The massless currentless Dirac equation in 2 + 2 dimensions.

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THE MASSLESS CURRENTLESS DIRAC EQUATION IN $2+2$ DIMENSIONS

Thesis submitted to the University of Wales Swansea
by

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in candidature for the degree of
Doctor of Philosophy

2002

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ACKNOWLEDGEMENTS

First and foremost, I would like to express my special gratitude and appreciation to my doctoral advisor, Dr Edwin J. Beggs, whose guidance and encouragement has been invaluable to me, and his unusual patience throughout the preparation of this thesis. Particularly, I would like to express my profound gratitude to him for teaching me many things; frankly I found it impossible to thank him enough for all his help, and this thesis could not see the light without his many contributions and encouragement.

All the subjects in this thesis have been taught to me by Dr Edwin J. Beggs, and he is the one who suggested the area of this research, I am very grateful for him indeed and I hope our link will continue.

My gratitude also to Professor A. Truman, Head of the Mathematics Department, and the lecturers and secretarial staff for their cooperation and friendly atmosphere during my stay, in particular Dr I. M. Davies, for his assistance and advice in using the computers.

I would like to take this opportunity to thank the Kingdom of Saudi Arabia government, and the Ministry of Higher Education, and my employer King Abdul Aziz University for their financial support and encouragement. I also thank the Saudi Arabian cultural Bureau in London for their help and support.

I thank the many people who have helped me in one way or another. In particular I thank my colleagues at the Department and my friends in Swansea and back home.

I am deeply indebted to my parents, brother and sisters, and father and mother in law, for their constant encouragement and support throughout my studies.

Finally and by no means least, I equally indebted to my dear wife, and my son for their constant encouragement and support which I cannot describe in words.

SUMMARY

This thesis is concerned with the massless Dirac equation in (2+2)-dimensional space-time. We start by presenting background material on spinors and the Dirac equation. Then we consider the modifications to the current which are necessary in $2+2$ dimensions, and spinors which imply zero current density. Then we consider plane wave solutions and superposition of solutions.
The action of the Lorentz group is considered, and we find solutions which are invariant to subgroup preserving the world line of a 'particle' solution. Then we find some more general solutions, together with the corresponding electromagnetic fields.

NOTATION.

In this thesis references are denoted by square brackets [ ] and equations by round brackets ( ) where (a.b) denotes Equation b in Chapter a.
This thesis has been typeset using L^AT_EX except the lightlike line solution (See the Appendix) and the graphs have been done using " Mathematica " program version 4.

Contents

Chapter 0

Preliminaries

0.1 Introduction

This thesis is concerned with the Dirac massless equation in (2+2)-dimensional space-time. The Dirac equation in $(3+1)$-dimensions was introduced by P.A.M. Dirac in 1928 to fix some problems with the Klein-Gordon equation. In $(n+1)$-dimensional space-time, the first difficulty faced by the Klein-Gordon equation was the problem of probability density. Dirac succeeded in producing a positive definite probability density ($j^0 = \psi^* \psi > 0$ for a non zero state ψ). The other difficulty faced by the Klein-Gordon equation was that of negative energy states. Dirac's solution to this problem relies on the fact that electrons have spin-$\frac{1}{2}$ and therefore obey Pauli's exclusion principle (identical particles cannot be in the same quantum state). For a more detailed discussion, see [13] and [21]. Dirac supposed that the negative energy states are already completely filled, and the exclusion principle prevents any more electrons being able to enter the 'sea' of negative states see [8].
This 'Dirac sea' is the vacuum, so in Dirac's theory, the vacuum is not 'nothing', but an infinite sea of negative energy electrons, protons, neutrinos and all other spin-$\frac{1}{2}$ particles! Now this ingenious theory makes an important prediction, for suppose there occurs one vacancy in the electron sea a 'hole' with energy $|E|$. An electron with energy E may fill this hole, emitting energy $2E$, and leaving the vacuum: $e^- + \text{hole} \longrightarrow \text{energy}$, so the 'hole' effectively has charge $+e$ and positive energy, and is called positron, the antiparticle of the electron. This theory of Dirac's in 1928 predicted the existence of antiparticle for all spin-$\frac{1}{2}$ particles. We conclude this account of antiparticles by

noting that, despite the successful resolution of problem of negative energies, the Dirac equation is no longer a single-particle equation, since it describes both particles and antiparticles. The only consistent philosophy is to regard the spinor ψ as a field, such that $|\psi|^2$ gives a measure of the number of particles at a particular point. This field is naturally a quantum field. For a more detailed discussion, see [8], [17], [18] and [19].

In 1994, Edwin J. Beggs considered the possibility of having non-trivial electromagnetic field with currentless spinors in $(4+2)$-dimensions. His purpose was to construct some exact solutions to the equations of quantum electrodynamics, treated as a classical field theory. In [2] there are currentless solutions of the Dirac equation in $(4+2)$-dimensions which have a non-trivial electromagnetic field. The field arises from a topological singularity in the solutions, centered on the supposed 'world line' of a particle.

In this thesis, the first question we put to ourselves is whether we can find more solutions, and try to work towards a general solution by studying a simpler problem in $(2+2)$-dimensions. Secondly we want to try to find traveling wave solutions, and solutions with the symmetry of a particle. Fortunately we did find lots of solutions, some of them having non-trivial electromagnetic fields with currentless spinors. The results are specific to $(2+2)$-dimensions. However it has been shown in [2] that results in $(4+2)$-dimensions are relevant to problems in $(3+1)$-dimensions. Also there are other examples of physical theories with more that one time dimension.

Spaces with several time components do appear among the solutions of some physically interesting systems in higher dimensions, and they have been studied in the compactification context of Kaluza-Klein and string theories [1].

The idea in Yang-Mills (YM) theory is to minimize the energy of the curvature of the connection, where the potentials A_μ live in a Lie algebra, for example the Lie algebra of $SU(2)$. In [10] the authors show that $N = 1$ Supersymmetric self-dual Yang-Mills theory (SDYM) in $2+2$ dimensions is an integrable system.

In [1] the authors show that, if we assume that the world is $n+1$ dimensional, and try to compactify, there are very rigid constraints on the compactification. Allowing more than one time dimension in the original world manifold allows a greater variety of compactifications, including some that they argue are physically useful.

Since the spionrs considered are currentless, the electromagnetic field is a solution of the currentless Maxwell equations in $2+2$ dimensions. However the Dirac equation can still place constraints on the electromagnetic potentials A_μ. Also the currentless Maxwell equations only apply on the domain of solution of the Dirac equation. If there is a singularity in the equation, then that singularity can behave as thought it had charge, e.g. [2]. We do not consider boundary conditions at infinity on the electromagnetic field in this thesis.

0.2 Quantum mechanics and the Schrödinger equation

The founding of Quantum Mechanics can be placed between the years of 1923 and 1927 [19]. Two equivalent formulations of quantum mechanics had been proposed almost simultaneously : Matrix Mechanics (due to Heisenberg and others) and Wave Mechanics (due to Schrödinger and others).
Wave Mechanics started between 1925 and 1926. The choice of a wave equation is restricted by a certain number of a priori conditions :

(I) The equation must be linear and homogeneous;
(II) The equation must be a differential equation of the first order with respect to time.

All these considerations will lead us to the Schrödinger equation in a very natural way.

$$i\hbar \frac{\partial}{\partial t} \psi(r,t) \ = \ -\frac{\hbar^2}{2m} \Delta \psi(r,t) \,. \tag{0.1}$$

This is the Schrödinger equation for a free particle; it satisfies conditions (I) and (II). It also satisfies the requirements of the correspondence principle. Indeed the formal analogy with Classical Mechanics is a actually realized, equation (0.1) is in a sense the quantum-mechanical translation of the classical equation which given by

$$E \ = \ \frac{P^2}{2m} \,, \tag{0.2}$$

the energy and momentum being represented in this quantum language by differential operators acting on the wave-function according to the correspondence rule

$$E \longrightarrow i\hbar \frac{\partial}{\partial t} \quad ; \quad p \longrightarrow \frac{\hbar}{i} \nabla \,. \tag{0.3}$$

The probability density for the Schrödinger equation is

$$\rho = \psi^* \psi , \tag{0.4}$$

and the probability current is

$$j = -\frac{i\hbar}{2m}(\psi^* \nabla \psi - \psi \nabla \psi^*) . \tag{0.5}$$

These obey a continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 . \tag{0.6}$$

For a single particle we would like to have probability one of the particle being somewhere, i.e.

$$\int_{\mathbb{R}^3} \psi^* \psi \, d^3x = 1 . \tag{0.7}$$

For a more detailed discussion, see [19] and [17].

0.3 Special relativity and Lorentz transformation

It is often said that special relativity is the theory of 4-dimensional spacetime. Consider two events in spacetime (x, y, z, t) and $(x+dx, y+dy, z+dz, t+dt)$. We may generalize the notion of the distance between two points in space to the'interval' between two points in spacetime; call it ds. In order that ds be the same for all (inertial) observers, it must be invariant under Lorentz transformations and rotations, and so is given by

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 . \tag{0.8}$$

With this definition, events which are separated by a timelike interval have $ds^2 > 0$; those separated by a spacelike interval $ds^2 < 0$; and those separated by a null or lightlike interval $ds^2 = 0$.

Let us introduce some convenient notation. Coordinates on spacetime will be denoted by letters with Greek superscript indices running from 0 to 3, with 0 generally denoting the time coordinate. Thus,

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) ,$$

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z)\,.$$

Furthermore, for the sake of simplicity we choose units in which $c = 1$. Empirically we know that c is the speed of light, 3×10^8 meters per second; Thus, we are working in units of seconds for time and light seconds for distance, for example.

It is also convenient to write the spacetime interval in a more compact form. We therefore introduce a 4×4 matrix, the metric, which we write using two lower indices:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Here rows and columns correspond to the 0, 1, 2 and 3 components. Since $g_{\mu\nu}$ has a non-zero determinant, its inverse exists, and is called $g^{\mu\nu}$. In fact it has the same value as $g_{\mu\nu}$ in Minkowski space but this equality does not hold in general. It is clear that $g_{\mu\nu}$ contains all the information about the geometry of the space. We then have the nice formula

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu\,.$$

Notice that we use the summation convention. Now we can consider linear coordinate transformations in spacetime. We would like the length to remain fixed,

$$\begin{aligned} ds^2 = (dx)^\top g\,(dx) &= (dx')^\top g\,(dx') \\ &= (dx)^\top \Lambda^\top g \Lambda\,(dx), \end{aligned} \tag{0.9}$$

and therefore

$$g = \Lambda^\top g \Lambda\,. \tag{0.10}$$

The matrices Λ which satisfy (0.10) are known as the Lorentz Transformations; the set of them forms a group under matrix multiplication, known as the Lorentz group. There is a close analog between this group and O(3), the rotation group in three-dimensional space.

Lorentz transformations fall into a number of categories. First there are the conventional rotations, such as a rotation in the $x - y$ plane:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The rotation angle θ is a periodic variable with period 2π. There are also boosts, which may be thought of as 'rotations between space and time directions'. An example is given by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The boost parameter ϕ, unlike the rotation angle, is defined from $-\infty$ to ∞. In general Lorentz transformations will not commute, so the Lorentz group is non-Abelian. For a more detailed discussion, see [4], [16], [17], [18] and [20].

0.4 The Klein-Gordon equation

We are now in a position to write down a wave equation for a particle with no spin, a scalar particle. Since it has no spin it has only one component, which we denote by ϕ. Note that (0.2) is the non-relativistic approximation to

$$E^2 = P^2 c^2 + m^2 c^4 . \qquad (0.11)$$

The wave equation is obtained from equation (0.11) by substituting differential operators for E and P in the standard fashion in quantum theory given by (0.3),

$$-\hbar^2 \frac{\partial^2}{\partial t^2} \phi = -\hbar^2 c^2 \Delta \phi + m^2 c^4 \phi ,$$

which may also be written for $\hbar = c = 1$ as

$$[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2] \phi = 0 . \qquad (0.12)$$

This is known as the Klein-Gordon equation. It then follows that the Schrödinger equation should be the non-relativistic approximation of the Klein-Gordon equation. What are the corresponding expressions for the probability density and the probability current for the Klein-Gordon equation? To be properly relativistic, ρ should not, as in (0.4), transform as a scalar, but the time component of 4-vector, whose space component is j, given by (0.5). Then ρ is given by

$$\rho = \frac{i\hbar}{2m}\,(\phi^* \frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\phi^*)\,,$$

with

$$\begin{aligned} j^\mu &= (\rho, J) = \frac{i\hbar}{m}\,\phi^*\,(\overleftrightarrow{\partial_0}\,, -\overleftrightarrow{\nabla}\,)\,\phi\,, \\ &= \frac{i\hbar}{m}\,\phi^*\,\overleftrightarrow{\partial^\mu}\,\phi\,, \end{aligned}$$

where the forward-backward arrow here means the derivatives act on both ϕ and ϕ^* [i.e. $\phi^*\,\overleftrightarrow{\partial^0}\,\phi = (\partial_0\,\phi^*)\,\phi + \phi^*\,(\partial_0\,\phi)$]. We have the continuity equation

$$\partial_\mu j^\mu = \frac{i\hbar}{2m}\,[\phi^*(\frac{\partial^2}{\partial t^2} - \Delta)\phi - \phi(\frac{\partial^2}{\partial t^2} - \Delta)\phi^*] = 0$$

since ϕ^* also obeys the Klein-Gordon equation. Then ρ and j are the probability density and current we want. But this immediately presents a problem, because ρ is not positive definite. Since the Klein-Gordon equation is second order, ϕ and $\partial\phi/\partial t$ can be fixed arbitrarily at a given time, so ρ may take on negative values, and its interpretation as a probability density has to be abandoned. There is another problem with the Klein-Gordon equation and that is the solution to (0.11), regarded as an equation for E,

$$E = \pm\sqrt{p^2 + m^2}$$

so a solution to the Klein-Gordon equation may contain negative energy terms as well as positive energy ones. For a more detailed discussion, see [8] and [17].

0.5 The Dirac equation in $1 + 3$ spacetime

Given the problems in the last section, it might have been better to treat space and time on a more equal footing in the Schrödinger equation. This what Dirac took as his starting point. In 1928 Dirac (see [7]) replaced the Klein-Gordon equation by a first-order equation,

$$i\,(\frac{\partial}{\partial t} + \alpha_i\,\frac{\partial}{\partial x^i}\,)\,\psi = \beta\,m\,\psi\,. \tag{0.13}$$

The question Dirac posed himself was to find the simplest choice for α_i and β, such that the square of the Schrödinger equation gives the Klein-Gordon equation

$$p_0^2 = (p.\alpha + \beta m)^2 = p^2 + m^2 .$$

Dirac noted that only when we allow α_i and β to be non-commuting objects (e.g. matrices), can one satisfy these equations. The above equation is equivalent to

$$\beta^2 = 1 \quad , \quad (\alpha_i\alpha_j + \alpha_j\alpha_i) = 2\delta_{ij} \quad \text{and} \quad \alpha_i\beta + \beta\alpha_i = 0 . \tag{0.14}$$

Historically, Dirac first considered $m \neq 0$, but the massless case ($m = 0$) is somewhat simpler. Here σ_i are the Pauli matrices, familiar from describing spin one-half particles

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{0.15}$$

For the massive case Dirac first incorrectly thought that the positive energy states described the electron and the negative energy states the proton. At that time antiparticles were unknown. Antiparticles were predicted by Dirac because the only way he could make the theory consistent was to invoke the Pauli exclusion principle and to fill all the negative energy states. A hole in this sea of positive energy states, called the Dirac sea, then corresponds to a state of positive energy. These holes describe the antiparticle with the same mass as the particle.
For the massive Dirac equation we need to find a matrix β that anticommutes with all α_i. For 2×2 matrices this is impossible, since the Pauli matrices form a complete set of anticommuting matrices. The smallest size turns out to be a 4×4 matrix. Dirac found a set of 4×4-matrices satisfying the relations (0.14) which are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} .$$

It will be profitable to introduce a 'four-vector' γ^μ of 4×4 matrices

$$\gamma^\mu \equiv (\gamma^0, \gamma^i) = (\beta, \beta\alpha_i) ,$$

such that the Dirac equation becomes

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 , \tag{0.16}$$

where $\partial_\mu = \partial/\partial x^\mu$. The Dirac gamma matrices satisfy anticommuting (Clifford algebra) relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu},$$

where $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. We have from the gamma matrices that

$$\gamma^{0*} = \gamma^0 \quad , \quad \gamma^{i*} = -\gamma^i .$$

This means that γ^1 is hermitian and γ^i is anti-hermitian for $i = 1, 2, 3$. An interaction with the electromagnetic field $F_{\mu\nu}$ is included via the spacetime potential (A_0, A_1, A_2, A_3) of $F_{\mu\nu}$ by employing the replacement $\partial_\mu \longrightarrow \partial_\mu - iqA_\mu$. The Dirac equation for the interacting case is given by

$$i\gamma^\mu(\partial_\mu - iqA_\mu)\psi = m\psi . \qquad (0.17)$$

0.6 The Clifford algebra and Dirac Spinors:

Take V to be any vector space with an inner product $<,>$: To each such vector space V with inner product $<,>$, we associate an associative algebra $Cl(V,<,>)$ called the Clifford algebra in such a way that $x^2 =< x, x >$, for all $x \in V$. If U is any invertible element of the algebra and $x' = UxU^{-1}$, then $(x')^2 =< x, x >$. Those elements U such that $x' = UxU^{-1}$ is in V for every x in V form a group, known as the Clifford group (see [3] and [6]). Those elements U give orthogonal transformations given by $x' = UxU^{-1}$. The center is the subgroup of the Clifford group which commutes with everything. If U is in the center, then we have

$$x' = UxU^{-1} = xUU^{-1} = x .$$

In Physics, elements of the vector space carrying an irreducible representation of the complexified Clifford algebra are termed Dirac spinors. To see the relation of spinors in $t + s$ dimensions (where t is the time dimension and s is the space dimension) to the classical division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ see [11].

Let us take $V = \mathbb{R}^{1+3}$, that is V is space-time with its Lorentz metric. There should be a linear map $\gamma : \mathbb{R}^{1+3} \longrightarrow Cl(\mathbb{R}^{1+3})$ such that
i) Every element of $Cl(\mathbb{R}^{1+3})$ can be written as a sum of products of elements

of $\gamma(\mathbb{R}^{1+3})$ (and of multiples of the identity I),
ii) $\gamma(v)\gamma(v') + \gamma(v')\gamma(v) = 2(v,v')I$ for every v and v' in $\mathbb{R}^{1+3}$,

Set

$$v = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \text{ and } \gamma(v) = \begin{pmatrix} 0 & 0 & t+z & x+iy \\ 0 & 0 & x-iy & t-z \\ t-z & -x-iy & 0 & 0 \\ -x+iy & t+z & 0 & 0 \end{pmatrix},$$

$$\gamma(v)^2 = (t^2 - x^2 - y^2 - z^2)I_4 .$$

The Clifford algebra of $\mathbb{R}^{1+3}$ is the complete algebra of these 4×4 matrices. The Clifford algebra is generated by

$$1, \gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^\mu\gamma^\nu \;\; (\mu < \nu), \gamma^\mu\gamma^\nu\gamma^\rho \;\; (\mu < \nu < \rho), \gamma^0\gamma^1\gamma^2\gamma^3 .$$

The last generator is of particular importance and we define

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

In general spinors are elements of minimal left ideals of Clifford algebras. A subalgebra S of algebra A is a left ideal if $as \in S$ for all $a \in A$ and $s \in S$ (see [6] and [14]). The word minimal in this context is equivalent to the word irreducible, meaning that there is not a smaller ideal in S. In four dimensional space-time the Dirac spinors lives in S, and the minimal left ideal is given by

$$S \;=\; Cl_4 f \;\simeq\; \left\{ \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \mid \psi_1 \,,\, \psi_2 \,,\, \psi_3 \,,\, \psi_4 \,\in\, \mathbb{C} \right\}.$$

The set of Dirac spinor is an irreducible representation of the Clifford algebra. $Spin(1,3)$ is the set of elements in the Clifford group which are even (i.e. sums of products of even number of gamma matrices) (see [14]). Irreducible representations of $Spin(1,3)$ are obtained by separating S according to the eigenvalues of γ^5. Since $(\gamma^5)^2 = I$, the eigenvalues of γ^5, called the chirality, must be ± 1 (see [15]). Then S is separated into two eigenspaces

$$S \;=\; S^+ \oplus S^- ,$$

where $\gamma^5 \, \psi^\pm \;=\; \pm \, \psi^\pm$ for $\psi^\pm \,\in\, S^\pm$.

0.7 Abelian gauge theories

At present many physically sensible theories of fundamental interactions are based on gauge theories. Here we give a brief summary of classical aspects of abelian theories. Maxwell's equations are given in terms of the magnetic field **B** and the electric field **E** by

$$\mathrm{div}\, \mathbf{B} = 0\,, \tag{0.18}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathrm{curl}\, \mathbf{E} = 0\,, \tag{0.19}$$

$$\mathrm{div}\, \mathbf{E} = j^0\,, \tag{0.20}$$

$$\frac{\partial \mathbf{E}}{\partial t} - \mathrm{curl}\, \mathbf{B} = -\mathbf{j}\,. \tag{0.21}$$

The magnetic field **B** and the electric field **E** are expressed in terms of the vector potential $A_\mu = (A_0, \mathbf{A})$ as

$$\mathbf{B} = \mathrm{curl}\, \mathbf{A}\,, \tag{0.22}$$

$$\mathbf{E} = \frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\, A_0\,, \tag{0.23}$$

respectively. Maxwell's equations are invariant under the gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu \chi \tag{0.24}$$

where χ is a scalar function. This invariance is manifest if we define the electromagnetic field tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{0.25}$$

From the construction, F is invariant under (0.24). Now Maxwell's equations could be expressed in terms of the electromagnetic field tensor $F_{\mu\nu}$ as

$$\partial_\xi F_{\mu\nu} + \partial_\mu F_{\nu\xi} + \partial_\nu F_{\xi\mu} = 0\,, \tag{0.26}$$

$$\partial_\nu F^{\mu\nu} = j^\mu\,, \tag{0.27}$$

where equation (0.26) includes the equations (0.18) and (0.19), and equation (0.27) includes the equations (0.20) and (0.21). The raising and lowering of spacetime indices are carried out with the metric $g^{\mu\nu} = diag(+1, +1, -1, -1)$.

Let ψ be a Dirac spinor field with electric charge q. The Dirac field ψ is invariant under the global gauge transformation

$$\psi \longrightarrow \psi' = e^{-iq\theta}\,\psi \quad ; \quad \bar{\psi} \longrightarrow \bar{\psi}' = \bar{\psi}\,e^{iq\theta}\,,$$

where $\theta \in \mathbb{R}$ is a constant. We elevate this symmetry to invariance under the local gauge transformation,

$$\psi \longrightarrow \psi' = e^{-iq\theta(x)}\,\psi \quad ; \quad \bar{\psi} \longrightarrow \bar{\psi}' = \bar{\psi}\,e^{iq\theta(x)}.$$

Let us introduce the covariant derivatives,

$$\nabla_\mu = \partial_\mu - iqA_\mu \quad ; \quad \nabla_\mu{}' = \partial_\mu - iqA_\mu{}' \tag{0.28}$$

where $A_\mu{}' = A_\mu - \partial_\mu\theta(x)$, and $\nabla_\mu\psi$ transforms in a nice way as

$$\nabla_\mu{}'\psi' = e^{-iq\theta(x)}\,\nabla_\mu\,\psi\,.$$

For a more detailed discussion, see [15] and [21].

Chapter 1

The representation of the gamma matrices in $2+2$ dimensions and currentless spinors

1.1 Introduction:

This chapter contains the following: In section 1.2 we give irreducible representation of the Dirac gamma matrices in $2+2$ dimensions. In section 1.3 we write the massless Dirac equation in $2+2$ dimensions. In section 1.4 we talk about the conserved current by showing first that the current is conserved, then we give the form for the conjugation matrix B in terms of the representation of the Dirac's γ matrices in $2+2$ dimensions. Finally in section 1.5 we find the general form of the currentless spinors. We treat the Dirac equation purely as a classical field theory, and do not try to look for a particle interpretation.

1.2 An irreducible representation of the gamma matrices in $2+2$ dimensions

Consider four dimensional flat space-time with the signature $(2,2)$, and the flat metric $g^{\mu\nu} = diag(+1,+1,-1,-1)$. The representation of the Dirac gamma matrices with (2×2)-dimensional entries is given by

$$\gamma^1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \quad , \quad \gamma^2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \quad , \quad \gamma^4 = \begin{pmatrix} 0 & i\,I_2 \\ i\,I_2 & 0 \end{pmatrix}, \tag{1.1}$$

where I_2 is the (2×2) identity matrix. These σ matrices are the Pauli spin matrices, and they are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This representation of the Dirac gamma matrices is particularly useful in that the γ^5 matrix is diagonal. It gives a preferred basis for introducing the two component notation for spinors in four space-time dimensions [18]. This representation satisfies the Clifford algebra condition

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \tag{1.2}$$

and in particular

$$\begin{aligned} (\gamma^\mu)^2 &= +1 \qquad \text{for } \ \mu = 1, 2\,, \\ (\gamma^\mu)^2 &= -1 \qquad \text{for } \ \mu = 3, 4. \end{aligned}$$

The γ^5 matrix is given by

$$\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{1.3}$$

1.3 The massless Dirac equation in $2+2$ dimensions

The massless Dirac equation with the real field (potential) A_μ, where the potentials A_μ live in the Lie algebra of group $U(1)$, is given by

$$\gamma^\mu(\partial_\mu - iqA_\mu)\psi \;=\; 0\,. \tag{1.4}$$

Here we use summations over μ for $\mu = 1, \ldots, 4$. The last equation can be written as

$$[\gamma^1(\partial_1 - iqA_1) + \gamma^2(\partial_2 - iqA_2) + \gamma^3(\partial_3 - iqA_3) + \gamma^4(\partial_4 - iqA_4)]\psi = 0\,.$$

In general we set $\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, where a, b, c and d are complex valued functions of the coordinates system (x^1, x^2, x^3, x^4). Now by substituting this value for ψ and by using the representation for gamma matrices given by (1.1) in equation (1.4), we can write equation (1.4) as two equations which are given by

$$\begin{pmatrix} i(\partial_4 - iqA_4) + (\partial_3 - iqA_3) & i(\partial_1 - iqA_1) + (\partial_2 - iqA_2) \\ i(\partial_1 - iqA_1) - (\partial_2 - iqA_2) & -(\partial_3 - iqA_3) + i(\partial_4 - iqA_4) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0\,,$$

$$\begin{pmatrix} i(\partial_4 - iqA_4) - (\partial_3 - iqA_3) & -i(\partial_1 - iqA_1) - (\partial_2 - iqA_2) \\ -i(\partial_1 - iqA_1) + (\partial_2 - iqA_2) & i(\partial_4 - iqA_4) + (\partial_3 - iqA_3) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0\,.$$

Note that the last two equations show how the Dirac operator act on spinors, where in the first one it acts on the top two components of the spinor, and in the second equation it acts on the last two components of the spinor.

1.4 The Conserved Current :

1.4.1 The Conservation Law

Suppose that the current with signature $(2,2)$ is given by

$$j^\mu = \psi^* B \gamma^\mu \psi\,, \tag{1.5}$$

(ψ^* is the complex conjugate of ψ^T), where B is a given 4×4 constant matrix. We interpret j^μ as being the charge current acting as a source in Maxwell's equations. Now we want to show that if the condition $B\,\gamma^\mu = (\gamma^\mu)^*\,B$ holds for the matrix B, then j^μ is a conserved current, i.e.

$$\partial_\mu\, j^\mu \;=\; 0\,.$$

We begin with

$$\partial_\mu\,(\psi^* B\gamma^\mu\psi) = (\partial_\mu\psi^*)B\gamma^\mu\,\psi + \psi^* B\gamma^\mu(\partial_\mu\psi)\,.$$

By using the condition which related B and γ^μ in the last equation, it becomes

$$\partial_\mu\,(\psi^* B\gamma^\mu\psi) = (\partial_\mu\psi^*)\gamma^{\mu^*} B\,\psi + \psi^* B\gamma^\mu(\partial_\mu\psi)\,.$$

Now we are in position to use the Dirac equation (1.4), and then the last equation becomes

$$\partial_\mu\,(\psi^* B\gamma^\mu\psi) = i\psi^* B\gamma^\mu A_\mu\psi - i\psi^*\bar{A}_\mu\gamma^{\mu^*} B\,\psi\,,$$

and we can rewrite the last equation as

$$\partial_\mu\,(\psi^* B\gamma^\mu\psi) = i\psi^* B\gamma^\mu A_\mu\psi - i\psi^*\bar{A}_\mu B\gamma^\mu\,\psi = 0\,,$$

where we remember that A_μ is real valued function, so $\bar{A}_\mu = A_\mu$. This means that the current is conserved.

1.4.2 The conjugation matrix

The idea now is to find the conjugation matrix B, which in general has this form

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

This matrix B should satisfy the following properties:

i) $\gamma^\mu\,B \;=\; B\,\gamma^\mu$ for $\mu = 1, 2$,

ii) $\gamma^\mu\,B = -B\,\gamma^\mu$ for $\mu = 3, 4$.

From (i) we can write B like this

$$B = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 \\ 0 & b_{22} & 0 & b_{24} \\ -b_{24} & 0 & b_{22} & 0 \\ 0 & -b_{13} & 0 & b_{11} \end{pmatrix},$$

and from (ii) we can write it as

$$B = b_{11} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where in general b_{11} is a complex number. It follows from the representation of the Dirac gamma matrices given by (1.1) that

$$\gamma^3 \gamma^4 = -i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since b_{11} is a complex number, we deduce that the matrix B is a multiple of $\gamma^3\gamma^4$.

Supposing that $B = \alpha \gamma^3 \gamma^4$, to find the value for α, let first take the complex conjugate for the current equation (1.5),

$$\bar{j^{\mu}} = \psi^* \gamma^{\mu^*} B^* \psi \,, \tag{1.6}$$

by substituting the value for B in equation (1.6), it becomes

$$\bar{j^{\mu}} = -\frac{\bar{\alpha}}{\alpha} \psi^* B \gamma^{\mu} \psi \,,$$

and by using the current equation (1.5) the last equation becomes

$$\bar{j^{\mu}} = -\frac{\bar{\alpha}}{\alpha} j^{\mu} \,.$$

But since j^{μ} is real, the last equation shows that α is pure imaginary. This fixes B upto a real multiple, and we chose to set $B = i \gamma^3 \gamma^4$.

1.5 The currentless spinors

The aim of this section is to find the general form of the currentless spinors which are given by putting $j^\mu \equiv 0$ in equation (1.5).
The zero current equation is

$$\psi^* B\gamma^\mu \psi = 0\,, \qquad \text{for } \mu = 1, \cdots, 4\,. \tag{1.7}$$

For $\mu = 1$, it becomes

$$\psi^* B\gamma^1 \psi = 0\,.$$

If we set $\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, where a, b, c and d are complex valueds, and ψ^* is just the complex conjugate of $\psi^\top$, then the last equation gives

$$\bar{a}d - \bar{b}c + \bar{c}b - \bar{d}a = 0\,. \tag{1.8}$$

For $\mu = 2$,

$$\psi^* B\gamma^2 \psi = 0\,,$$

which gives

$$\bar{a}d + \bar{b}c + \bar{c}b + \bar{d}a = 0\,. \tag{1.9}$$

For $\mu = 3$,

$$\psi^* B\gamma^3 \psi = 0\,,$$

which gives

$$\bar{a}c + \bar{b}d + \bar{c}a + \bar{d}b = 0\,. \tag{1.10}$$

For $\mu = 4$,

$$\psi^* B\gamma^4 \psi = 0\,,$$

which gives

$$\bar{a}c - \bar{b}d - \bar{c}a + \bar{d}b = 0\,. \tag{1.11}$$

The equations (1.8), (1.9), (1.10) and (1.11) are called the zero current equations.

Now let us simplify the zero current equations. There are two simpler equations given by (1.10) and (1.11), the first one is given by adding equations (1.10) and (1.11), and the second by subtracting them,

$$\bar{a}c + \bar{d}b = 0\,, \tag{1.12}$$

$$\bar{b}d + \bar{c}a = 0\,, \tag{1.13}$$

respectively. Similarly there are two simpler equations given by equations (1.8) and (1.9), the first is given by adding equations (1.8) and (1.9) and the second by subtracting them,

$$\bar{a}d + \bar{c}b = 0\,, \tag{1.14}$$

$$\bar{b}c + \bar{d}a = 0\,, \tag{1.15}$$

respectively. This simplification reduces the number of the currentless equations to two, and they are given by the equations (1.12) and (1.14). If we take the complex conjugate of equations (1.12) and (1.14), then we will just get the equations (1.13) and (1.15) respectively.

Proposition(1.1):

There are three possible general solutions for the equations (1.12) and (1.14) that give zero current, and they are given by the following two cases:

Case (I): When a vanishes, then the general solution for the equations (1.12) and (1.14) is

I1) $b = 0$, where c and d are complex valueds.

Case (II): When a is non-vanishing, then the second and the third general solutions for the equations (1.12) and (1.14) are

II1) $c = d = 0$, where a and b are any complex valueds,

II2) $b = a\,e^{is}$, $c = i\,a\,r\,e^{i\frac{1}{2}(s-t)}$, $d = i\,a\,r\,e^{i\frac{1}{2}(s+t)}$,

where r is a non zero real valued, and s and t are real valueds, and a is a complex valued.

Case(I): Suppose that a vanishes:

In this case we have from the equations (1.12) and (1.14) these results

$$\bar{d}\,b = 0 \quad \text{and} \quad \bar{c}\,b = 0\,.$$

From this we have to say that $b = 0$ or $d = c = 0$.

I1)If $b \;=\; 0$, then we have this solution

$$\psi = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}, \tag{1.16}$$

where c and d are complex valueds.
I2)If $b \;\neq\; 0$, then we will have this solution

$$\psi = \begin{pmatrix} 0 \\ b \\ 0 \\ 0 \end{pmatrix}, \tag{1.17}$$

where b is a complex valued.

Case(II): Suppose that a is non-vanishing:

In this case we can write the equations (1.12) and (1.14) as

$$c \;=\; -\frac{\bar{d}\,b}{\bar{a}} \quad , \quad d \;=\; -\frac{\bar{c}\,b}{\bar{a}}\,,$$

respectively.

Now by using the first equation of the last two in the other one, giving

$$d \;=\; \frac{b\,\bar{b}}{a\,\bar{a}}\,d\,. \tag{1.18}$$

II1)If $d \;=\; 0$, then from equation (1.12) we need as well $c \;=\; 0$, so the solution is given by

$$\psi = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}, \tag{1.19}$$

where a and b are complex valued. Note that the solution (1.17) is a special case of the solution (1.19).

II2)If $d \neq 0$, then equation (1.18) becomes

$$b\,\bar{b} = a\,\bar{a}\,,$$

since $a \neq 0$, this means that $b \neq 0$, and this requires as well that $c \neq 0$. The last equation can be written as

$$\frac{b}{a} = \frac{\bar{a}}{\bar{b}}\,, \tag{1.20}$$

and if we set $b/a = \lambda$, then equation (1.20) becomes

$$\lambda\,\bar{\lambda} = 1 \qquad \text{so} \qquad \lambda = e^{is}\,,$$

for some real s. Now by substituting this value for b in the equations (1.12) and (1.14), they become

$$\bar{a}c + \bar{d}ae^{is} = 0\,, \tag{1.21}$$

$$\bar{a}d + \bar{c}ae^{is} = 0\,, \tag{1.22}$$

respectively. Since c and d are non zero, by multiplying equation (1.21) by $\bar{c}$ and equation (1.22) by $\bar{d}$ and then subtracting the results,

$$\frac{d}{c} = \frac{\bar{c}}{\bar{d}} \qquad \text{so} \qquad d = e^{it}c\,, \tag{1.23}$$

for some real t. Similarly by substituting this value for d in the equations (1.21) and (1.22), they become

$$\bar{a}c + \bar{c}ae^{-it}e^{is} = 0\,, \tag{1.24}$$

$$\bar{a}ce^{it} + \bar{c}ae^{is} = 0\,, \tag{1.25}$$

respectively. Note that the last two equations are just the same, and we can rewrite them as

$$\frac{c}{a} = -\,\frac{\bar{c}}{\bar{a}}e^{i(s-t)}\,. \tag{1.26}$$

We set $c/a = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$, and by substituting this value in the equation (1.26), it becomes

$$re^{i\theta} = -\,re^{i(s-t-\theta)}\,.$$

The last equation requires that

$$\theta \,=\, n\pi + \frac{\pi}{2} + \frac{s-t}{2} \qquad \text{where} \quad n \in \mathbb{N}\,,$$

then θ is, up to a multiple of 2π,

$$\theta \,=\, \frac{\pi}{2} + \frac{s-t}{2} \qquad \text{or} \qquad \theta \,=\, -\,\frac{\pi}{2} + \frac{s-t}{2}\,.$$

Now by allowing r to be positive or negative, we can write the solution as

$$\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,re^{\frac{i}{2}(s-t)} \\ i\,re^{\frac{i}{2}(s+t)} \end{pmatrix} a\,, \tag{1.27}$$

where s and t are real valueds, r is a non zero real valued, and a is a complex valued, as required.

Chapter 2

The plane wave solutions and superposition of the currentless spinors

2.1 Introduction

This chapter contains two main sections, and in both these two sections we work with the currentless spinors from the previous chapter. In section 2.2 we find the plane wave solutions for the massless Dirac equation with potential A_μ, and in this section we shall allow generalized plane waves, including exponentially increasing or decreasing terms. The periodic plane wave solutions for the massless Dirac equation are special cases of our solutions where the wave vector is real. Finally in section 2.3 we study superposition for the currentless spinors.

2.2 The plane wave solutions

The starting point in this part is to write the massless Dirac equation with potential A_μ, which is given by

$$\gamma^\mu \left(\partial_\mu - i\, A_\mu \right) \psi \;=\; 0\,. \tag{2.1}$$

Let us look for a plane wave solution of the form

$$\psi \;=\; \psi_0 \cdot e^{i\,\underline{k}\cdot\underline{x}}\,,$$

where the vector $\underline{k}$ is called the wave vector, and ψ_0 is a constant four-vector. We shall allow generalized plane waves, including exponentially increasing or decreasing terms, corresponding to $\underline{k}$ having complex entries. So wé put

$$\underline{k} = \underline{k'} + i\underline{k''},$$

where $\underline{k'}$ and $\underline{k''}$ are real constants.

Now by using the representation for the γ matrices given by (1.1) and by substituting the value for ψ in equation (2.1), giving

$$\begin{pmatrix} 0 & 0 & i\,\partial_4 - \partial_3 & -i\,\partial_1 - \partial_2 \\ 0 & 0 & \partial_2 - i\,\partial_1 & \partial_3 + i\,\partial_4 \\ \partial_3 + i\,\partial_4 & \partial_2 + i\,\partial_1 & 0 & 0 \\ i\,\partial_1 - \partial_2 & i\,\partial_4 - \partial_3 & 0 & 0 \end{pmatrix} \psi_0 \cdot e^{i\,k_\mu . x^\mu} =$$

$$i \begin{pmatrix} 0 & 0 & i\,A_4 - A_3 & -i\,A_1 - A_2 \\ 0 & 0 & A_2 - i\,A_1 & A_3 + i\,A_4 \\ A_3 + i\,A_4 & A_2 + i\,A_1 & 0 & 0 \\ i\,A_1 - A_2 & i\,A_4 - A_3 & 0 & 0 \end{pmatrix} \psi_0 \cdot e^{i\,k_\mu . x^\mu} .$$

In this section we put ψ_0 equal to $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. We chose not to write ψ_0 as $\begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}$ to avoid too many subscripts. Then the last equation gives the following four equations

$$a\,(k_3 + i\,k_4) + b\,(k_2 + i\,k_1) \;=\; a\,(A_3 + i\,A_4) + b\,(A_2 + i\,A_1)\,, \tag{2.2}$$

$$a\,(i\,k_1 - k_2) + b\,(i\,k_4 - k_3) \;=\; a\,(i\,A_1 - A_2) + b\,(i\,A_4 - A_3)\,, \tag{2.3}$$

$$c\,(i\,k_4 - k_3) - d\,(i\,k_1 + k_2) \;=\; c\,(i\,A_4 - A_3) - d\,(i\,A_1 + A_2)\,, \tag{2.4}$$

$$c\,(k_2 - i\,k_1) + d\,(k_3 + i\,k_4) \;=\; c\,(A_2 - i\,A_1) + d\,(A_3 + i\,A_4)\,. \tag{2.5}$$

Next we are going to set ψ_0 to be of the form of the currentless spinors given in the previous chapter, and then find the plane wave solutions for each one by solving the last four equations. We additionally assume (if A_1 is not a constant) that

$$A_1 = M \sin(\underline{L} \cdot \underline{x}), \tag{2.6}$$

where $\underline{L}$ is a constant vector, and M is a real constant, and that $\underline{A}$ is a solution for the zero current field for the electromagnetic field,

$$\partial_\mu F^{\mu\nu} = 0.$$

If A_2 is not determined by A_1, we assume that

$$A_2 = N \sin(\underline{L} \cdot \underline{x} + r), \tag{2.7}$$

for a real constant N, and an arbitrary real constant r.

Proposition (2.1):

All the plane wave solutions of the massless Dirac equation (2.1), subject to the preceding restrictions, are given by the following cases :

case(I) :

If we set $\psi_0 = \begin{pmatrix} a \\ e^{is}a \\ ira\, e^{\frac{i}{2}(s-t)} \\ ira\, e^{\frac{i}{2}(s+t)} \end{pmatrix}$ where s and t are real constants, r is a non zero real constant, and a is a complex constant, then the plane wave solutions of (2.1) are given by :

I1) For $\cos(t) - \cos(s) \neq 0$. We have k''_1 an arbitrary constant, and

$$\begin{aligned} k''_2 &= \frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))}\, k''_1, \\ k''_3 &= \frac{-\sin(t-s)}{(\cos(t) - \cos(s))}\, k''_1. \\ k''_4 &= \frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))}\, k''_1, \end{aligned}$$

A_1 is given by (2.6), where M is an arbitrary constant and $\underline{L}$ is constrained by

$$L_1 = \pm \frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))} L_2 ,$$

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0 .$$

Then A_2, A_3 and A_4 are given by

$$A_2 = k_2' - \frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)} (k_1' - A_1) ,$$

$$A_3 = k_3' + \frac{-\sin(t-s)}{\cos(t) - \cos(s)} (k_1' - A_1) ,$$

$$A_4 = k_4' - \frac{1 - \cos(t-s)}{\cos(t) - \cos(s)} (k_1' - A_1) .$$

There is a non vanishing solution for the electromagnetic field, given by (2.44).

I2) For $\cos(t) - \cos(s) = 0$, we have the following:

I2a) If $t = s + 2n\pi$, then there are two possibilities:

I2a(i)) For $\sin(s) \neq 0$. Then k_1'' is an arbitrary constant, and

$$k_2'' = -\frac{\cos(s)}{\sin(s)} k_1'' \quad , \quad k_3'' = \frac{1}{\sin(s)} k_1'' \quad , \quad k_4'' = 0 .$$

A_1 is given by (2.6), where M is an arbitrary constant and $\underline{L}$ is constrained by

$$L_3 = L_1 \sin(s) - L_2 \cos(s) ,$$

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0 .$$

Then A_2, A_3 and A_4 are given by

$$A_2 = k_2' + \frac{\cos(s)}{\sin(s)} (k_1' - A_1) ,$$

$$A_3 = k_3' - \frac{1}{\sin(s)} (k_1' - A_1) ,$$

$$A_4 = 0 .$$

There is a non vanishing solution for the electromagnetic field, given by (2.60).

I2a(ii)) For $\sin(s) = 0$. We have k_2'' arbitrary, and

$$k_1'' = 0 \quad , \quad k_3'' = -k_2'' \quad , \quad k_4'' = 0 \, .$$

In this case A_1 and A_4 arbitrary constants, and A_2 is given by (2.7), where N and r arbitrary constants and $\underline{L}$ is constrained by

$$L_3 = -L_2 \, ,$$

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0 \, .$$

Then A_1, A_3 and A_4 are given by

$$A_1 = k_1' \quad , \quad A_3 = k_3' + (k_2' - A_2) \quad , \quad A_4 = k_4' \, .$$

There is a non vanishing solution for the electromagnetic field, given by (2.73).

I2b) If $t = -s + 2n\pi$, then we have k_2'' arbitrary, and

$$k_1'' = 0 \quad , \quad k_3'' = -k_2'' \cos(s) \quad , \quad k_4'' = -k_2'' \sin(s) \, .$$

In this case A_1 is an arbitrary constant, A_2 is given by (2.7), where N and r arbitrary constants, and $\underline{L}$ is constrained by

$$L_1 = \pm\Big(L_4 \cos(s) - L_3 \sin(s)\Big)$$

$$L_2 = -\Big(L_3 \cos(s) + L_4 \sin(s)\Big) \, ,$$

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0 \, .$$

Then A_1, A_3 and A_4 are given by

$$\begin{aligned} A_1 &= k_1' \, , \\ A_3 &= k_3' + (k_2' - A_2) \cos(s) \, , \\ A_4 &= k_4' + (k_2' - A_2) \sin(s) \, . \end{aligned}$$

There is a non vanishing solution for the electromagnetic field, given by (2.88).

case(II) :

If we set $\psi_0 = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$ where c and d are complex constants. We take $c = R_1 e^{i\theta_1}$ and $d = R_2 e^{i\theta_2}$, where R_1 and R_2 are positive real numbers, and θ_1 and θ_2 are real numbers. Define $\lambda = R_1/R_2$ and $\phi = \theta_1 - \theta_2$. The plane wave solutions of (2.1) are given by the following:

II1) For $1 - \lambda^2 \neq 0$, we have

$$(k_3' - A_3) = \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big)k_4'' + \Big(\frac{2\lambda}{1-\lambda^2}\Big)\, k_2'' \sin\phi$$

$$- \Big(\frac{2\lambda}{1-\lambda^2}\Big)\, k_1'' \cos\phi\,,$$

$$(k_4' - A_4) = -\Big(\frac{1+\lambda^2}{1-\lambda^2}\Big)k_3'' - \Big(\frac{2\lambda}{1-\lambda^2}\Big)\, k_2'' \cos\phi$$

$$- \Big(\frac{2\lambda}{1-\lambda^2}\Big)\, k_1'' \sin\phi\,,$$

$$(k_2' - A_2) = \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big)\, k_1'' - \Big(\frac{2\lambda}{1-\lambda^2}\Big)\, (k_4'' \cos\phi - k_3'' \sin\phi)\,,$$

$$(k_1' - A_1) = -\Big(\frac{1+\lambda^2}{1-\lambda^2}\Big)\, k_2'' - \Big(\frac{2\lambda}{1-\lambda^2}\Big)\, (k_4'' \sin\phi + k_3'' \cos\phi)\,.$$

The vector potential is constant as shown, this means that the electromagnetic field is vanishing.

II2) For $1 - \lambda^2 = 0$, so that when $\lambda = 1$, $\underline{k''}$ obeys the relations

$$k_4'' = k_1'' \cos\phi - k_2'' \sin\phi\,,$$

$$k_3'' = -(k_2'' \cos\phi + k_1'' \sin\phi).$$

The potentials A_1 and A_2 are given by (2.6) and (2.7), and A_3 and A_4 are given by

$$A_3 = k_3' + (k_2' - A_2)\cos\phi + (k_1' - A_1)\sin\phi,$$

$$A_4 = k_4' + (k_2' - A_2)\sin\phi - (k_1' - A_1)\cos\phi.$$

The detailed cases for A_1 and A_2 are:

II2a) M, N and r are arbitrary constants, and $\underline{L}$ is constrained by

$$L_3 = -\Big(L_1 \sin\phi + L_2 \cos\phi\Big),$$

$$L_4 = L_1 \cos\phi - L_2 \sin\phi,$$

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0.$$

There is a non vanishing solution for the electromagnetic field, given by (2.118).

II2b(i)) M, N and r are arbitrary constants, and $\underline{L}$ is constrained by

$$L_1 = L_4 \cos\phi - L_3 \sin\phi,$$

$$L_2 = -\Big(L_4 \sin\phi + L_3 \cos\phi\Big).$$

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0.$$

There is a non vanishing solution for the electromagnetic field, given by (2.123).

II2b(ii)) M and r are arbitrary constants, and N and $\underline{L}$ are constrained by

$$N = -M \frac{(L_1 + \sin\phi\, L_3 - \cos\phi\, L_4)}{(L_2 + \cos\phi\, L_3 + \sin\phi\, L_4)},$$

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0 .$$

There is a non vanishing solution for the electromagnetic field, given by (2.127).

II2b(iii)) If $N = M = 0$, then A_1 and A_2 vanish, and A_3 and A_4 are arbitrary constants.

case(III) :

If we set $\psi_0 = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$ where a and b are complex constants, the plane wave solutions of (2.1) are similar to Case(II).

2.2.1 The plane wave solutions, case (I):

In this case we have $\psi_0 = \begin{pmatrix} a \\ e^{is}a \\ ira\, e^{\frac{i}{2}(s-t)} \\ ira\, e^{\frac{i}{2}(s+t)} \end{pmatrix}$ where s and t are real constants, r is a non zero real constant, and a is a complex constant, so the equations (2.2), (2.3), (2.4) and (2.5) become

$$(k_3 + i\,k_4) + e^{is}\,(k_2 + i\,k_1) = (A_3 + i\,A_4) + e^{is}\,(A_2 + i\,A_1)\,, \tag{2.8}$$

$$e^{-is}\,(i\,k_1 - k_2) + (i\,k_4 - k_3) = e^{-is}(i\,A_1 - A_2) + (i\,A_4 - A_3)\,, \tag{2.9}$$

$$(i\,k_4 - k_3) - e^{it}\,(i\,k_1 + k_2) = (i\,A_4 - A_3) - e^{it}\,(i\,A_1 + A_2)\,, \tag{2.10}$$

$$e^{-it}\,(k_2 - i\,k_1) + (k_3 + i\,k_4) = e^{-it}\,(A_2 - i\,A_1) + (A_3 + i\,A_4)\,. \tag{2.11}$$

respectively. There are two simpler equations we can get from the equations (2.8) and (2.9), where the first one is given by adding the equations (2.8) and (2.9), and the second by subtracting them,

$$k_4 + k_1\,\cos(s) + k_2\,\sin(s) = A_4 + A_1\,\cos(s) + A_2\,\sin(s)\,, \tag{2.12}$$

$$k_3 - k_1 \sin(s) + k_2 \cos(s) = A_3 - A_1 \sin(s) + A_2 \cos(s), \tag{2.13}$$

respectively. Similarly there are two simpler equations we can get from the equations (2.10) and (2.11), where the first one is given by adding the equations (2.10) and (2.11), and the second by subtracting them,

$$k_4 - k_1 \cos(t) - k_2 \sin(t) = A_4 - A_1 \cos(t) - A_2 \sin(t), \tag{2.14}$$

$$k_3 - k_1 \sin(t) + k_2 \cos(t) = A_3 - A_1 \sin(t) + A_2 \cos(t), \tag{2.15}$$

respectively. Now to work with the last four equations we need first to know that s, t and A_μ are real for $\mu = 1, \cdots, 4$, and that k_j are complex numbers for $j = 1, \cdots, 4$. Set $k_j = k'_j + ik''_j$, so each equation of the last four equations provides two equations, where one comes from the real part and the other comes from the imaginary part. Let us start with equation (2.12), which gives us the following:

$$(k'_4 - A_4) + (k'_1 - A_1)\cos(s) + (k'_2 - A_2)\sin(s) = 0, \tag{2.16}$$

$$k''_4 + k''_1 \cos(s) + k''_2 \sin(s) = 0. \tag{2.17}$$

From equation (2.13),

$$(k'_3 - A_3) - (k'_1 - A_1)\sin(s) + (k'_2 - A_2)\cos(s) = 0, \tag{2.18}$$

$$k''_3 - k''_1 \sin(s) + k''_2 \cos(s) = 0. \tag{2.19}$$

And from equation (2.14),

$$(k'_4 - A_4) - (k'_1 - A_1)\cos(t) - (k'_2 - A_2)\sin(t) = 0, \tag{2.20}$$

$$k''_4 - k''_1 \cos(t) - k''_2 \sin(t) = 0. \tag{2.21}$$

Finally from equation (2.15),

$$(k'_3 - A_3) - (k'_1 - A_1)\sin(t) + (k'_2 - A_2)\cos(t) = 0, \tag{2.22}$$

$$k''_3 - k''_1 \sin(t) + k''_2 \cos(t) = 0. \tag{2.23}$$

Now by subtracting equation (2.17) from (2.21), and equation (2.19) from (2.23), gives

$$k_1'' \Big(\cos(t) + \cos(s) \Big) + k_2'' \Big(\sin(t) + \sin(s) \Big) = 0 , \qquad (2.24)$$

$$k_1'' \Big(\sin(t) - \sin(s) \Big) - k_2'' \Big(\cos(t) - \cos(s) \Big) = 0 , \qquad (2.25)$$

respectively. Similarly by subtracting equation (2.16) from (2.20), and equation (2.18) from (2.22), gives

$$(k_1' - A_1)\Big(\cos(t) + \cos(s)\Big) + (k_2' - A_2)\Big(\sin(t) + \sin(s)\Big) = 0 , \qquad (2.26)$$

$$(k_1' - A_1)\Big(\sin(t) - \sin(s)\Big) - (k_2' - A_2)\Big(\cos(t) - \cos(s)\Big) = 0 , \qquad (2.27)$$

respectively. To work out the last four equations there are two main cases we have to consider, which are :

I1) <u>For $(\cos(t) - \cos(s)) \neq 0$</u>,

the equations (2.25) and (2.27) give

$$k_2'' = \frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)} \, k_1'' , \qquad (2.28)$$

$$(k_2' - A_2) = \frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)} \, (k_1' - A_1) , \qquad (2.29)$$

respectively. These values for k_2'' and $(k_2' - A_2)$ satisfy the equations (2.24) and (2.26). By substituting this value for k_2'' in the equations (2.17) and (2.19), they become

$$k_4'' = \frac{1 - \cos(t-s)}{\cos(t) - \cos(s)} \, k_1'' \quad , \quad k_3'' = \frac{-\sin(t-s)}{\cos(t) - \cos(s)} \, k_1'' \qquad (2.30)$$

respectively. Similarly by substituting the value for $(k_2' - A_2)$ in the equations (2.16) and (2.18), they become

$$(k_4' - A_4) = \frac{1 - \cos(t-s)}{\cos(t) - \cos(s)} \, (k_1' - A_1) ,$$

$$(k_3' - A_3) = \frac{-\sin(t-s)}{\cos(t) - \cos(s)} \, (k_1' - A_1) , \qquad (2.31)$$

respectively.

Note that the periodic plane wave solution is exactly the same as the solution we have here; the only difference is that k_1 is real.

The electromagnetic field for (I1)

The electromagnetic field tensor is given by

$$F_{\mu\nu} \; = \; \partial_\mu A_\nu \; - \; \partial_\nu A_\mu \, . \tag{2.32}$$

The values for A_2, A_3 and A_4 are given in terms of A_1 and the k's as in (2.29) and (2.31). Now by substituting these values in equation (2.32), giving

$$\begin{aligned}
F_{12} &= \Big(\frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))} \, \partial_1 \; - \; \partial_2 \Big) \, A_1 \, , \\
F_{13} &= -\Big(\frac{\sin(t-s)}{(\cos(t) - \cos(s))} \, \partial_1 \; + \; \partial_3 \Big) \, A_1 \, , \\
F_{14} &= \Big(\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))} \, \partial_1 \; - \; \partial_4 \Big) \, A_1 \, , \\
F_{23} &= -\Big(\frac{\sin(t-s)}{(\cos(t) - \cos(s))} \, \partial_2 + \frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))} \, \partial_3 \Big) \, A_1 \, , \\
F_{24} &= \Big(\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))} \, \partial_2 - \frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))} \, \partial_4 \Big) \, A_1 \, , \\
F_{34} &= \Big(\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))} \, \partial_3 + \frac{\sin(t-s)}{(\cos(t) - \cos(s))} \, \partial_4 \Big) \, A_1 \, .
\end{aligned}$$

The idea now is to find out what the zero current equation will give us,

$$\partial_\mu \, F^{\mu\nu} \; = \; 0 \, . \tag{2.33}$$

For $\nu \, = \, 1$,

$$\begin{aligned}
&(\cos(t) - \cos(s)) \, (\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2) \, A_1 - \\
&\Big[(\cos(t) - \cos(s)) \, \partial_1 \; + \; (\sin(t) - \sin(s)) \, \partial_2 + \\
&\sin(t-s) \, \partial_3 \; - \; (1 - \cos(t-s)) \, \partial_4 \Big] \, \partial_1 \, A_1 \; = 0 \, .
\end{aligned} \tag{2.34}$$

For $\nu = 2$,

$$\begin{aligned} &(\sin(t) - \sin(s))\,(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 - \\ &\Big[(\cos(t) - \cos(s))\,\partial_1 + (\sin(t) - \sin(s))\,\partial_2 + \\ &\sin(t-s)\,\partial_3 - (1 - \cos(t-s))\,\partial_4\Big]\,\partial_2 A_1 = 0\,. \end{aligned} \tag{2.35}$$

For $\nu = 3$,

$$\begin{aligned} &\sin(t-s)\,(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 + \\ &\Big[(\cos(t) - \cos(s))\,\partial_1 + (\sin(t) - \sin(s))\,\partial_2 + \\ &\sin(t-s)\,\partial_3 - (1 - \cos(t-s))\,\partial_4\Big]\,\partial_3 A_1 = 0\,. \end{aligned} \tag{2.36}$$

For $\nu = 4$,

$$\begin{aligned} &(1 - \cos(t-s))\,(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 - \\ &\Big[(\cos(t) - \cos(s))\,\partial_1 + (\sin(t) - \sin(s))\,\partial_2 + \\ &\sin(t-s)\,\partial_3 - (1 - \cos(t-s))\,\partial_4\Big]\,\partial_4 A_1 = 0\,. \end{aligned} \tag{2.37}$$

The last four equations imply another equation, which is given by

$$\begin{aligned} &\Big[(\cos(t) - \cos(s))\,\partial_1 + (\sin(t) - \sin(s))\,\partial_2 + \\ &\sin(t-s)\,\partial_3 - (1 - \cos(t-s))\,\partial_4\Big]^2 A_1 = 0\,. \end{aligned} \tag{2.38}$$

Under the given assumption for A_1 (2.6), by substituting this value for A_1 in equation (2.38), it becomes

$$\begin{aligned} &(\cos(t) - \cos(s))\,L_1 + (\sin(t) - \sin(s))\,L_2 + \\ &\sin(t-s)\,L_3 - (1 - \cos(t-s))\,L_4 = 0\,. \end{aligned} \tag{2.39}$$

By using the result given by (2.39) in equation (2.34),

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0\,. \tag{2.40}$$

And by using equation (2.39) in equation (2.40),

$$L_4 = \pm\sqrt{2\,\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2}\,L_2^2 - L_3^2}\,, \tag{2.41}$$

where L_4 is real. This means that we have to chose the values for L_2 and L_3 for L_4 to be real.
Finally substituting this value for L_4 in equation (2.40) gives us

$$L_1 = \pm \frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))} L_2 . \tag{2.42}$$

Now we are able to write A_1 as

$$\begin{aligned} A_1 &= M \sin\Big(\Big[x^2 \pm \frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))} x^1\Big] L_2 + x^3 L_3 \\ &\pm \Big[2 \frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2} L_2^2 - L_3^2\Big]^{\frac{1}{2}} x^4\Big) . \end{aligned} \tag{2.43}$$

Now by substituting this value for A_1 in the electromagnetic field, we get

$$\begin{aligned} F_{12} &= M\Big(\pm \frac{(\sin(t) - \sin(s))^2}{(\cos(t) - \cos(s))^2} - 1\Big) L_2 \cos(L \cdot X) , \\ F_{13} &= -M\Big(L_3 \pm \frac{\sin(t-s)(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))^2} L_2 \Big) \cos(L \cdot X) , \\ F_{14} &= \pm M\Big(\frac{(1 - \cos(t-s))(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))^2} L_2 - \\ &\sqrt{2 \frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2} L_2^2 - L_3^2} \Big) \cos(L \cdot X) , \\ F_{23} &= -M\Big(\frac{\sin(t-s)}{(\cos(t) - \cos(s))} L_2 + \frac{(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))} L_3 \Big) A_1 , \\ F_{24} &= M\Big(\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))} L_2 \mp \\ &\sqrt{2 \frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2} L_2^2 - L_3^2} \Big) \cos(L \cdot X) , \\ F_{34} &= M\Big(\pm \frac{\sin(t-s)}{(\cos(t) - \cos(s))} \sqrt{2 \frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2} L_2^2 - L_3^2} \\ &+ \frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))} L_3 \Big) \cos(L \cdot X) . \end{aligned} \tag{2.44}$$

I2) For $(\cos(t) - \cos(s)) = 0$, this means that

$$\cos(t) = \cos(s)\,, \quad \text{so} \quad t = \pm s + 2n\pi, \quad \text{where } n \in \mathbb{N}\,.$$

I2a) When $t = s + 2n\pi$, this value for t satisfies the equations (2.25) and (2.27), and the equations (2.24) and (2.26) become

$$k_1'' \cos(s) + k_2'' \sin(s) = 0\,, \tag{2.45}$$

$$(k_1' - A_1) \cos(s) + (k_2' - A_2) \sin(s) = 0\,, \tag{2.46}$$

respectively.

I2a(i)) For $\sin(s) \neq 0$, the equations (2.45) and (2.46) give

$$k_2'' = -\frac{\cos(s)}{\sin(s)}\, k_1''\,, \tag{2.47}$$

$$(k_2' - A_2) = -\frac{\cos(s)}{\sin(s)}\, (k_1' - A_1)\,, \tag{2.48}$$

respectively. Now by substituting this value for k_2'' in the equations (2.17) and (2.19), we get

$$k_4'' = 0 \quad , \quad k_3'' = \frac{1}{\sin(s)}\, k_1''\,, \tag{2.49}$$

respectively. Similarly by substituting this value for $(k_2' - A_2)$ in the equations (2.16) and (2.18), we get

$$(k_4' - A_4) = 0 \quad , \quad (k_3' - A_3) = \frac{1}{\sin(s)}\, (k_1' - A_1)\,, \tag{2.50}$$

respectively.

Note that the periodic plane wave solution is exactly the same as the solution we have here; the only difference is that k_1 is real.

The electromagnetic field for I2a(i)

The electromagnetic field tensor is given by (2.32). The values for A_2, A_3 and A_4 are given in terms of A_1 and the k's as in (2.48) and (2.50). Now by substituting these values in equation (2.32), giving

$$\begin{aligned}
F_{12} &= -\left(\frac{\cos(s)}{\sin(s)}\,\partial_1 + \partial_2 \right) A_1\,, \\
F_{13} &= \left(\frac{1}{\sin(s)}\,\partial_1 - \partial_3 \right) A_1\,, \\
F_{14} &= -\,\partial_4\, A_1\,, \\
F_{23} &= \left(\frac{1}{\sin(s)}\,\partial_2 + \frac{\cos(s)}{\sin(s)}\,\partial_3 \right) A_1\,, \\
F_{24} &= \frac{\cos(s)}{\sin(s)}\,\partial_4\, A_1\,, \\
F_{34} &= \frac{-1}{\sin(s)}\,\partial_4\, A_1\,.
\end{aligned}$$

The idea now is find out what the zero current equation (2.33) will give us.

For $\nu = 1$,

$$\sin(s)\,(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 + \Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\,\partial_1\,A_1 = 0\,. \tag{2.51}$$

For $\nu = 2$,

$$\cos(s)\,(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 - \Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\,\partial_2\,A_1 = 0\,. \tag{2.52}$$

For $\nu = 3$,

$$(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 + \Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\,\partial_3\,A_1 = 0\,. \tag{2.53}$$

For $\nu = 4$,

$$\Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\,\partial_4\,A_1 = 0\,. \tag{2.54}$$

The last four equations imply another equation, which is given by

$$\Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)^2\,A_1 = 0\,. \tag{2.55}$$

Similarly under the given assumption for A_1 (2.6), by substituting this value for A_1 in equation (2.55),

$$L_3 + \cos(s)\,L_2 - \sin(s)\,L_1 = 0\,. \tag{2.56}$$

By using the result given by (2.56) in equation (2.51), it becomes

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0\,. \tag{2.57}$$

Now by substituting equation (2.56) in equation (2.57), it gives

$$L_4 = \pm\Big(\cos(s)\,L_1 + \sin(s)\,L_2\Big)\,. \tag{2.58}$$

Finally we could write A_1 as

$$\begin{aligned} A_1 = {} & M\sin\Big(\big[x^1 + x^3\sin(s) \pm x^4\cos(s)\big]\,L_1 + \\ & \big[x^2 - x^3\cos(s) \pm x^4\sin(s)\big]\,L_2\Big)\,. \end{aligned} \tag{2.59}$$

Now by substituting this value for A_1 in the electromagnetic field,

$$\begin{aligned}
F_{12} &= -M\Big(\frac{\cos(s)}{\sin(s)}\,L_1 + L_2\Big)\cos(L\cdot X)\,,\\
F_{13} &= M\,\frac{\cos(s)}{\sin(s)}\Big(\cos(s)\,L_1 + \sin(s)\,L_2\Big)\cos(L\cdot X)\,,\\
F_{14} &= \mp M\Big(\cos(s)\,L_1 + \sin(s)\,L_2\Big)\cos(L\cdot X)\,,\\
F_{23} &= M\Big(\cos(s)\,L_1 + \sin(s)\,L_2\Big)\cos(L\cdot X)\,,\\
F_{24} &= \pm M\,\frac{\cos(s)}{\sin(s)}\Big(\cos(s)\,L_1 + \sin(s)\,L_2\Big)\cos(L\cdot X)\,,\\
F_{34} &= \mp M\,\frac{1}{\sin(s)}\Big(\cos(s)\,L_1 + \sin(s)\,L_2\Big)\cos(L\cdot X)\,.
\end{aligned} \tag{2.60}$$

I2a(ii)) For $\sin(s) = 0$, this implies that $s = \pm\pi$. Now by substituting this value for s in the equations (2.45) and (2.46), they become

$$k_1'' \cos(s) = 0, \quad \text{so} \quad k_1'' = 0, \tag{2.61}$$

$$(k_1' - A_1) \cos(s) = 0, \quad \text{so} \quad (k_1' - A_1) = 0, \tag{2.62}$$

respectively. By putting $k_1'' = 0$ in the equations (2.17) and (2.19), we get

$$k_4'' = 0 \quad , \quad k_3'' = -k_2'', \tag{2.63}$$

respectively. Similarly by putting $(k_1' - A_1) = 0$ in the equations (2.16) and (2.18), we get

$$(k_4' - A_4) = 0 \quad , \quad (k_3' - A_3) = -(k_2' - A_2), \tag{2.64}$$

respectively.

Note that the periodic plane wave solution is exactly the same as the solution we have here; the only difference is that k_2 is real.

The electromagnetic field for I2a(ii)

The electromagnetic field tensor is given by (2.32). Now by substituting the values for A_2, A_3, A_4 and A_1 we have from (2.62) and (2.64) in equation (2.32), giving

$$\begin{aligned} F_{12} &= \partial_1 A_2, \\ F_{13} &= -\partial_1 A_2, \\ F_{14} &= 0, \\ F_{23} &= -(\partial_2 + \partial_3) A_2, \\ F_{24} &= -\partial_4 A_2, \\ F_{34} &= \partial_4 A_2. \end{aligned}$$

The idea now is find out what the zero current equation (2.33) gives us.

For $\nu = 1$,

$$\partial_1 (\partial_2 + \partial_3) A_2 = 0. \tag{2.65}$$

For $\nu = 2$,

$$(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\, A_1 \; - \partial_2\, (\partial_2 \; + \partial_3\,)\, A_2 \;\; = 0\,. \tag{2.66}$$

For $\nu = 3$,

$$(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\, A_1 \; + \partial_3\, (\partial_2 \; + \partial_3\,)\, A_2 \;\; = 0\,. \tag{2.67}$$

For $\nu = 4$,

$$\partial_4\, (\partial_2 \; + \partial_3\,)\, A_2 \;\; = 0\,. \tag{2.68}$$

Now by subtracting the equations (2.66) and (2.67), we have

$$(\partial_2 \; + \partial_3\,)^2\, A_2 \;\; = 0\,. \tag{2.69}$$

Under the given assumption for A_2 (2.7), by substituting this value for A_2 in equation (2.69), it gives

$$L_2 + \, L_3 \;\; = 0\,. \tag{2.70}$$

By using this last result equation (2.66) becomes

$$L_4^2 - L_1^2 \;=\; 0 \qquad \text{so} \qquad L_4 \;=\; \pm\, L_1\,. \tag{2.71}$$

Finally we could write A_2 as

$$A_2 \;=\; N \,\sin\Big(L_1(x^1 \,\pm\, x^4) \;+\; L_2(x^2 \,-\, x^3) + r \Big)\,. \tag{2.72}$$

Now by substituting this value for A_2 in the electromagnetic field, we get

$$
\begin{aligned}
F_{12} \;&=\; N\, L_1 \,\cos\Big(L_1(x^1 \,\pm\, x^4) \;+\; L_2(x^2 \,-\, x^3) + r \Big)\,, \\
F_{13} \;&=\; -\,N\, L_1 \,\cos\Big(L_1(x^1 \,\pm\, x^4) \;+\; L_2(x^2 \,-\, x^3) + r \Big)\,, \\
F_{14} \;&=\; 0 \quad , \quad F_{14} \;=\; 0\,, \\
F_{24} \;&=\; \mp\, N\, L_1 \,\cos\Big(L_1(x^1 \,\pm\, x^4) \;+\; L_2(x^2 \,-\, x^3) + r \Big)\,, \\
F_{34} \;&=\; \pm\, N\, L_1 \,\cos\Big(L_1(x^1 \,\pm\, x^4) \;+\; L_2(x^2 \,-\, x^3) + r \Big)\,.
\end{aligned}
\tag{2.73}
$$

I2b) <u>When $t = -s + 2n\pi$</u>, for this value of t the equations (2.24) and (2.25) give

$$k_1'' \cos(s) = 0 \quad , \quad k_1'' \sin(s) = 0 , \tag{2.74}$$

respectively, so $k_1'' = 0$. Then the equations (2.17) and (2.19) become

$$k_4'' = -k_2'' \sin(s) \quad , \quad k_3'' = -k_2'' \cos(s) , \tag{2.75}$$

respectively. Similarly equations (2.26) and (2.27) give us

$$k_1' - A_1 = 0 . \tag{2.76}$$

Then by using this last result the equations (2.16) and (2.18) become

$$(k_4' - A_4) = -(k_2' - A_2) \sin(s) , \tag{2.77}$$

$$(k_3' - A_3) = -(k_2' - A_2) \cos(s) , \tag{2.78}$$

respectively.

Note that the periodic plane wave solution is exactly the same as the solution we have here, the only difference is that k_2 is real.

The electromagnetic field for (I2b)

The electromagnetic field tensor is given by (2.32). Now by substituting the values for A_2, A_3, A_4 and A_1 we have from (2.77) and (2.78) in equation (2.32), giving

$$
\begin{aligned}
F_{12} &= \partial_1 A_2 , \\
F_{13} &= -\cos(s)\, \partial_1 A_2 , \\
F_{14} &= -\sin(s)\, \partial_1 A_2 , \\
F_{23} &= -\Big(\cos(s)\, \partial_2 + \partial_3 \Big) A_2 , \\
F_{24} &= -\Big(\sin(s)\, \partial_2 + \partial_4 \Big) A_2 , \\
F_{34} &= \Big(\cos(s)\, \partial_4 - \sin(s)\, \partial_3 \Big) A_2 .
\end{aligned}
$$

The idea now is find out what the zero current equation (2.33) will give us.

For $\nu = 1$,

$$\partial_1 \Big(\partial_2 + \partial_3 \cos(s) + \partial_4 \sin(s) \Big) A_2 = 0 . \tag{2.79}$$

For $\nu = 2$,

$$(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2) A_2 - \\ \partial_2 \Big(\partial_2 + \partial_3 \cos(s) + \partial_4 \sin(s) \Big) A_2 = 0 . \tag{2.80}$$

For $\nu = 3$,

$$\cos(s) (\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2) A_2 + \\ \partial_3 \Big(\partial_2 + \partial_3 \cos(s) + \partial_4 \sin(s) \Big) A_2 = 0 . \tag{2.81}$$

For $\nu = 4$,

$$\sin(s) (\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2) A_2 + \\ \partial_4 \Big(\partial_2 + \partial_3 \cos(s) + \partial_4 \sin(s) \Big) A_2 = 0 . \tag{2.82}$$

The last three equations imply another equation, which is given by

$$\Big(\partial_2 + \partial_3 \cos(s) + \partial_4 \sin(s) \Big)^2 A_2 = 0 . \tag{2.83}$$

Under the given assumption for A_2 (2.7), by substituting this value for A_2 in equation (2.83), it gives

$$L_2 + L_3 \cos(s) + L_4 \sin(s) = 0 . \tag{2.84}$$

By using equation (2.84) in equation (2.80), we get

$$L_1^2 + L_2^2 - L_3^2 - L_4^2 = 0 . \tag{2.85}$$

Now if we substitute equation (2.84) in equation (2.85), it becomes

$$L_1 = \pm \Big(L_4 \cos(s) - L_3 \sin(s) \Big) . \tag{2.86}$$

Finally we could write A_2 as

$$A_2 = N \sin\Big([x^3 \mp x^1 \sin(s) - x^2 \cos(s)]\, L_3 + [x^4 \pm x^1 \cos(s) - x^2 \sin(s)]\, L_4 + r \Big). \tag{2.87}$$

Now by substituting this value for A_2 in the electromagnetic field, we get

$$\begin{aligned}
F_{12} &= \pm N \Big(L_4 \cos(s) - L_3 \sin(s) \Big) \cos(L \cdot X + r), \\
F_{13} &= \mp N \Big(L_4 \cos(s) - L_3 \sin(s) \Big) \cos(s) \cos(L \cdot X + r), \\
F_{14} &= \mp N \Big(L_4 \cos(s) - L_3 \sin(s) \Big) \sin(s) \cos(L \cdot X + r), \\
F_{23} &= N \sin(s) \Big(\cos(s)\, L_4 - \sin(s)\, L_3 \Big) \cos(L \cdot X + r), \\
F_{24} &= -N \Big(L_2 \sin(s) + L_4 \Big) A_2, \\
F_{34} &= N \Big(\cos(s)\, L_4 - \sin(s)\, L_3 \Big) \cos(L \cdot X + r).
\end{aligned} \tag{2.88}$$

2.2.2 The plane wave solutions, case (II) :

In this case we have $\psi_0 = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$ where c and d are complex constants. Then the equations (2.4) and (2.5) become

$$(i\, k_4 - k_3) - \frac{d}{c}(i\, k_1 + k_2) = (i\, A_4 - A_3) - \frac{d}{c}(i\, A_1 + A_2), \tag{2.89}$$

$$\frac{c}{d}(k_2 - i\, k_1) + (k_3 + i\, k_4) = \frac{c}{d}(A_2 - i\, A_1) + (A_3 + i\, A_4), \tag{2.90}$$

respectively. We know that c and d are any non zero complex numbers. We set $c = R_1 e^{i\theta_1}$ and $d = R_2 e^{i\theta_2}$, where R_1 and R_2 are positive real numbers, and θ_1 and θ_2 are real numbers. Define $\lambda = R_1/R_2$ and $\phi = \theta_1 - \theta_2$. By substituting these values for c and d in the equations (2.89) and (2.90), they become

$$(i\,k_4 - k_3) - \frac{1}{\lambda}\,e^{-i\phi}\,(i\,k_1 + k_2) \;=\; (iA_4 - A_3) - \frac{1}{\lambda}\,e^{-i\phi}\,(iA_1 + A_2)\,, \quad (2.91)$$

$$\lambda\,e^{i\phi}\,(k_2 - i\,k_1) + (k_3 + i\,k_4) \;=\; \lambda\,e^{i\phi}\,(A_2 - iA_1) + (A_3 + iA_4)\,. \quad (2.92)$$

Now by multiplying (2.91) by λ^2 and adding the result to (2.92), giving

$$\begin{aligned}(1-\lambda^2)(k_3 - A_3) \;+\; i(1+\lambda^2)(k_4 - A_4) \;+&\\ 2i\lambda\sin\phi(k_2 - A_2) \;-2i\lambda\cos\phi(k_1 - A_1) \;=\; 0\,. &\quad (2.93)\end{aligned}$$

Similarly by multiplying (2.91) by λ^2 and subtracting the result from (2.92), giving

$$\begin{aligned}(1+\lambda^2)(k_3 - A_3) \;+\; i(1-\lambda^2)(k_4 - A_4) \;+&\\ 2\lambda\cos\phi(k_2 - A_2) \;+2\lambda\sin\phi(k_1 - A_1) \;=\; 0\,. &\quad (2.94)\end{aligned}$$

We know that ϕ and A_μ are real for $\mu = 1, \cdots, 4$, and k_j are complex numbers for $j = 1, \cdots, 4$. We set $k_j = k'_j + ik''_j$, so each of the last two equations will spilt into two equations, where one comes from the real part and the other comes from the imaginary part. Let us start with equation (2.93), which give us the following:

$$\begin{aligned}(1-\lambda^2)\,k''_3 \;+\; (1+\lambda^2)\,(k'_4 - A_4) \;+&\\ 2\lambda\,(k'_2 - A_2)\,\sin\phi - \;2\lambda\,(k'_1 - A_1)\,\cos\phi \;=\; 0\,, &\quad (2.95)\end{aligned}$$

$$(1-\lambda^2)(k'_3 - A_3) - (1+\lambda^2)k''_4 \;-\; 2\lambda\,k''_2\,\sin\phi + \;2\lambda\,k''_1\,\cos\phi \;=\; 0\,. \quad (2.96)$$

From equation (2.94),

$$\begin{aligned}(1+\lambda^2)\,(k'_3 - A_3) \;-\; (1-\lambda^2)\,k''_4 \;+&\\ 2\lambda\,(k'_2 - A_2)\,\cos\phi + \;2\lambda\,(k'_1 - A_1)\,\sin\phi \;=\; 0\,, &\quad (2.97)\end{aligned}$$

$$(1-\lambda^2)(k'_4 - A_4) + (1+\lambda^2)k''_3 \;+\; 2\lambda\,k''_2\,\cos\phi + \;2\lambda\,k''_1\,\sin\phi \;=\; 0\,. \quad (2.98)$$

We can solve the last four equations by considering the following:

II1) For $1-\lambda^2 \neq 0$, so we can write the equations (2.96) and (2.98) as

$$(k_3' - A_3) = \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_4'' + \Big(\frac{2\lambda}{1-\lambda^2}\Big) k_2'' \sin\phi - \Big(\frac{2\lambda}{1-\lambda^2}\Big) k_1'' \cos\phi\,, \tag{2.99}$$

$$(k_4' - A_4) = -\Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_3'' - \Big(\frac{2\lambda}{1-\lambda^2}\Big) k_2'' \cos\phi - \Big(\frac{2\lambda}{1-\lambda^2}\Big) k_1'' \sin\phi\,. \tag{2.100}$$

Now by using the equations (2.99) and (2.100) in the equations (2.97) and (2.95) respectively, they become

$$\Big(\frac{2\lambda}{1-\lambda^2}\Big) k_4'' + \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_2'' \sin\phi - \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_1'' \cos\phi + (k_2' - A_2)\cos\phi + (k_1' - A_1)\sin\phi = 0\,, \tag{2.101}$$

$$\Big(\frac{2\lambda}{1-\lambda^2}\Big) k_3'' + \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_2'' \cos\phi + \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_1'' \sin\phi - (k_2' - A_2)\sin\phi + (k_1' - A_1)\cos\phi = 0\,. \tag{2.102}$$

We can simplify the last two equations and that by multiplying (2.101) by $\cos\phi$ and (2.102) by $\sin\phi$, by subtracting the results

$$(k_2' - A_2) = \Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_1'' - \Big(\frac{2\lambda}{1-\lambda^2}\Big) (k_4''\cos\phi - k_3''\sin\phi)\,. \tag{2.103}$$

Similarly by multiplying (2.101) by $\sin\phi$ and (2.102) by $\cos\phi$, by adding the results

$$(k_1' - A_1) = -\Big(\frac{1+\lambda^2}{1-\lambda^2}\Big) k_2'' - \Big(\frac{2\lambda}{1-\lambda^2}\Big) (k_4''\sin\phi + k_3''\cos\phi)\,. \tag{2.104}$$

As a result all the information in this case are given by (2.99), (2.100), (2.103) and (2.104). The vector potential is constant as shown, this means that the electromagnetic field is vanishing.

Note that the periodic plane wave solution is exactly the same as the solution we have here, the only difference is that the wave vector $\underline{k}$ is real.

II2) For $1 - \lambda^2 = 0$, so that $\lambda = \pm 1$. But we know that R_1 and R_2 are both positive and real numbers. This means that the only valid possibility here is that $\lambda = 1$. By substituting this value for λ in the equations (2.95), (2.96), (2.97) and (2.98), they become

$$(k_4' - A_4) + (k_2' - A_2)\sin\phi - (k_1' - A_1)\cos\phi = 0\,, \tag{2.105}$$

$$k_4'' + k_2''\sin\phi - k_1''\cos\phi = 0\,, \tag{2.106}$$

$$(k_3' - A_3) + (k_2' - A_2)\cos\phi + (k_1' - A_1)\sin\phi = 0\,, \tag{2.107}$$

$$k_3'' + k_2''\cos\phi + k_1''\sin\phi = 0\,. \tag{2.108}$$

Note that the periodic plane wave solution is exactly the same as the solution we have here; the only difference is that k_1 and k_2 are real.

The electromagnetic field for case (II2)

The electromagnetic field tensor is given by (2.32). Now by substituting the values for A_3 and A_4 we have from (2.105) and (2.107) in equation (2.32), giving

$$
\begin{aligned}
F_{12} &= \partial_1 A_2 - \partial_2 A_1 \,, \\
F_{13} &= -\cos\phi\partial_1 A_2 - \sin\phi\,\partial_1 A_1 - \partial_3 A_1 \,, \\
F_{14} &= -\sin\phi\,\partial_1 A_2 + \cos\phi\,\partial_1 A_1 - \partial_4 A_1 \,, \\
F_{23} &= -\cos\phi\partial_2 A_2 - \sin\phi\,\partial_2 A_1 - \partial_3 A_2 \,, \\
F_{24} &= -\sin\phi\,\partial_2 A_2 + \cos\phi\,\partial_2 A_1 - \partial_4 A_2 \,, \\
F_{34} &= -\sin\phi\,\partial_3 A_2 + \cos\phi\,\partial_3 A_1 \,\cos\phi\partial_4 A_2 + \sin\phi\,\partial_4 A_1 \,.
\end{aligned}
$$

The idea now is find out what the zero current equation (2.33) will give us.

For $\nu = 1$,

$$
\begin{aligned}
(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 - (\partial_2 + \cos\phi\,\partial_3 + \sin\phi\,\partial_4)\partial_1 A_2 & \\
- (\partial_1 + \sin\phi\,\partial_3 - \cos\phi\,\partial_4)\partial_1 A_1 &= 0 \quad (2.109)
\end{aligned}
$$

For $\nu = 2$,

$$
\begin{aligned}
(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_2 - (\partial_2 + \cos\phi\,\partial_3 + \sin\phi\,\partial_4)\partial_2 A_2 & \\
- (\partial_1 + \sin\phi\,\partial_3 - \cos\phi\,\partial_4)\partial_2 A_1 &= 0 \quad (2.110)
\end{aligned}
$$

For $\nu = 3$,

$$
\begin{aligned}
&\sin\phi((\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 + (\partial_1 + \sin\phi\,\partial_3 - \cos\phi\,\partial_4)\partial_3 A_1 + \\
&\cos\phi(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_2 + (\partial_2 + \cos\phi\,\partial_3 + \sin\phi\,\partial_4)\partial_3 A_2 = 0 \quad (2.111)
\end{aligned}
$$

For $\nu = 4$,

$$
\begin{aligned}
&\sin\phi((\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_2 + (\partial_1 + \sin\phi\,\partial_3 - \cos\phi\,\partial_4)\partial_4 A_1 - \\
&\cos\phi(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)\,A_1 + (\partial_2 + \cos\phi\,\partial_3 + \sin\phi\,\partial_4)\partial_4 A_2 = 0 \quad (2.112)
\end{aligned}
$$

Now by multiplying equation (2.109) by $\sin\phi$ and equation (2.110) by $\cos\phi$, and then subtracting the results from equation (2.111),

$$
\begin{aligned}
&(\partial_2 + \cos\phi\,\partial_3 + \sin\phi\,\partial_4)(\sin\phi\partial_1 + \cos\phi\,\partial_2 + \partial_3)\,A_2 \;+ \\
&(\partial_1 + \sin\phi\,\partial_3 - \cos\phi\,\partial_4)(\sin\phi\partial_1 + \cos\phi\,\partial_2 + \partial_3)\,A_1 \;=\; 0\,. \qquad (2.113)
\end{aligned}
$$

Similarly by multiplying equation (2.109) by $\cos\phi$ and equation (2.110) by $\sin\phi$, and then subtracting the results from equation (2.112),

$$
\begin{aligned}
&(\partial_2 + \cos\phi\,\partial_3 + \sin\phi\,\partial_4)(\cos\phi\partial_1 - \sin\phi\,\partial_2 - \partial_4)\,A_2 \;+ \\
&(\partial_1 + \sin\phi\,\partial_3 - \cos\phi\,\partial_4)(\cos\phi\partial_1 - \sin\phi\,\partial_2 - \partial_4)\,A_1 \;=\; 0\,. \qquad (2.114)
\end{aligned}
$$

In this case A_2 is not determined by A_1, so under the assumptions for A_1 and A_2 given by (2.6) and (2.7) respectively, we find by substituting these values for A_1 and A_2 in the equations (2.113) and (2.114), that

$$
\begin{aligned}
&\Big[N\,\sin\Big(L\cdot X + r\Big)(L_2 + \cos\phi\,L_3 + \sin\phi\,L_4) + M\,\sin\Big(L\cdot X\Big)(L_1 + \\
&\qquad \sin\phi\,L_3 - \cos\phi\,L_4)\Big](\sin\phi L_1 + \cos\phi\,L_2 + L_3) \;= 0\,, \qquad (2.115)
\end{aligned}
$$

$$
\begin{aligned}
&\Big[N\,\sin\Big(L\cdot X + r\Big)(L_2 + \cos\phi\,L_3 + \sin\phi\,L_4) + M\,\sin\Big(L\cdot X\Big)(L_1 + \\
&\qquad \sin\phi\,L_3 - \cos\phi\,L_4)\Big](\cos\phi L_1 - \sin\phi\,L_2 - L_4) \;= 0\,, \qquad (2.116)
\end{aligned}
$$

respectively. The last two equations include two main cases which are :

II2a) If we have the following :

$$
\begin{aligned}
\sin\phi L_1 + \cos\phi\,L_2 + L_3 &= 0\,, \\
\cos\phi L_1 - \sin\phi\,L_2 - L_4 &= 0\,,
\end{aligned}
$$

then we can write A_1 and A_2 as

$$
\begin{aligned}
A_1 &= M\,\sin\Big(x^1\,L_1 + x^2\,L_2 - x^3[\sin\phi L_1 + \cos\phi\,L_2] \\
&\qquad +x^4[\cos\phi L_1 - \sin\phi\,L_2]\Big)\,, \\
A_2 &= N\,\sin\Big(x^1\,L_1 + x^2\,L_2 - x^3[\sin\phi L_1 + \cos\phi\,L_2] \\
&\qquad +x^4[\cos\phi L_1 - \sin\phi\,L_2] + r\Big)\,. \qquad (2.117)
\end{aligned}
$$

The electromagnetic field is given by

$$
\begin{aligned}
F_{12} &= N\cos\big(L\cdot X + r\big)\,L_1 - M\cos\big(L\cdot X\big)\,L_2\,,\\
F_{13} &= -\cos\phi\big[N\cos\big(L\cdot X + r\big)\,L_1 - M\cos\big(L\cdot X\big)\,L_2\big]\,,\\
F_{14} &= \sin\phi\big[N\cos\big(L\cdot X + r\big)\,L_1 - M\cos\big(L\cdot X\big)\,L_2\big]\,,\\
F_{23} &= \sin\phi\big[N\cos\big(L\cdot X + r\big)\,L_1 - M\cos\big(L\cdot X\big)\,L_2\big]\,,\\
F_{24} &= \cos\phi\,\big[N\cos\big(L\cdot X + r\big)\,L_1 - M\cos\big(L\cdot X\big)\,L_2\big]\,,\\
F_{34} &= N\cos\big(L\cdot X + r\big)\,L_1 - M\cos\big(L\cdot X\big)\,L_2\,. \qquad (2.118)
\end{aligned}
$$

II2b) If we have

$$
\begin{aligned}
\sin\big(L\cdot X\big)\big[N\cos(r)(L_2 + \cos\phi\,L_3 + \sin\phi\,L_4) + {}&\\
M\,(L_1 + \sin\phi\,L_3 - \cos\phi\,L_4)\big] + {}&\\
\cos\big(L\cdot X\big)\big[N\sin(r)(L_2 + \cos\phi\,L_3 + \sin\phi\,L_4)\big] &= 0\,.
\end{aligned}
$$

The last equation can be written as two equations as

$$
\begin{aligned}
N\cos(r)(L_2 + \cos\phi\,L_3 + \sin\phi\,L_4) + {}&\\
M\,(L_1 + \sin\phi\,L_3 - \cos\phi\,L_4) &= 0\,, \qquad (2.119)\\
N\sin(r)(L_2 + \cos\phi\,L_3 + \sin\phi\,L_4) &= 0\,. \qquad (2.120)
\end{aligned}
$$

By multiplying equation (2.119) by $\sin(r)$ and equation (2.120) by $\cos(r)$, and then subtracting the results,

$$
M\sin(r)\,(L_1 + \sin\phi\,L_3 - \cos\phi\,L_4) = 0 \qquad (2.121)
$$

There are many possibilities included in the equations (2.120) and (2.121). Next we will consider the most general possibilities, which include the others as special cases.

II2b(i)) If we have

$$\begin{aligned} L_2 &= -(\cos\phi\, L_3 + \sin\phi\, L_4) \quad \text{and} \\ L_1 &= \cos\phi\, L_4 - \sin\phi\, L_3\,, \end{aligned}$$

then we can write A_1 and A_2 as

$$\begin{aligned} A_1 &= M \sin\Big(x^1(\cos\phi\, L_4 - \sin\phi\, L_3) - x^2(\cos\phi\, L_3 + \sin\phi\, L_4) \\ &\quad + x^3 L_3 + x^4 L_4\Big)\,, \\ A_2 &= N \sin\Big(x^1(\cos\phi\, L_4 - \sin\phi\, L_3) - x^2(\cos\phi\, L_3 + \sin\phi\, L_4) \\ &\quad + x^3 L_3 + x^4 L_4 + r\Big)\,. \end{aligned} \tag{2.122}$$

The electromagnetic field is given by

$$\begin{aligned} F_{12} &= N \cos\Big(L\cdot X + r\Big)(\cos\phi\, L_4 - \sin\phi\, L_3 \\ &\quad + M \cos\Big(L\cdot X\Big)(\cos\phi\, L_3 + \sin\phi\, L_4)\,, \\ F_{13} &= -\cos\phi\Big[N \cos\Big(L\cdot X + r\Big)(\cos\phi\, L_4 - \sin\phi\, L_3 \\ &\quad + M \cos\Big(L\cdot X\Big)(\cos\phi\, L_3 + \sin\phi\, L_4)\Big]\,, \\ F_{14} &= -\sin\phi\Big[N \cos\Big(L\cdot X + r\Big)(\cos\phi\, L_4 - \sin\phi\, L_3 \\ &\quad + M \cos\Big(L\cdot X\Big)(\cos\phi\, L_3 + \sin\phi\, L_4)\Big]\,, \\ F_{23} &= \sin\phi\Big[N \cos\Big(L\cdot X + r\Big)(\cos\phi\, L_4 - \sin\phi\, L_3 \\ &\quad + M \cos\Big(L\cdot X\Big)(\cos\phi\, L_3 + \sin\phi\, L_4)\Big]\,, \end{aligned}$$

$$F_{24} = -\cos\phi\Big[N\cos\Big(L\cdot X + r\Big)(\cos\phi\, L_4 - \sin\phi\, L_3 + M\cos\Big(L\cdot X\Big)(\cos\phi\, L_3 + \sin\phi\, L_4)\Big],$$

$$F_{34} = N\cos\Big(L\cdot X + r\Big)(\cos\phi\, L_4 - \sin\phi\, L_3 + M\cos\Big(L\cdot X\Big)(\cos\phi\, L_3 + \sin\phi\, L_4). \qquad (2.123)$$

II2b(ii)) If $\sin(r) = 0$,

this means that $r = n\pi$, and then equation (2.119) becomes

$$M(L_1 + \sin\phi\, L_3 - \cos\phi\, L_4) \pm N(L_2 + \cos\phi\, L_3 + \sin\phi\, L_4) = 0.$$

By allowing N to be positive or negative, we can write the last equation as

$$N = -M\frac{(L_1 + \sin\phi\, L_3 - \cos\phi\, L_4)}{(L_2 + \cos\phi\, L_3 + \sin\phi\, L_4)}. \qquad (2.124)$$

Now by substituting this value for N in equation (2.109), it becomes

$$L_4 = \pm\sqrt{L_1^2 + L_2^2 - L_3^2}. \qquad (2.125)$$

Then we can write A_1 and A_2 as

$$A_1 = M\sin\Big(x^1 L_1 + x^2 L_2 + x^3 L_3 \pm x^4\sqrt{L_1^2 + L_2^2 - L_3^2}\Big),$$

$$A_2 = -M\frac{(L_1 + \sin\phi\, L_3 - \cos\phi\, L_4)}{(L_2 + \cos\phi\, L_3 + \sin\phi\, L_4)}\sin\Big(x^1 L_1 + x^2 L_2 + x^3 L_3 \pm x^4\sqrt{L_1^2 + L_2^2 - L_3^2}\Big). \qquad (2.126)$$

The electromagnetic field is given by

$$
\begin{aligned}
F_{12} &= -M\cos\big(L\cdot X\big)\big[L_2+L_1\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\big]\\
F_{13} &= M\cos\big(L\cdot X\big)\big[L_1\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\\
&\quad -L_1\sin\phi-L_3\big],\\
F_{14} &= M\cos\big(L\cdot X\big)\big[L_1\sin\phi\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\\
&\quad +L_1\cos\phi-L_4\big],\\
F_{23} &= M\cos\big(L\cdot X\big)\big[L_2\cos\phi\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\\
&\quad -L_2\sin\phi+L_3\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\big],\\
F_{24} &= M\cos\big(L\cdot X\big)\big[L_2\sin\phi\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\\
&\quad +L_2\cos\phi+L_4\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\big],\\
F_{34} &= M\cos\big(L\cdot X\big)\big[L_3\sin\phi\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\\
&\quad -L_4\cos\phi\big(\frac{L_1+\sin\phi\,L_3-\cos\phi\,L_4}{L_2+\cos\phi\,L_3+\sin\phi\,L_4}\big)\\
&\quad +L_3\cos\phi+L_4\sin\phi\big],
\end{aligned}
\tag{2.127}
$$

where the vector L satisfies

$$L_4 \;=\; \pm\sqrt{L_1^2+L_2^2-L_3^2}\,.$$

II2b(iii)) <u>If $M = N = 0$</u>,

then A_1 and A_2 are vanishing. This means that A_3 and A_4 are constants, given by

$$A_3 = k_3' + k_2' \cos\phi + k_1' \sin\phi,$$

$$A_4 = k_4' + k_2' \sin\phi - k_1' \cos\phi. \qquad (2.128)$$

The electromagnetic field is vanishing.

2.3 Superposition of the currentless spinors

If we had a linear equation for ψ, we could super-impose solutions to get new solutions. The complication is that the spinors are connected back to the electromagnetic field by the equation for the current. This means that the whole system of the spinors and the electromagnetic field is nonlinear. Even in the zero current case, the zero current spinors do not form a vector space, so it is not possible to add any two solutions of the Dirac equation and get another zero current spinor solution. The idea in this section is to try to see when a limited version of superposition does hold, that is when the sum of zero current spinors is still zero current. We consider the case of adding currenless constants spinors, and find the conditions for the sum to be zero current. For the case of functions, we can do the superposition provided the values of the two fields at any given point satisfy the conditions.

In general if $\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, then the conditions for ψ to be currentless are given by

$$\overline{a}\,d + \overline{c}\,b = 0 \quad , \quad \overline{a}\,c + \overline{d}\,b = 0. \qquad (2.129)$$

Proposition (2.2):

The cases where the sum of two currentless spinors is another currentless spinor are given by the following:

Case I):

Consider the following addition

$$\begin{pmatrix} a_1 \\ a_1\, e^{is_1} \\ i\, a_1\, r_1\, e^{\frac{i}{2}(s_1-t_1)} \\ i\, a_1\, r_1\, e^{\frac{i}{2}(s_1+t_1)} \end{pmatrix} + \begin{pmatrix} a_2 \\ a_2\, e^{is_2} \\ i\, a_2\, r_2\, e^{\frac{i}{2}(s_2-t_2)} \\ i\, a_2\, r_2\, e^{\frac{i}{2}(s_2+t_2)} \end{pmatrix},$$

where s_i and t_i are real constants, r_i is a non zero real constant, and a_i is a complex constant for $i = 1, 2$. This spinor will be currentless and satisfy the conditions (2.129), if and only if

I1) For $r_2\, e^{\frac{i}{2}(s_2+t_2)} - r_1\, e^{i(s_2-\frac{1}{2}(s_1-t_1)} \neq 0$, we have

$$a_2\, \overline{a_1} \;=\; -\,\overline{a_2}\, a_1\, \frac{\left(r_1\, e^{\frac{i}{2}(s_1+t_1)} - r_2\, e^{i(s_1-\frac{1}{2}(s_2-t_2)}\right)}{\left(r_2\, e^{\frac{i}{2}(s_2+t_2)} - r_1\, e^{i(s_2-\frac{1}{2}(s_1-t_1)}\right)}\,.$$

I2) For $r_2\, e^{\frac{i}{2}(s_2+t_2)} - r_1\, e^{i(s_2-\frac{1}{2}(s_1-t_1)} = 0$, we set $a_2 = a_1\, R\, e^{i\lambda}$, where $R > 0$ and $\lambda \in \mathbb{R}$. There are two possibilities:

I2a)

$$\begin{aligned} \lambda &= \tfrac{1}{2}\,(s_1 - s_2)\,, \\ s_2 - s_1 - t_1 - t_2 &= 2n\pi\,,\ n \in \mathbb{N}\,, \\ r_2 &= \pm\, r_1\,. \end{aligned}$$

I2b)

$$\begin{aligned} s_2 &= s_1 + 2n\pi\,,\ n \in \mathbb{N}\,, \\ t_2 &= t_1 + 2m\pi\,,\ m \in \mathbb{N}\,, \\ r_2 &= \pm\, r_1\,. \end{aligned}$$

Case II):

II1) Add any spinor of the form $\begin{pmatrix} 1 \\ e^{is} \\ ir\, e^{\frac{i}{2}(s-t)} \\ ir\, e^{\frac{i}{2}(s+t)} \end{pmatrix} a$, where s and t are real constants, r is a non zero real constant, and a is a complex constant, to any spinor of the form $\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, where c and d are complex constants. Then the resultant spinor will be a currentless spinor and satisfy the conditions (2.129), if and only if

$$d = c\,e^{i\lambda} \quad , \quad c = a\,R\,e^{i\theta} \quad \text{and} \quad \theta = \frac{1}{2}\,(s - \lambda \pm \pi)\,,$$

where R, λ and θ are real constants, and $R > 0$.

II2) Add any spinor of the form $\begin{pmatrix} 1 \\ e^{is} \\ ir\,e^{\frac{i}{2}(s-t)} \\ ir\,e^{\frac{i}{2}(s+t)} \end{pmatrix} a$, where s and t are real constants, r is a non zero real constant, and a is a complex constant, to any spinor of the form $\begin{pmatrix} c \\ d \\ 0 \\ 0 \end{pmatrix}$, where c and d are complex numbers. Then the resultant spinor will be a currentless spinor and satisfy the conditions (2.129), if and only if

$$a = d\,R\,e^{i\lambda} \quad , \quad d = c\,L\,e^{i\theta} \quad \text{and} \quad L = \pm R\,,$$
$$\lambda + s + \theta = n\pi\,, \qquad n \in \mathbb{N}\,,$$

where R, λ and θ are real constants, and $R > 0$.

Case III):

III1) If we add any two spinors of the form $\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$, or of the form $\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, then the resultant spinor will be trivially currentless, where a, b, c and d are any complex constants.

III2) If we add any spinor of the form $\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$ to any spinor of the form $\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, where a, b, c and d are any complex constants, then the resultant spinor will be a currentless spinor and satisfy the conditions (2.129), if and only if

III2a) $c = d = 0,$

III2b) $a = b = 0,$

III2c) $b = a\,e^{is} \quad , \quad c = a\,ir\,e^{\frac{i}{2}(s-t)} \quad , \quad d = a\,ir\,e^{\frac{i}{2}(s+t)},$

where t and s are real constants, r is a non zero real constant.

From now until the end of this section we are going to prove the superposition of the currentless spinors for each case.

2.3.1 Superposition of currentless spinors, case(I)

Let ψ be of the form $\begin{pmatrix} 1 \\ e^{is} \\ ir\,e^{\frac{i}{2}(s-t)} \\ ir\,e^{\frac{i}{2}(s+t)} \end{pmatrix} a$, where s and t are real constants and r is a non zero constant, and a is any complex constant. Then we can add any two spinors of this form to get

$$\begin{pmatrix} 1 \\ e^{is_1} \\ ir_1\,e^{\frac{i}{2}(s_1-t_1)} \\ ir_1\,e^{\frac{i}{2}(s_1+t_1)} \end{pmatrix} a_1 + \begin{pmatrix} 1 \\ e^{is_2} \\ ir_2\,e^{\frac{i}{2}(s_2-t_2)} \\ ir_2\,e^{\frac{i}{2}(s_2+t_2)} \end{pmatrix} a_2$$

$$= \begin{pmatrix} a_1 + a_2 \\ a_1 e^{is_1} + a_2 e^{is_2} \\ i(a_1 \, r_1 \, e^{\frac{i}{2}(s_1 - t_1)} + a_2 \, r_2 \, e^{\frac{i}{2}(s_2 - t_2)}) \\ i(a_1 \, r_1 \, e^{\frac{i}{2}(s_1 + t_1)} + a_2 \, r_2 \, e^{\frac{i}{2}(s_2 + t_2)}) \end{pmatrix}.$$

Now by substituting this resultant spinor in the two conditions given by (2.129), giving

$$a_2 \, \overline{a_1} \left(r_2 \, e^{\frac{i}{2}(s_2 + t_2)} - r_1 \, e^{i(s_2 - \frac{1}{2}(s_1 - t_1))} \right) = - \overline{a_2} \, a_1 \left(r_1 \, e^{\frac{i}{2}(s_1 + t_1)} - r_2 \, e^{i(s_1 - \frac{1}{2}(s_2 - t_2))} \right), \quad (2.130)$$

$$a_2 \, \overline{a_1} \left(r_2 \, e^{\frac{i}{2}(s_2 - t_2)} - r_1 \, e^{i(s_2 - \frac{1}{2}(s_1 + t_1))} \right) = - \overline{a_2} \, a_1 \left(r_1 \, e^{\frac{i}{2}(s_1 - t_1)} - r_2 \, e^{i(s_1 - \frac{1}{2}(s_2 + t_2))} \right), \quad (2.131)$$

respectively. The last two equations contain two possibilities, given by

I1) <u>If $r_2 \, e^{\frac{i}{2}(s_2 + t_2)} - r_1 \, e^{i(s_2 - \frac{1}{2}(s_1 - t_1)} \neq 0$,</u>

Then we could write (2.130) as

$$a_2 \, \overline{a_1} = - \overline{a_2} \, a_1 \left(\frac{r_1 \, e^{\frac{i}{2}(s_1 + t_1)} - r_2 \, e^{i(s_1 - \frac{1}{2}(s_2 - t_2)}}{r_2 \, e^{\frac{i}{2}(s_2 + t_2)} - r_1 \, e^{i(s_2 - \frac{1}{2}(s_1 - t_1)}} \right). \quad (2.132)$$

This value for $a_2 \, \overline{a_1}$, also satisfies equation (2.131).

I2) <u>If $r_2 \, e^{\frac{i}{2}(s_2 + t_2)} - r_1 \, e^{i(s_2 - \frac{1}{2}(s_1 - t_1)} = 0$,</u>

then we have

$$r_2 = r_1 \, e^{\frac{i}{2}(s_2 - s_1 - t_2 + t_1)}. \quad (2.133)$$

But since r_1 and r_2 are non zero real constants, then

$$\frac{1}{2} \left(s_2 - s_1 - t_2 + t_1 \right) = n\pi, \quad \text{where } n \in \mathbb{N}. \quad (2.134)$$

Now by using (2.133) equation (2.131) becomes

$$\frac{a_2}{a_1} \left(e^{i(s_2 - t_2 - \frac{1}{2}(s_1 - t_1)} - e^{i(s_2 - \frac{1}{2}(s_1 + t_1)} \right) = - \frac{\overline{a_2}}{\overline{a_1}} \left(e^{\frac{i}{2}(s_1 - t_1)} - e^{i(\frac{1}{2}(s_1 + t_1) - t_2)} \right). \quad (2.135)$$

If we set $a_2/a_1 = R\,e^{i\lambda}$, where $R > 0$ and λ is real, then we can write equation (2.135) as

$$e^{2i\lambda}\left(e^{i(s_2-t_2-\frac{1}{2}(s_1-t_1)} - e^{i(s_2-\frac{1}{2}(s_1+t_1))}\right) = \left(e^{i(\frac{1}{2}(s_1+t_1)-t_2)} - e^{\frac{i}{2}(s_1-t_1)}\right).$$

By multiplying both sides in the last equation by $e^{i(\frac{s_1-t_1}{2})}$, it becomes

$$e^{2i\lambda}\left(e^{i(s_2-t_2)} - e^{i(s_2-t_1)}\right) = \left(e^{i(s_1-t_2)} - e^{i(s_1-t_1)}\right).$$

Now by using (2.134) the last equation becomes

$$e^{2i\lambda}\left(e^{i(s_1-t_1)} - e^{i(s_2-t_1)}\right) = \left(e^{i(s_1-t_2)} - e^{i(s_1-t_1)}\right),$$

and on multiplying both sides by $e^{i(t_1-s_1)}$,

$$e^{2i\lambda}\left(1 - e^{i(s_2-s_1)}\right) = \left(e^{i(t_1-t_2)} - 1\right).$$

Again by using (2.134) the last equation becomes

$$e^{2i\lambda}\left(1 - e^{i(s_2-s_1)}\right) = \left(e^{i(s_1-s_2)} - 1\right). \qquad (2.136)$$

There are two solutions for equation (2.136) which we denote by *I2a*) and *I2b*) :

$$\textbf{I2a)} \qquad \lambda = \frac{1}{2}(s_1 - s_2). \qquad (2.137)$$

Then from equation (2.134),

$$t_1 - t_2 = 2\lambda + 2n\pi, \qquad (2.138)$$

where $n \in \mathbb{N}$. Finally from equation (2.133),

$$r_2 = \pm r_1. \qquad (2.139)$$

Therefore the resultant spinor can be written as, remembering that we allowed r_1 to be positive or negative :

$$\begin{pmatrix} \left(1 + R\,e^{\frac{i}{2}(s_1-s_2)}\right) \\ e^{is_1}\left(1 + R\,e^{\frac{i}{2}(s_2-s_1)}\right) \\ ir_1\,e^{\frac{i}{2}(s_1-t_1)}\left(1 + R\,e^{\frac{i}{2}(s_1-s_2)}\right) \\ ir_1\,e^{\frac{i}{2}(s_1+t_1)}\left(1 + R\,e^{\frac{i}{2}(s_2-s_1)}\right) \end{pmatrix} a_1, \qquad (2.140)$$

where s_i and t_i are real constants for $i = 1, 2$, R and r_1 are non zero real constants, and a_1 is a complex constant.

$$\textbf{I2b)} \qquad s_2 \;=\; s_1 + 2n\pi\,, \tag{2.141}$$

where $n \in \mathbb{N}$. Then from equation (2.134),

$$t_2 \;=\; t_1 + 2m\pi\,. \tag{2.142}$$

Similarly from equation (2.133),

$$r_2 \;=\; \pm r_1\,. \tag{2.143}$$

Therefore the result can be written as, remembering that we allowed r_1 to be positive or negative :

$$\begin{pmatrix} 1 \\ e^{is_1} \\ ir_1\, e^{\frac{i}{2}(s_1-t_1)} \\ ir_1\, e^{\frac{i}{2}(s_1+t_1)} \end{pmatrix} \Big(1 + R e^{i\lambda}\Big) a_1\,, \tag{2.144}$$

where s_1 and t_1 are real constants, R and r_1 are non zero real constants, and a_1 is a complex constant.

2.3.2 Superposition of currentless spinors, case(II)

In this part we are going to add two different forms of currentless spinors.

II1) We add any spinor of the form $\begin{pmatrix} 1 \\ e^{is} \\ ir\, e^{\frac{i}{2}(s-t)} \\ ir\, e^{\frac{i}{2}(s+t)} \end{pmatrix} a$, where s and t are real constants, r is a non zero constant, and a is any complex constant, to any spinor of the form $\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, where c and d are any complex numbers. Then the resultant spinor will be a currentless spinor, if it satisfies the conditions given by (2.129). Then

$$\overline{a}\, d \;=\; -\, e^{is}\, a\, \overline{c}\,, \tag{2.145}$$

$$\bar{a}\,c \;=\; -\,e^{is}\,a\,\bar{d}\,. \tag{2.146}$$

By multiplying equation (2.145) by $\bar{d}$ and equation (2.146) by $\bar{c}$, subtracting the results

$$c\,\bar{c} \;=\; d\,\bar{d}\,. \tag{2.147}$$

We set $d = e^{i\lambda}\,c$, where $\lambda \in \mathbb{R}$. Then by substituting this value for d in equation (2.145), it becomes

$$\bar{a}\,c\,e^{i\lambda} \;=\; -\,e^{is}\,a\,\bar{c}\,. \tag{2.148}$$

Similarly we set $c = a\,R\,e^{i\theta}$, where $R > 0$ and $\theta \in \mathbb{R}$. Now by substituting this value for c in the equation (2.148), it becomes

$$e^{i\theta} \;=\; -\,e^{-i\theta}\,e^{i(s-\lambda)}\,,$$

and we can write this last equation as

$$e^{i\theta} \;=\; e^{i(s-\lambda+\pi-\theta+2n\pi)} \qquad (n \in \mathbb{N})\,. \tag{2.149}$$

By comparing both sides in the last equation,

$$\theta \;=\; (2\,n+1)\,\frac{\pi}{2} \;+\; \frac{s-\lambda}{2}\,. \tag{2.150}$$

Now by allowing R to be positive or negative, we can write the values for c and d as

$$\begin{aligned} c &= i\,R\,a\,e^{\frac{i}{2}(s-\lambda)}\,, \\ d &= i\,R\,a\,e^{\frac{i}{2}(s+\lambda)}\,. \end{aligned} \tag{2.151}$$

Finally the resultant spinor can be written as

$$\begin{pmatrix} 1 \\ e^{is} \\ i(r\,e^{\frac{i}{2}(s-t)} \;+\; R\,e^{\frac{i}{2}(s-\lambda)}) \\ i(r\,e^{\frac{i}{2}(s+t)} \;+\; R\,e^{\frac{i}{2}(s+\lambda)}) \end{pmatrix} a\,, \tag{2.152}$$

where s and t are real constants, R and r are non zero real constants, and a is a complex constant.

II2) We add any spinor of the form $\begin{pmatrix} 1 \\ e^{is} \\ ir\, e^{\frac{i}{2}(s-t)} \\ ir\, e^{\frac{i}{2}(s+t)} \end{pmatrix} a$, where s and t are real constants, r is a non zero constant, and a is any complex number, to any spinor of the form $\begin{pmatrix} c \\ d \\ 0 \\ 0 \end{pmatrix}$, where c and d are complex. Then the resultant spinor will be a currentless spinor, if it satisfies the conditions given by (2.129). Now by using the resultant spinor in the two conditions given by (2.129), both of them will give just one equation which is

$$\overline{a}\, d \;=\; e^{is}\, a\, \overline{c}\,. \tag{2.153}$$

We set

$$a \;=\; d\, R\, e^{i\lambda} \quad \text{and} \quad a \;=\; c\, L\, e^{ih}\,, \tag{2.154}$$

where λ and h are real constants, and R and L are non zero constants. Now by substituting these in equation (2.155), it becomes

$$L \;=\; R\, e^{i(\lambda+s+h)}\,. \tag{2.155}$$

Since R and L are non zero real constants, we have

$$\lambda + s + h \;=\; n\pi \qquad (n \in \mathbb{N})\,, \tag{2.156}$$

so $L \;=\; \pm R$. Finally we can write the resultant spinor as

$$\begin{pmatrix} (\, L\, e^{ih} + 1\,) \\ e^{i(s+h)}(\, L + e^{ih}\,) \\ ir\, L\, e^{i(h+\frac{1}{2}(s-t))} \\ ir\, L\, e^{i(h+\frac{1}{2}(s+t))} \end{pmatrix} c\,, \tag{2.157}$$

where L, h and s are real constants, r is a non zero real constant, and c is any complex constant.

2.3.3 Superposition of currentless spinors, case(III)

III1) We add any two spinors of the form $\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$ or two of the form $\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, where a, b, c and d are any complex constants. Then the resultant spinor will be trivially currentless.

III2) We add any spinor of the form $\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$ to any spinor of the form $\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, where a, b, c and d are any complex constants. In fact the resultant spinor here is just a general spinor $\psi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, and we proved in chapter(1) that this spinor is a currentless spinor and satisfy the conditions (2.129), if and only if

III2a) $c = d = 0$,

III2b) $a = b = 0$,

III2c) $b = a\,e^{is} \quad , \quad c = a\,ir\,e^{\frac{i}{2}(s-t)} \quad , \quad d = a\,ir\,e^{\frac{i}{2}(s+t)}$,

where t and s are real constants, and r is a non zero constant.

Chapter 3

Lorentz transformations and world lines

3.1 Introduction

In this chapter we find the solutions of the massless Dirac equation in $2+2$ dimensions with certain symmetries.
In section 3.2 we consider the action of the Lorentz group on the spinors in $\mathbb{R}^{2+2}$, and then we identify the Lie algebra of this group. Then we give the Lie algebra representation described in [9] in the section on relativistic covariance (p51-52).
In section 3.3 we find those elements of Lie algebra of the group $SO(2,2)$ that fix the lightlike line $(t,0,t,0)$ for $t \in \mathbb{R}$, and then we write the corresponding symmetry equations in this case. We are looking for light speed particles traveling along the line $(t,0,t,0)$, so we will postulate that $(\partial_1+\partial_3)\psi = iK\psi$, where K is 4×4 matrix that takes the spinors to the tangent space of the spinors. Note that the spinors we are dealing with are the zero current spinors we got from chapter 1. Next in this section we consider the action of the Lorentz group on the vector potential, and then we write the corresponding symmetry equations for the vector potential. At this stage we move to the electromagnetic field, and find the zero current equations ($\partial_\mu F^{\mu\nu} = 0$). Now we use all the information we found previously in this section to write the massless Dirac equation in $2+2$ dimensions with potential A_μ. Finally in this section we find the solution for the lightlike case, and the corresponding vector potential for the electromagnetic field.

Similarly in section 3.4 we find those elements of Lie algebra of the group $SO(2,2)$ that fix the timelike line $(t,0,0,0)$ for $t \in \mathbb{R}$, and then we write the corresponding symmetry equations in this case. Here we look for a slower than light (massive) particle. Next we consider the action of the Lorentz group on the vector potential, and then we write the corresponding symmetry equations for the vector potential. Similarly at this point we want to study the electromagnetic field, and to find the zero current equations. Now we use all the information we found previously in this section to write the massless Dirac equation in $2+2$ dimensions with potential A_μ. Finally in this section we find the solution for the timelike case, and the corresponding vector potential for the electromagnetic field.

3.2 The action of Lorentz group on spinors

We take the Lorentz group to act on vectors in $\mathbb{R}^{2+2}$ by the formula

$$x^a \longmapsto \Lambda^a{}_b\, x^b\,, \tag{3.1}$$

where $\mathbb{R}^{2+2}$ is just $\mathbb{R}^4$ with the metric $g_{ab} = \mathrm{diag}(+1,+1,-1,-1)$. The condition that the metric g_{ab} is preserved is given by

$$g_{ab}\,\Lambda^a{}_c\,\Lambda^b{}_d = g_{cd}\,. \tag{3.2}$$

If we move to the Lie algebra, we write

$$\Lambda^a{}_b = g^a{}_b + h\,\omega^a{}_b\,,$$

where $g^a{}_b$ is the identity matrix, h is a small parameter, and ω is an element of Lie algebra. By using this in the metric preserving condition, we get

$$\omega_{ab} = -\,\omega_{ba}\,. \tag{3.3}$$

Thus, the Lie algebra consists of anti-symmetric matrices.
The double cover of the Lorentz group acts on the Dirac spinors in the following way

$$(\omega)(\psi) = \frac{1}{8}\,\omega_{ab}\,.\,[\gamma^a,\gamma^b]\,.\,\psi\,, \tag{3.4}$$

where we only give the Lie algebra representation.
The action of the double cover of Lorentz group on spinor valued functions in $\mathbb{R}^{2+2}$ is given by

$$(\Lambda\,\psi)(X) = \Lambda(\psi(\Lambda^{-1}X))\,. \tag{3.5}$$

To show that this is an action, let us take another element Λ', and

$$\begin{aligned}(\Lambda'(\Lambda\,\psi))(X) &= \Lambda'((\Lambda\psi)(\Lambda'^{-1}X)) \\ &= \Lambda'\,\Lambda\,\psi(\Lambda^{-1}\Lambda'^{-1}X)) \\ &= \Lambda'\,\Lambda\,\psi((\Lambda'\Lambda)^{-1}X) \\ &= ((\Lambda'\,\Lambda)\,\psi)(X)\,,\end{aligned}$$

as required. To look for solutions with certain symmetries means specifying a subgroup of the Lorentz transformations, and looking for spinor valued functions ψ which are unchanged or invariant under these Lorentz transformations, that is

$$(\Lambda\,\psi)\,(X) = \psi(X)\,.$$

If we move to the Lie algebra,

$$\Lambda = 1 + h\,\omega\,,$$

and

$$\Lambda^{-1} = 1 - h\,\omega\,.$$

For Λ in the subgroup,

$$\Lambda(\psi(\Lambda^{-1}X)) = \psi(X)\,,$$

and by differentiating this equation with respect to h,

$$\omega\,\psi(X) - \psi'(X;\omega\,X) = 0\,,$$

or

$$\omega\,\psi(X) = \psi'(X;\omega\,X)\,.$$

By using equation (3.4) the last equation becomes

$$\psi'(X;\omega X) = \frac{1}{8}\omega_{ab} \cdot [\gamma^a, \gamma^b] \cdot \psi \,. \tag{3.6}$$

To consider solutions which might look like particles, we consider the subgroups which preserve a lightlike line (for massless particles) and a timelike line (for massive particles).

3.3 The lightlike line

The line $(t, 0, t, 0)$ for t real is a lightlike line. Now we are in position to look for the elements of the Lie algebra of the group $SO(2,2)$ that fix this line. There are two linearly independent elements which fix the lightlike line, and they are given by (i) and (ii) below :

i) $\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}$, and by acting by the metric g^{ab}, and

$$\omega^c{}_b = g^{ca}\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}.$$

We can get the Lie group elements that fix the lightlike line from these Lie algebra elements by using the exponential map

$$Exp(\omega^c{}_b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(\beta) & 0 & \sinh(\beta) \\ 0 & 0 & 1 & 0 \\ 0 & \sinh(\beta) & 0 & \cosh(\beta) \end{pmatrix}.$$

We see that this matrix fixes x^1 and x^3, and that x^2 and x^4 move along a hyperbola.

Now $\omega^c{}_b$ acts on a point X like this,

$$\omega^c{}_b \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 0 \\ x^4\beta \\ 0 \\ x^2\beta \end{pmatrix} .$$

The differential of ψ in the direction ωX is given by

$$\psi'(X; \omega^c{}_b X) = \beta(x^4 \frac{\partial\psi}{\partial x^2} + x^2 \frac{\partial\psi}{\partial x^4}),$$

and by substituting this result in equation (3.6), the first symmetry equation in this case is given by

$$x^4 \frac{\partial\psi}{\partial x^2} + x^2 \frac{\partial\psi}{\partial x^4} = \frac{1}{2}\gamma^2\gamma^4 \cdot \psi. \tag{3.7}$$

ii) $\omega_{ab} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and

$$\omega^c{}_b = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

The Lie group element here is given by

$$Exp(\omega^c{}_b) = \begin{pmatrix} \cosh(\alpha) & 0 & \sinh(\alpha) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(\alpha) & 0 & \cosh(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\omega^c{}_b$ acts on X like this,

$$\omega^c{}_b \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} x^3\alpha \\ 0 \\ x^1\alpha \\ 0 \end{pmatrix} .$$

The differential of ψ in the direction ωX is given by

$$\psi'(X; {\omega^c}_b X) = \alpha(x^3 \frac{\partial\psi}{\partial x^1} + x^1 \frac{\partial\psi}{\partial x^3}) .$$

Now by using equation (3.6), the second symmetry equation in this case is given by

$$x^3 \frac{\partial\psi}{\partial x^1} + x^1 \frac{\partial\psi}{\partial x^3} = \frac{1}{2}\gamma^1\gamma^3\psi . \qquad (3.8)$$

To look for a 'particle' traveling on the line $(t, 0, t, 0)$, we suppose that we have a particularly simple equation for translation along the line, and postulate that

$$\frac{\partial\psi}{\partial x^1} + \frac{\partial\psi}{\partial x^3} = iK\psi , \qquad (3.9)$$

where K is a 4×4 matrix whose entries are functions of x^1 and x^3 only. There are three symmetry equations (3.7), (3.8) and (3.9).

3.3.1 The K matrix and currentless spinors

In this part we want to find the general form for the matrix K which takes the currentless spinors to the tangent space of the currentless spinors. We find K so that $\psi + i\,K\,\psi\,h + O(h^2)$ is a currentless spinor for all currentless spinors ψ, i.e.

$$\psi' = i\,K.\psi . \qquad (3.10)$$

is in the tangent space to the currentless spinors for all ψ.

Let us take one of the zero current spinors, $\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,re^{\frac{i}{2}(s-t)} \\ i\,re^{\frac{i}{2}(s+t)} \end{pmatrix} a$, where s and t are real functions, r is a non zero real function, and a is a complex function. We consider a variation in ψ, by introducing a rate of change of a, r, s and t with respect to a parameter. By using this in (3.10),

$$\begin{pmatrix} 1 \\ e^{is} \\ i\,re^{\frac{i}{2}(s-t)} \\ i\,re^{\frac{i}{2}(s+t)} \end{pmatrix} a' + \begin{pmatrix} 0 \\ 0 \\ i\,ae^{\frac{i}{2}(s-t)} \\ i\,ae^{\frac{i}{2}(s+t)} \end{pmatrix} r' + \begin{pmatrix} 0 \\ i\,ae^{is} \\ \frac{-1}{2}are^{\frac{i}{2}(s-t)} \\ \frac{-1}{2}are^{\frac{i}{2}(s+t)} \end{pmatrix} s' + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}are^{\frac{i}{2}(s-t)} \\ \frac{-1}{2}are^{\frac{i}{2}(s+t)} \end{pmatrix} t'$$

$$= i \begin{pmatrix} k_{11} + e^{is}k_{12} + i\,re^{\frac{i}{2}(s-t)}k_{13} + i\,re^{\frac{i}{2}(s+t)}k_{14} \\ k_{21} + e^{is}k_{22} + i\,re^{\frac{i}{2}(s-t)}k_{23} + i\,re^{\frac{i}{2}(s+t)}k_{24} \\ k_{31} + e^{is}k_{32} + i\,re^{\frac{i}{2}(s-t)}k_{33} + i\,re^{\frac{i}{2}(s+t)}k_{34} \\ k_{41} + e^{is}k_{42} + i\,re^{\frac{i}{2}(s-t)}k_{43} + i\,re^{\frac{i}{2}(s+t)}k_{44} \end{pmatrix} a\,, \tag{3.11}$$

where the L.H.S of (3.11) is an element of the tangent space to the zero charge spinors at ψ. By comparing the two sides in the last equation we can write the equations that show the change in a, s, t and r as

$$a' = i(k_{11} + e^{is}k_{12} + i\,re^{\frac{i}{2}(s-t)}k_{13} + i\,re^{\frac{i}{2}(s+t)}k_{14})\,a\,, \tag{3.12}$$

$$\begin{aligned} s' = {} & k_{22} - k_{11} + ir(e^{\frac{-i}{2}(s+t)}k_{23} - e^{\frac{i}{2}(s+t)}k_{14}) \\ & (e^{-is}k_{21} - e^{is}k_{12}) + ir(e^{\frac{-i}{2}(s-t)}k_{24} - e^{\frac{i}{2}(s-t)}k_{13})\,, \end{aligned} \tag{3.13}$$

$$\begin{aligned} t' = {} & \frac{i}{r}\Big[(e^{\frac{-i}{2}(s-t)}k_{31} - e^{\frac{i}{2}(s-t)}k_{42}) + (e^{\frac{i}{2}(s+t)}k_{32} - e^{\frac{-i}{2}(s+t)}k_{42}) \\ & + ir(k_{33} - k_{44}) + ir(e^{it}\,k_{34} - e^{-it}\,k_{43})\Big]\,, \end{aligned} \tag{3.14}$$

$$\begin{aligned} r' = {} & \frac{-ir}{2}\Big[k_{22} + k_{11} + (e^{-is}\,k_{21} + e^{is}\,k_{12}) + \\ & ir(e^{\frac{-i}{2}(s+t)}k_{23} + e^{\frac{i}{2}(s+t)}k_{14}) + ir(e^{\frac{-i}{2}(s-t)}k_{24} + e^{\frac{i}{2}(s-t)}k_{13})\Big] \\ & + \frac{1}{2}\Big[(e^{\frac{-i}{2}(s-t)}k_{31} + e^{\frac{i}{2}(s-t)}k_{42}) + (e^{\frac{i}{2}(s+t)}k_{32} + e^{\frac{-i}{2}(s+t)}k_{41}) \\ & + ir(k_{33} + k_{44}) + ir(e^{it}\,k_{34} + e^{-it}\,k_{43})\Big]\,. \end{aligned} \tag{3.15}$$

The idea now is to find out the general form for the matrix K that satisfies the equations (3.12)-(3.15) by using the information we have about the other variables. As we know that r, s and t are real, from equation (3.13),

$$k_{22} - k_{11} \in \mathbb{R}\,, \tag{3.16}$$

$$e^{-is}k_{21} - e^{is}k_{12} \in \mathbb{R} \;\; \forall s\,, \tag{3.17}$$

$$e^{\frac{-i}{2}(s+t)}(ik_{23}) - e^{\frac{i}{2}(s+t)}(ik_{14}) \in \mathbb{R} \;\; \forall s,t\,, \tag{3.18}$$

$$e^{\frac{-i}{2}(s-t)}(ik_{24}) - e^{\frac{i}{2}(s-t)}(ik_{13}) \in \mathbb{R} \ \ \forall s,t. \tag{3.19}$$

Let first consider equation (3.17), which gives

$$k_{21} = -\overline{k_{12}}. \tag{3.20}$$

Similarly from (3.18) and (3.19),

$$k_{23} = \overline{k_{14}} \quad \text{and} \quad k_{24} = \overline{k_{13}}, \tag{3.21}$$

respectively. Now from equation (3.14),

$$k_{44} - k_{33} \in \mathbb{R}, \tag{3.22}$$

$$e^{-it}k_{43} - e^{it}k_{34} \in \mathbb{R} \ \ \forall t, \tag{3.23}$$

$$e^{\frac{-i}{2}(s-t)}(ik_{31}) - e^{\frac{i}{2}(s-t)}(ik_{42}) \in \mathbb{R} \ \ \forall s,t, \tag{3.24}$$

$$e^{\frac{i}{2}(s+t)}(ik_{41}) - e^{\frac{-i}{2}(s+t)}(ik_{32}) \in \mathbb{R} \ \ \forall s,t. \tag{3.25}$$

The last three equations give us the following relations

$$k_{43} = -\overline{k_{34}} \ , \ k_{42} = \overline{k_{31}} \quad \text{and} \quad k_{41} = \overline{k_{32}}. \tag{3.26}$$

Finally equation(3.15) will provide one more relation which is

$$k_{44} = \overline{k_{22}} + \overline{k_{11}} - k_{33}. \tag{3.27}$$

The form for the K matrix can be written by using the relations we had above as

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ -\overline{k_{12}} & k_{22} & \overline{k_{14}} & \overline{k_{13}} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ \overline{k_{32}} & \overline{k_{31}} & -\overline{k_{34}} & (\overline{k_{22}} + \overline{k_{11}} - k_{33}) \end{pmatrix}. \tag{3.28}$$

3.3.2 The K matrix and symmetry

Now we see what matrices K in (3.9) are consistent with the symmetry equations (3.7) and (3.8). By acting by $(\partial_1 + \partial_3)$ on the first symmetry equation (3.7), it becomes

$$(\partial_1 + \partial_3)(x^4\,\partial_2 + x^2\,\partial_4)\psi = \frac{1}{2}(\partial_1 + \partial_3)\gamma^2\gamma^4\,\psi\,,$$

and we can write the last equation as

$$(x^4\,\partial_2 + x^2\,\partial_4)(\partial_1 + \partial_3)\psi = \frac{1}{2}\gamma^2\gamma^4\,(\partial_1 + \partial_3)\psi\,.$$

By using equation (3.9) the last equation becomes

$$(x^4\,\partial_2 + x^2\,\partial_4)K\,\psi = \frac{1}{2}\gamma^2\gamma^4\,K\,\psi\,.$$

Since the matrix K is independent of x^2 and x^4, the last equation becomes

$$K\,\gamma^2\gamma^4\,\psi = \gamma^2\gamma^4\,K\,\psi\,. \tag{3.29}$$

This equation is true for all ψ, so that K commutes with $\gamma^2\gamma^4$, and this fact gives us

$$k_{22} = k_{11} \quad , \quad k_{33} = k_{44}\,, \tag{3.30}$$

$$k_{12} = \overline{k_{12}} \;\; , \;\; k_{14} = \overline{k_{14}} \;\; , \;\; k_{32} = \overline{k_{32}} \;\; , \;\; k_{34} = \overline{k_{34}}\,, \tag{3.31}$$

$$k_{13} \;=\; -\overline{k_{13}} \quad , \quad k_{31} \;=\; -\overline{k_{31}}\,. \tag{3.32}$$

From (3.30) and by using (3.27) we have $k_{33} = \overline{k_{11}}$, and from equation (3.31) it follows that k_{12}, k_{14}, k_{32} and k_{34} are real. Finally from (3.32) we deduce that k_{13} and k_{31} are pure imaginary.

The form for the matrix K at this stage is given by

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ -k_{12} & k_{11} & k_{14} & -k_{13} \\ k_{31} & k_{32} & \overline{k_{11}} & k_{34} \\ k_{32} & -k_{31} & -k_{34} & \overline{k_{11}} \end{pmatrix}, \tag{3.33}$$

where k_{12}, k_{14}, k_{32} and k_{34} are real, k_{13} and k_{31} are pure imaginary, and k_{11} is complex.

Similarly, by acting by $(\partial_1 + \partial_3)$ on the second symmetry equation (3.8),

$$(\partial_1 + \partial_3)(x^3\,\partial_1 + x^1\,\partial_3)\psi = \frac{1}{2}(\partial_1 + \partial_3)\gamma^1\gamma^3\,\psi\,,$$

and this equation can be written as

$$(x^3\,\partial_1^2 + \partial_3 + x^1\,\partial_1\partial_3 + \partial_1 + x^3\,\partial_1\partial_3 + x^1\,\partial_3^2)\psi = \frac{1}{2}\gamma^1\gamma^3(\partial_1 + \partial_3)\psi\,,$$

and by simplifying this equation it becomes

$$\Big[(x^3\,\partial_1 + x^1\,\partial_3) + 1\Big](\partial_1 + \partial_3)\psi = \frac{1}{2}\gamma^1\gamma^3(\partial_1 + \partial_3)\psi\,.$$

By using (3.9) in the last equation, it becomes

$$\Big[(x^3\,\partial_1 + x^1\,\partial_3) + 1\Big]\,(K\,\psi) = \frac{1}{2}\gamma^1\gamma^3 K\,\psi\,, \tag{3.34}$$

and this equation can be written as

$$\Big[(x^3\,\partial_1 + x^1\,\partial_3)K\Big]\psi + K\Big[(x^3\,\partial_1 + x^1\,\partial_3)\psi\Big] + K\,\psi = \frac{1}{2}\gamma^1\gamma^3 K\,\psi\,.$$

Now by using equation (3.8) in the last equation, it becomes

$$\Big[(x^3\,\partial_1 + x^1\,\partial_3)K\Big]\cdot\psi = \frac{1}{2}(\gamma^1\gamma^3\,K - K\,\gamma^1\gamma^3)\cdot\psi - K\,\cdot\psi\,.$$

If we suppose that this equation is true for all ψ, then K satisfies

$$(x^3\,\partial_1 + x^1\,\partial_3)K = \frac{1}{2}(\gamma^1\gamma^3\,K - K\,\gamma^1\gamma^3) - K\,. \tag{3.35}$$

From equation (3.35) we deduce that

$$(x^3\partial_1 + x^1\partial_3)k_{11} = -k_{11} \quad , \quad (x^3\partial_1 + x^1\partial_3)k_{12} = -k_{12}\,,$$

$$(x^3\partial_1 + x^1\partial_3)k_{34} = -k_{34}\,, \tag{3.36}$$

$$(x^3\partial_1 + x^1\partial_3)k_{13} = -k_{13} + ik_{14}\ , \ (x^3\partial_1 + x^1\partial_3)k_{14} = -ik_{13} - k_{14}\, , \tag{3.37}$$

$$(x^3\partial_1 + x^1\partial_3)k_{31} = -k_{31} + ik_{32}\ , \ (x^3\partial_1 + x^1\partial_3)k_{32} = -ik_{31} - k_{32}\, . \tag{3.38}$$

Next we change variables by

$$x^1(L,J) = J\ \sinh(L) \quad \text{and} \quad x^3(L,J) = J\ \cosh(L)\,, \tag{3.39}$$

where L and J are defined by

$$L = \frac{1}{2}\ \ln(\frac{x^3 + x^1}{x^3 - x^1}) \quad \text{and} \quad J = \left((x^3)^2 - (x^1)^2\right)^{\frac{1}{2}}. \tag{3.40}$$

Also we define

$$x^4(p,q) = q\ \sinh(p) \quad \text{and} \quad x^2(p,q) = q\ \cosh(p)\,, \tag{3.41}$$

where p and q are defined by

$$p = \frac{1}{2}\ \ln(\frac{x^2 + x^4}{x^2 - x^4}) \quad \text{and} \quad q = \left((x^2)^2 - (x^4)^2\right)^{\frac{1}{2}}. \tag{3.42}$$

We can consider these new variables as abstract substitutions, which may take complex values, and return to our reality conditions at the end of this case (solutions). Alternatively these new variables may be taken to be real valued on a subset of space-time, with slightly different substitutions with changed signs used on other subsets.

First take equation (3.7) and change variables,

$$\frac{\partial}{\partial p}\,\psi = \frac{1}{2}\gamma^2\gamma^4\psi\,. \tag{3.43}$$

Now assume ψ to be one of the zero current spinors (1.16), (1.19) or (1.27). If we set $\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,re^{\frac{i}{2}(s-t)} \\ i\,re^{\frac{i}{2}(s+t)} \end{pmatrix} a$, where s and t are real functions, r is a non zero real function, and a is a complex function, substituting this zero current spinor in (3.43) gives

$$\frac{\partial}{\partial p}\begin{pmatrix}1\\ e^{is}\\ i\,re^{\frac{i}{2}(s-t)}\\ i\,re^{\frac{i}{2}(s+t)}\end{pmatrix} a = \frac{1}{2}\gamma^2\gamma^4\begin{pmatrix}1\\ e^{is}\\ i\,re^{\frac{i}{2}(s-t)}\\ i\,re^{\frac{i}{2}(s+t)}\end{pmatrix} a\,.$$

The last equation gives us the following:

$$\frac{\partial}{\partial p}\log a \;=\; \frac{-i}{2}\,e^{is}\,, \tag{3.44}$$

$$\frac{\partial}{\partial p}s \;=\; \cos(s)\,, \tag{3.45}$$

$$\frac{\partial}{\partial p}t \;=\; -\cos(t)\,, \tag{3.46}$$

$$\frac{\partial}{\partial p}\log r \;=\; \frac{-1}{2}(\sin(s)+\sin(t))\,. \tag{3.47}$$

Changing variables in (3.8) gives

$$\frac{\partial}{\partial L}\psi = \frac{1}{2}\gamma^1\gamma^3\psi\,. \tag{3.48}$$

Similarly by substituting $\psi = \begin{pmatrix}1\\ e^{is}\\ i\,re^{\frac{i}{2}(s-t)}\\ i\,re^{\frac{i}{2}(s+t)}\end{pmatrix} a$ in equation (3.48), it becomes

$$\frac{\partial}{\partial L}\begin{pmatrix}1\\ e^{is}\\ i\,re^{\frac{i}{2}(s-t)}\\ i\,re^{\frac{i}{2}(s+t)}\end{pmatrix} a = \frac{1}{2}\gamma^1\gamma^3\begin{pmatrix}1\\ e^{is}\\ i\,re^{\frac{i}{2}(s-t)}\\ i\,re^{\frac{i}{2}(s+t)}\end{pmatrix} a\,.$$

The last equation gives us the following:

$$\frac{\partial}{\partial L}\log a = \frac{i}{2}e^{is}\,, \tag{3.49}$$

$$\frac{\partial}{\partial L}s = -\cos(s)\,, \tag{3.50}$$

$$\frac{\partial}{\partial L}t = -\cos(t)\,, \tag{3.51}$$

$$\frac{\partial}{\partial L}\log r = \frac{1}{2}(\sin(s) - \sin(t))\,. \tag{3.52}$$

Finally change the variables in (3.9) as

$$z_1 = x^3 + x^1 \quad \text{and} \quad z_2 = x^3 - x^1\,.$$

Then it becomes

$$2\frac{\partial}{\partial z_1}\psi = iK\psi\,. \tag{3.53}$$

By using the chain rule we can write (3.53) in the new variables as

$$2\left(\frac{\partial L}{\partial z_1}\frac{\partial}{\partial L} + \frac{\partial J}{\partial z_1}\frac{\partial}{\partial J}\right)\psi = iK\psi\,.$$

From (3.40) and by using equation (3.48) the last equation becomes

$$\frac{\partial}{\partial J}\psi = (ie^L K - \frac{1}{2J}\gamma^1\gamma^3)\,\psi\,. \tag{3.54}$$

Similarly by substituting $\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,re^{\frac{i}{2}(s-t)} \\ i\,re^{\frac{i}{2}(s+t)} \end{pmatrix} a$ in equation (3.54), it becomes

$$\frac{\partial}{\partial J}\begin{pmatrix} 1 \\ e^{is} \\ i\,re^{\frac{i}{2}(s-t)} \\ i\,re^{\frac{i}{2}(s+t)} \end{pmatrix} a = (ie^L K - \frac{1}{2J}\gamma^1\gamma^3)\begin{pmatrix} 1 \\ e^{is} \\ i\,re^{\frac{i}{2}(s-t)} \\ i\,re^{\frac{i}{2}(s+t)} \end{pmatrix} a\,.$$

The last equation gives :

$$\frac{\partial}{\partial J}\log a = ie^L\Big(k_{11}+k_{12}e^{is}+ire^{\frac{i}{2}(s-t)}k_{13} + ire^{\frac{i}{2}(s+t)}k_{14}\Big) - \frac{i}{2J}e^{is}\,, \tag{3.55}$$

$$\frac{\partial}{\partial J}s = \frac{1}{J}\cos(s) - 2e^L\Big(k_{12}\cos(s) - rk_{14}\sin(\frac{1}{2}(s+t)) + irk_{13}\cos(\frac{1}{2}(s-t))\Big)\,, \tag{3.56}$$

$$\frac{\partial}{\partial J}t = \frac{2i}{r}e^L\Big(ik_{32}\sin(\frac{1}{2}(s+t)) + k_{31}\cos(\frac{1}{2}(s-t)) + irk_{34}\cos(t)\Big) + \frac{1}{J}\cos(t)\,, \tag{3.57}$$

$$\begin{aligned}\frac{\partial}{\partial J}r = {} & -ire^L(k_{11}-\overline{k_{11}}) + ir^2e^Lk_{13}\sin(\frac{1}{2}(s-t)) \\ & +r^2e^Lk_{14}\cos(\frac{1}{2}(s+t)) - \frac{r}{2J}(\sin(s)-\sin(t)) \\ & +e^Lk_{32}\cos(\frac{1}{2}(s+t)) - ie^Lk_{31}\sin(\frac{1}{2}(s-t)) \\ & +rk_{12}e^L\sin(s) - re^Lk_{34}\sin(t)\,.\end{aligned} \tag{3.58}$$

Now for s, integrating the equations (3.45) and (3.50) give

$$\log(\sec(s)+\tan(s)) = p + \log E_1(L,J,q)\,,$$

$$\log(\sec(s)+\tan(s)) = -L + \log E_2(p,J,q)\,,$$

respectively. The last two equations can be written as

$$\log(\sec(s)+\tan(s)) = p - L + \log E(J,q)\,, \tag{3.59}$$

where $E(J,q)$ is a function in J and q. The last equation gives us the values for $\sin(s)$ and $\cos(s)$ as

$$\sin(s) = \frac{E^2(J,q)\,e^{2(p-L)}-1}{E^2(J,q)\,e^{2(p-L)}+1}\,,$$

$$\cos(s) = \frac{2E(J,q)\,e^{p-L}}{E^2(J,q)\,e^{2(p-L)}+1}\,. \tag{3.60}$$

Similarly for t, integrating the equations (3.46) and (3.51) give

$$\log(\sec(t) + \tan(t)) = -p + \log D_1(L, J, q),$$

$$\log(\sec(s) + \tan(s)) = -L + \log D_2(p, J, q),$$

respectively. The last two equations can be written as

$$\log(\sec(t) + \tan(t)) = -p - L + \log D(J, q), \tag{3.61}$$

where $D(J, q)$ is a function in J and q. The last equation gives us the values for $\sin(t)$ and $\cos(t)$ as

$$\begin{aligned} \sin(t) &= \frac{D^2(J,q)\, e^{-2(p+L)} - 1}{D^2(J,q)\, e^{-2(p+L)} + 1}, \\ \cos(t) &= \frac{2D(J,q)\, e^{-(p+L)}}{D^2(J,q)\, e^{-2(p+L)} + 1}. \end{aligned} \tag{3.62}$$

3.3.3 The action of the Lorentz group on the vector potential

In this part we see how the Lorentz group acts on the connection field A. The action of the Lie algebra of the Lorentz group on the covector A_a is given by $A_a \longrightarrow g_{ac}\, w^c{}_b\, g^{bd}\, A_d$, so the symmetry equation is given by

$$A'(X; \omega^c{}_b X) = g_{ac}\, \omega^c{}_b\, g^{bd}\, A_d, \tag{3.63}$$

where ω_{ab} are the same elements of the Lie algebra of $SO(2,2)$ which fixes the lightlike line $(t, 0, t, 0)$, and $g_{ab} = (g^{ab})^{-1}$. We know that there are two linearly independent elements which fix the lightlike line. Next we want to find the symmetry equations on the connection field A which are given by (3.63) for each element.

(i) $\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}$ then $\omega^c{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}$, then by substituting this value for $\omega^{c}{}_{b}$ equation (3.63) becomes

$$A'\left(\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix} A,$$

which gives the first symmetry equation

$$(x^4 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^4})\, A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A. \tag{3.64}$$

Now we can write equation (3.64) in the new variables as

$$\frac{\partial}{\partial p} A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A, \tag{3.65}$$

where p is given by (3.42).

(ii) $\omega_{ab} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ then $\omega^{c}{}_{b} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, .

then by substituting this value for $\omega^{c}{}_{b}$ equation (3.63) becomes

$$A'\left(\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}; \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A,$$

which gives the second symmetry equation

$$x^3 \frac{\partial}{\partial x^1} A + x^1 \frac{\partial}{\partial x^3} A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A. \tag{3.66}$$

Finally we can write equation (3.66) in the new variables as

$$\frac{\partial}{\partial L} A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A \,, \tag{3.67}$$

where L is given by (3.40).

Next we want to find out what values the vector potential takes by substituting $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$ in the equations (3.65) and (3.67). Let us first consider equation (3.65), which gives :

$$\frac{\partial}{\partial p} A_1 = 0 \qquad \text{so} \qquad A_1 = f_1{}'(L, J, q) \,, \tag{3.68}$$

$$\frac{\partial}{\partial p} A_3 = 0 \qquad \text{so} \qquad A_3 = f_3{}'(L, J, q) \,, \tag{3.69}$$

$$\frac{\partial}{\partial p} A_2 = -A_4 \,, \tag{3.70}$$

$$\frac{\partial}{\partial p} A_4 = -A_2 \,, \tag{3.71}$$

where $f_1{}'(L, J, q)$ and $f_3{}'(L, J, q)$ are any real valued functions. From the equations (3.70) and (3.71), we have

$$\frac{\partial^2}{\partial p^2} A_2 = A_2 \quad \text{so} \quad A_2 = f_2(L, J, q)\, e^{p} + g_2(L, J, q)\, e^{-p} \,, \tag{3.72}$$

$$\frac{\partial^2}{\partial p^2} A_4 = A_4 \quad \text{so} \quad A_4 = f_4(L, J, q)\, e^{p} + g_4(L, J, q)\, e^{-p} \,, \tag{3.73}$$

where $f_2(L, J, q)$, $g_2(L, J, q)$, $f_4(L, J, q)$ and $g_4(L, J, q)$ are any real valued functions, and L, J, p and q are given by (3.40) and (3.42) respectively.

Equation (3.67) gives

$$\frac{\partial}{\partial L} A_2 = 0 \qquad \text{so that} \qquad A_2 = f_2^{'}(J, q, p)\,, \tag{3.74}$$

$$\frac{\partial}{\partial L} A_4 = 0 \qquad \text{so that} \qquad A_4 = f_4^{'}(J, q, p)\,, \tag{3.75}$$

$$\frac{\partial}{\partial L} A_1 = -A_3\,, \tag{3.76}$$

$$\frac{\partial}{\partial L} A_3 = -A_1\,. \tag{3.77}$$

where $f_2^{'}(J, q, p)$ and $f_4^{'}(J, q, p)$ are any real valued functions. From the equations (3.76) and (3.77),

$$\frac{\partial^2}{\partial L^2} A_1 = A_1 \quad \text{so} \quad A_1 = f_1(J, q, p)e^{L} + g_1(J, q, p)e^{-L}\,, \tag{3.78}$$

$$\frac{\partial^2}{\partial L^2} A_3 = A_3 \quad \text{so} \quad A_3 = f_3(J, q, p)e^{L} + g_3(J, q, p)e^{-L}\,, \tag{3.79}$$

where $f_1(L, J, q)$, $g_1(L, J, q)$, $f_3(L, J, q)$ and $g_3(L, J, q)$ are any real valued functions. Now by comparing the values for A_1 and A_3, and by substituting them in (3.76) and (3.77),

$$A_1 = f_1(J, q)\, e^{L} + g_1(J, q)\, e^{-L}\,, \tag{3.80}$$

$$A_3 = -f_1(J, q)\, e^{L} + g_1(J, q)\, e^{-L}\,, \tag{3.81}$$

respectively, where $f_1(J, q)$ and $g_1(J, q)$ are any real valued functions. Similarly by comparing the values for A_2 and A_4 and by substituting them in (3.70) and (3.71),

$$A_2 = f_2(J, q)\, e^{p} + g_2(J, q)\, e^{-p}\,, \tag{3.82}$$

$$A_4 = -f_2(J, q)\, e^{p} + g_2(J, q)\, e^{-p}\,, \tag{3.83}$$

respectively, where $f_2(J, q)$ and $g_2(J, q)$ are any real valued functions.

3.3.4 The electromagnetic field

In this part more about the vector potential, which is a solution for the electromagnetic field, where the electromagnetic field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{3.84}$$

Now using the chain rule, the derivatives in the new variables become:

$$\begin{aligned}
\frac{\partial}{\partial x^1} &= -\sinh(L)\,\frac{\partial}{\partial J} + \frac{1}{J}\cosh(L)\,\frac{\partial}{\partial L}\,, \\
\frac{\partial}{\partial x^2} &= \cosh(p)\,\frac{\partial}{\partial q} - \frac{1}{q}\sinh(p)\,\frac{\partial}{\partial p}\,, \\
\frac{\partial}{\partial x^3} &= \cosh(L)\,\frac{\partial}{\partial J} - \frac{1}{J}\sinh(L)\,\frac{\partial}{\partial L}\,, \\
\frac{\partial}{\partial x^4} &= -\sinh(p)\,\frac{\partial}{\partial q} + \frac{1}{q}\cosh(p)\,\frac{\partial}{\partial p},
\end{aligned}$$

where L, J, p and q are given by (3.40) and (3.42) respectively. The electromagnetic field components (3.84) are then

$$\begin{aligned}
F_{12} &= -\sinh(L)\,\frac{\partial}{\partial J}(f_2(J,q)e^p + g_2(J,q)e^{-p}) \\
&\quad - \cosh(p)\,\frac{\partial}{\partial q}(f_1(J,q)e^L + g_1(J,q)e^{-L})\,, \\
F_{13} &= -(\frac{\partial}{\partial J} + \frac{1}{J})\,(f_1(J,q) + g_1(J,q))\,, \\
F_{14} &= -\sinh(L)\,\frac{\partial}{\partial J}(-f_2(J,q)e^p + g_2(J,q)e^{-p}) \\
&\quad + \sinh(p)\,\frac{\partial}{\partial q}(f_1(J,q)e^L + g_1(J,q)e^{-L})\,, \\
F_{23} &= \cosh(p)\,\frac{\partial}{\partial q}(-f_1(J,q)\,e^L + g_1(J,q)\,e^{-L}) \\
&\quad - \cosh(L)\,\frac{\partial}{\partial J}(f_2(J,q)e^p + g_2(J,q)e^{-p})\,, \\
F_{24} &= -(\frac{\partial}{\partial q} + \frac{1}{q})\,(f_2(J,q) - g_2(J,q))\,, \\
F_{34} &= \cosh(L)\,\frac{\partial}{\partial J}(-f_2(J,q)e^p + g_2(J,q)e^{-p}) \\
&\quad + \sinh(p)\,\frac{\partial}{\partial q}(-f_1(J,q)e^L + g_1(J,q)e^{-L})\,.
\end{aligned} \tag{3.85}$$

Next let us substitute the electromagnetic field components in the currentless equation

$$\partial_\mu F^{\nu\mu} = 0 . \tag{3.86}$$

For $\nu = 1$,

$$\begin{aligned}
&\sinh(L)(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial J}(f_2(J,q) + g_2(J,q)) \\
&-\cosh(L)(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}(f_1(J,q) + g_1(J,q)) \\
&+(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}(f_1(J,q)\, e^{L} + g_1(J,q)\, e^{-L}) \\
&\quad + \cosh(L)\,\frac{1}{J^2}\,(f_1(J,q) + g_1(J,q)) = 0 .
\end{aligned} \tag{3.87}$$

For $\nu = 2$,

$$\begin{aligned}
&\sinh(p)(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}(f_2(J,q) - g_2(J,q)) \\
&-\cosh(p)(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial q}(f_1(J,q) - g_1(J,q)) \\
&-(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}(f_2(J,q)\, e^{p} + g_2(J,q)\, e^{-p}) \\
&\quad - \sinh(p)\,\frac{1}{q^2}\,(f_2(J,q) - g_2(J,q)) = 0 .
\end{aligned} \tag{3.88}$$

For $\nu = 3$,

$$\begin{aligned}
&\cosh(L)(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial J}(f_2(J,q) + g_2(J,q)) \\
&-\sinh(L)(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}(f_1(J,q) + g_1(J,q)) \\
&-(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}(-f_1(J,q)\, e^{L} + g_1(J,q)\, e^{-L}) \\
&\quad + \sinh(L)\,\frac{1}{J^2}\,(f_1(J,q) + g_1(J,q)) = 0 .
\end{aligned} \tag{3.89}$$

For $\nu = 4$,

$$\cosh(p)(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}(f_2(J,q) - g_2(J,q))$$

$$-\sinh(p)(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial q}(f_1(J,q) - g_1(J,q))$$

$$+(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}(-f_2(J,q)\,e^{p} + g_2(J,q)\,e^{-p})$$

$$-\cosh(p)\,\frac{1}{q^2}\,(f_2(J,q) - g_2(J,q)) = 0\,. \tag{3.90}$$

To solve the last four equations. We multiply equation (3.87) by $\sinh(L)$ and equation (3.89) by $\cosh(L)$, and subtract, giving

$$(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial J}(f_2(J,q) + g_2(J,q)) +$$

$$(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}(f_1(J,q) - g_1(J,q)) = 0\,. \tag{3.91}$$

By multiplying equation (3.87) by $\cosh(L)$ and equation (3.89) by $\sinh(L)$, and subtract, giving

$$(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}(f_1(J,q) + g_1(J,q)) + \frac{1}{J^2}\,(f_1(J,q) + g_1(J,q))$$

$$-(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}(f_1(J,q) + g_1(J,q)) = 0\,. \tag{3.92}$$

Similarly by multiplying equation (3.88) by $\cosh(p)$ and equation (3.90) by $\sinh(p)$, and subtract, giving

$$(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial q}(f_1(J,q) - g_1(J,q)) +$$

$$(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}(f_2(J,q) + g_2(J,q)) = 0\,. \tag{3.93}$$

Finally by multiplying equation (3.88) by $\sinh(p)$ and equation (3.90) by $\cosh(p)$, and subtract, giving

$$(\frac{\partial}{\partial J}+\frac{1}{J})\frac{\partial}{\partial J}(f_2(J,q)-g_2(J,q))+\frac{1}{q^2}(f_2(J,q)-g_2(J,q))$$

$$-(\frac{\partial}{\partial q}+\frac{1}{q})\frac{\partial}{\partial q}(f_2(J,q)-g_2(J,q))=0\,. \tag{3.94}$$

From the equations (3.91) and (3.93),

$$\frac{\partial}{\partial J}(f_2(J,q)+g_2(J,q))+\frac{\partial}{\partial q}(f_1(J,q)-g_1(J,q))=b(Jq)^{-1}\,, \tag{3.95}$$

where b is a constant.
We solve (3.92) by the method of separation of variables. If we set $f_1(J,q)+g_1(J,q)=C(J)D(q)$, equation (3.92) becomes

$$C(J)(\frac{\partial}{\partial q}+\frac{1}{q})\frac{\partial}{\partial q}D(q)+\frac{1}{J^2}C(J)D(q)$$

$$-D(q)(\frac{\partial}{\partial J}+\frac{1}{J})\frac{\partial}{\partial J}C(J)=0\,.$$

We can write the last equation as

$$\underbrace{\frac{1}{D(q)}(\frac{\partial}{\partial q}+\frac{1}{q})\frac{\partial}{\partial q}D(q)}_{-H}+\underbrace{\frac{1}{J^2}-\frac{1}{C(J)}(\frac{\partial}{\partial J}+\frac{1}{J})\frac{\partial}{\partial J}C(J)}_{H}=0\,. \tag{3.96}$$

We see that this equation splits into two parts, the first being a function of q only, and the second a function of J only. This means that

$$\Big(\frac{d^2}{d\,q^2}+\frac{1}{q}\frac{d}{d\,q}+H\Big)D(q)=0\,, \tag{3.97}$$

$$\Big(\frac{d^2}{d\,J^2}+\frac{1}{J}\frac{d}{d\,J}+(H-\frac{1}{J^2})\Big)C(J)=0\,, \tag{3.98}$$

where H is an arbitrary constant.

Note that the equations (3.97) and (3.98) are of the form of **Bessel's equation**, which is given by

$$\Big(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+(1-\frac{n^2}{x^2})\Big)R(x)=0\,. \tag{3.99}$$

This is called **Bessel's equation** of order n, where n is a constant. The general solution of Bessel's equation is given by

$$R(x) \;=\; uJ_n(x)+vY_n(x) \tag{3.100}$$

where u and v are constants. The function $J_n(x)$ is called **Bessel's function** of the first kind of order n, and is given by

$$J_n(x) \;=\; \sum_{p=0}^{\infty}\frac{(-1)^p}{p!(p+n)!}(\frac{x}{2})^{2p+n}\,. \tag{3.101}$$

The function $Y_n(x)$ is called **Bessel's function** of the second kind of order n, and is given by

$$\begin{aligned} Y_n(x) \;=\; & \frac{2}{\pi}\Big[\Big(\ln\frac{x}{2}+\gamma\Big)J_n(x)-\frac{1}{2}\sum_{p=0}^{n-1}\frac{(n-p-1)!}{p!}(\frac{x}{2})^{2p-n} \\ & +\frac{1}{2}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{p!(p+n)!}(\frac{x}{2})^{2p+n}\Big(\sum_{k=1}^{p}\frac{1}{k}+\sum_{k=1}^{p+n}\frac{1}{k}\Big)\Big]\,. \end{aligned} \tag{3.102}$$

To solve the equations (3.97) and (3.98), first let us take equation (3.97), which is just Bessel's equation of order zero. The solution for this equation is given in terms of Bessel functions of the first and the second kind of order zero as

$$D(q) \;=\; d_1J_0(q\sqrt{H})+d_2Y_0(q\sqrt{H})\,,$$

where d_1 and d_2 are arbitrary constants, and it can be written as

$$\begin{aligned} D(q) \;=\; & d_1\Big[\sum_{p=0}^{\infty}\frac{(-1)^p}{(p!)^2}(\frac{q\sqrt{H}}{2})^{2p}\Big]+ \\ & d_2\Big[\frac{2}{\pi}\Big(\Big(\ln(\frac{q\sqrt{H}}{2})+\gamma\Big)J_0(q\sqrt{H})+ \\ & \sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{(p!)^2}(\frac{q\sqrt{H}}{2})^{2p}(1+\frac{1}{2}+\cdots+\frac{1}{p})\Big)\Big]\,. \end{aligned} \tag{3.103}$$

Similarly for equation (3.98), which is just Bessel's equation of the order one, where the solution for this equation will given in terms of Bessel function of the first and the second kind of order one as

$$C(J) = c_1 J_1(J\sqrt{H}) + c_2 Y_1(J\sqrt{H}),$$

where c_1 and c_2 are arbitrary constants. It can be written as

$$\begin{aligned} C(J) &= c_1\Big[\sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}(\frac{J\sqrt{H}}{2})^{2m+1}\Big] + \\ & c_2\Big[\frac{2}{\pi}\Big((\ln(\frac{J\sqrt{H}}{2}) + \gamma)J_1(J\sqrt{H}) + \\ & \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{m!(m+1)!}(\frac{J\sqrt{H}}{2})^{2m+1}\Big(\sum_{k=1}^{m}\frac{1}{k} + \sum_{k=1}^{m+1}\frac{1}{k}\Big)\Big)\Big], \quad (3.104) \end{aligned}$$

where γ denotes Euler's constant, defined by

$$\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n) = 0,5772\cdots.$$

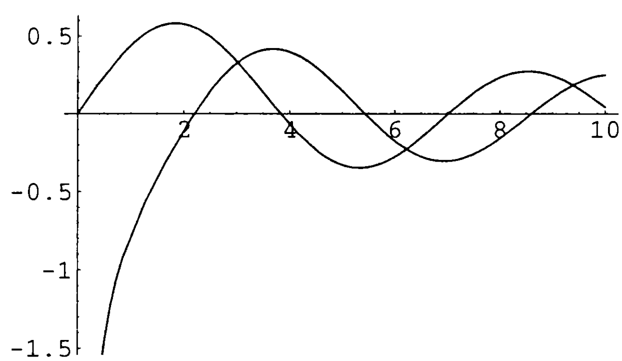

Figure 3.1: This figure shows $J_0(x)$ and $Y_0(x)$, where $Y_0(x)$ has a singularity at $x = 0$.

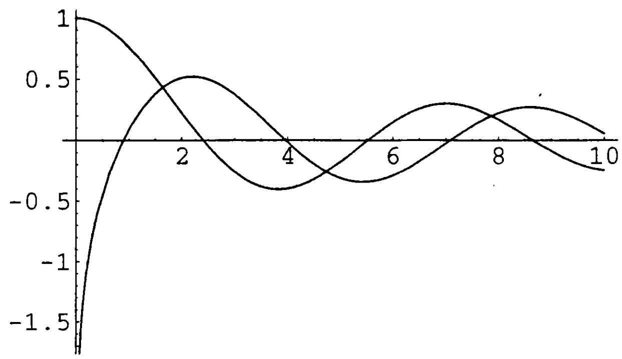

Figure 3.2: This figure shows $J_1(x)$ and $Y_1(x)$, where $Y_1(x)$ has a singularity at $x = 0$.

Now we can write $f_1(J,q) + g_1(J,q)$ from (3.92) as

$$\begin{aligned} f_1(J,q) + g_1(J,q) &= c_1 d_1 J_0(q\sqrt{H}) J_1(J\sqrt{H}) + \\ &\quad c_2 d_2 Y_0(q\sqrt{H}) Y_1(J\sqrt{H}) + \\ &\quad c_2 d_1 J_0(q\sqrt{H}) Y_1(J\sqrt{H}) + \\ &\quad c_1 d_2 Y_0(q\sqrt{H}) J_1(J\sqrt{H}) , \end{aligned} \tag{3.105}$$

where c_i and d_i are constants for $i = 1, 2$.

Similarly if we set $f_2(J,q) - g_2(J,q) = U(J)V(q)$, then equation (3.94) becomes

$$\begin{aligned} V(q)(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}U(J) + \frac{1}{q^2}U(J)V(q) \\ -U(J)(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}V(q) = 0 . \end{aligned}$$

We can write the last equation as

$$\underbrace{\frac{1}{U(J)}(\frac{\partial}{\partial J} + \frac{1}{J})\frac{\partial}{\partial J}U(J)}_{-h} + \underbrace{\frac{1}{q^2} - \frac{1}{V(q)}(\frac{\partial}{\partial q} + \frac{1}{q})\frac{\partial}{\partial q}V(q)}_{h} = 0 . \tag{3.106}$$

We see that this equation splits into two parts, the first being a function of J only, and the second a function of q only. This means that

$$\Big(\frac{d^2}{d\,J^2} + \frac{1}{J}\frac{d}{d\,J} + h\Big)U(J) = 0 , \tag{3.107}$$

$$\Big(\frac{d^2}{d\,q^2}+\frac{1}{q}\frac{d}{d\,q}+(h-\frac{1}{q^2})\Big)V(q)=0\,, \tag{3.108}$$

where h is an arbitrary constant.

The equations (3.105) and (3.107) are just of the form of **Bessel's equation** of order zero and order one respectively. Their solutions are given by

$$\begin{aligned} U(J) \;=\; & u_1\Big[\sum_{p=0}^{\infty}\frac{(-1)^p}{(p!)^2}\,(\frac{J\sqrt{h}}{2})^{2p}\Big]+ \\ & u_2\Big[\frac{2}{\pi}\Big(\Big(\ln(\frac{J\sqrt{h}}{2})+\gamma\Big)J_0(J\sqrt{h})+ \\ & \sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{(p!)^2}\,(\frac{J\sqrt{h}}{2})^{2p}(1+\frac{1}{2}+\cdots+\frac{1}{p})\Big)\Big]\,, \end{aligned} \tag{3.109}$$

$$\begin{aligned} V(q) \;=\; & v_1\Big[\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+1)!}\,(\frac{q\sqrt{h}}{2})^{2m+1}\Big]+ \\ & v_2\Big[\frac{2}{\pi}\Big((\ln(\frac{q\sqrt{h}}{2})+\gamma)J_1(q\sqrt{h})+ \\ & \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{m!(m+1)!}\,(\frac{q\sqrt{h}}{2})^{2m+1}\Big(\sum_{k=1}^{m}\frac{1}{k}+\sum_{k=1}^{m+1}\frac{1}{k}\Big)\Big)\Big]\,, \end{aligned} \tag{3.110}$$

respectively, where u_1, u_2, v_1 and v_2 are arbitrary constants. Now we can write $f_2(J,q)-g_2(J,q)$ from (3.94) as

$$\begin{aligned} f_2(J,q)-g_2(J,q) \;=\; & u_1v_1J_0(J\sqrt{h})J_1(q\sqrt{h})+ \\ & u_2v_2Y_0(J\sqrt{h})Y_1(q\sqrt{h})+ \\ & u_2v_1J_0(J\sqrt{h})Y_1(q\sqrt{h})+ \\ & u_1v_2Y_0(J\sqrt{h})J_1(q\sqrt{h})\,, \end{aligned} \tag{3.111}$$

where u_i and v_i are constants for $i=1,2$.

We now consider the case where $H=h=0$, because the formulas (3.105) and (3.111) are quite difficult to deal with. Then by substituting $H=0$ in equations (3.97) and (3.98), they become

$$\Big(\frac{d^2}{d\,q^2}+\frac{1}{q}\frac{d}{d\,q}\Big)D(q)=0\,, \tag{3.112}$$

$$\Big(\frac{d^2}{d\,J^2} + \frac{1}{J}\frac{d}{d\,J} - \frac{1}{J^2}\Big)C(J) = 0\,, \tag{3.113}$$

respectively. The solutions of the equations (3.112) and (3.113) are

$$D(q) \;=\; d_1\,\ln q + d_2\,, \tag{3.114}$$

$$C(J) \;=\; c_1\,J^{-1} + c_2\,J\,, \tag{3.115}$$

respectively, where c_1, c_2, d_1 and d_2 are arbitrary constants.
Now we are able to write $f_1(J,q) + g_1(J,q)$ from (3.92) as

$$f_1(J,q) + g_1(J,q) \;= d_1\,(c_1\,J^{-1} + c_2\,J)\ln{(q)} + d_2(c_1\,J^{-1} + c_2\,J)\,. \tag{3.116}$$

Similarly by substituting $h = 0$ in the equations (3.107) and (3.108), they become

$$\Big(\frac{d^2}{d\,J^2} + \frac{1}{J}\frac{d}{d\,J}\Big)U(J) = 0\,, \tag{3.117}$$

$$\Big(\frac{d^2}{d\,q^2} + \frac{1}{q}\frac{d}{d\,q} - \frac{1}{q^2}\Big)V(q) = 0\,, \tag{3.118}$$

respectively. The solutions for the equations (3.112) and (3.113) are given by

$$U(J) \;=\; u_1\,\ln J + u_2\,, \tag{3.119}$$

$$V(q) \;=\; v_1\,q^{-1} + v_2\,q\,, \tag{3.120}$$

respectively, where u_1, u_2, v_1 and v_2 are arbitrary constants.
Now we are able to write $f_2(J,q) - g_2(J,q)$ from (3.94) as

$$f_2(J,q) - g_2(J,q) \;= u_1\,(v_1\,q^{-1} + v_2\,q)\ln{(J)} + u_2(v_1\,q^{-1} + v_2\,q)\,. \tag{3.121}$$

Summary of 3.3.4 :

In this subsection we have obtained the following equations

$$\begin{aligned}
(3.95) \Rightarrow \quad & \tfrac{\partial}{\partial J}(f_2(J,q) + g_2(J,q)) + \tfrac{\partial}{\partial q}(f_1(J,q) - g_1(J,q)) = b(Jq)^{-1}\,, \\
(3.116) \Rightarrow \quad & f_1(J,q) + g_1(J,q) \ = d_1\,(c_1\,J^{-1} + c_2\,J)\ln(q) + d_2(c_1\,J^{-1} + c_2\,J)\,, \\
(3.121) \Rightarrow \quad & f_2(J,q) - g_2(J,q) \ = u_1\,(v_1\,q^{-1} + v_2\,q)\ln(J) + u_2(v_1\,q^{-1} + v_2\,q)\ ,
\end{aligned}$$

where b, c_i, d_i, u_i and v_i are arbitrary constants and f_i and g_i are arbitrary functions for all $i = 1, 2$. The last two equations are found on the assumption that $H = h = 0$ in (3.96) and (3.106).

The electromagnetic field is given in terms of these equations by (3.85). The J and q coordinates are defined by (3.40) and (3.42) respectively.

3.3.5 The solution of the lightlike line case

In this part of the section we first write the massless Dirac equation in $2+2$ dimensions in the new variables p, q, J and L. Next we use the symmetry equations. By doing that we will be left just with one derivative in the massless Dirac equation. The massless Dirac equation with potential A_μ is given by

$$\gamma^\mu\,(\partial_\mu - iA_\mu)\,.\psi = 0$$

Now we are in position to write the massless Dirac equation in $2+2$ dimensions in the new variables as

$$\begin{aligned}
\gamma^1 \ \ (-\sinh(L)\,\frac{\partial}{\partial J} + \frac{1}{J}\cosh(L)\,\frac{\partial}{\partial L})\,\psi \ \ + \\
\gamma^2 \ \ (\cosh(p)\,\frac{\partial}{\partial q} - \frac{1}{q}\sinh(p)\,\frac{\partial}{\partial p})\,\psi \ \ + \\
\gamma^3 \ \ (\cosh(L)\,\frac{\partial}{\partial J} - \frac{1}{J}\sinh(L)\,\frac{\partial}{\partial L})\,\psi \ \ + \\
\gamma^4 \ \ (-\sinh(p)\,\frac{\partial}{\partial q} + \frac{1}{q}\cosh(p)\,\frac{\partial}{\partial p})\,\psi \ \ = i\,\gamma^\mu\,A_\mu\cdot\psi\,.
\end{aligned}$$

At this point we substitute the symmetry equations (3.43), (3.48) and (3.54) in the last equation, and it becomes

$$\begin{aligned}
&\gamma^1 \ \Big(-\sinh(L)\,(ie^L K - \frac{1}{2J}\gamma^1\gamma^3) + \frac{1}{2J}\cosh(L)\,\gamma^1\gamma^3\Big)\,\psi \ + \\
&\gamma^3 \ \Big(\cosh(L)\,(ie^L K - \frac{1}{2J}\gamma^1\gamma^3) - \frac{1}{2J}\sinh(L)\,\gamma^1\gamma^3\Big)\,\psi \ + \\
&\gamma^2 \ \Big(\cosh(p)\,\frac{\partial}{\partial q} - \frac{1}{2q}\sinh(p)\,\gamma^2\gamma^4\Big)\,\psi \ + \\
&\gamma^4 \ \Big(-\sinh(p)\,\frac{\partial}{\partial q} + \frac{1}{2q}\cosh(p)\,\gamma^2\gamma^4\Big)\,\psi = i\,\gamma^\mu\,A_\mu\cdot\psi\,.
\end{aligned}$$

We could simplify the last equation like this

$$\begin{aligned}
&\Big(-\sinh(L)\,\gamma^1(ie^L K) \ + \ \frac{1}{2J}\sinh(L)\,\gamma^3 + \frac{1}{2J}\cosh(L)\,\gamma^3\Big)\,\psi \ + \\
&\Big(\cosh(L)\,\gamma^3(ie^L K) \ - \ \frac{1}{2J}\cosh(L)\,\gamma^1 - \frac{1}{2J}\sinh(L)\,\gamma^1\Big)\,\psi \ + \\
&\Big(\cosh(p)\,\frac{\partial}{\partial q}\gamma^2 - \frac{1}{2q}\sinh(p)\,\gamma^4\Big)\,\psi \ + \\
&\Big(-\sinh(p)\,\frac{\partial}{\partial q}\gamma^4 + \frac{1}{2q}\cosh(p)\,\gamma^2\Big)\,\psi = i\,\gamma^\mu\,A_\mu\cdot\psi\,.
\end{aligned}$$

Let us now arrange the last equation as

$$\begin{aligned}
&ie^L\Big(\cosh(L)\,\gamma^3 - \sinh(L)\,\gamma^1\Big)K\,\psi + \frac{1}{2J}\,e^L(\gamma^3 \ - \gamma^1\,)\psi + \\
&\Big(\cosh(p)\,\gamma^2 - \sinh(p)\,\gamma^4\Big)\,(\frac{\partial}{\partial q}\psi + \frac{1}{2q})\,\psi = i\,\gamma^\mu\,A_\mu\cdot\psi\,.
\end{aligned}$$

Finally the massless Dirac equation has become

$$\begin{aligned}
&\frac{1}{2}\,(e^L(\gamma^3 - \gamma^1) + e^{-L}(\gamma^3 + \gamma^1))(ie^L K)\,\psi + \frac{1}{2J}\,e^L(\gamma^3 - \gamma^1)\,\psi + \\
&\frac{1}{2}\,(e^p(\gamma^2 - \gamma^4) + e^{-p}(\gamma^2 + \gamma^4))(\frac{\partial}{\partial\, q} + \frac{1}{2q})\,\psi = i\,\gamma^\mu\,A_\mu\cdot\psi. \qquad (3.122)
\end{aligned}$$

Now we substitute the values for the matrix K and the vector potential that given by (3.33) and (3.80)-(3.83) respectively in equation (3.122), where

$\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,r e^{\frac{i}{2}(s-t)} \\ i\,r e^{\frac{i}{2}(s+t)} \end{pmatrix}$ a is a zero current spinor. Next by using Mathematica [See the Appendix], equation (3.122) gives

$$\begin{aligned}
\frac{\partial}{\partial q}s &= -2irk_{13}e^{L}\Big[\cos(\frac{s+t}{2})\cosh(L-p)+\sin(\frac{s-t}{2})\sinh(L-p)\Big]- \\
&\quad 2rk_{14}e^{L}\Big[\sin(\frac{s-t}{2})\cosh(L-p)+\cos(\frac{s+t}{2})\sinh(L-p)\Big]+ \\
&\quad 2ik_{11}e^{L}\Big[\cosh(L-p)\,\sin(s)+\sinh(L-p)\Big]- \\
&\quad 2k_{12}e^{L}\Big[\cosh(L-p)+\sin(s)\,\sinh(L-p)\Big]+ \\
&\quad 2if_1(J,q)e^{L-p}(1+\sin(s))-2ig_1(J,q)(\sin(s)-1)- \\
&\quad 2i\cos(s)(f_2(J,q)-g_2(J,q))+\frac{1}{J}\,e^{L-p}(1+\sin(s))\,,
\end{aligned} \tag{3.123}$$

$$\begin{aligned}
\frac{\partial}{\partial q}t &= \frac{2i}{r}k_{13}e^{L}\Big[\cos(\frac{s+t}{2})\cosh(L+p)-\sin(\frac{s-t}{2})\sinh(L+p)\Big]+ \\
&\quad \frac{2}{r}k_{32}e^{L}\Big[\cos(\frac{s+t}{2})\sinh(L+p)-\sin(\frac{s-t}{2})\cosh(L+p)\Big]- \\
&\quad 2k_{34}e^{L}\Big[\cosh(L+p)+\sin(t)\,\sinh(L+p)\Big]+ \\
&\quad 2i\overline{k_{11}}e^{L}\Big[\cosh(L+p)\,\sin(t)+\sinh(L+p)\Big]+ \\
&\quad 2if_1(J,q)e^{L-p}(1+\sin(t))-2ig_1(J,q)(\sin(t)-1)+ \\
&\quad 2i\cos(t)(f_2(J,q)-g_2(J,q))+\frac{1}{J}\,e^{L+p}(1+\sin(t))\,,
\end{aligned} \tag{3.124}$$

$$
\begin{aligned}
\frac{\partial}{\partial q} r \;=\; & r^2 k_{14} e^L \Big[\sin\big(\frac{s+t}{2}\big) \sinh(L-p) - \cos\big(\frac{s-t}{2}\big) \cosh(L-p) \Big] + \\
& i r^2 k_{13} e^L \Big[\sin\big(\frac{s+t}{2}\big) \cosh(L-p) - \cos\big(\frac{s-t}{2}\big) \sinh(L-p) \Big] + \\
& i k_{31} e^L \Big[\sin\big(\frac{s+t}{2}\big) \cosh(L+p) - \cos\big(\frac{s-t}{2}\big) \sinh(L+p) \Big] + \\
& k_{32} e^L \Big[\sin\big(\frac{s+t}{2}\big) \sinh(L+p) - \cos\big(\frac{s-t}{2}\big) \cosh(L+p) \Big] + \\
& i r k_{11} e^L \cos(s) \cosh(L-p) - r k_{12} e^L \cos(s) \sinh(L-p) - \\
& i r \overline{k_{11}} e^L \cos(t) \cosh(L+p) + r k_{34} e^L \cos(t) \sinh(L+p) - \\
& i r e^L f_1(J,q) (\cos(t) e^p - \cos(s) e^{-p}) - \\
& i r e^{-L} g_1(J,q) \Big(\cos(s) e^p - \cos(t) e^{-p} \Big) + \\
& i r \Big(\sin(s) + \sin(t) \Big) \Big(f_2(J,q) - g_2(J,q) \Big) + \\
& \frac{r}{2J} e^L \Big[\cos(s)\, e^{-p} - \cos(t)\, e^p \Big], \qquad (3.125)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial q} \log a \;=\; & \frac{1}{2} k_{11} e^{-p} \Big[e^{2L} (1 + i e^{is}) - e^{2p} (1 - i e^{is}) \Big] + \\
& \frac{i}{2} k_{12} \Big[e^p (1 + i e^{is}) + e^{2L-p} (1 - i e^{is}) \Big] + \\
& \frac{ir}{2} k_{13} e^{\frac{i}{2}(s-t)-p} \Big[e^{2L} (1 + i e^{is}) - e^{2p} (1 - i e^{is}) \Big] + \\
& \frac{r}{2} k_{14} e^{\frac{i}{2}(s-t)-p} \Big[e^{2L} (1 + i e^{is}) + e^{2p} (1 - i e^{is}) \Big] + \\
& e^{L-p} (1 - i e^{is}) f_1(J,q) + e^{p-L} g_1(J,q) (1 + i e^{is}) + \\
& i (1 + i e^{is})\, f_2(J,q) + i (1 - i e^{is})\, g_2(J,q) - \\
& \frac{1}{2Jq}\, e^{-p} \Big[e^p J + i q e^L (1 - i e^{is}) \Big]. \qquad (3.126)
\end{aligned}
$$

Since we know that s, t and r are real, the equations (3.123), (3.124) and (3.125) will split into two parts, where the first parts equal the change of s, t and r and the second parts vanish. We know from (3.30)-(3.32) that k_{12}, k_{14}, k_{32} and k_{34} are real, k_{13} and k_{31} are pure imaginary, and k_{11} is complex. Define

$$
k_{13} = i k'_{13} \quad , \quad k_{31} = i k'_{31}
$$

$$
k_{11} = k''_{11} + i k'_{11}
$$

where k'_{13}, k'_{31}, k''_{11} and k'_{11} are real. From equation (3.123),

$$\begin{aligned}\frac{\partial}{\partial q}s &= 2rk'_{13}e^{L}\Big[\cos\big(\frac{s+t}{2}\big)\cosh(L-p)+\sin\big(\frac{s-t}{2}\big)\sinh(L-p)\Big]-\\ &\quad 2rk_{14}e^{L}\Big[\sin\big(\frac{s-t}{2}\big)\cosh(L-p)+\cos\big(\frac{s+t}{2}\big)\sinh(L-p)\Big]-\\ &\quad 2k'_{11}e^{L}\Big[\cosh(L-p)\,\sin(s)+\sinh(L-p)\Big]-\\ &\quad 2k_{12}e^{L}\Big[\cosh(L-p)+\sin(s)\,\sinh(L-p)\Big]+\\ &\quad \frac{1}{J}\,e^{L-p}(1+\sin(s))\,, \end{aligned} \tag{3.127}$$

$$\begin{aligned}&2f_1(J,q)e^{L-p}\Big(1+\sin(s)\Big)-2g_1(J,q)\Big(\sin(s)-1\Big)\\ &\qquad -2\cos(s)\Big(f_2(J,q)-g_2(J,q)\Big)+\\ &\qquad 2k''_{11}e^{L}\Big[\cosh(L-p)\,\sin(s)+\sinh(L-p)\Big]\;=0\,. \end{aligned} \tag{3.128}$$

From equation (3.124),

$$\begin{aligned}\frac{\partial}{\partial q}t &= \frac{-2}{r}k'_{13}e^{L}\Big[\cos\big(\frac{s+t}{2}\big)\cosh(L+p)-\sin\big(\frac{s-t}{2}\big)\sinh(L+p)\Big]+\\ &\quad \frac{2}{r}k_{32}e^{L}\Big[\cos\big(\frac{s+t}{2}\big)\sinh(L+p)-\sin\big(\frac{s-t}{2}\big)\cosh(L+p)\Big]-\\ &\quad 2k_{34}e^{L}\Big[\cosh(L+p)+\sin(t)\,\sinh(L+p)\Big]+\\ &\quad 2k'_{11}e^{L}\Big[\cosh(L+p)\,\sin(t)+\sinh(L+p)\Big]+\\ &\quad \frac{1}{J}\,e^{L+p}(1+\sin(t))\,, \end{aligned} \tag{3.129}$$

$$\begin{aligned}&2f_1(J,q)e^{L-p}(1+\sin(t))-2g_1(J,q)(\sin(t)-1)\\ &\qquad +2\cos(t)(f_2(J,q)-g_2(J,q))+\\ &\qquad 2k''_{11}e^{L}\Big[\cosh(L+p)\,\sin(t)+\sinh(L+p)\Big]\;=0\,. \end{aligned} \tag{3.130}$$

Finally from equation (3.125),

$$\begin{aligned}\frac{\partial}{\partial q}r \;=\; & r^2 k_{14} e^L \Big[\sin\big(\frac{s+t}{2}\big)\sinh(L-p) - \cos\big(\frac{s-t}{2}\big)\cosh(L-p)\Big] - \\ & r^2 k'_{13} e^L \Big[\sin\big(\frac{s+t}{2}\big)\cosh(L-p) - \cos\big(\frac{s-t}{2}\big)\sinh(L-p)\Big] - \\ & k'_{31} e^L \Big[\sin\big(\frac{s+t}{2}\big)\cosh(L+p) - \cos\big(\frac{s-t}{2}\big)\sinh(L+p)\Big] + \\ & k_{32} e^L \Big[\sin\big(\frac{s+t}{2}\big)\sinh(L+p) - \cos\big(\frac{s-t}{2}\big)\cosh(L+p)\Big] - \\ & r k'_{11} e^L \cos(s)\cosh(L-p) - r k_{12} e^L \cos(s)\sinh(L-p) - \\ & r k'_{11} e^L \cos(t)\cosh(L+p) + r k_{34} e^L \cos(t)\sinh(L+p) + \\ & \frac{r}{2J}\, e^L \Big[\cos(s)\, e^{-p} - \cos(t)\, e^{p}\Big], \end{aligned} \tag{3.131}$$

$$\begin{aligned} & r e^L f_1(J,q)(\cos(t)e^p - \cos(s)e^{-p}) + r e^{-L} g_1(J,q)\Big(\cos(s)e^p - \\ & \quad \cos(t)e^{-p}\Big) - r\Big(\sin(s) + \sin(t)\Big)\Big(f_2(J,q) - g_2(J,q)\Big) - \\ & \quad\quad r k''_{11} e^L \cos(s)\cosh(L-p) + r k''_{11}\cos(t)\cosh(L+p) \;=\; 0\,. \end{aligned} \tag{3.132}$$

At this stage we will assume that $k_{13} = k_{31} = 0$. This implies by the equations (3.37) and (3.38) that $k_{14} = k_{32} = 0$ as well, and then the matrix K becomes

$$K = \begin{pmatrix} k_{11} & k_{12} & 0 & 0 \\ -k_{12} & k_{11} & 0 & 0 \\ 0 & 0 & \overline{k_{11}} & k_{34} \\ 0 & 0 & -k_{34} & \overline{k_{11}} \end{pmatrix}. \tag{3.133}$$

where k_{12} and k_{34} are real, and k_{11} is complex. From equation (3.36),

$$k_{11} = e^{-L}(N''_{11}(J) + iN'_{11}(J)) \quad , \quad k_{12} = e^{-L}N_{12}(J)\,,$$

$$k_{34} = e^{-L}N_{34}(J)\,, \tag{3.134}$$

where N''_{11}, N'_{11}, N_{13}, and N_{34} are real functions of J only. In this case we can write the equations (3.127), (3.129), (3.131) and (3.126) as

$$\begin{aligned}\frac{\partial}{\partial q}s &= -N'_{11}(J)\Big[\cosh(L-p)\,\sin(s)+\sinh(L-p)\Big]\\&\quad -2N_{12}(J)\Big[\cosh(L-p)+\sin(s)\,\sinh(L-p)\Big]\\&\quad +\frac{1}{J}\,e^{L-p}(1+\sin(s)),\end{aligned}\tag{3.135}$$

$$\begin{aligned}\frac{\partial}{\partial q}t &= 2N'_{11}(J)\Big[\cosh(L+p)\,\sin(t)+\sinh(L+p)\Big]-\\&\quad 2N_{34}(J)\Big[\cosh(L+p)+\sin(t)\,\sinh(L+p)\Big]+\\&\quad \frac{1}{J}\,e^{L+p}(1+\sin(t)),\end{aligned}\tag{3.136}$$

$$\begin{aligned}\frac{\partial}{\partial q}r &= -rN'_{11}(J)\cos(s)\cosh(L-p)-rN_{12}(J)\cos(s)\sinh(L-p)\\&\quad -rN'_{11}(J)\cos(t)\cosh(L+p)+rN_{34}(J)\cos(t)\sinh(L+p)\\&\quad +\frac{r}{2J}\,e^{L}\Big[\cos(s)\,e^{-p}-\cos(t)\,e^{p}\Big].\end{aligned}\tag{3.137}$$

$$\begin{aligned}\frac{\partial}{\partial q}\log a &= \frac{1}{2}N_{11}(J)e^{-(L+p)}\Big[e^{2L}(1+ie^{is})-e^{2p}(1-ie^{is})\Big]+\\&\quad \frac{i}{2}N_{12}(J)e^{-L}\Big[e^{p}(1+ie^{is})+e^{2L-p}(1-ie^{is})\Big]+\\&\quad e^{L-p}(1-ie^{is})f_1(J,q)+e^{p-L}g_1(J,q)(1+ie^{is})+\\&\quad i(1+ie^{is})\,f_2(J,q)+i(1-ie^{is})\,g_2(J,q)-\\&\quad \frac{1}{2Jq}\,e^{-p}\Big[e^{p}J+iqe^{L}(1-ie^{is})\Big],\end{aligned}\tag{3.138}$$

respectively. There are as well the equations (3.56), (3.57), (3.58) and (3.55), which can be written in this case as

$$\frac{\partial}{\partial J}s = \frac{1}{J}\cos(s)-2N_{12}(J)\cos(s),\tag{3.139}$$

$$\frac{\partial}{\partial J}t = \cos(t)\Big(\frac{1}{J}-2N_{34}(J)\Big),\tag{3.140}$$

$$\begin{aligned}\frac{\partial}{\partial J}\log r &= N'_{11}(J)+N_{12}(J)\sin(s)-N_{34}(J)\sin(t)\\ &\quad -\frac{1}{2J}(\sin(s)-\sin(t)),\end{aligned}\tag{3.141}$$

$$\frac{\partial}{\partial J}\log a = i\Big[N''_{11}(J)+\Big(N_{12}(J)-\frac{1}{2J}\Big)e^{is}\Big].\tag{3.142}$$

There are two equations for s, (3.135) and (3.139), which we need to solve. Now we use the symmetry of double derivatives,

$$\frac{\partial}{\partial q}\frac{\partial}{\partial J}s=\frac{\partial}{\partial J}\frac{\partial}{\partial q}s.$$

The last equation gives

$$\begin{aligned}&\Big[\Big(\frac{1}{J}-2N'_{11}(J)\Big)\Big(\frac{1}{J}-2N_{12}(J)\Big)+\frac{\partial}{\partial J}\Big(\frac{1}{J}-2N_{12}(J)\Big)\Big]\\ &\qquad\Big(\cosh(L-p)+\sin(s)\sinh(L-p)\Big) \quad +\\ &\Big[\Big(\frac{1}{J}-2N_{12}(J)\Big)^2+\frac{\partial}{\partial J}\Big(\frac{1}{J}-2N'_{11}(J)\Big)\Big]\\ &\qquad\Big(\sinh(L-p)+\sin(s)\cosh(L-p)\Big) \quad =0.\end{aligned}$$

By substituting the value for $\sin(s)$ from (3.60), the last equation becomes

$$\begin{aligned}&\Big[\Big(\Big(\frac{1}{J}-2N'_{11}(J)\Big)\Big(\frac{1}{J}-2N_{12}(J)\Big)+\frac{\partial}{\partial J}\Big(\frac{1}{J}-2N_{12}(J)\Big)\Big)\\ &\quad+\Big(\Big(\frac{1}{J}-2N_{12}(J)\Big)^2+\frac{\partial}{\partial J}\Big(\frac{1}{J}-2N'_{11}(J)\Big)\Big)\Big]E^2(J,q)+\\ &\Big[\Big(\Big(\frac{1}{J}-2N'_{11}(J)\Big)\Big(\frac{1}{J}-2N_{12}(J)\Big)+\frac{\partial}{\partial J}\Big(\frac{1}{J}-2N_{12}(J)\Big)\Big)\\ &\quad-\Big(\Big(\frac{1}{J}-2N_{12}(J)\Big)^2+\frac{\partial}{\partial J}\Big(\frac{1}{J}-2N'_{11}(J)\Big)\Big)\Big] \;=\; 0.\end{aligned}\tag{3.143}$$

If $E(J,q)$ varies as a function of q, then the first term varies with q whereas the second is a constant when q varies. We deduce that the two square brackets vanish. Define

$$R_1(J)=\Big(\frac{1}{J}-2N'_{11}(J)\Big)\quad\text{and}\quad R_2(J)=\Big(\frac{1}{J}-2N_{12}(J)\Big).\tag{3.144}$$

Hence

$$R_1(J)R_2(J) + \frac{d}{dJ}R_2(J) = 0, \tag{3.145}$$

and

$$R_2^2(J) + \frac{d}{dJ}R_1(J) = 0. \tag{3.146}$$

From the equations (3.145) and (3.146),

$$\frac{d}{dJ}\left(R_1^2(J) - R_2^2(J)\right) = 0 \quad \text{so that} \quad R_1^2(J) - R_2^2(J) = G^2, \tag{3.147}$$

where G is an arbitrary constant. We rewrite equation (3.146) as

$$\frac{d}{dJ}R_1(J) = G^2 - R_1^2(J).$$

By integrating the last equation, for $G \neq 0$ it becomes

$$\frac{R_1(J) - G}{R_1(J) + G} = e^{-2GJ+2c},$$

where c an arbitrary constant. From the last equation we find

$$R_1(J) = \frac{\cosh(GJ - c)}{\sinh(GJ - c)}\,G, \tag{3.148}$$

where $G \neq 0$ and c are arbitrary constants. Now by substituting this value for $R_1(J)$ in equation (3.145), it becomes

$$\frac{d}{dJ}R_2(J) = -\Big(\frac{\cosh(GJ - c)}{\sinh(GJ - c)}\,G\Big)\,R_2(J).$$

By integrating the last equation,

$$R_2(J) = \frac{\alpha}{\sinh(GJ - c)}, \tag{3.149}$$

where α is constant. Now substitute the values for $R_1(J)$ and $R_2(J)$ which given by (3.148) and (3.149) in equation (3.146), to find the relation between G and α, which is $\alpha^2 = G^2$, so that $\alpha = \pm G$.

From (3.144), $N'_{11}(J)$ and $N_{12}(J)$ are given by

$$N'_{11}(J) = \frac{1}{2}\Big[\frac{1}{J} - \frac{\cosh(GJ-c)}{\sinh(GJ-c)}G\Big], \tag{3.150}$$

$$N_{12}(J) = \frac{1}{2}\Big[\frac{1}{J} - \frac{\alpha}{\sinh(GJ-c)}\Big], \tag{3.151}$$

where $\alpha = \pm G$.

We also need to solve two equations for t, (3.136) and (3.140). Using the symmetry of double derivatives,

$$\frac{\partial}{\partial q}\frac{\partial}{\partial J}t = \frac{\partial}{\partial J}\frac{\partial}{\partial q}t.$$

The last equation gives us

$$\begin{aligned}\Big[\Big(\frac{1}{J} - 2N_{34}(J)\Big)\Big(\frac{1}{J} + 2N'_{11}(J)\Big) + \frac{\partial}{\partial J}\Big(\frac{1}{J} - 2N_{12}(J)\Big)\Big]&\\ \Big(\cosh(L+p) + \sin(t)\sinh(L+p)\Big) \quad &+\\ \Big[\Big(\frac{1}{J} - 2N_{34}(J)\Big)^2 + \frac{\partial}{\partial J}\Big(\frac{1}{J} + 2N'_{11}(J)\Big)\Big]&\\ \Big(\cosh(L+p)\sin(t) + \sinh(L+p)\Big) \quad &= 0.\end{aligned}$$

By substituting the value for $\sin(t)$ from (3.62), the last equation becomes

$$\begin{aligned}&\Big[\Big(\Big(\frac{1}{J} - 2N_{34}(J)\Big)\Big(\frac{1}{J} + 2N'_{11}(J)\Big) + \frac{\partial}{\partial J}\Big(\frac{1}{J} - 2N_{12}(J)\Big)\Big)\\ &+\Big(\Big(\frac{1}{J} - 2N_{34}(J)\Big)^2 + \frac{\partial}{\partial J}\Big(\frac{1}{J} + 2N'_{11}(J)\Big)\Big)\Big]D^2(J,q) +\\ &\Big[\Big(\Big(\frac{1}{J} - 2N_{34}(J)\Big)\Big(\frac{1}{J} + 2N'_{11}(J)\Big) + \frac{\partial}{\partial J}\Big(\frac{1}{J} - 2N_{12}(J)\Big)\Big)\\ &-\Big(\Big(\frac{1}{J} - 2N_{34}(J)\Big)^2 + \frac{\partial}{\partial J}\Big(\frac{1}{J} + 2N'_{11}(J)\Big)\Big)\Big] = 0.\end{aligned} \tag{3.152}$$

If $D(J,q)$ varies as a function of q, then the first term varies with q whereas the second is a constant when q varies. Then we deduce that the two square brackets vanish. Define

$$T_1(J) = \Big(\frac{1}{J} + 2N'_{11}(J)\Big) \quad \text{and} \quad T_2(J) = \Big(\frac{1}{J} - 2N_{34}(J)\Big). \tag{3.153}$$

Thus the result is given by the following two equations

$$T_1(J)T_2(J) + \frac{d}{dJ}T_2(J) = 0\,, \tag{3.154}$$

$$T_2^2(J) + \frac{d}{dJ}T_1(J) = 0\,. \tag{3.155}$$

From (3.150) the value for $N'_{11}(J)$, and by substituting it in $T_1(J)$, it becomes

$$T_1(J) = \Big(\frac{2}{J} - \frac{\cosh(GJ-c)}{\sinh(GJ-c)}\,G\Big)\,.$$

Now by substituting this value for $T_1(J)$ in equation (3.154), it becomes

$$\frac{d}{dJ}T_2(J) = -T_2(J)\Big(\frac{2}{J} - \frac{\cosh(GJ-c)}{\sinh(GJ-c)}\,G\Big)\,.$$

Integrating the last equation gives

$$T_2(J) = \delta\,J^{-2}\sinh(GJ-c)\,, \tag{3.156}$$

where δ is an arbitrary constant. Substituting the value for $T_1(J)$ in equation (3.155) gives

$$\frac{d}{dJ}\Big(\frac{2}{J} - \frac{\cosh(GJ-c)}{\sinh(GJ-c)}\,G\Big) = -T_2^2(J)\,,$$

which becomes

$$T_2(J) = \pm\Big(\frac{2}{J^2} - \frac{G^2}{\sinh^2(GJ-c)}\Big)^{\frac{1}{2}}\,. \tag{3.157}$$

Note that we have two different values, (3.156) and (3.157), for $T_2(J)$, and there is no possibility of chose of constants to make them equal. The fact is that the equations (3.145), (3.146), (3.154) and (3.155) are all true if and only if $G = 0$. We conclude that $G \neq 0$ is not a valid possibility, and continue with $G = 0$.

Now substitute $G = 0$ in equation (3.147), giving

$$R_1^2(J) - R_2^2(J) = 0\,, \quad \text{so} \quad R_1^2(J) = R_2^2(J)\,.$$

By using this result in equation (3.146), it becomes

$$\frac{d}{dJ} R_1(J) = -R_1^2(J) .$$

Now integrating the last equation,

$$R_1(J) = \frac{1}{J+c}, \quad \text{so} \quad R_2(J) = \pm\frac{1}{J+c}, \tag{3.158}$$

where c is an arbitrary constant. The values for $N'_{11}(J)$ and $N_{12}(J)$ are given by

$$N'_{11}(J) = \frac{1}{2}\Big[\frac{1}{J} - \frac{1}{J+c}\Big] \quad \text{and} \quad N_{12}(J) = \frac{1}{2}\Big[\frac{1}{J} - \frac{(\pm 1)}{J+c}\Big] . \tag{3.159}$$

We have from (3.159) the value for $N'_{11}(J)$, and by substituting it in $T_1(J)$ which given by (3.153), we find

$$T_1(J) = \Big(\frac{2}{J} - \frac{1}{J+c}\Big) . \tag{3.160}$$

Now by substituting this value for $T_1(J)$ in equation (3.154), it becomes

$$\frac{d}{dJ} T_2(J) = -T_2(J)\Big(\frac{2}{J} - \frac{1}{J+c}\Big) .$$

Integrating the last equation,

$$T_2(J) = \delta\, J^{-2}(J+c) , \tag{3.161}$$

where δ is an arbitrary constant. Substituting the value for $T_1(J)$ given by (3.160) in equation (3.155), it becomes

$$\frac{d}{dJ}\Big(\frac{2}{J} - \frac{1}{J+c}\Big) = -T_2^2(J) ,$$

and we have

$$T_2^2(J) = \Big(\frac{2}{J^2} - \frac{1}{(J+c)^2}\Big) . \tag{3.162}$$

Now by comparing the two values for $T_2(J)$,

$$\delta^2 = \frac{2J^2}{(J+c)^2} - \frac{J^4}{(J+c)^4} ,$$

but to make the last equation true for all J we have to put $c = 0$, and we deduce that $\delta = \pm 1$.

Putting $c = 0$ and $\delta = \pm 1$ in the equations (3.159), (3.161) and (3.153), then they become

$$N'_{11}(J) = 0 , \tag{3.163}$$

$$N_{12}(J) = 0 \qquad \text{or} \qquad N_{12}(J) = \frac{1}{J} , \tag{3.164}$$

$$N_{34}(J) = 0 \qquad \text{or} \qquad N_{34}(J) = \frac{1}{J} . \tag{3.165}$$

Therefore there are four cases. We are not going to do them all because they are much like each other. We will consider the following two cases:

Case (I): For $N'_{11}(J) = N_{12}(J) = N_{34}(J) = 0$

Let us start with s, by substituting the value for $N_{12}(J)$ in equation (3.139),

$$\frac{\partial}{\partial J} s = \frac{1}{J} \cos(s) . \tag{3.166}$$

We can write the last equation as

$$\frac{d\,s}{\cos(s)} = \frac{d\,J}{J} .$$

By integrating this last equation,

$$\log\Big(\sec(s) + \tan(s)\Big) = p - L + \log J + \log E(q) , \tag{3.167}$$

where $E(q)$ is any function of q, and we can rewrite the last equation as

$$\sec(s) + \tan(s) = J\, E(q)\, e^{p-L} .$$

Then we can write $\sin(s)$ and $\cos(s)$ as

$$\sin(s) = \frac{J^2 E^2(q) e^{2(p-L)} - 1}{J^2 E^2(q) e^{2(p-L)} + 1} ,$$

$$\cos(s) = \frac{2 J E(q) e^{p-L}}{J^2 E^2(q) e^{2(p-L)} + 1} . \tag{3.168}$$

Now by differentiating (3.167) with respect to q, it becomes

$$\frac{\partial}{\partial q} \log\Big(\sec(s) + \tan(s)\Big) = \frac{1}{\cos(s)} \frac{\partial s}{\partial q} = \frac{d}{dq} \log E(q) .$$

By substituting the values for $\partial s/\partial q$ given by (3.135) and $\cos(s)$ in the last equation, it becomes

$$\frac{J^2 E^2(q) e^{2(p-L)} + 1}{2 J E(q) e^{p-L}} \Big[\frac{1}{J} e^{L-p}\Big(1 + \frac{J^2 E^2(q) e^{2(p-L)} - 1}{J^2 E^2(q) e^{2(p-L)} + 1}\Big)\Big] = \frac{d}{dq} \log E(q) ,$$

and by simplifying the last equation,

$$\frac{d}{dq} E(q) = E^2(q) .$$

Now by integrating the last equation,

$$E(q) = \frac{-1}{q + \alpha} , \tag{3.169}$$

where α is an arbitrary constant.
Finally we can write $\sin(s)$ and $\cos(s)$ as

$$\sin(s) = \frac{\Big(\frac{J^2}{(q+\alpha)^2}\Big) e^{2(p-L)} - 1}{\Big(\frac{J^2}{(q+\alpha)^2}\Big) e^{2(p-L)} + 1} ,$$

$$\cos(s) = \frac{-2\Big(\frac{J}{(q+\alpha)}\Big) e^{p-L}}{\Big(\frac{J^2}{(q+\alpha)^2}\Big) e^{2(p-L)} + 1} . \tag{3.170}$$

Similarly for t, by substituting the value for $N_{34}(J)$ in equation (3.140), it becomes

$$\frac{\partial}{\partial J} t = \frac{1}{J} \cos(t) , \tag{3.171}$$

and we can write the last equation as

$$\frac{dt}{\cos(t)} = \frac{dJ}{J} .$$

By integrating this last equation,

$$\log\Big(\sec(t)+\tan(t)\Big) = -p-L+\log J+\log D(q)\,, \tag{3.172}$$

where $D(q)$ is any function of q, and we can rewrite the last equation as

$$\sec(t)+\tan(t) = J\,D(q)\,e^{-(p+L)}\,.$$

Then we can write $\sin(t)$ and $\cos(t)$ as

$$\sin(t) = \frac{J^2D^2(q)e^{-2(p+L)}-1}{J^2D^2(q)e^{-2(p+L)}+1}\,,$$

$$\cos(t) = \frac{2JD(q)e^{-(p+L)}}{J^2D^2(q)e^{-2(p+L)}+1}\,.$$

Now by differentiating (3.172) with respect to q, it becomes

$$\frac{\partial}{\partial q}\log\Big(\sec(t)+\tan(t)\Big) = \frac{1}{\cos(t)}\frac{\partial t}{\partial q} = \frac{d}{dq}\log D(q)\,.$$

By substituting the values for $\partial t/\partial q$ given by (3.136) and $\cos(t)$ in the last equation, it becomes

$$\frac{J^2D^2(q)e^{-2(p+L)}+1}{2JD(q)e^{-(p+L)}}\Big[\frac{1}{J}e^{-(p+L)}\Big(1+\frac{J^2D^2(q)e^{-2(p+L)}-1}{J^2(q)e^{-2(p+L)}+1}\Big)\Big] = \frac{d}{dq}\log D(q)\,,$$

and by simplifying the last equation,

$$\frac{d}{dq}D(q) = D^2(q)\,.$$

Now by integrating the last equation,

$$D(q) = \frac{-1}{q+\beta}\,, \tag{3.173}$$

where β is an arbitrary constant. Finally we can write $\sin(t)$ and $\cos(t)$ as

$$\sin(t) = \frac{\Big(\frac{J^2}{(q+\beta)^2}\Big)e^{-2(p+L)}-1}{\Big(\frac{J^2}{(q+\beta)^2}\Big)e^{-2(p+L)}+1}\,,$$

$$\cos(t) = \frac{-2\Big(\frac{J}{(q+\beta)}\Big)e^{-(p+L)}}{\Big(\frac{J^2}{(q+\beta)^2}\Big)e^{-2(p+L)}+1}\,. \tag{3.174}$$

Now if we substitute the values for $N'_{11}(J)$, $N_{12}(J)$ and $N_{34}(J)$ in equation (3.137), then it becomes

$$\frac{\partial}{\partial q}\log r = \frac{1}{2J}e^{L}\left[\cos(s)\,e^{-p} - \cos(t)\,e^{p}\right]. \tag{3.175}$$

By substituting the values for $\cos(s)$ and $\cos(t)$ in (3.175), it becomes

$$\frac{d}{dq}\log r = \frac{(q+\beta)}{(q+\beta)^2 + J^2e^{-2(p+L)}} - \frac{(q+\alpha)}{(q+\alpha)^2 + J^2e^{2(p-L)}}.$$

Now by integrating both sides with respect to q gives

$$r = \left(\frac{(q+\beta)^2 + J^2e^{-2(p+L)}}{(q+\alpha)^2 + J^2e^{2(p-L)}}\right)^{\frac{1}{2}}, \tag{3.176}$$

where α and β are arbitrary constants.

Next substitute the values $\cos(s)$, $\sin(s)$, $\cos(t)$ and $\sin(t)$ in the equations (3.128), (3.130) and (3.132). Let us first start with the equations (3.128) and (3.130)

$$\begin{aligned} 2f_1(J,q)\frac{J^2}{(q+\alpha)^2} &+ 2\frac{J}{(q+\alpha)}\big(f_2(J,q) - g_2(J,q)\big) \\ +2g_1(J,q) &+ N''_{11}(J)\Big(\frac{J^2}{(q+\alpha)^2} - 1\Big) = 0, \end{aligned} \tag{3.177}$$

$$\begin{aligned} 2f_1(J,q)\frac{J^2}{(q+\beta)^2} &- 2\frac{J}{(q+\beta)}\big(f_2(J,q) - g_2(J,q)\big) \\ +2g_1(J,q) &+ N''_{11}(J)\Big(\frac{J^2}{(q+\beta)^2} - 1\Big) = 0. \end{aligned} \tag{3.178}$$

From equation (3.132),

$$
\begin{aligned}
e^{-2(p+L)} &\Big[\big(2f_1(J,q)+N''_{11}(J)\big)\frac{J^3}{(q+\alpha)(q+\beta)^2}+\\
&\big(2g_1(J,q)-N''_{11}(J)\big)\frac{J}{(q+\beta)}\Big]-\\
e^{2(p-L)} &\Big[\big(2f_1(J,q)+N''_{11}(J)\big)\frac{J^3}{(q+\alpha)^2(q+\beta)}+\\
&\big(2g_1(J,q)-N''_{11}(J)\big)\frac{J}{(q+\alpha)}\Big]+\\
e^{-4L} &\Big[\big(2g_1(J,q)-N''_{11}(J)\big)\frac{J^3}{(q+\alpha)^2(q+\beta)}-\\
&\big(2g_1(J,q)-N''_{11}(J)\big)\frac{J^3}{(q+\alpha)(q+\beta)^2}-\\
&2\big(f_2(J,q)-g_2(J,q)\big)\frac{J^4}{(q+\alpha)^2(q+\beta)^2}\Big]+\\
&\Big[\big(2f_1(J,q)+N''_{11}(J)\big)\Big(\frac{J}{(q+\alpha)}-\frac{J}{(q+\beta)}\Big)\\
&+2\big(f_2(J,q)-g_2(J,q)\big)\Big]=0\,.
\end{aligned}
$$

This equation must be true for all values of L and p, so we deduce that the following equations are true:

$$\big(2f_1(J,q)+N''_{11}(J)\big)\frac{J^2}{(q+\alpha)(q+\beta)}+\big(2g_1(J,q)-N''_{11}(J)\big)=0\,, \quad (3.179)$$

$$\big(2g_1(J,q)-N''_{11}(J)\big)\,(\beta-\alpha)-2\big(f_2(J,q)-g_2(J,q)\big)\,J=0\,, \quad (3.180)$$

$$\big(2f_1(J,q)+N''_{11}(J)\big)\,\frac{J(\beta-\alpha)}{(q+\alpha)(q+\beta)}-2\big(f_2(J,q)-g_2(J,q)\big)=0\,. \quad (3.181)$$

By rearranging the equation (3.181) and adding it to equation (3.180),

$$f_2(J,q)-g_2(J,q) \;=\; \frac{J(\beta-\alpha)}{J^2-(q+\alpha)(q+\beta)}\big(f_1(J,q)+g_1(J,q)\big)\,. \quad (3.182)$$

Next we substitute the values for $f_2(J,q) - g_2(J,q)$ and $f_1(J,q) + g_1(J,q)$ that we had from (3.121) and (3.116) respectively in equation (3.182), and it becomes

$$\frac{J(\beta-\alpha)}{J^2-(q+\alpha)(q+\beta)}\left[d_1\,(c_1\,J^{-1}+c_2\,J)\ln(q)+d_2(c_1\,J^{-1}+c_2\,J)\right]$$
$$= u_1\,(v_1\,q^{-1}+v_2\,q)\ln(J)+u_2(v_1\,q^{-1}+v_2\,q)\,,$$

where the only variables in the last equation are J and q, and the rest are all arbitrary constants. To compare both sides in the last equation, first start with the terms that have logarithms in them, and we find these two terms vanish. This leaves

$$\left[\frac{J(\beta-\alpha)}{J^2-(q+\alpha)(q+\beta)}\right]d_2(c_1\,J^{-1}+c_2\,J) = u_2(v_1\,q^{-1}+v_2\,q)\,.$$

By comparing coefficients

$$\begin{aligned} f_2(J,q)-g_2(J,q) &= 0 \\ \text{and either}\quad f_1(J,q)+g_1(J,q) &= 0 \ \text{ or }\ \beta = \alpha\,. \end{aligned} \qquad (3.183)$$

In fact we found that if we started with assuming $\beta = \alpha$ at some stage we will get $f_1(J,q) + g_1(J,q) = 0$. This means one case is a special case of the other. Next we are going to consider the general possibility which is $f_1(J,q) + g_1(J,q) = 0$.

Suppose that $f_1(J,q) + g_1(J,q) = 0$, and we have

$$f_2(J,q) - g_2(J,q) = 0\,.$$

Now by substituting the last result in the equations (3.181) and (3.180), we get the values for $f_1(J,q)$ and $g_1(J,q)$ which are

$$f_1(J,q) = \frac{-1}{2}N''_{11}(J) \quad \text{and} \quad g_1(J,q) = \frac{1}{2}N''_{11}(J)\,. \qquad (3.184)$$

Note that from these values for $f_1(J,q)$ and $g_1(J,q)$, it seem that f_1 and g_1 are functions of J only. Now by using all these results in equation (3.95), it becomes

$$\frac{\partial}{\partial J}\,f_2(J,q) = \frac{b}{2}\,(J\,q)^{-1}\,. \qquad (3.185)$$

By integrating equation (3.185) with respect to J,

$$f_2(J,q) = \frac{b}{2} q^{-1} \ln J + C(q) \tag{3.186}$$

where $C(q)$ is a function of q and b is an arbitrary constant.

Now to work out the equations (3.138) and (3.142), by differentiating (3.138) with respect to J and (3.142) with respect to q, and then by substituting the results in the following equation

$$\frac{\partial}{\partial J}(\frac{\partial}{\partial q} \log a) = \frac{\partial}{\partial q}(\frac{\partial}{\partial J} \log a),$$

it gives

$$\begin{aligned} &2i \cosh(L-p) \cos(s) \Big(\frac{\partial}{\partial J} N''_{11}(J) - \frac{1}{J} \sin(s) N''_{11}(J)\Big) + \frac{b}{Jq} \\ &-2 \cosh(L-p) \Big(\sin(s) \frac{\partial}{\partial J} N''_{11}(J) + \frac{1}{J} \cos^2(s) N''_{11}(J)\Big) = 0 . \end{aligned} \tag{3.187}$$

Now by taking the real and imaginary parts of this equation, we will find that

$$N''_{11}(J) = 0 \quad \text{and} \quad b = 0, \tag{3.188}$$

so that

$$f_1 = g_1 = 0 \quad , \quad f_2 = g_2 = C(q) .$$

Finally by substituting these values for f_1, g_1 and f_2 in equation (3.138), it becomes

$$\begin{aligned} \frac{\partial}{\partial q} \log a &= 2i\, C(q) - i \frac{J e^{p-L}}{J^2 e^{2(p-L)} + (q+\alpha)^2} \\ &\quad - \frac{1}{2q} + \frac{(q+\alpha)}{J^2 e^{2(p-L)} + (q+\alpha)^2}, \end{aligned} \tag{3.189}$$

and by integrating the last equation with respect to q,

$$a = \frac{1}{\sqrt{q}} \sqrt{(J^2 e^{2(p-L)} + (q+\alpha)^2)}\; e^{-i\, tan^{-1}\left(\frac{(q+\alpha)}{J} e^{L-p}\right) + 2iC(q)} . \tag{3.190}$$

A_1 and A_3 are vanishing, and A_2 and A_4 are given by

$$
\begin{aligned}
A_2 &= 2\,C(q)\,\cosh{(p)}\,, \\
A_4 &= -\,2\,C(q)\,\sinh{(p)}\,,
\end{aligned}
$$

where $C(q)$ is any function of q. The electromagnetic field in this case is vanishing.

Summary of case (I) :

The solution for the Dirac equation is given by

$$
\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,r e^{\frac{i}{2}\,(s-t)} \\ i\,r e^{\frac{i}{2}\,(s+t)} \end{pmatrix} a \quad \text{where}\,,
$$

$$
\begin{aligned}
a &= \frac{1}{\sqrt{q}}\sqrt{\left(J^2 e^{2(p-L)} + (q+\alpha)^2\right)}\; e^{-i\,tan^{-1}\left(\frac{(q+\alpha)}{J}e^{L-p}\right)+2iC(q)}\,, \\
r &= \left(\frac{(q+\beta)^2 + J^2 e^{-2(p+L)}}{(q+\alpha)^2 + J^2 e^{2(p-L)}}\right)^{\frac{1}{2}},
\end{aligned}
$$

$$
\sin{(s)} = \frac{\left(\frac{J^2}{(q+\alpha)^2}\right)e^{2(p-L)} - 1}{\left(\frac{J^2}{(q+\alpha)^2}\right)e^{2(p-L)} + 1}\,, \qquad \cos{(s)} = \frac{-2\left(\frac{J}{(q+\alpha)}\right)e^{p-L}}{\left(\frac{J^2}{(q+\alpha)^2}\right)e^{2(p-L)} + 1}\,,
$$

$$
\sin{(t)} = \frac{\left(\frac{J^2}{(q+\beta)^2}\right)e^{-2(p+L)} - 1}{\left(\frac{J^2}{(q+\beta)^2}\right)e^{-2(p+L)} + 1}\,, \qquad \cos{(t)} = \frac{-2\left(\frac{J}{(q+\beta)}\right)e^{-(p+L)}}{\left(\frac{J^2}{(q+\beta)^2}\right)e^{-2(p+L)} + 1}\,,
$$

where α and β are arbitrary constants.

A_1 and A_3 are vanishing, and A_2 and A_4 are given by

$$
\begin{aligned}
A_2 &= 2\,C(q)\,\cosh{(p)}\,, \\
A_4 &= -\,2\,C(q)\,\sinh{(p)}\,,
\end{aligned}
$$

where $C(q)$ is any function of q. The electromagnetic field in this case is vanishing.

The variables L, J, p and q are written in the original variables as

$$L = \frac{1}{2}\ln(\frac{x^3+x^1}{x^3-x^1}) \quad \text{and} \quad J = \left((x^3)^2-(x^1)^2\right)^{\frac{1}{2}},$$

$$p = \frac{1}{2}\ln(\frac{x^2+x^4}{x^2-x^4}) \quad \text{and} \quad q = \left((x^2)^2-(x^4)^2\right)^{\frac{1}{2}}.$$

For more details about these variables see p76.

Note that there could be problems with imaginary values for s and t for general α and β. However if we put $\alpha=\beta=0$, we get

$$\sin(s) = \frac{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2-1}{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1}, \quad \cos(s) = \frac{-2\left(\frac{x^3-x^1}{x^2-x^4}\right)}{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1},$$

$$\sin(t) = \frac{\left(\frac{x^3-x^1}{x^2+x^4}\right)^2-1}{\left(\frac{x^3-x^1}{x^2+x^4}\right)^2+1}, \quad \cos(t) = \frac{-2\left(\frac{x^3-x^1}{x^2+x^4}\right)}{\left(\frac{x^3-x^1}{x^2+x^4}\right)^2+1}.$$

The value for a and r become

$$a = \left((x^2)^2-(x^4)^2\right)^{-3/4}\sqrt{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1}\; e^{-i\,tan^{-1}\left(\frac{x^2-x^4}{x^3-x^1}\right)+2iC(q)},$$

$$r = \left(\frac{\left(\frac{x^3-x^1}{x^2+x^4}\right)^2+1}{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1}\right)^{\frac{1}{2}}.$$

Note that since $C(q)$ is an arbitrary function of q, then we can choose it to cancel the multiple $q^{-3/2}$ in the value of a.

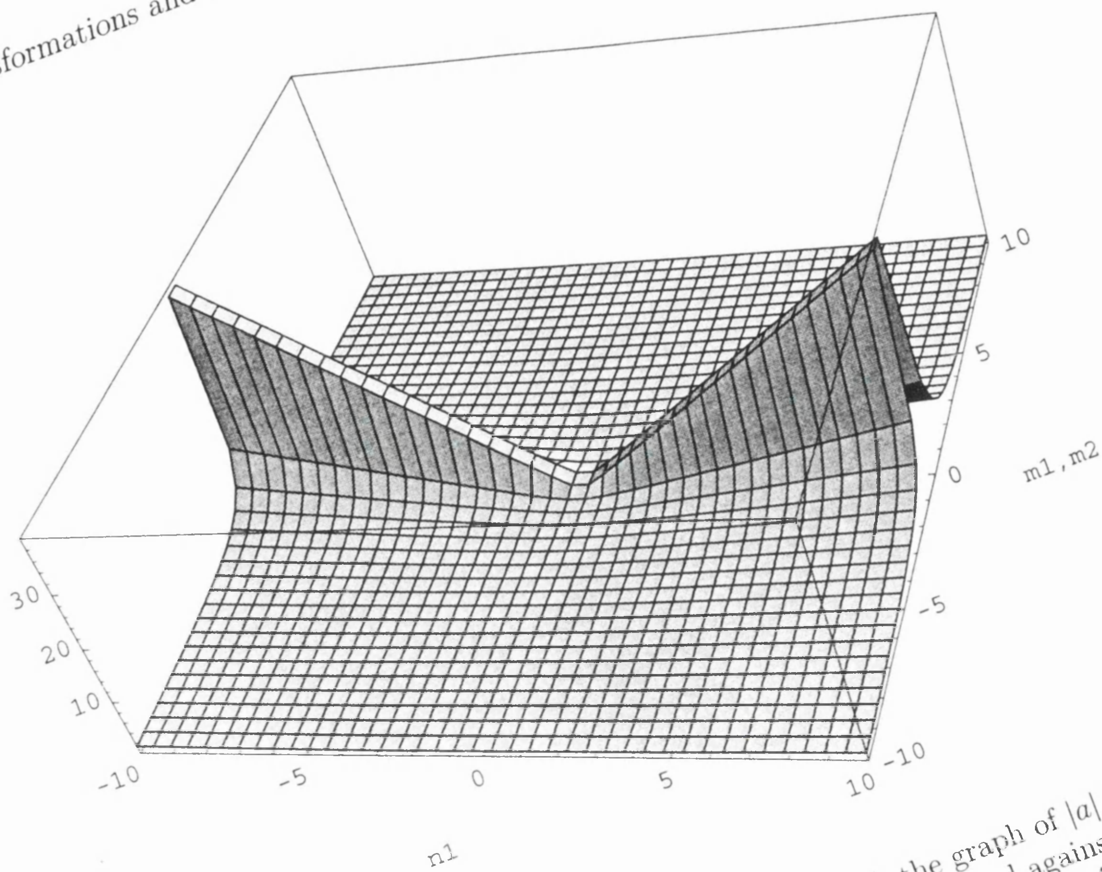

Figure 3.3: Shows the graphs of both $|a|$ and $|ra|$, the graph of $|a|$ is plotted against n_1 and m_1, and the graph of $|ra|$ is plotted against n_1 and m_2, where $n_1 = x^3 - x^1$, $m_1 = x^2 - x^4$, and $m_2 = x^2 + x^4$. The world line of the particle is given by $n_1 = 0$.

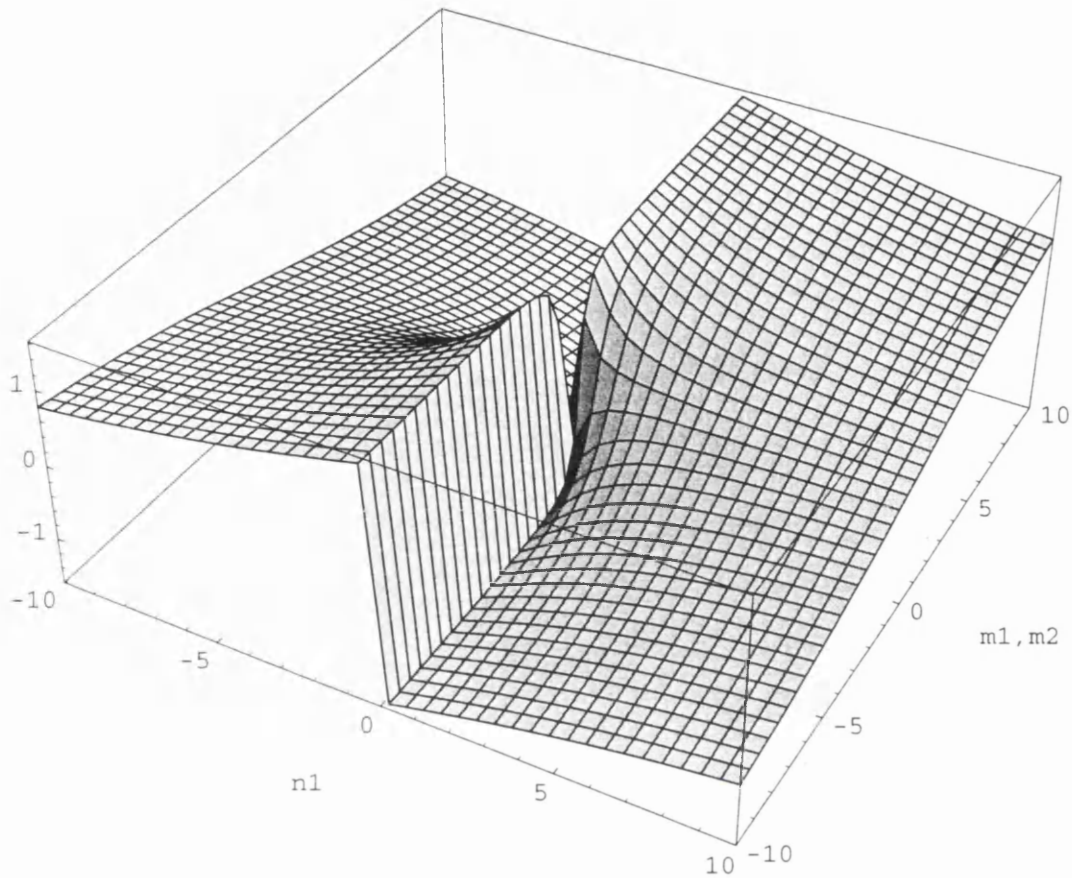

Figure 3.4: Shows the graphs of both $Arg(a)$ and $Arg(ra)$, the graph of $Arg(a)$ is plotted against n_1 and m_1, and the graph of $Arg(ra)$ is plotted against n_1 and m_2, where $n_1 = x^3 - x^1$, $m_1 = x^2 - x^4$, and $m_2 = x^2 + x^4$. The world line of the particle is given by $n_1 = 0$.

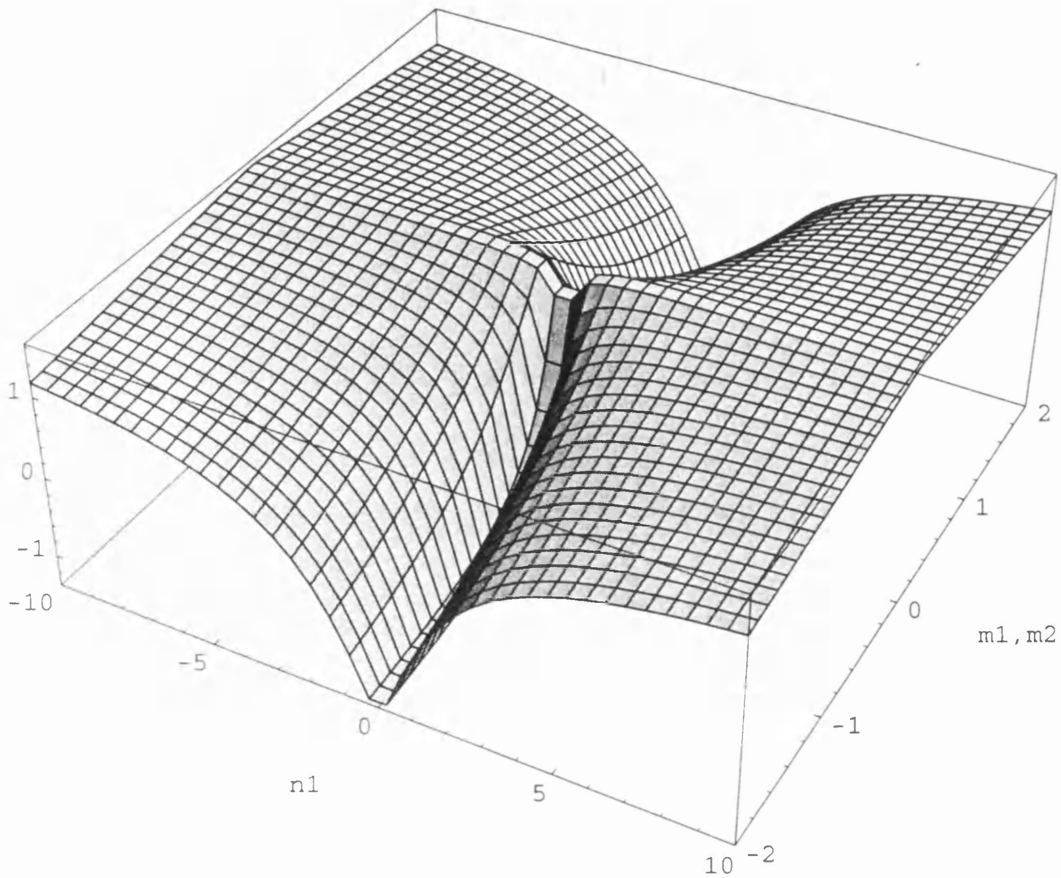

Figure 3.5: Shows the graphs of both s and t, the graph of s is plotted against n_1 and m_1 and the graph of t is plotted against n_1 and m_2, where $n_1 = x^3 - x^1$, $m_1 = x^2 - x^4$, and $m_2 = x^2 + x^4$. The world line of the particle is given by $n_1 = 0$.

Case (II): For $N'_{11}(J) = N_{12}(J) = 0$ and $N_{34}(J) = 1/J$

In this case if we substitute the value for $N_{12}(J)$ in equation (3.139), then for s we will have the same as in the first case. But if we substitute the value for $N_{34}(J)$ in equation (3.140), it becomes

$$\frac{\partial}{\partial J} t = \frac{-1}{J} \cos(t), \tag{3.191}$$

and we can write the last equation as

$$\frac{d\,t}{\cos(t)} = -\frac{d\,J}{J}.$$

By integrating the last equation,

$$\log\Big(\sec(t)+\tan(t)\Big) = -p - L - \log J + \log D(q), \tag{3.192}$$

where $D(q)$ is any function of q, and we rewrite the last equation as

$$\sec(t)+\tan(t) = \frac{1}{J}\,D(q)\,e^{-(p+L)}.$$

Then we write $\sin(t)$ and $\cos(t)$ as

$$\sin(t) = \frac{\frac{D^2(q)}{J^2}e^{-2(p+L)}-1}{\frac{D^2(q)}{J^2}e^{-2(p+L)}+1},$$

$$\cos(t) = \frac{2\frac{D(q)}{J}e^{-(p+L)}}{\frac{D^2(q)}{J^2}e^{-2(p+L)}+1}.$$

Now by differentiating (3.192) with respect to q,

$$\frac{\partial}{\partial q}\log\Big(\sec(t)+\tan(t)\Big) = \frac{1}{\cos(t)}\frac{\partial\,t}{\partial\,q} = \frac{d}{d\,q}\log D(q).$$

Substituting the values for $\partial t/\partial q$ given by (3.136) and $\cos(t)$ the last equation becomes

$$\frac{\frac{D^2(q)}{J^2}e^{-2(p+L)}+1}{2\frac{D(q)}{J}e^{-(p+L)}}\left[\frac{1}{J}e^{-(p+L)}\Big(\frac{\frac{D^2(q)}{J^2}e^{-2(p+L)}-1}{\frac{D^2(q)}{J^2}e^{-2(p+L)}+1}-1\Big)\right] = \frac{d}{d\,q}\log D(q),$$

and by simplifying the last equation,

$$\frac{d}{d\,q}D(q) = -1.$$

Now by integrating both sides,

$$D(q) = -1\,(q+\beta), \tag{3.193}$$

where β is an arbitrary constant.
Finally we can write $\sin(t)$ and $\cos(t)$ as

$$\begin{aligned} \sin(t) &= \frac{\left(\frac{(q+\beta)^2}{J^2}\right)e^{-2(p+L)} - 1}{\left(\frac{(q+\beta)^2}{J^2}\right)e^{-2(p+L)} + 1}, , \\ \cos(t) &= \frac{-2\left(\frac{(q+\beta)}{J}\right)e^{-(p+L)}}{\left(\frac{(q+\beta)^2}{J^2}\right)e^{-2(p+L)} + 1}. \end{aligned} \tag{3.194}$$

Now substitute the values for $N'_{11}(J)$, $N_{12}(J)$ and $N_{34}(J)$ in equation (3.137), it becomes

$$\frac{\partial}{\partial q}\log r = \frac{1}{2J}\left[\cos(s)\, e^{L-p} - \cos(t)\, e^{-(L+p)}\right]. \tag{3.195}$$

By substituting the values for $\cos(s)$ and $\cos(t)$ equation (3.195) becomes

$$\frac{d}{dq}\log r = \frac{(q+\beta)}{(q+\beta)^2 + J^2 e^{2(p+L)}} - \frac{(q+\alpha)}{(q+\alpha)^2 + J^2 e^{2(p-L)}}.$$

Now by integrating both sides with respect to q,

$$r = \left(\frac{(q+\beta)^2 + J^2 e^{2(p+L)}}{(q+\alpha)^2 + J^2 e^{2(p-L)}}\right)^{\frac{1}{2}}, \tag{3.196}$$

where α and β are arbitrary constants.

Next substitute the values for $\cos(s)$, $\sin(s)$, $\cos(t)$ and $\sin(t)$ in the equations (3.128), (3.130) and (3.132). Let us first start with the equations (3.128) and (3.130), they become

$$\begin{aligned} 2f_1(J,q)\frac{J^2}{(q+\alpha)^2} &+ 2\frac{J}{(q+\alpha)}\big(f_2(J,q) - g_2(J,q)\big) \\ +2g_1(J,q) &+ N''_{11}(J)\Big(\frac{J^2}{(q+\alpha)^2} - 1\Big) = 0, \end{aligned} \tag{3.197}$$

$$\begin{aligned} 2f_1(J,q)\frac{(q+\beta)^2}{J^2} &- 2\frac{(q+\beta)}{J}\big(f_2(J,q) - g_2(J,q)\big) \\ +2g_1(J,q) &+ N''_{11}(J)\Big(\frac{(q+\beta)^2}{J^2} - 1\Big) = 0. \end{aligned} \tag{3.198}$$

From equation (3.132),

$$
\begin{aligned}
e^{-2(p+L)} \Big[& \Big(2f_1(J,q) + N''_{11}(J)\Big)\ \frac{(q+\beta)^2}{J\,(q+\alpha)} + \\
& \Big(2g_1(J,q) - N''_{11}(J)\Big)\ \frac{(q+\beta)}{J}\ \Big] - \\
e^{2(p-L)} \Big[& \Big(2f_1(J,q) + N''_{11}(J)\Big)\ \frac{J(q+\beta)^2}{(q+\alpha)} + \\
& \Big(2g_1(J,q) - N''_{11}(J)\Big)\ \frac{J}{(q+\alpha)}\ \Big] + \\
e^{-4L} \Big[& \Big(2g_1(J,q) - N''_{11}(J)\Big)\ \frac{J(q+\beta)}{(q+\alpha)^2} - \\
& \Big(2g_1(J,q) - N''_{11}(J)\Big)\ \frac{(q+\beta)^2}{J\,(q+\alpha)} - \\
& 2\Big(f_2(J,q) - g_2(J,q)\Big)\frac{(q+\beta)^2}{(q+\alpha)^2}\ \Big] + \\
\Big[& \Big(2f_1(J,q) + N''_{11}(J)\Big)\Big(\frac{J}{(q+\alpha)} - \frac{(q+\beta)}{J}\Big) \\
& +2\Big(f_2(J,q) - g_2(J,q)\Big)\ \Big] = 0\,.
\end{aligned}
$$

This equation must be true for all L and p. We deduce that the following equations are true:

$$
\Big(2f_1(J,q) + N''_{11}(J)\Big)\ \frac{(q+\beta)}{(q+\alpha)} + \ \Big(2g_1(J,q) - N''_{11}(J)\Big) = 0\,, \tag{3.199}
$$

$$
\begin{aligned}
\Big(2g_1(J,q) - N''_{11}(J)\Big)\ \Big(\frac{J^2 - (q+\beta)(q+\alpha)}{(q+\alpha)}\Big) & \\
-\ 2\Big(f_2(J,q) - g_2(J,q)\Big)\, J \ &= \ 0\,,
\end{aligned} \tag{3.200}
$$

$$
\begin{aligned}
\Big(2f_1(J,q) + N''_{11}(J)\Big)\ \Big(\frac{J^2 - (q+\beta)(q+\alpha)}{(q+\alpha)}\Big) & \\
+\ 2\Big(f_2(J,q) - g_2(J,q)\Big) &= 0\,.
\end{aligned} \tag{3.201}
$$

Now by rearranging the equation (3.201) and by adding it to equation (3.200),

$$\Big(\frac{J^2-(q+\beta)(q+\alpha)}{J}\Big)\Big(f_1(J,q)+g_1(J,q)\Big)$$
$$= \Big(f_2(J,q)-g_2(J,q)\Big)\,(\beta-\alpha)\,. \qquad (3.202)$$

Next we substitute the values for $f_2(J,q)-g_2(J,q)$ and $f_1(J,q)+g_1(J,q)$ that we had from (3.121) and (3.116) respectively in equation (3.182),

$$\Big(\frac{J^2-(q+\beta)(q+\alpha)}{J}\Big)\Big[d_1\,(c_1\,J^{-1}+\,c_2\,J)\ln(q)+\,d_2(c_1\,J^{-1}+\,c_2\,J)\Big]$$
$$= \Big[u_1\,(v_1\,q^{-1}+\,v_2\,q)\ln(J)+\,u_2(v_1\,q^{-1}+\,v_2\,q)\Big](\beta-\alpha)\,,$$

where the only variables in the last equation are J and q, and the rest are all arbitrary constants. To compare both sides in the last equation, first start with the terms that have logarithms. We find these two terms vanish. This leaves

$$\Big(\frac{J^2-(q+\beta)(q+\alpha)}{J}\Big)\,d_2(c_1\,J^{-1}+\,c_2\,J)\;=\;u_2(v_1\,q^{-1}+\,v_2\,q)\,(\beta-\alpha)\,.$$

By comparing coefficients,

$$f_1(J,q)+g_1(J,q) \;=\; 0$$
$$\text{and either}\quad f_2(J,q)-g_2(J,q) \;=\; 0\;\text{ or }\;\beta\;=\;\alpha\,. \qquad (3.203)$$

In fact we found that if we started with assuming $\beta=\alpha$ at some stage we will get $f_2(J,q)-g_2(J,q)=0$, but if we start with the possibility $f_2(J,q)-g_2(J,q)=0$. This means one case is a special case of the other. Next we are going to consider the general possibility which is $f_2(J,q)-g_2(J,q)=0$.

Suppose that $f_2(J,q)-g_2(J,q)\;=\;0$, and we have

$$f_1(J,q)\;+\;g_1(J,q)\;=\;0\,.$$

In fact in this case we will have the same results as in case(I). We are not going to repeat this, but we will write the results as

$$N''_{11}(J)\;=\;0\quad\text{and}\quad b\;=\;0\,,$$

so that

$$f_1 = g_1 = 0 \quad , \quad f_2 = g_2 = C(q) .$$

Finally in this case we will have the same value for a, which is given by

$$a = \frac{1}{\sqrt{q}} \sqrt{(J^2 e^{2(p-L)} + (q+\alpha)^2)} \; e^{-i \tan^{-1}\left(\frac{(q+\alpha)}{J} e^{L-p}\right) + 2iC(q)} .$$

A_1 and A_3 are vanishing, and A_2 and A_4 are given by

$$A_2 = 2\,C(q) \cosh(p) ,$$

$$A_4 = -\,2\,C(q) \sinh(p) ,$$

where $C(q)$ is any function of q. The electromagnetic field in this case is vanishing.

Summary of case (II) :

The Dirac spinor for the solution in this case is given by

$$\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\, r e^{\frac{i}{2}(s-t)} \\ i\, r e^{\frac{i}{2}(s+t)} \end{pmatrix} a \quad \text{where},$$

$$a = \frac{1}{\sqrt{q}} \sqrt{(J^2 e^{2(p-L)} + (q+\alpha)^2)} \; e^{-i \tan^{-1}\left(\frac{(q+\alpha)}{J} e^{L-p}\right) + 2iC(q)},$$

$$r = \left(\frac{(q+\beta)^2 + J^2 e^{2(p+L)}}{(q+\alpha)^2 + J^2 e^{2(p-L)}} \right)^{\frac{1}{2}},$$

$$\sin(s) = \frac{\left(\frac{J^2}{(q+\alpha)^2}\right) e^{2(p-L)} - 1}{\left(\frac{J^2}{(q+\alpha)^2}\right) e^{2(p-L)} + 1} \quad , \quad \cos(s) = \frac{-2\left(\frac{J}{(q+\alpha)}\right) e^{p-L}}{\left(\frac{J^2}{(q+\alpha)^2}\right) e^{2(p-L)} + 1} ,$$

$$\sin(t) = \frac{\left(\frac{(q+\beta)^2}{J^2}\right) e^{-2(p+L)} - 1}{\left(\frac{(q+\beta)^2}{J^2}\right) e^{-2(p+L)} + 1} \quad , \quad \cos(t) = \frac{-2\left(\frac{(q+\beta)}{J}\right) e^{-(p+L)}}{\left(\frac{(q+\beta)^2}{J^2}\right) e^{-2(p+L)} + 1} ,$$

where α and β are arbitrary constants.

A_1 and A_3 are vanishing, and A_2 and A_4 are given by

$$A_2 = 2\,C(q)\,\cosh(p)\,,$$

$$A_4 = -\,2\,C(q)\,\sinh(p)\,,$$

where $C(q)$ is any function of q. The electromagnetic field in this case is vanishing.

The variables L, J, p and q are written in the original variables as

$$L = \frac{1}{2}\ln(\frac{x^3+x^1}{x^3-x^1}) \quad \text{and} \quad J = \left((x^3)^2-(x^1)^2\right)^{\frac{1}{2}},$$

$$p = \frac{1}{2}\ln(\frac{x^2+x^4}{x^2-x^4}) \quad \text{and} \quad q = \left((x^2)^2-(x^4)^2\right)^{\frac{1}{2}}.$$

Note that there could be problems with imaginary values for s and t for general α and β. However if we put $\alpha=\beta=0$, we get

$$\sin(s) = \frac{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2-1}{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1}\,, \quad \cos(s) = \frac{-2\left(\frac{x^3-x^1}{x^2-x^4}\right)}{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1}\,,$$

$$\sin(t) = \frac{\left(\frac{x^2-x^4}{x^3+x^1}\right)^2-1}{\left(\frac{x^2-x^4}{x^3+x^1}\right)^2+1}\,, \quad \cos(t) = \frac{-2\left(\frac{x^2-x^4}{x^3+x^1}\right)}{\left(\frac{x^2-x^4}{x^3+x^1}\right)^2+1}\,.$$

The value for a and r become

$$a = \left((x^2)^2-(x^4)^2\right)^{-3/4}\sqrt{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1}\;e^{-i\,tan^{-1}\left(\frac{x^2-x^4}{x^3-x^1}\right)+2iC(q)}\,,$$

$$r = \left(\frac{\left(\frac{x^3+x^1}{x^2-x^4}\right)^2+1}{\left(\frac{x^3-x^1}{x^2-x^4}\right)^2+1}\right)^{\frac{1}{2}}.$$

3.4 The timelike line

Consider a timelike line $(t, 0, 0, 0)$ for t a real number. As in the lightlike case we want to find the elements of the Lie algebra of the group $SO(2,2)$ that fix this line. In the process of solving this case we find (3.221) and (3.241), which means that the spinor field must be zero outside the cone

$$(x^2)^2 = (x^3)^2 + (x^4)^2 .$$

This is an unusual circumstance, but we shall continue to solve the equations, as the solutions restricted to the cone seem to be well behaved.

There are three linearly independent elements that fix the given timelike line, given by

i) $\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}$, and

$$\omega^c{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta \\ 0 & 0 & \delta & 0 \end{pmatrix} .$$

The Lie group element is given by

$$Exp(\omega^c{}_b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\delta) & -\sin(\delta) \\ 0 & 0 & \sin(\delta) & \cos(\delta) \end{pmatrix} ,$$

and $\omega^c{}_b$ acts on X like this

$$\omega^c{}_b \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -x^4\delta \\ x^3\delta \end{pmatrix} .$$

The differential of ψ in the direction ωX is given by

$$\psi'(X; \omega^c{}_b X) = \gamma(x^3 \frac{\partial\psi}{\partial x^4} - x^4 \frac{\partial\psi}{\partial x^3}) .$$

Now by substituting this in equation (3.6), the first symmetry equation in this case is given by

$$x^3\frac{\partial\psi}{\partial x^4} - x^4\frac{\partial\psi}{\partial x^3} = \frac{1}{2}\gamma^3\gamma^4\psi\,. \tag{3.204}$$

ii) $\omega_{a\,b} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}$, and

$$\omega^c{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}.$$

The Lie algebra element is given by

$$Exp(\omega^c{}_b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(\beta) & 0 & \sinh(\beta) \\ 0 & 0 & 1 & 0 \\ 0 & \sinh(\beta) & 0 & \cosh(\beta) \end{pmatrix},$$

and $\omega^c{}_b$ acts on X like this

$$\omega^c{}_b \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 0 \\ x^4\beta \\ 0 \\ x^2\beta \end{pmatrix}.$$

The differential of ψ in the direction ωX is given by

$$\psi'(X;\omega^c{}_b X) = \beta(x^4\frac{\partial\psi}{\partial x^2} + x^2\frac{\partial\psi}{\partial x^4})\,.$$

Now by substituting this in equation (3.6), the second symmetry equation in this case is given by

$$x^4\frac{\partial\psi}{\partial x^2} + x^2\frac{\partial\psi}{\partial x^4} = \frac{1}{2}\gamma^2\gamma^4\psi\,. \tag{3.205}$$

iii) $\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and

$$\omega^c{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Lie algebra element is given by

$$Exp(\omega^c{}_b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(\alpha) & \sinh(\alpha) & 0 \\ 0 & \sinh(\alpha) & \cosh(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\omega^c{}_b$ acts on X like this

$$\omega^c{}_b \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} 0 \\ x^3\alpha \\ x^2\alpha \\ 0 \end{pmatrix}.$$

The differential of ψ in the direction ωX is given by

$$\psi'(X; \omega^c{}_b X) = \alpha(x^3 \frac{\partial \psi}{\partial x^2} + x^2 \frac{\partial \psi}{\partial x^3}).$$

Now by substituting this in equation (3.6), the third symmetry equation in this case is given by

$$x^3 \frac{\partial \psi}{\partial x^2} + x^2 \frac{\partial \psi}{\partial x^3} = \frac{1}{2}\gamma^2\gamma^3\psi. \qquad (3.206)$$

We find a total of three symmetry equations in this case, which are equations (3.204), (3.205) and (3.206).

By multiplying equation (3.204) by x^2, it becomes

$$x^2 x^3 \frac{\partial \psi}{\partial x^4} - x^2 x^4 \frac{\partial \psi}{\partial x^3} = \frac{1}{2} x^2\gamma^3\gamma^4\psi, \qquad (3.207)$$

and by multiplying equation (3.205) by x^3, it becomes

$$x^2\,x^3\frac{\partial\psi}{\partial x^4}+x^3\,x^4\frac{\partial\psi}{\partial x^2}=\frac{1}{2}x^3\gamma^2\gamma^4\psi\,. \tag{3.208}$$

Finally by multiplying equation (3.206) by x^4, it becomes

$$x^4\,x^3\frac{\partial\psi}{\partial x^2}+x^2\,x^4\frac{\partial\psi}{\partial x^3}=\frac{1}{2}x^4\gamma^2\gamma^3\psi\,. \tag{3.209}$$

Now by adding equations (3.207) and (3.209), and then by subtracting the result from equation (3.208),

$$(x^2\gamma^3\gamma^4+x^4\gamma^2\gamma^3-x^3\gamma^2\gamma^4)\,\psi=0\,. \tag{3.210}$$

By multiplying equation (3.210) by $\gamma^2\gamma^3\gamma^4$, it becomes

$$(x^3\,\gamma^3+x^4\,\gamma^4-x^2\,\gamma^2)\,\psi=0\,. \tag{3.211}$$

3.4.1 The symmetry equations and currentless spinors

In this part we examine the currentless spinors, first by applying condition (3.211), and secondly by applying the symmetry equations we had in the previous part. Let us now consider these currentless spinors:

I) When $\psi=\begin{pmatrix}1\\ e^{is}\\ i\,re^{\frac{i}{2}(s-t)}\\ i\,re^{\frac{i}{2}(s+t)}\end{pmatrix}a$, by substituting this spinor in equation (3.211), it gives

$$x^2 = (x^3-i\,x^4)\;e^{-it}\,, \tag{3.212}$$

$$x^2 = (x^3+i\,x^4)\;e^{it}\,, \tag{3.213}$$

$$x^2 = (x^3+i\,x^4)\;e^{-is}\,, \tag{3.214}$$

$$x^2 = (x^3-i\,x^4)\;e^{is}\,. \tag{3.215}$$

From the last four equations, there is a relation between s and t which given by

$$t \;=\; -s + 2n\pi\,. \tag{3.216}$$

Then this currentless spinor becomes

$$\psi \;=\; \begin{pmatrix} 1 \\ e^{is} \\ i\,re^{is} \\ i\,r \end{pmatrix} a\,. \tag{3.217}$$

Now if we subtract (3.214) from (3.215),

$$x^4 \;=\; \frac{\sin(s)}{\cos(s)}\,x^3 \;\text{ so that }\; s \;=\; \tan^{-1}(\frac{x^4}{x^3}) \quad (\text{for } \cos(s) \neq 0)\,. \tag{3.218}$$

By adding (3.214) to (3.215),

$$x^2 \;=\; x^4\,\sin(s) + x^3\,\cos(s)\,. \tag{3.219}$$

Now by substituting the value for x^4 given by (3.218) equation (3.219) becomes

$$x^3 \;=\; x^2\,\cos(s) \quad \text{and} \quad x^4 \;=\; x^2\,\sin(s)\,. \tag{3.220}$$

This means well,

$$(x^2)^2 \;=\; (x^3)^2 + (x^4)^2\,. \tag{3.221}$$

Now we are in position to start to apply the symmetry equations on this currentless spinor, but before that we change the variables as

$$x^4(p,q) = q\,\sinh(p) \quad \text{and} \quad x^2(p,q) = q\,\cosh(p)\,, \tag{3.222}$$

where p and q are defined by

$$p = \frac{1}{2}\,\ln(\frac{x^2+x^4}{x^2-x^4}) \quad \text{and} \quad q = \Big((x^2)^2 - (x^4)^2\Big)^{\frac{1}{2}}\,. \tag{3.223}$$

Now the symmetry equation given by (3.205) becomes

$$\frac{\partial}{\partial p}\psi \;=\; \frac{1}{2}\,\gamma^2\gamma^4\,\psi\,. \tag{3.224}$$

By substituting the value for ψ given by (3.217) in equation (3.224), it gives

$$\frac{\partial}{\partial p} \log a = \frac{-i}{2} e^{is}, \tag{3.225}$$

$$\frac{\partial}{\partial p} s = \cos(s), \tag{3.226}$$

$$\frac{\partial}{\partial p} r = 0. \tag{3.227}$$

Now the symmetry equation which given by (3.206) will become

$$\Big[x^3 \Big(\frac{-\sinh(p)}{q}\frac{\partial}{\partial p} + \cosh(p)\frac{\partial}{\partial q}\Big) + q\cosh(p)\frac{\partial}{\partial x^3}\Big]\psi = \frac{1}{2}\gamma^2\gamma^3\psi. \tag{3.228}$$

Note from the equations (3.221) and (3.223),

$$(x^3)^2 = q^2 \quad \text{so that} \quad x^3 = \pm q. \tag{3.229}$$

This means that there are two possibilities. Next we will consider the possibility where $x^3 = q$, and the other will be similar. For $x^3 = q$, we know from (3.218) that

$$\tan(s) = \frac{x^4}{x^3} \quad \text{so} \quad \tan(s) = \sinh(p). \tag{3.230}$$

This means that s is a function of p only, and we write it as

$$s = \tan^{-1}\big(\sinh(p)\big). \tag{3.231}$$

Now by using (3.229) in equation (3.228), it becomes

$$\Big(2q\cosh(p)\frac{\partial}{\partial q} - \sinh(p)\frac{\partial}{\partial p}\Big)\psi = \frac{1}{2}\gamma^2\gamma^3\psi. \tag{3.232}$$

By substituting the value for the ψ given by (3.217) in equation (3.232), it gives

$$\Big(2q\cosh(p)\frac{\partial}{\partial q} - \sinh(p)\frac{\partial}{\partial p}\Big)\log a = \frac{1}{2}e^{is}, \tag{3.233}$$

$$\Big(2q \cosh(p) \frac{\partial}{\partial q} - \sinh(p) \frac{\partial}{\partial p} \Big) r = 0 . \tag{3.234}$$

From equation (3.227) we know that r is independent of p, and by using this in equation (3.234), we find that r is independent of q as well. This means that r is a function only of x^1. Now by using equation (3.225) in equation (3.233), it becomes

$$2q \cosh(p) \frac{\partial}{\partial q} \log a = \frac{1}{2} \Big(1 - i \sinh(p) \Big) e^{is} ,$$

and the last equation can be written as

$$2q \cosh(p) \frac{\partial}{\partial q} \log a = \frac{1}{2} \Big(1 - i \sinh(p) \Big) \Big[\frac{1}{\cosh(p)} + i \frac{\sinh(p)}{\cosh(p)} \Big] .$$

Finally this equation becomes

$$\frac{\partial}{\partial q} \log a = \frac{1}{4 q} . \tag{3.235}$$

By integrating equation (3.235) with respect to q,

$$\log a = \frac{1}{4} \log q + f(p, x^1) . \tag{3.236}$$

Now we can write equation (3.225) as

$$\frac{\partial}{\partial p} \log a = \frac{-i}{2} \Big(\frac{1}{\cosh(p)} + i \frac{\sinh(p)}{\cosh(p)} \Big) . \tag{3.237}$$

By integrating equation (3.237) with respect to p,

$$\log a = \frac{1}{4} \log q + \frac{1}{2} \log(\cosh(p)) - i \tan^{-1}(e^p) + f(x^1) . \tag{3.238}$$

Summary of (I):

We found that r is a function of x^1 only, and for $x^3 = q$, the values for s and $\log a$ are given by

$$s = \tan^{-1} \Big(\sinh(p) \Big) ,$$

$$a = q^{\frac{1}{4}} (\cosh(p))^{\frac{1}{2}} e^{-i \tan^{-1}(e^p) + f(x^1)},$$

where p and q are given by (3.223).

II) When $\psi = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, by substituting this currentless spinor in equation (3.211), there are two possibilities:

$$x^2 d = (x^3 - ix^4) c \quad \text{so that} \quad d = \frac{(x^3 - ix^4)}{x^2} c, \tag{3.239}$$

and

$$x^2 c = (x^3 + ix^4) d \quad \text{so that} \quad d = \frac{x^2}{(x^3 + ix^4)} c. \tag{3.240}$$

From these two values for d, there is a condition which is

$$(x^2)^2 = (x^3)^2 + (x^4)^2 \tag{3.241}$$

Then the currentless spinor becomes

$$\psi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{(x^3 - ix^4)^{\frac{1}{2}}}{(x^3 + ix^4)^{\frac{1}{2}}} \end{pmatrix} c, \tag{3.242}$$

Now we are in position to change the variables as in (3.222). Similarly in this case we will consider the possibility where $x^3 = q$ and the other will be similar. By substituting the value for ψ given by (3.242) in equation (3.224), it gives

$$\frac{\partial}{\partial p} c = \frac{i}{2} \left(\frac{1 - i \sinh(p)}{1 + i \sinh(p)} \right)^{\frac{1}{2}} c. \tag{3.243}$$

By integrating the last equation with respect to p,

$$\log c = \frac{1}{2} \log(1 + i \sinh(p)) + g(q, x^1). \tag{3.244}$$

By substituting the value for the ψ given by (3.242) in equation (3.232) and by using equation (3.243), it becomes

$$\frac{\partial}{\partial q} \log c = \frac{1}{4q}. \tag{3.245}$$

By integrating equation (3.245) with respect to q,

$$\log c = \frac{1}{2} \log(1 + i \sinh(p)) + \frac{1}{4} \log q + g(x^1). \tag{3.246}$$

Summary of (II):

In this case we found

$$\log c = \frac{1}{2} \log(1 + i \sinh(p)) + \frac{1}{4} \log q + g(x^1),$$

where p and q are given by (3.223).

3.4.2 The action of Lorentz group on vector potential

In this part to see how the Lorentz group acts on the connection field A. We will just give the Lie algebra representation which is given by

$$A_c{}'(X^c; \omega^c{}_b X^b) = g_{ac} \omega^c{}_b g^{bd} A_d. \tag{3.247}$$

The idea now is to get the symmetry equations for the connection field A given by equation (3.247). We know that there are three linearly independent elements of the Lie algebra of the group $SO(2,2)$ that fix the timelike line, and they are:

$$\text{i)}\ \omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \\ 0 & 0 & -\delta & 0 \end{pmatrix}, \quad \omega^c{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta \\ 0 & 0 & \delta & 0 \end{pmatrix}.$$

By substituting this value for $\omega^c{}_b$ in equation (3.247), the first symmetry equation is given by

$$(x^3 \frac{\partial}{\partial x^4} - x^4 \frac{\partial}{\partial x^3}) A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A. \tag{3.248}$$

$$\text{ii)}\,\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}, \quad \omega^c{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}.$$

By substituting this value for $\omega^c{}_b$ in equation (3.247), the second symmetry equation is given by

$$(x^4 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^4})\, A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A\,. \tag{3.249}$$

$$\text{iii)}\,\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \omega^c{}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By using this value for $\omega^c{}_b$ in (3.247),

$$(x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3})\, A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A\,. \tag{3.250}$$

Now by multiplying equation (3.248) by x^2, it becomes

$$x^2(x^3 \frac{\partial}{\partial x^4} - x^4 \frac{\partial}{\partial x^3})\, A = x^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A\,, \tag{3.251}$$

and by multiplying equation (3.249) by x^3, it becomes

$$x^3(x^4 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^4})\, A = x^3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A\,. \tag{3.252}$$

Similarly if we multiply equation (3.250) by x^4, it becomes

$$x^4(x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3})\, A = x^4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A\,. \tag{3.253}$$

Now add equations (3.251) and (3.253), and subtracting the result from equation (3.252),

$$(x^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + x^4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - x^3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}) A = 0 .$$

By putting $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$ in the last equation,

$$\begin{pmatrix} 0 \\ -x^4 A_3 + x^3 A_4 \\ -x^4 A_2 - x^2 A_4 \\ x^3 A_2 + x^2 A_3 \end{pmatrix} = 0 .$$

From this last equation,

$$A_3 = \frac{-x^3}{x^2} A_2 \quad , \quad A_4 = \frac{-x^4}{x^2} A_2 . \tag{3.254}$$

Now by changing the variables as in (3.222), equation (3.249) becomes

$$\frac{\partial}{\partial p} A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} A .$$

We considered that $x^3 = q$. by using (3.254), the last equation gives

$$\frac{\partial}{\partial p} A_1 = 0 \quad , \quad \frac{\partial}{\partial p} A_2 = \frac{\sinh(p)}{\cosh(p)} A_2 . \tag{3.255}$$

Similarly for equation (3.250), by changing variables

$$\left(2q \cosh(p) \frac{\partial}{\partial q} - \sinh(p) \frac{\partial}{\partial p} \right) A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} A .$$

This last equation gives

$$\frac{\partial}{\partial q} A_1 = 0 \quad , \quad \frac{\partial}{\partial q} A_2 = \frac{1}{2\,q} A_2 \,. \tag{3.256}$$

From the equations (3.255) and (3.256) we deduce that A_1 is a function of x^1 only, and for A_2

$$A_2 = q^{\frac{1}{2}} \cosh(p)\, h(x^1)\,, \tag{3.257}$$

where h is an arbitrary function

3.4.3 The electromagnetic field :

From the previous part we found that A_1 is a function of x^1 only, and A_2, A_3 and A_4 are given by

$$\begin{aligned} A_2 &= q^{\frac{1}{2}} \cosh(p)\, h(x^1)\,, \\ A_3 &= -\,q^{\frac{1}{2}}\, h(x^1)\,, \\ A_4 &= -\,q^{\frac{1}{2}} \sinh(p)\, h(x^1)\,. \end{aligned}$$

The electromagnetic field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.$$

The electromagnetic fields are given by

$$\begin{aligned} F_{12} &= q^{\frac{1}{2}} \cosh(p) \frac{\partial}{\partial x^1} h(x^1)\,, \\ F_{13} &= -q^{\frac{1}{2}} \frac{\partial}{\partial x^1} h(x^1)\,, \\ F_{14} &= -q^{\frac{1}{2}} \sinh(p) \frac{\partial}{\partial x^1} h(x^1)\,, \\ F_{23} &= q^{\frac{-1}{2}} \cosh(p)\, h(x^1)\,, \\ F_{24} &= 0\,, \\ F_{34} &= q^{\frac{-1}{2}} \sinh(p)\, h(x^1)\,. \end{aligned}$$

Now the zero current equation is given by

$$\partial_\mu F^{\mu\nu} = 0\,.$$

For $\nu = 1$,

$$\frac{\partial}{\partial x^1} h(x^1) = 0, \tag{3.258}$$

and this means that $h(x^1)$ is a constant.

For $\nu = 2$,

$$h(x^1) = 0, \tag{3.259}$$

and this means that $A_i = 0$ for $i = 2, 3, 4$. Therefore the electromagnetic field vanishes.

3.4.4 The solution of the timelike line case

The Dirac equation with zero mass is given by

$$(\gamma^1 \frac{\partial}{\partial x^1} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} + \gamma^4 \frac{\partial}{\partial x^4})\, \psi = i\, \gamma^\mu A_\mu\, \psi\,.$$

Now by using equations (3.204), (3.205) and (3.206),

$$\gamma^1 \frac{\partial}{\partial x^1}\, \psi + \gamma^2 (\frac{1}{2x^4} \gamma^2 \gamma^4 - \frac{x^2}{x^4} \frac{\partial}{\partial x^4})\, \psi + \gamma^3 (\frac{x^3}{x^4} \frac{\partial}{\partial x^4} - \frac{1}{2x^4} \gamma^3 \gamma^4)\, \psi$$
$$+ \gamma^4 \frac{\partial}{\partial x^4}\, \psi = i\, \gamma^\mu A_\mu\, \psi\,,$$

which simplifies to

$$\gamma^1 \frac{\partial}{\partial x^1}\, \psi + \frac{1}{x^4} \gamma^4\, \psi + \frac{1}{x^4} (x^4 \gamma^4 + x^3 \gamma^3 - x^2 \gamma^2) \frac{\partial}{\partial x^4}\, \psi = i\, \gamma^\mu A_\mu\, \psi\,.$$

Now by differentiating equation (3.211) with respect to x^4,

$$\gamma^4\, \psi + (x^4 \gamma^4 + x^3 \gamma^3 - x^2 \gamma^2) \frac{\partial}{\partial x^4}\, \psi = 0\,.$$

By using this result in the Dirac equation, it becomes

$$\gamma^1 \frac{\partial}{\partial x^1}\, \psi = i\, \gamma^\mu A_\mu\, \psi\,. \tag{3.260}$$

Now we use the value for the vector potential we had in the previous subsection in equation (3.260), it becomes

$$\gamma^1 \frac{\partial}{\partial x^1} \psi = i\gamma^1 A_1 \psi .$$

Now by multiplying both sides by γ^1, the last equation becomes

$$\frac{\partial}{\partial x^1} \psi = i A_1 \psi . \tag{3.261}$$

Now by substituting the value for ψ given by (3.217) in equation (3.261),

$$\frac{\partial}{\partial x^1} \log a = i A_1 \quad \text{and} \quad \frac{\partial}{\partial x^1} \log r = 0 . \tag{3.262}$$

The last two equations tell us first that r is a constant, and that a is given by

$$a = q^{\frac{1}{4}} (\cosh (p))^{\frac{1}{2}} e^{-i \tan^{-1} (e^p) + i \int A_1 dx^1} . \tag{3.263}$$

Summary of (I):

The solution for the Dirac equation is given by

$$\psi = \begin{pmatrix} 1 \\ e^{is} \\ i r e^{is} \\ i r \end{pmatrix} a ,$$

where r is constant, and for $x^3 = q$ the values for s and a are given by

$$s = \tan^{-1} \Big(\sinh (p) \Big) ,$$

$$a = q^{\frac{1}{4}} (\cosh (p))^{\frac{1}{2}} e^{-i \tan^{-1} (e^p) + i \int A_1 dx^1} .$$

Note that the spinor field is vanishing out sides the cone

$$(x^2)^2 = (x^3)^2 + (x^4)^2 .$$

The variables p and q are given by

$$p = \frac{1}{2} \ln(\frac{x^2 + x^4}{x^2 - x^4}) \quad \text{and} \quad q = \Big((x^2)^2 - (x^4)^2\Big)^{\frac{1}{2}} .$$

A_1 is a function of x^1 only, and A_2, A_3 and A_4 are vanishing, which means that the electromagnetic field vanishes as well.

Similarly by substituting the value for ψ given by (3.242) in equation (3.261),

$$\frac{\partial}{\partial x^1} \log c \;=\; i\, A_1 \,. \tag{3.264}$$

Now by integrating the last equation, we can write c finally as

$$c \;=\; q^{\frac{1}{4}} \left(1 + i \sinh(p)\right)^{\frac{1}{2}} e^{i \int A_1\, dx^1} \,. \tag{3.265}$$

Summary of (II):

The solution for the Dirac equation is given by

$$\psi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{(x^3 - i x^4)^{\frac{1}{2}}}{(x^3 + i x^4)^{\frac{1}{2}}} \end{pmatrix} c \,,$$

where for $x^3 = q$ the value for c is given by

$$c \;=\; q^{\frac{1}{4}} \left(1 + i \sinh(p)\right)^{\frac{1}{2}} e^{i \int A_1\, dx^1} \,.$$

Note that the spinor field is vanishing out sides the cone

$$(x^2)^2 \;=\; (x^3)^2 \;+\; (x^4)^2 \,.$$

The variables p and q are given by

$$p = \frac{1}{2} \ln\left(\frac{x^2 + x^4}{x^2 - x^4}\right) \quad \text{and} \quad q = \left((x^2)^2 - (x^4)^2\right)^{\frac{1}{2}} .$$

A_1 is a function of x^1 only, and A_2, A_3 and A_4 are vanishing as well as the electromagnetic field.

Chapter 4

Solutions for the massless Dirac equation in $2+2$ dimensions

4.1 Introduction

The aim of this chapter is to find more general solutions for the massless Dirac equation with potential A_μ in $2+2$ dimensions. These solutions are not constrained by the sort of symmetries we described in the last chapter. In this chapter we want to consider one of the currentless spinors,

$$\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,r\,e^{\frac{i}{2}(s-t)} \\ i\,r\,e^{\frac{i}{2}(s+t)} \end{pmatrix} a\,,$$

where s and t are real functions, r is a non zero real function, and a is a complex function. For spinors of the form $\begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$, we have found plane wave solutions already in chapter 2. As the sum of any spinors of this form is also currentless, we can write quite general solutions as the sums (or integrals) of such plane waves. Similarly we can consider spinors of the form $\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$.

In section 4.2 we write the Dirac massless equation with potential A_μ in $2+2$ dimensions. Then write the corresponding four equations for the currentless spinor, and find that we have two cases to study. In section 4.3 we want to show that some of the equations are equivalent to equations written in terms of differential forms. In section 4.4 we study the first case. For convenience we make an assumption that the vector potential components satisfy two relations to simplify the equations, then we study the electromagnetic field. Similarly in the last section we study the second case.

4.2 The massless Dirac equation in $2+2$ dimensions

The massless Dirac equation with potential A_μ in $2+2$ dimensions is given by

$$\gamma^\mu(\partial_\mu - iA_\mu)\psi = 0\,, \tag{4.1}$$

where A_μ is a real valued function. Now we substitute the representation for the gamma matrices given by (1.1) in equation (4.1), to give

$$\begin{pmatrix} 0 & 0 & i\partial_4 - \partial_3 & -i\partial_1 - \partial_2 \\ 0 & 0 & -i\partial_1 + \partial_2 & i\partial_4 + \partial_3 \\ i\partial_4 + \partial_3 & i\partial_1 + \partial_2 & 0 & 0 \\ i\partial_1 - \partial_2 & i\partial_4 - \partial_3 & 0 & 0 \end{pmatrix} \psi =$$

$$i \begin{pmatrix} 0 & 0 & iA_4 - A_3 & -iA_1 - A_2 \\ 0 & 0 & -iA_1 + A_2 & iA_4 + A_3 \\ iA_4 + A_3 & iA_1 + A_2 & 0 & 0 \\ iA_1 - A_2 & iA_4 - A_3 & 0 & 0 \end{pmatrix} \psi\,.$$

The last equation, by putting ψ in general equal to $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, provides the following four equations

$$a\,(k_3 + i\,k_4) + b\,(k_2 + i\,k_1) \;=\; a\,(A_3 + i\,A_4) + b\,(A_2 + i\,A_1)\,, \tag{4.2}$$

$$a\,(i\,k_1 - k_2) + b\,(i\,k_4 - k_3) \;=\; a\,(i\,A_1 - A_2) + b\,(i\,A_4 - A_3)\,, \tag{4.3}$$

$$c\,(i\,k_4 - k_3) - d\,(i\,k_1 + k_2) \;=\; c\,(i\,A_4 - A_3) - d\,(i\,A_1 + A_2)\,, \tag{4.4}$$

$$c\,(k_2 - i\,k_1) + d\,(k_3 + i\,k_4) \;=\; c\,(A_2 - i\,A_1) + d\,(A_3 + i\,A_4)\,. \tag{4.5}$$

Next we set ψ to be the currentless spinor we chose earlier, and then write the corresponding four equations for the equations (4.2)-(4.5).

Set

$$b \;=\; a\,e^{is} \quad , \quad c \;=\; i\,a\,r\,e^{\frac{i}{2}(s-t)} \quad , \quad d \;=\; i\,a\,r\,e^{\frac{i}{2}(s+t)}\,,$$

where s and t are real functions, r is a non zero real function, and a is a complex function. The corresponding four equations for the equations (4.2)-(4.5) are given by

$$(\,\partial_3 \,+\, \partial_4\,)\log a \,+\, e^{is}\,(\,\partial_2 \,+\, i\partial_1\,)\log a \,+\, ie^{is}\,(\,\partial_2 \,+\, i\partial_1\,)s$$

$$= \; i[\,(\,A_3 \,+\, iA_4\,) \,+\, (\,A_2 \,+\, iA_1\,)\,e^{is}\,]\,, \tag{4.6}$$

$$(\,\partial_3 \,+\, \partial_4\,)\log a \,+\, e^{is}\,(\,\partial_2 \,+\, i\partial_1\,)\log a \,+\, ie^{is}\;\partial_2 \,+\, i\partial_1\,)s$$

$$= \; i[\,(\,iA_1 \,-\, A_2\,) \,+\, (\,iA_4 \,-\, A_3\,)\,e^{is}\,]\,, \tag{4.7}$$

$$e^{\frac{i}{2}(s-t)}\,(\,i\partial_4 \,-\, \partial_3\,)\log{(ra)} \,-\, e^{\frac{i}{2}(s+t)}\,(\,\partial_2 \,+\, i\partial_1\,)\log{(ra)} \,+$$

$$\frac{i}{2}\,e^{\frac{i}{2}(s-t)}\,(\,i\partial_4 \,-\, \partial_3\,)s \,-\, \frac{i}{2}\,e^{\frac{i}{2}(s-t)}\,(\,i\partial_4 \,-\, \partial_3\,)t \,-$$

$$\frac{i}{2}\,e^{\frac{i}{2}(s+t)}\,(\,\partial_2 \,+\, i\partial_1\,)s \,-\, \frac{i}{2}\,e^{\frac{i}{2}(s+t)}\,(\,\partial_2 \,+\, i\partial_1\,)t \;=$$

$$i[\,(\,iA_4 \,-\, A_3\,)\,e^{\frac{i}{2}(s-t)} \,-\, (\,A_2 \,+\, iA_1\,)e^{\frac{i}{2}(s+t)}\,]\,, \tag{4.8}$$

$$e^{\frac{i}{2}(s-t)}\,(\,\partial_2 \,-\, i\partial_1\,)\log{(ra)} \,-\, e^{\frac{i}{2}(s+t)}\,(\,\partial_3 \,+\, i\partial_4\,)\log{(ra)} \,+$$

$$\frac{i}{2}\,e^{\frac{i}{2}(s-t)}\,(\,\partial_2 \,-\, i\partial_1\,)s \,-\, \frac{i}{2}\,e^{\frac{i}{2}(s-t)}\,(\,\partial_2 \,-\, i\partial_1\,)t \,+$$

$$\frac{i}{2}\,e^{\frac{i}{2}(s+t)}\,(\,\partial_3 \,+\, i\partial_4,)s \,+\, \frac{i}{2}\,e^{\frac{i}{2}(s+t)}\,(\,\partial_3 \,+\, i\partial_4\,)t \;=$$

$$i[\,(\,A_2 \,-\, iA_1\,)\,e^{\frac{i}{2}(s-t)} \,+\, (\,A_3 \,+\, iA_4\,)e^{\frac{i}{2}(s+t)}\,]\,. \tag{4.9}$$

We can simplify the last two equations giving by multiplying equation (4.8) by $e^{\frac{i}{2}(t-s)}$, to give

$$(i\partial_4 - \partial_3)\log(ra) - e^{it}(\partial_2 + i\partial_1)\log(ra) +$$

$$\frac{i}{2}(i\partial_4 - \partial_3)s - \frac{i}{2}(i\partial_4 - \partial_3)t -$$

$$\frac{i}{2}e^{it}(\partial_2 + i\partial_1)s - \frac{i}{2}e^{it}(\partial_2 + i\partial_1)t =$$

$$i[(iA_4 - A_3) - (A_2 + iA_1)e^{it}]. \qquad (4.10)$$

Similarly by multiplying equation (4.9) by $e^{\frac{-i}{2}(s+t)}$,

$$e^{-it}(\partial_2 - i\partial_1)\log(ra) - (\partial_3 + i\partial_4,)\log(ra) +$$

$$\frac{i}{2}e^{-it}(\partial_2 - i\partial_1)s - \frac{i}{2}e^{-it}(\partial_2 - i\partial_1)t +$$

$$\frac{i}{2}(\partial_3 + i\partial_4,)s + \frac{i}{2}(\partial_3 + i\partial_4,)t =$$

$$i[(A_2 - iA_1)e^{-it} + (A_3 + iA_4)]. \qquad (4.11)$$

The system of equations we have from the Dirac equation which we want to solve are given by (4.6), (4.7), (4.10) and (4.11). Next we want to show that each equation of the system provides two new equations, one from the real part and the other from the imaginary part. Note that r, s and t are real, and we can use a gauge transformation to make a positive, so that $\log a$ is real as well. Let us start with equation (4.6), giving

$$\partial_3 \log a + (\cos(s)\partial_2 - \sin(s)\partial_1)\log a - (\sin(s)\partial_2 + \cos(s)\partial_1)s$$

$$= -[A_4 + \sin(s)A_2 + \cos(s)A_1], \qquad (4.12)$$

$$\partial_4 \log a + (\cos(s)\partial_1 + \sin(s)\partial_2)\log a - (\sin(s)\partial_1 - \cos(s)\partial_2)s$$

$$= A_3 - \sin(s)A_1 + \cos(s)A_2. \qquad (4.13)$$

From equation (4.7),

$$- \partial_3 \log a + (\sin(s)\,\partial_1 - \cos(s)\,\partial_2)\log a - \partial_4 s$$

$$= -[A_4 + \sin(s)\,A_2 + \cos(s)\,A_1], \tag{4.14}$$

$$\partial_4 \log a + (\cos(s)\,\partial_1 + \sin(s)\,\partial_2)\log a - \partial_3 s$$

$$= -[A_3 - \sin(s)\,A_1 + \cos(s)\,A_2]. \tag{4.15}$$

From equation (4.10),

$$- \partial_3 \log(ra) + (\sin(t)\,\partial_1 - \cos(t)\,\partial_2)\log(ra) -$$

$$\frac{1}{2}\partial_4(s - t) + \frac{1}{2}(\sin(t)\,\partial_2 + \cos(t)\,\partial_1)(s + t)$$

$$= -A_4 + \sin(t)\,A_2 + \cos(t)\,A_1, \tag{4.16}$$

$$\partial_4 \log(ra) - (\sin(t)\,\partial_2 + \cos(t)\,\partial_1)\log(ra) -$$

$$\frac{1}{2}\partial_3(s - t) + \frac{1}{2}(\sin(t)\,\partial_1 - \cos(t)\,\partial_2)(s + t)$$

$$= -A_3 + \sin(t)\,A_1 - \cos(t)\,A_2. \tag{4.17}$$

Finally from equation (4.11),

$$\partial_3 \log(ra) + (\cos(t)\,\partial_2 - \sin(t)\,\partial_1)\log(ra) -$$

$$\frac{1}{2}\partial_4(s + t) + \frac{1}{2}(\sin(t)\,\partial_2 + \cos(t)\,\partial_1)(s - t)$$

$$= -A_4 + \sin(t)\,A_2 + \cos(t)\,A_1, \tag{4.18}$$

$$- \partial_4 \log(ra) + (\sin(t)\,\partial_2 + \cos(t)\,\partial_1)\log(ra) -$$

$$\frac{1}{2}\partial_3(s + t) + \frac{1}{2}(\sin(t)\,\partial_1 - \cos(t)\,\partial_2)(s - t)$$

$$= -A_3 + \sin(t)\,A_1 - \cos(t)\,A_2. \tag{4.19}$$

Now we can simplify the last eight equations by adding and subtracting one from the other. This gives another eight equations, given by the following :

The first two equations are given by subtracting (4.12) from (4.14) and by adding them,

$$2\Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\log a = \Big(\cos(s)\,\partial_1 + \sin(s)\,\partial_2 - \partial_4\Big)\,s\,, \tag{4.20}$$

$$\Big(\partial_4 + \cos(s)\,\partial_1 + \sin(s)\,\partial_2\Big)\,s = 2\Big(A_4 + A_1\cos(s) + A_2\sin(s)\Big)\,, \tag{4.21}$$

respectively. The second two equations are given by adding (4.13) to (4.15) and by subtracting them,

$$2\Big(\partial_4 + \cos(s)\,\partial_1 + \sin(s)\,\partial_2\Big)\log a = \Big(\partial_3 - \cos(s)\,\partial_2 + \sin(s)\,\partial_1\Big)\,s\,, \tag{4.22}$$

$$\Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\,s = 2\Big(A_3 + A_2\cos(s) - A_1\sin(s)\Big)\,, \tag{4.23}$$

respectively. The third two equations are given by subtracting (4.16) from (4.18) and by adding them,

$$2\Big(\partial_3 + \cos(t)\,\partial_2 - \sin(t)\,\partial_1\Big)\log(ra) = \Big(\partial_4 + \cos(t)\,\partial_1 + \sin(t)\,\partial_2\Big)\,t\,, \tag{4.24}$$

$$\Big(\partial_4 - \cos(t)\,\partial_1 - \sin(t)\,\partial_2\Big)\,s = 2\Big(A_4 - A_1\cos(t) - A_2\sin(t)\Big)\,, \tag{4.25}$$

respectively. The last two equations are given by subtracting (4.17) from (4.19) and by adding them,

$$2\left(\partial_4 - \cos(t)\,\partial_1 - \sin(t)\,\partial_2\right)\log(ra) = -\left(\partial_3 - \cos(t)\,\partial_2 + \sin(t)\,\partial_1\right)t\,, \qquad (4.26)$$

$$\left(\partial_3 + \cos(t)\,\partial_2 - \sin(t)\,\partial_1\right)s = 2\left(A_3 + A_2\cos(t) - A_1\sin(t)\right), \qquad (4.27)$$

respectively. The general solution for the Dirac massless equation is given by solving the last eight equations.

We can replace the equations (4.21), (4.23), (4.25) and (4.27) by simpler four equations

By subtracting (4.21) from (4.25),

$$\left(\cos(t) + \cos(s)\right)(\partial_1 s - 2A_1) + \left(\sin(t) + \sin(s)\right)(\partial_2 s - 2A_2) = 0\,. \quad (4.28)$$

And by subtracting (4.23) from (4.27),

$$\left(\sin(t) - \sin(s)\right)(\partial_1 s - 2A_1) - \left(\cos(t) - \cos(s)\right)(\partial_2 s - 2A_2) = 0\,. \quad (4.29)$$

And by adding (4.23) from (4.27),

$$2(\partial_3 s - 2A_3) - \left(\sin(t) + \sin(s)\right)(\partial_1 s - 2A_1) + \left(\cos(t) + \cos(s)\right)(\partial_2 s - 2A_2) = 0\,. \quad (4.30)$$

Finally by adding (4.21) from (4.25),

$$2(\partial_4 s - 2A_4) - \left(\cos(t) - \cos(s)\right)(\partial_1 s - 2A_1) - \left(\sin(t) - \sin(s)\right)(\partial_2 s - 2A_2) = 0\,. \quad (4.31)$$

4.3 Using differential forms

There are two equations that involve derivatives of t which are (4.24) and (4.26). Now we want to look at these two equations to get more information about t. Let us consider this equation for $\log(ra)$:

$$\Big[H\,,\,G\Big]\,\log(ra) \;=\; H\Big(G\,\log(ra)\Big)\;-\;G\Big(H\,\log(ra)\Big)\,, \qquad (4.32)$$

where H and G are given by

$$H \;=\; 2\Big(\partial_4\;-\;\sin(t)\,\partial_2\;-\;\cos(t)\,\partial_1\Big)\,,$$

$$G \;=\; 2\Big(\partial_3\;-\;\sin(t)\,\partial_1\;+\;\cos(t)\,\partial_2\Big)\,.$$

Then by using the equations (4.24) and (4.26) in equation (4.32),we find

$$(\,{\partial_4}^2\;+\;{\partial_3}^2\;-\;{\partial_2}^2\;-\;{\partial_1}^2\,)\,t \;=\; 2\,[\,(\partial_1\,t)\,\partial_1\;+$$

$$(\partial_2\,t)\,\partial_2\;-\;(\partial_3\,t)\,\partial_3\;-\;(\partial_4\,t)\,\partial_4\,]\;\log(ra)\,. \qquad (4.33)$$

The last equation can be written as

$$(ra)^2\,(\,{\partial_4}^2\;+\;{\partial_3}^2\;-\;{\partial_2}^2\;-\;{\partial_1}^2\,)\,t \;=\; [\,(\partial_1\,t)\,\partial_1\;+$$

$$(\partial_2\,t)\,\partial_2\;-\;(\partial_3\,t)\,\partial_3\;-\;(\partial_4\,t)\,\partial_4\,]\,(ra)^2\,.$$

Let set $p \;=\; (ra)^2$, then we can write the last equation in a compact form as

$$\partial_4\,(\,p\,\partial_4\,t\,)\;+\partial_3\,(\,p\,\partial_3\,t\,)\;-\partial_2\,(\,p\,\partial_2\,t\,)\;-\partial_1\,(\,p\,\partial_1\,t\,)\;=\;0\,. \qquad (4.34)$$

If we can solve equation (4.34) for t and p, this might help to solve the other equations. Note that an equation like (4.34) could be solved by using differential forms, where the second order partial differential equation (4.34) is equivalent to this equation in differential forms

$$d\,(\,-\,W_4\;dx^1\wedge dx^2\wedge dx^3\;+\;W_3\;dx^1\wedge dx^2\wedge dx^4\;+$$

$$W_2\;dx^1\wedge dx^3\wedge dx^4\;-\;W_1\;dx^2\wedge dx^3\wedge dx^4\,)\;=\;0\,, \quad (4.35)$$

where $W_i \equiv p\,\partial_i t$ for $i = 1, \cdots, 4$. The 3-form

$$\sigma = -W_4\, dx^1 \wedge dx^2 \wedge dx^3 + W_3\, dx^1 \wedge dx^2 \wedge dx^4 + W_2\, dx^1 \wedge dx^3 \wedge dx^4 - W_1\, dx^2 \wedge dx^3 \wedge dx^4 \,,$$

is called a closed 3-form, since $d\sigma = 0$. Note that every closed form on $\mathbb{R}^4$ is exact, which means that there is a 2-form U which satisfies

$$\sigma = dU \,.$$

The problem now is for a given W_i satisfying (4.35), can we find p and t? If the answer is yes, then we can use this to try to solve the equations (4.24) and (4.26) for p and t. Now the equations we want start with are given by this formula

$$\partial_j W_i - \partial_i W_j = \frac{\partial_j p}{p} W_i - \frac{\partial_i p}{p} W_j \,, \tag{4.36}$$

for $1 \le i < j \le 4$. This means that we have six equations,

$$\begin{aligned}
\partial_2 W_1 - \partial_1 W_2 &= \frac{\partial_2 p}{p} W_1 - \frac{\partial_1 p}{p} W_2 \,, \\
\partial_3 W_1 - \partial_1 W_3 &= \frac{\partial_3 p}{p} W_1 - \frac{\partial_1 p}{p} W_3 \,, \\
\partial_4 W_1 - \partial_1 W_4 &= \frac{\partial_4 p}{p} W_1 - \frac{\partial_1 p}{p} W_4 \,, \\
\partial_3 W_2 - \partial_2 W_3 &= \frac{\partial_3 p}{p} W_2 - \frac{\partial_2 p}{p} W_3 \,, \\
\partial_4 W_2 - \partial_2 W_4 &= \frac{\partial_4 p}{p} W_2 - \frac{\partial_2 p}{p} W_4 \,, \\
\partial_4 W_3 - \partial_3 W_4 &= \frac{\partial_4 p}{p} W_3 - \frac{\partial_3 p}{p} W_4 \,.
\end{aligned}$$

which we would like to solve for $\partial_i \log p$, $i = 1, 2, 3, 4$.

Treating these as six linear equations in four variables (the W_i) we find

$$b_4 W_1 - b_2 W_2 + b_1 W_3 = 0,$$

$$b_5 W_1 - b_3 W_2 + b_1 W_4 = 0,$$

$$b_6 W_1 - b_3 W_3 + b_2 W_4 = 0,$$

$$b_6 W_2 - b_5 W_3 + b_4 W_4 = 0,$$

where b_k for $k = 1, \cdots, 6$ are given by the left hand sides of (4.36) respectively, so for example $b_1 = \partial_2 W_1 - \partial_1 W_2$. From these equations we derive

$$b_3 b_4 - b_2 b_5 + b_1 b_6 = 0.$$

We can write the last equation in terms of Ws as

$$\epsilon^{ijkl} (\partial_i W_j)(\partial_k W_l) = 0, \tag{4.37}$$

where $\epsilon^{1234} = +1$. This tensor ϵ is anti-symmetric, that is every time we swap two indices the sign will change.
We see that (4.37) is an additional relation to impose on W to find the value of p and t. So far I have not been able to solve this, or to find any additional conditions.

4.4 Case (I), For $\cos(t) - \cos(s) \neq 0$

We begin with just the assumption $\cos(t) - \cos(s) \neq 0$, but it will be convenient later to make further assumptions, namely (4.41) and that t is constant. In this case we have from the equations (4.28)-(4.31) the following relations :

$$(\partial_2 s - 2A_2) = \left(\frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)}\right)(\partial_1 s - 2A_1), \tag{4.38}$$

$$(\partial_3 s - 2A_3) = \left(\frac{-\sin(t-s)}{\cos(t) - \cos(s)}\right)(\partial_1 s - 2A_1), \tag{4.39}$$

$$(\partial_4 s \;-\; 2A_4) \;=\; \Big(\frac{1 \;-\; \cos(t-s)}{\cos(t) \;-\; \cos(s)}\Big)\,(\partial_1 s \;-\; 2A_1)\,. \qquad (4.40)$$

At this stage we need to simplify our equations, because they are quite difficult to deal with without simplifying them, so we **assume that the following two equations are true**

$$\begin{aligned} A_4 &= -\,(A_1 \cos(s) + A_2 \sin(s))\,, \\ A_3 &= A_1 \sin(s) - A_2 \cos(s)\,. \end{aligned} \qquad (4.41)$$

Then the equations (4.21) and (4.23) will become

$$\Big(\partial_4 + \cos(s)\,\partial_1 + \sin(s)\,\partial_2\Big)\, s \;=\; 0\,, \qquad (4.42)$$

$$\Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\, s \;=\; 0\,, \qquad (4.43)$$

respectively. Consider a point $\underline{X}$ in space-time. Equations (4.42) and (4.43) show that s is constant on the plane through $\underline{X}$ spanned by the vectors $(\cos(s), \sin(s), 0, 1)$ and $(-\sin(s), \cos(s), 1, 0)$, where $s = s(X)$. If we take another point $\underline{X}'$, we also find that the function s is constant on the plane through $\underline{X}'$ spanned by the vectors $(\cos(s'), \sin(s'), 0, 1)$ and $(-\sin(s'), \cos(s'), 1, 0)$, where $s' = s(X')$.

If these planes intersect, we then have $s(\underline{X}) = s(\underline{X}')$.

The planes must intersect if the four vectors $(\cos(s), \sin(s), 0, 1)$, $(-\sin(s), \cos(s), 1, 0)$, $(\cos(s'), \sin(s'), 0, 1)$ and $(-\sin(s'), \cos(s'), 1, 0)$ form a basis for $\mathbb{R}^{2+2}$.

If they do not give a basis, then

$$\begin{vmatrix} \cos(s(\underline{X}')) & \sin(s(\underline{X}')) & 0 & 1 \\ -\sin(s(\underline{X}')) & \cos(s(\underline{X}')) & 1 & 0 \\ \cos(s(\underline{X})) & \sin(s(\underline{X})) & 0 & 1 \\ -\sin(s(\underline{X})) & \cos(s(\underline{X})) & 1 & 0 \end{vmatrix} \;=\; 0\,.$$

The condition we have from the last equation is

$$\cos(s(\underline{X}))\,\cos(s(\underline{X}')) + \sin(s(\underline{X}))\,\sin(s(\underline{X}')) \;=\; 1\,,$$

and we could write this condition as

$$\cos\left(s(\underline{X}) - s(\underline{X}')\right) = 1. \tag{4.44}$$

This means that $s(\underline{X}) - s(\underline{X}') = 2n\pi$, where $n \in \mathbb{N}$, and as s is a continuous function, then we deduce that s is a constant.

Now by considering s as a constant in the equations (4.20) and (4.22), we get

$$\Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\,a = 0\,, \tag{4.45}$$

$$\Big(\partial_4 + \cos(s)\,\partial_1 + \sin(s)\,\partial_2\Big)\,a = 0\,, \tag{4.46}$$

respectively. By changing the variables in the equations (4.45) and (4.46), we can solve them and write a as

$$a \equiv h\Big(x^4 - x^1\cos(s) - x^2\sin(s)\,,\, x^3 + x^1\sin(s) - x^2\cos(s)\Big)\,. \tag{4.47}$$

where h is an arbitrary function of two variables.
We can write the relations (4.38), (4.39) and (4.40) as

$$A_2 = \Big(\frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)}\Big)\,A_1\,, \tag{4.48}$$

$$A_3 = \Big(\frac{-\sin(t-s)}{\cos(t) - \cos(s)}\Big)\,A_1\,, \tag{4.49}$$

$$A_4 = \Big(\frac{1 - \cos(t-s)}{\cos(t) - \cos(s)}\Big)\,A_1\,. \tag{4.50}$$

4.4.1 The electromagnetic field

We know that the electromagnetic field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{4.51}$$

By using the values for A_2, A_3 and A_4 given by (4.48), (4.49) and (4.50) in terms of A_1 in (4.51),

$$
\begin{aligned}
F_{12} &= \frac{\big(\sin(t) - \sin(s)\big)}{\big(\cos(t) - \cos(s)\big)}\,(\partial_1 A_1) - \partial_2 A_1 - \\
&\quad A_1 \frac{\big(1 - \cos(t-s)\big)}{\big(\cos(t) - \cos(s)\big)^2}\,(\partial_1 t),
\end{aligned}
$$

$$
\begin{aligned}
F_{13} &= \frac{-\sin(t-s)}{\big(\cos(t) - \cos(s)\big)}\,(\partial_1 A_1) - \partial_3 A_1 + \\
&\quad A_1 \frac{\cos(s)\big(1 - \cos(t-s)\big)}{\big(\cos(t) - \cos(s)\big)^2}\,(\partial_1 t),
\end{aligned}
$$

$$
\begin{aligned}
F_{14} &= \frac{\big(1 - \cos(t-s)\big)}{\big(\cos(t) - \cos(s)\big)}\,(\partial_1 A_1) - \partial_4 A_1 - \\
&\quad A_1 \frac{\cos(s)\,\sin(t-s)}{\big(\cos(t) - \cos(s)\big)^2}\,(\partial_1 t),
\end{aligned}
$$

$$
\begin{aligned}
F_{23} &= \frac{-\sin(t-s)}{\big(\cos(t) - \cos(s)\big)}\,(\partial_2 A_1) - \frac{\big(\sin(t) - \sin(s)\big)}{\big(\cos(t) - \cos(s)\big)}\,(\partial_3 A_1) \\
&\quad - A_1 \frac{\big(1 - \cos(t-s)\big)}{\big(\cos(t) - \cos(s)\big)^2}\Big(\partial_3 t + \cos(s)\,\partial_2 t\Big),
\end{aligned}
$$

$$
\begin{aligned}
F_{24} &= \frac{\big(1 - \cos(t-s)\big)}{\big(\cos(t) - \cos(s)\big)}\,(\partial_2 A_1) - \frac{\big(\sin(t) - \sin(s)\big)}{\big(\cos(t) - \cos(s)\big)}\,(\partial_4 A_1) - \\
&\quad A_1 \frac{1}{\big(\cos(t) - \cos(s)\big)^2}\Big(\big(1 - \cos(t-s)\big)\partial_4 t \\
&\quad + \cos(s)\,\sin(t-s)\,\partial_2 t\Big),
\end{aligned}
$$

$$F_{34} = \frac{\Big(1 - \cos(t-s)\Big)}{\Big(\cos(t) - \cos(s)\Big)} (\partial_3 A_1) + \frac{\sin(t-s)}{\Big(\cos(t) - \cos(s)\Big)} (\partial_4 A_1) + A_1 \frac{\cos(s)}{\Big(\cos(t) - \cos(s)\Big)^2} \Big(\Big(1 - \cos(t-s)\Big)\partial_4 t - \sin(t-s)\,\partial_3 t\Big).$$

The next thing we want to do is to find out what the currentless equation gives us in terms of the electromagnetic field components. The currentless equation is given by

$$\partial_\mu F^{\mu\nu} = 0. \tag{4.52}$$

When $\nu = 1$.

$$(\partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2) A_1 + \frac{1}{\Big(\cos(t) - \cos(s)\Big)}\Big[(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\partial_3 - \Big(\sin(t) - \sin(s)\Big)\partial_2\Big]\,\partial_1 A_1 -$$
$$\frac{A_1}{\Big(\cos(t) - \cos(s)\Big)^2}\Big[(1 - \cos(t-s))\partial_2 + \cos(s)(1 - \cos(t-s))\partial_3 + \cos(s)\sin(t-s)\partial_4\Big]\,\partial_1 t - \frac{\partial_1 A_1}{\Big(\cos(t) - \cos(s)\Big)^2}\Big[(1 - \cos(t-s))\partial_2 +$$
$$\cos(s)(1 - \cos(t-s))\partial_3 + \cos(s)\sin(t-s)\partial_4\Big]\,t - \frac{\partial_1 t}{\Big(\cos(t) - \cos(s)\Big)^2}\Big[(1 - \cos(t-s))\partial_2 + \cos(s)(1 - \cos(t-s))\partial_3$$
$$+ \cos(s)\sin(t-s)\partial_4\Big]\,A_1 + \frac{A_1(\partial_1 t)}{\Big(\cos(t) - \cos(s)\Big)^3}\Big[\Big(\sin(t)\cos(t-s) + \sin(s)(1 - \cos(t-s))\Big)(\partial_2 + \cos(s)\partial_3)\,t + \Big(\sin(t)\sin(t-s) +$$
$$\cos(s)(1 - \cos(t-s))\Big)\,\partial_4 t\Big] = 0. \tag{4.53}$$

When $\nu = 2$,

$$\Big(\frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)}\Big)(\partial_1{}^2 - \partial_3{}^2 - \partial_4{}^2)A_1 + 2\Big[(\partial_1 A_1)(\partial_1 t) - (\partial_3 A_1)(\partial_3 t) - (\partial_4 A_1)(\partial_4 t)\Big]\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2} +$$
$$\frac{1}{(\cos(t) - \cos(s))}\Big[(1 - \cos(t-s))\,\partial_4 - \sin(t-s)\,\partial_3 - (\cos(t) - \cos(s))\,\partial_1\Big]\partial_2 A_1 + A_1\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2}(\partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)\,t$$
$$- \frac{(\partial_2 A_1)\cos(s)}{(\cos(t) - \cos(s))^2}\Big[(1 - \cos(t-s))\partial_3 + \sin(t-s)\,\partial_4\Big]t +$$
$$A_1\frac{(1 - \cos(t-s))(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))^3}\Big((\partial_1 t)^2 - (\partial_3 t)^2 - (\partial_4 t)^2\Big)$$
$$- \frac{A_1\cos(s)}{(\cos(t) - \cos(s))^2}\Big[(1 - \cos(t-s))\partial_3 + \sin(t-s)\,\partial_4\Big]\partial_2 t -$$
$$A_1\Big[\frac{\sin(t-s)(\sin(t) - \sin(s)) - (\cos(t) - \cos(s))}{(\cos(t) - \cos(s))^3}\cos(s)(\partial_4 t) +$$
$$\frac{\cos(s)(1 - \cos(t-s))(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))^3}(\partial_3 t)\Big](\partial_2 t) = 0. \quad (4.54)$$

When $\nu = 3$,

$$\Big(\frac{\sin(t-s)}{\cos(t) - \cos(s)}\Big)(\partial_1{}^2 + \partial_2{}^2 - \partial_4{}^2)A_1 + 2\Big[(\partial_1 A_1)(\partial_1 t) + (\partial_2 A_1)(\partial_2 t) - (\partial_4 A_1)(\partial_4 t)\Big]\frac{\cos(s)(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2} +$$

$$\frac{1}{(\cos(t) - \cos(s))}\Big[(\cos(t) - \cos(s))\partial_1 - (1 - \cos(t-s))\partial_4 + (\sin(t) - \sin(s))\partial_1\Big]\partial_3 A_1 + A_1\frac{\cos(s)(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2}(\partial_1{}^2 + \partial_2{}^2 - \partial_4{}^2)t$$

$$+\frac{(\partial_3 A_1)}{(\cos(t) - \cos(s))^2}\Big[(1 - \cos(t-s))\partial_2 + \cos(s)\sin(t-s)\partial_4\Big]t +$$

$$A_1\frac{\cos(s)(1 - \cos(t-s))(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))^3}\Big((\partial_1 t)^2 + (\partial_2 t)^2 - (\partial_4 t)^2\Big)$$

$$+\frac{A_1}{(\cos(t) - \cos(s))^2}\Big[(1 - \cos(t-s))\partial_2 + \cos(s)\sin(t-s)\partial_4\Big]\partial_3 t +$$

$$A_1\Big[\frac{\sin(t-s)(\sin(t) - \sin(s)) - (\cos(t) - \cos(s))}{(\cos(t) - \cos(s))^3}\cos(s)(\partial_4 t) + \frac{(1 - \cos(t-s))(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))^3}(\partial_2 t)\Big](\partial_3 t) = 0\,. \quad (4.55)$$

When $\nu = 4$,

$$\Big(\frac{1 - \cos(t-s)}{\cos(t) - \cos(s)}\Big)(\partial_3{}^2 - \partial_2{}^2 - \partial_1{}^2)A_1 + 2\Big[(\partial_1 A_1)(\partial_1 t) +$$

$$(\partial_2 A_1)(\partial_2 t) - (\partial_3 A_1)(\partial_3 t)\Big]\frac{\cos(s)\sin(t-s)}{(\cos(t) - \cos(s))^2} +$$

$$\frac{1}{(\cos(t) - \cos(s))}\Big[(\cos(t) - \cos(s))\partial_1 +$$

$$(\sin(t) - \sin(s))\partial_2 + \sin(t-s)\partial_3\Big]\partial_4 A_1 +$$

$$A_1\frac{\cos(s)\sin(t-s)}{(\cos(t) - \cos(s))^2}(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2)t +$$

$$\frac{(\partial_4 A_1)}{(\cos(t) - \cos(s))^2}\Big[(1 - \cos(t-s))\partial_2 +$$

$$\cos(s)(1 - \cos(t-s))\partial_3\Big]t +$$

$$A_1\cos(s)\frac{\sin(t-s)(\sin(t) - \sin(s)) - (\cos(t) - \cos(s))}{(\cos(t) - \cos(s))^3}$$

$$\Big((\partial_1 t)^2 + (\partial_2 t)^2 - (\partial_3 t)^2\Big) + A_1\frac{(1 - \cos(t-s))}{(\cos(t) - \cos(s))^2}\Big[\partial_2 +$$

$$\cos(s)\partial_3\Big]\partial_4 t + A_1\frac{(1-\cos(t-s))(\sin(t) - \sin(s))}{(\cos(t) - \cos(s))^3}$$

$$\Big[(\partial_2 t) + \cos(s)(\partial_3 t)\Big](\partial_4 t) = 0. \qquad (4.56)$$

Now as we see all these equations are still quite difficult to work with. The next idea is to simplify them by making an assumption.

Assume that t is a constant, then as we showed that s was a constant (4.44), equations (4.24) and (4.26) become

$$\Big(\partial_3 + \cos(t)\partial_2 - \sin(t)\partial_1\Big)(ra) = 0, \qquad (4.57)$$

$$\Big(\partial_4 - \cos(t)\partial_1 - \sin(t)\partial_2\Big)(ra) = 0, \qquad (4.58)$$

respectively. By changing the variables in the equations (4.57) and (4.58), we can solve them and we can write (ra) as

$$ra \equiv g\Big(x^4 + x^1\cos(t) + x^2\sin(t)\,,\; x^3 + x^1\sin(t) - x^2\cos(t)\Big)\,, \qquad (4.59)$$

where g is an arbitrary function of two variables.
Now we can write the equations (4.53), (4.54), (4.55) and (4.56) as the following

$$\begin{aligned}&\Big(\cos(t) - \cos(s)\Big)(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)A_1 + \\ &\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 - \Big(\sin(t) - \\ &\quad \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]\partial_1 A_1 = 0\,, \end{aligned} \qquad (4.60)$$

$$\begin{aligned}&\Big(\sin(t) - \sin(s)\Big)(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)A_1 + \\ &\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 - \Big(\sin(t) - \\ &\quad \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]\partial_2 A_1 = 0\,, \end{aligned} \qquad (4.61)$$

$$\begin{aligned}&\sin(t-s)\;(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)A_1 - \\ &\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 - \Big(\sin(t) - \\ &\quad \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]\partial_3 A_1 = 0\,, \end{aligned} \qquad (4.62)$$

$$\begin{aligned}&\Big(1 - \cos(t-s)\Big)(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)A_1 + \\ &\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 - \Big(\sin(t) - \\ &\quad \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]\partial_4 A_1 = 0\,, \end{aligned} \qquad (4.63)$$

respectively. There are another four interesting equations which are derived from the last four equations, and which we will solve first. To get the first equation let us multiply (4.60) by $\Big(\cos(t) - \cos(s)\Big)$, (4.61) by $\Big(\sin(t) - \sin(s)\Big)$, (4.62) by $\Big(-\sin(t-s)\Big)$ and finally (4.63) by $\Big(\cos(t-s) - 1\Big)$, and add all the results together, giving

$$\begin{aligned}&\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 - \Big(\sin(t) - \\ &\quad \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]^2 A_1 = 0\,. \end{aligned} \qquad (4.64)$$

The second equation is given by multiplying (4.60) by $\Big(\sin(t) - \sin(s)\Big)$ and (4.61) by $\Big(\cos(t) - \cos(s)\Big)$, then subtracting the results, giving

$$\Big[\Big(\sin(t) - \sin(s)\Big)\partial_1 - \Big(\cos(t) - \cos(s)\Big)\partial_2\Big]$$

$$\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 -$$

$$\Big(\sin(t) - \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]A_1 = 0. \tag{4.65}$$

The third equation is given by multiplying (4.60) by $\sin(t-s)$ and (4.62) by $\Big(\cos(t) - \cos(s)\Big)$, then adding the results, giving

$$\Big[\sin(t-s)\,\partial_1 - \Big(\cos(t) - \cos(s)\Big)\partial_3\Big]$$

$$\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 -$$

$$\Big(\sin(t) - \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]A_1 = 0. \tag{4.66}$$

Finally the last equation is given by multiplying (4.60) by $\Big(\cos(t-s) - 1\Big)$ and (4.63) by $\Big(\cos(t) - \cos(s)\Big)$, and subtracting the results, giving

$$\Big[\Big(\cos(t-s) - 1\Big)\partial_1 - \Big(\cos(t) - \cos(s)\Big)\partial_4\Big]$$

$$\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 -$$

$$\Big(\sin(t) - \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]A_1 = 0. \tag{4.67}$$

To solve the last four equations we set

$$f_1 = \Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 -$$

$$\Big(\sin(t) - \sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]A_1. \tag{4.68}$$

Now we can write the equations (4.64)-(4.67) as the following

$$\Big[\Big(1 - \cos(t-s)\Big)\partial_4 - \sin(t-s)\,\partial_3 - \Big(\sin(t) -$$

$$\sin(s)\Big)\partial_2 - \Big(\cos(t) - \cos(s)\Big)\partial_1\Big]f_1 = 0, \tag{4.69}$$

$$\Big[\Big(\sin(t) - \sin(s)\Big)\,\partial_1 - \Big(\cos(t) - \cos(s)\Big)\,\partial_2\Big]\,f_1 = 0\,, \qquad (4.70)$$

$$\Big[\sin(t-s)\,\partial_1 - \Big(\cos(t) - \cos(s)\Big)\,\partial_3\Big]\,f_1 = 0\,, \qquad (4.71)$$

$$\Big[\Big(1 - \cos(t-s)\Big)\,\partial_1 - \Big(\cos(t) - \cos(s)\Big)\,\partial_4\Big]\,f_1 = 0\,, \qquad (4.72)$$

respectively. The last four equations are linearly dependent, implying that f_1 is a function of one variable,

$$f_1\Big(a_1\,x^1 + a_1\,\frac{\sin(t) - \sin(s)}{\cos(t) + \cos(s)}\,x^2 - a_1\,\frac{\sin(t-s)}{\cos(t) + \cos(s)}\,x^3 + a_1\,\frac{1 - \cos(t-s)}{\cos(t) + \cos(s)}\,x^4\Big)\,, \qquad (4.73)$$

where a_1 is an arbitrary constant. By substituting this value for f_1 in equation (4.68), it becomes

$$\Big[\Big(1 - \cos(t-s)\Big)\,\partial_4 - \sin(t-s)\,\partial_3 - \Big(\sin(t) - \sin(s)\Big)\,\partial_2 - \Big(\cos(t) - \cos(s)\Big)\,\partial_1\Big]\,A_1 = f_1\Big(a_1\,x^1 + a_1\,\frac{\sin(t) - \sin(s)}{\cos(t) + \cos(s)}\,x^2 - a_1\,\frac{\sin(t-s)}{\cos(t) + \cos(s)}\,x^3 + a_1\,\frac{1 - \cos(t-s)}{\cos(t) + \cos(s)}\,x^4\Big)\,. \qquad (4.74)$$

To work out the last equation we need to change the variables first:

For x^1 and x^2,

$$\begin{aligned} p &= x^1 + \Big(\frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)}\Big)\,x^2\,, \\ q &= \Big(\cos(t) + \cos(s)\Big)\,x^1 + \Big(\sin(t) + \sin(s)\Big)\,x^2\,. \end{aligned} \qquad (4.75)$$

And for x^3 and x^4,

$$\begin{aligned} L &= \Big(\frac{1 - \cos(t-s)}{\cos(t) - \cos(s)}\Big)\,x^4 - \Big(\frac{\sin(t-s)}{\cos(t) - \cos(s)}\Big)\,x^3\,, \\ J &= \Big(1 + \cos(t-s)\Big)\,x^4 + \sin(t-s)\,x^3\,. \end{aligned} \qquad (4.76)$$

Now we can write equation (4.74) in the new variables as

$$\frac{2\cos(t-s)}{\cos(t) - \cos(s)} \left(\frac{\partial}{\partial p} - \frac{\partial}{\partial L}\right) A_1 = f_1(a_1(p+L)). \qquad (4.77)$$

Again here we need to change the variables as

$$R = p - L \qquad \text{and} \qquad K = p + L. \qquad (4.78)$$

Then equation (4.77) becomes

$$\frac{\partial}{\partial R} A_1 = \frac{\cos(t) - \cos(s)}{2\cos(t-s)} f_1(a_1 K).$$

By integrating the last equation with respect to R, we can write A_1 as

$$A_1 = R\frac{\cos(t) - \cos(s)}{2\cos(t-s)} f_1(a_1 K) + g_1(K, q, J),$$

or

$$A_1 = \frac{\cos(t) - \cos(s)}{2\cos(t-s)} (p - L) f_1(a_1(p+L)) + g_1(p+L, q, J), \qquad (4.79)$$

where L, J, p and q are as given in (4.65) and (4.66), and g_1 is an arbitrary function of three variables..

The idea now is to substitute this value for A_1 in the equations (4.60)-(4.63). In fact by doing this we will have the same result from all of them,

$$2a_1\left(\frac{1 - \cos(t-s)}{\cos(t-s)}\right) f_1' + a_1 f_1' + (\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2) g_1 = 0. \qquad (4.80)$$

Since g_1 is a function of $p + L$, q and J only, we have

$$(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2) g_1 = 2(1 + \cos(t-s))\left(\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial J^2}\right) g_1. \qquad (4.81)$$

Now by using equation (4.81) in equation (4.80), it becomes

$$\left(\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial J^2}\right) g_1 = \frac{\cos(t-s) - 2}{2\cos(t-s)(1+\cos(t-s))}\, a_1\, f_1'(a_1 K)\,. \qquad (4.82)$$

To solve the last equation we need first to change the variables as

$$h_1 = q + J \quad , \quad h_2 = q - J\,. \qquad (4.83)$$

Now we can write equation (4.82) as

$$4\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2} g_1 = \frac{\cos(t-s) - 2}{2\cos(t-s)(1+\cos(t-s))}\, a_1\, f_1'(a_1 K)\,.$$

Since g_1 is a function of $p+L$, h_1 and h_2 only, and f_1 is function of K, where K is $p+L$, now we are able to write g_1 as

$$g_1 = \frac{\cos(t-s) - 2}{8\cos(t-s)(1+\cos(t-s))}\, a_1 h_1\, h_2\, f_1'(a_1 K) + g_2(h_1, h_2, K)\,, \qquad (4.84)$$

where g_2 is any solution of

$$\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2} g_2 = 0\,,$$

where h_1 and h_2 are given by (4.83).

Finally we can write A_1 as

$$A_1 = \frac{\cos(t) - \cos(s)}{2\cos(t-s)}\,(p - L)\, f_1(a_1(p + L)) + \frac{\cos(t-s) - 2}{8\cos(t-s)(1+\cos(t-s))}\, a_1 h_1\, h_2\, f_1'(a_1(p + L)) + g_2(h_1, h_2, p + L)\,. \qquad (4.85)$$

There are non vanishing solutions for the electromagnetic field, given by

$$
\begin{aligned}
F_{12} &= \frac{2\sin(t-s)}{\cos(t) - \cos(s)}\left(\frac{\partial}{\partial h_1} + \frac{\partial}{\partial h_2}\right) g_2 + \\
&\quad \frac{\sin(t-s)(\cos(t-s)-2)}{4\cos(t-s)(1+\cos(t-s))} a_1 (h_1+h_2) f_1' , \\
F_{13} &= -\frac{\sin(t-s)}{\cos(t) - \cos(s)}\left(2\cos(t)\frac{\partial}{\partial h_1} + 2\cos(s)\frac{\partial}{\partial h_2}\right) g_2 - \\
&\quad \frac{\sin(t-s)(\cos(t-s)-2)}{8\cos(t-s)(1+\cos(t-s))}\Big((h_1+h_2)(\cos(t)+\cos(s)) - \\
&\quad (h_1-h_2)\Big) a_1 f_1' - \frac{\sin(t-s)}{\cos(t-s)} f_1 , \\
F_{14} &= \frac{1}{\cos(t) - \cos(s)}\left(2\cos(s)\frac{\partial}{\partial h_1} + 2\cos(t)\frac{\partial}{\partial h_2}\right) g_2 - \\
&\quad \frac{\cos(t-s)}{\cos(t) - \cos(s)}\left(2\cos(t)\frac{\partial}{\partial h_1} + 2\cos(s)\frac{\partial}{\partial h_2}\right) g_2 + \\
&\quad \frac{\cos(t-s)-2}{4\cos(t-s)(1+\cos(t-s))(\cos(t)-\cos(s))}\Big((\cos(t) - \\
&\quad \cos(s)\cos(t-s))h_1 + (\cos(s) - \cos(t)\cos(t-s))h_2\Big) a_1 f_1' + \\
&\quad \frac{1 - \cos(t-s)}{\cos(t-s)} f_1 , \\
F_{23} &= -\frac{\sin(t-s)}{\cos(t) - \cos(s)}\left(2\sin(t)\frac{\partial}{\partial h_1} + 2\sin(s)\frac{\partial}{\partial h_2}\right) g_2 \\
&\quad - \frac{\sin(t-s)(\cos(t-s)-2)}{4\cos(t-s)(1+\cos(t-s))(\cos(t)-\cos(s))}(\sin(t)h_2 + \\
&\quad \sin(s)h_1) a_1 f_1' - \frac{\sin(t-s)(\sin(t) - \sin(s))}{\cos(t-s)(\cos(t) - \cos(s))} f_1 ,
\end{aligned}
$$

$$\begin{aligned} F_{24} &= \frac{1}{\cos(t) - \cos(s)} \Big(2\sin(s) \frac{\partial}{\partial h_1} + 2\sin(t) \frac{\partial}{\partial h_2} \Big) g_2 - \\ &\quad \frac{\cos(t-s)}{\cos(t) - \cos(s)} \Big(2\sin(t) \frac{\partial}{\partial h_1} + 2\sin(s) \frac{\partial}{\partial h_2} \Big) g_2 + \\ &\quad \frac{\cos(t-s) - 2}{4\cos(t-s)(1+\cos(t-s))(\cos(t) - \cos(s))} \Big((\sin(t) - \\ &\quad \sin(s)\cos(t-s))h_1 + (\sin(s) - \sin(t)\cos(t-s))h_2 \Big) a_1 f_1' + \\ &\quad \frac{(1 - \cos(t-s))(\sin(t) - \sin(s))}{\cos(t-s)(\cos(t) - \cos(s))} f_1 , \\ F_{34} &= \frac{\sin(t-s)(\cos(t-s) - 2)}{4\cos(t-s)(1+\cos(t-s))(\cos(t) - \cos(s))} a_1 (h_1 - h_2) f_1' \\ &\quad + \frac{\sin(t-s)}{\cos(t) - \cos(s)} \Big(\frac{\partial}{\partial h_1} - \frac{\partial}{\partial h_2} \Big) g_2 . \end{aligned} \tag{4.86}$$

4.4.2 Summary of Case (I)

In this case we have that

$$\cos(t) - \cos(s) \neq 0 .$$

The solution for the Dirac massless equation is given by

$$\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\, r\, e^{\frac{i}{2}(s-t)} \\ i\, r\, e^{\frac{i}{2}(s+t)} \end{pmatrix} a ,$$

where s and t are real constants, and

$$\begin{aligned} a &\equiv h\Big(x^4 - x^1 \cos(s) - x^2 \sin(s) ,\, x^3 + x^1 \sin(s) - x^2 \cos(s) \Big) , \\ ra &\equiv g\Big(x^4 + x^1 \cos(t) + x^2 \sin(t) ,\, x^3 + x^1 \sin(t) - x^2 \cos(t) \Big) , \end{aligned}$$

where h and g are arbitrary functions.

The vector potential is given by

$$A_2 = \Big(\frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)}\Big) A_1 ,$$

$$A_3 = \Big(\frac{-\sin(t-s)}{\cos(t) - \cos(s)}\Big) A_1 ,$$

$$A_4 = \Big(\frac{1 - \cos(t-s)}{\cos(t) - \cos(s)}\Big) A_1 ,$$

where A_1 is given by

$$\begin{aligned} A_1 = {} & \frac{\cos(t) - \cos(s)}{2\cos(t-s)} (p - L) f_1(a_1(p + L)) + \\ & \frac{\cos(t-s) - 2}{8\cos(t-s)(1+\cos(t-s))} a_1 h_1 h_2 f_1'(a_1(p + L)) \\ & + g_2(h_1, h_2, p + L), \end{aligned}$$

where f_1 is an arbitrary function of $p + L$, a_1 is an arbitrary constant, and g_2 is any solution of

$$\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2} g_2 = 0,$$

where h_1 and h_2 are given by

$$h_1 = q + J \quad , \quad h_2 = q - J .$$

There are non vanishing solutions for the electromagnetic field, given by (4.86).

The variables p, L, q and J are given by

$$p = x^1 + \Big(\frac{\sin(t) - \sin(s)}{\cos(t) - \cos(s)}\Big) x^2 ,$$

$$q = \Big(\cos(t) + \cos(s)\Big) x^1 + \Big(\sin(t) + \sin(s)\Big) x^2 .$$

$$L = \Big(\frac{1 - \cos(t-s)}{\cos(t) - \cos(s)}\Big) x^4 - \Big(\frac{\sin(t-s)}{\cos(t) - \cos(s)}\Big) x^3 ,$$

$$J = \Big(1 + \cos(t-s)\Big) x^4 + \sin(t-s)\, x^3 .$$

4.5 Case (II), For $\cos(t) - \cos(s) = 0$

We begin with just the assumption $\cos(t) - \cos(s) = 0$. It will be convenient now to make further assumptions, namely (4.91) and that t is constant. In this case we have

$$\cos(t) = \cos(s) \qquad \text{so that} \qquad t = \pm s + 2n\pi\,, \quad n \in \mathbb{N}\,. \tag{4.87}$$

Next we are going to do one of these possibilities. The other will be similar. The one we want to do now is **when $t = s + 2n\pi$**:

Then we have from the equations (4.28), (4.29), (4.30) and (4.31) for $\sin(s) \neq 0$ the following relations :

$$(\partial_2 s - 2A_2) = -\frac{\cos(s)}{\sin(s)}(\partial_1 s - 2A_1)\,, \tag{4.88}$$

$$(\partial_3 s - 2A_3) = \frac{1}{\sin(s)}(\partial_1 s - 2A_1)\,, \tag{4.89}$$

$$(\partial_4 s - 2A_4) = 0\,. \tag{4.90}$$

At this stage we need to simplify our equations (4.21)-(4.27), because it is quite difficult to deal with them without simplifying them, so now we want to **assume that the following are true**

$$\begin{aligned} A_4 &= -(A_1\cos(s) + A_2\sin(s))\,, \\ A_3 &= A_1\sin(s) - A_2\cos(s)\,. \end{aligned} \tag{4.91}$$

Then the equations (4.21) and (4.23) become

$$\Big(\partial_4 + \cos(s)\,\partial_1 + \sin(s)\,\partial_2\Big)\,s = 0\,, \tag{4.92}$$

$$\Big(\partial_3 + \cos(s)\,\partial_2 - \sin(s)\,\partial_1\Big)\,s = 0\,, \tag{4.93}$$

respectively. Similarly from the last two equations we can deduce that s is constant, then from the equations (4.20) and (4.22) we will have the same value for a as in (4.47), and from the equations (4.24) and (4.26) the value for (ra) as in (4.59) after replacing each t by $s + 2n\pi$, so that

$$ra \equiv g\Big(x^4 + x^1 \cos(s) + x^2 \sin(s)\,,\, x^3 + x^1 \sin(s) - x^2 \cos(s)\Big)\,, \tag{4.94}$$

where g is an arbitrary function of two variables.
Now we can write the relations (4.88), (4.89) and (4.90) as

$$A_2 = -\frac{\cos(s)}{\sin(s)} A_1 \quad , \quad A_3 = \frac{1}{\sin(s)} A_1 \quad , \quad A_4 = 0\,. \tag{4.95}$$

4.5.1 The electromagnetic field

We know that the electromagnetic field tensor is given by (4.51). Now by using the values for A_2, A_3 and A_4 given by (4.95) in (4.51), we have following

$$\begin{aligned}
F_{12} &= -\Big(\frac{\cos(s)}{\sin(s)}\partial_1 + \partial_2\Big) A_1\,,\\
F_{13} &= \Big(\frac{1}{\sin(s)}\partial_1 - \partial_3\Big) A_1\,,\\
F_{14} &= -\partial_4 A_1\,,\\
F_{23} &= \frac{1}{\sin(s)}\Big(\partial_2 + \cos(s)\,\partial_3\Big) A_1\,,\\
F_{24} &= \frac{\cos(s)}{\sin(s)}\partial_4 A_1\,,\\
F_{34} &= -\frac{1}{\sin(s)}\partial_4 A_1\,.
\end{aligned}$$

The next thing we want to do is to find what the currentless equation (4.52) will give us by using the electromagnetic field components.

When $\nu = 1$,

$$\sin(s)\,(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)\,A_1 + \Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3\Big)\,\partial_1 A_1 = 0\,. \tag{4.96}$$

When $\nu = 2$,

$$\cos(s)\,(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)\,A_1 - \Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3\Big)\,\partial_2 A_1 = 0\,. \tag{4.97}$$

When $\nu = 3$,

$$(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)\,A_1 + \Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3 \Big)\,\partial_3\,A_1 = 0\,. \qquad (4.98)$$

When $\nu = 4$,

$$\Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3 \Big)\,\partial_4\,A_1 = 0\,. \qquad (4.99)$$

There are another three interesting equations which are derived from the last four equations, and which we will solve first. To get the first equation, let us multiply (4.96) by $\sin(s)$ and (4.97) by $\cos(s)$, and add the results to equation (4.98), giving

$$\Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3 \Big)^2 A_1 = 0\,. \qquad (4.100)$$

The second equation is given by multiplying (4.96) by $\cos(s)$ and (4.97) by $\sin(s)$, and by subtracting the results,

$$\Big(\cos(s)\,\partial_1 + \sin(s)\,\partial_2 \Big)\Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3 \Big)\,A_1 = 0\,. \qquad (4.101)$$

The last equation is given by multiplying (4.98) by $\sin(s)$ then by subtracting the result from (4.96).

$$\Big(\partial_1 + \sin(s)\,\partial_3 \Big)\Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3 \Big)\,A_1 = 0\,. \qquad (4.102)$$

Now before solving these equations let us first set

$$f_1 = \Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3 \Big)\,A_1\,. \qquad (4.103)$$

Then using this we write the equations (4.99)-(4.102) as

$$\partial_4\,f_1 = 0\,, \qquad (4.104)$$

$$\Big(\cos(s)\,\partial_2 - \sin(s)\,\partial_1 + \partial_3 \Big)\,f_1 = 0\,, \qquad (4.105)$$

$$\Big(\cos(s)\,\partial_1 + \sin(s)\,\partial_2 \Big)\,f_1 = 0\,, \qquad (4.106)$$

$$\Big(\partial_1 + \sin(s)\, \partial_3 \Big) f_1 = 0 , \tag{4.107}$$

respectively. The last four equations are linearly dependent equations, implying that f_1 is a function of one variable,

$$f_1 \Big(a_1 x^1 - \frac{\cos(s)}{\sin(s)} a_1 x^2 + \frac{1}{\sin(s)} a_1 x^3 \Big) . \tag{4.108}$$

By substituting this value for f_1 in equation (4.103), it becomes

$$\Big(\cos(s)\, \partial_2 - \sin(s)\, \partial_1 + \partial_3 \Big) A_1 = f_1 \Big(a_1 x^1 - \frac{\cos(s)}{\sin(s)} a_1 x^2 + \frac{1}{\sin(s)} a_1 x^3 \Big) . \tag{4.109}$$

To work out the last equation we need to change the variables first as the following

$$\begin{aligned} p &= x^1 - \frac{\cos(s)}{\sin(s)} x^2 , \\ q &= x^1 \sin(s) + x^2 \cos(s) . \end{aligned} \tag{4.110}$$

Now we can write equation (4.109) in the new variables as

$$\frac{-1}{\sin(s)} \Big(\frac{\partial}{\partial p} - \sin(s)\, \partial_3 \Big) A_1 = f_1 \Big(a_1 p + \frac{1}{\sin(s)} a_1 x^3 \Big) .$$

Again we change the variables for the last equation as

$$\begin{aligned} R &= p - \frac{1}{\sin(s)} x^3 , \\ K &= p + \frac{1}{\sin(s)} x^3 . \end{aligned} \tag{4.111}$$

By using this change of variables, we write the last equation as

$$2 \frac{\partial}{\partial R} A_1 = - \sin(s)\, f_1(a_1 K) .$$

Now by integrating both sides in the last equation with respect to R,

$$A_1 = \frac{-1}{2} \sin(s)\, f_1(a_1 K)\, R + g_1(K , q , x^4) .$$

or

$$A_1 = \frac{-1}{2} \sin(s) f_1\Big(a_1 (p + \frac{1}{\sin(s)} x^3)\Big) (p - \frac{1}{\sin(s)} x^3) + g_1\Big(p + \frac{1}{\sin(s)} x^3 , q , x^4\Big), \qquad (4.112)$$

where p and q are given by (4.110), and g_1 is an arbitrary function of three variables.

The idea now is to substitute this value for A_1 in the equations (4.96)-(4.99). In fact by doing this we will have the same result coming from all of them, giving

$$(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2) g_1 (K , q , x^4) = 0. \qquad (4.113)$$

Next we want to know what g_1 is independent of. Starting with

$$(a\partial_1 + b\partial_2 + c\partial_3) K = a - b\frac{\cos(s)}{\sin(s)} + \frac{c}{\sin(s)} = 0,$$

$$(a\partial_1 + b\partial_2 + c\partial_3)(\frac{q}{\sin(s)}) = a + b\frac{\cos(s)}{\sin(s)} = 0,$$

we find values of a. b and c so that

$$(a\partial_1 + b\partial_2 + c\partial_3) g_1 = 0,$$

If we set $b = -\sin(s)$. then

$$(\cos(s)\partial_1 - \sin(s)\partial_2 - \sin(2s)\partial_3) g_1 = 0. \qquad (4.114)$$

Now unless $\sin(2s) = \pm 1$. this equation is a derivative in a spacelike direction. We can do a Lorentz transformation to new coordinates (y_1, y_2, y_3, y_4) for $\mathbb{R}^{2+2}$ with signature $(+, +, -, -)$, in which (4.114) becomes

$$\frac{\partial}{\partial y_1} g_1 (y_1, y_2, y_3, y_4) = 0,$$

i.e. g_1 is a function of (y_2, y_3, y_4) and

$$(\partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2) g_1 (y_2, y_3, y_4) = 0.$$

This equation has lots of solutions, one way to solve an equation like this is by separation of variables.

If $\sin(2s) = \pm 1$, similarly we can get new coordinates in which (4.114) becomes

$$(\partial_1 - \partial_3)\, g_1(y_1, y_2, y_3, y_4) = 0.$$

Now by changing the variables to

$$p' = y_1 + y_3 \quad , \quad q' = y_1 - y_3 ,$$

the last equation says that

$$({\partial_2}^2 - {\partial_4}^2)\, g_1(y_1 - y_3, y_2, y_4) = 0.$$

There are non vanishing solutions for the electromagnetic field, given by

$$\begin{aligned}
F_{12} &= -2\cos(s)\frac{\partial}{\partial q} g_1 , \\
F_{13} &= -\sin(s) f_1 + \frac{1}{\sin(s)}\left(\frac{\partial}{\partial p} - \frac{\partial}{\partial x^3}\right) g_1 + \frac{\partial}{\partial q} g_1 , \\
F_{14} &= -\frac{\partial}{\partial x^4} g_1 , \\
F_{23} &= \frac{\cos(s)}{\sin(s)} f_1 - \cos(s)\frac{\partial}{\partial p} g_1 + \frac{\cos(s)}{\sin(s)}\frac{\partial}{\partial q} g_1 + \frac{\cos(s)}{\sin(s)^2}\frac{\partial}{\partial x^3} g_1 , \\
F_{24} &= \frac{\cos(s)}{\sin(s)}\frac{\partial}{\partial x^4} g_1 , \\
F_{34} &= \frac{-1}{\sin(s)}\frac{\partial}{\partial x^4} g_1 .
\end{aligned} \tag{4.115}$$

4.5.2 Summary of Case (II)

In this case

$$\cos(t) - \cos(s) = 0.$$

The solution for the Dirac massless equation is given by

$$\psi = \begin{pmatrix} 1 \\ e^{is} \\ i\,r\,e^{\frac{i}{2}(s-t)} \\ i\,r\,e^{\frac{i}{2}(s+t)} \end{pmatrix} a ,$$

where s and t are real constants, and

$$a \equiv h\Big(x^4 - x^1 \cos(s) - x^2 \sin(s), x^3 + x^1 \sin(s) - x^2 \cos(s)\Big),$$

$$ra \equiv g\Big(x^4 + x^1 \cos(s) + x^2 \sin(s), x^3 + x^1 \sin(s) - x^2 \cos(s)\Big),$$

where h and g are arbitrary functions.

The vector potential is given by

$$A_2 = -\frac{\cos(s)}{\sin(s)} A_1 \quad , \quad A_3 = \frac{1}{\sin(s)} A_1 \quad , \quad A_4 = 0,$$

where A_1 is given by

$$\begin{aligned} A_1 &= \frac{-1}{2} \sin(s) f_1\Big(a_1 (p + \frac{1}{\sin(s)} x^3)\Big) (p - \frac{1}{\sin(s)} x^3) \\ &\quad + g_1\Big(p + \frac{1}{\sin(s)} x^3, q, x^4\Big), \end{aligned}$$

where f_1 is an arbitrary function of $(p + x^3/\sin(s))$, a_1 is an arbitrary constant, and g_1 is any solution of

$$(\partial_1{}^2 + \partial_2{}^2 - \partial_3{}^2 - \partial_4{}^2)\, g_1 = 0,$$

where g_1 is a function of $(p + x^3/\sin(s))$, q and x^4. There are non vanishing solutions for the electromagnetic field, given by (4.115).

The variables p and q are given by

$$\begin{aligned} p &= x^1 - \frac{\cos(s)}{\sin(s)} x^2, \\ q &= x^1 \sin(s) + x^2 \cos(s). \end{aligned}$$

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APPENDIX

The lightlike line solution (light speed particle)

In this case we are looking for light speed particle, in other word we are looking for particles traveling along the line (t,0,t,0). The representation of the Dirac gamma matrices are given by

```
γ1 = {{0, 0, 0, -I}, {0, 0, -I, 0}, {0, I, 0, 0}, {I, 0, 0, 0}}

{{0, 0, 0, -i}, {0, 0, -i, 0}, {0, i, 0, 0}, {i, 0, 0, 0}}
```

```
γ2 = {{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}}

{{0, 0, 0, -1}, {0, 0, 1, 0}, {0, 1, 0, 0}, {-1, 0, 0, 0}}
```

```
γ3 = {{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}}

{{0, 0, -1, 0}, {0, 0, 0, 1}, {1, 0, 0, 0}, {0, -1, 0, 0}}
```

```
γ4 = {{0, 0, I, 0}, {0, 0, 0, I}, {I, 0, 0, 0}, {0, I, 0, 0}}

{{0, 0, i, 0}, {0, 0, 0, i}, {i, 0, 0, 0}, {0, i, 0, 0}}
```

The currentless spinor we want to consider in this case is given by

$$\psi = \{\{a\}, \{a\, e^{I s}\}, \{I\, r\, a\, e^{I((s-t)/2)}\}, \{I\, r\, a\, e^{I((s+t)/2)}\}\}$$

$$\left\{\{a\},\ \{a\, e^{i s}\},\ \left\{i\, a\, e^{\frac{1}{2} i (s-t)}\, r\right\},\ \left\{i\, a\, e^{\frac{1}{2} i (s+t)}\, r\right\}\right\}$$

There are two linearly independent elements of the Lie algebra of the group SO(2,2) which preserve a lightlike line (for massless particles), and they will provide two symmetry equations by substituting them in the Lie algebra representation (3.4). The symmetry equations we have here are given by

$$x4 \frac{\partial \psi}{\partial x2} + x2 \frac{\partial \psi}{\partial x4} = \frac{1}{2} \gamma 2\, \gamma 4 \,.\psi \,,$$

$$x3 \frac{\partial \psi}{\partial x1} + x1 \frac{\partial \psi}{\partial x3} = \frac{1}{2} \gamma 1\, \gamma 3 \,.\psi \,.$$

To look for particles travelling on the line (t,0,t,0), we suppose that we have a particularly simple equation for translation along the line,we guess that

$$\frac{\partial \psi}{\partial x1} + \frac{\partial \psi}{\partial x3} = iK\psi \,.$$

where K is a 4x4 matrix, and its entries are all functions of x1 and x3 only. The matrix K takes the currentless spinor to the tangent space of the currentless spinor. By working out the last equation together with the symmetry equations we find the form for the matrix K which is

```
K = {{k11, k12, k13, k14}, {-k12, k11, k14, -k13},
  {k31, k32, Conjugate[k11], k34},
  {k32, -k31, -k34, Conjugate[k11]}}

{{k11, k12, k13, k14}, {-k12, k11, k14, -k13},
 {k31, k32, Conjugate[k11], k34},
 {k32, -k31, -k34, Conjugate[k11]}}
```

The massless Dirac equation is given by

$$\gamma^{\mu} (\partial_{\mu} - i A_{\mu}) \psi = 0 .$$

By using the symmetry equations and by changing the variables we write could the Dirac equation in 2+2 dimensions as

$$(1/2)\ (e^{L}\ (\gamma3 - \gamma1) + e^{-L}\ (\gamma3 + \gamma1))\ (i\ e^{L}\ K)\ \psi + (1/(2\ J))\ e^{L}\ (\gamma3 - \gamma1)\ \psi + (1/2)\ (e^{p}\ (\gamma2 - \gamma4) + e^{-p}\ (\gamma2 + \gamma4)) \left(\frac{\partial}{\partial q} + \frac{1}{2\ q}\right) \psi = i\ \gamma^{\mu} A_{\mu}\ \psi,$$

where p, q, J and L are given by (3.40) and (3.42). To work out the last equation we need first to find the values for the following operators:

```
Oprator1 = I (A1 γ1 + A2 γ2 + A3 γ3 + A4 γ4)

{{0, 0, i (-A3 + i A4), i (-i A1 - A2)},
 {0, 0, i (-i A1 + A2), i (A3 + i A4)},
 {i (A3 + i A4), i (i A1 + A2), 0, 0},
 {i (i A1 - A2), i (-A3 + i A4), 0, 0}}
```

Oprator2 = (1 / 2) (e^{L} (γ3 - γ1) + e^{-L} (γ3 + γ1))

$$\left\{\left\{0,\ 0,\ \frac{1}{2}\ (-e^{-L} - e^{L}),\ \frac{1}{2}\ (-i\ e^{-L} + i\ e^{L})\right\},\ \left\{0,\ 0,\ \frac{1}{2}\ (-i\ e^{-L} + i\ e^{L}),\ \frac{1}{2}\ (e^{-L} + e^{L})\right\},\ \left\{\frac{1}{2}\ (e^{-L} + e^{L}),\ \frac{1}{2}\ (i\ e^{-L} - i\ e^{L}),\ 0,\ 0\right\},\ \left\{\frac{1}{2}\ (i\ e^{-L} - i\ e^{L}),\ \frac{1}{2}\ (-e^{-L} - e^{L}),\ 0,\ 0\right\}\right\}$$

Oprator3 = I (Oprator2.K)

$$\{\{i\left(\frac{1}{2}(-e^{-L}-e^{L})\,k31+\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,k32\right),$$
$$i\left(-\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,k31+\frac{1}{2}(-e^{-L}-e^{L})\,k32\right),$$
$$i\left(-\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,k34+\frac{1}{2}(-e^{-L}-e^{L})\,\text{Conjugate}[k11]\right),$$
$$i\left(\frac{1}{2}(-e^{-L}-e^{L})\,k34+\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,\text{Conjugate}[k11]\right)\},$$
$$\{i\left(\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,k31+\frac{1}{2}(e^{-L}+e^{L})\,k32\right),$$
$$i\left(-\frac{1}{2}(e^{-L}+e^{L})\,k31+\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,k32\right),$$
$$i\left(-\frac{1}{2}(e^{-L}+e^{L})\,k34+\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,\text{Conjugate}[k11]\right),$$
$$i\left(\frac{1}{2}(-i\,e^{-L}+i\,e^{L})\,k34+\frac{1}{2}(e^{-L}+e^{L})\,\text{Conjugate}[k11]\right)\},$$
$$\{i\left(\frac{1}{2}(e^{-L}+e^{L})\,k11-\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k12\right),$$
$$i\left(\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k11+\frac{1}{2}(e^{-L}+e^{L})\,k12\right),$$
$$i\left(\frac{1}{2}(e^{-L}+e^{L})\,k13+\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k14\right),$$
$$i\left(-\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k13+\frac{1}{2}(e^{-L}+e^{L})\,k14\right)\},$$
$$\{i\left(\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k11-\frac{1}{2}(-e^{-L}-e^{L})\,k12\right),$$
$$i\left(\frac{1}{2}(-e^{-L}-e^{L})\,k11+\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k12\right),$$
$$i\left(\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k13+\frac{1}{2}(-e^{-L}-e^{L})\,k14\right),$$
$$i\left(-\frac{1}{2}(-e^{-L}-e^{L})\,k13+\frac{1}{2}(i\,e^{-L}-i\,e^{L})\,k14\right)\}\}$$

Oprator4 = (1 / (2 J)) e^L (γ3 - γ1)

$$\{\{0,0,-\frac{e^{L}}{2J},\frac{i\,e^{L}}{2J}\},\{0,0,\frac{i\,e^{L}}{2J},\frac{e^{L}}{2J}\},$$
$$\{\frac{e^{L}}{2J},-\frac{i\,e^{L}}{2J},0,0\},\{-\frac{i\,e^{L}}{2J},-\frac{e^{L}}{2J},0,0\}\}$$

Oprator5 = (1 / 2) (e^p (γ2 - γ4) + e^-p (γ2 + γ4))

$$\{\{0,0,\frac{1}{2}(i\,e^{-p}-i\,e^{p}),\frac{1}{2}(-e^{-p}-e^{p})\},$$
$$\{0,0,\frac{1}{2}(e^{-p}+e^{p}),\frac{1}{2}(i\,e^{-p}-i\,e^{p})\},$$
$$\{\frac{1}{2}(i\,e^{-p}-i\,e^{p}),\frac{1}{2}(e^{-p}+e^{p}),0,0\},$$
$$\{\frac{1}{2}(-e^{-p}-e^{p}),\frac{1}{2}(i\,e^{-p}-i\,e^{p}),0,0\}\}$$

Tearm1 = (e^L Oprator3).ψ

$$\Big\{\Big\{i\,a\,e^{L+i s}\left(-\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)k31+\frac{1}{2}\left(-e^{-L}-e^{L}\right)k32\right)+i\,a\,e^{L}\left(\frac{1}{2}\left(-e^{-L}-e^{L}\right)k31+\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)k32\right)-a\,e^{L+\frac{1}{2}i(s-t)}\,r\left(-\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)k34+\frac{1}{2}\left(-e^{-L}-e^{L}\right)\text{Conjugate}[k11]\right)-a\,e^{L+\frac{1}{2}i(s+t)}\,r\left(\frac{1}{2}\left(-e^{-L}-e^{L}\right)k34+\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)\text{Conjugate}[k11]\right)\Big\},$$

$$\Big\{i\,a\,e^{L+i s}\left(-\frac{1}{2}\left(e^{-L}+e^{L}\right)k31+\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)k32\right)+i\,a\,e^{L}\left(\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)k31+\frac{1}{2}\left(e^{-L}+e^{L}\right)k32\right)-a\,e^{L+\frac{1}{2}i(s-t)}\,r\left(-\frac{1}{2}\left(e^{-L}+e^{L}\right)k34+\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)\text{Conjugate}[k11]\right)-a\,e^{L+\frac{1}{2}i(s+t)}\,r\left(\frac{1}{2}\left(-i\,e^{-L}+i\,e^{L}\right)k34+\frac{1}{2}\left(e^{-L}+e^{L}\right)\text{Conjugate}[k11]\right)\Big\},$$

$$\Big\{i\,a\,e^{L}\left(\frac{1}{2}\left(e^{-L}+e^{L}\right)k11-\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k12\right)+i\,a\,e^{L+i s}\left(\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k11+\frac{1}{2}\left(e^{-L}+e^{L}\right)k12\right)-a\,e^{L+\frac{1}{2}i(s-t)}\left(\frac{1}{2}\left(e^{-L}+e^{L}\right)k13+\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k14\right)r-a\,e^{L+\frac{1}{2}i(s+t)}\left(-\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k13+\frac{1}{2}\left(e^{-L}+e^{L}\right)k14\right)r\Big\},$$

$$\Big\{i\,a\,e^{L}\left(\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k11-\frac{1}{2}\left(-e^{-L}-e^{L}\right)k12\right)+i\,a\,e^{L+i s}\left(\frac{1}{2}\left(-e^{-L}-e^{L}\right)k11+\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k12\right)-a\,e^{L+\frac{1}{2}i(s-t)}\left(\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k13+\frac{1}{2}\left(-e^{-L}-e^{L}\right)k14\right)r-a\,e^{L+\frac{1}{2}i(s+t)}\left(-\frac{1}{2}\left(-e^{-L}-e^{L}\right)k13+\frac{1}{2}\left(i\,e^{-L}-i\,e^{L}\right)k14\right)r\Big\}\Big\}$$

Tearm2 = Oprator4 . ψ

$$\left\{\left\{-\frac{i\,a\,e^{L+\frac{1}{2}i(s-t)}\,r}{2J}-\frac{a\,e^{L+\frac{1}{2}i(s+t)}\,r}{2J}\right\},\left\{-\frac{a\,e^{L+\frac{1}{2}i(s-t)}\,r}{2J}+\frac{i\,a\,e^{L+\frac{1}{2}i(s+t)}\,r}{2J}\right\},\left\{\frac{a\,e^{L}}{2J}-\frac{i\,a\,e^{L+i s}}{2J}\right\},\left\{-\frac{i\,a\,e^{L}}{2J}-\frac{a\,e^{L+i s}}{2J}\right\}\right\}$$

```
Tearm3 = Oprator1 .ψ
```

$$\{\{-a\,(-A3 + i\,A4)\,e^{\frac{1}{2} i (s-t)}\,r - a\,(-i\,A1 - A2)\,e^{\frac{1}{2} i (s+t)}\,r\},$$
$$\{-a\,(-i\,A1 + A2)\,e^{\frac{1}{2} i (s-t)}\,r - a\,(A3 + i\,A4)\,e^{\frac{1}{2} i (s+t)}\,r\},$$
$$\{i\,a\,(A3 + i\,A4) + i\,a\,(i\,A1 + A2)\,e^{i s}\},$$
$$\{i\,a\,(i\,A1 - A2) + i\,a\,(-A3 + i\,A4)\,e^{i s}\}\}$$

Tearm4 = Tearm3 - Tearm2 - Tearm1

$$\Big\{\Big\{-i\,a\,e^{L+i\,s}\left(-\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,k31+\frac{1}{2}\,(-e^{-L}-e^{L})\,k32\right)-i\,a\,e^{L}\left(\frac{1}{2}\,(-e^{-L}-e^{L})\,k31+\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,k32\right)-a\,(-A3+i\,A4)\,e^{\frac{1}{2}\,i\,(s-t)}\,r-a\,(-i\,A1-A2)\,e^{\frac{1}{2}\,i\,(s+t)}\,r+\frac{i\,a\,e^{L+\frac{1}{2}\,i\,(s-t)}\,r}{2\,J}+\frac{a\,e^{L+\frac{1}{2}\,i\,(s+t)}\,r}{2\,J}+a\,e^{L+\frac{1}{2}\,i\,(s-t)}\,r\left(-\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,k34+\frac{1}{2}\,(-e^{-L}-e^{L})\,\text{Conjugate}[k11]\right)+a\,e^{L+\frac{1}{2}\,i\,(s+t)}\,r\left(\frac{1}{2}\,(-e^{-L}-e^{L})\,k34+\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,\text{Conjugate}[k11]\right)\Big\},$$

$$\Big\{-i\,a\,e^{L+i\,s}\left(-\frac{1}{2}\,(e^{-L}+e^{L})\,k31+\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,k32\right)-i\,a\,e^{L}\left(\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,k31+\frac{1}{2}\,(e^{-L}+e^{L})\,k32\right)-a\,(-i\,A1+A2)\,e^{\frac{1}{2}\,i\,(s-t)}\,r-a\,(A3+i\,A4)\,e^{\frac{1}{2}\,i\,(s+t)}\,r+\frac{a\,e^{L+\frac{1}{2}\,i\,(s-t)}\,r}{2\,J}-\frac{i\,a\,e^{L+\frac{1}{2}\,i\,(s+t)}\,r}{2\,J}+a\,e^{L+\frac{1}{2}\,i\,(s-t)}\,r\left(-\frac{1}{2}\,(e^{-L}+e^{L})\,k34+\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,\text{Conjugate}[k11]\right)+a\,e^{L+\frac{1}{2}\,i\,(s+t)}\,r\left(\frac{1}{2}\,(-i\,e^{-L}+i\,e^{L})\,k34+\frac{1}{2}\,(e^{-L}+e^{L})\,\text{Conjugate}[k11]\right)\Big\},$$

$$\Big\{i\,a\,(A3+i\,A4)+i\,a\,(i\,A1+A2)\,e^{i\,s}-\frac{a\,e^{L}}{2\,J}+\frac{i\,a\,e^{L+i\,s}}{2\,J}-i\,a\,e^{L}\left(\frac{1}{2}\,(e^{-L}+e^{L})\,k11-\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k12\right)-i\,a\,e^{L+i\,s}\left(\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k11+\frac{1}{2}\,(e^{-L}+e^{L})\,k12\right)+a\,e^{L+\frac{1}{2}\,i\,(s-t)}\left(\frac{1}{2}\,(e^{-L}+e^{L})\,k13+\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k14\right)r+a\,e^{L+\frac{1}{2}\,i\,(s+t)}\left(-\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k13+\frac{1}{2}\,(e^{-L}+e^{L})\,k14\right)r\Big\},$$

$$\Big\{i\,a\,(i\,A1-A2)+i\,a\,(-A3+i\,A4)\,e^{i\,s}+\frac{i\,a\,e^{L}}{2\,J}+\frac{a\,e^{L+i\,s}}{2\,J}-i\,a\,e^{L}\left(\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k11-\frac{1}{2}\,(-e^{-L}-e^{L})\,k12\right)-i\,a\,e^{L+i\,s}\left(\frac{1}{2}\,(-e^{-L}-e^{L})\,k11+\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k12\right)+a\,e^{L+\frac{1}{2}\,i\,(s-t)}\left(\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k13+\frac{1}{2}\,(-e^{-L}-e^{L})\,k14\right)r+a\,e^{L+\frac{1}{2}\,i\,(s+t)}\left(-\frac{1}{2}\,(-e^{-L}-e^{L})\,k13+\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k14\right)r\Big\}\Big\}$$

```
Oprator6 = Inverse[Oprator5]
```

$$\left\{\left\{0,\ 0,\ \frac{i\,e^{-p}}{2}-\frac{i\,e^{p}}{2},\ -\frac{e^{-p}}{2}-\frac{e^{p}}{2}\right\},\right.$$
$$\left\{0,\ 0,\ \frac{e^{-p}}{2}+\frac{e^{p}}{2},\ \frac{i\,e^{-p}}{2}-\frac{i\,e^{p}}{2}\right\},$$
$$\left\{\frac{i\,e^{-p}}{2}-\frac{i\,e^{p}}{2},\ \frac{e^{-p}}{2}+\frac{e^{p}}{2},\ 0,\ 0\right\},$$
$$\left.\left\{-\frac{e^{-p}}{2}-\frac{e^{p}}{2},\ \frac{i\,e^{-p}}{2}-\frac{i\,e^{p}}{2},\ 0,\ 0\right\}\right\}$$

```
Oprator7 = FullSimplify[Oprator6]
```

```
{{0, 0, -i Sinh[p], -Cosh[p]}, {0, 0, Cosh[p], -i Sinh[p]},
 {-i Sinh[p], Cosh[p], 0, 0}, {-Cosh[p], -i Sinh[p], 0, 0}}
```

```
Tearm6 = Oprator7 . Tearm4
```

$$\left\{\left\{-\left(i\,a\,(i\,A1-A2)+i\,a\,(-A3+i\,A4)\,e^{i\,s}+\frac{i\,a\,e^{L}}{2\,J}+\frac{a\,e^{L+i\,s}}{2\,J}-\right.\right.\right.$$
$$i\,a\,e^{L}\left(\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k11-\frac{1}{2}\,(-e^{-L}-e^{L})\,k12\right)-$$
$$i\,a\,e^{L+i\,s}\left(\frac{1}{2}\,(-e^{-L}-e^{L})\,k11+\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k12\right)+$$
$$a\,e^{L+\frac{1}{2}\,i\,(s-t)}\left(\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k13+\frac{1}{2}\,(-e^{-L}-e^{L})\,k14\right)r+$$
$$a\,e^{L+\frac{1}{2}\,i\,(s+t)}\left(-\frac{1}{2}\,(-e^{-L}-e^{L})\,k13+\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k14\right)$$
$$\left.r\right)\mathrm{Cosh}[p]-$$
$$i\left(i\,a\,(A3+i\,A4)+i\,a\,(i\,A1+A2)\,e^{i\,s}-\frac{a\,e^{L}}{2\,J}+\frac{i\,a\,e^{L+i\,s}}{2\,J}-\right.$$
$$i\,a\,e^{L}\left(\frac{1}{2}\,(e^{-L}+e^{L})\,k11-\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k12\right)-$$
$$i\,a\,e^{L+i\,s}\left(\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k11+\frac{1}{2}\,(e^{-L}+e^{L})\,k12\right)+$$
$$a\,e^{L+\frac{1}{2}\,i\,(s-t)}\left(\frac{1}{2}\,(e^{-L}+e^{L})\,k13+\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k14\right)r+$$
$$a\,e^{L+\frac{1}{2}\,i\,(s+t)}\left(-\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k13+\frac{1}{2}\,(e^{-L}+e^{L})\,k14\right)$$
$$\left.\left.r\right)\mathrm{Sinh}[p]\right\},$$
$$\left\{\left(i\,a\,(A3+i\,A4)+i\,a\,(i\,A1+A2)\,e^{i\,s}-\frac{a\,e^{L}}{2\,J}+\frac{i\,a\,e^{L+i\,s}}{2\,J}-\right.\right.$$
$$i\,a\,e^{L}\left(\frac{1}{2}\,(e^{-L}+e^{L})\,k11-\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k12\right)-$$
$$i\,a\,e^{L+i\,s}\left(\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k11+\frac{1}{2}\,(e^{-L}+e^{L})\,k12\right)+$$
$$a\,e^{L+\frac{1}{2}\,i\,(s-t)}\left(\frac{1}{2}\,(e^{-L}+e^{L})\,k13+\frac{1}{2}\,(i\,e^{-L}-i\,e^{L})\,k14\right)r+$$

$$
\begin{aligned}
&a\, e^{L+\frac{1}{2}\, i\, (s+t)} \left(-\frac{1}{2} \left(i\, e^{-L} - i\, e^{L}\right) k13 + \frac{1}{2} \left(e^{-L} + e^{L}\right) k14\right) r\Big) \\
&\text{Cosh}[p] - i \Big(i\, a\, (i\, A1 - A2) + i\, a\, (-A3 + i\, A4)\, e^{i s} + \\
&\quad \frac{i\, a\, e^{L}}{2 J} + \frac{a\, e^{L+i s}}{2 J} - \\
&\quad i\, a\, e^{L} \left(\frac{1}{2} \left(i\, e^{-L} - i\, e^{L}\right) k11 - \frac{1}{2} \left(-e^{-L} - e^{L}\right) k12\right) - \\
&\quad i\, a\, e^{L+i s} \left(\frac{1}{2} \left(-e^{-L} - e^{L}\right) k11 + \frac{1}{2} \left(i\, e^{-L} - i\, e^{L}\right) k12\right) + \\
&\quad a\, e^{L+\frac{1}{2}\, i\, (s-t)} \left(\frac{1}{2} \left(i\, e^{-L} - i\, e^{L}\right) k13 + \frac{1}{2} \left(-e^{-L} - e^{L}\right) k14\right) r + \\
&\quad a\, e^{L+\frac{1}{2}\, i\, (s+t)} \left(-\frac{1}{2} \left(-e^{-L} - e^{L}\right) k13 + \frac{1}{2} \left(i\, e^{-L} - i\, e^{L}\right) k14\right) \\
&\qquad r\Big)\, \text{Sinh}[p]\Big\}, \\
&\Big\{\Big(-i\, a\, e^{L+i s} \left(-\frac{1}{2} \left(e^{-L} + e^{L}\right) k31 + \frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k32\right) - \\
&\quad i\, a\, e^{L} \left(\frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k31 + \frac{1}{2} \left(e^{-L} + e^{L}\right) k32\right) - \\
&\quad a\, (-i\, A1 + A2)\, e^{\frac{1}{2}\, i\, (s-t)}\, r - a\, (A3 + i\, A4)\, e^{\frac{1}{2}\, i\, (s+t)}\, r + \\
&\quad \frac{a\, e^{L+\frac{1}{2}\, i\, (s-t)}\, r}{2 J} - \frac{i\, a\, e^{L+\frac{1}{2}\, i\, (s+t)}\, r}{2 J} + a\, e^{L+\frac{1}{2}\, i\, (s-t)}\, r \\
&\qquad \left(-\frac{1}{2} \left(e^{-L} + e^{L}\right) k34 + \frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) \text{Conjugate}[k11]\right) + \\
&\quad a\, e^{L+\frac{1}{2}\, i\, (s+t)}\, r \Big(\frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k34 + \\
&\qquad \frac{1}{2} \left(e^{-L} + e^{L}\right) \text{Conjugate}[k11]\Big)\Big)\, \text{Cosh}[p] - \\
&\ i \Big(-i\, a\, e^{L+i s} \left(-\frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k31 + \frac{1}{2} \left(-e^{-L} - e^{L}\right) k32\right) - \\
&\quad i\, a\, e^{L} \left(\frac{1}{2} \left(-e^{-L} - e^{L}\right) k31 + \frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k32\right) - \\
&\quad a\, (-A3 + i\, A4)\, e^{\frac{1}{2}\, i\, (s-t)}\, r - a\, (-i\, A1 - A2)\, e^{\frac{1}{2}\, i\, (s+t)}\, r + \\
&\quad \frac{i\, a\, e^{L+\frac{1}{2}\, i\, (s-t)}\, r}{2 J} + \frac{a\, e^{L+\frac{1}{2}\, i\, (s+t)}\, r}{2 J} + \\
&\quad a\, e^{L+\frac{1}{2}\, i\, (s-t)}\, r \Big(-\frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k34 + \frac{1}{2} \left(-e^{-L} - e^{L}\right) \\
&\qquad \text{Conjugate}[k11]\Big) + a\, e^{L+\frac{1}{2}\, i\, (s+t)}\, r \Big(\frac{1}{2} \left(-e^{-L} - e^{L}\right) k34 + \\
&\qquad \frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) \text{Conjugate}[k11]\Big)\Big)\, \text{Sinh}[p]\Big\}, \\
&\Big\{-\Big(-i\, a\, e^{L+i s} \left(-\frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k31 + \frac{1}{2} \left(-e^{-L} - e^{L}\right) k32\right) - \\
&\quad i\, a\, e^{L} \left(\frac{1}{2} \left(-e^{-L} - e^{L}\right) k31 + \frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k32\right) - \\
&\quad a\, (-A3 + i\, A4)\, e^{\frac{1}{2}\, i\, (s-t)}\, r - a\, (-i\, A1 - A2)\, e^{\frac{1}{2}\, i\, (s+t)}\, r + \\
&\quad \frac{i\, a\, e^{L+\frac{1}{2}\, i\, (s-t)}\, r}{2 J} + \frac{a\, e^{L+\frac{1}{2}\, i\, (s+t)}\, r}{2 J} + \\
&\quad a\, e^{L+\frac{1}{2}\, i\, (s-t)}\, r \Big(-\frac{1}{2} \left(-i\, e^{-L} + i\, e^{L}\right) k34 + \frac{1}{2} \left(-e^{-L} - e^{L}\right) \\
&\qquad \text{Conjugate}[k11]\Big) + a\, e^{L+\frac{1}{2}\, i\, (s+t)}\, r \Big(\frac{1}{2} \left(-e^{-L} - e^{L}\right)
\end{aligned}
$$

$$k34 + \frac{1}{2}\left(-i\, e^{-L} + i\, e^{L}\right) \text{Conjugate}[k11]\Big)\Big) \text{Cosh}[p] -$$

$$i \Big(-i\, a\, e^{L+i\, s} \left(-\frac{1}{2}\left(e^{-L} + e^{L}\right) k31 + \frac{1}{2}\left(-i\, e^{-L} + i\, e^{L}\right) k32\right) -$$

$$i\, a\, e^{L} \left(\frac{1}{2}\left(-i\, e^{-L} + i\, e^{L}\right) k31 + \frac{1}{2}\left(e^{-L} + e^{L}\right) k32\right) -$$

$$a\,(-i\, A1 + A2)\, e^{\frac{1}{2} i\,(s-t)}\, r - a\,(A3 + i\, A4)\, e^{\frac{1}{2} i\,(s+t)}\, r +$$

$$\frac{a\, e^{L+\frac{1}{2} i\,(s-t)}\, r}{2\, J} - \frac{i\, a\, e^{L+\frac{1}{2} i\,(s+t)}\, r}{2\, J} + a\, e^{L+\frac{1}{2} i\,(s-t)}\, r$$

$$\left(-\frac{1}{2}\left(e^{-L} + e^{L}\right) k34 + \frac{1}{2}\left(-i\, e^{-L} + i\, e^{L}\right) \text{Conjugate}[k11]\right) +$$

$$a\, e^{L+\frac{1}{2} i\,(s+t)}\, r \Big(\frac{1}{2}\left(-i\, e^{-L} + i\, e^{L}\right) k34 +$$

$$\frac{1}{2}\left(e^{-L} + e^{L}\right) \text{Conjugate}[k11]\Big)\Big) \text{Sinh}[p]\}\}$$

Equa1 = (1 / (2 q)) ψ

$$\left\{\left\{\frac{a}{2\, q}\right\}, \left\{\frac{a\, e^{i\, s}}{2\, q}\right\}, \left\{\frac{i\, a\, e^{\frac{1}{2} i\,(s-t)}\, r}{2\, q}\right\}, \left\{\frac{i\, a\, e^{\frac{1}{2} i\,(s+t)}\, r}{2\, q}\right\}\right\}$$

****

****

$$\frac{\partial}{\partial q}\psi = \text{Equa2}$$

```
Equa2 = FullSimplify[Tearm6 - Equa1]
```

$$
\begin{aligned}
&\Big\{\Big\{\frac{1}{2\,J\,q}\\
&\quad\Big(a\,e^{-\frac{i t}{2}}\,\Big(e^{-p+\frac{1}{2}\,i\,(2\,s+t)}\,(J\,(-i\,A1-A2+i\,A3+A4+e^{2\,p}\,(i\,A1+\\
&\qquad\qquad A2+i\,A3+A4-i\,k11-k12))+\\
&\qquad\quad e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12)))\,q+e^{-p+\frac{i s}{2}}\,J\,q\\
&\qquad r\,(e^{2\,p}\,(-i\,k13+k14)+e^{2\,L}\,(i\,k13+k14)-\\
&\qquad\quad 2\,e^{L+p+i\,t}\,(k13\,\mathrm{Cosh}[L-p]-i\,k14\,\mathrm{Sinh}[L-p]))+\\
&\qquad e^{\frac{i t}{2}}\,(-i\,e^{L-p}\,q+e^{2\,L-p}\,J\,(k11+i\,k12)\,q+\\
&\qquad\quad J\,(-1+(2\,A1+i\,(2\,A2+i\,k11+k12))\,q\,\mathrm{Cosh}[p]+\\
&\qquad\qquad (2\,A3+2\,i\,A4-k11+i\,k12)\,q\,\mathrm{Sinh}[p])))\Big)\Big)\Big\},\\
&\Big\{\frac{1}{2\,J\,q}\,\Big(a\,e^{-\frac{i t}{2}}\,\Big(e^{-p+\frac{i t}{2}}\,(e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12))+\\
&\qquad\quad J\,(-i\,A1+A2+i\,A3-A4-\\
&\qquad\qquad e^{2\,p}\,(-i\,A1+A2-i\,A3+A4+i\,k11+k12)))\,q+\\
&\qquad e^{-p+\frac{i s}{2}}\,J\,q\,r\,(e^{2\,L}\,(-i+e^{i\,t})\,(i\,k13+k14)+\\
&\qquad\quad e^{2\,p}\,(k13+i\,k14)\,(1-i\,\mathrm{Cos}[t]+\mathrm{Sin}[t]))+\\
&\qquad e^{\frac{1}{2}\,i\,(2\,s+t)}\,(e^{-p}\,(-e^{p}\,J+i\,e^{L}\,q-e^{2\,L}\,J\,(k11+i\,k12)\,q)+\\
&\qquad\quad J\,q\,((-2\,A1+2\,i\,A2+k11-i\,k12)\,\mathrm{Cosh}[p]+\\
&\qquad\qquad (-2\,A3+2\,i\,A4+k11-i\,k12)\,\mathrm{Sinh}[p]))\Big)\Big)\Big\},\\
&\Big\{\frac{1}{2\,J\,q}\,\Big(a\,e^{-\frac{i t}{2}}\,\Big(e^{\frac{1}{2}\,i\,(s+2\,t)}\,q\,r\,(i\,e^{L+p}\,(-1+e^{L}\,J\,k34)-\\
&\qquad\quad J\,(2\,A3+i\,(2\,A4+k34))\,\mathrm{Cosh}[p]+\\
&\qquad\quad 2\,e^{L}\,J\,\mathrm{Conjugate}[k11]\,\mathrm{Cosh}[L+p]+\\
&\qquad\quad J\,(2\,A1-2\,i\,A2+i\,k34)\,\mathrm{Sinh}[p])+\\
&\qquad 2\,e^{L+\frac{i t}{2}}\,J\,q\,(i\,(e^{i\,s}\,k31-k32)\,\mathrm{Cosh}[L+p]+\\
&\qquad\quad (k31+e^{i\,s}\,k32)\,\mathrm{Sinh}[L+p])-\\
&\qquad e^{\frac{i s}{2}}\,r\,(e^{L+p}\,(-1+e^{L}\,J\,k34)\,q+J\,(i+(-2\,i\,A1+2\,A2+k34)\\
&\qquad\qquad q\,\mathrm{Cosh}[p]+(2\,i\,A3+2\,A4-k34)\,q\,\mathrm{Sinh}[p])-\\
&\qquad\quad 2\,i\,e^{L}\,J\,q\,\mathrm{Conjugate}[k11]\,\mathrm{Sinh}[L+p])\Big)\Big)\Big\},\\
&\Big\{\frac{1}{2\,J\,q}\,\Big(a\,e^{-\frac{i t}{2}}\,\Big(e^{\frac{i s}{2}}\,q\,r\,(i\,e^{L+p}\,(-1+e^{L}\,J\,k34)+\\
&\qquad\quad J\,(-2\,A3+2\,i\,A4-i\,k34)\,\mathrm{Cosh}[p]+\\
&\qquad\quad 2\,e^{L}\,J\,\mathrm{Conjugate}[k11]\,\mathrm{Cosh}[L+p]+\\
&\qquad\quad J\,(2\,A1+i\,(2\,A2+k34))\,\mathrm{Sinh}[p])+\\
&\qquad 2\,e^{L+\frac{i t}{2}}\,J\,q\,(-i\,(k31+e^{i\,s}\,k32)\,\mathrm{Cosh}[L+p]+\\
&\qquad\quad (e^{i\,s}\,k31-k32)\,\mathrm{Sinh}[L+p])+e^{\frac{1}{2}\,i\,(s+2\,t)}\,r\\
&\qquad\ (e^{L+p}\,(-1+e^{L}\,J\,k34)\,q+J\,(-2\,i\,A1-2\,A2+k34)\,q\\
&\qquad\qquad \mathrm{Cosh}[p]-J\,(i+(-2\,i\,A3+2\,A4+k34)\,q\,\mathrm{Sinh}[p])-\\
&\qquad\quad 2\,i\,e^{L}\,J\,q\,\mathrm{Conjugate}[k11]\,\mathrm{Sinh}[L+p])\Big)\Big)\Big\}\Big\}
\end{aligned}
$$

****

$$\frac{\partial}{\partial q}\,a = a1$$

$$a1 = \frac{1}{2 J q}\left(a e^{-\frac{i t}{2}}\left(e^{-p+\frac{1}{2} i (2 s+t)} (J (-i A1 - A2 + i A3 + A4 + e^{2 p} (i A1 + A2 + i A3 + A4 - i k11 - k12)) + e^{L} (-1 + e^{L} J (-i k11 + k12))) q + e^{-p+\frac{i s}{2}} J q\, r (e^{2 p} (-i k13 + k14) + e^{2 L} (i k13 + k14) - 2 e^{L+p+i t} (k13 \operatorname{Cosh}[L - p] - i k14 \operatorname{Sinh}[L - p])) + e^{\frac{i t}{2}} (-i e^{L-p} q + e^{2 L-p} J (k11 + i k12) q + J (-1 + (2 A1 + i (2 A2 + i k11 + k12)) q \operatorname{Cosh}[p] + (2 A3 + 2 i A4 - k11 + i k12) q \operatorname{Sinh}[p]))\right)\right)$$

$$\frac{1}{2 J q}\left(a e^{-\frac{i t}{2}}\left(e^{-p+\frac{1}{2} i (2 s+t)} (J (-i A1 - A2 + i A3 + A4 + e^{2 p} (i A1 + A2 + i A3 + A4 - i k11 - k12)) + e^{L} (-1 + e^{L} J (-i k11 + k12))) q + e^{-p+\frac{i s}{2}} J q\, r (e^{2 p} (-i k13 + k14) + e^{2 L} (i k13 + k14) - 2 e^{L+p+i t} (k13 \operatorname{Cosh}[L - p] - i k14 \operatorname{Sinh}[L - p])) + e^{\frac{i t}{2}} (-i e^{L-p} q + e^{2 L-p} J (k11 + i k12) q + J (-1 + (2 A1 + i (2 A2 + i k11 + k12)) q \operatorname{Cosh}[p] + (2 A3 + 2 i A4 - k11 + i k12) q \operatorname{Sinh}[p]))\right)\right)$$

****

$$\frac{\partial}{\partial q} s = s1$$

```
s1 = ((-I) / a) e^(-(I s))
  (1/(2 J q) (a e^(-(i t)/2) (e^(-p+(i t)/2) (e^L (-1 + e^L J (-i k11 + k12)) +
        J (-i A1 + A2 + i A3 - A4 - e^(2 p)
           (-i A1 + A2 - i A3 + A4 + i k11 + k12))) q +
      e^(-p+(i s)/2) J q r (e^(2 L) (-i + e^(i t)) (i k13 + k14) + e^(2 p)
         (k13 + i k14) (1 - i Cos[t] + Sin[t])) + e^(1/2 i (2 s+t))
       (e^(-p) (-e^p J + i e^L q - e^(2 L) J (k11 + i k12) q) + J q
         ((-2 A1 + 2 i A2 + k11 - i k12) Cosh[p] + (-2 A3 + 2
              i A4 + k11 - i k12) Sinh[p]))))) + (I / a) a1

- 1/(2 J q) (i e^(-i s-(i t)/2)
     (e^(-p+(i t)/2) (e^L (-1 + e^L J (-i k11 + k12)) + J (-i A1 + A2 + i A3 -
            A4 - e^(2 p) (-i A1 + A2 - i A3 + A4 + i k11 + k12))) q +
      e^(-p+(i s)/2) J q r (e^(2 L) (-i + e^(i t)) (i k13 + k14) +
         e^(2 p) (k13 + i k14) (1 - i Cos[t] + Sin[t])) +
      e^(1/2 i (2 s+t)) (e^(-p) (-e^p J + i e^L q - e^(2 L) J (k11 + i k12) q) +
         J q ((-2 A1 + 2 i A2 + k11 - i k12) Cosh[p] +
            (-2 A3 + 2 i A4 + k11 - i k12) Sinh[p])))) +
 1/(2 J q) (i e^(-(i t)/2) (e^(-p+1/2 i (2 s+t)) (J (-i A1 - A2 + i A3 + A4 +
              e^(2 p) (i A1 + A2 + i A3 + A4 - i k11 - k12)) +
         e^L (-1 + e^L J (-i k11 + k12))) q +
      e^(-p+(i s)/2) J q r (e^(2 p) (-i k13 + k14) + e^(2 L) (i k13 + k14) -
         2 e^(L+p+i t) (k13 Cosh[L - p] - i k14 Sinh[L - p])) +
      e^((i t)/2) (-i e^(L-p) q + e^(2 L-p) J (k11 + i k12) q +
         J (-1 + (2 A1 + i (2 A2 + i k11 + k12)) q Cosh[p] +
            (2 A3 + 2 i A4 - k11 + i k12) q Sinh[p]))))
```

****

```
Equa3 = (1 / (r a)) e^(1/2 i (t-s))
  (1/(2 J q) (a e^(-(i t)/2) (e^(1/2 i (s+2 t)) q r (i e^(L+p) (-1 + e^L J k34) - J (2 A3 +
              i (2 A4 + k34)) Cosh[p] + 2 e^L J Conjugate[k11]
            Cosh[L + p] + J (2 A1 - 2 i A2 + i k34) Sinh[p]) +
      2 e^(L+(i t)/2) J q (i (e^(i s) k31 - k32) Cosh[L + p] +
         (k31 + e^(i s) k32) Sinh[L + p]) - e^((i s)/2) r
       (e^(L+p) (-1 + e^L J k34) q + J (i + (-2 i A1 + 2 A2 + k34)
             q Cosh[p] + (2 i A3 + 2 A4 - k34) q Sinh[p]) -
         2 i e^L J q Conjugate[k11] Sinh[
            L + p])))) + ((-I) / a) a1 + (1 / 2) s1
```

$$
\begin{aligned}
&-\frac{1}{2\,J\,q}\\
&\quad\Big(i\,e^{-\frac{i t}{2}}\,\Big(e^{-p+\frac{1}{2} i (2 s+t)}\,(J\,(-i\,A1-A2+i\,A3+A4+e^{2 p}\,(i\,A1+A2+\\
&\qquad\qquad i\,A3+A4-i\,k11-k12))+\\
&\qquad\quad e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12)))\,q+e^{-p+\frac{i s}{2}}\,J\,q\\
&\qquad\quad r\,(e^{2 p}\,(-i\,k13+k14)+e^{2 L}\,(i\,k13+k14)-\\
&\qquad\qquad 2\,e^{L+p+i t}\,(k13\,\mathrm{Cosh}[L-p]-i\,k14\,\mathrm{Sinh}[L-p]))+\\
&\qquad\quad e^{\frac{i t}{2}}\,(-i\,e^{L-p}\,q+e^{2 L-p}\,J\,(k11+i\,k12)\,q+\\
&\qquad\qquad J\,(-1+(2\,A1+i\,(2\,A2+i\,k11+k12))\,q\,\mathrm{Cosh}[p]+\\
&\qquad\qquad\quad (2\,A3+2\,i\,A4-k11+i\,k12)\,q\,\mathrm{Sinh}[p]))\Big)\Big)+\\
&\frac{1}{2}\,\Big(-\frac{1}{2\,J\,q}\,\Big(i\,e^{-i s-\frac{i t}{2}}\,\Big(e^{-p+\frac{i t}{2}}\,(e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12))+\\
&\qquad\qquad J\,(-i\,A1+A2+i\,A3-A4-\\
&\qquad\qquad\quad e^{2 p}\,(-i\,A1+A2-i\,A3+A4+i\,k11+k12)))\,q+\\
&\qquad\quad e^{-p+\frac{i s}{2}}\,J\,q\,r\,(e^{2 L}\,(-i+e^{i t})\,(i\,k13+k14)+\\
&\qquad\qquad e^{2 p}\,(k13+i\,k14)\,(1-i\,\mathrm{Cos}[t]+\mathrm{Sin}[t]))+\\
&\qquad\quad e^{\frac{1}{2} i (2 s+t)}\,(e^{-p}\,(-e^{p}\,J+i\,e^{L}\,q-e^{2 L}\,J\,(k11+i\,k12)\,q)+\\
&\qquad\qquad J\,q\,((-2\,A1+2\,i\,A2+k11-i\,k12)\,\mathrm{Cosh}[p]+\\
&\qquad\qquad\quad (-2\,A3+2\,i\,A4+k11-i\,k12)\,\mathrm{Sinh}[p]))\Big)\Big)+\\
&\quad\frac{1}{2\,J\,q}\,\Big(i\,e^{-\frac{i t}{2}}\,\Big(e^{-p+\frac{1}{2} i (2 s+t)}\,(J\,(-i\,A1-A2+i\,A3+A4+\\
&\qquad\qquad e^{2 p}\,(i\,A1+A2+i\,A3+A4-i\,k11-k12))+\\
&\qquad\quad e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12)))\,q+\\
&\qquad\quad e^{-p+\frac{i s}{2}}\,J\,q\,r\,(e^{2 p}\,(-i\,k13+k14)+e^{2 L}\,(i\,k13+k14)-\\
&\qquad\qquad 2\,e^{L+p+i t}\,(k13\,\mathrm{Cosh}[L-p]-i\,k14\,\mathrm{Sinh}[L-p]))+\\
&\qquad\quad e^{\frac{i t}{2}}\,(-i\,e^{L-p}\,q+e^{2 L-p}\,J\,(k11+i\,k12)\,q+\\
&\qquad\qquad J\,(-1+(2\,A1+i\,(2\,A2+i\,k11+k12))\,q\,\mathrm{Cosh}[p]+\\
&\qquad\qquad\quad (2\,A3+2\,i\,A4-k11+i\,k12)\,q\,\mathrm{Sinh}[p]))\Big)\Big)\Big)+\\
&\frac{1}{2\,J\,q\,r}\,\Big(e^{-\frac{i t}{2}+\frac{1}{2} i (-s+t)}\,\Big(e^{\frac{1}{2} i (s+2 t)}\,q\,r\,(i\,e^{L+p}\,(-1+e^{L}\,J\,k34)-\\
&\qquad J\,(2\,A3+i\,(2\,A4+k34))\,\mathrm{Cosh}[p]+\\
&\qquad 2\,e^{L}\,J\,\mathrm{Conjugate}[k11]\,\mathrm{Cosh}[L+p]+\\
&\qquad J\,(2\,A1-2\,i\,A2+i\,k34)\,\mathrm{Sinh}[p])+\\
&\quad 2\,e^{L+\frac{i t}{2}}\,J\,q\,(i\,(e^{i s}\,k31-k32)\,\mathrm{Cosh}[L+p]+\\
&\qquad (k31+e^{i s}\,k32)\,\mathrm{Sinh}[L+p])-\\
&\quad e^{\frac{i s}{2}}\,r\,(e^{L+p}\,(-1+e^{L}\,J\,k34)\,q+J\,(i+(-2\,i\,A1+2\,A2+k34)\\
&\qquad\quad q\,\mathrm{Cosh}[p]+(2\,i\,A3+2\,A4-k34)\,q\,\mathrm{Sinh}[p])-\\
&\qquad 2\,i\,e^{L}\,J\,q\,\mathrm{Conjugate}[k11]\,\mathrm{Sinh}[L+p])\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\text{Equa4} = (1 / (r\,a))\, e^{\frac{1}{2} i (-t-s)} \\
&\quad \Bigg(\frac{1}{2\,J\,q} \Big(a\, e^{-\frac{i t}{2}} \Big(e^{\frac{i s}{2}}\, q\, r\, (i\, e^{L+p}\, (-1 + e^{L}\, J\, k34) + J\, (-2\, A3 + 2\, i \\
&\qquad\qquad A4 - i\, k34)\, \text{Cosh}[p] + 2\, e^{L}\, J\, \text{Conjugate}[k11] \\
&\qquad\qquad \text{Cosh}[L + p] + J\, (2\, A1 + i\, (2\, A2 + k34))\, \text{Sinh}[p]) + \\
&\qquad 2\, e^{L+\frac{i t}{2}}\, J\, q\, (-i\, (k31 + e^{i s}\, k32)\, \text{Cosh}[L + p] + \\
&\qquad\quad (e^{i s}\, k31 - k32)\, \text{Sinh}[L + p]) + e^{\frac{1}{2} i (s+2 t)}\, r \\
&\qquad (e^{L+p}\, (-1 + e^{L}\, J\, k34)\, q + J\, (-2\, i\, A1 - 2\, A2 + k34)\, q \\
&\qquad\quad \text{Cosh}[p] - J\, (i + (-2\, i\, A3 + 2\, A4 + k34)\, q\, \text{Sinh}[p]) - \\
&\qquad 2\, i\, e^{L}\, J\, q\, \text{Conjugate}[k11] \\
&\qquad\quad \text{Sinh}[L + p]) \Big) \Big) \Bigg) + ((-I) / a)\, a1 + (1 / 2)\, s1
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\,J\,q}\\
&\quad (i\,e^{-\frac{i t}{2}}\,(e^{-p+\frac{1}{2}i\,(2 s+t)}\,(J\,(-i\,A1-A2+i\,A3+A4+e^{2 p}\,(i\,A1+A2+\\
&\qquad\qquad i\,A3+A4-i\,k11-k12))+\\
&\qquad\quad e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12)))\,q+e^{-p+\frac{i s}{2}}\,J\,q\\
&\qquad r\,(e^{2 p}\,(-i\,k13+k14)+e^{2 L}\,(i\,k13+k14)-\\
&\qquad\quad 2\,e^{L+p+i t}\,(k13\,\mathrm{Cosh}[L-p]-i\,k14\,\mathrm{Sinh}[L-p]))+\\
&\qquad e^{\frac{i t}{2}}\,(-i\,e^{L-p}\,q+e^{2 L-p}\,J\,(k11+i\,k12)\,q+\\
&\qquad\quad J\,(-1+(2\,A1+i\,(2\,A2+i\,k11+k12))\,q\,\mathrm{Cosh}[p]+\\
&\qquad\qquad (2\,A3+2\,i\,A4-k11+i\,k12)\,q\,\mathrm{Sinh}[p]))))+\\
&\frac{1}{2}\,(-\frac{1}{2\,J\,q}\,(i\,e^{-i s-\frac{i t}{2}}\,(e^{-p+\frac{i t}{2}}\,(e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12))+\\
&\qquad\qquad J\,(-i\,A1+A2+i\,A3-A4-\\
&\qquad\qquad\quad e^{2 p}\,(-i\,A1+A2-i\,A3+A4+i\,k11+k12)))\,q+\\
&\qquad e^{-p+\frac{i s}{2}}\,J\,q\,r\,(e^{2 L}\,(-i+e^{i t})\,(i\,k13+k14)+\\
&\qquad\quad e^{2 p}\,(k13+i\,k14)\,(1-i\,\mathrm{Cos}[t]+\mathrm{Sin}[t]))+\\
&\qquad e^{\frac{1}{2}i\,(2 s+t)}\,(e^{-p}\,(-e^{p}\,J+i\,e^{L}\,q-e^{2 L}\,J\,(k11+i\,k12)\,q)+\\
&\qquad\quad J\,q\,((-2\,A1+2\,i\,A2+k11-i\,k12)\,\mathrm{Cosh}[p]+\\
&\qquad\qquad (-2\,A3+2\,i\,A4+k11-i\,k12)\,\mathrm{Sinh}[p]))))+\\
&\quad\frac{1}{2\,J\,q}\,(i\,e^{-\frac{i t}{2}}\,(e^{-p+\frac{1}{2}i\,(2 s+t)}\,(J\,(-i\,A1-A2+i\,A3+A4+\\
&\qquad\qquad e^{2 p}\,(i\,A1+A2+i\,A3+A4-i\,k11-k12))+\\
&\qquad\quad e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12)))\,q+\\
&\qquad e^{-p+\frac{i s}{2}}\,J\,q\,r\,(e^{2 p}\,(-i\,k13+k14)+e^{2 L}\,(i\,k13+k14)-\\
&\qquad\quad 2\,e^{L+p+i t}\,(k13\,\mathrm{Cosh}[L-p]-i\,k14\,\mathrm{Sinh}[L-p]))+\\
&\qquad e^{\frac{i t}{2}}\,(-i\,e^{L-p}\,q+e^{2 L-p}\,J\,(k11+i\,k12)\,q+\\
&\qquad\quad J\,(-1+(2\,A1+i\,(2\,A2+i\,k11+k12))\,q\,\mathrm{Cosh}[p]+\\
&\qquad\qquad (2\,A3+2\,i\,A4-k11+i\,k12)\,q\,\mathrm{Sinh}[p])))))+\\
&\frac{1}{2\,J\,q\,r}\,(e^{\frac{1}{2}i\,(-s-t)-\frac{i t}{2}}\,(e^{\frac{i s}{2}}\,q\,r\,(i\,e^{L+p}\,(-1+e^{L}\,J\,k34)+\\
&\qquad J\,(-2\,A3+2\,i\,A4-i\,k34)\,\mathrm{Cosh}[p]+\\
&\qquad 2\,e^{L}\,J\,\mathrm{Conjugate}[k11]\,\mathrm{Cosh}[L+p]+\\
&\qquad J\,(2\,A1+i\,(2\,A2+k34))\,\mathrm{Sinh}[p])+\\
&\quad 2\,e^{L+\frac{i t}{2}}\,J\,q\,(-i\,(k31+e^{i s}\,k32)\,\mathrm{Cosh}[L+p]+\\
&\qquad (e^{i s}\,k31-k32)\,\mathrm{Sinh}[L+p])+\\
&\quad e^{\frac{1}{2}i\,(s+2 t)}\,r\,(e^{L+p}\,(-1+e^{L}\,J\,k34)\,q+\\
&\qquad J\,(-2\,i\,A1-2\,A2+k34)\,q\,\mathrm{Cosh}[p]-\\
&\qquad J\,(i+(-2\,i\,A3+2\,A4+k34)\,q\,\mathrm{Sinh}[p])-\\
&\qquad 2\,i\,e^{L}\,J\,q\,\mathrm{Conjugate}[k11]\,\mathrm{Sinh}[L+p])))
\end{aligned}
$$

****

$$\frac{\partial}{\partial q}\,t = t1$$

t1 = Equa3 - Equa4

$$
\begin{aligned}
&\frac{1}{2}\,(\frac{1}{2\,J\,q}\,(i\,e^{-i s-\frac{i t}{2}}\,(e^{-p+\frac{i t}{2}}\\
&\qquad (e^{L}\,(-1+e^{L}\,J\,(-i\,k11+k12))+J\,(-i\,A1+A2+i\,A3-
\end{aligned}
$$

$$A4 - e^{2p} (-i\,A1 + A2 - i\,A3 + A4 + i\,k11 + k12)))$$
$$q + e^{-p+\frac{is}{2}} J q r (e^{2L} (-i + e^{it}) (i\,k13 + k14) +$$
$$e^{2p} (k13 + i\,k14) (1 - i\,\mathrm{Cos}[t] + \mathrm{Sin}[t])) +$$
$$e^{\frac{1}{2} i (2s+t)} (e^{-p} (-e^{p} J + i\,e^{L} q - e^{2L} J (k11 + i\,k12) q) +$$
$$J q ((-2\,A1 + 2\,i\,A2 + k11 - i\,k12) \mathrm{Cosh}[p] +$$
$$(-2\,A3 + 2\,i\,A4 + k11 - i\,k12) \mathrm{Sinh}[p])))) -$$
$$\frac{1}{2Jq} (i\,e^{-\frac{it}{2}} (e^{-p+\frac{1}{2} i (2s+t)} (J (-i\,A1 - A2 + i\,A3 + A4 +$$
$$e^{2p} (i\,A1 + A2 + i\,A3 + A4 - i\,k11 - k12)) +$$
$$e^{L} (-1 + e^{L} J (-i\,k11 + k12))) q +$$
$$e^{-p+\frac{is}{2}} J q r (e^{2p} (-i\,k13 + k14) + e^{2L} (i\,k13 + k14) -$$
$$2\,e^{L+p+it} (k13\,\mathrm{Cosh}[L-p] - i\,k14\,\mathrm{Sinh}[L-p])) +$$
$$e^{\frac{it}{2}} (-i\,e^{L-p} q + e^{2L-p} J (k11 + i\,k12) q +$$
$$J (-1 + (2\,A1 + i (2\,A2 + i\,k11 + k12)) q\,\mathrm{Cosh}[p] +$$
$$(2\,A3 + 2\,i\,A4 - k11 + i\,k12) q\,\mathrm{Sinh}[p]))))) +$$
$$\frac{1}{2} (-\frac{1}{2Jq} (i\,e^{-is-\frac{it}{2}} (e^{-p+\frac{it}{2}} (e^{L} (-1 + e^{L} J (-i\,k11 + k12)) +$$
$$J (-i\,A1 + A2 + i\,A3 - A4 -$$
$$e^{2p} (-i\,A1 + A2 - i\,A3 + A4 + i\,k11 + k12))) q +$$
$$e^{-p+\frac{is}{2}} J q r (e^{2L} (-i + e^{it}) (i\,k13 + k14) +$$
$$e^{2p} (k13 + i\,k14) (1 - i\,\mathrm{Cos}[t] + \mathrm{Sin}[t])) +$$
$$e^{\frac{1}{2} i (2s+t)} (e^{-p} (-e^{p} J + i\,e^{L} q - e^{2L} J (k11 + i\,k12) q) +$$
$$J q ((-2\,A1 + 2\,i\,A2 + k11 - i\,k12) \mathrm{Cosh}[p] +$$
$$(-2\,A3 + 2\,i\,A4 + k11 - i\,k12) \mathrm{Sinh}[p])))) +$$
$$\frac{1}{2Jq} (i\,e^{-\frac{it}{2}} (e^{-p+\frac{1}{2} i (2s+t)} (J (-i\,A1 - A2 + i\,A3 + A4 +$$
$$e^{2p} (i\,A1 + A2 + i\,A3 + A4 - i\,k11 - k12)) +$$
$$e^{L} (-1 + e^{L} J (-i\,k11 + k12))) q +$$
$$e^{-p+\frac{is}{2}} J q r (e^{2p} (-i\,k13 + k14) + e^{2L} (i\,k13 + k14) -$$
$$2\,e^{L+p+it} (k13\,\mathrm{Cosh}[L-p] - i\,k14\,\mathrm{Sinh}[L-p])) +$$
$$e^{\frac{it}{2}} (-i\,e^{L-p} q + e^{2L-p} J (k11 + i\,k12) q +$$
$$J (-1 + (2\,A1 + i (2\,A2 + i\,k11 + k12)) q\,\mathrm{Cosh}[p] +$$
$$(2\,A3 + 2\,i\,A4 - k11 + i\,k12) q\,\mathrm{Sinh}[p]))))) +$$
$$\frac{1}{2Jqr} (e^{-\frac{it}{2}+\frac{1}{2} i (-s+t)} (e^{\frac{1}{2} i (s+2t)} q r (i\,e^{L+p} (-1 + e^{L} J\,k34) -$$
$$J (2\,A3 + i (2\,A4 + k34)) \mathrm{Cosh}[p] +$$
$$2\,e^{L} J\,\mathrm{Conjugate}[k11]\,\mathrm{Cosh}[L+p] +$$
$$J (2\,A1 - 2\,i\,A2 + i\,k34) \mathrm{Sinh}[p]) +$$
$$2\,e^{L+\frac{it}{2}} J q (i (e^{is} k31 - k32) \mathrm{Cosh}[L+p] +$$
$$(k31 + e^{is} k32) \mathrm{Sinh}[L+p]) -$$
$$e^{\frac{is}{2}} r (e^{L+p} (-1 + e^{L} J\,k34) q + J (i + (-2\,i\,A1 + 2\,A2 + k34)$$
$$q\,\mathrm{Cosh}[p] + (2\,i\,A3 + 2\,A4 - k34) q\,\mathrm{Sinh}[p]) -$$
$$2\,i\,e^{L} J q\,\mathrm{Conjugate}[k11]\,\mathrm{Sinh}[L+p]))) -$$
$$\frac{1}{2Jqr} (e^{\frac{1}{2} i (-s-t)-\frac{it}{2}} (e^{\frac{is}{2}} q r (i\,e^{L+p} (-1 + e^{L} J\,k34) +$$
$$J (-2\,A3 + 2\,i\,A4 - i\,k34) \mathrm{Cosh}[p] +$$
$$2\,e^{L} J\,\mathrm{Conjugate}[k11]\,\mathrm{Cosh}[L+p] +$$
$$J (2\,A1 + i (2\,A2 + k34)) \mathrm{Sinh}[p]) +$$
$$2\,e^{L+\frac{it}{2}} J q (-i (k31 + e^{is} k32) \mathrm{Cosh}[L+p] +$$
$$(e^{is} k31 - k32) \mathrm{Sinh}[L+p]) +$$

$$e^{\frac{1}{2} i (s+2t)} r (e^{L+p} (-1 + e^{L} J k34) q + J (-2 i A1 - 2 A2 + k34) q \text{Cosh}[p] - J (i + (-2 i A3 + 2 A4 + k34) q \text{Sinh}[p]) - 2 i e^{L} J q \text{Conjugate}[k11] \text{Sinh}[L + p]))$$

****

$$\frac{\partial}{\partial q} r = r1$$

```
r1 = ((-I r) / 2) (Equa3 + Equa4)
```

```
-1/2 i r
 (-1/(2 J q) (i e^(-i s - i t/2) (e^(-p + i t/2) (e^L (-1 + e^L J (-i k11 + k12)) + J
            (-i A1 + A2 + i A3 - A4 - e^(2 p)
               (-i A1 + A2 - i A3 + A4 + i k11 + k12))) q +
       e^(-p + i s/2) J q r (e^(2 L) (-i + e^(i t)) (i k13 + k14) +
          e^(2 p) (k13 + i k14) (1 - i Cos[t] + Sin[t])) +
       e^(1/2 i (2 s + t)) (e^(-p) (-e^p J + i e^L q - e^(2 L) J (k11 + i k12) q) +
          J q ((-2 A1 + 2 i A2 + k11 - i k12) Cosh[p] +
             (-2 A3 + 2 i A4 + k11 - i k12) Sinh[p])))) -
   1/(2 J q) (i e^(-i t/2) (e^(-p + 1/2 i (2 s + t)) (J (-i A1 - A2 + i A3 + A4 +
             e^(2 p) (i A1 + A2 + i A3 + A4 - i k11 - k12)) +
          e^L (-1 + e^L J (-i k11 + k12))) q +
       e^(-p + i s/2) J q r (e^(2 p) (-i k13 + k14) + e^(2 L) (i k13 + k14) -
          2 e^(L + p + i t) (k13 Cosh[L - p] - i k14 Sinh[L - p])) +
       e^(i t/2) (-i e^(L - p) q + e^(2 L - p) J (k11 + i k12) q +
          J (-1 + (2 A1 + i (2 A2 + i k11 + k12)) q Cosh[p] +
             (2 A3 + 2 i A4 - k11 + i k12) q Sinh[p])))) +
   1/(2 J q r) (e^(-i t/2 + 1/2 i (-s + t)) (e^(1/2 i (s + 2 t)) q r (i e^(L + p) (-1 + e^L J k34) -
          J (2 A3 + i (2 A4 + k34)) Cosh[p] +
          2 e^L J Conjugate[k11] Cosh[L + p] +
          J (2 A1 - 2 i A2 + i k34) Sinh[p]) +
       2 e^(L + i t/2) J q (i (e^(i s) k31 - k32) Cosh[L + p] +
          (k31 + e^(i s) k32) Sinh[L + p]) - e^(i s/2) r
        (e^(L + p) (-1 + e^L J k34) q + J (i + (-2 i A1 + 2 A2 + k34)
               q Cosh[p] + (2 i A3 + 2 A4 - k34) q Sinh[p]) -
          2 i e^L J q Conjugate[k11] Sinh[L + p]))) +
   1/(2 J q r) (e^(1/2 i (-s - t) - i t/2) (e^(i s/2) q r (i e^(L + p) (-1 + e^L J k34) +
          J (-2 A3 + 2 i A4 - i k34) Cosh[p] +
          2 e^L J Conjugate[k11] Cosh[L + p] +
          J (2 A1 + i (2 A2 + k34)) Sinh[p]) +
       2 e^(L + i t/2) J q (-i (k31 + e^(i s) k32) Cosh[L + p] +
          (e^(i s) k31 - k32) Sinh[L + p]) + e^(1/2 i (s + 2 t)) r
        (e^(L + p) (-1 + e^L J k34) q + J (-2 i A1 - 2 A2 + k34) q
           Cosh[p] - J (i + (-2 i A3 + 2 A4 + k34) q Sinh[p]) -
          2 i e^L J q Conjugate[k11] Sinh[L + p]))))
```

****

****

Now we want to give the values for the vector potential A_μ as the following :

```
A1 = e^L f1 + e^-L g1
```

$e^{L}\,f1 + e^{-L}\,g1$

```
A2 = e^p f2 + e^-p g2
```

$e^{p}\,f2 + e^{-p}\,g2$

```
A3 = -e^L f1 + e^-L g1
```

$-e^{L}\,f1 + e^{-L}\,g1$

```
A4 = -e^p f2 + e^-p g2
```

$-e^{p}\,f2 + e^{-p}\,g2$

where f_i and g_i are all real functions of q and J only, for i = 1, 2 .

$$\frac{\partial}{\partial q}a = a[1]$$

```
a[1] = FullSimplify[Factor[a1]]
```

$$\frac{1}{2\,J\,q}\Big(a\,e^{-L-p-\frac{i\,t}{2}}\Big(e^{\frac{1}{2}i\,(2\,s+t)}\,(-2\,e^{L+p}\,(f2-g2)\,J + i\,e^{2\,p}\,J\,(2\,g1 - e^{L}\,(k11 - i\,k12)) + e^{2\,L}\,(-1 - 2\,i\,f1\,J + e^{L}\,J\,(-i\,k11 + k12)))\,q + e^{\frac{i\,t}{2}}\,(e^{2\,p}\,J\,(2\,g1 - e^{L}\,(k11 - i\,k12))\,q + e^{2\,L}\,(-i + 2\,f1\,J + e^{L}\,J\,(k11 + i\,k12))\,q + i\,e^{L+p}\,J\,(i + 2\,(f2+g2)\,q)) + e^{L+\frac{i\,s}{2}}\,J\,(-e^{2\,L}\,(-i + e^{i\,t})\,(k13 - i\,k14) - e^{2\,p}\,(i + e^{i\,t})\,(k13 + i\,k14))\,q\,r\Big)\Big)$$

****

$$\frac{\partial}{\partial q}s = s[1]$$

```
s[1] = FullSimplify[Factor[s1]]
```

$$\frac{1}{2J}\left(e^{-L-p-\frac{1}{2}i(2s+t)}\left(-2\,i\,e^{L+p+\frac{it}{2}}\left(1+e^{2is}\right)(f2-g2)\,J+e^{2L}\left(e^{\frac{it}{2}}\left(i+e^{is}\right)^2\left(-i+2\,f1\,J+e^{L}J\,(k11+i\,k12)\right)-e^{L+\frac{is}{2}}\left(i+e^{is}\right)J\,(k13-i\,k14)\,r+e^{L+\frac{1}{2}i(s+2t)}J\,(k13-i\,k14)\,r\,(1-i\,Cos[s]+Sin[s])\right)+e^{2p}\left(-i+e^{is}\right)J\left(-2\,e^{\frac{it}{2}}\left(-i+e^{is}\right)g1+(1+i)\,e^{L}\left((k11-i\,k12)\left(Cos\left[\frac{s}{2}\right]-Sin\left[\frac{s}{2}\right]\right)+(k13+i\,k14)\,r\left(Cos\left[\frac{t}{2}\right]+Sin\left[\frac{t}{2}\right]\right)\right)\left(-i\,Cos\left[\frac{s+t}{2}\right]+Sin\left[\frac{s+t}{2}\right]\right)\right)\right)\right)$$

We know that s is real, so now we want to find the following coefficients to enable us to get the right formula for s[1]

```
FullSimplify[Coefficient[s[1], f1, 1]]
```

$$e^{L-p-is}\left(i+e^{is}\right)^2$$

```
FullSimplify[Coefficient[s[1], g1, 1]]
```

$$-2\,i\,e^{-L+p}\,(-1+Sin[s])$$

```
FullSimplify[Coefficient[s[1], f2, 1]]
```

$$-2\,i\,Cos[s]$$

```
FullSimplify[Coefficient[s[1], g2, 1]]
```

$$2\,i\,Cos[s]$$

```
FullSimplify[Coefficient[s[1], k11, 1]]
```

$$2\,i\,e^{L}\,(Cosh[L-p]\,Sin[s]+Sinh[L-p])$$

```
FullSimplify[Coefficient[s[1], k12, 1]]
```

$$-2\, e^{L} \left(\text{Cosh}[L-p] + \text{Sin}[s]\, \text{Sinh}[L-p]\right)$$

```
FullSimplify[Coefficient[s[1], k13, 1]]
```

$$-2\, i\, e^{L}\, r \left(\text{Cos}\left[\frac{s+t}{2}\right] \text{Cosh}[L-p] + \text{Sin}\left[\frac{s-t}{2}\right] \text{Sinh}[L-p]\right)$$

```
FullSimplify[Coefficient[s[1], k14, 1]]
```

$$-2\, e^{L}\, r \left(\text{Cosh}[L-p]\, \text{Sin}\left[\frac{s-t}{2}\right] + \text{Cos}\left[\frac{s+t}{2}\right] \text{Sinh}[L-p]\right)$$

$$0\ \text{tearms1} = \text{FullSimplify}\Big[s[1] - f1\left(e^{L-p-i s}\,(i + e^{i s})^2\right) - g1\,(-2\, i\, e^{-L+p}\,(-1+\text{Sin}[s])) - f2\,(-2\, i\, \text{Cos}[s]) - g2\,(2\, i\, \text{Cos}[s]) - k11\,(2\, i\, e^{L}\,(\text{Cosh}[L-p]\,\text{Sin}[s] + \text{Sinh}[L-p])) - k12\,(-2\, e^{L}\,(\text{Cosh}[L-p] + \text{Sin}[s]\,\text{Sinh}[L-p])) - k13\left(-2\, i\, e^{L}\, r\left(\text{Cos}\left[\frac{s+t}{2}\right]\text{Cosh}[L-p] + \text{Sin}\left[\frac{s-t}{2}\right]\text{Sinh}[L-p]\right)\right) - k14\left(-2\, e^{L}\, r\left(\text{Cosh}[L-p]\,\text{Sin}\left[\frac{s-t}{2}\right] + \text{Cos}\left[\frac{s+t}{2}\right]\text{Sinh}[L-p]\right)\right)\Big]$$

$$\frac{e^{L-p}\,(1+\text{Sin}[s])}{J}$$

****

$$\frac{\partial}{\partial q} t = t[1]$$

```
t[1] = FullSimplify[Factor[t1]]
```

$$\frac{1}{2\,J\,r}\Big(e^{-L-p-\frac{1}{2}i(s+2t)}\Big(2\,i\,e^{L+p+\frac{is}{2}}(1+e^{2it})(f2-g2)\,J\,r+J\Big(e^{L+\frac{1}{2}i(2s+t)}(1+i\,e^{it})(k31+i\,k32)-e^{L+\frac{it}{2}}(-i+e^{it})(k31+i\,k32)-e^{\frac{is}{2}}(-i+e^{it})^2\,r\,(2\,g1+i\,e^{L}(k34+i\,\text{Conjugate}[k11]))\Big)+e^{2(L+p)}(i+e^{it})\Big(e^{L+\frac{it}{2}}J(k31-i\,k32)+e^{L+\frac{1}{2}i(2s+t)}J(i\,k31+k32)+e^{\frac{is}{2}}(i+e^{it})\,r\,(-i+2\,f1\,J+e^{L}J(i\,k34+\text{Conjugate}[k11]))\Big)\Big)\Big)$$

Similarly we know that t is real, so

```
FullSimplify[Coefficient[t[1], f1, 1]]
```

$$e^{L+p-it}(i+e^{it})^2$$

```
FullSimplify[Coefficient[t[1], g1, 1]]
```

2 i (-1 + Sin[t]) (-Cosh[L + p] + Sinh[L + p])

```
FullSimplify[Coefficient[t[1], f2, 1]]
```

2 i Cos[t]

```
FullSimplify[Coefficient[t[1], g2, 1]]
```

-2 i Cos[t]

```
FullSimplify[Coefficient[t[1], k11, 1]]
```

0

```
FullSimplify[Coefficient[t[1], k12, 1]]
```

0

```
FullSimplify[Coefficient[t[1], k13, 1]]
```

0

```
FullSimplify[Coefficient[t[1], k14, 1]]
```

0

```
FullSimplify[Coefficient[t[1], k31, 1]]
```

$$\frac{2\, i\, e^{L}\, (\text{Cos}[\frac{s+t}{2}]\ \text{Cosh}[L+p] - \text{Sin}[\frac{s-t}{2}]\ \text{Sinh}[L+p])}{r}$$

```
FullSimplify[Coefficient[t[1], k32, 1]]
```

$$\frac{2\, e^{L}\, (-\text{Cosh}[L+p]\ \text{Sin}[\frac{s-t}{2}] + \text{Cos}[\frac{s+t}{2}]\ \text{Sinh}[L+p])}{r}$$

```
FullSimplify[Coefficient[t[1], k34, 1]]
```

$$-2\, e^{L}\, (\text{Cosh}[L+p] + \text{Sin}[t]\ \text{Sinh}[L+p])$$

```
FullSimplify[Coefficient[t[1], Conjugate[k11], 1]]
```

$$2\, i\, e^{L}\, (\text{Cosh}[L+p]\ \text{Sin}[t] + \text{Sinh}[L+p])$$

```
0 tearmt1 = FullSimplify[t[1] - f1 (e^(L+p-i t) (i + e^(i t))^2) -
    g1 (2 i (-1 + Sin[t]) (-Cosh[L + p] + Sinh[L + p])) -
    f2 (2 i Cos[t]) - g2 (-2 i Cos[t]) -
    k31 (1/r (2 i e^L (Cos[(s + t)/2] Cosh[L + p] -
          Sin[(s - t)/2] Sinh[L + p]))) - k32 (1/r (2 e^L
        (-Cosh[L + p] Sin[(s - t)/2] + Cos[(s + t)/2] Sinh[L + p]))) -
    k34 (-2 e^L (Cosh[L + p] + Sin[t] Sinh[L + p])) -
    Conjugate[k11] (2 i e^L (Cosh[L + p] Sin[t] + Sinh[L + p]))]
```

$$\frac{e^{L+p}\,(1+\mathrm{Sin}[t])}{J}$$

****

$$\frac{\partial}{\partial q} r = r[1]$$

```
r[1] = FullSimplify[Factor[r1]]
```

$$\frac{1}{4J}\Big(e^{-L-p-i(s+t)}\Big(e^{it}(1+e^{2is})(e^{2L}(1+iJ(2f1+e^{L}(k11+i\,k12)))+e^{2p}J(-2i\,g1+e^{L}(i\,k11+k12)))\,r+e^{is}(1+e^{2it})(2i\,g1\,J-e^{L}J\,k34+e^{2(L+p)}(-1-2i\,f1\,J+e^{L}J\,k34))\,r-4e^{2L+p}J(\mathrm{Cos}[s+t]+i\,\mathrm{Sin}[s+t])\Big(k32\,\mathrm{Cos}\Big[\frac{s-t}{2}\Big]\mathrm{Cosh}[L+p]+i\,r\,\mathrm{Conjugate}[k11]\,\mathrm{Cos}[t]\,\mathrm{Cosh}[L+p]-i\,k31\,\mathrm{Cosh}[L+p]\,\mathrm{Sin}\Big[\frac{s+t}{2}\Big]+r^{2}\,\mathrm{Cosh}[L-p]\Big(k14\,\mathrm{Cos}\Big[\frac{s-t}{2}\Big]-i\,k13\,\mathrm{Sin}\Big[\frac{s+t}{2}\Big]\Big)+i\,(f2-g2)\,r\,(\mathrm{Sin}[s]+\mathrm{Sin}[t])(-\mathrm{Cosh}[L]+\mathrm{Sinh}[L])+i\,k13\,r^{2}\,\mathrm{Cos}\Big[\frac{s-t}{2}\Big]\mathrm{Sinh}[L-p]-k14\,r^{2}\,\mathrm{Sin}\Big[\frac{s+t}{2}\Big]\mathrm{Sinh}[L-p]+\Big(i\,k31\,\mathrm{Cos}\Big[\frac{s-t}{2}\Big]-k32\,\mathrm{Sin}\Big[\frac{s+t}{2}\Big]\Big)\mathrm{Sinh}[L+p]\Big)\Big)\Big)$$

```
FullSimplify[Coefficient[r[1], f1, 1]]
```

$$-i\,e^{L}\,r\,((-\mathrm{Cos}[s]+\mathrm{Cos}[t])\,\mathrm{Cosh}[p]+(\mathrm{Cos}[s]+\mathrm{Cos}[t])\,\mathrm{Sinh}[p])$$

```
FullSimplify[Coefficient[r[1], g1, 1]]

-i e^-L r
 ((Cos[s] - Cos[t]) Cosh[p] + (Cos[s] + Cos[t]) Sinh[p])
```

```
FullSimplify[Coefficient[r[1], f2, 1]]

i r (Sin[s] + Sin[t])
```

```
FullSimplify[Coefficient[r[1], g2, 1]]

-i r (Sin[s] + Sin[t])
```

```
FullSimplify[Coefficient[r[1], k11, 1]]

i e^L r Cos[s] Cosh[L - p]
```

```
FullSimplify[Coefficient[r[1], k12, 1]]

-e^L r Cos[s] Sinh[L - p]
```

```
FullSimplify[Coefficient[r[1], k13, 1]]
```

$$i\, e^{L}\, r^{2} \left(\text{Cosh}[L-p]\ \text{Sin}\left[\frac{s+t}{2}\right] - \text{Cos}\left[\frac{s-t}{2}\right]\ \text{Sinh}[L-p]\right)$$

```
FullSimplify[Coefficient[r[1], k14, 1]]
```

$$e^{L}\, r^{2} \left(-\text{Cos}\left[\frac{s-t}{2}\right]\ \text{Cosh}[L-p] + \text{Sin}\left[\frac{s+t}{2}\right]\ \text{Sinh}[L-p]\right)$$

```
FullSimplify[Coefficient[r[1], k31, 1]]
```

$$i\, e^{L} \left(\text{Cosh}[L+p]\ \text{Sin}\left[\frac{s+t}{2}\right] - \text{Cos}\left[\frac{s-t}{2}\right]\ \text{Sinh}[L+p]\right)$$

```
FullSimplify[Coefficient[r[1], k32, 1]]
```

$$e^{L}\left(-\mathrm{Cos}\left[\frac{s-t}{2}\right]\mathrm{Cosh}[L+p]+\mathrm{Sin}\left[\frac{s+t}{2}\right]\mathrm{Sinh}[L+p]\right)$$

```
FullSimplify[Coefficient[r[1], k34, 1]]
```

$$e^{L}\, r\, \mathrm{Cos}[t]\, \mathrm{Sinh}[L+p]$$

```
FullSimplify[Coefficient[r[1], Conjugate[k11], 1]]
```

$$-i\, e^{L}\, r\, \mathrm{Cos}[t]\, \mathrm{Cosh}[L+p]$$

```
0 tearmr1 =
 FullSimplify[r[1] - f1 (-i e^L r ((-Cos[s] + Cos[t]) Cosh[p] +
        (Cos[s] + Cos[t]) Sinh[p])) -
   g1 (-i e^-L r ((Cos[s] - Cos[t]) Cosh[p] +
        (Cos[s] + Cos[t]) Sinh[p])) -
   f2 (i r (Sin[s] + Sin[t])) - g2 (-i r (Sin[s] + Sin[t])) -
   k11 (i e^L r Cos[s] Cosh[L - p]) -
   k12 (-e^L r Cos[s] Sinh[L - p]) - k13 (i e^L r^2
      (Cosh[L - p] Sin[(s + t)/2] - Cos[(s - t)/2] Sinh[L - p])) - k14
    (e^L r^2 (-Cos[(s - t)/2] Cosh[L - p] + Sin[(s + t)/2] Sinh[L - p])) -
   k31 (i e^L (Cosh[L + p] Sin[(s + t)/2] -
        Cos[(s - t)/2] Sinh[L + p])) - k32
    (e^L (-Cos[(s - t)/2] Cosh[L + p] + Sin[(s + t)/2] Sinh[L + p])) -
   k34 (e^L r Cos[t] Sinh[L + p]) -
   Conjugate[k11] (-i e^L r Cos[t] Cosh[L + p])]
```

$$\frac{e^{L}\, r\,((\mathrm{Cos}[s]-\mathrm{Cos}[t])\,\mathrm{Cosh}[p]-(\mathrm{Cos}[s]+\mathrm{Cos}[t])\,\mathrm{Sinh}[p])}{2\,J}$$