tag{5.21}$$

By (5.20) and (5.21), we obtain

$$\begin{aligned} |g_n^{(2)}(x,t)| &\leq \frac{1}{n}\sum_{i=0}^{n-1} LK_h(X_{t_i}-x)e^{L\Delta}\Big(|\mu(X_{t_i},t_i)|\Delta + \sup_{t_i\leq t\leq t_{i+1}}\Big|\int_{t_i}^{t} \sigma(X_s,s)dB_s\Big|\Big) \\ &\leq \Delta e^{L\Delta}\frac{1}{n}\sum_{i=0}^{n-1} LK_h(X_{t_i}-x)|\mu(X_{t_i},t_i)| \\ &\quad + e^{L\Delta}\frac{1}{n}\sum_{i=0}^{n-1} LK_h(X_{t_i}-x)\sup_{t_i\leq t\leq t_{i+1}}\Big|\int_{t_i}^{t} \sigma(X_s,s)dB_s\Big| \\ &:= g_n^{(2)}(1)(x,t) + g_n^{(2)}(2)(x,t). \end{aligned} \tag{5.22}$$

For $g_n^{(2)}(1)(x,t)$, it is clear that

$$\frac{1}{n}\sum_{i=0}^{n-1} K_h(X_{t_i}-x)|\mu(X_{t_i},t_i)| \to_P |\mu(x,t)|f(x,t) \quad as \quad n\to\infty.$$

This implies that $g_n^{(2)}(1)(x,t) \to_P 0$ as $n\to\infty$.

For $g_n^{(2)}(2)(x,t)$, by Markov Inequality and Lemma 2.3.1, we have

$$\begin{aligned}
&P\Big(\frac{1}{n}\sum_{i=0}^{n-1} LK_h(X_{t_i}-x)\sup_{t_i\le t\le t_{i+1}}\Big|\int_{t_i}^{t}\sigma(X_s,s)dB_s\Big| > \zeta\Big)\\
\le&\frac{1}{n\zeta}\sum_{i=0}^{n-1} LE\Big[\sup_{t_i\le t\le t_{i+1}}\Big|\int_{t_i}^{t} K_h(X_{t_i}-x)\sigma(X_s,s)dB_s\Big|\Big]\\
\le&\frac{4\sqrt{2}}{n\zeta}\sum_{i=0}^{n-1} LE\Big[\Big|\int_{t_i}^{t} K_h^2(X_{t_i}-x)\sigma^2(X_s,s)ds\Big|^{\frac{1}{2}}\Big]\\
\le&\frac{4\sqrt{2}}{n\zeta}\sum_{i=0}^{n-1} LE[K_h(X_{t_i}-x)\sigma_1\Delta^{\frac{1}{2}}]
\end{aligned} \tag{5.23}$$

So it is clear that $g_n^{(2)}(2)(x,t)\to_P 0$ as $n\to\infty$. And we get $g_n^{(2)}(x,t)\to_P 0$ as $n\to\infty$.

Now for $g_n^{(3)}(x,t)$, we define an adapted process $\varphi_n(x,t)$ by

$$\varphi_n(x,t) = \sum_{i=0}^{n-1}\frac{1}{h^{\frac{1}{2}}}K\Big(\frac{X_{t_i}-x}{h}\Big)\sigma(X_t,t)\mathbb{1}_{(t_i,t_{i+1}]}(t), \tag{5.24}$$

Then we have

$$g_n^{(3)}(x,t) = \frac{1}{n\Delta h^{\frac{1}{2}}}\int_0^{t_n}\varphi_n(x,t)dB_t.$$

By Markov Inequality and Lemma 2.3.1, we get

$$\begin{aligned}
P(|g_n^{(3)}(x,t)|>\zeta) &\le \frac{1}{n\Delta h^{\frac{1}{2}}\zeta}E\Big|\int_0^{t_n}\varphi_n(x,t)dB_t\Big|\\
&\le \frac{4\sqrt{2}}{n\Delta h^{\frac{1}{2}}\zeta}E\Big[\Big(\int_0^{t_n}|\varphi_n(x,t)|^2dt\Big)^{\frac{1}{2}}\Big]\\
&\le \frac{4\sqrt{2}}{n\Delta h^{\frac{1}{2}}\zeta}\Big(E\Big[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sigma^2(X_t,t)dt\Big]\Big)^{\frac{1}{2}}\\
&\le \frac{4\sqrt{2}}{n\Delta h^{\frac{1}{2}}\zeta}\Big(n\Delta\sigma_1^2E\Big[\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\Big]\Big)^{\frac{1}{2}}
\end{aligned} \tag{5.25}$$

which goes to zero under condition (5*).

5.3 Asymptotic of The Nonparametric Drift Estimator

We will impose some new conditions for this section:

(8*) $\mu(.,.)$ is $C^{2,1}$-function with bounded first and second order derivatives.

(9*) The density function $f(x,t)$ of the stationary distribution ψ is continuously differentiable.

Since the set of new conditions is stronger than the set of condition in Section 5.2, all the results in Section 5.2 are valid under new conditions.

Theorem 5.3.1. *Let*

$$\Pi(x,t) = \frac{f(x,t)^{\frac{1}{2}}}{\sigma(x,t)\Big(\int_{-\infty}^{\infty} K^2(u)du\Big)^{\frac{1}{2}}}$$

and

$$\Xi_\mu(x,t) = \Big[\mu'(x,t)\frac{f'(x,t)}{f(x,t)} + \frac{1}{2}\mu''(x,t)\Big]\int_{-\infty}^{\infty} u^2 K(u)du.$$

Assume that $f(x,t) > 0$, under conditions (1)-(6*) and (8*)-(9*), we have*

(1) If $(n\Delta h)^{\frac{1}{2}}h^2 = o(1)$ and $(n\Delta h)^{\frac{1}{2}}\Delta^{\frac{1}{\kappa}} = O(1)$ for some $\kappa > 2$, then

$$(n\Delta h)^{\frac{1}{2}}\Pi(x,t)(\hat{\mu}_n(x,t) - \mu(x,t)) \Rightarrow N(0,1). \tag{5.26}$$

(2) If $(n\Delta h)^{\frac{1}{2}}h^2 = O(1)$ and $(n\Delta h)^{\frac{1}{2}}\Delta^{\frac{1}{\kappa}} = O(1)$ for some $\kappa > 2$, then

$$(n\Delta h)^{\frac{1}{2}}\Pi(x,t)(\hat{\mu}_n(x,t) - \mu(x,t) - h^2\Xi_\mu(x,t)) \Rightarrow N(0,1). \tag{5.27}$$

Before we give a proof of Theorem 5.3.1, we should consider some conditions on the bandwidth h and the time frequency Δ. First, we consider the (1) of Theorem 5.3.1, when $0 < \delta < 4$, with the condition (5*), to ensure satisfying $(n\Delta h)^{\frac{1}{2}}h^2 = o(1)$ and condition (5*), we should have $h = (n\Delta\log(n\Delta))^{-\frac{1}{5}}$ or $h = (n\Delta)^{-\frac{1+\delta}{5}}$ for any $0 < \delta < 4$. After the calculation, we find if $\Delta = O\Big((\log n)\beta n^{-\frac{2\kappa}{2\kappa+5}}\Big)$, where

$$\beta = \frac{\kappa}{4\kappa + 10}$$

or $\Delta = O(n^{-\gamma})$ with

$$\gamma = \frac{40\kappa - 10\delta\kappa}{40\kappa + 100 - 10\delta\kappa},$$

then the condition $(n\Delta h)^{\frac{1}{2}}\Delta^{\frac{1}{\kappa}} = O(1)$ is satisfied. If $\delta = 0$, we get the condition of h and Δ for (2) of Theorem of 5.3.1.

Proof. Proof of Theorem 5.3.1-(1). Since

$$\begin{aligned}(n\Delta h)^{\frac{1}{2}}\Pi(x,t)(\hat{\mu}_n(x,t)-\mu(x,t)) &= \frac{(n\Delta h)^{\frac{1}{2}}\Pi(x,t)[\hat{g}_n(x,t)-\mu(x,t)\hat{f}_n(x,t)]}{\hat{f}_n(x,t)} \\ &:= \frac{\Upsilon_n(x,t)}{\hat{f}_n(x,t)}.\end{aligned} \tag{5.28}$$

By (5.15), we get

$$\begin{aligned}\Upsilon_n(x,t) &= (n\Delta h)^{\frac{1}{2}}\Pi(x,t)[g_n^{(1)}(x,t)-\mu(x,t)\hat{f}_n(x,t)] \\ &\quad + (n\Delta h)^{\frac{1}{2}}\Pi(x,t)g_n^{(2)}(x,t) \\ &\quad + (n\Delta h)^{\frac{1}{2}}\Pi(x,t)g_n^{(3)}(x,t) \\ &:= \Upsilon_n^{(1)}(x,t)+\Upsilon_n^{(2)}(x,t)+\Upsilon_n^{(3)}(x,t).\end{aligned} \tag{5.29}$$

For $\Upsilon_n^{(1)}(x,t)$

$$\Upsilon_n^{(1)}(x,t) = (n\Delta h)^{\frac{1}{2}}\Pi(x,t)\frac{1}{n}\sum_{i=0}^{n-1}(\mu(X_{t_i},t_i)-\mu(x,t))K_h(X_{t_i}-x).$$

By Taylor's Expansion, we have

$$\mu(X_{t_i},t_i)-\mu(x,t) = \mu'(x,t)(X_{t_i}-x)+\frac{1}{2}\mu''(x+\theta_i(X_{t_i}-x),t)(X_{t_i}-x)^2,$$

where θ_i is some random variable satisfying $\theta_i \in [0,1]$. So that, we have

$$\begin{aligned}\Upsilon_n^{(1)}(x,t) &= (n\Delta h)^{\frac{1}{2}}\Pi(x,t)\frac{1}{n}\mu'(x,t)\sum_{i=0}^{n-1}(X_{t_i}-x)K_h(X_{t_i}-x) \\ &\quad + (n\Delta h)^{\frac{1}{2}}\Pi(x,t)\frac{1}{2n}\mu''(x,t)\sum_{i=0}^{n-1}(X_{t_i}-x)^2K_h(X_{t_i}-x) \\ &\quad + (n\Delta h)^{\frac{1}{2}}\Pi(x,t)\frac{1}{2n}\sum_{i=0}^{n-1}[\mu''(x+\theta_i(X_{t_i}-x),t)-\mu''(x,t)](X_{t_i}-x)^2K_h(X_{t_i}-x) \\ &:= \Upsilon_n^{(1)}(1)(x,t)+\Upsilon_n^{(1)}(2)(x,t)+\Upsilon_n^{(1)}(3)(x,t).\end{aligned} \tag{5.30}$$

For $\Upsilon_n^{(1)}(1)(x,t)$, for $i = 1,2,...,n$, we set

$$\xi_{n,i}(x,t) = (n\Delta h)^{\frac{1}{2}}((X_{t_{i-1}}-x)K_h(X_{t_{i-1}}-x)-E[(X_{t_{i-1}}-x)K_h(X_{t_{i-1}}-x)]).$$

By the stationary of X_t, we have

$$\begin{aligned}\Upsilon_n^{(1)}(1)(x,t) &= \Pi(x,t)\mu'(x,t)\frac{1}{n}\sum_{i=1}^{n}\xi_{n,i}(x,t) \\ &\quad + \Pi(x,t)\mu(x,t)(n\Delta h)^{\frac{1}{2}}E[(X_0-x)K_h(X_0-x)] \\ &:= A_n^1(x,t)+A_n^2(x,t).\end{aligned}$$

For $A_n^1(x,t)$, note that

$$\sup_{1\le i\le n}|\xi_{n,i}(x,t)|\le M_0(n\Delta h)^{\frac{1}{2}} \quad a.s.$$

where $M_0<\infty$ is some positive constant. Then we apply Theorem 1.3 of Bosq [4], for each integer $q\in[1,\frac{n}{2}]$ and $\delta>0$, we have

$$\begin{aligned}P\Big(\frac{1}{n}\Big|\sum_{i=0}^{n-1}\xi_{n,i}(x,t)\Big|>\delta\Big) &\le 4\exp(-\frac{\delta^2 q}{8v^2(q)}) \\ &\quad + 22\Big(1+\frac{4M_0(n\Delta t_i h)^{\frac{1}{2}}}{\delta}\Big)^{\frac{1}{2}}q\alpha_X([p]\Delta),\end{aligned}$$

where

$$v^2(q)=\frac{2}{p^2}s(q)+\frac{M_0(n\Delta h)^{\frac{1}{2}}\delta}{2}$$

with $p=\frac{n}{2q}$ and

$$\begin{aligned}s(q) &= \max_{0\le j\le 2q-1}E[([jp]+1-jp)\xi_{n,[jp]+1}(x,t)+\xi_{n,[jp]+2}(x,t) \\ &\quad +\ldots+\xi_{n,[(j+1)p]}(x,t)+((j+1)p-[(j+1)p])\xi_{n,[(j+1)p]+1}(x,t)]^2.\end{aligned}$$

By Billingsley's Inequality (see Corollary 1.1 of Bosq [4]) and stationary of $\xi_{n,i}(x,t)$, we find that

$$s(q)=O(pnh).$$

Under the GSM condition on X_t, we have

$$\sum_{k=0}^{[p]}\alpha_X(k\Delta)=O(\Delta t_i^{-1}).$$

Then, we get

$$\frac{\delta^2 q}{8v^2(q)}=\frac{\delta^2 n}{O(nh)+O(\delta p(n\Delta h)^{\frac{1}{2}})}$$

which goes to ∞ by choosing $q = [\frac{\sqrt{n\Delta}}{\sqrt{h}}]$ and $p = \frac{n}{2q} = O(\frac{\sqrt{nh}}{\sqrt{\Delta}})$. By GSM property of X_t, we have

$$22\Big(1 + \frac{4M_0(n\Delta h)^{\frac{1}{2}}}{\delta}\Big) q\alpha_X([p]\Delta) \to 0.$$

Then we get

$$\frac{1}{n}\sum_{i=1}^{n} \xi_{n,i}(x,t) = o_p(1).$$

For $A_n^2(x,t)$, since

$$\begin{aligned} E[(X_0 - x)K_h(X_0 - x)] &= \int_{-\infty}^{\infty} \frac{y-x}{h} K\Big(\frac{y-x}{h}\Big) f(y,s)dy \\ &= \int_{-\infty}^{\infty} uK(u)f(x+uh,s)duh \\ &= hf(x,t)\int_{-\infty}^{\infty} uK(u)du + h^2\int_{-\infty}^{\infty} K(u)u^2 f'(x+\theta uh,s)du \\ &= f'(x,t)\int_{-\infty}^{\infty} u^2K(u)duh^2(1+o(1)). \end{aligned}$$

Then we have

$$A_n^2(x,t) = \Pi(x,t)\mu'(x,t)f'(x,t)\int_{-\infty}^{\infty} u^2K(u)du(n\Delta h)^{\frac{1}{2}}h^2(1+o(1)).$$

By the proof of $A_n^1(x,t)$ and $A_n^2(x,t)$, we get

$$\Upsilon_n^{(1)}(1)(x,t) = o_P(1) + \Pi(x,t)\mu'(x,t)f'(x,t)\int_{-\infty}^{\infty} u^2K(u)du(n\Delta h)^{\frac{1}{2}}h^2(1+o(1)).$$

Then, for $\Upsilon_n^{(1)}(2)(x,t)$, we set

$$\zeta_{n,i+1}(x,t) = (n\Delta h)^{\frac{1}{2}}((X_{t_i} - x)^2K_h(X_{t_i} - x) - E[(X_{t_i} - x)^2K_h(X_{t_i} - x)]),$$

where $i = 0,...,n-1$. By the stationary of X_t, we have

$$\begin{aligned} \Upsilon_n^{(1)}(2)(x,t) &= \frac{1}{2}\Pi(x,t)\mu''(x,t)\frac{1}{n}\sum_{i=0}^{n-1}\zeta_{n,i+1}(x,t) \\ &\quad + \frac{1}{2}\Pi(x,t)\mu'(x,t)(n\Delta h)^{\frac{1}{2}}E[(X_0 - x)^2K_h(X_0 - x)] \\ &:= A_n^3(x,t) + A_n^4(x,t). \end{aligned}$$

Note that

$$\sup_{1\le i\le n}|\zeta_{n,i}(x,t)| \le M_1(n\Delta h)^{\frac{1}{2}}h$$

for some positive constant $M_1 < \infty$. Then we apply Theorem 1.3 of Bosq [4], for each integer $q \in [1, \frac{n}{2}]$ and $\delta > 0$

$$P\Big(\frac{1}{n}\Big|\sum_{i=0}^{n-1}\zeta_{n,i}(x,t)\Big| > \delta\Big) \le 4\exp(-\frac{\delta^2 q}{8\tilde{v}^2(q)})$$
$$+ 22\Big(1 + \frac{4M_1(n\Delta t_i h)^{\frac{1}{2}}h}{\delta}\Big)^{\frac{1}{2}} q\alpha_X([p]\Delta t_i),$$

where

$$\tilde{v}^2(q) = \frac{2}{p^2}\tilde{s}(q) + \frac{M_1(n\Delta h)^{\frac{1}{2}}\delta h}{2}$$

with $p = \frac{n}{2q}$ and

$$\tilde{s}(q) = \max_{0\le j\le 2q-1} E[([jp] + 1 - jp)\zeta_{n,[jp]+1}(x,t) + \zeta_{n,[jp]+2}(x,t)$$
$$+ \dots + \zeta_{n,[(j+1)p]}(x,t) + ((j+1)p - [(j+1)p])\zeta_{n,[(j+1)p]+1}(x,t)]^2.$$

By Billingsley's Inequality and stationary of $\xi_{n,i}(x,t)$, we find that

$$\tilde{s}(q) = O(pnh^3).$$

Then, we get

$$\frac{\delta^2 q}{8v^2(q)} = \frac{\delta^2 n}{O(nh^3) + O(\delta p(n\Delta h)^{\frac{1}{2}}h)}$$

which goes to ∞ by choosing $q = [\frac{\sqrt{n\Delta}}{\sqrt{h}}]$ and $p = \frac{n}{2q} = O(\frac{\sqrt{nh}}{\sqrt{\Delta}})$. By GSM property of X_t, we have

$$22\Big(1 + \frac{4M_1(n\Delta h)^{\frac{1}{2}}h}{\delta}\Big)q\alpha_X([p]\Delta) \to 0.$$

Therefore, we obtain $A_n^3(x,t) = o_P(1)$.

For $A_n^4(x,t)$, we have

$$E[(X_0 - x)^2K_h(X_0 - x)] = f(t,x)\int_{-\infty}^{\infty} u^2K(u)du \cdot h^2(1 + o(1)).$$

Then, we have

$$A_n^4(x,t) = \frac{1}{2}\Pi(x,t)\mu'(x,t)f(x,t)\int_{-\infty}^{\infty} u^2K(u)du \cdot (n\Delta h)^{\frac{1}{2}}h^2(1 + o(1)).$$

So that we obtain

$$\Upsilon_n^{(1)}(2)(x,t) = o_P(1) + \frac{1}{2}\Pi(x,t)\mu''(x,t)f(x,t)\int_{-\infty}^{\infty} u^2K(u)du \cdot (n\Delta h)^{\frac{1}{2}}h^2(1 + o(1)).$$

For $\Upsilon_n^{(1)}(3)(x,t)$, by the uniform continuity of $\mu''(\cdot)$ and the bounded support of the kernel function $K(\cdot)$, we assume that $K(x) = 0$ if $|x| > M$ for some finite positive number M. Then we have

$$|\Upsilon_n^{(1)}(3)(x,t)| \le \frac{1}{2}\Pi(x,t) \sup_{|x-y|\le Mh} |\mu''(x,t) - \mu''(y,t)|(n\Delta h)^{\frac{1}{2}} \frac{1}{n}\sum_{i=0}^{n-1}(X_{t_i} - x)K_h(X_{t_i} - x)$$
$$= o(1) \cdot (n\Delta h)^{\frac{1}{2}} h^2 \frac{1}{n}\sum_{i=0}^{n-1} K_h(X_{t_i} - x)$$
$$= o_P(1) \cdot (n\Delta h)^{\frac{1}{2}} h^2$$

since $\frac{1}{n}\sum\limits_{i=0}^{n-1} K_h(X_{t_i} - x) \to f(x)$ in probability. Then we get

$$\Upsilon_n^{(1)}(3)(x,t) = o_P(1) \cdot O((n\Delta h)^{\frac{1}{2}} h^2).$$

So by (5.30), we get

$$\Upsilon_n^{(1)}(x,t) = o_P(1) + o_P(1) \cdot O((n\Delta h)^{\frac{1}{2}} h^2)$$
$$+ \Pi(x,t)[\mu'(x,t)f'(x,t) + \frac{1}{2}\mu''(x,t)f(x,t)] \int_{-\infty}^{\infty} u^2 K(u)du \cdot (n\Delta h)^{\frac{1}{2}} h^2 (1 + o(1)). \tag{5.31}$$

For $\Upsilon_n^{(2)}(x,t)$, we have

$$\Upsilon_n^{(2)}(x,t) = (n\Delta h)^{\frac{1}{2}} \Pi(x,t) g_n^{(2)}(x,t)$$
$$\le (n\Delta h)^{\frac{1}{2}} \Pi(x,t) \Big[L\Delta e^{L\Delta} \cdot \frac{1}{n}\sum_{i=0}^{n-1} K_h(X_{t_i} - x)|\mu(X_{t_i}, t_i)|$$
$$+ Ke^{K\Delta}\Delta^{\frac{1}{\kappa}} \cdot \frac{1}{n\Delta^{\frac{1}{\kappa}}} \sum_{i=0}^{n-1} K_h(X_{t_i} - x) \sup_{t_i \le t \le t_{i+1}} \Big| \int_{t_i}^{t} \sigma(X_s, t_s)dB_s \Big| \Big].$$

We use the same method in the proof of $\Upsilon_n^{(1)}(x,t)$, under the given conditions, we get

$$\frac{1}{n}\sum_{i=0}^{n-1} K_h(X_{t_i} - x)|\mu(X_{t_i}, t_i)| \to_P |\mu(x,t)| f(x,t).$$

By the proof of $g_n^{(2)}(x,t)$ and Lemma 2.3.1, we can get

$$P\Big(\frac{1}{n\Delta^{\frac{1}{\kappa}}} \sum_{i=0}^{n-1} K_h(X_{t_i} - x) \sup_{t_i \le t \le t_{i+1}} \Big| \int_{t_i}^{t} \sigma(X_s, s)dB_s \Big| > \delta \Big) \le O(\Delta^{\frac{1}{2} - \frac{1}{\kappa}}).$$

Then we obtain that

$$\Upsilon_n^{(2)}(x,t) = O_P(1)\cdot(n\Delta h)^{\frac{1}{2}}\Delta + o_P(1)\cdot(n\Delta h)^{\frac{1}{2}}\Delta^{\frac{1}{\kappa}}. \tag{5.32}$$

For $\Upsilon_n^{(3)}(x,t)$, we define

$$\chi_{t_n} = \Big(t_n\sigma^2(x,t)f(x,t)\int_{-\infty}^{\infty}K^2(u)du\Big)^{-\frac{1}{2}}$$

and recall that

$$\varphi_n(x,t) = \sum_{i=0}^{n-1}\frac{1}{h^{\frac{1}{2}}}K\Big(\frac{X_{t_i}-x}{h}\Big)\sigma(X_t,t)\mathbb{1}_{(t_i,t_{i+1}]}(t).$$

Then we have

$$\begin{aligned}
\chi_{t_n}^2\int_0^{t_n}\varphi_n^2(x,t)dt &= \chi_{t_n}^2\cdot\int_0^{t_n}\sum_{i=0}^{n-1}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sigma^2(X_t,t)\mathbb{1}_{(t_i,t_{i+1}]}(t)dt\\
&= \chi_{t_n}^2\sum_{i=0}^{n-1}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\int_{t_i}^{t_{i+1}}\sigma^2(X_t,t)dt\\
&= \chi_{t_n}^2\sum_{i=0}^{n-1}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sigma^2(X_{t_i},t_i)\Delta\\
&\quad+\chi_{t_n}^2\sum_{i=0}^{n-1}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\int_{t_i}^{t_{i+1}}(\sigma^2(X_t,t)-\sigma^2(X_{t_i},t_i))dt\\
&=: B + D.
\end{aligned}\tag{5.33}$$

For B, by Lemma 5.2.3, we can prove that

$$\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sigma^2(X_{t_i},t_i)\to_P\sigma^2(x,t)f(x,t)\int_{-\infty}^{\infty}K^2(u)du.$$

Then we have

$$B = \frac{1}{\sigma^2(x,t)f(x,t)\int_{-\infty}^{\infty}K^2(u)du}\cdot\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sigma^2(X_{t_i},t_i)\to_P 1. \tag{5.34}$$

For D, by condition (1*), (5.21) and inequality $||u+v|^q-|v|^q|\le|u|^q$ for $u,v\in\mathbb{R}$ and

$q \in (0,1]$, we have

$$\begin{aligned}
|D| &= \chi_{t_n}^2 \Big| \sum_{i=0}^{n-1} \frac{1}{h} K^2\Big(\frac{X_{t_i}-x}{h}\Big) \int_{t_i}^{t_{i+1}} (\sigma^2(X_t,t) - \sigma^2(X_{t_i},t_i))dt \Big| \\
&\le \chi_{t_n}^2 \sum_{i=0}^{n-1} \frac{1}{h} K^2\Big(\frac{X_{t_i}-x}{h}\Big) \int_{t_i}^{t_{i+1}} |\sigma^2(X_t,t) - \sigma^2(X_{t_i},t_i)|dt \\
&\le \chi_{t_n}^2 \sum_{i=0}^{n-1} \frac{1}{h} K^2\Big(\frac{X_{t_i}-x}{h}\Big) \int_{t_i}^{t_{i+1}} |\sigma(X_t,t) - \sigma(X_{t_i},t_i)|^2 dt \\
&\le \chi_{t_n}^2 \sum_{i=0}^{n-1} \frac{1}{h} K^2\Big(\frac{X_{t_i}-x}{h}\Big) \mathrm{L}^2 \Delta \sup_{t_i \le t \le t_{i+1}} |X_t - X_{t_i}|^2 \\
&\le \frac{L^2 e^{2L_s\Delta}\Delta^3}{\sigma^2(x,t) f(x,t) \int_{-\infty}^{\infty} K^2(u)du} \cdot \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^2\Big(\frac{X_{t_i}-x}{h}\Big) |\mu(X_{t_i},t_i)|^2 \\
&\quad + \frac{L^2 \Delta e^{2L_s\Delta}}{\sigma^2(x,t) f(x,t) \int_{-\infty}^{\infty} K^2(u)du} \cdot \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^2\Big(\frac{X_{t_i}-x}{h}\Big) \sup_{t_i \le t \le t_{i+1}} \Big| \int_{t_i}^{t} \sigma(X_s,s)dB_s \Big|^2 \\
&:= D_1 + D_2
\end{aligned} \tag{5.35}$$

For D_1, since

$$\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{h} K^2\Big(\frac{X_{t_i}-x}{h}\Big) |\mu(X_{t_i},t_i)|^2 \to_P |\mu(x,t)|^2 f(x,t) \int_{-\infty}^{\infty} K^2(u)du \tag{5.36}$$

by the method of proof Lemma 5.2.3, it is clear $D_1 \to_P 0$.

For D_2, by Lemma 2.3.1, Markov Inequality and condition (3*), we have

$$\begin{aligned}
&P\Big(\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{h}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sup_{t_i\le t\le t_{i+1}}\Big|\int_{t_i}^{t}\sigma(X_s,s)dB_s\Big|^2>\zeta\Big)\\
\le&\frac{1}{nh\zeta}E\Big|\sum_{i=0}^{n-1}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sup_{t_i\le t\le t_{i+1}}\Big|\int_{t_i}^{t}\sigma(X_s,s)dB_s\Big|^2\Big|\\
\le&\frac{1}{nh\zeta}\sum_{i=0}^{n-1}E\Big[\Big(\sup_{t_i\le t\le t_{i+1}}\Big|\int_{t_i}^{t}K\Big(\frac{X_{t_i}-x}{h}\Big)\sigma(X_s,s)dB_s\Big|\Big)^2\Big]\\
\le&\frac{4}{nh\zeta}\sum_{i=0}^{n-1}E\Big[\int_{t_i}^{t_{i+1}}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sigma^2(X_s,s)ds\Big] \qquad (5.37)\\
\le&\frac{4}{nh\zeta}\sum_{i=0}^{n-1}E\Big[K^2\Big(\frac{X_{t_i}-x}{h}\Big)\sigma_1^2\Delta\Big]\\
=&\frac{4\sigma_1^2\Delta}{\zeta}E\Big[h^{-1}K^2\Big(\frac{X_{t_i}-x}{h}\Big)\Big]\\
=&\frac{4\sigma_1^2\Delta}{\zeta}\int_{-\infty}^{\infty}h^{-1}K^2\Big(\frac{y-x}{h}\Big)f(y)dy
\end{aligned}$$

which goes to zero as $\Delta\to 0$. Then by (5.33), (5.34), (5.35), (5.36) and (5.37), we have

$$\chi_{t_n}^2\int_0^{t_n}\varphi_n^2(x,t)dt\to_P 1. \qquad (5.38)$$

Since

$$\begin{aligned}
\Upsilon_n^{(3)}(x,t)&=(n\Delta h)^{\frac{1}{2}}\Pi(x,t)g_n^{(3)}(x,t)\\
&=(n\Delta h)^{\frac{1}{2}}\Pi(x,t)\frac{1}{n\Delta}\sum_{i=0}^{n-1}K_h(X_{t_i}-x)\int_{t_i}^{t_{i+1}}\sigma(X_s,s)dB_s\\
&=f(x,t)\cdot\chi_{t_n}\int_0^{t_n}\varphi_n(x,t)dB_t.
\end{aligned}$$

By (5.38) and Lemma 5.2.2, we have

$$\Upsilon_n^{(3)}(x,t)\Rightarrow f(x,t)N(0,1). \qquad (5.39)$$

By (5.29), we get

$$\Upsilon_n(x,t)\Rightarrow f(x,t)N(0,1),$$

this, together with (5.28), Lemma 5.2.3 and Slutsky's Theorem, we obtain

$$(n\Delta h)^{\frac{1}{2}}\Pi(x,t)(\hat{\mu}_n(x,t)-\mu(x,t))\Rightarrow N(0,1)$$

where we complete the proof of Theorem 5.3.1-(1). □

Proof. Proof of Theorem 5.3.1-(2). If $(n\Delta h)^{\frac{1}{2}}h^2 = O(1)$, with (5.31), we have

$$\Upsilon_n^{(1)}(x,t) = (n\Delta h)^{\frac{1}{2}}h^2\Pi(x,t)\Xi_\mu(x,t)f(x,t) + o_P(1).$$

By (5.32) and (5.39), we also get

$$\Upsilon_n^{(2)}(x,t) = o_P(1)$$

and

$$\Upsilon_n^{(3)}(x,t) \Rightarrow f(x,t)N(0,1).$$

Then by (5.29), we have

$$\Upsilon_n(x,t) - (n\Delta h)^{\frac{1}{2}}h^2\Pi(x,t)\Xi_\mu(x,t)f(x,t) \Rightarrow f(x,t)N(0,1).$$

By Lemma 5.2.3 and Slutsky's Theorem we have

$$\begin{aligned}
&(n\Delta h)^{\frac{1}{2}}\Pi(x,t)(\mu_n(x,t) - \mu(x,t) - h^2\Xi_\mu(x,t)) \\
=&\frac{\Upsilon_n(x,t)}{\hat{f}_n(x,t)} - (n\Delta h)^{\frac{1}{2}}h^2\Pi(x,t)\Xi_\mu(x,t) \\
=&\frac{\Upsilon_n(x,t) - (n\Delta h)^{\frac{1}{2}}h^2\Pi(x,t)\Xi_\mu(x,t)f(x,t)}{\hat{f}_n(x,t)} + (n\Delta h)^{\frac{1}{2}}h^2\Pi(x,t)\Xi_\mu(x,t)\Big(\frac{f(x,t)}{\hat{f}_n(x,t)} - 1\Big) \\
\Rightarrow& N(0,1).
\end{aligned}$$

Where we complete the proof of Theorem 5.3.1-(2). □

5.4 Example

In this section, we are aiming to give a mean-reversion type example for (5.1). Let $\mu(X_t, t) = [a(X_t,t) + br(X_t,t)]$ and $\sigma(X_t,t) = b$ in (5.1). Consider the following equation:

$$dX_t = [a(X_t,t) + br(X_t,t)]dt + bdB_t. \tag{5.40}$$

where $r(x,t) : \mathbb{R} \times [0,T] \to \mathbb{R}$ and $a(x,t) : \mathbb{R} \times [0,T] \to \mathbb{R}$ are $C^{2,1}-function$, and b is a positive constant. Then we make some consumptions: for $0 \le t \le T$, let

(1)

$$|a(x,t) - a(y,t)| \le L_1|x-y|$$

and

$$|r(x,t) - r(y,t)| \le L_2|x-y|$$

for $L_1 > 0$ and $L_2 > 0$, $x, y \in \mathbb{R}$.

(2)

$$\frac{a(x,t) - a(y,t)}{x-y} \le \alpha_1$$

and

$$\frac{r(x,t) - r(y,t)}{x-y} \le \alpha_2$$

where $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$.

Now we will prove that the equation (5.40) satisfies Lipschitz condition. We have

$$\begin{aligned}
&\Big|[a(x,t) + br(x,t)] - [a(y,t) + br(y,t)]\Big| \\
=&\Big|[a(x,t) - a(y,t)] + [b(r(x,t) - r(y,t))]\Big| \\
\le&\Big|a(x,t) - a(y,t)\Big| + |b|\Big|r(x,t) - r(y,t)\Big| \\
\le&L_1(t)|x-y| + |b|L_2(t)|x-y| \\
=&(L_1(t) + |b|L_2(t))|x-y|
\end{aligned}$$

which satisfies Lipschitz condition. Then, we will verify the dissipative condition. We get

$$\begin{aligned}
&\frac{[a(x,t) + br(x,t)] - [a(y,t) + br(y,t)]}{x-y} \\
=&\frac{[a(x,t) - a(y,t)]}{x-y} + \frac{b[r(x,t) - r(y,t)]}{x-y} \\
\le&\alpha_1(t) + b\alpha_2(t).
\end{aligned}$$

Since $[\alpha_1(t) + b\alpha_2(t)] < 0$, it satisfies the dissipative condition. So that the equation (5.40) meets the overall conditions mentioned in previous parts and obeys our main results in Section 5.2 and Section 5.3. That is, if we let $\hat{r}_n(x,t)$ express the estimator of $r(x,t)$, under the conditions and notations in previous sections in this chapter, we formulate $\hat{r}_n(x,t) \rightarrow_P r(x,t)$ as $n \rightarrow 0$ and

(1) If $(n\Delta h)^{\frac{1}{2}} h^2 = o(1)$ and $(n\Delta h)^{\frac{1}{2}} \Delta^{\frac{1}{\kappa}} = O(1)$ for some $\kappa > 2$, then

$$(n\Delta h)^{\frac{1}{2}} \Pi(x,t)(\hat{r}_n(x,t) - r(x,t)) \Rightarrow N(0,1).$$

(2) If $(n\Delta h)^{\frac{1}{2}} h^2 = O(1)$ and $(n\Delta h)^{\frac{1}{2}} \Delta^{\frac{1}{\kappa}} = O(1)$ for some $\kappa > 2$, then

$$(n\Delta h)^{\frac{1}{2}} \Pi(x,t)(\hat{r}_n(x,t) - r(x,t) - h^2 \Xi_r(x,t)) \Rightarrow N(0,1).$$

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