The orbits of periodic solutions of many body problems.

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THE ORBITS OF PERIODIC SOLUTIONS OF MANY BODY PROBLEMS

Thesis submitted to the University of Wales Swansea
by

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To

My Parents
and
My Late Brother

OVERVIEW OF THESIS

This thesis is concerned with the equal mass many-body problem and the stability of periodic solutions, with Keplerian (Coulombic) potential and other potentials. The classical n-body problem is a system of ordinary differential equations that describes the motion of n particles moving under Newton/Coulomb laws of motion, where the forces acting are the mutual gravitational attractions, Coulombic interaction with the presence of a constant magnetic field. In Chapter 1 we give some preliminaries of Kepler's laws of planetary motion, Newton's gravitational law and Coulomb's law. Very few solutions are known for the general many-body problem, one only has one and two body solutions (because they are completely integrable) and the three body problem has the Lagrange, restricted three body solutions and some planar polygonal solutions. The many-body problem for more than three bodies remains completely unsolved (a generic Hamiltonian system is not integrable). Just a few planar solutions are known as the central configurations an aspect of the regular n-gon solutions which are initial arrangements of the bodies that lead to special solutions of the n-body problem called relative equilibrium solutions, in which all the bodies rotate around the central of mass, keeping the mutual distances between the bodies remain constant with constant angular velocity. (The solutions of this kind remain self-similar for all time), and the other aspect of the regular n-gon solutions, in which all bodies move periodically tracing the same curve on the plane. There is just a time shift in the position to pass from point to the next one. Many interesting Physical systems are nearly integrable systems (they are small perturbations of completely integrable systems). For the periodic solutions for a Hamiltonian system with two degrees of freedom one can use the KAM (Kolmogorov, Arnold, and Moser) theory in which to make a small perturbation to the Hamiltonian system to study the stability, but for a Hamiltonian system with more than two degrees of freedom, KAM theory does not guarantee stability. Also in Chapter 1 we start to investigate the method of Davies, Truman, Williams to obtain classical periodic solutions of the equal-mass $2n$-body problem, $2n$-ion problem and the n-electron atom problem in $\mathbb{R}^3$ with Newtonian (Coulombic) potential. (That method reduces such systems with n degrees of freedom to a system with just three degrees of freedom by using rotational symmetry and these three degrees of freedom system can be reduced to two degrees of freedom using Cylindrical polar coordinates. We may continue the reduction to one degree of freedom, resulting in the f-equation which is a highly non-linear second order differential equation for f in terms of r only). In this Chapter we apply the method of DTW for the system of a $2n$ electron atom problem with constant magnetic field. That means our system has $12n$ degrees of freedom and this can be reduced using the method to that of one particle with three degrees of freedom. We give examples of the four node solution of the four ion problem, four electron atom problem without (with) constant magnetic field and four body gravitational problem. These examples we consider as our standard four node solution of the DTW-periodic set up and we show how the DTW-solutions were found. We refer to some recently related works for planar and non-planar periodic solutions by Chenciner et al and Simó. In Chapter 2, we continue the investigation of the method of DTW by showing how to continue to reduce our systems by using the Cylindrical polar coordinates (a result of this is the linear terms from the constant magnetic field cancelling each other). Thus reduced, giving two-degrees of freedom, we continue the reduction by obtaining the derivation of the f-equation which is a highly non-linear second order differential equation for f in terms of r only. We give examples of the harmonic oscillator and the two particle problem to show how to calculate the important characteristics of the solution of the equation of motion generated by the f-equation. We give the numerical solutions (Runge-Kutta scheme) of the f-equation with the initial conditions for our standard examples. These solution agree with the numerical solutions (Runge-Kutta scheme) of the reduced system of our standard examples of the same problems.

The main contributions of the present work are:

1. The ϵ-equation which characterises the linear stability of the f-equation of the DTW-solutions and provides information about the stability of the non-planar periodic solutions of the many-body problems. To note that the method can be applied to any conservative Hamiltonian system with three degrees of freedom which can be reduced to two degrees of freedom using the Cylindrical polar coordinates (Chapter 3).

2. A numerical approach to solving the ϵ-equation for our standard examples providing a set of illustrative systems that show that the general solution to the ϵ-equation is well behaved. The comparison of numerical and approximate analytical solutions of the ϵ-solution for the four body gravitational problem appears to be good. (Chapter 4).

3. New periodic solutions, weaving styles and chasing styles, with axial symetry and non-collison of the bodies, we describe the algebra and symmetry that allows us to reduce a full system of equations to just those for essentially one particle. Some of these styles provided the figure eight periodic solutions (Chapter 5).

4. We try to give approximate solutions for the new families of the weaving periodic solutions (Chapter 5).

5. DTW-periodic solutions with a Logarithmic potential energy. One interesting feature of these solutions is the appearance of double points in the initial data space corresponding to specified nodal structures. We also have the appearance of periodic orbits with the same nodal structure but different winding numbers. In the work of DTW these were denoted by use of a notation like "11/7", ie. 11 nodes with 7 revolutions required to complete the orbit (Chapter 5).
6. The extention of the use of the f-equation, the ϵ-equation and the numerical approach to other potentials (Chapter 5).
7. New style of periodic solution, the weaving style with the Logarithmic potential energy, this gave the figure eight periodic solution (Chapter 5).
8. Suggestion for further research areas in which one could continue this investigation (Chapter 5).

NOTATION.

In this thesis references are indicated by square brackets [ ] and equations are numbered in round brackets ( ), where (a.b) denotes equation b in chapter a.
This thesis has been typeset using LaTeX, except Chapter 6 using Mathematica.

Declaration

This work has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree.

Ahmed Eid Al-Saedi. (Candidate)

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Date ....2 / 10 / 2001.......

Statement 1

This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by citations giving explicit references. A bibliography is appended.

Ahmed Eid Al-Saedi. (Candidate)

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Statement 2

I hereby give consent for this thesis, if accepted, to be available for photocopying and for interlibrary loan, and for the title and summary to be made available to outside organisations.

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Contents

Chapter 1

THE MANY BODY PROBLEM

1.1 Introduction

In this Chapter we give some preliminaries, namely Kepler's laws of planetary motion and Newton's gravitational law and Coulomb's law of motions. In it we show that very few solutions are known for the general many-body problem. Here we apply the method of DTW for the system of a $2n$ electron atom problem with constant magnetic field. That means our system has $12n$ degrees of freedom. This can be reduced to our system using the method to solve for just one particle with three degrees of freedom. We give examples of the four node solution of the four ion problem, four electron atom problem without (with) constant magnetic field and four body gravitational problem. These examples we consider as our standard four node solutions in the DTW-periodic set up. We shall show how the DTW-solutions were found. We shell also refer to some recently related works for planar and non-planar periodic solutions by Chenciner et al and Simó.

1.2 Many body problems

1.2.1 History

Throughout the ages man has looked at the heavens and made many attempts to arrive at an explanation of the motions perceived. A complete and thorough treatment of the nature and date of the main contributions to the theory of dynamics is not appropriate herein but the significant contributions from our point of view are due to Kepler$(1571-1630)$, Newton$(1642-1727)$, Lagrange$(1736-1813)$ and Hamilton$(1805-1865)$.

1.2.2 Laws of planetary motion

The motion of celestial bodies is governed by Kepler's laws of planetary motion, the first law stating that the path of each planet is an ellipse with the Sun at a focus. The second law states that the radius vector (a straight line joining the Sun and a planet) sweeps out equal areas in equal times. This law implies that a planet moves more rapidly when it is close to the Sun. The third law states that the square of each planet's period is proportional to the cube of the semi-major axis of its elliptical orbit. These laws explain geometrically how the planets move in relation to the Sun. See [8], [19], [25], [29] and [76] for more details.

1.2.3 Newton

Newton's theory of universal gravitation produced a theoretical principle on which to base Kepler's three observational laws. From these laws, Newton deduced that the acceleration of any planet in its orbit is proportional to the inverse square of its distance from the Sun. For further details see [1], [4], [9] and [10], [42].

1.2.4 Solution of inverse square law motion

In this section we shall deal with Newton's gravitational law of attraction and Coulomb's law of motions. In both laws the force between two particles is inversely proportional to the square of the distance between them, acts along the line joining the particles, is proportional to the product of some constants α and β, determined in the case of Newton's gravitational law by means of masses or in the case of Coulombic law by means of charges, and in the latter case may be attractive or repulsive. Then

$\mathbf{F} = \frac{\mu}{r^2}\frac{\mathbf{r}}{r}$, where $\mu = k\alpha\beta$ is a force constant, for k is usually a constant.

This is a central force field dependent only on r, where r is the distance between the particles. It is also a conservative field because there exists a potential function $V(r)$, $V(r) = -\frac{\mu}{r}$ such that

$$\mathbf{F} = -\nabla V$$

and energy is conserved. That is,

$$E = \frac{1}{2}|\dot{\mathbf{r}}|^2 + V(r), \qquad \text{for unit mass,}$$

is a constant of the motion being a function of the state variables $\mathbf{q} = \mathbf{r}$ and $\mathbf{p} = \dot{\mathbf{r}}$ independent of time, $E = H(\mathbf{q}, \mathbf{p})$. The energy is an important state variable even for a macroscopic oscillator, but for an atomic oscillator it is indispensable. Here $\mu = k\alpha\beta$, where k, α and β are constants, determining whether we have Newton's gravitational law or Coulomb's law. In the Newtonian case $k = G$, $\alpha = m_1$ and $\beta = m_2$, where G is the universal gravitational constant and m_1, m_2 are the masses of the particles. In this case the force is always attractive. In the Coulombic case we have $k = \frac{1}{4\pi\epsilon_0}$, $\alpha = Q_1$ and $\beta = Q_2$, where ϵ_0 is known as the permittivity of free space, and Q_1, Q_2 are the charges of the particles. In this case the force is attractive or repulsive according as Q_1, Q_2 are of unlike or like sign. Let us consider two particles $\mathbf{P}_1$ and $\mathbf{P}_2$ moving according to the inverse square law of motion in an inertial frame with origin O. Locate them by means of radius vectors $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ in cartesian coordinates, where the constant unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to the rectangular axes x, y and z respectively. A vector along the line from $\mathbf{P}_1$ to $\mathbf{P}_2$ is $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, then for $r = |\mathbf{r}_2 - \mathbf{r}_1|$ a unit vector in this direction is $\mathbf{r}_{12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$. Then the force on $\mathbf{P}_2$, $\mathbf{F}_2$, is defined by

$$\mathbf{F}_2 = \frac{\mu}{|\mathbf{r}_2 - \mathbf{r}_1|^2}\,\mathbf{r}_{12} = \frac{\mu}{|\mathbf{r}_2 - \mathbf{r}_1|^3}\,(\mathbf{r}_2 - \mathbf{r}_1),$$

and analogously the force on $\mathbf{P}_1$, $\mathbf{F}_1$, satisfies $\mathbf{F}_1 = -\mathbf{F}_2$, or more explicitly

$$\mathbf{F}_1 = \frac{\mu}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\,\mathbf{r}_{21}$$

$$= \frac{\mu}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\,(\mathbf{r}_1 - \mathbf{r}_2).$$

Refer to [17], [36], [57] for further details.

1.2.5 One and two body problems

Consider one particle with unit mass moving according to the inverse square law of motion in an inertial frame of reference. With respect to some 3-dimensional cartesian coordinate system, the equation of motion is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\,\mathbf{r} \quad \text{where } \mu \text{ is a constant.}$$

This equation can be solved by using the constants of the motion,

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}, \quad \text{the angular momentum of the particle ,}$$
$$E = \frac{1}{2}|\dot{\mathbf{r}}|^2 + V(r), \quad \text{the energy of the particle and}$$
$$\mathbf{A} = \mathbf{h} \times \dot{\mathbf{r}} + \frac{\mu}{r}\,\mathbf{r}, \quad \text{the Hamilton Lenz Runge vector of the particle.}$$

By using these constants we can obtain

$$r = \frac{l}{1 - e\cos\theta} \, .$$

One recognises this as the equation of a conic section with one focus at the origin, eccentricity e and semi latus rectum l, where $l = h^2/\mu$ and $e = A/\mu$. These constants are determined by the initial conditions. We will consider the case where we have a bounded orbit, that is, for $0 \leq e < 1$. The two body problem can be reduced to a one body central force problem by considering the motion relative to the centre of mass which can be shown to move rectilinearly. As an example one could consider the motion of a planet around the Sun. Since the gravitational force of the Sun on a planet is so much larger than the forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except for the Sun and the one planet moving about it. We use a cartesian coordinate system with the centre of the Sun as origin in an inertial frame of reference and treat the planet as particle with position vector $\mathbf{r}$. Then the equation of the motion of the planet is

$$m\,\ddot{\mathbf{r}} = -\frac{GmM}{r^3}\,\mathbf{r}, \text{ or}$$

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\,\mathbf{r}, \qquad \mu = GM,$$

where G is the universal gravitational constant and M, m are the masses of the Sun and the planet respectively. For another example, consider two particles $\mathbf{P}_1$, $\mathbf{P}_2$ with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and masses m_1, m_2 respectively, moving according to the inverse square law of motion in an inertial frame with origin O. Suppose the only force on each is the force due to the other. Then, the equation of motion of particle two relative to the centre of mass is

$$\ddot{\mathbf{r}} = -\frac{m_1^3 G}{(m_1 + m_2)^2 r^3}\,\mathbf{r}, \text{ or}$$

$$\ddot{\mathbf{r}} = \frac{\mu}{r^3}\,\mathbf{r}, \qquad \mu = -m_1 G \left(\frac{m_1}{m_1 + m_2}\right)^2.$$

See [36], [47], [61], [63] for more details.

1.2.6 Restricted three body problem

The restricted problem of three gravitating bodies considers the case in which the mass of one of the bodies is negligibly small compared to both of the other two masses, and in which the two large masses move in circular orbits about their common center of mass. Thus, the small mass is assumed not to disturb the motion of the larger masses. See [44], [61] and [75].

1.2.7 The Lagrange solution

In 1772, Lagrange showed that three masses positioned at the vertices of an equilateral triangle, rotating about their common centre of mass with an appropriate angular velocity, constitutes a periodic solution of the three body problem. The solution of the restricted three body problem is just a version of this exact solution of Lagrange. This particular solution exists in the real world as the Trojan asteroids are to be found at the Lagrange point for the Sun-Jupiter system. For more details see [36], [61], [75] and [76].

1.2.8 Planar polygonal solutions

We may construct exact explicit many body solutions for gravitational attraction in special situations only. One way is to position the n bodies (particles) at the vertices of a regular polygon with n sides. If this system then rotates about its centre with a critical angular velocity a periodic solution will be achieved. See [62], [68] and [69] for further details.

1.2.9 Applications

When a satellite is in orbit round the Earth, the main force acting on the satellite is the attractive force of the Earth's gravitation. Provided the satellite does not orbit near the Moon, and stays in the vicinity of the Earth. The equation of motion of the satellite is approximately

$$m\ddot{\mathbf{r}} = -\frac{GmM_E}{r^3}\mathbf{r}, \text{ or}$$

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}, \qquad \mu = GM_E,$$

where m and M_E are the masses of the satellite and the Earth respectivly, and $\mathbf{r}$ the position vector of the satellite relative to the centre of the Earth. See [8], [43] and [65].
Another situation where a detailed knowledge of the periodic solutions of a Hamiltonian operator is required is in semi-classical quantum mechanics. In semi-classical quantum mechanics one can express the Green's function in terms of the classical action computed along the periodic paths for the system. A knowledge of the stability angles for these paths enables one to compute the spectrum for the quantum mechanical Hamiltonian operator. This technique has its origins in the work of Bohr and Sommerfeld [45], [46]. Both employed a knowledge of the periodic solutions of an electron around a point nucleus to calculate the energy levels of the Hydrogen atom. Bohr concentrated on the circular orbits whilst Sommerfeld extended the study to the elliptical orbits. A comprehensive account of these semi-classical techniqices can be found in the excellent papers of M. Gutzwiller [32], [33], [34], [35].

1.3 Coulombic/Keplerian many particle systems

1.3.1 n particle problem

In the n-particle (electron) problem we are concerned with a many particle problem where the forces involved are due to Coulombic interaction with the presence of a constant magnetic field (only non-relativistic dynamics will be considered). Suppose there are n particles with charges distributed at fixed positions in free space and let each charge Q_i have position vector $\mathbf{q}_i$, where $i = 1, 2, ..., n$, moving in the presence of a constant magnetic field, $\mathbf{B}$, and an infinitely heavy positively charged nucleus, under their mutual attractions and repulsions in pairs with the force of magnitude $kQ_iQ_j|\mathbf{q}_j - \mathbf{q}_i|^{-2}$ where $|\mathbf{q}_j - \mathbf{q}_i|$ is the distance between the i^{th} and j^{th} particle and k is a constant. We suppose that $n \geq 2$. Let O an origin fixed in space be the position of the infinitely heavy positively charged nucleus and let $\mathbf{q}_i$, $\dot{\mathbf{q}}_i$ denote the position and velocity vectors of the i^{th} particle. Then, the i^{th} particle satisfies the equation

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} k_1 \mathbf{F}(\mathbf{q}_i, \mathbf{q}_j) - nk_2 \frac{\mathbf{q}_i}{|\mathbf{q}_i|^3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_i), \tag{1.1}$$

where $\mathbf{F}(\mathbf{q}_i, \mathbf{q}_j) = kQ_iQ_j(\mathbf{q}_j - \mathbf{q}_i)|\mathbf{q}_j - \mathbf{q}_i|^{-3}$. In the case of $2n$-bodies, $2n$-ions the forces involved are due to gravitational interaction, Coulombic interaction respectively. Also k_1, k_2 and k_3 are either constant or functions of the indexes i and j determining the case of gravitating particles, electrons or ions. See [17], [20] and [63]. There is no loss of generality in assuming that any bounded solution takes place in the unit sphere since the n-particle problem (1.1) is invariant under the transformation $\mathbf{q}(t) \to \mu^{\frac{2}{3}}\, \mathbf{q}(\mu t)$, $\mu > 0$. See [21], [62].

The system (1.1) may be represented by $3n$ second order differential equations in the Newtonian formulation and by $6n$ nonlinear, simultaneous first order differential equations in the Hamiltonian formulation, a Hamiltonian system with the coordinates and momenta together defining the system's instantaneous state. See [36], [44], [47] and [50], [51], [52] for more details.

1.3.2 Pseudo rotational symmetry

Consider $2n$ particles (electrons) with unit mass with positions $\mathbf{q}_i \in \mathbb{R}^3$, $i = 1, 2, ..., 2n$, moving about a core of $2n$ protons (an infinitely heavy positively charged nucleus) in the presence of a constant magnetic field, moving under their mutual attractions and repulsions. The Hamiltonian $H(\mathbf{q}, \mathbf{p})$, $H : \mathbb{R}^{12n} \longrightarrow \mathbb{R}$ is defined by

$$H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{2n} \frac{1}{2}\left(\mathbf{p}_i - \frac{1}{2}\mathbf{B} \times \mathbf{q}_i\right)^2 + \sum_{i<j}^{2n} \frac{k}{|\mathbf{q}_i - \mathbf{q}_j|} - 2n\alpha \sum_{i=1}^{2n} \frac{1}{|\mathbf{q}_i|},$$

where $\mathbf{p}_i \in \mathbb{R}^3$, $\mathbf{q}_i \in \mathbb{R}^3$, $i = 1, 2, ..., 2n$, and k, α are constants. The equations of motion reduce to

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1\\ j\neq i}}^{2n} k_1 \frac{(\mathbf{q}_j - \mathbf{q}_i)}{|\mathbf{q}_j - \mathbf{q}_i|^3} - 2nk_2 \frac{\mathbf{q}_i}{|\mathbf{q}_i|^3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_i) \quad , i = 1, 2, ..., 2n,$$

where $\mathbf{B}$ is a constant magnetic field, and k_1, k_2 and k_3 are either constant or functions of the indices $(i - j + 1)$, determining the case of gravitating particles, electrons or ions. The problem can be reduced from that of solving $6n$ second order non linear differential equations to that of solving three second order non-linear differential equations by demanding that the positions of the particles are related by the pseudo rotational matrix $P \in O(3)$ such that $P^{2n} = I$ and $\det P = -1$. The equations of motion admit a solution of the form $\mathbf{q}_{i+1} = P\mathbf{q}_i$. By applying P to the equations of motion of $\mathbf{q}_i$ we obtain the equation of motion for $\mathbf{q}_{i+1}$. This way allows one to solve the system of $6n$ equations by solving the equations of motion of $\mathbf{q}_1$ alone. If we choose $\mathbf{B}$ parallel to the z axis, the pseudo rotational matrix for the $2n$ particles is represented by

$$P = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & 0 \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

1.3.3 $2n$ particles with pseudo rotational symmetry

In this section we shall give the complete description of how one tackles the $2n$ particle problem See [21], [22]. We esentially study the $2n$-electron problem with a magnetic field as this allows us to develop the most difficult case. The $2n$-ion problem differs most from this setup in that we need to deal with $k_1 = (-1)^{i-j+1}$. This causes no great difficulty and so we do not develop a separate argument. Consider the equation of motion of $\mathbf{q}_i$

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1\\j\neq i}}^{2n} k_1(\mathbf{q}_j - \mathbf{q}_i)|\mathbf{q}_j - \mathbf{q}_i|^{-3} - 2nk_2\,\mathbf{q}_i|\mathbf{q}_i|^{-3} + k_3(\mathbf{B} \times \mathbf{q}_i)\ , \quad i = 1, 2, ...2n.$$

If we applying P to the equation of motion of $\mathbf{q}_i$, where $\mathbf{q}_{i+1} = P\mathbf{q}_i$, we obtain

$$\begin{aligned}
P\ddot{\mathbf{q}}_i &= P\Big(\sum_{\substack{j=1\\j\neq i}}^{2n} k_1(\mathbf{q}_j - \mathbf{q}_i)|\mathbf{q}_j - \mathbf{q}_i|^{-3} - 2nk_2\,\mathbf{q}_i|\mathbf{q}_i|^{-3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_i)\Big) \\
P\ddot{\mathbf{q}}_i &= \sum_{\substack{j=1\\j\neq i}}^{2n} k_1(P\mathbf{q}_j - P\mathbf{q}_i)|\mathbf{q}_j - \mathbf{q}_i|^{-3} - 2nk_2\,P\mathbf{q}_i|\mathbf{q}_i|^{-3} + k_3P(\mathbf{B} \times \dot{\mathbf{q}}_i) \\
P\ddot{\mathbf{q}}_i &= \sum_{\substack{j=1\\j\neq i}}^{2n} k_1(P\mathbf{q}_j - P\mathbf{q}_i)|P\mathbf{q}_j - P\mathbf{q}_i|^{-3} - 2nk_2\,P\mathbf{q}_i|P\mathbf{q}_i|^{-3} + k_3P(\mathbf{B} \times \dot{\mathbf{q}}_i) \\
\ddot{\mathbf{q}}_{i+1} &= \sum_{\substack{j+1=2\\j\neq i}}^{2n+1} k_1(\mathbf{q}_{j+1} - \mathbf{q}_{i+1})|\mathbf{q}_{j+1} - \mathbf{q}_{i+1}|^{-3} - 2nk_2\,\mathbf{q}_{i+1}|\mathbf{q}_{i+1}|^{-3} + k_3\mathbf{B} \times \dot{\mathbf{q}}_{i+1} \\
\ddot{\mathbf{q}}_{i+1} &= \sum_{\substack{k=1\\k\neq i+1}}^{2n} k_1(\mathbf{q}_k - \mathbf{q}_{i+1})|\mathbf{q}_k - \mathbf{q}_{i+1}|^{-3} - 2nk_2\,\mathbf{q}_{i+1}|\mathbf{q}_{i+1}|^{-3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_{i+1})\ ,
\end{aligned}$$

which is the equation of motion of $\mathbf{q}_{i+1}$. Here we have used

$$P(\mathbf{B} \times \dot{\mathbf{q}}_i) = \det P(P\mathbf{B} \times P\dot{\mathbf{q}}_i) = \det P((\det P)\mathbf{B} \times \dot{\mathbf{q}}_{i+1}) = \mathbf{B} \times \dot{\mathbf{q}}_{i+1},$$

$$P\mathbf{q}_i|P\mathbf{q}_i|^{-3} = \mathbf{q}_{i+1}|\mathbf{q}_{i+1}|^{-3}$$

and

$$P\mathbf{B} = (\det P)\mathbf{B}, \ \ \det P = -1, \ \ P^{2n} = I.$$

Now, we shall just consider the equation of motion of one particle with position vector, $\mathbf{q}_1$. Then, the equation of motion of $\mathbf{q}_1$ is

$$
\begin{aligned}
\ddot{\mathbf{q}}_1 &= \sum_{j=2}^{2n} k_1(\mathbf{q}_j - \mathbf{q}_1)|\mathbf{q}_j - \mathbf{q}_1|^{-3} - 2nk_2\mathbf{q}_1|\mathbf{q}_1|^{-3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_1) \\
&= \sum_{j=2}^{n} \alpha_1(\mathbf{q}_j + \mathbf{q}_{2n+2-j} - 2\mathbf{q}_1)|\mathbf{q}_j - \mathbf{q}_1|^{-3} + \alpha_2(\mathbf{q}_{n+1} - \mathbf{q}_1)|\mathbf{q}_{n+1} - \mathbf{q}_1|^{-3} \\
&\quad - 2nk_2\mathbf{q}_1|\mathbf{q}_1|^{-3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_1) \\
&= \sum_{m=1}^{n-1} \alpha_1(\mathbf{q}_{m+1} + \mathbf{q}_{2n+2-(m+1)} - 2\mathbf{q}_1)|\mathbf{q}_{m+1} - \mathbf{q}_1|^{-3} \\
&\quad + \alpha_2(\mathbf{q}_{n+1} - \mathbf{q}_1)|\mathbf{q}_{n+1} - \mathbf{q}_1|^{-3} - 2nk_2\mathbf{q}_1|\mathbf{q}_1|^{-3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_1) \\
&= \sum_{m=1}^{n-1} \alpha_1(P^m + P^{2n-m} - 2I)\mathbf{q}_1|(P^m - I)\mathbf{q}_1|^{-3} \\
&\quad + \alpha_2(P^n - I)\mathbf{q}_1|(P^n - I)\mathbf{q}_1|^{-3} - 2nk_2\mathbf{q}_1|\mathbf{q}_1|^{-3} + k_3(\mathbf{B} \times \dot{\mathbf{q}}_1),
\end{aligned}
$$

with initial position $\mathbf{q}_1(0) = (1, 0, 0)$ and initial velocity $\dot{\mathbf{q}}_1(0) = (0, \dot{y}_0, \dot{z}_0)$, where α_1, α_2 are constants determined by the nature of the problem in hand. If we want to know how $P^m\mathbf{q}$ is related to $\mathbf{q}$, we will need some simple algebraic properties of P. Now consider again the pseudo rotation matrix, represented by the matrix.

$$
P = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} & 0 \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$

with respect to some orthonormal basis $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The eigenvalues of P are -1, $\exp(\pm i\pi/n)$, with corresponding eigenvectors

$$
\mathbf{v}_1 = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{i}{\sqrt{2}}\mathbf{j}, \ \mathbf{v}_2 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{i}{\sqrt{2}}\mathbf{j} \text{ and } \mathbf{v}_3 = \mathbf{k} \text{ respectively, where } i = \sqrt{-1}.
$$

Note that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis. It is convenient to consider vectors with respect to both sets of basis vectors, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ as this eases the calculation of the effect of P. Now, we are going to write the position vector $\mathbf{q}_1$ with respect to the orthonormal basis, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, where $\mathbf{q}_1$ is some vector in $\mathbb{R}^3$. Then, $\mathbf{q}_1 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with respect to the original basis, and so it has components $\dfrac{x+iy}{\sqrt{2}}$, $\dfrac{x-iy}{\sqrt{2}}$ and z, with respect to the eigenvectors of P, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. We shall use the following notation for $\mathbf{q}_1$

$$\mathbf{q}_1 \equiv (x, y, z) \equiv [a, b, z],$$

where

$$a = \frac{x+iy}{\sqrt{2}} \quad \text{and} \quad b = \frac{x-iy}{\sqrt{2}},$$

noting

$$|a|^2 + |b|^2 = x^2 + y^2.$$

We now calculate the components of $P^m\mathbf{q}_1$ with respect to $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, with the notation $\mathbf{q}_1 \equiv [a, b, z]$. First we will let $\omega = \exp(\pm i\pi/n)$, then $P^m\mathbf{q}_1$ has components $[\omega^m a, \omega^{-m} b, (-1)^m z]$ for $\mathbf{q}_1 \equiv [a, b, z]$. We may now calculate the distance between $\mathbf{q}_{m+1}$ and $\mathbf{q}_1$ more easily. First of all consider,

$$\begin{aligned}
\mathbf{q}_{m+1} - \mathbf{q}_1 &= P^m\mathbf{q}_1 - \mathbf{q}_1 \\
&= (P^m - I)\mathbf{q}_1 \\
&\equiv [(\omega^m - 1)a, (\omega^{-m} - 1)b, ((-1)^m - 1)z] \ ,
\end{aligned}$$

which gives,

$$\begin{aligned}
|\mathbf{q}_{m+1} - \mathbf{q}_1|^2 &= |(P^m - I)\mathbf{q}_1|^2 \\
&= |\omega^m - 1|^2(|a|^2 + |b|^2) + ((-1)^m - 1)^2 z^2 \\
&= |\omega^m - 1|^2(x^2 + y^2) + ((-1)^m - 1)^2 z^2.
\end{aligned}$$

Now, calculate $|\omega^m - 1|$,

$$\begin{aligned}
\omega^m - 1 &= (\cos\frac{\pi}{n} \pm i\sin\frac{\pi}{n})^m - 1 \\
&= 2\sin\frac{m\pi}{2n}(-\sin\frac{m\pi}{2n} \pm i\cos\frac{m\pi}{2n}) \, ,
\end{aligned}$$

and so,

$$|\omega^m - 1|^2 = 4\sin^2\frac{m\pi}{2n} \, ,$$

yielding

$$\begin{aligned}
|\mathbf{q}_{m+1} - \mathbf{q}_1|^2 &= |(P^m - I)\mathbf{q}_1|^2 \\
&= 4\sin^2(\frac{m\pi}{2n})\,(x^2 + y^2) + ((-1)^m - 1)^2 z^2.
\end{aligned}$$

Next we want the components of $(P^m + P^{2n-m} - 2I)\mathbf{q}_1$. With $\mathbf{q}_1 \equiv [a, b, z]$, we have $P^m\mathbf{q}_1 \equiv [\omega^m a, \omega^{-m} b, (-1)^m z]$ for $\mathbf{q}_1 \equiv [a, b, z]$, and so

$$\begin{aligned}(P^m + P^{2n-m} - 2I)\mathbf{q}_1 &\equiv [(\omega^m + \omega^{2n-m} - 2)a, (\omega^{-m} + \omega^{-2n+m} - 2)b, 2((-1)^m - 1)z] \\ &\equiv [(\omega^m + \omega^{-m} - 2)a, (\omega^m + \omega^{-m} - 2)b, 2((-1)^m - 1)z] \\ &\equiv [-4\sin^2(\frac{m\pi}{2n})a, -4\sin^2(\frac{m\pi}{2n})b, 2((-1)^m - 1)z],\end{aligned}$$

since $\omega^{2n} = 1$ and $\omega^m + \omega^{-m} - 2 = -4\sin^2(\frac{m\pi}{2n})$. We also have

$$(P^n - I)\mathbf{q}_1 \equiv [-2a, -2b, ((-1)^n - 1)z],$$

yielding

$$\begin{aligned}|(P^n - I)\mathbf{q}_1|^{-3} &= \left(4(\,|a^2| + |b^2|\,) + ((-1)^n - 1)^2 z^2\right)^{-\frac{3}{2}} \\ &= \left(4(\,x^2 + y^2\,) + ((-1)^n - 1)^2 z^2\right)^{-\frac{3}{2}}.\end{aligned}$$

For the ions problem note that the ion with position $P^m\mathbf{q}_1$ has the same charge as the ion with position $P^{2n-m}\mathbf{q}_1$.

1.3.4 Reduced system of equations with constant magnetic field

Having simplified the system we now have the equations of the motion for one of the $2n$ particles, given by

$$\begin{aligned}\ddot{x} = &\sum_{m=1}^{n-1} \alpha_1\Big(-4\sin^2(\frac{m\pi}{2n})\Big)x\left(4\sin^2(\frac{m\pi}{2n})\,(x^2 + y^2) + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}} \\ &+ \alpha_2(-2)x\left(4(x^2 + y^2) + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2 x\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}} \\ &- k_3 B\dot{y},\end{aligned}$$

$$\ddot{y} = \sum_{m=1}^{n-1} \alpha_1\Big(-4\sin^2(\frac{m\pi}{2n})\Big)y\Bigg(4\sin^2(\frac{m\pi}{2n})\,(x^2+y^2)+\Big((-1)^m-1\Big)^2z^2\Bigg)^{-\frac{3}{2}}$$
$$+\alpha_2(-2)y\Bigg(4(x^2+y^2)+\Big((-1)^n-1\Big)^2z^2\Bigg)^{-\frac{3}{2}} - 2nk_2y\Bigg(x^2+y^2+z^2\Bigg)^{-\frac{3}{2}}$$
$$+k_3B\dot{x}\ ,$$

$$\ddot{z} = \sum_{m=1}^{n-1} \alpha_1 2\Big((-1)^m-1\Big)z\Bigg(4\sin^2(\frac{m\pi}{2n})\,(x^2+y^2)+\Big((-1)^m-1\Big)^2z^2\Bigg)^{-\frac{3}{2}}$$
$$+\alpha_2\Big((-1)^n-1\Big)z\Bigg(4(x^2+y^2)+\Big((-1)^n-1\Big)^2z^2\Bigg)^{-\frac{3}{2}} - 2nk_2z\Bigg(x^2+y^2+z^2\Bigg)^{-\frac{3}{2}}.$$

Note that the energy equation is

$$E = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \sum_{m=1}^{n-1}\alpha_1\Bigg(4\sin^2(\frac{m\pi}{2n})\,(x^2+y^2)+\Big((-1)^m-1\Big)^2z^2\Bigg)^{-\frac{1}{2}}$$
$$-\alpha_2\frac{1}{2}\Bigg(4(x^2+y^2)+\Big((-1)^n-1\Big)^2z^2\Bigg)^{-\frac{1}{2}} - 2nk_2\Bigg(x^2+y^2+z^2\Bigg)^{-\frac{1}{2}}.$$

Where $\det P = -1$ and $P^{2n} = 1$. (Note that in the $2n$ electron atom case we could also handle $\det P = +1$ and $P^{2n} = I$. In the $(2n+1)$ electron atom case we can hande $\det P = \pm 1$ and $P^{2n+1} = I$, in a similar manner).

1.3.5 Four ion problem

Consider the motion of four ions of unit mass of charges ± 1, moving under their mutual attractions and repulsions. Denote their positions by $\mathbf{q}_i \in \mathbb{R}^3$, with charge $(-1)^i$ for $\mathbf{q}_i$, $i = 1, 2, 3, 4$. One obtains the equations of motion by setting $k_1 = (-1)^{i-j+1}$, $k_2 = 0$, $k_3 = 0$ and $\alpha_1 = (-1)^{m+1}$, $\alpha_2 = (-1)^{n+1}$ and $n = 2$ in the general reduced system,

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1\\ j\neq i}}^{4}(-1)^{i-j+1}(\mathbf{q}_j - \mathbf{q}_i)|\mathbf{q}_j - \mathbf{q}_i|^{-3} \quad i = 1, 2, 3, 4.$$

The problem can be reduced from that of solving twelve second order nonlinear differential equation by using the matrix P represented by

$$P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

to that of solving three second order nonlinear differential equations. That is, it simplifies the system of equations to

$$\ddot{x} = -2x(2x^2 + 2y^2 + 4z^2)^{-\frac{3}{2}} + 2x(4x^2 + 4y^2)^{-\frac{3}{2}},$$

$$\ddot{y} = -2y(2x^2 + 2y^2 + 4z^2)^{-\frac{3}{2}} + 2y(4x^2 + 4y^2)^{-\frac{3}{2}},$$

$$\ddot{z} = -4z(2x^2 + 2y^2 + 4z^2)^{-\frac{3}{2}}.$$

The energy equation is

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - (2x^2 + 2y^2 + 4z^2)^{-\frac{1}{2}} + \frac{1}{2}(4x^2 + 4y^2)^{-\frac{1}{2}}.$$

The basic initial data for the problem is $x(0) = 1,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0,\ \dot{z}(0) = \dot{z}_0$. For the four node solution for the four ion problem, let the initial velocity be as below

$$\dot{y}_0 = 0.419778768 \quad \text{and} \quad \dot{z}_0 = 0.600918801.$$

This gives the angular momentum and the energy as

$$|\mathbf{h}| = \dot{y}_0 = 0.419778768\ , \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) - \frac{1}{\sqrt{2}} + \frac{1}{4} = -0.188448.$$

The motion of one of the ions is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

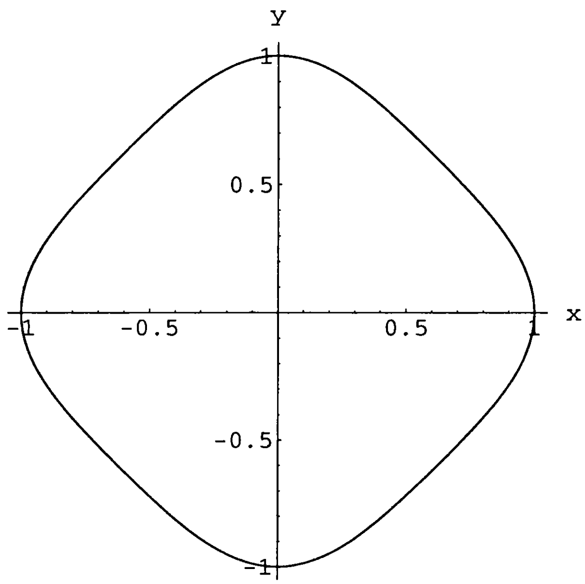


Figure 1.1:(x, y)-projection.

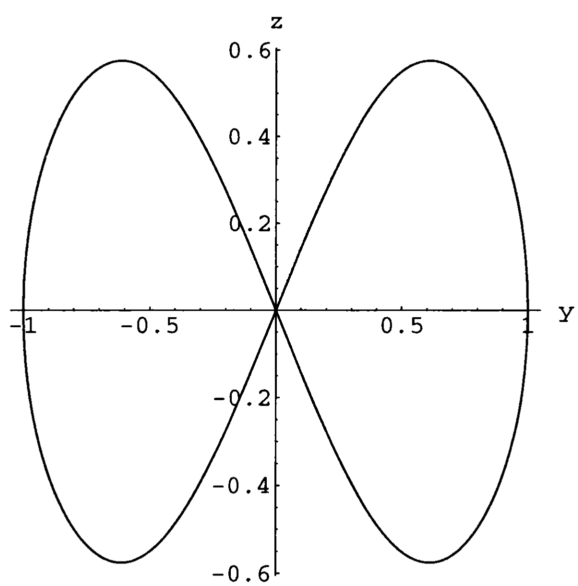


Figure 1.2:(y, z)-projection.

1.3.6 Four electron atom problem with (without) constant magnetic field

Consider the motion of four electrons of unit mass each with charge (-1), and position $\mathbf{q}_i \in \mathbb{R}^3$, $i = 1,2,3,4$ moving in the presence of a constant magnetic field and an infinitely heavy positively charged nucleus which has charge $+4$, moving under their mutual attractions and repulsions. By setting $k_1 = -1$, $k_2 = 1$, $k_3 = 1$ and $\alpha_1 = -1$, $\alpha_2 = -1$ and $n = 2$ in the general reduced system, one obtains the equations of motion

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1\\ j\neq i}}^{4}(-1)(\mathbf{q}_j - \mathbf{q}_i)|\mathbf{q}_j - \mathbf{q}_i|^{-3} - 4\mathbf{q}_i|\mathbf{q}_i|^{-3} + \mathbf{B}\times\dot{\mathbf{q}}_i\ , \quad i = 1,2,3,4.$$

The problem can be reduced from that of solving twelve second order nonlinear differential equation by using the matrix P represented by

$$P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \det P = -1,$$

to that of solving three second order nonlinear differential equations. Chosing an orthonormal coordinate system in which $\mathbf{B} = B\mathbf{k}$ gives us particularly simple equations. That is, it simplifies the system of equations to

$$\ddot{x} = 2x(2x^2+2y^2+4z^2)^{-\frac{3}{2}} + 2x(4x^2+4y^2)^{-\frac{3}{2}} - 4x(x^2+y^2+z^2)^{-\frac{3}{2}} - B\dot{y},$$

$$\ddot{y} = 2y(2x^2+2y^2+4z^2)^{-\frac{3}{2}} + 2y(4x^2+4y^2)^{-\frac{3}{2}} - 4y(x^2+y^2+z^2)^{-\frac{3}{2}} + B\dot{x},$$

$$\ddot{z} = 4z(2x^2+2y^2+4z^2)^{-\frac{3}{2}} - 4z(x^2+y^2+z^2)^{-\frac{3}{2}}.$$

The energy equation is now

$$E = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + (2x^2+2y^2+4z^2)^{-\frac{1}{2}} + \frac{1}{2}(4x^2+4y^2)^{-\frac{1}{2}} - 4(x^2+y^2+z^2)^{-\frac{1}{2}}.$$

The basic initial data for the problem is $x(0) = 1,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0,\ \dot{z}(0) = \dot{z}_0$. For the four node solution for the four electron problem, let the initial velocity be as below

$$\dot{y}_0 = 0.126889159 \quad \text{and} \quad \dot{z}_0 = 1.816605346 \quad \text{and} \quad B = 0.$$

This gives the angular momentum and the energy as

$$|\mathbf{h}| = \dot{y}_0 = 0.126889159 \ , \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) + \frac{1}{\sqrt{2}} + \frac{1}{4} - 4 = -1.38482.$$

The motion of one of the electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

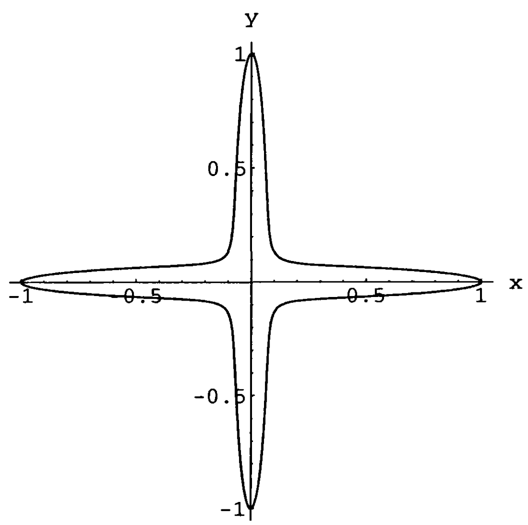


Figure 1.3:(x, y)-projection.

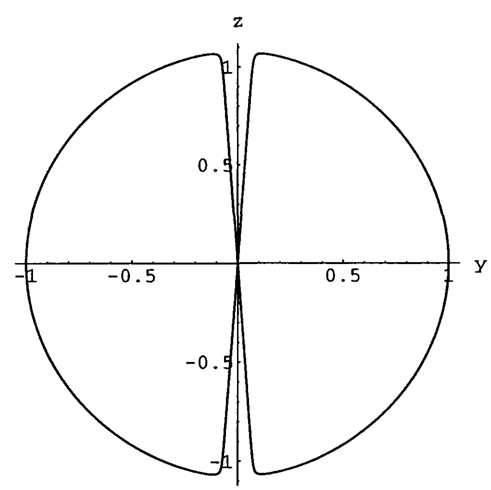


Figure 1.4:(y, z)-projection.

For the four node solution for the four electron problem with non-zero magnetic field. Now let the initial velocity be as below

$$\dot{y}_0 = 0.328117187 \quad \text{and} \quad \dot{z}_0 = 1.823429687 \quad \text{and} \quad B = \frac{1}{2}.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}| = \dot{y}_0 = 0.328117187\ , \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) + \frac{1}{\sqrt{2}} + \frac{1}{4} - 4 = -1.32661.$$

The motion of one of the electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

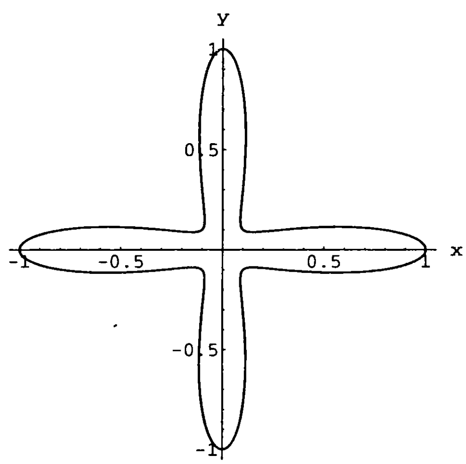


Figure 1.5:(x, y)-projection.

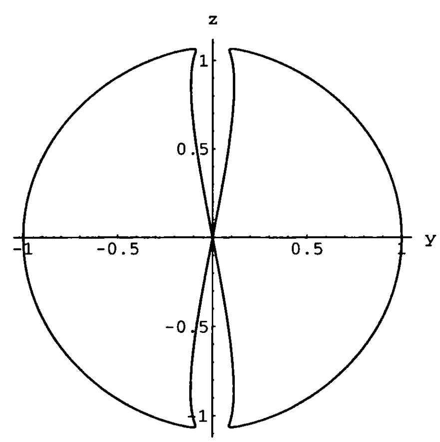


Figure 1.6:(y, z)-projection.

1.3.7 Four body gravitational problem

Consider the motion of four gravitating bodies of unit mass subject to their mutual attractions with positions $\mathbf{q}_i \in \mathbb{R}^3$, $i = 1, 2, 3, 4$. By setting $k_1 = 1$, $k_2 = 0$, $k_3 = 0$ and $\alpha_1 = 1$, $\alpha_2 = 1$ and $n = 2$ in the general reduced system, one obtains the equations of motion, which are

$$\ddot{\mathbf{q}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{4} (\mathbf{q}_j - \mathbf{q}_i)|\mathbf{q}_j - \mathbf{q}_i|^{-3} \quad i = 1, 2, 3, 4.$$

We let the matrix P be

$$P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and as before, simplify the system of equations to

$$\begin{aligned}
\ddot{x} &= -2x(2x^2 + 2y^2 + 4z^2)^{-\frac{3}{2}} - 2x(4x^2 + 4y^2)^{-\frac{3}{2}}, \\
\ddot{y} &= -2y(2x^2 + 2y^2 + 4z^2)^{-\frac{3}{2}} - 2y(4x^2 + 4y^2)^{-\frac{3}{2}}, \\
\ddot{z} &= -4z(2x^2 + 2y^2 + 4z^2)^{-\frac{3}{2}}.
\end{aligned}$$

The energy equation is now

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - (2x^2 + 2y^2 + 4z^2)^{-\frac{1}{2}} - \frac{1}{2}(4x^2 + 4y^2)^{-\frac{1}{2}}.$$

The basic initial data for the problem is $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0$, $\dot{z}(0) = \dot{z}_0$. For the four node solution for the four body gravitational problem, let the initial velocity be as below

$$\dot{y}_0 = 0.333250244 \quad \text{and} \quad \dot{z}_0 = 0.841783691.$$

This gives the angular momentum and the energy as

$$|\mathbf{h}| = \dot{y}_0 = 0.333250244 \ , \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) - \frac{1}{\sqrt{2}} - \frac{1}{4} = -0.547279.$$

The motion of one of the electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

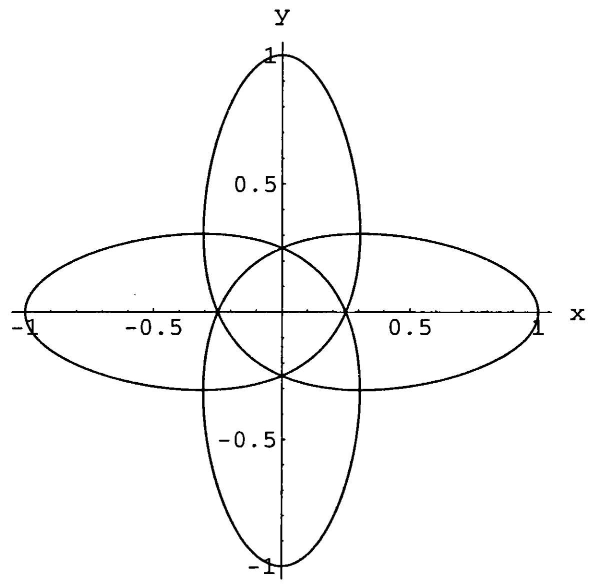


Figure 1.7: (x, y)-projection.

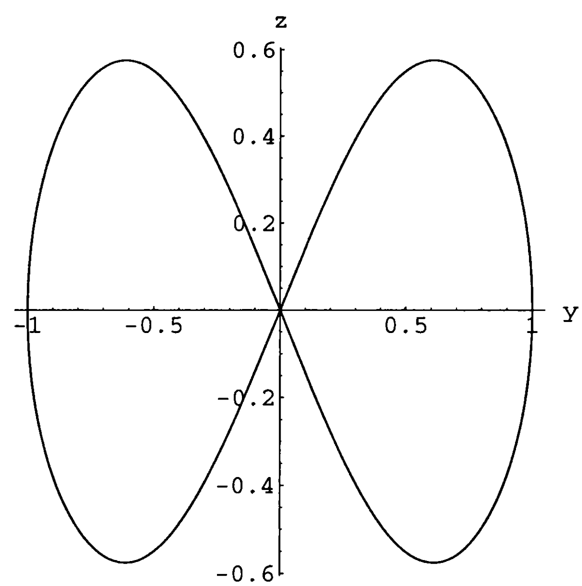


Figure 1.8: (y, z)-projection.

1.4 Mathematica code

The diagrams above were created using Mathematica to firstly apply a fourth order Runge-Kutta scheme to numerically solve the reduced system of equations of motion and then plot a subset of the points generated. The Mathematica code used throughout the thesis will be presented in chapter 6 along with a brief descriptions.

1.5 How solutions were found

1.5.1 Search procedures

In the papers of Ian Davies, Aubrey Truman and David Williams [21], [22] solutions like those above were found by employing computer based search routines. These routines are documented in [21]. We briefly describe the basic ideas below. Suppose we wish to obtain a periodic solution, periodic in both $r = \sqrt{x^2 + y^2}$ and z with the period, P_z, for the z-motion being twice the period, P_r, for the r-motion. Let the minimum value of r be r_m, where $r_m \leq r \leq 1$ and let the maximum value of z be z_m, where $0 \leq z \leq z_m$. First one obtains a planar polygonal periodic solution by initially choosing $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_c$, $\dot{z}(0) = 0$, and using the equation of the motion and the energy equation to ensure that $\dot{r} = 0$. Now say we have the planar periodic solution subject to the equation of the motion with negative energy, E_c. Now initially choose small $\dot{y} = \dot{y}_c \pm m$, $\dot{z} = \dot{z}_c \pm m$ subject to the bound state, where m is a very small mesh. And try to find position and velocity for first time $z = 0$ (when $r = 1$, $\dot{r} = 0$). Let P_θ be the angular distance between the starting point and the first point at which $z = 0$ (when $r = 1$ hopefully). Let R denote the matrix representing a rotation of P_θ about the z-axes. If the initial position and velocity were $(1, 0, 0)$ and $(0, \dot{y}_0, \dot{z}_0)$ respectively then the corresponding values after angular distance(period) P_θ would have to be $R(1, 0, 0)$ and $R(0, \dot{y}_0, -\dot{z}_0)$ respectively(the minus of the velocity of z because the motion goes up and down in z-direction). Given some initial data, one solves the equations of motion numerically until $z \leq 0\,(\dot{z}_0 > 0)$. We then interpolate between the last two computed positions of z (one is negative and the otheir is positive) to find the time at which $z = 0$. The position and velocity for this time, T, $(x_T, y_T, 0)$ and $(\dot{x}_T, \dot{y}_T, \dot{z}_T)$ respectively, are then used to compute

$$d(\dot{y}_0, \dot{z}_0) = |(x_T, y_T, 0) - R(1, 0, 0)|^2 + |(\dot{x}_T, \dot{y}_T, \dot{z}_T) - R(0, \dot{y}_0, -\dot{z}_0)|^2.$$

For a fixed $\dot{z}_0$ we calculate $d(\dot{y}_0, \dot{z}_0)$ for a variety of values of $\dot{y}_0$ and find that $\dot{z}_0$ which minimises $d(\dot{y}_0, \dot{z}_0)$. We then begin to work with a finer mesh and refine the calculations proceeding in the same manner as before. Ian Davies, Aubrey Truman and David Williams were very successful in obtaining periodic solution of several types of many-body problems by employing variants of the above method. We must emphasize that as the differential equations were solved numerically we could not expect to find $(\dot{y}_0, \dot{z}_0)$ yielding $d(\dot{y}_0, \dot{z}_0) = 0$, we would only be able to consider $d(\dot{y}_0, \dot{z}_0)$, small with regard to the errors inherent in the chosen numerical scheme.
Note that one can make use of the naturally occurring constants of the motion to continually check that the numerical solution is not behaving badly. One could also solve the full(unreduced) system and check on the agreement of both systems.

1.6 Recent related works

We are of course, aware that one may easily confuse periodic solutions and quasi-periodic solutions when one is employing the numerical techniques to solve the differential equation. However the recent works of Chenciner et al [11], [12], [13],[14] especially [15], give the existence of periodic orbits of these form. There have been other numerically based developments see Georges Hoynant [26], Moore [54] and Simó [71], [72].

Chapter 2

DERIVATION OF THE f-EQUATION

2.1 Introduction

In this chapter we will give the complete description of how to study the $2n$ particle (electron) problem. We can represent the motion in Cylindrical polar coordinates (r, θ, z) where $x = r\cos\theta$ and $y = r\sin\theta$. Then, the system reduces to a pair of second order non linear differential equations for r and z. We are going to see how to derive the f-equation and approach the solution of the system of equations from an analytical point of view as in the papers of Davies, Truman and Williams [21], [22]. To do this we express z in terms of r alone and then obtain an equation for $f(r) = z^2$ in terms of r only by eliminating the explicit dependence of the derivatives on time. We obtain a non-linear differential equation for f in terms of r only, a highly non-linear second order differential equation. The boundary conditions on f are found from the initial position and the initial velocity which are dependent upon $\dot{y}_0 = h$ and $\dot{z}_0$. The solutions in which we are interested are those which are periodic in both z and r such that the period for the z-motion is twice the period for the r-motion. Also, we will calculate the important characteristics of the solution of the equations of motion generated by the f-equation as follows. We let the maximum and minimum values of z and r be z_m and r_m respectively. We will define the periods of the respective coordinate motions, P_z and P_r and the angular distance P_θ, in terms of $\dot{y}_0$ and $\dot{z}_0$. Furthermore in this chapter we shall give some examples of the harmonic oscillator and the two particle problem, where we study the geometry of the elliptic periodic solutions of the two particle problem. We will state, explicity, $f(r)$ for the two particle problem. We will study the numerical solutions of the f-equation with the initial conditions for the four examples respectively as in Chapter 1, demonstrating the agreement between the two systems.

2.2 Cylindrical polar form

Firstly rewrite the equations of motion in the previous chapter in Cylindrical polar coordinates, (r, θ, z). The equation of motion of one of the $2n$ particles becomes

$$\begin{aligned}\frac{d^2}{dt^2}(r\mathbf{e}_r) = \Bigg[\sum_{m=1}^{n-1} \alpha_1\Big(-4\sin^2(\frac{m\pi}{2n})\Big)r\Big(4\sin^2(\frac{m\pi}{2n})\,r^2 + \big((-1)^m - 1\big)^2 z^2\Big)^{-\frac{3}{2}} \\ + \alpha_2(-2)r\Big(4r^2 + \big((-1)^n - 1\big)^2 z^2\Big)^{-\frac{3}{2}} - 2nk_2\Big(r^2 + z^2\Big)^{-\frac{3}{2}} \\ + k_3 B r\dot{\theta}\Bigg]\mathbf{e}_r + k_3 B \dot{r}\mathbf{e}_\theta,\end{aligned} \tag{2.1}$$

$$\begin{aligned}\ddot{z} = \sum_{m=1}^{n-1} \alpha_1 2\big((-1)^m - 1\big)z\Big(4\sin^2(\frac{m\pi}{2n})\,r^2 + \big((-1)^m - 1\big)^2 z^2\Big)^{-\frac{3}{2}} \\ + \alpha_2\big((-1)^n - 1\big)z\Big(4r^2 + \big((-1)^n - 1\big)^2 z^2\Big)^{-\frac{3}{2}} \\ - 2nk_2 z\Big(r^2 + z^2\Big)^{-\frac{3}{2}}.\end{aligned} \tag{2.2}$$

The position vector $\mathbf{r}(t)$ in Cylindrical polar coordinates is

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{k}, \tag{2.3}$$

where $\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ and $\mathbf{e}_r$, $\mathbf{e}_\theta$ are the usual unit tangent vectors in the direction of increasing r, θ. Differentiating (2.3), gives

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{k}, \tag{2.4}$$

showing that the velocity of the particle, in general, has components in both $\mathbf{e}_r$ and $\mathbf{e}_\theta$ directions, called respectively the radial resolute $\dot{r}$ and the cross-radial resolute (transverse) $r\dot{\theta}$ components. Differentiating (2.4), we have

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{k}. \tag{2.5}$$

Hence the acceleration has, in general, the radial component $(\ddot{r} - r\dot{\theta}^2)$ and the cross-radial (transverse) component $(2\dot{r}\dot{\theta} + r\ddot{\theta})$, which can be written as $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$. Substitute the right hand side of (2.5) in (2.1) to get

$$\begin{aligned}
&(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \\
&\quad = \Bigg[\sum_{m=1}^{n-1} \alpha_1\Big(-4\sin^2(\frac{m\pi}{2n})\Big)r\left(4\sin^2(\frac{m\pi}{2n})\,r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}} \\
&\qquad + \alpha_2(-2)r\left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2 r\left(r^2 + z^2\right)^{-\frac{3}{2}} \\
&\qquad + k_3 B r\,\dot{\theta}\Bigg]\mathbf{e}_r + k_3 B\dot{r}\,\mathbf{e}_\theta,
\end{aligned}$$

$$\begin{aligned}
\ddot{z} = &\sum_{m=1}^{n-1} \alpha_1 2\Big((-1)^m - 1\Big)z\left(4\sin^2(\frac{m\pi}{2n})\,r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}} \\
&+ \alpha_2\Big((-1)^n - 1\Big)z\left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2 z\left(r^2 + z^2\right)^{-\frac{3}{2}}.
\end{aligned}$$

Equate coefficients of $\mathbf{e}_r$, $\mathbf{e}_\theta$, since they are orthogonal coordinates, to obtain

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 = &\sum_{m=1}^{n-1} \alpha_1\Big(-4\sin^2(\frac{m\pi}{2n})\Big)r\left(4\sin^2(\frac{m\pi}{2n})\,r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}} \\
&+ \alpha_2(-2)r\left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2\left(r^2 + z^2\right)^{-\frac{3}{2}} + k_3 B r\dot{\theta},
\end{aligned} \tag{2.6}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = k_3 B\dot{r}, \tag{2.7}$$

$$\ddot{z} = \sum_{m=1}^{n-1} \alpha_1 2\Big((-1)^m - 1\Big) z \left(4\sin^2(\frac{m\pi}{2n})\, r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}}$$
$$+ \alpha_2\Big((-1)^n - 1\Big) z \left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2 z \left(r^2 + z^2\right)^{-\frac{3}{2}}.$$

From (2.7), we get,

$$\frac{d}{dt}(r^2\dot{\theta} - \frac{k_3 B r^2}{2}) = 0,$$

then

$$r^2\dot{\theta} - \frac{k_3 B r^2}{2} = h,$$

giving

$$\dot{\theta} = \frac{h}{r^2} + \frac{k_3 B}{2}.$$

Then, by eliminating $\dot{\theta}$ from (2.6) and noting that this results in a cancellation of the linear terms in the constant magnetic field, the equations of motion become a pair of second order non linear differential equations for r and z. We have

$$\ddot{z} = \sum_{m=1}^{n-1} \alpha_1\, 2\Big((-1)^m - 1\Big)\, z \left(4\sin^2(\frac{m\pi}{2n})\, r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}}$$
$$+ \alpha_2\Big((-1)^n - 1\Big)\, z \left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2 z \left(r^2 + z^2\right)^{-\frac{3}{2}},$$

$$\ddot{r} = \frac{h^2}{r^3} - \frac{k_3^2 B^2 r}{4} + \sum_{m=1}^{n-1} \alpha_1\Big(-4\sin^2(\frac{m\pi}{2n})\Big)\, r \left(4\sin^2(\frac{m\pi}{2n})\, r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}}$$
$$+ \alpha_2\, 2r \left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2 r \left(r^2 + z^2\right)^{-\frac{3}{2}}.$$

The energy equation becomes

$$
\begin{aligned}
E =& \frac{1}{2}\Big(\dot{r}^2 + \dot{z}^2 + \frac{h^2}{r^2} + \frac{k_3^2 B^2 r^2}{4}\Big) - \sum_{m=1}^{n-1} \alpha_1 \left(4\sin^2(\frac{m\pi}{2n})\, r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{1}{2}} \\
& - \alpha_2 \frac{1}{2} \left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{1}{2}} - 2nk_2 \Big(r^2 + z^2\Big)^{-\frac{1}{2}}.
\end{aligned}
$$

2.3 The derivation of the f-equation

In this section we shall rewrite the equation of motion and the energy equation as below

$$
\begin{aligned}
\ddot{z} &= zF_1(z^2, r) \\
\ddot{r} &= F_2(z^2, r, h) \\
E &= \frac{1}{2}(\dot{r}^2 + \dot{z}^2) + F_3(z^2, r, h), \text{ or} \\
g(z^2, r, h, E) &= 2E - 2F_3(z^2, r, h) = \dot{r}^2 + \dot{z}^2
\end{aligned}
$$

where F_1, F_2 and F_3 or g are functions of z, r and h. Here

$$
\begin{aligned}
F_1(z^2, r) =& \sum_{m=1}^{n-1} \alpha_1 2\Big((-1)^m - 1\Big)\left(4\sin^2(\frac{m\pi}{2n})\, r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}} \\
& + \alpha_2\Big((-1)^n - 1\Big)\left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2\Big(r^2 + z^2\Big)^{-\frac{3}{2}},
\end{aligned}
$$

$$
\begin{aligned}
F_2(z^2, r, h) =& \frac{h^2}{r^3} - \frac{k_3^2 B^2 r}{4} \\
& + \sum_{m=1}^{n-1} \alpha_1\Big(-4\sin^2(\frac{m\pi}{2n})\Big)\, r \left(4\sin^2(\frac{m\pi}{2n})\, r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{3}{2}} \\
& + \alpha_2\, 2r \left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{3}{2}} - 2nk_2 r\Big(r^2 + z^2\Big)^{-\frac{3}{2}}
\end{aligned}
$$

$$F_3(z^2,r,h) = \frac{1}{2}\Big(\frac{h^2}{r^2} + \frac{k_3^2 B^2 r^2}{4}\Big) + \sum_{m=1}^{n-1} \alpha_1 \left(4\sin^2(\frac{m\pi}{2n})\, r^2 + \Big((-1)^m - 1\Big)^2 z^2\right)^{-\frac{1}{2}}$$
$$+ \alpha_2 \frac{1}{2}\left(4r^2 + \Big((-1)^n - 1\Big)^2 z^2\right)^{-\frac{1}{2}} - 2nk_2\left(r^2 + z^2\right)^{-\frac{1}{2}}.$$

We shall now see how to approach the solution of the system of equations from an analytical point of view. We let

$$z^2 = f(r), \quad \text{where} \quad r^2 = x^2 + y^2. \tag{2.8}$$

Differentiating (2.8) with respect to t, gives

$$2z\dot{z} = f'(r)\dot{r} \ ,$$

so

$$\dot{z}^2 = \frac{f'^2(r)\dot{r}^2}{4f(r)}.$$

Differentiating again yields

$$2\dot{z}^2 + 2z\ddot{z} = f''(r)\dot{r}^2 + f'(r)\ddot{r}, \tag{2.9}$$

where

$$\dot{r}^2 = \frac{[2E - 2F_3(f(r),r,h)]}{1 + \big(\frac{f'^2(r)}{4f(r)}\big)}.$$

By substituting, we obtain

$$\Big(\frac{f'^2(r)}{2f(r)} - f''(r)\Big)\frac{[2E - 2F_3(f(r),r,h)]}{1 + \big(\frac{f'^2(r)}{4f(r)}\big)} = f'(r)F_2(f(r),r,h) - 2f(r)F_1(f(r),r). \tag{2.10}$$

Equation (2.10) is a second-order nonlinear ordinary differential equation which we call the f-equation. The equations of motion have now been reduced to a second order non linear ordinary differential equation, where F_1, F_2, and F_3 are all given. We now find the initial conditions on f corresponding to the initial conditions $\mathbf{q}_1(0) = \mathbf{i}$ and $\dot{\mathbf{q}}_1(0) = \dot{y}_0\mathbf{j} + \dot{z}_0\mathbf{k}$.

Now the initial conditions on f are

$$f(1) = 0, \qquad f'(1) = \frac{2\dot{z}_0^2}{F_2(0,1,h)}.$$

From the energy equation, we obtain

$$E = \frac{1}{4} f'(1) F_2(0,1,h) + F_3(0,1,h).$$

Consider $z^2 = f(r)$, giving, $2z\dot{z} = f'(r)\dot{r}$ and $\ddot{z} = zF_1(z^2, r)$. We have therefor

$$2\dot{z}\ddot{z} = 2z\dot{z}F_1(z^2, r), \quad \frac{d}{dt}(\dot{z}^2) = f'(r)\dot{r}F_1(f(r), r).$$

Now integrate on both sides, to get

$$\dot{z}^2(t(z)) - \dot{z}^2(0) = \int_1^{r(z)} f'(r)F_1(f(r), r)\, dr,$$

where $t(z)$ is the inverse function of $z(t)$ and $r(z)$ is the inverse function of $z = \sqrt{f(r)}$. Now, define $F(z)$ by

$$F(z) = \dot{z}^2(0) + \int_1^{r(z)} f'(r)F_1(f(r), r)\, dr.$$

The period of the z-motion is

$$P_z = 4\int_0^{z_m} \Big(F(z)\Big)^{-\frac{1}{2}} dz,$$

where z_m satisfies

$$\dot{z}^2(0) + 2\int_0^{z_m} zF_1(z^2, r(z))\, dz = 0.$$

The radial period is

$$P_r = 2\int_{r_m}^1 \Big(G(r)\Big)^{-\frac{1}{2}} dr,$$

where

$$G(r) = \frac{4f(r)[2E - 2F_3(f(r), r, h)]}{\Big(f'^2(r) + 4f(r)\Big)}.$$

The angular distance is

$$P_\theta = h\int_0^{P_r} (r(t))^{-2}\,dt,$$

where $r(t)$ satisfies

$$t = \int_{r(t)}^{1} \Big(G(r)\Big)^{-\frac{1}{2}}\,dr.$$

2.4 Examples

2.4.1 Harmonic oscillator

We shall give an example to show how to calculate the important characteristics of the solution of the equations of motion generated by the f-equation by employing the simple harmonic oscillator. Consider the equations of motion

$$\ddot{x} = -\omega_1^2 x,\ \ \ddot{y} = -\omega_1^2 y,\ \ \ddot{z} = -\omega_2^2 z,$$

with initial position and velocity $x(0) = 1,\ y(0) = 0,\ z(0) = 0$, and $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0,\ \ \dot{z}(0) = \dot{z}_0$, respectively. The energy equation is

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}(\omega_1^2 x^2 + \omega_1^2 y^2 + \omega_2^2 z^2).$$

By using cylindrical polar coordinates, and resolving into radial and cross-radial resolutes, we obtain

$$\ddot{r} = \frac{h^2}{r^3} - \omega_1^2 r,$$

$$\ddot{z} = -\omega_2^2 z.$$

The energy equation becomes

$$E = \frac{1}{2}(\dot{r}^2 + \dot{z}^2 + \frac{h^2}{r^2}) + \frac{1}{2}(\omega_1^2 r^2 + \omega_2^2 z^2).$$

In this case we have

$$F_1(z^2, r) = -\omega_2^2,\ \ F_2(z^2, r, h) = \frac{h^2}{r^3} - \omega_1^2 r,\ \ F_3(z^2, r, h) = \frac{1}{2}\left(\frac{h^2}{r^2} + \omega_1^2 r^2 + \omega_2^2 z^2\right),$$

where F_1, F_2 and F_3 or g are functions of z, r, h. As before we obtain the f-equation with the usual initial conditions. That is

$$\Big(\frac{f'^2(r)}{2f(r)} - f''(r)\Big)\Big(h^2 + \omega_1^2 - \omega_1^2 r^2 - \frac{h^2}{r^2}\Big) = f'(r)\Big(\frac{h^2}{r^3} - \omega_1^2 r\Big) - 2f(r)\Big(-\omega_2^2\Big),$$

with

$$f(1) = 0, \qquad f'(1) = \frac{2\dot{z}_0^2}{(h^2 - \omega_1^2)} \quad \text{where} \quad h = \dot{y}_0.$$

In this simple case

$$\begin{aligned} F(z) =& \dot{z}^2(t(z)) \\ =& \dot{z}_0^2 - \omega_2^2 \int_1^{r(z)} f'(r)\,dr = \dot{z}_0^2 - \omega_2^2\big[f(r(z)) - f(1)\big] = \dot{z}_0^2 - \omega_2^2 z^2. \end{aligned}$$

The period of the z-motion is

$$\begin{aligned} P_z &= 4\int_0^{z_m} (F(z))^{-\frac{1}{2}}\,dz = 4\int_0^{z_m} \Big(\dot{z}_0^2 - \omega_2^2 \int_1^{r(z)} f'(r)\,dr\Big)^{-\frac{1}{2}} dz \\ &= 4\int_0^{z_m} \Big(\dot{z}_0^2 - \omega_2^2 z^2\Big)^{-\frac{1}{2}} dz = \frac{4}{\omega_2}\Big[\sin^{-1}\big(\frac{\omega_2 z_m}{\dot{z}_0}\big)\Big] \\ &= \frac{2\pi}{\omega_2} \quad \text{as expected,} \end{aligned}$$

since z_m is defined by

$$\dot{z}_0^2 = -2\int_0^{z_m} zF_1(z^2, r)\,dz = \omega_2^2\, z_m^2.$$

The radial period is

$$P_r = 2\int_{r_m}^1 \Big(G(r)\Big)^{-\frac{1}{2}} dr = \frac{\pi}{\omega_1},$$

where

$$G(r) = \dot{r}^2 = h^2 + \omega_1^2 - \frac{h^2}{r^2} - \omega_1^2 r^2.$$

The angular period is

$$P_\theta = h\int_0^{P_r} \Big(r(t)\Big)^{-2} dt = \omega_1 \pi,$$

where

$$r(t) = \cos^2(\omega_1 t) + \frac{h^2}{\omega_1^2}\sin^2(\omega_1 t).$$

In this simple example we easily calculated the important characteristics of the solution of the equation of motion generated by the f-equation which are P_z, P_r and P_θ.

2.4.2 The two particle problem

Consider the motion of two particles of unit masses subject to their mutual gravitational attractions with positions $\mathbf{q}_i \in \mathbb{R}^3$, $i = 1, 2$. The equation of motion of $\mathbf{q}_1$ subject to a suitable choice of units is

$$\ddot{\mathbf{q}}_1 = \frac{\mathbf{q}_2 - \mathbf{q}_1}{|\mathbf{q}_2 - \mathbf{q}_1|^3},$$

with $P = -I$, $P^2 = I$, $\det P = -1$. P has an eigenvalue -1 of algebraic multiplicity 3, and so we can take the eigenvectors corresponding to -1 as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. The equation of motion of one particle can be writen as

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \text{ where } \mu = \frac{1}{4},$$

with initial conditions $\mathbf{r}(0) = \mathbf{i}$ and $\dot{\mathbf{r}}(0) = \dot{y}_0\,\mathbf{j} + \dot{z}_0\,\mathbf{k}$. As one expects the path of the particle is an ellipse with semi-major and semi-minor axes a and b respectively. For periodic orbits we get,

$$r = \frac{l}{1 - e\cos\theta}, \qquad l = \frac{\dot{y}_0{}^2 + \dot{z}_0{}^2}{\mu} \quad \text{where} \quad \mu = \frac{1}{4}, \ e = 1 - l,$$

we have $a = \frac{1}{2-l}$, $b^2 = al = r_{min}$ (since $r_{max} = 1$). By using cylindrical polar coordinates, and resolving into radial and cross-radial resolutes, we obtain

$$\ddot{z} = -\mu z\left(r^2 + z^2\right)^{-\frac{3}{2}},$$

$$\ddot{r} = \frac{h^2}{r^3} - \mu r\left(r^2 + z^2\right)^{-\frac{3}{2}}.$$

The energy equation is

$$E = \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2\right) + \frac{1}{2}\frac{h^2}{r^2} - \mu\left(r^2 + z^2\right)^{-\frac{1}{2}}.$$

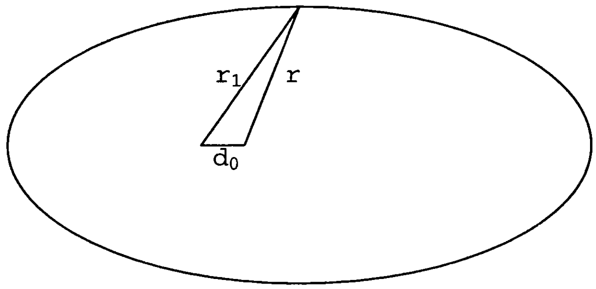


Figure 2.1.

Now let us now concentrate on the relation between $r = \sqrt{x^2+y^2}$ and z. The path of the particle projected onto the plane $z = 0$ is again an ellipse as in figure 2.1, with semi-major axis equal to that of the physical path but different semi-minor axis. That is, differing e and l. Let a_1 be semi-major axis, b_1 be semi-minor axis, l_1 be the semi latus rectum of the ellipse projected onto the plane $z = 0$. We have

$$a_1 = a \quad \text{but} \quad b_1 = \frac{\dot{y_0}}{\sqrt{\dot{y_0}^2 + \dot{z_0}^2}}\, b,$$
$$\text{and} \quad l_1 = \frac{b_1^2}{a_1} = \frac{\dot{y_0}^2}{\dot{y_0}^2 + \dot{z_0}^2}\, l = \frac{\dot{y_0}^2}{\mu}.$$

The f-equation in this case is

$$\Big(\frac{f'^2(r)}{2f(r)} - f''(r)\Big)\frac{\Big(2E - \frac{h^2}{r^2} + 2\mu\big(r^2 + f(r)\big)^{-\frac{1}{2}}\Big)}{1 + \Big(\frac{f'^2(r)}{4f(r)}\Big)}$$
$$= f'(r)\Big(\frac{h^2}{r^3} - \mu r\big(r^2 + f(r)\big)^{-\frac{3}{2}}\Big) - 2f(r)\Big(- \mu\big(r^2 + f(r)\big)^{-\frac{3}{2}}\Big),$$

with

$$f(1) = 0, \qquad f'(1) = \frac{2\dot{z}_0^2}{(h^2 - \mu)} \quad \text{where } h = \dot{y}_0, \ \mu = \frac{1}{4}.$$

From figure 2.1, to state, explicity, $f(r)$ for this example, given r_1 we shall get z as

$$z = \frac{\dot{z_0}}{\dot{y_0}}\, r_1 \sin\theta,$$

where r_1 satisfies

$$r^2 = d_0^2 + r_1^2 - 2r_1 d_0 \cos\theta, \tag{2.11}$$

or

$$r_1 = d_0 \cos\theta + \sqrt{d_0^2 \cos\theta + r^2 - d_0^2},$$

Equation (2.11) can be written as

$$r_1^2 - 2\frac{d_0}{e_1} r_1 + (d_0^2 - r^2 + 2\frac{d_0\, l_1}{e_1}) = 0, \quad \text{where } d_0 = a(e_1 - e).$$

Note that

$$\frac{dr_1}{dr} = \frac{r}{r_1 - d_0/e_1}.$$

We observe that

$$f(r) = \frac{\dot{y_0}^2}{\dot{z_0}^2}\Big(r_1^2 - (\frac{r_1 - l_1}{e_1})^2\Big),$$

where

$$r_1 = d_0/e_1 + \sqrt{(d_0/e_1)^2 + r^2 - d_0^2 - 2l_1 d_0/e_1} \text{ with } l_1 = \dot{y}_0^2/\mu,$$
$$d_0 = a(e_1 - e),\ e_1 = 1 - b_1^2/a^2,\ b_1 = \dot{y}_0\, b/\sqrt{\dot{y}_0^2 + \dot{z}_0^2},$$

which satisfies

$$f(1) = 0 \quad \text{and} \quad f'(1) = \frac{2\dot{z_0}^2}{\dot{y_0}^2 - \mu}.$$

Note that from the geometry of the ellipse we calculate that,

$$r_{min} = \frac{h^2}{2\mu - h^2},$$

where $h^2 = \dot{y_0}^2 + \dot{z_0}^2$ and $r_1 = r_{min} - d_0$, we get,

$$\begin{aligned} f'(r_{min}) &= \frac{2\dot{z_0}^2}{\dot{y_0}^2}\left(\frac{1}{\dfrac{1}{r_{min}} - \dfrac{1}{l_1}}\right) = \frac{2\dot{z_0}^2}{\dot{y_0}^2}\left(\frac{r_{min} l_1}{l_1 - r_{min}}\right) \\ &= \frac{2\dot{z_0}^2}{\dot{y_0}^2}\left(\frac{\dot{y_0}^2 h^2}{\dot{y_0}^2(2\mu - h^2) - h^2\mu}\right). \end{aligned}$$

This leads to several different zeros, which means $f'(r_{min})$ could be negative, infinity, or positive.

2.5 Numerical solution of f-equation

The numerical solutions (Runge-Kutta scheme) of the f-equation with the initial conditions for the four examples respectively as in chapter one are illustrated below. These diagrams agree with the numerical solutions (Runge-Kutta scheme) of the reduced system of the same problems. These diagrams were created using Mathematica code and will be presented in Chapter 6.

2.5.1 Four ion problem

For the four node solution we had $\dot{y}_0 = 0.419778769$ and $\dot{z}_0 = 0.600918801$ giving $f_0'(1) = -2.571113998$ ($f(1) = 0$ as standard). The diagram below illustrates $f(r)$ for $r_m \leq r \leq 1$, $r_m = r_{min} = 0.866$.

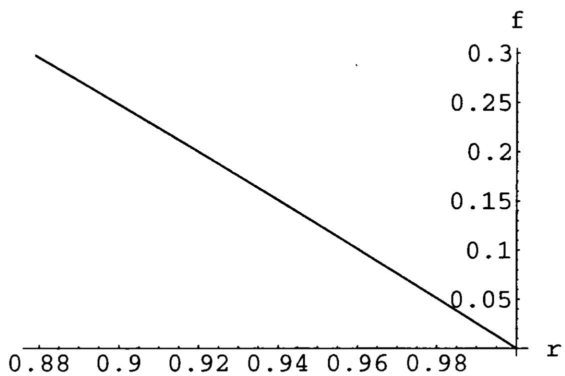


Figure 2.2:the f-solution for the 4ip

2.5.2 Four electron atom problem

For the four node solution we had $\dot{y}_0 = 0.126889159$ and $\dot{z}_0 = 1.816605346$ giving $f_0'(1) = -2.180573661$ ($f(1) = 0$ as standard). The diagram below illustrates $f(r)$ for $r_m \leq r \leq 1$, $r_m = r_{min} = 0.171$.

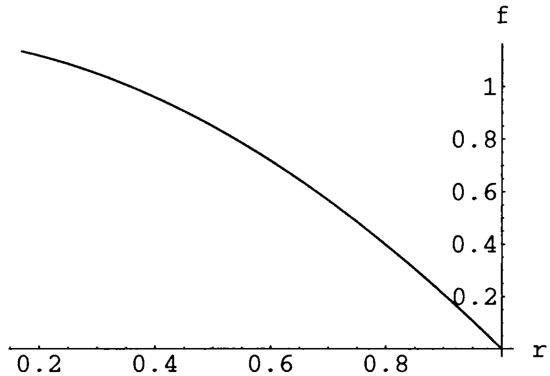


Figure 2.3:the f-solution for the 4eap

2.5.3 Four electron atom problem with constant magnetic field

For the four node solution we had $\dot{y}_0 = 0.328117187$ and $\dot{z}_0 = 1.823429687$ giving $f_0'(1) = -2.218274657$ ($f(1) = 0$ as standard). The diagram below illustrates $f(r)$ for $r_m \leq r \leq 1$, $r_m = r_{min} = 0.157$.

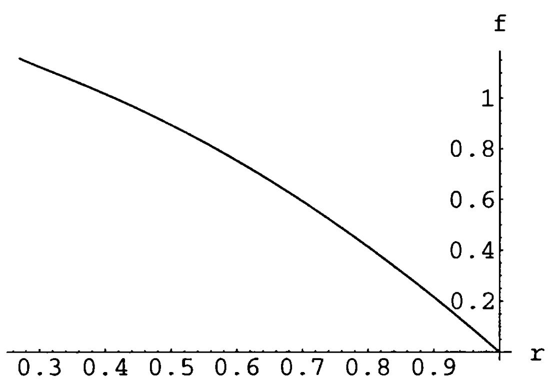


Figure 2.4:the f-solution for the 4eapB

2.5.4 Four body gravitational problem

For the four node solution we had $\dot{y}_0 = 0.33250244$ and $\dot{z}_0 = 0.841783691$ giving $f_0'(1) = -1.675076607$ ($f(1) = 0$ as standard). The diagram below illustrates $f(r)$ for $r_m \leq r \leq 1$, $r_m = r_{min} = 0.219$.

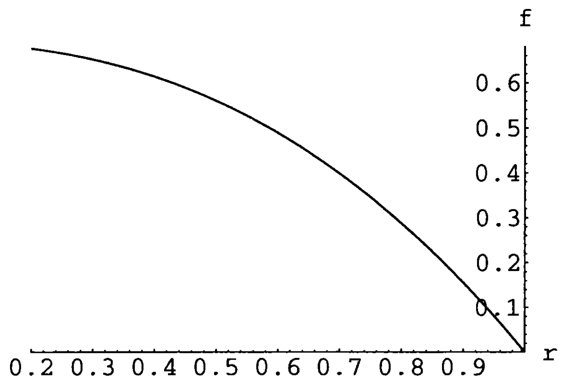


Figure 2.5:the f-solution for the 4bgp

Chapter 3

LINEAR STABILITY OF THE f-EQUATION

3.1 Introduction

In this chapter we are going to study the linear stability of the f-equation. We assume a known solution, a positive function, $f_0(r)$, where $f_0(1) = 0$, $f_0'(1) < 0$, investigating whether there is a nearby solution of the same kind with the same E and h and we will consider a small displacement from $f_0(r)$, $\epsilon(r)$. We will obtain the general form of the ϵ-equation from the f-equation by employing the customary linearisation techniques. We shall give examples of the four electron atom problem with constant magnetic field and without constant magnetic field, the four body gravitational problem and the four ion problem. We will also develop the local behaviour of $\epsilon(r)$ near and at $r = 1$. This will result in a convergent series solutions (Frobenius series solutions) where the leading terms in the series solutions are solutions of the Cauchy-Euler equation. We will also give two special cases, one of them can be expressed in terms of Bessel functions, the other can be expressed in terms of Hypergeometric functions. These techniques can be applied to the n-electron atom problem with (without) constant magnetic field, the $2n$-gravitational body problem and the $2n$-ion problem. They may also applied to the restricted three body problem and the Lagrange solutions, the hip-hop solution. In general it can be applied to any conservative Hamiltonian system that can be reduced to Cylindrical polar form and that can be reduced using the DTW-method (the f-equation).

3.2 The linearisation of the f-equation

3.2.1 The general ϵ-equation

We shall write the f-equation in the form

$$(\frac{1}{2}f'^2 - ff'')[2E - 2F_3] = (\frac{1}{4}f'^2 + f)[f'F_2 - 2fF_1] \qquad (3.1)$$

Suppose f_0 is a solution for (3.1), where f_0 is a positive function and $f_0(1) = 0$, $f_0'(1) < 0$. Now consider a small displacement from $f = f_0$, then, $f = f_0 + \epsilon$, where ϵ is small. Differentiate twice with respect to r, to get $f' = f_0' + \epsilon'$ and $f'' = f_0'' + \epsilon''$. We will use the traditional linearisation argument for equation (3.1) where $f = f_0 + \epsilon$. We shall do this by considering each of the four parts separately. First part of the linearsation becomes

$$(\frac{1}{2}f'^2 - ff'') = (\frac{1}{2}f_0'^2 - f_0f_0'') + f_0'\epsilon' - f_0\epsilon'' - f_0''\epsilon + O(\epsilon^2).$$

Here $O(\epsilon^2)$ means terms of second order or above in ϵ, ϵ', ϵ'' and products thereof. The second part becomes

$$[2E - 2F_3(f(r), r, h)] = [2E - 2F_3(f_0(r), r, h)] + \left.\frac{\partial[2E - 2F_3(f(r), r, h)]}{\partial f}\right|_{f_0} \epsilon + O(\epsilon^2).$$

The third part becomes

$$(\frac{1}{4}f'^2 + f) = (\frac{1}{4}f_0'^2 + f_0) + \frac{1}{2}f_0'\epsilon' + \epsilon + O(\epsilon^2).$$

The last part becomes

$$[f'F_2(f(r), r, h) - 2fF_1(f(r), r)] = [f_0'F_2(f_0(r), r, h) - 2f_0F_1(f_0(r), r)]$$

$$+ F_2(f_0(r), r, h)\epsilon' + \left[f_0'\left.\frac{\partial F_2(f(r), r, h)}{\partial f}\right|_{f_0} - 2F_1(f_0(r), r) - 2f_0\left.\frac{\partial F_1(f(r), r)}{\partial f}\right|_{f_0}\right]\epsilon + O(\epsilon^2).$$

Let us gather the terms together beginning with the l.h.s. of (3.1),

$$
\begin{aligned}
(\frac{1}{2}f'^2 - ff'')[2E - 2F_3] &= (\frac{1}{2}f_0'^2 - f_0 f_0'')[2E - 2F_3(f_0(r), r, h)] \\
&- f_0[2E - 2F_3(f_0(r), r, h)]\epsilon'' + f_0'[2E - 2F_3(f_0(r), r, h)]\epsilon' \\
&+ \left\{ (\frac{1}{2}f_0'^2 - f_0 f_0'') \frac{\partial[2E - 2F_3(f(r), r, h)]}{\partial f}\bigg|_{f_0} - f_0''[2E - 2F_3(f_0(r), r, h)] \right\}\epsilon \\
&+ O(\epsilon^2).
\end{aligned}
\tag{3.2}
$$

Now the right hand side of the f-equation (3.1) becomes

$$
\begin{aligned}
(\frac{1}{4}f'^2 + f)[f'F_2 - 2fF_1] &= (\frac{1}{4}f_0'^2 + f_0)[f_0'F_2(f_0(r), r, h) - 2f_0 F_1(f_0(r), r)] \\
&+ \left[f_0 F_2(f_0(r), r, h) + \frac{3}{4}f_0'^2 F_2(f_0(r), r, h) - f_0 f_0' F_1(f_0(r), r) \right]\epsilon' \\
&+ \Bigg\{ [f_0' F_2(f_0(r), r, h) - 2f_0 F_1(f_0(r), r)] \\
&+ (\frac{1}{4}f_0'^2 + f_0)\left[f_0' \frac{\partial F_2(f(r), r, h)}{\partial f}\bigg|_{f_0} - 2F_1(f_0(r), r) - 2f_0 \frac{\partial F_1(f(r), r)}{\partial f}\bigg|_{f_0} \right] \Bigg\}\epsilon \\
&+ O(\epsilon^2).
\end{aligned}
\tag{3.3}
$$

The coefficient of ϵ' in (3.3) is

$$
f_0 F_2(f_0(r), r, h) + \frac{3}{4}f_0'^2 F_2(f_0(r), r, h) - f_0 f_0' F_1(f_0(r), r).
$$

By adding and subtracting $\frac{1}{2}f_0 f_0' F_1(f_0(r), r)$, we get

$$
\begin{aligned}
f_0 &\left[F_2(f_0(r), r, h) + \frac{1}{2}f_0' F_1(f_0(r), r) \right] \\
&+ \frac{3}{4}f_0'^2 F_2(f_0(r), r, h) - \frac{3}{2}f_0 f_0' F_1(f_0(r), r),
\end{aligned}
$$

or

$$f_0\left[F_2(f_0(r),r,h)+\frac{1}{2}f_0'F_1(f_0(r),r)\right]$$
$$+\frac{3}{4}f_0'\left[f_0'F_2(f_0(r),r,h)-2f_0F_1(f_0(r),r)\right].$$

Recall the equation of motion and the energy equation from previous chapter,

$$\ddot{z}=zF_1(z^2,r)$$
$$\ddot{r}=F_2(z^2,r,h)$$
$$E=\frac{1}{2}(\dot{r}^2+\dot{z}^2)+F_3(z^2,r,h),\text{ or}$$
$$g=g(z^2,r,h,E)=2E-2F_3(z^2,r,h)=\dot{r}^2+\dot{z}^2.$$

Then

$$\dot{r}\ddot{r}+\dot{z}\ddot{z}=\dot{r}F_2(z^2,r,h)+z\dot{z}F_1(z^2,r),$$

where $z^2=f(r)$ gives $z\dot{z}=\frac{1}{2}f'(r)\dot{r}$, yielding

$$\dot{r}\ddot{r}+\dot{z}\ddot{z}=\left[F_2(f(r),r,h)+\frac{1}{2}f'(r)F_1(f(r),r)\right]\dot{r},$$

$$\frac{1}{2}(\dot{r}^2+\dot{z}^2)=\int\left[F_2(f(r),r,h)+\frac{1}{2}f'(r)F_1(f(r),r)\right]dr,$$

and so

$$\frac{1}{2}g(f(r),r,h,E)=\int\left[F_2(f(r),r,h)+\frac{1}{2}f'(r)F_1(f(r),r)\right]dr.$$

Then

$$g'=\frac{dg(f(r),r,h,E)}{dr}=2\left[F_2(f(r),r,h)+\frac{1}{2}f'(r)F_1(f(r),r)\right],$$

or

$$\frac{dF_3(f(r),r,h)}{dr}=-\left[F_2(f(r),r,h)+\frac{1}{2}f'(r)F_1(f(r),r)\right].$$

Also we have

$$g' = \frac{dg(f(r), r, h, E)}{dr} = \frac{\partial g(f(r), r, h, E)}{\partial r} + f'(r)\frac{\partial g(f(r), r, h, E)}{\partial f}.$$

Then,

$$\frac{\partial g(f(r), r, h, E)}{\partial r} = 2F_2(f(r), r, h),$$
$$\frac{\partial g(f(r), r, h, E)}{\partial f} = F_1(f(r), r).$$

This gives

$$\frac{\partial^2 g(f(r), r, h, E)}{\partial r \partial f} = 2\frac{\partial F_2(f(r), r, h)}{\partial f},$$
$$\frac{\partial^2 g(f(r), r, h, E)}{\partial r^2} = 2\frac{\partial F_2(f(r), r, h)}{\partial r},$$
$$\frac{\partial^2 g(f(r), r, h, E)}{\partial f^2} = \frac{\partial F_1(f(r), r)}{\partial f},$$
$$\frac{\partial^2 g(f(r), r, h, E)}{\partial f \partial r} = \frac{\partial F_1(f(r), r)}{\partial r}.$$

Then we have

$$2\frac{\partial F_2(f(r), r, h)}{\partial f} = \frac{\partial F_1(f(r), r)}{\partial r} = r\,\frac{\partial F_1(f(r), r)}{\partial f},$$
$$\frac{\partial g(f(r), r, h, E)}{\partial r \partial f} = r\,\frac{\partial^2 g(f(r), r, h, E)}{\partial f^2},$$

and

$$F_2' = \frac{dF_2(f(r), r, h)}{dr} = \frac{\partial F_2(f(r), r, h)}{\partial r} + f'(r)\frac{\partial F_2(f(r), r, h)}{\partial f},$$
$$F_1' = \frac{dF_1(f(r), r)}{dr} = \frac{\partial F_1(f(r), r)}{\partial r} + f'(r)\frac{\partial F_1(f(r), r)}{\partial f},$$

giving

$$g'' = \frac{d^2 g(f(r), r, h, E)}{dr^2} = \frac{\partial^2 g(f(r), r, h, E)}{\partial r^2} + f_0''(r)\frac{\partial g(f(r), r, h, E)}{\partial f}$$
$$+ f_0'(r)\big(2r + f_0'(r)\big)\frac{\partial^2 g(f(r), r, h, E)}{\partial f^2}.$$

The equality of (3.2) and (3.3) gives a linear differential equation of second order of the form,

$$f_0[2E-2F_3(f_0(r),r,h)]\epsilon'' + \Big[f_0[F_2(f_0(r),r,h)+\frac{1}{2}f_0'F_1(f_0(r),r)]$$

$$-f_0'[2E-2F_3(f_0(r),r,h)]+\frac{3}{4}f_0'[f_0'F_2(f_0(r),r,h)-2f_0F_1(f_0(r),r)]\Big]\epsilon'$$

$$+\Big\{f_0''[2E-2F_3(f_0(r),r,h)]+[f_0'F_2(f_0(r),r,h)-2f_0F_1(f_0(r),r)]$$

$$+(\frac{1}{4}f_0'^2+f_0)\Big[f_0'\frac{\partial F_2(f(r),r,h)}{\partial f}\Big|_{f_0}-2F_1(f_0(r),r)-2f_0\frac{\partial F_1(f(r),r)}{\partial f}\Big|_{f_0}\Big]$$

$$-(\frac{1}{2}f_0'^2-f_0f_0'')\frac{\partial[2E-2F_3(f(r),r,h)]}{\partial f}\Big|_{f_0}\Big\}\epsilon=0.$$

This last equation can be written in the form

$$f_0(r)g(f_0(r),r,h,E)\epsilon''+\Big[\frac{1}{2}f_0(r)g'(f_0(r),r,h,E)-f_0'(r)g(f_0(r),r,h,E)$$

$$+\frac{3}{4}f_0'(r)\Big(\frac{1}{2}f_0'(r)\frac{\partial g(f(r),r,h,E)}{\partial r}\Big|_{f_0}-2f_0(r)\frac{\partial g(f(r),r,h,E)}{\partial f}\Big|_{f_0}\Big)\Big]\epsilon'$$

$$+\frac{1}{f_0(r)g(f_0(r),r,h,E)}\Big\{\Big(\frac{1}{2}f_0'^2(r)g(f_0(r),r,h,E)-\big(\frac{1}{4}f_0'^2(r)+f_0(r)\big)$$

$$\Big(\frac{1}{2}f_0'(r)\frac{\partial g(f(r),r,h,E)}{\partial r}\Big|_{f_0}-2f_0(r)\frac{\partial g(f(r),r,h,E)}{\partial f}\Big|_{f_0}\Big)\Big)$$

$$\Big(g(f_0(r),r,h,E)+f_0(r)\frac{\partial g(f(r),r,h,E)}{\partial f}\Big|_{f_0}\Big)$$

$$+f_0(r)g(f_0(r),r,h,E)\Big(\frac{1}{2}f_0'(r)\frac{\partial g(f(r),r,h,E)}{\partial r}\Big|_{f_0}$$

$$+(\frac{1}{4}f_0'^2(r)+f_0(r))\Big[\big(\frac{1}{2}rf_0'(r)-2f_0(r)\big)\frac{\partial^2 g(f(r),r,h,E)}{\partial f^2}\Big|_{f_0}$$

$$-4\frac{\partial g(f(r),r,h,E)}{\partial f}\Big|_{f_0}\Big]\Big)\Big\}\epsilon=0. \tag{3.4}$$

From the f-equation we have (since f_0 is a solution)

$$\frac{1}{g(f_0(r),r,h,E)}\left(\frac{1}{2}f_0'(r)\frac{\partial g(f(r),r,h,E)}{\partial r}\bigg|_{f_0} - 2f_0(r)\frac{\partial g(f(r),r,h,E)}{\partial f}\bigg|_{f_0}\right) = \frac{(\frac{1}{2}{f_0}'^2(r) - f_0(r){f_0}''(r))}{(\frac{1}{4}f_0'^2(r) + f_0(r))}.$$

Since ${f_0}'^2(r)/(4f_0(r)) > 0$, we can write,

$$-\frac{1}{2}\frac{f_0'(r)}{f_0(r)}\left(\frac{\frac{1}{2}{f_0}'^2(r) - f_0(r){f_0}''(r)}{\frac{1}{4}f_0'^2(r) + f_0(r)}\right) = \frac{(1+\frac{{f_0}'^2(r)}{4f_0(r)})'}{(1+\frac{{f_0}'^2(r)}{4f_0(r)})} = \frac{d}{dr}\left(\ln(1+\frac{{f_0}'^2(r)}{4f_0(r)})\right).$$

Now we are able to write equation (3.4) in the form

$$\begin{aligned}
&\epsilon'' + \left[\frac{g'(f_0(r),r,h,E)}{2g(f_0(r),r,h,E)} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1+\frac{{f_0}'^2(r)}{4f_0(r)})'}{(1+\frac{{f_0}'^2(r)}{4f_0(r)})}\right]\epsilon' \\
&+ \frac{1}{f_0^2(r)g^2(f_0(r),r,h,E)}\Bigg\{\Bigg(\frac{1}{2}f_0'^2(r)g(f_0(r),r,h,E) - \Big(\frac{1}{4}f_0'^2(r) + f_0(r)\Big) \\
&\qquad\qquad \left(\frac{1}{2}f_0'(r)\frac{\partial g(f(r),r,h,E)}{\partial r}\bigg|_{f_0} - 2f_0(r)\frac{\partial g(f(r),r,h,E)}{\partial f}\bigg|_{f_0}\right)\Bigg) \\
&\qquad \left(g(f_0(r),r,h,E) + f_0(r)\frac{\partial g(f(r),r,h,E)}{\partial f}\bigg|_{f_0}\right) \\
&+ f_0(r)g(f_0(r),r,h,E)\Bigg(\frac{1}{2}f_0'(r)\frac{\partial g(f(r),r,h,E)}{\partial r}\bigg|_{f_0} \\
&+ (\frac{1}{4}f_0'^2(r) + f_0(r))\Bigg[\Big(\frac{1}{2}rf_0'(r) - 2f_0(r)\Big)\frac{\partial^2 g(f(r),r,h,E)}{\partial f^2}\bigg|_{f_0} \\
&\quad - 4\frac{\partial g(f(r),r,h,E)}{\partial f}\bigg|_{f_0} - 2f_0\frac{\partial^2 g(f(r),r,h,E)}{\partial f^2}\bigg|_{f_0}\Bigg]\Bigg)\Bigg\}\epsilon = 0.
\end{aligned} \tag{3.5}$$

Multiplying equation (3.5) by the integrating factor

$$\psi_1(r) = \left(\frac{g(f_0(r), r, h, E)}{f_0^2(r)(1 + \frac{{f_0}'^2(r)}{4f_0(r)})^3} \right)^{\frac{1}{2}},$$

where

$$\phi(r) = \left[\frac{g'}{2g} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2} \frac{(1 + \frac{{f_0}'^2(r)}{4f_0(r)})'}{(1 + \frac{{f_0}'^2(r)}{4f_0(r)})} \right] = \frac{d}{dr}\left(\ln(\frac{g^{\frac{1}{2}}}{f_0(r)\left(1 + \frac{{f_0}'^2(r)}{4f_0(r)}\right)^{\frac{3}{2}}}) \right),$$

$$g = g(f_0(r), r, h, E) = [2E - 2F_3(f_0, r, h)] \geq 0, \ \left(E \geq F_3(f_0(r), r, h) \right),$$

the sum of the first two terms on the left of (3.5) can be written as an exact derivative, that is,

$$\begin{aligned}
&\frac{d}{dr}\left(\left(\frac{g(f_0(r), r, h, E)}{f_0^2(r)(1 + \frac{{f_0}'^2(r)}{4f_0(r)})^3} \right)^{\frac{1}{2}} \frac{d\epsilon}{dr} \right) + \left(\frac{g(f_0(r), r, h, E)}{f_0^2(r)(1 + \frac{{f_0}'^2(r)}{4f_0(r)})^3} \right)^{\frac{1}{2}} \\
&\quad + \frac{1}{f_0(r) g(f_0(r), r, h, E)} \Bigg\{ \Bigg(\frac{1}{2} f_0'^2(r) g(f_0(r), r, h, E) - \left(\frac{1}{4} f_0'^2(r) + f_0(r) \right) \\
&\qquad \left(\frac{1}{2} f_0'(r) \frac{\partial g(f(r), r, h, E)}{\partial r}\bigg|_{f_0} - 2 f_0(r) \frac{\partial g(f(r), r, h, E)}{\partial f}\bigg|_{f_0} \right) \Bigg) \\
&\qquad \left(g(f_0(r), r, h, E) + f_0(r) \frac{\partial g(f(r), r, h, E)}{\partial f}\bigg|_{f_0} \right) \\
&\quad + f_0(r) g(f_0(r), r, h, E) \Bigg(\frac{1}{2} f_0'(r) \frac{\partial g(f(r), r, h, E)}{\partial r}\bigg|_{f_0} \\
&\quad + (\frac{1}{4} f_0'^2(r) + f_0(r)) \Bigg[\left(\frac{1}{2} r f_0'(r) - 2 f_0(r) \right) \frac{\partial^2 g(f(r), r, h, E)}{\partial f^2}\bigg|_{f_0} \\
&\qquad - 4 \frac{\partial g(f(r), r, h, E)}{\partial f}\bigg|_{f_0} - 2 f_0 \frac{\partial^2 g(f(r), r, h, E)}{\partial f^2}\bigg|_{f_0} \Bigg] \Bigg) \Bigg\} \epsilon = 0.
\end{aligned} \tag{3.6}$$

Equation (3.6) can be written in the form

$$\frac{d}{dr}\left(\psi_1(r)\frac{d\epsilon(r)}{dr}\right) + \psi(r)\epsilon(r) = 0, \tag{3.7}$$

where $\psi_1(r)$, $\psi(r)$ are functions of r such that

$$\psi(r) = \psi_1(r)\psi_2(r),$$

where

$$\psi_1(r) = \left(\frac{g(f_0(r), r, h, E)}{f_0^2(r)(1 + \frac{{f_0}'^2(r)}{4f_0(r)})^3}\right)^{\frac{1}{2}},$$

with

$$g = g(f_0(r), r, h, E) = [2E - 2F_3(f_0(r), r, h)] \text{ and}$$

$$\psi_2(r) = \frac{b(r)}{f_0^2(r)g^2(f_0(r), r, h, E)},$$

with

$$b(r) = \left(\frac{1}{2}f_0'^2(r)g(f_0(r), r, h, E) - \left(\frac{1}{4}f_0'^2(r) + f_0(r)\right)\right.$$

$$\left(f_0'(r)g_1(f_0(r), r, h) - 2f_0(r)g_2(f_0(r), r)\right)\Bigg)\left(g(f_0(r), r, h, E) + f_0(r)g_2(f_0(r), r)\right)$$

$$+ f_0(r)g(f_0(r), r, h, E)\Bigg(f_0'(r)g_1(f_0(r), r, h)$$

$$\left. + (\frac{1}{4}f_0'^2(r) + f_0(r))\left[\left(\frac{1}{2}rf_0'(r) - 2f_0(r)\right)g_3(f_0(r), r) - 4g_2(f_0(r), r)\right]\right),$$

so that,

$$
\begin{aligned}
g_1 &= g_1(f_0(r), r, h) = \frac{1}{2}\frac{\partial g(f(r), r, h, E)}{\partial r}\bigg|_{f_0},\\
g_2 &= g_2(f_0(r), r) = \frac{\partial g(f(r), r, h, E)}{\partial f}\bigg|_{f_0},\\
g_3 &= g_3(f_0(r), r) = \frac{\partial^2 g(f(r), r, h, E)}{\partial f^2}\bigg|_{f_0},\\
g_4 &= g_4(f_0(r), r, h) = \frac{1}{2}\frac{\partial^2 g(f(r), r, h, E)}{\partial r^2}\bigg|_{f_0}.
\end{aligned}
$$

3.3 Examples

The purpose of this section is to give some examples of ϵ-equation for our standard examples, the four electron atom problem with (without) constant magnetic field, four body gravitation problem and four ion problem.

3.3.1 Four electron atom problem with constant magnetic field

First consider the four electron atom problem with constant magnetic field. We have,

$$F_1(f(r), r) = 4\left(2r^2 + 4f(r)\right)^{-\frac{3}{2}} - 4\left(r^2 + f(r)\right)^{-\frac{3}{2}},$$

$$F_2(f(r), r, h) = \frac{h^2}{r^3} + 2r\left(2r^2 + 4f(r)\right)^{-\frac{3}{2}} + \frac{1}{4r^2} - 4r\left(r^2 + f(r)\right)^{-\frac{3}{2}} - \frac{B^2 r}{4},$$

$$
\begin{aligned}
g(f(r), r, h, E) = 2E - 2F_3(f(r), r, h) =& 2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f(r)\right)^{-\frac{1}{2}} - \frac{1}{2r}\\
&+ 8\left(r^2 + f(r)\right)^{-\frac{1}{2}} - \frac{B^2 r^2}{4}.
\end{aligned}
$$

Then the ϵ-equation becomes

$$\epsilon'' + \left[\frac{g'(f_0(r), r, h, E)}{2g(f_0(r), r, h, E)} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})'}{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})}\right]\epsilon' + \frac{b(r)}{f_0^2(r)g^2(f_0(r), r, h, E)}\epsilon = 0, \tag{3.8}$$

where

$$b(r) = \Bigg(\frac{1}{2}f_0'^2(r)g(f_0(r), r, h, E) - \Big(\frac{1}{4}f_0'^2(r) + f_0(r)\Big)$$
$$\Big(f_0'(r)g_1(f_0(r), r, h) - 2f_0(r)g_2(f_0(r), r)\Big)\Bigg)\Big(g(f_0(r), r, h, E) + f_0(r)g_2(f_0(r), r)\Big)$$
$$+ f_0(r)g(f_0(r), r, h, E)\Bigg(f_0'(r)g_1(f_0(r), r, h)$$
$$+ (\frac{1}{4}f_0'^2(r) + f_0(r))\left[\Big(\frac{1}{2}rf_0'(r) - 2f_0(r)\Big)g_3(f_0(r), r) - 4g_2(f_0(r), r)\right]\Bigg),$$

so that,

$$g(f_0(r), r, h, E) = 2E - \frac{h^2}{r^2} - 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2 + f_0(r)\Big)^{-\frac{1}{2}} - \frac{B^2r^2}{4},$$

$$
\begin{aligned}
g_1(f_0(r), r, h) &= \frac{1}{2}\frac{\partial g(f(r), r, h, E)}{\partial r}\bigg|_{f_0} = F_2(f_0(r), r, h) \\
&= \frac{h^2}{r^3} + 2r\Big(2r^2 + 4f_0(r)\Big)^{-\frac{3}{2}} + \frac{1}{4r^2} - 4r\Big(r^2 + f_0(r)\Big)^{-\frac{3}{2}} - \frac{B^2 r}{4}, \\
g_2(f_0(r), r) &= \frac{\partial g(f(r), r, h, E)}{\partial f}\bigg|_{f_0} = F_1(f_0(r), r) \\
&= 4\Big(2r^2 + 4f_0(r)\Big)^{-\frac{3}{2}} - 4\Big(r^2 + f_0(r)\Big)^{-\frac{3}{2}} \\
g_3(f_0(r), r) &= \frac{\partial^2 g(f(r), r, h, E)}{\partial f^2}\bigg|_{f_0} = \frac{\partial F_1(f(r), r)}{\partial f}\bigg|_{f_0} \\
&= -24\Big(2r^2 + 4f_0(r)\Big)^{-\frac{5}{2}} + 6\Big(r^2 + f_0(r)\Big)^{-\frac{5}{2}}.
\end{aligned}
$$

By multiplying equation (3.8) by the integrating factor

$$
\psi_1(r) = \left(\frac{2E - \frac{h^2}{r^2} - 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2 + f_0(r)\Big)^{-\frac{1}{2}} - \frac{B^2 r^2}{4}}{f_0^2(r)\Big(1 + \frac{f_0'^2(r)}{4f_0(r)}\Big)^3} \right)^{\frac{1}{2}},
$$

where

$$\phi(r) = \left\{ \frac{\left[2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}} - \frac{B^2r^2}{4}\right]'}{2\left[2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}} - \frac{B^2r^2}{4}\right]} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})'}{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})} \right\}$$

$$= \frac{d}{dr}\left(\ln \left(\frac{\left[2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}} - \frac{B^2r^2}{4}\right]^{\frac{1}{2}}}{f_0(r)(1 + \frac{{f_0'}^2(r)}{4f_0(r)})^{\frac{3}{2}}} \right)\right).$$

The sum of the first two terms on the left of (3.8) can be written as an exact derivative, that is,

$$\frac{d}{dr}\left(\left(\frac{2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}} - \frac{B^2r^2}{4}}{f_0^2(r)\left(1 + \frac{f_0'^2(r)}{4f_0(r)}\right)^3} \right)^{\frac{1}{2}} \frac{d\epsilon}{dr} \right)$$

$$+\left\{ \frac{\left(\frac{2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}} - \frac{B^2r^2}{4}}{f_0^2(r)\left(1 + \frac{f_0'^2(r)}{4f_0(r)}\right)^3} \right)^{\frac{1}{2}} b(r)}{f_0^2\left[2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}} - \frac{B^2r^2}{4}\right]^2} \right\}\epsilon = 0$$

$$
\begin{aligned}
b(r) = \Bigg(\frac{1}{2}f_0'^2(r)\Bigg[2E - \frac{h^2}{r^2} - 2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2+f_0(r)\Big)^{-\frac{1}{2}} - \frac{B^2r^2}{4}\Bigg] \\
- (\frac{1}{4}f_0'^2(r) + f_0(r))\Bigg(f_0'(r)\Bigg[\frac{h^2}{r^3} + 2r\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}} + \frac{1}{4r^2} \\
- 4r\Big(r^2+f_0(r)\Big)^{-\frac{3}{2}} - \frac{B^2r}{4}\Bigg] - 2f_0(r)\Bigg[4\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}} - 4\Big(r^2+f_0(r)\Big)^{-\frac{3}{2}}\Bigg]\Bigg)\Bigg) \\
\Bigg(\Bigg[2E - \frac{h^2}{r^2} - 2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2+f_0(r)\Big)^{-\frac{1}{2}} - \frac{B^2r^2}{4}\Bigg] \\
+ f_0(r)\Bigg[4\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}} - 4\Big(r^2+f_0(r)\Big)^{-\frac{3}{2}}\Bigg]\Bigg) + (\frac{1}{4}f_0'^2(r) + f_0(r)) \\
\Bigg[(\frac{1}{2}rf_0'(r) - 2f_0(r))\Big(-24\Big(2r^2+4f_0(r)\Big)^{-\frac{5}{2}} + 6\Big(r^2+f_0(r)\Big)^{-\frac{5}{2}}\Big) \\
- 4\Big(4\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}} - 4\Big(r^2+f_0(r)\Big)^{-\frac{3}{2}}\Big)\Bigg].
\end{aligned}
$$

3.3.2 Four electron atom problem

Secondly consider the four electron atom problem without constant magnetic field. We have,

$$
F_1(f(r), r) = 4\Big(2r^2+4f(r)\Big)^{-\frac{3}{2}} - 4\Big(r^2+f(r)\Big)^{-\frac{3}{2}},
$$

$$
F_2(f(r), r, h) = \frac{h^2}{r^3} + 2r\Big(2r^2+4f(r)\Big)^{-\frac{3}{2}} + \frac{1}{4r^2} - 4r\Big(r^2+f(r)\Big)^{-\frac{3}{2}},
$$

$$
\begin{aligned}
g(f(r), r, h, E) = 2E - 2F_3(f(r), r, h) = 2E - \frac{h^2}{r^2} - 2\Big(2r^2+4f(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} \\
+ 8\Big(r^2+f(r)\Big)^{-\frac{1}{2}}.
\end{aligned}
$$

Then the ϵ-equation becomes

$$\epsilon'' + \left[\frac{g'(f_0(r), r, h, E)}{2g(f_0(r), r, h, E)} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1 + \frac{{f_0}'^2(r)}{4f_0(r)})'}{(1 + \frac{{f_0}'^2(r)}{4f_0(r)})}\right]\epsilon' + \frac{b(r)}{f_0^2(r)g^2(f_0(r), r, h, E)}\epsilon = 0. \tag{3.9}$$

where

$$b(r) = \left(\frac{1}{2}f_0'^2(r)g(f_0(r), r, h, E) - \left(\frac{1}{4}f_0'^2(r) + f_0(r)\right)\right.$$
$$\left.\left(f_0'(r)g_1(f_0(r), r, h) - 2f_0(r)g_2(f_0(r), r)\right)\right)\left(g(f_0(r), r, h, E) + f_0(r)g_2(f_0(r), r)\right)$$
$$+ f_0(r)g(f_0(r), r, h, E)\left(f_0'(r)g_1(f_0(r), r, h)\right.$$
$$\left. + (\frac{1}{4}f_0'^2(r) + f_0(r))\left[\left(\frac{1}{2}rf_0'(r) - 2f_0(r)\right)g_3(f_0(r), r) - 4g_2(f_0(r), r)\right]\right).$$

Hence, we obtain

$$\begin{aligned} g(f_0(r), r, h, E) &= 2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}}, \\ g_1(f_0(r), r, h) &= \frac{1}{2}\frac{\partial g(f(r), r, h, E)}{\partial r}\bigg|_{f_0} = F_2(f_0(r), r, h) \\ &= \frac{h^2}{r^3} + 2r\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} + \frac{1}{4r^2} - 4r\left(r^2 + f_0(r)\right)^{-\frac{3}{2}} \\ g_2(f_0(r), r) &= \frac{\partial g(f(r), r, h, E)}{\partial f}\bigg|_{f_0} = F_1(f_0(r), r) \\ &= 4\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} - 4\left(r^2 + f_0(r)\right)^{-\frac{3}{2}}, \end{aligned}$$

$$g_3(f_0(r), r) = \left.\frac{\partial^2 g(f(r), r, h, E)}{\partial f^2}\right|_{f_0} = \left.\frac{\partial F_1(f(r), r)}{\partial f}\right|_{f_0}$$
$$= -24\Big(2r^2 + 4f_0(r)\Big)^{-\frac{5}{2}} + 6\Big(r^2 + f_0(r)\Big)^{-\frac{5}{2}}.$$

By multiplying equation (3.9) by the integrating factor

$$\psi_1(r) = \left(\frac{2E - \frac{h^2}{r^2} - 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2 + f_0(r)\Big)^{-\frac{1}{2}}}{f_0^2(r)\big(1 + \frac{f_0'^2(r)}{4f_0(r)}\big)^3}\right)^{\frac{1}{2}}$$

where

$$\phi(r) = \left\{\frac{\left[2E - \frac{h^2}{r^2} - 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2 + f_0(r)\Big)^{-\frac{1}{2}}\right]'}{2\left[2E - \frac{h^2}{r^2} - 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2 + f_0(r)\Big)^{-\frac{1}{2}}\right]}\right.$$
$$\left. - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})'}{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})}\right\}$$

$$= \frac{d}{dr}\left(\ln\left(\frac{\left[2E - \frac{h^2}{r^2} - 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r} + 8\Big(r^2 + f_0(r)\Big)^{-\frac{1}{2}}\right]^{\frac{1}{2}}}{f_0(r)\big(1 + \frac{{f_0'}^2(r)}{4f_0(r)}\big)^{\frac{3}{2}}}\right)\right).$$

The sum of the first two terms on the left of (3.9) can be written as an exact derivative, that is,

$$\frac{d}{dr}\left(\left(\frac{2E-\frac{h^2}{r^2}-2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}+8\left(r^2+f_0(r)\right)^{-\frac{1}{2}}}{f_0^2(r)\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)^3}\right)^{\frac{1}{2}}\frac{d\epsilon}{dr}\right)$$

$$+\left\{\left(\frac{2E-\frac{h^2}{r^2}-2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}+8\left(r^2+f_0(r)\right)^{-\frac{1}{2}}}{f_0^2(r)\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)^3}\right)^{\frac{1}{2}}\frac{b(r)}{f_0^2\left[2E-\frac{h^2}{r^2}-2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}+8\left(r^2+f_0(r)\right)^{-\frac{1}{2}}\right]^2}\right\}\epsilon=0,$$

$$\begin{aligned}
b(r) = {} & \left(\frac{1}{2}f_0'^2(r)\left[2E-\frac{h^2}{r^2}-2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}+8\left(r^2+f_0(r)\right)^{-\frac{1}{2}}\right]\right. \\
& -(\frac{1}{4}f_0'^2(r)+f_0(r))\left(f_0'(r)\left[\frac{h^2}{r^3}+2r\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}+\frac{1}{4r^2}\right.\right. \\
& \left.\left.\left.-4r\left(r^2+f_0(r)\right)^{-\frac{3}{2}}\right]-2f_0(r)\left[4\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}-4\left(r^2+f_0(r)\right)^{-\frac{3}{2}}\right]\right)\right) \\
& \left(\left[2E-\frac{h^2}{r^2}-2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}+8\left(r^2+f_0(r)\right)^{-\frac{1}{2}}\right]\right. \\
& \left.+f_0(r)\left[4\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}-4\left(r^2+f_0(r)\right)^{-\frac{3}{2}}\right]\right)+(\frac{1}{4}f_0'^2(r)+f_0(r)) \\
& \left[(\frac{1}{2}rf_0'(r)-2f_0(r))\left(-24\left(2r^2+4f_0(r)\right)^{-\frac{5}{2}}+6\left(r^2+f_0(r)\right)^{-\frac{5}{2}}\right)\right. \\
& \qquad -4\left(4\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}-4\left(r^2+f_0(r)\right)^{-\frac{3}{2}}\right).
\end{aligned}$$

3.3.3 Four body gravitation problem

Consider the four body gravitation problem. We have,

$$F_1(f(r),r) = -4\left(2r^2 + 4f(r)\right)^{-\frac{3}{2}},$$

$$F_2(f(r),r,h) = \frac{h^2}{r^3} - 2r\left(2r^2 + 4f(r)\right)^{-\frac{3}{2}} - \frac{1}{4r^2},$$

$$g(f_0(r),r,h,E) = 2E - 2F_3(f(r),r,h) = 2E - \frac{h^2}{r^2} + 2\left(2r^2 + 4f(r)\right)^{-\frac{1}{2}} + \frac{1}{2r}.$$

Then the ϵ-equation becomes

$$\epsilon'' + \left[\frac{g'(f_0(r),r,h,E)}{2g(f_0(r),r,h,E)} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1+\frac{f_0'^2(r)}{4f_0(r)})'}{(1+\frac{f_0'^2(r)}{4f_0(r)})}\right]\epsilon' + \frac{b(r)}{f_0^2(r)g^2(f_0(r),r,h,E)}\epsilon = 0. \tag{3.10}$$

where

$$b(r) = \left(\frac{1}{2}f_0'^2(r)g(f_0(r),r,h,E) - \left(\frac{1}{4}f_0'^2(r) + f_0(r)\right)\right.$$
$$\left.\left(f_0'(r)g_1(f_0(r),r,h) - 2f_0(r)g_2(f_0(r),r)\right)\right)\left(g(f_0(r),r,h,E) + f_0(r)g_2(f_0(r),r)\right)$$
$$+ f_0(r)g(f_0(r),r,h,E)\left(f_0'(r)g_1(f_0(r),r,h)\right.$$
$$\left. + (\frac{1}{4}f_0'^2(r) + f_0(r))\left[\left(\frac{1}{2}rf_0'(r) - 2f_0(r)\right)g_3(f_0(r),r) - 4g_2(f_0(r),r)\right]\right),$$

so that,

$$g(f_0(r),r,h,E) = 2E - \frac{h^2}{r^2} + 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} + \frac{1}{2r},$$

$$g_1(f_0(r),r,h) = \frac{1}{2}\frac{\partial g(f(r),r,h,E)}{\partial r}\bigg|_{f_0} = F_2(f_0(r),r,h)$$
$$= \frac{h^2}{r^3} - 2r\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}} - \frac{1}{4r^2},$$
$$g_2(f_0(r),r) = \frac{\partial g(f(r),r,h,E)}{\partial f}\bigg|_{f_0} = F_1(f_0(r),r)$$
$$= -4\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}$$
$$g_3(f_0(r),r) = \frac{\partial^2 g(f(r),r,h,E)}{\partial f^2}\bigg|_{f_0} = \frac{\partial F_1(f(r),r)}{\partial f}\bigg|_{f_0}$$
$$= 24\left(2r^2+4f_0(r)\right)^{-\frac{5}{2}}.$$

By multiplying equation (3.10) by the integrating factor

$$\psi_1(r) = \left(\frac{2E - \frac{h^2}{r^2} + 2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}} + \frac{1}{2r}}{f_0^2(r)\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)^3}\right)^{\frac{1}{2}}$$

where

$$\phi(r) = \left\{\frac{\left[2E - \frac{h^2}{r^2} + 2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}} + \frac{1}{2r}\right]'}{2\left[2E - \frac{h^2}{r^2} + 2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}} + \frac{1}{2r}\right]} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)'}{\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)}\right\}$$
$$= \frac{d}{dr}\left(\ln\left(\frac{\left[2E - \frac{h^2}{r^2} + 2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}} + \frac{1}{2r}\right]^{\frac{1}{2}}}{f_0(r)\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)^{\frac{3}{2}}}\right)\right).$$

The sum of the first two terms on the left of (3.10) can be written as an exact derivative, that is,

$$\frac{d}{dr}\left(\left(\frac{2E-\frac{h^2}{r^2}+2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}}+\frac{1}{2r}}{f_0^2(r)\big(1+\frac{f_0'^2(r)}{4f_0(r)}\big)^3}\right)^{\frac{1}{2}}\frac{d\epsilon}{dr}\right)$$

$$+\left\{\left(\frac{2E-\frac{h^2}{r^2}+2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}}+\frac{1}{2r}}{f_0^2(r)\big(1+\frac{f_0'^2(r)}{4f_0(r)}\big)^3}\right)^{\frac{1}{2}}\right) \frac{b(r)}{f_0^2\left[2E-\frac{h^2}{r^2}+2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}}+\frac{1}{2r}\right]^2}\right\}\epsilon=0,$$

$$\begin{aligned}
b(r) = &\Big(\frac{1}{2}f_0'^2(r)\Big[2E-\frac{h^2}{r^2}+2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}}+\frac{1}{2r}\Big]\\
&-(\frac{1}{4}f_0'^2(r)+f_0(r))\Big(f_0'(r)\Big[\frac{h^2}{r^3}-2r\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}}-\frac{1}{4r^2}\Big]\\
&-2f_0(r)\Big[-4\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}}\Big]\Big)\Big)\\
&\Big(\Big[2E-\frac{h^2}{r^2}+2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}}+\frac{1}{2r}\Big]\\
&+f_0(r)\Big[-4\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}}\Big]\Big)+(\frac{1}{4}f_0'^2(r)+f_0(r))\\
&\Big[\Big(\frac{1}{2}rf_0'(r)-2f_0(r)\Big)\Big(24\Big(2r^2+4f_0(r)\Big)^{-\frac{5}{2}}\Big)-4\Big(-4\Big(2r^2+4f_0(r)\Big)^{-\frac{3}{2}}\Big).
\end{aligned}$$

3.3.4 Four ion problem

Consider the four ion problem. We have,

$$F_1(f(r),r) = -4\Big(2r^2+4f(r)\Big)^{-\frac{3}{2}},$$

$$F_2(f(r),r,h) = \frac{h^2}{r^3} - 2r\Big(2r^2+4f(r)\Big)^{-\frac{3}{2}} + \frac{1}{4r^2},$$

$$g(f(r),r,h,E) = 2E - 2F_3(f(r),r,h) = 2E - \frac{h^2}{r^2} + 2\Big(2r^2+4f(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r}.$$

Then the ϵ-equation becomes

$$\epsilon'' + \left[\frac{g'(f_0(r),r,h,E)}{2g(f_0(r),r,h,E)} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1+\frac{f_0'^2(r)}{4f_0(r)})'}{(1+\frac{f_0'^2(r)}{4f_0(r)})}\right]\epsilon' \\ + \frac{b(r)}{f_0^2(r)g^2(f_0(r),r,h,E)}\epsilon = 0, \tag{3.11}$$

where

$$b(r) = \Bigg(\frac{1}{2}f_0'^2(r)g(f_0(r),r,h,E) - \Big(\frac{1}{4}f_0'^2(r)+f_0(r)\Big) \\ \Big(f_0'(r)g_1(f_0(r),r,h) - 2f_0(r)g_2(f_0(r),r)\Big)\Bigg)\Big(g(f_0(r),r,h,E) + f_0(r)g_2(f_0(r),r)\Big) \\ + f_0(r)g(f_0(r),r,h,E)\Bigg(f_0'(r)g_1(f_0(r),r,h) \\ + (\frac{1}{4}f_0'^2(r)+f_0(r))\Big[\Big(\frac{1}{2}rf_0'(r) - 2f_0(r)\Big)g_3(f_0(r),r) - 4g_2(f_0(r),r)\Big]\Bigg),$$

so that,

$$g(f_0(r),r,h,E) = 2E - \frac{h^2}{r^2} + 2\Big(2r^2+4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r},$$

$$g_1(f_0(r),r,h) = \frac{1}{2}\frac{\partial g(f(r),r,h,E)}{\partial r}\bigg|_{f_0} = F_2(f_0(r),r,h)$$

$$= \frac{h^2}{r^3} - 2r\Big(2r^2 + 4f_0(r)\Big)^{-\frac{3}{2}} + \frac{1}{4r^2},$$

$$g_2(f_0(r),r) = \frac{\partial g(f(r),r,h,E)}{\partial f}\bigg|_{f_0} = F_1(f_0(r),r)$$

$$= -4\Big(2r^2 + 4f_0(r)\Big)^{-\frac{3}{2}},$$

$$g_3(f_0(r),r) = \frac{\partial^2 g(f(r),r,h,E)}{\partial f^2}\bigg|_{f_0} = \frac{\partial F_1(f(r),r)}{\partial f}\bigg|_{f_0}$$

$$= 24\Big(2r^2 + 4f_0(r)\Big)^{-\frac{5}{2}}.$$

By multiplying equation (3.11) by the integrating factor

$$\psi_1(r) = \left(\frac{2E - \frac{h^2}{r^2} + 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r}}{f_0^2(r)\big(1 + \frac{f_0'^2(r)}{4f_0(r)}\big)^3}\right)^{\frac{1}{2}}$$

where

$$\phi(r) = \left\{\frac{\left[2E - \frac{h^2}{r^2} + 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r}\right]'}{2\left[2E - \frac{h^2}{r^2} + 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r}\right]} - \frac{f_0'(r)}{f_0(r)} - \frac{3}{2}\frac{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})'}{(1 + \frac{{f_0'}^2(r)}{4f_0(r)})}\right\}$$

$$= \frac{d}{dr}\left(\ln\left(\frac{\left[2E - \frac{h^2}{r^2} + 2\Big(2r^2 + 4f_0(r)\Big)^{-\frac{1}{2}} - \frac{1}{2r}\right]^{\frac{1}{2}}}{f_0(r)\big(1 + \frac{{f_0'}^2(r)}{4f_0(r)}\big)^{\frac{3}{2}}}\right)\right).$$

The sum of the first two terms on the left of (3.11) can be written as an exact derivative, that is,

$$\frac{d}{dr}\left(\left(\frac{2E-\frac{h^2}{r^2}+2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}}{f_0^2(r)\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)^3}\right)^{\frac{1}{2}}\frac{d\epsilon}{dr}\right)$$

$$+\left\{\left(\left(\frac{2E-\frac{h^2}{r^2}+2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}}{f_0^2(r)\left(1+\frac{f_0'^2(r)}{4f_0(r)}\right)^3}\right)^{\frac{1}{2}}\right)\right.$$

$$\left.\frac{b(r)}{f_0^2\left[2E-\frac{h^2}{r^2}+2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}\right]^2}\right\}\epsilon=0,$$

$$b(r)=\left(\frac{1}{2}f_0'^2(r)\left[2E-\frac{h^2}{r^2}+2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}\right]\right.$$

$$-\left(\frac{1}{4}f_0'^2(r)+f_0(r)\right)\left(f_0'(r)\left[\frac{h^2}{r^3}-2r\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}+\frac{1}{4r^2}\right]\right.$$

$$\left.\left.-2f_0(r)\left[-4\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}\right]\right)\right)$$

$$\left(\left[2E-\frac{h^2}{r^2}+2\left(2r^2+4f_0(r)\right)^{-\frac{1}{2}}-\frac{1}{2r}\right]\right.$$

$$\left.+f_0(r)\left[-4\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}\right]\right)+\left(\frac{1}{4}f_0'^2(r)+f_0(r)\right)$$

$$\left[\left(\frac{1}{2}rf_0'(r)-2f_0(r)\right)\left(24\left(2r^2+4f_0(r)\right)^{-\frac{5}{2}}\right)-4\left(-4\left(2r^2+4f_0(r)\right)^{-\frac{3}{2}}\right)\right].$$

3.4 Local behaviour near and at $r = 1$ for the general ϵ-equation

The aim of this section is to describe the solution's behaviour near and at $r = 1$ for the general ϵ-equation. The f-equation can be written in the form

$$\begin{aligned} f''(r) &= F(f(r), r, h, E) \\ &= \frac{1}{f(r)g(f(r), r, h, E)}\left(\frac{1}{2}f'^2(r)g(f(r), r, h, E) - \left(\frac{1}{4}f'^2(r) + f(r)\right)\right. \\ &\quad \left.\left(\frac{1}{2}f'(r)\frac{\partial g(f(r), r, h, E)}{\partial r}\bigg|_{f_0} - 2f(r)\frac{\partial g(f(r), r, h, E)}{\partial f}\bigg|_{f_0}\right)\right), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} f(1) &= 0, \\ g(1) = g(0, 1, h, E) &= \frac{1}{2}f'(1)F_2(0, 1, h) = \dot{z}_0^2, \\ \frac{\partial g(f(r), r, h, E)}{\partial r}\bigg|_{f_0} &= 2F_2(0, 1, h). \end{aligned}$$

Now when $r = 1$, the left hand side of (3.12) is $f''(1)$. Although F is not defined when $r = 1$, we need to know how F behaves near and at 1. The value of the limit of F is not obvious because both numerator and denominator approach 0 and 0/0 is not defined. From (3.12) using L'Hopital's Rule we get the value of the limit when $r = 1$, which is

$$\begin{aligned} f''(1) = \frac{f'(1)}{3F_2(0, 1, h)}\left(4F_1(0, 1) - f'(1)\frac{\partial F_2(f(r), r, h)}{\partial f}\bigg|_{(0,1,h)}\right. \\ \left. - \frac{\partial F_2(f(r), r, h)}{\partial r}\bigg|_{(0,1,h)}\right). \end{aligned}$$

This is the same value we can obtain by take the derivatives with respect to t of the equation

$$z^2 = f(r)$$

We will develop the local behaviour of $\epsilon(r)$ near and at $r = 1$, from equation (3.4), in the form

$$f_0^2(r)g^2(f_0(r), r, h, E)\epsilon''(r) + f_0(r)g(f_0(r), r, h, E)a(r)\epsilon'(r) + b(r)\epsilon(r) = 0, \tag{3.13}$$

where

$$
\begin{aligned}
g &= g(f_0(r), r, h, E) = [2E - 2F_3(f_0(r), r, h)],\\
a(r) &= \frac{1}{2} f_0(r) g'(f_0(r), r, h, E) - f_0'(r) g(f_0(r), r, h, E)\\
&\quad + \frac{3}{4} f_0'(r)\Big(f_0'(r) g_1(f_0(r), r, h) - 2 f_0(r) g_2(f_0(r), r)\Big),\\
b(r) &= \Bigg(\frac{1}{2} f_0'^2(r) g(f_0(r), r, h, E) - \Big(\frac{1}{4} f_0'^2(r) + f_0(r)\Big)\\
&\Bigg(f_0'(r) g_1(f_0(r), r, h) - 2 f_0(r) g_2(f_0(r), r)\Bigg)\Bigg)\Bigg(g(f_0(r), r, h, E) + f_0(r) g_2(f_0(r), r)\Bigg)\\
&\quad + f_0(r) g(f_0(r), r, h, E)\Bigg(f_0'(r) g_1(f_0(r), r, h)\\
&\quad + \Big(\frac{1}{4} f_0'^2(r) + f_0(r)\Big)\Bigg[\Big(\frac{1}{2} r f_0'(r) - 2 f_0(r)\Big) g_3(f_0(r), r) - 4 g_2(f_0(r), r)\Bigg]\Bigg).
\end{aligned}
$$

Now since $f_0(1) = 0$, $b(1) = 0$, we will study equation (3.13) at $r = 1$, which can be written in the form

$$
f_0(r) g(r) \epsilon''(r) = -\frac{1}{f_0(r) g(r)}\Big(f_0(r) g(r) a(r) \epsilon'(r) + b(r)\epsilon(r)\Big). \tag{3.14}
$$

Now when $r = 1$, assuming $\epsilon''(1)$ is a finite, the left hand side of (3.14) is 0, but the right hand side of (3.14) is 0/0, since $f_0(1) = 0$ and $b(1) = 0$. Then we have to use L'Hopital's Rule with the right hand side of (3.14) and we obtain

$$
\epsilon'(1) = -\frac{b'(1)}{f_0'(1) g(1) a(1)}\epsilon(1).
$$

Also from (3.13) we get

$$
\epsilon''(r) = -\frac{1}{f_0^2(r) g^2(r)}\Big(f_0(r) g(r) a(r) \epsilon'(r) + b(r)\epsilon(r)\Big). \tag{3.15}
$$

Now when $r = 1$, assuming $\epsilon'''(1)$ is a finite, the left hand side of (3.15) is $\epsilon''(1)$, but the right hand side of (3.15) is 0/0, since $f_0(1) = 0$, $b(1) = 0$ and $\epsilon'(1) = -b'(1)\epsilon(1)/f_0'(1)g(1)a(1)$. Then we have to use L'Hopital's Rule with the right hand side of (3.15) and we obtain

$$\epsilon''(1) = -\frac{\Big(\big(f_0''(1)g(1)a(1) + 2f_0'(1)g'(1)a(1) + 2f_0'(1)g(1)a'(1) + 2b'(1)\big)\epsilon'(1) + b''(1)\epsilon(1)\Big)}{2f_0'^2(1)g^2(1) + 2f_0'(1)g(1)a(1)}.$$

Now when $\epsilon(1) = 0$, then $\epsilon'(1) = 0$ and $\epsilon''(1) = 0$ and when $\epsilon(1) = \delta_1$, then $\epsilon'(1) = A(1)\delta_1$ and $\epsilon''(1) = B(1)\delta_1$ where δ_1, $A(1)$ and $B(1)$ are non-zero constants, we see that $\epsilon''(1)$ and $\epsilon'(1)$ are linear dependent on $\epsilon(1)$, where

$$\begin{aligned}
g(1) &= \frac{1}{2}f_0'(1)g_1(1) = \frac{1}{2}f_0'(1)F_2(0,1,h),\\
a(1) &= \frac{1}{2}f_0'(1)g(1) = \frac{1}{4}f_0'^2(1)g_1(1) = \frac{1}{4}f_0'^2(1)F_2(0,1,h),\\
b(1) &= 0,\\
g'(1) &= 2g_1(1) + f_0'(1)g_2(1) = 2F_2(0,1,h) + f_0'(1)F_1(0,1),\\
a'(1) &= -f_0''(1)g(1) - \frac{1}{2}f_0'(1)g'(1) + \frac{3}{4}f_0'(1)f_0''(1)g_1(1)\\
&\quad + \frac{3}{4}f_0'(1)\Big(f_0''(1)g_1(1) + f_0'(1)g_1'(1) - 2f_0'(1)g_2(1)\Big),\\
b'(1) &= f_0'(1)f_0''(1)g^2(1) + \frac{1}{2}f_0'^2(1)g(1)g'(1) - f_0'^2(1)g(1)g_1(1)\big(1 + \frac{1}{2}f_0''(1)\big)\\
&\quad - \frac{1}{4}f_0'^2(1)g(1)\Big(f_0''(1)g_1(1) + f_0'(1)g_1'(1) - 2f_0'(1)g_2(1)\Big)\\
&\quad - \frac{1}{2}f_0'^2(1)\big(g(1) - \frac{1}{2}f_0'(1)g_1(1)\big)\big(g'(1) + f_0'(1)g_2(1)\big)\\
&\quad + f_0'(1)g(1)\Big(f_0'(1)g_1(1) + \frac{1}{4}f_0'^2(1)\big(\frac{1}{2}f_0'(1)g_3(1) - 4g_2(1)\big)\Big),\\
f_0''(1) &= \frac{f_0'(1)}{3F_2(0,1,h)}\left(4F_1(0,1) - f_0'(1)\frac{\partial F_2(f(r),r,h)}{\partial f}\bigg|_{(0,1,h)} - \frac{\partial F_2(f(r),r,h)}{\partial r}\bigg|_{(0,1,h)}\right),\\
f_0'(1) &= \frac{2\dot{z}_0^2}{F_2(0,1,h)},
\end{aligned}$$

$$
\begin{aligned}
b''(1) = & f_0''^2(1)g^2(1) + f_0'(1)f_0'''(1)g^2(1) + 2f_0'(1)f_0''(1)g(1)g'(1) \\
& + \frac{1}{2}f_0'^2(1)g(1)g''(1) - f_0'(1)f_0''(1)g(1)g_1(1)\Big(\frac{1}{2}f_0''(1) + \frac{1}{2}f_0'(1) + 1\Big) \\
& - 2f_0'(1)g(1)\Big(\frac{1}{2}f_0''(1) + 1\Big)\Big(f_0''(1)g_1(1) + f_0'(1)g_1'(1) - 2f_0'(1)g_2(1)\Big) \\
& - \frac{1}{4}f_0'^2(1)g(1)\Big(f_0'''(1)g_1(1) + 2f_0''(1)g_1'(1) + f_0'(1)g_1''(1) - 2f_0''(1)g_2'(1) \\
& - 4f_0'(1)g_2'(1)\Big) \\
& + 2\Bigg(f_0'(1)f_0''(1)g(1)\frac{1}{2}f_0'^2(1)g'(1) - f_0'^2(1)g_1(1)\Big(1 + \frac{1}{2}f_0''(1)\Big) \\
& - \frac{1}{4}f_0'^2(1)\Big(f_0''(1)g_1(1) + f_0'(1)g_1'(1) - 2f_0'(1)g_2(1)\Big)\Bigg)\Big(g'(1) + f_0'(1)g_2(1)\Big) \\
& + \frac{1}{2}\Big(f_0'^2(1)g(1) - \frac{1}{2}f_0'^3(1)g_1(1)\Big)\Big(g''(1) + f_0''(1)g_2(1) + 2f_0'(1)g_2'(1)\Big) \\
& + \Big(f_0''(1)g(1) + 2f_0'(1)g'(1)\Big)\Big(f_0'(1)g_1(1) + \frac{1}{8}f_0'^3(1)g_3(1) - 4g_4(1)\Big) \\
& + 2f_0'(1)g(1)\Bigg(f_0''(1)g_1(1) + 2f_0'(1)g_1'(1) \\
& + f_0'(1)\Big(1 + \frac{1}{2}f_0''(1)\Big)\Big(\frac{1}{2}f_0'(1)g_3(1) - 4g_4(1)\Big) \\
& + \frac{1}{8}f_0'^2(1)\Big(\Big(f_0''(1) - 3f_0'(1)\Big)g_3(1) + \frac{1}{2}f_0'(1)g_3'(1) - 4g_2'(1)\Big)\Bigg).
\end{aligned}
$$

with

$$
\begin{aligned}
g_1(1) &= F_2(0,1,h), \\
g_2(1) &= F_1(0,1), \\
g_3(1) &= \left.\frac{\partial F_1(f(r),r)}{\partial f}\right|_{(0,1)}, \\
g_1'(1) &= F_2'(0.1.h), \\
g_2'(1) &= F_1'(0,1), \\
g_3'(1) &= \left.\frac{\partial^2 F_1(f(r),r)}{\partial f \partial r}\right|_{(0,1)} + f_0'(1)\left.\frac{\partial^2 F_1(f(r),r)}{\partial f^2}\right|_{(0,1)}, \\
g''(1) &= g_4(1) + f_0'(1)\Big(2 + f_0'(1)\Big)g_3(1) + f_0''(1)g_2(1),
\end{aligned}
$$

$$
\begin{aligned}
f_0'''(1) = \frac{2}{15F_2^5(0,1,h)} \Bigg(& 16\dot{z}_0^2\, F_1^2(0,1)\, F_2^2(0,1,h) \\
& + 18\dot{z}_0^2\, F_2^3(0,1,h)\, \frac{\partial F_1(f(r),r)}{\partial r}\bigg|_{(0,1)} + 36\dot{z}_0^4\, F_2^2(0,1,h)\, \frac{\partial F_1(f(r),r)}{\partial f}\bigg|_{(0,1)} \\
& - 20\dot{z}_0^2\, F_1(0,1)\, F_2^2(0,1,h)\, \frac{\partial F_2(f(r),r,h)}{\partial r}\bigg|_{(0,1,h)} \\
& + 4\dot{z}_0^2\, F_2^2(0,1,h) \left(\frac{\partial F_2(f(r),r,h)}{\partial r}\bigg|_{(0,1,h)} \right)^2 - 3\dot{z}_0^2\, F_2^3(0,1,h)\, \frac{\partial^2 F_2(f(r),r,h)}{\partial r^2}\bigg|_{(0,1,h)} \\
& - 48\dot{z}_0^4\, F_1(0,1)\, F_2(0,1,h) \frac{\partial F_2(f(r),r,h)}{\partial f}\bigg|_{(0,1,h)} \\
& + 18\dot{z}_0^4\, F_2(0,1,h)\, \frac{\partial F_2(f(r),r,h)}{\partial r}\bigg|_{(0,1,h)} \frac{\partial F_2(f(r),r,h)}{\partial f}\bigg|_{(0,1,h)} \\
& + 20\dot{z}_0^6 \left(\frac{\partial F_2(f(r),r,h)}{\partial f}\bigg|_{(0,1,h)} \right)^2 - 12\dot{z}_0^4\, F_2^2(0,1,h)\, \frac{\partial^2 F_2(f(r),r,h)}{\partial f \partial r}\bigg|_{(0,1,h)} \\
& - 12\dot{z}_0^6\, F_2(0,1,h)\, \frac{\partial^2 F_2(f(r),r,h)}{\partial f^2}\bigg|_{(0,1,h)} \Bigg).
\end{aligned}
$$

Now consider equation (3.13) which can be written in the form

$$
f_0^2(r) g^2(r) \epsilon''(r) + f_0(r) g(r) a(r) \epsilon'(r) + b(r)\epsilon(r) = 0. \tag{3.16}
$$

Now, we shall take a Taylor expansion for $f_0(r)$, $g(r)$ and $a(r)$, $b(r)$ about $r = 1$ and we will let $u = 1 - r$ in that expansion since we are interested in $r \leq 1$. Suppose there exists a Taylor expansion for $f_0(r)$ about $r = 1$ in the form

$$
\begin{aligned}
f_0(r) &= \sum_{n=0}^{\infty} \frac{f_0^{(n)}(1)}{n!} (r-1)^n \\
&= \sum_{n=0}^{\infty} c_n u^n.
\end{aligned}
$$

where $u = 1 - r$, $c_n = (-1)^n f_0^{(n)}(1)/n!$, $c_0 = f_0(1) = 0$, $c_1 = -f_0'(1) \neq 0$, since $\dot{z}_0 \neq 0$, we see that the constant term of the Taylor series expansion for

$f_0(r)$ about $r = 1$ is zero and the coefficient of the linear term is non-zero. Therefore

$$f_0^2(r) = \left(\sum_{n=0}^{\infty} c_n u^n\right)^2 = \sum_{n=0}^{\infty} d_n u^n.$$

where $d_0 = c_0^2 = 0$, $d_1 = 2c_0c_1 = 0$, $d_2 = c_1^2 = f_0'^2(1) \neq 0$. We see that the constant term and the coefficient of the linear term are zero and the coefficient of the quadratic term is non-zero for the Taylor series expansion for $f_0^2(r)$ about $r = 1$. Also a Taylor expansion for $g(r)$ about $r = 1$ can be written in the form

$$g(r) = \sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!}(r-1)^n = \sum_{n=0}^{\infty} m_n u^n.$$

where $m_n = (-1)^n g^{(n)}(1)/n!$, $m_0 = g(1) \neq 0$, $m_1 = -g'(1) \neq 0$, $m_2 = g''(1)/2 \neq 0$. We see that the constant term of the Taylor series expansion for $g(r)$ about $r = 1$, the coefficient of the linear term and the coefficient of the quadratic term are non-zero. Therefore

$$g^2(r) = \left(\sum_{n=0}^{\infty} m_n u^n\right)^2 = \sum_{n=0}^{\infty} s_n u^n.$$

where $s_0 = m_0^2 = g^2(1)$, $s_1 = 2m_0m_1 = -2g(1)g'(1) \neq 0$, $s_2 = 2m_0m_2 + m_1^2 = g(1)g''(1) + g'^2(1) \neq 0$. We also have that the constant term of the Taylor series expansion for $g^2(r)$ about $r = 1$, the coefficient of the linear term and the coefficient of the quadratic term are non-zero. A Taylor expansion for $a(r)$ about $r = 1$ can be written in the form

$$a(r) = \sum_{n=0}^{\infty} \frac{a^{(n)}(1)}{n!}(r-1)^n = \sum_{n=0}^{\infty} a_n u^n.$$

where $a_n = (-1)^n a^{(n)}(1)/n!$, $a_0 = a(1) \neq 0$, $a_1 = -a'(1) \neq 0$, $a_2 = a''(1)/2 \neq 0$. Again we see that the constant term of Taylor series expansion for $a(r)$ about $r = 1$, the coefficient of the linear term and the coefficient of the quadratic term are non-zero. Also we can write the Taylor expansion for $b(r)$ about $r = 1$ in the form

$$\begin{aligned} b(r) &= \sum_{n=0}^{\infty} \frac{b^{(n)}(1)}{n!}(r-1)^n \\ &= \sum_{n=0}^{\infty} b_n u^n. \end{aligned}$$

where $b_n = (-1)^n b^{(n)}(1)/n!$, $b_0 = b(1) = 0$, $b_1 = -b'(1) \neq 0$, $b_2 = b''(1)/2 \neq 0$. We have that the constant term of Taylor series expansion for $b(r)$ about $r = 1$ is zero and the coefficient of the linear term and the coefficient of the quadratic term are non-zero. Let

$$\begin{aligned} P(u) = f_0(r)g(r)a(r) &= \left(\sum_{n=0}^{\infty} c_n u^n\right)\left(\sum_{n=0}^{\infty} m_n u^n\right)\left(\sum_{n=0}^{\infty} a_n u^n\right) \\ &= \sum_{n=0}^{\infty} P_n u^n. \end{aligned}$$

where $P_0 = c_0 m_0 a_0 = 0$, $P_1 = c_1 m_0 a_0 = -f_0'^2(1)g^2(1)/2 \neq 0$, $P_2 = c_1 m_0 a_1 + c_1 m_1 a_0 + c_2 m_0 a_0 \neq 0$, where the constant term is zero and the coefficient of the linear term and the coefficient of the quadratic term are non-zero. Let

$$\begin{aligned} R(u) = f_0^2(r)g^2(r) &= \left(\sum_{n=0}^{\infty} d_n u^n\right)\left(\sum_{n=0}^{\infty} s_n u^n\right) \\ &= \sum_{n=0}^{\infty} R_n u^n. \end{aligned}$$

where $R_0 = d_0 s_0 = 0$, $R_1 = d_0 s_1 + d_1 s_0 = 0$, $R_2 = d_2 s_0 = f_0'^2(1)g^2(1) \neq 0$, with the constant term and the coefficient of the linear term are zero and the coefficient of the quadratic term is non-zero. Let

$$\begin{aligned} Q(u) = b(r) &= \left(\sum_{n=0}^{\infty} b_n u^n\right) \\ &= \sum_{n=0}^{\infty} Q_n u^n. \end{aligned}$$

where $Q_0 = b_0 = b(1) = 0$, $Q_1 = b_1 = -b'(1) \neq 0$, $Q_2 = b_2 = b''(1)/2 \neq 0$. We see that the constant term is zero and the coefficient of the linear term and the coefficient of the quadratic term are non-zero. To solve equation (3.16) which can be written in the form

$$R(u)\epsilon''(u) + P(u)\epsilon'(u) + Q(u)\epsilon(u) = 0. \tag{3.17}$$

where $R(u)$, $P(u)$ and $Q(u)$ have Taylor series expansion, we multiply all the coefficients of $P(u)$ by -1 since $du/dr = -1$ and $R(0) = 0$. This means $u = 0$, $(r = 1)$ is a regular singular point of equation (3.17). The fact that $u = 0$ is a regular singular point of equation (3.17) means that $uP(u)/R(u) = up(u)$ and $u^2Q(u)/R(u) = u^2q(u)$ have finite limits as $u \to 0$, and are analytic at $u = 0$. Thus they have convergent power series expansions of the form $up(u) = \sum_{n=0}^{\infty} p_n u^n$ and $u^2q(u) = \sum_{n=0}^{\infty} q_n u^n$. To see this divide equation (3.17) by $R(u)$ and then multiply by u^2, obtaining

$$u^2\epsilon''(u) + u\big(up(u)\big)\epsilon'(u) + \big(u^2q(u)\big)\epsilon(u) = 0, \tag{3.18}$$

or

$$u^2\epsilon''(u) + u\Big(\sum_{n=0}^{\infty} p_n u^n\Big)\epsilon'(u) + \Big(\sum_{n=0}^{\infty} q_n u^n\Big)\epsilon(u) = 0.$$

Since

$$\frac{uP(u)}{R(u)} = up(u) = p_0 + p_1u + p_2u^2 + \cdots = \sum_{n=0}^{\infty} p_n u^n,$$

$$\frac{u^2Q(u)}{R(u)} = u^2q(u) = q_0 + q_1u + q_2u^2 + \cdots = \sum_{n=0}^{\infty} q_n u^n.$$

We have

$$p_0 = \lim_{u\to 0} \frac{uP(u)}{R(u)} = \frac{1}{2},$$

$$q_0 = \lim_{u\to 0} \frac{u^2Q(u)}{R(u)} = 0.$$

Hence, for u near 0 we have, $up(u) \approx p_0$ and $u^2p(u) \approx q_0$. Therefore it is reasonable to expect that the solutions to (3.18) will behave (for u near 0) like the solutions to the Cauchy-Euler equation. We thus have

$$u^2\frac{d\epsilon(u)}{du^2} + p_0u\,\frac{d\epsilon(u)}{du} + q_0\epsilon(u) = 0, \tag{3.19}$$

the Cauchy-Euler equation corresponding to equation (3.18). The general solution of the Cauchy-Euler equation (3.19) in any interval not containing the origin is determined by the roots λ_1 and λ_2 of the equation

$$\chi(\lambda) = \lambda(\lambda - 1) + p_0\lambda + q_0 = 0.$$

Thus $\lambda_1 = 0$, $\lambda_2 = 1/2$, since the roots are real and different and $\lambda_2 - \lambda_1$ is not zero or positive integer, then

$$\epsilon(u) = \delta_1 + \delta_2 u^{\frac{1}{2}}, \text{ where } \delta_1,\ \delta_2 \text{ are constants, } u > 0.$$

To solve equation (3.18) we assume that there is a solution of the form

$$\epsilon(u) = \Psi(\lambda, u) = \sum_{n=0}^{\infty} \alpha_n u^{\lambda+n}, \tag{3.20}$$

where $\alpha_0 \neq 0$, and we have written $\epsilon = \Psi(\lambda, u)$ to emphasize that Ψ depends on λ as well as on u. Then $\epsilon'(u)$, $\epsilon''(u)$ are given by

$$\epsilon'(u) = \sum_{n=0}^{\infty} \alpha_n(\lambda + n)u^{\lambda+n-1},$$

$$\epsilon''(u) = \sum_{n=0}^{\infty} \alpha_n(\lambda + n)(\lambda + n - 1)u^{\lambda+n-2}.$$

By substitution the expressions for $\epsilon(u)$, $\epsilon'(u)$ and $\epsilon''(u)$ in the equation (3.18) we obtain

$$\sum_{n=0}^{\infty} \alpha_n(\lambda + n)(\lambda + n - 1)u^{\lambda+n} + \left(\sum_{n=0}^{\infty} p_n u^n\right)\left(\sum_{n=0}^{\infty} \alpha_n(\lambda + n)u^{\lambda+n}\right)$$
$$+ \left(\sum_{n=0}^{\infty} q_n u^n\right)\left(\sum_{n=0}^{\infty} \alpha_n u^{\lambda+n}\right) = 0.$$

Which can be written in the form

$$\sum_{n=0}^{\infty} \left[(\lambda + n)(\lambda + n - 1)\alpha_n + \sum_{j=0}^{n}(\lambda + j)\alpha_j p_{n-j} + \sum_{j=0}^{n} \alpha_j q_{n-j}\right] u^{\lambda+n} = 0,$$

or

$$\alpha_0\chi(\lambda)u^\lambda + \sum_{n=1}^{\infty}\left[\chi(\lambda+n)\alpha_n + \sum_{j=0}^{n-1}\left((\lambda+j)p_{n-j} + q_{n-j}\right)\alpha_j\right]u^{\lambda+n} = 0.$$

The last equation to be satisfied for all u, the coefficient of each power of u in the last equation must be zero. Since $\alpha_0 \neq 0$, the coefficient of u^λ becomes

$$\chi(\lambda) = \lambda(\lambda-1) + p_0\lambda + q_0 = \lambda(\lambda-1) + \frac{1}{2}\lambda = 0.$$

Note that it is exactly the polynomial equation we would obtain for the Euler equation (3.19) associated with equation (3.18). The roots are $\lambda_1 = 0$, $\lambda_2 = 1/2$, these values of λ are called the exponents at the singularity for the regular singular point $u = 0 (r = 1)$. They determine the qualitative (nature) behavior of the solution (3.20) in the neighborhood of the singular point. Now let the coefficient of $u^{\lambda+n}$ equal to zero. This gives the relation

$$\chi(\lambda+n)\alpha_n + \sum_{j=0}^{n-1}\left((\lambda+j)p_{n-j} + q_{n-j}\right)\alpha_j = 0, \ n \geq 1.$$

The last equation shows that α_n depends on the value of λ and all of the preceding coefficients $\alpha_0, \alpha_1, ..., \alpha_{n-1}$, and the coefficients in the series for $up(u)$ and $u^2q(u)$ provided that $\chi(\lambda+n) \neq 0$, $n \geq 1$. Then the last equation can be written in the form

$$\alpha_n(\lambda) = -\frac{1}{\chi(\lambda+n)}\sum_{j=0}^{n-1}\left((\lambda+j)p_{n-j} + q_{n-j}\right)\alpha_j = 0, \ n \geq 1. \quad (3.21)$$

Since these roots are real, different and do not differ by an integer, there exist two linearly independent solutions of equation (3.18) of the form

$$\epsilon_1(u) = 1 + \sum_{n=1}^{\infty}\alpha_n(0)u^n,$$

$$\epsilon_1(u) = u^{\frac{1}{2}}\left(1 + \sum_{n=1}^{\infty}\alpha_n(\frac{1}{2})u^n\right),$$

where $\alpha_n(0)$ and $\alpha_n(\frac{1}{2})$ are given by relation (3.21) with $\alpha_0 = 1$.
Since equation (3.17) has the form

$$\Big(R_2 u^2 + R_3 u^3 + O(u^4)\Big)\epsilon''(u) + \Big(-P_1 u - P_2 u^2 - P_3 u^3 + O(u^4)\Big)\,\epsilon'(u)$$
$$+\Big(Q_0 + Q_1 u + Q_2 u^2 + Q_3 u^3 + O(u^4)\Big)\epsilon(u) = 0.$$

We could choose special case of the form

$$R_2 u^2\epsilon''(u) + (-P_1)u\epsilon'(u) + \big(Q_0 + Q_1 u\big)\epsilon(u) = 0.$$

or

$$u^2\epsilon''(u) + \frac{(-P_1)}{R_2}u\epsilon'(u) + \frac{(Q_0 + Q_1 u)}{R_2}\,\epsilon(u) = 0.$$

or

$$u^2\epsilon''(u) + p_0 u\epsilon'(u) + (q_0 + q_1 u)\,\epsilon(u) = 0. \qquad (3.22)$$

where

$$p_0 = \frac{-P_1}{R_2} = -\frac{\Big(-1/2 f_0'^2(1)g^2(1)\Big)}{f_0'^2(1)g^2(1)} = \frac{1}{2},$$
$$q_0 = 0,$$
$$q_1 = \frac{Q_1}{R_2} = -\frac{b'(1)}{f_0'^2(1)g^2(1)},$$

To solve equation (3.22) which special case of equation (3.18), we obtain from the general relation (3.21) the special relation in the form

$$\alpha_n = -\frac{q_1\alpha_{n-1}}{(\lambda + n)(\lambda + n - \frac{1}{2})}, \quad n \geq 1, \qquad (3.23)$$

where

$$q_1 = \frac{Q_1}{R_2} = -\frac{b'(1)}{f_0'^2(1)g^2(1)}.$$

For each root λ_1 and λ_2 we use equation (3.23) to determine the coeficients $\alpha_1, \alpha_2, ...\alpha_n....$ For $\lambda_1 = 0$, we obtain

$$\alpha_n = -\frac{q_1 \alpha_{n-1}}{n(n-\frac{1}{2})}, \quad n \geq 1$$

Thus

$$\begin{aligned} \alpha_1 &= -\frac{2q_1\alpha_0}{1 \cdot 1} \\ \alpha_2 &= \frac{(2q_1)^2\alpha_0}{(1\cdot 2)(1 \cdot 3)} \\ \alpha_3 &= -\frac{(2q_1)^3\alpha_0}{(1 \cdot 2 \cdot 3)(1 \cdot 3 \cdot 5)}. \end{aligned}$$

In general we get

$$\alpha_n = \frac{(-1)^n (2q_1)^n \alpha_0}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]n!}, \quad n > 1.$$

Therefore, when $\alpha_0 = 1$, one solution of equation (3.21) is

$$\epsilon_1(u) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2q_1)^n \alpha_0 \, u^n}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]n!}, \quad u > 0$$

This series converges for all u because

$$\lim_{n \to \infty} \left| \frac{\alpha_{n+1} u^{n+1}}{\alpha_n u^n} \right| = \lim_{n \to \infty} \frac{2|q_1 u|}{(2n+1)(n+1)} = 0.$$

Corresponding to the second root $\lambda = \lambda_2 = 1/2$, we have

$$\alpha_n = -\frac{q_1 \alpha_{n-1}}{n(n+\frac{1}{2})}, \quad n \geq 1$$

Thus

$$\begin{aligned} \alpha_1 &= -\frac{2q_1\alpha_0}{1 \cdot 3} \\ \alpha_2 &= \frac{(2q_1)^2\alpha_0}{(1\cdot 2)(3 \cdot 5)} \\ \alpha_3 &= -\frac{(2q_1)^3\alpha_0}{(1 \cdot 2 \cdot 3)(3 \cdot 5 \cdot 7)}. \end{aligned}$$

In general we get

$$\alpha_n = \frac{(-1)^n (2q_1)^n \alpha_0}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!}, \; n > 1.$$

Therefore, when $\alpha_0 = 1$, another solution of equation (3.21) is

$$\epsilon_2(u) = u^{\frac{1}{2}} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2q_1)^n \alpha_0 \, u^n}{[1 \cdot 3 \cdot 5 \cdots (2n+1)]n!}\right), \; u > 0$$

Also this series converges for all u because

$$\lim_{n\to\infty} \left| \frac{\alpha_{n+1} u^{n+1}}{\alpha_n u^n} \right| = \lim_{n\to\infty} \frac{2q_1 |u|}{(2n+3)(n+1)} = 0.$$

Since the leading terms in the series solutions $\epsilon_1(u)$ and $\epsilon_2(u)$ are u^0 and $u^{\frac{1}{2}}$, respectively, it follows that the solutions are linearly independent. Hence the general solution of equation (3.21) in the form

$$\epsilon(u) = \beta_1 \epsilon_1(u) + \beta_2 \epsilon_2(u), \text{ where } \beta_1, \, \beta_2 \text{ are constants, } u > 0.$$

The above solution are recognised as Bessel functions and we have

$$\epsilon(u) = u^{\frac{1}{4}} \left[c_1 J_{\frac{1}{2}}(2q_1 u^{\frac{1}{2}}) + c_2 J_{-\frac{1}{2}}(2q_1 u^{\frac{1}{2}}) \right], \; u > 0$$

where

$$J_{\frac{1}{2}}(2q_1 u^{\frac{1}{2}}) = \sqrt{\frac{1}{\pi q_1 u^{\frac{1}{2}}}} \, \sin(2q_1 u^{\frac{1}{2}}), \; u > 0$$

$$J_{-\frac{1}{2}}(2q_1 u^{\frac{1}{2}}) = \sqrt{\frac{1}{\pi q_1 u^{\frac{1}{2}}}} \, \cos(2q_1 u^{\frac{1}{2}}), \; u > 0.$$

Thus

$$\epsilon(u) = \frac{1}{\sqrt{\pi q_1}} \left[c_1 \sin(2q_1 \sqrt{u}\,) + c_2 \cos(2q_1 \sqrt{u}\,) \right], \; u > 0.$$

If we chose to keep some more terms in equation (3.17) we could consider the form

$$\left(R_2 u^2 + R_3 u^3 \right) \epsilon''(u) + u \left(-P_1 - P_2 u \right) \epsilon'(u) + \left(Q_0 + Q_1 u \right) \epsilon(u) = 0.$$

This equation is a Hypergeometric equation

$$w\Big(1-w\Big)\epsilon''(w)+\Big(c-\frac{\beta}{\beta_1}w\Big)\,\epsilon'(w)-\frac{\beta_2}{\beta_1}\epsilon(w)=0,$$

where

$$\begin{aligned}
w &= \beta_1 u,\\
c &= -\frac{P_1}{R_2}=\frac{1}{2}\\
\beta &= \frac{P_2}{R_2}=\frac{a'(1)}{f_0'(1)g(1)}+\frac{1}{4}\beta_1,\\
\beta_1 &= -\frac{R_3}{R_2}=\frac{2g'(1)}{g(1)}+\frac{f_0''(1)}{f_0'(1)},\\
\beta_2 &= -\frac{Q_1}{R_2}=\frac{b'(1)}{f_0'^2(1)g^2(1)},
\end{aligned}$$

and so has a solution of the form

$$\epsilon(w)=A\;{}_2F_1(a,b;c;w)+B\,w^{1-c}\,{}_2F_1(a-c+1,b-c+1;2-c;w)$$

where A and B being arbitrary constants and the constants a, b satisfy the equations

$$b^2+(1-\frac{\beta}{\beta_1})b+\frac{\beta_2}{\beta_1}=0,$$

$$a=\frac{\beta_2}{b\,\beta_1}.$$

3.5 Conclusion

In this chapter we have obtained the ϵ-equation which characterises the linear stability of solutions of the f-equation and consequently provides information about the stability of the periodic solutions of the many-body problems highlighted in earlier chapters. The ϵ-equation is of second order with complicated coefficients depending on f_0, a known solution, and other known functions. We may express this equation in a particularly pleasing form where the leading terms are written $(\psi\epsilon')'$. Given a series representation, suitably well behaved, for f_0 we have obtained a solution in series form and commented on its region of convergence. The complicated nature of the ϵ-equation has led us to develop an approximate analytical solution in the region of $r = 1$, $u = 0$. This exact analytical solution can be cast in several forms all exhibiting the same basic behaviour. Two of these can be written in terms of familiar special functions, Bessel and Hypergeometric. In order to study and exhibit the generic behaviour of solution to the ϵ-equation we must procede to a numerical solution of the equation for a particular system and specific f_0. We do so in the next chapter for our now familiar set of illustrative examples.

Chapter 4

NUMERICAL APPROACH

4.1 Introduction

The ϵ-equation as developed in Chapter 3, is best studied from a numerical point of view due to the lack of exact solutions to the f-equation. We will obtain numerical solutions of the ϵ-equation for our now familiar standard problems; four body gravitational, four body ionic, four electron atom and four electron atom with magnetic field. In each of these cases we will use the solution of the f-equation corresponding to the four node periodic solution. We have chosen to use a fourth order Runge Kutta scheme to solve the ϵ-equation as this is both familiar and highly accurate. We do not use native Mathematica code to solve our differential equations due to the somewhat obscure behaviour of the ϵ-equation at $r = 1\,(u = 0)$. We have chosen to solve for both the known solution of the f-equation, f_0, and the sought solution of the ϵ-equation simultaneously in order to avoid having to precompute and store a large amount of data for our numerical representation of f_0. With this approach we also have the freedom to vary the step length associated with the Runge Kutta method without having to ensure that it is an integer multiple of some datum step length associated with a precomputed representation of f_0. We have to exercise some care in starting the numerical solution due to the nature of the ϵ-equation and must compute the values of the coefficients of ϵ and ϵ' at $r = 1\,(u = 0)$. We have one algorithm to perform the calculations in the first instance and then the usual algorithm is used to handle the computation thereon. Note that the nature of the ϵ-equation forces ϵ and ϵ' to be related at $r = 1\,(u = 0)$.

4.2 The general ϵ-equation

To study numerically the generic behaviour of solutions to the ϵ-equation for our standard examples, we consider the general ϵ-equation in the form

$$f_0^2(r)g^2(f_0(r),r,h,E)\epsilon''(r) + f_0(r)g(f_0(r),r,h,E)a(r)\epsilon'(r) + b(r)\epsilon(r) = 0, \tag{4.1}$$

where

$$\begin{aligned}
g &= g(f_0(r),r,h,E) = [2E - 2F_3(f_0(r),r,h)],\\
a(r) &= \frac{1}{2}f_0(r)g'(f_0(r),r,h,E) - f_0'(r)g(f_0(r),r,h,E)\\
&\quad + \frac{3}{4}f_0'(r)\Big(f_0'(r)g_1(f_0(r),r,h) - 2f_0(r)g_2(f_0(r),r)\Big),\\
b(r) &= \Bigg(\frac{1}{2}f_0'^2(r)g(f_0(r),r,h,E) - \Big(\frac{1}{4}f_0'^2(r) + f_0(r)\Big)\\
&\quad \Big(f_0'(r)g_1(f_0(r),r,h) - 2f_0(r)g_2(f_0(r),r)\Big)\Bigg)\Big(g(f_0(r),r,h,E) + f_0(r)g_2(f_0(r),r)\Big)\\
&\quad + f_0(r)g(f_0(r),r,h,E)\Bigg(f_0'(r)g_1(f_0(r),r,h)\\
&\quad + (\frac{1}{4}f_0'^2(r) + f_0(r))\Big[\Big(\frac{1}{2}rf_0'(r) - 2f_0(r)\Big)g_3(f_0(r),r) - 4g_2(f_0(r),r)\Big]\Bigg),\\
g' &= \left.\frac{dg(f(r),r,h,E)}{dr}\right|_{f_0} = 2\Big(g_1(f_0(r),r,h) + \frac{1}{2}f_0'(r)g_2(f_0(r),r)\Big),
\end{aligned}$$

as before. This gives

$$\begin{aligned}
g_1(f_0(r),r,h) &= \left.\frac{1}{2}\frac{\partial g(f(r),r,h,E)}{\partial r}\right|_{f_0} = F_2(f_0(r),r,h),\\
g_2(f_0(r),r) &= \left.\frac{\partial g(f(r),r,h,E)}{\partial f}\right|_{f_0} = F_1(f_0(r),r),\\
g_3(f_0(r),r) &= \left.\frac{\partial^2 g(f(r),r,h,E)}{\partial f^2}\right|_{f_0} = \left.\frac{\partial F_1(f(r),r)}{\partial f}\right|_{f_0}.
\end{aligned}$$

As mentioned previously $\epsilon(1)$ and $\epsilon'(1)$ are linearly dependent and so we choose to solve for $\epsilon(r)$, $r_m \le r \le 1$, in the two cases $\epsilon(1) = 1$ and $\epsilon''(1) = 0$ and $\epsilon(1) = 0$ and $\epsilon''(1) = 1$.

4.3 The four body gravitational problem

In this section we consider the case of the four body gravitational problem. We have the general form of the ϵ-equation (4.1) with,

$$g(f_0(r), r, h, E) = 2E - \frac{h^2}{r^2} + 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} + \frac{1}{2r},$$

$$g_1(f_0(r), r, h) = \frac{h^2}{r^3} - 2r\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} - \frac{1}{4r^2},$$

$$g_2(f_0(r), r) = -4\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}},$$

$$g_3(f_0(r), r) = 24\left(2r^2 + 4f_0(r)\right)^{-\frac{5}{2}}.$$

The initial data for the four node solution of this system gives us $f_0'(1) = -1.675076608$ with $f_0(1) = 0$ as usual and $r_{min} = 0.219$.

4.3.1 Case $\epsilon(1) = 1$ and $\epsilon''(1) = 0$

The numerical solution of the ϵ-equation under these initial conditions is plotted below. Note that the solution is well behaved with no excessive growth.

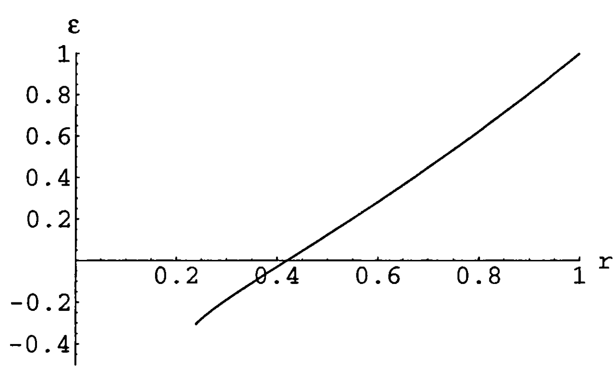


Figure 4.1:the ϵ-solution for the 4nbgp

4.3.2 Case $\epsilon(1) = 0$ and $\epsilon''(1) = 1$

Again note that after an initial period of rapid growth the solution settles down and remains very small.

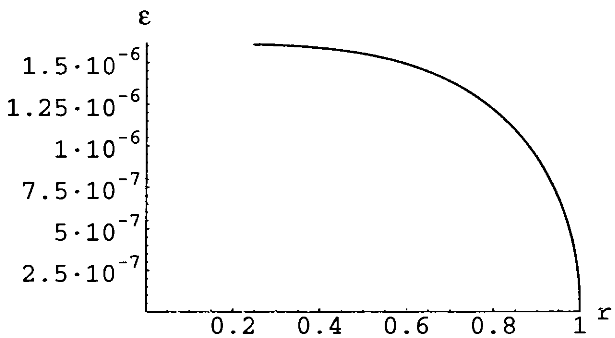


Figure 4.2:the ϵ-solution for the 4nbgp

4.4 The four body ionic problem

We consider the case of the four body ionic problem. We have the general form of the ϵ-equation (4.1) with,

$$g(f_0(r), r, h, E) = 2E - \frac{h^2}{r^2} + 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r},$$

$$g_1(f_0(r), r, h) = \frac{h^2}{r^3} - 2r\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} + \frac{1}{4r^2},$$

$$g_2(f_0(r), r) = -4\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}},$$

$$g_3(f_0(r), r) = 24\left(2r^2 + 4f_0(r)\right)^{-\frac{5}{2}}.$$

The initial data for the four node solution of this system gives us $f_0'(1) = -2.571113999$ with $f_0(1) = 0$ as usual and $r_{min} = 0.866$.

4.4.1 Case $\epsilon(1) = 1$ and $\epsilon''(1) = 0$

The numerical solution of the ϵ-equation under these initial conditions is plotted below. Note that the solution is well behaved with no excessive growth.

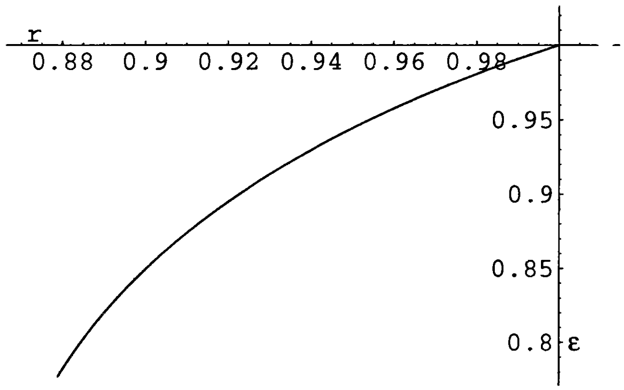


Figure 4.3:the ϵ-solution for the 4nbip

4.4.2 Case $\epsilon(1) = 0$ and $\epsilon''(1) = 1$

Again note that after an initial period of rapid growth the solution settles down and remains very small.

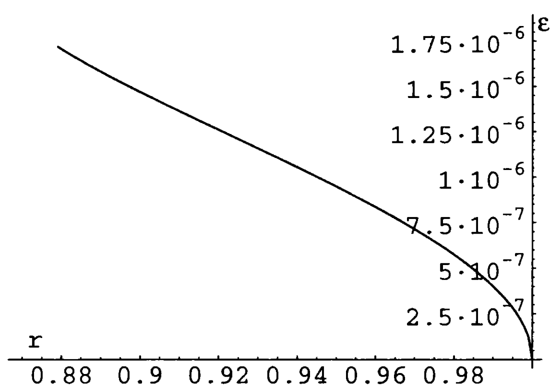


Figure 4.4:the ϵ-solution for the 4nbip

4.5 The four electron atom problem

In this section we consider the case of the four electron atom problem. We have the general form of the ϵ-equation (4.1) with,

$$g(f_0(r), r, h, E) = 2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}},$$

$$g_1(f_0(r), r, h) = \frac{h^2}{r^3} + 2r\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} + \frac{1}{4r^2} - 4r\left(r^2 + f_0(r)\right)^{-\frac{3}{2}},$$

$$g_2(f_0(r), r) = 4\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} - 4\left(r^2 + f_0(r)\right)^{-\frac{3}{2}},$$

$$g_3(f_0(r), r) = -24\left(2r^2 + 4f_0(r)\right)^{-\frac{5}{2}} + 6\left(r^2 + f_0(r)\right)^{-\frac{5}{2}}.$$

The initial data for the four node solution of this system gives us $f_0'(1) = -2.180573661$ with $f_0(1) = 0$ as usual and $r_{min} = 0.171$.

4.5.1 Case $\epsilon(1) = 1$ and $\epsilon''(1) = 0$

The numerical solution of the ϵ-equation under these initial conditions is plotted below. Note that the solution is well behaved with no excessive growth.

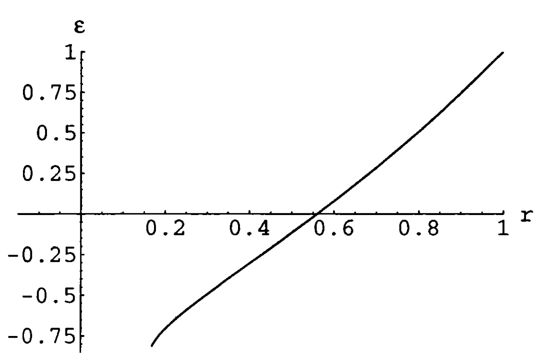


Figure 4.5:the ϵ-solution for the 4neap

4.5.2 Case $\epsilon(1) = 0$ and $\epsilon''(1) = 1$

Again note that after an initial period of rapid growth the solution settles down and remains very small.

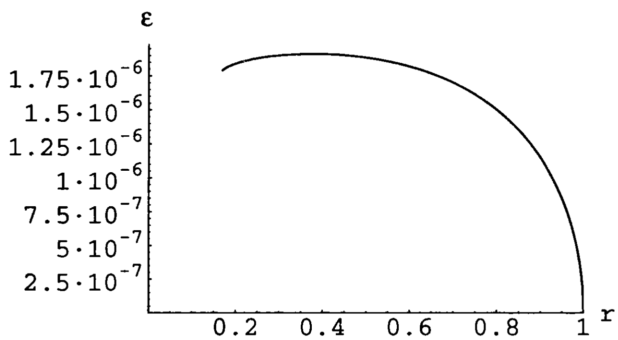


Figure 4.6:the ϵ-solution for the 4neap

4.6 The four electron atom with magnetic field problem

We consider the case of the four electron atom problem with constant magnetic field. We have the general form of the ϵ-equation (4.1) with,

$$g(f_0(r), r, h, E) = 2E - \frac{h^2}{r^2} - 2\left(2r^2 + 4f_0(r)\right)^{-\frac{1}{2}} - \frac{1}{2r} + 8\left(r^2 + f_0(r)\right)^{-\frac{1}{2}} - \frac{B^2r^2}{4},$$

$$g_1(f_0(r), r, h) = \frac{h^2}{r^3} + 2r\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} + \frac{1}{4r^2} - 4r\left(r^2 + f_0(r)\right)^{-\frac{3}{2}} - \frac{B^2r}{4},$$

$$g_2(f_0(r), r) = 4\left(2r^2 + 4f_0(r)\right)^{-\frac{3}{2}} - 4\left(r^2 + f_0(r)\right)^{-\frac{3}{2}},$$

$$g_3(f_0(r), r) = -24\left(2r^2 + 4f_0(r)\right)^{-\frac{5}{2}} + 6\left(r^2 + f_0(r)\right)^{-\frac{5}{2}}.$$

The initial data for the four node solution of this system gives us $f_0'(1) = -2.218274657$ with $f_0(1) = 0$ as usual and $r_{min} = 0.157$.

4.6.1 Case $\epsilon(1) = 1$ and $\epsilon''(1) = 0$

The numerical solution of the ϵ-equation under these initial conditions is plotted below. Note that the solution is well behaved with no excessive growth.

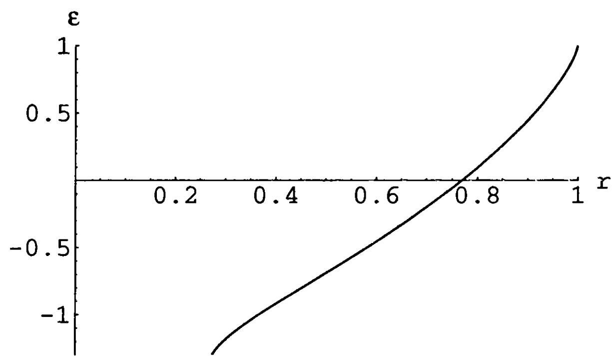


Figure 4.7:the ϵ-solution for the 4neapB

4.6.2 Case $\epsilon(1) = 0$ and $\epsilon''(1) = 1$

Again note that after an initial period of rapid growth the solution settles down and remains very small.

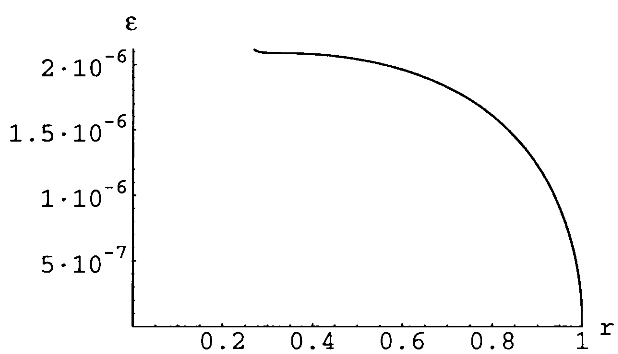


Figure 4.8:the ϵ-solution for the 4neapB

4.7 Comparisons with approximate analytic solution

It is interesting to compare the approximate analytical solution, $J(u)$, of Chapter 3 with the numerical solution $\epsilon(u)$ as follows.
We have from the approximate analytical solution for the four node solution of the four body gravitional problem, $b'(1) = -1.388353518$,
$f'(1) = -1.675076608$, and $g(1) = 0.708600303$, that gives

$$q_1 = -\frac{b'(1)}{f'^2(1)g^2(1)} = 0.985435153,$$

we also have

$$J(u) = \cos(2q_1\sqrt{u}),$$

that gives $J(0) = 1$, $J'(0) = -2q_1^2 = -1.9421564$, $J''(0) = 0$.
We have from the numerical solution for the four node solution of the four body gravitional problem, $\epsilon(0) = 1$, $\epsilon'(0) = -1.9708701$, $\epsilon''(0) = 0$.
The diagram of the comparison of numerical and approximate analytical solutions of the ϵ-solution for the four body gravitational problem is illustrated below

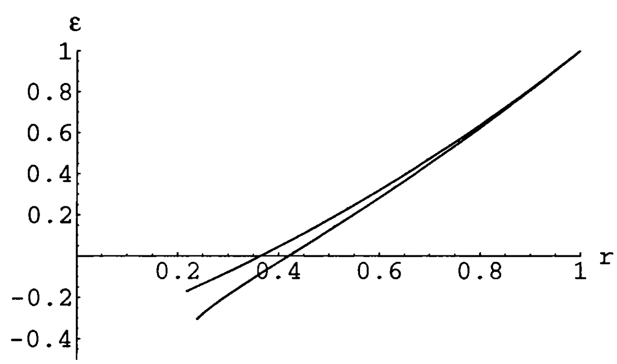


Figure 4.9:comparison of solutions of the ϵ-solution for the 4nbgp

4.8 Conclusion

First let us emphasize that the Mathematica code used in the calculations illustrated herein is detailed in Chapter 6. Many of the necessary calculations performed could only be easily done with access to software like Mathematica.

The subsequent manipulation of the data strings representing the solutions of the ϵ-equation was again eased by use of Mathematica. Note that in all the cases above the general behaviour of the solution to the ϵ-equation is well behaved. The comparison of numerical and approximate analytical solutions of the ϵ-solution for the four body gravitational problem appears to be good.

Chapter 5

New styles of Periodic Solutions and DTW-Periodic Solutions for new potentials

5.1 Introduction

In the process of studying the f-equation and the ϵ-equation we began to comprehend that there could be other styles of periodic solution accessible by our standard methods. The papers of MR [49] and DTW [21], [22] were crucial in prompting our initial thoughts in this new direction. As often happens our attention had been brought, by I Stewart, to the recent work of Chenciner et al and we were pleased to note that while the DTW solutions were considered by Chenciner our new structures were not. The planar periodic solutions of Chenciner and Montgomery [14] have a little more in common with our new style of periodic solution. The initial work of DTW concentrated on periodic solutions where each particle followed a path which was that of one particle subject to some fixed rotation. Other forms of solution were accessible to this ansatz but essentially satisfied the description above. The new types of periodic solution fall into two categories; weaving and chasing. The weaving periodic solutions are characterized by the particles moving towards each other in pairs and following one of two paths related by rotation and (or) reflection. The axial symmetry of the solutions is however preserved. The chasing periodic solutions are characterised by the particles pairwise chasing each other. Let us also note that the original DTW style of solutions can be obtained with a logarithmic pair potential replacing the Kepler/Coulomb potential. The papers of DTW [21], [22] and the book of Bradbury [4] were crucial in prompting our initial thoughts in this way by changing potential energy to obtain DTW-solutions. One interesting feature of the logarithmic solutions is the appearance of double points in the initial data space corresponding to specified nodal structures. We also have the appearance of periodic orbits with the same nodal structure but defferent winding numbers. In the work of DTW these were denotes by use of a notation like "11/7", ie. 11 nodes with 7 revolutions required to complete the orbit. We also obtain the new style, weaving solution, with the logarithmic potential energy.

5.2 The new styles

In this section we will describe the algebra and symmetry that allows us to reduce a full system of equations to just those for essentially one body in the new styles of motion. The two main points to keep track of are the preservation of the axial symmetries and the non-collision of the bodies. In the weaving case the bodies are initially situated at the vertices of a regular polygon as in the case of the DTW periodic solutions. The bodies are then considered in adjacent pairs to move towards each other rotationally about a common axis but with opposing signs for their angular velocities and linear velocities orthogonal to the initial reference plane of the polygonal arrangement. It is not too difficult to envisage this style of motion for the four body problems but it is a bit more awkward in the case of the six body problems. In the chasing style, for the six body problem, the bodies are initially positioned on the coordinate axes each at the same distance from the center of mass (the origin). The positions of the bodies are related essentially by shifting the coordinates to the left or to the right by one place and possibly reflecting in the origin. This ensures that the configuration can never be planar. The full flavour of the algebra and symmetry is presented below.

5.3 The algebra of P_1 , P_2 and $P = P_1P_2$, $Q = P_2P_1$

5.3.1 Four electron case $(2,2)$

We shall start by studying the algebra of P_1 , P_2 and P, Q in the case of $(2,2)$-electron atom problem where $P = Q$. These are represented by

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $P_1^2 = I$, $P_2^2 = I$, $P^2 = I$, $\det P_1 = 1$, $\det P_2 = 1$, the eigenvalues of P_1, P_2 and P are -1 with algebraic multiplicity 2, 1, with corresponding eigenvectors for P_1,

$$v_1 = (0,0,1),\; v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \text{and}\, v_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \text{ respectively,}$$

with corresponding eigenvectors for P_2 ,

$$v_1 = (0,0,1),\; v_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), \text{and}\, v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \text{ respectively.}$$

with corresponding eigenvectors for P,

$$u_1 = (1,0,0),\; u_2 = (0,1,0), \text{and}\, u_3 = (0,0,1) \text{ respectively.}$$

Now applying P_1 to the equation of motion of $\mathbf{q}_1$, to obtain the equation of motion of $\mathbf{q}_2$ and then we will apply P_2 to this equation to get the equation of motion of $\mathbf{q}_3$ and by applying P_1 to it to obtain the equation of motion of $\mathbf{q}_4$ and by applying P_2 to last equation we will return to the first equation. Now, let $\mathbf{q}_1 = (x,y,z)$, then, $\mathbf{q}_2 = P_1\mathbf{q}_1 = (y,x,-z)$, and so $\mathbf{q}_3 = P_2\mathbf{q}_2 = (-x,-y,z)$ and $\mathbf{q}_4 = P_1\mathbf{q}_3 = (-y,-x,-z)$. We choose two families, the first is up which is $(1,3)$ and the second is down which is $(2,4)$. Also we have $1 \overset{P_1}{\Longleftrightarrow} 2$, $3 \overset{P_1}{\Longleftrightarrow} 4$ and $1 \overset{P_2}{\Longleftrightarrow} 4$, $2 \overset{P_2}{\Longleftrightarrow} 3$, also we get $1 \overset{P}{\Longleftrightarrow} 3$, $2 \overset{P}{\Longleftrightarrow} 4$. Then, we shall solve the equation of motion of $\mathbf{q}_1$. The equation of motion of $\mathbf{q}_1$ is

$$\begin{aligned}
\ddot{\mathbf{q}}_1 &= \sum_{j=2}^{4} k_1(\mathbf{q}_j - \mathbf{q}_1)|\mathbf{q}_j - \mathbf{q}_1|^{-3} - 4\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\left[\sum_{j=2}^{4}(\mathbf{q}_j - \mathbf{q}_1)|\mathbf{q}_j - \mathbf{q}_1|^{-3}\right] - 4\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\Big[(\mathbf{q}_2 - \mathbf{q}_1)|\mathbf{q}_2 - \mathbf{q}_1|^{-3} + (\mathbf{q}_3 - \mathbf{q}_1)|\mathbf{q}_3 - \mathbf{q}_1|^{-3} + (\mathbf{q}_4 - \mathbf{q}_1)|\mathbf{q}_4 - \mathbf{q}_1|^{-3}\Big] \\
&\quad - 4\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\Big[(P_1 - I)\mathbf{q}_1|(P_1 - I)\mathbf{q}_1|^{-3} + (P - I)\mathbf{q}_1|(P - I)\mathbf{q}_1|^{-3} \\
&\quad + (P_2 - I)\mathbf{q}_1|(P_2 - I)\mathbf{q}_1|^{-3}\Big] - 4\mathbf{q}_1|\mathbf{q}_1|^{-3},
\end{aligned}$$

where $k = -1$, with initial position $\mathbf{q}_1(0) = (2, 0, 0)$ and initial velocity $\dot{\mathbf{q}}_1(0) = (0, \dot{y}_0, \dot{z}_0)$. Now, we want to know the coordinates of $(P_i - I)\mathbf{q}_1$ with $\mathbf{q}_1 = (x, y, z)$. Consider

$$\begin{aligned}
\mathbf{q}_2 - \mathbf{q}_1 &= (P_1 - I)\mathbf{q}_1 \\
&= (y - x, x - y, -2z),
\end{aligned}$$

which gives

$$|(P_1 - I)\mathbf{q}_1|^{-3} = \left(2(x - y)^2 + 4z^2\right)^{-\frac{3}{2}}.$$

Consider

$$\begin{aligned}
\mathbf{q}_3 - \mathbf{q}_1 &= (P - I)\mathbf{q}_1 \\
&= (-2x, -2y, 0),
\end{aligned}$$

which gives

$$|(P - I)\mathbf{q}_1|^{-3} = \left(4(x^2 + y^2)\right)^{-\frac{3}{2}}.$$

Consider

$$\begin{aligned}
\mathbf{q}_4 - \mathbf{q}_1 &= (P_2 - I)\mathbf{q}_1 \\
&= (-(x + y), -(x + y), -2z),
\end{aligned}$$

which gives

$$|(P_2 - I)\mathbf{q}_1|^{-3} = \Big(4(x+y)^2 + 4z^2\Big)^{-\frac{3}{2}}.$$

5.3.2 Six electron case $(3,3)$

We shall start by studying the algebra of P_1, P_2 and P, Q in the case of $(3,3)$-electron atom problem where $P \neq Q$. These are represented by

$$P_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and}$$

$$P = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $P_1^3 = I$, $P_2^2 = I$, $P^2 = I$, $Q^2 = I$, $\det P_1 = 1$, $\det P_2 = 1$, $\det P = 1$, $\det Q = 1$. The eigenvalues of P_1 are $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, the eigenvalues of P_2 is -1 with algebraic multiplicity 2 , 1 and the eigenvalues of P and Q are -1 with algebraic multiplicity 2 , 1, with corresponding eigenvectors for P_1,

$$v_1 = (0,0,1),\ v_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} i, 0), \text{and}\ v_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} i, 0) \text{ respectively,}$$

and with corresponding eigenvectors for P_2,

$$u_1 = (1,0,0), \quad u_2 = (0,1,0), \text{and}\ u_3 = (0,0,1) \text{ respectively.}$$

with corresponding eigenvectors for P,

$$w_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0),\ u_3 = (0,0,1), \text{and}\ w_2 = (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0) \text{ respectively.}$$

with corresponding eigenvectors for Q,

$$w_0 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0), \; u_3 = (0,0,1), \text{and}\, w_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0) \text{ respectively.}$$

Now applying P_1 to the equation of motion of $\mathbf{q}_1$, to obtain the equation of motion of $\mathbf{q}_2$ and then we will apply P_1 to the second equation to get the equation of motion of $\mathbf{q}_3$ and by applying P_2 to the equation of motion of $\mathbf{q}_3$ to obtain the equation of motion of $\mathbf{q}_4$ and by applying P_1 to fourth equation to get fifth equation and by applying P_1 to fifth equation to obtain the equation of motion of $\mathbf{q}_6$ and by applying P_2 to the sixth equation we will return to the first equation. Now, let $\mathbf{q}_1 = (x, y, z)$, then, $\mathbf{q}_2 = P_1\mathbf{q}_1 = (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y, z)$, and so $\mathbf{q}_3 = P_1\mathbf{q}_1 = (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x - \frac{1}{2}y, z)$ and $\mathbf{q}_4 = P_2\mathbf{q}_3 = (\frac{1}{2}x - \frac{\sqrt{3}}{2}y, -\frac{\sqrt{3}}{2}x - \frac{1}{2}y, -z)$, $\mathbf{q}_5 = P_1\mathbf{q}_4 = (\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y, -z)$ and $\mathbf{q}_6 = P_1\mathbf{q}_5 = (-x, y, -z)$. We choose two family the first is up which is $(1,2,3)$ and the second is down which is $(4,5,6)$ suth that $1 \overset{P_1}{\Longrightarrow} 2 \overset{P_1}{\Longrightarrow} 3 \overset{P_1}{\Longrightarrow} 1$ and $4 \overset{P_1}{\Longrightarrow} 5 \overset{P_1}{\Longrightarrow} 6 \overset{P_1}{\Longrightarrow} 4$ and $1 \overset{P_2}{\Longleftrightarrow} 6$, $2 \overset{P_2}{\Longleftrightarrow} 5$, $3 \overset{P_2}{\Longleftrightarrow} 4$, also we get $1 \overset{P}{\Longleftrightarrow} 4$, $2 \overset{P}{\Longleftrightarrow} 6$, $3 \overset{P}{\Longleftrightarrow} 5$ and $1 \overset{Q}{\Longleftrightarrow} 5$, $2 \overset{Q}{\Longleftrightarrow} 4$, $3 \overset{Q}{\Longleftrightarrow} 6$. Then, we shall solve the equation of motion of $\mathbf{q}_1$, The equation of motion $\mathbf{q}_1$ is

$$\begin{aligned}
\ddot{\mathbf{q}}_1 &= \sum_{j=2}^{6} k_1(\mathbf{q}_j - \mathbf{q}_1)|\mathbf{q}_j - \mathbf{q}_1|^{-3} - 6\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\left[\sum_{j=2}^{6}(\mathbf{q}_j - \mathbf{q}_1)|\mathbf{q}_j - \mathbf{q}_1|^{-3}\right] - 6\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\Big[(\mathbf{q}_2 - \mathbf{q}_1)|\mathbf{q}_2 - \mathbf{q}_1|^{-3} + (\mathbf{q}_3 - \mathbf{q}_1)|\mathbf{q}_3 - \mathbf{q}_1|^{-3} + (\mathbf{q}_4 - \mathbf{q}_1)|\mathbf{q}_4 - \mathbf{q}_1|^{-3} \\
&\quad + (\mathbf{q}_5 - \mathbf{q}_1)|\mathbf{q}_5 - \mathbf{q}_1|^{-3} + (\mathbf{q}_6 - \mathbf{q}_1)|\mathbf{q}_6 - \mathbf{q}_1|^{-3}\Big] - 6\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\Big[(P_1 - I)\mathbf{q}_1|(P_1 - I)\mathbf{q}_1|^{-3} + (P_1^2 - I)\mathbf{q}_1|(P_1^2 - I)\mathbf{q}_1|^{-3} \\
&\quad + (P - I)\mathbf{q}_1|(P - I)\mathbf{q}_1|^{-3} + (Q - I)\mathbf{q}_1|(Q - I)\mathbf{q}_1|^{-3} \\
&\quad + (P_2 - I)\mathbf{q}_1|(P_2 - I)\mathbf{q}_1|^{-3}\Big] - 6\mathbf{q}_1|\mathbf{q}_1|^{-3},
\end{aligned}$$

where $k = -1$, with initial position $\mathbf{q}_1(0) = (2,0,0)$ and initial velocity $\dot{\mathbf{q}}_1(0) = (0, \dot{y}_0, \dot{z}_0)$. Now, we want to know the coordinates of $(P_i - I)\mathbf{q}_1$ with

$\mathbf{q}_1 = (x, y, z)$. Consider

$$\begin{aligned}\mathbf{q}_2 - \mathbf{q}_1 &= (P_1 - I)\mathbf{q}_1 \\ &= (-\frac{3}{2}x - \frac{\sqrt{3}}{2}y\,, \frac{\sqrt{3}}{2}x - \frac{3}{2}y\,, 0),\end{aligned}$$

which gives

$$|(P_1 - I)\mathbf{q}_1|^{-3} = \Big(3(x^2 + y^2)\Big)^{-\frac{3}{2}},$$

also

$$\begin{aligned}\mathbf{q}_3 - \mathbf{q}_1 &= (P_1^2 - I)\mathbf{q}_1 \\ &= (-\frac{3}{2}x + \frac{\sqrt{3}}{2}y\,, -\frac{\sqrt{3}}{2}x - \frac{3}{2}y\,, 0),\end{aligned}$$

which gives

$$|(P_1^2 - I)\mathbf{q}_1|^{-3} = \Big(3(x^2 + y^2)\Big)^{-\frac{3}{2}}.$$

Also

$$\begin{aligned}\mathbf{q}_4 - \mathbf{q}_1 &= (P - I)\mathbf{q}_1 \\ &= (-\frac{1}{2}x - \frac{\sqrt{3}}{2}y\,, -\frac{\sqrt{3}}{2}x - \frac{3}{2}y\,, -2z),\end{aligned}$$

which gives

$$|(P - I)\mathbf{q}_1|^{-3} = \Big((x + \sqrt{3}\,y)^2 + 4z^2\Big)^{-\frac{3}{2}},$$

also

$$\begin{aligned}\mathbf{q}_5 - \mathbf{q}_1 &= (Q - I)\mathbf{q}_1 \\ &= (-\frac{1}{2}x + \frac{\sqrt{3}}{2}y\,, \frac{\sqrt{3}}{2}x - \frac{3}{2}y\,, -2z),\end{aligned}$$

which gives

$$|(Q - I)\mathbf{q}_1|^{-3} = \Big((x - \sqrt{3}\,y)^2 + 4z^2\Big)^{-\frac{3}{2}},$$

also

$$\mathbf{q}_6 - \mathbf{q}_1 = (P_2 - I)\mathbf{q}_1$$
$$= (-2\,x\,, 0\,, -2z),$$

which gives

$$|(P_2 - I)\mathbf{q}_1|^{-3} = \left(4(\,x^2 + z^2\,)\right)^{-\frac{3}{2}}.$$

5.3.3 Six electron case $(2, 2, 2)$

In this subsection we shall start by studing the algebra of P_1 , P_2 and P, Q in the case of $(2, 2, 2)$-electron atom problem where $P = Q$. These are represented by

$$P_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \quad P = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

and $P_1^2 = I$, $P_2^3 = I$, $P^6 = I$, $\det P_1 = -1$; $\det P_2 = 1$, the eigenvalue of P_1 is -1 with algebraic multiplicity 3, the eigenvalues of P_2 are 1, $-1/2 \pm \sqrt{3}\,i/2$ and the eigenvalues of P are -1, $1/2 \pm \sqrt{3}\,i/2$, with corresponding eigenvectors for P_1,

$$v_1 = (1, 0, 0),\ v_2 = (0, 1, 0), \text{and}\, v_3 = (0, 0, 1) \text{ respectively,}$$

and with corresponding eigenvectors for P_2,

$$u_1 = \frac{1}{\sqrt{3}}(\,1, 1, 1),\ u_2 = \frac{1}{\sqrt{3}}(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i, -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i, 1) \text{ and}$$

$$u_3 = \frac{1}{\sqrt{3}}(-1/2 + \sqrt{3}\,i/2, -\frac{1}{2} - \frac{\sqrt{3}}{2}\,i, 1) \text{ respectively.}$$

with corresponding eigenvectors for P,

$$u_1 = \frac{1}{\sqrt{3}}(1,1,1),\ u_3 = \frac{1}{\sqrt{3}}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, 1), \quad \text{and}$$

$$u_2 = \frac{1}{\sqrt{3}}(-\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, 1) \text{ respectively.}$$

Now applying P_1 to the equation of motion of $\mathbf{q}_1$, to obtain the equation of motion of $\mathbf{q}_2$ and then we will apply P_1 to the first equation to get the equation of motion of $\mathbf{q}_3$ and by applying P_2 to the equation of motion of $\mathbf{q}_2$ to obtain the equation of motion of $\mathbf{q}_4$ and by applying P_1 to third equation to get fifth equation and by applying P_2 to fourth equation to obtain the equation of motion of $\mathbf{q}_6$ and by applying P_2 to the fifth equation we will return to the first equation. Now, let $\mathbf{q}_1 = (x,y,z)$, then, $\mathbf{q}_2 = P_1\mathbf{q}_1 = (-x,-y,-z)$, and so $\mathbf{q}_3 = P_2\mathbf{q}_1 = (z,x,y)$ and $\mathbf{q}_4 = P_1\mathbf{q}_3 = (-z,-x,-y)$, $\mathbf{q}_5 = P_2\mathbf{q}_3 = (y,z,x)$ and $\mathbf{q}_6 = P_1\mathbf{q}_5 = (-y,-z,-x)$ and We choose two family the first is up which is $(1,3,5)$ and the second is down which is $(2,4,6)$ suth that $1 \overset{P_1}{\Longleftrightarrow} 2$, $3 \overset{P_1}{\Longleftrightarrow} 4$, $5 \overset{P_1}{\Longleftrightarrow} 6$ and $1 \overset{P_2}{\Longrightarrow} 3 \overset{P_2}{\Longrightarrow} 5 \overset{P_2}{\Longrightarrow} 1$ and $2 \overset{P_2}{\Longrightarrow} 4 \overset{P_2}{\Longrightarrow} 6 \overset{P_2}{\Longrightarrow} 2$, also we get $1 \overset{P}{\Longrightarrow} 4$, $2 \overset{P}{\Longrightarrow} 3$, $3 \overset{P}{\Longrightarrow} 6$, $4 \overset{P}{\Longrightarrow} 5$ and $5 \overset{P}{\Longrightarrow} 2$, $6 \overset{P}{\Longrightarrow} 1$. Then, we shall solve the equation of motion of $\mathbf{q}_1$, The equation of motion $\mathbf{q}_1$ is

$$\begin{aligned}
\ddot{\mathbf{q}}_1 &= \sum_{j=2}^{6} k_1(\mathbf{q}_1 - \mathbf{q}_j)|\mathbf{q}_j - \mathbf{q}_1|^{-3} - 6\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\left[\sum_{j=2}^{6}(\mathbf{q}_j - \mathbf{q}_1)|\mathbf{q}_j - \mathbf{q}_1|^{-3}\right] - 6\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\Big[(\mathbf{q}_2 - \mathbf{q}_1)|\mathbf{q}_2 - \mathbf{q}_1|^{-3} + (\mathbf{q}_3 - \mathbf{q}_1)|\mathbf{q}_3 - \mathbf{q}_1|^{-3} + (\mathbf{q}_4 - \mathbf{q}_1)|\mathbf{q}_4 - \mathbf{q}_1|^{-3} \\
&\quad + (\mathbf{q}_5 - \mathbf{q}_1)|\mathbf{q}_5 - \mathbf{q}_1|^{-3} + (\mathbf{q}_6 - \mathbf{q}_1)|\mathbf{q}_6 - \mathbf{q}_1|^{-3}\Big] - 6\mathbf{q}_1|\mathbf{q}_1|^{-3} \\
&= k_1\Big[(P_1 - I)\mathbf{q}_1|(P_1 - I)\mathbf{q}_1|^{-3} + (P_2 - I)\mathbf{q}_1|(P_2 - I)\mathbf{q}_1|^{-3} \\
&\quad + (P - I)\mathbf{q}_1|(P - I)\mathbf{q}_1|^{-3} + (P_2^2 - I)\mathbf{q}_1|(P_2^2 - I)\mathbf{q}_1|^{-3} \\
&\quad + (P_1P_2^2 - I)\mathbf{q}_1|(P_1P_2^2 - I)\mathbf{q}_1|^{-3}\Big] - 6\mathbf{q}_1|\mathbf{q}_1|^{-3},
\end{aligned}$$

where $k = -1$, with initial position $\mathbf{q}_1(0) = (2,0,0)$ and initial velocity $\dot{\mathbf{q}}_1(0) = (0, \dot{y}_0, \dot{z}_0)$. Now, we want to know the coordinates of $(P_i - I)\mathbf{q}_1$ with $\mathbf{q}_1 = (x, y, z)$. Consider

$$\begin{aligned} \mathbf{q}_2 - \mathbf{q}_1 &= (P_1 - I)\mathbf{q}_1 \\ &= (-2x, -2y, -2z), \end{aligned}$$

which gives

$$|(P_1 - I)\mathbf{q}_1|^{-3} = \left(4x^2 + 4y^2 + 4z^2\right)^{-\frac{3}{2}},$$

also

$$\begin{aligned} \mathbf{q}_3 - \mathbf{q}_1 &= (P_2 - I)\mathbf{q}_1 \\ &= (z - x, x - y, y - z), \end{aligned}$$

which gives

$$|(P_2 - I)\mathbf{q}_1|^{-3} = \left(2(x^2 + y^2 + z^2) - 2(x(y+z) + yz)\right)^{-\frac{3}{2}},$$

also

$$\begin{aligned} \mathbf{q}_4 - \mathbf{q}_1 &= (P - I)\mathbf{q}_1 \\ &= (-(x+z), -(x+y), -(y+z)), \end{aligned}$$

which gives

$$|(P - I)\mathbf{q}_1|^{-3} = \left(2(x^2 + y^2 + z^2) + 2(x(y+z) + yz)\right)^{-\frac{3}{2}},$$

also

$$\begin{aligned} \mathbf{q}_5 - \mathbf{q}_1 &= (P_2^2 - I)\mathbf{q}_1 \\ &= (y - x, z - y, x - z), \end{aligned}$$

which gives

$$|(P_2^2 - I)\mathbf{q}_1|^{-3} = \left(2(x^2 + y^2 + z^2) - 2(x(y+z) + yz)\right)^{-\frac{3}{2}},$$

also

$$\begin{aligned}\mathbf{q}_6 - \mathbf{q}_1 &= (P_1P_2^2 - I)\mathbf{q}_1\\ &= (-(x+y), -(y+z), -(x+z)),\end{aligned}$$

which gives

$$|(P_1P_2^2 - I)\mathbf{q}_1|^{-3} = \Big(2(\,x^2+y^2+z^2\,) + 2(\,x(y+z)+yz\,)\Big)^{-\frac{3}{2}}.$$

5.4 Reduced system

5.4.1 Four electron case $(2,2)$

Having simplified the system we now have the equation of the motion for one of the 4-electrons case $(2,2)$, given by

$$\begin{aligned}\ddot{x} = k_1\Bigg[&\Big(y-x\Big)\Big(2(x-y)^2+4z^2\Big)^{-\frac{3}{2}} + (-2)x\Big(4(x^2+y^2)\Big)^{-\frac{3}{2}}\\ &-\Big(x+y\Big)\Big(2(x+y)^2+4z^2\Big)^{-\frac{3}{2}}\Bigg] - 4x\Big(x^2+y^2+z^2\Big)^{-\frac{3}{2}},\end{aligned}$$

$$\begin{aligned}\ddot{y} = k_1\Bigg[&\Big(x-y\Big)\Big(2(x-y)^2+4z^2\Big)^{-\frac{3}{2}} + (-2)y\Big(4(x^2+y^2)\Big)^{-\frac{3}{2}}\\ &-\Big(x+y\Big)\Big(2(x+y)^2+4z^2\Big)^{-\frac{3}{2}}\Bigg] - 4y\Big(x^2+y^2+z^2\Big)^{-\frac{3}{2}},\end{aligned}$$

$$\begin{aligned}\ddot{z} = k_1&\Bigg[(-2)z\Big(2(x-y)^2+4z^2\Big)^{-\frac{3}{2}} + (-2)z\Big(2(x+y)^2+4z^2\Big)^{-\frac{3}{2}}\Bigg]\\ &-4z\Big(x^2+y^2+z^2\Big)^{-\frac{3}{2}}.\end{aligned}$$

Note that the energy equation is

$$\begin{aligned}E = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) &+ \frac{1}{2}\Big(2(x-y)^2+4z^2\Big)^{-\frac{1}{2}} + \frac{1}{2}\Big(4(x^2+y^2)\Big)^{-\frac{1}{2}}\\ &+\frac{1}{2}\Big(2(x+y)^2+4z^2\Big)^{-\frac{1}{2}} + 4\Big(x^2+y^2+z^2\Big)^{-\frac{1}{2}}.\end{aligned}$$

5.4.2 Six electron case $(3,3)$

Having simplified the system we now have the equation of the motion for one of the 6-electrons case $(3,3)$, given by

$$\ddot{x} = k_1\Big[\Big(-3x\Big)\Big(3(x^2+y^2)\Big)^{-\frac{3}{2}} + (-2x)\Big(4(x^2+z^2)\Big)^{-\frac{3}{2}}$$
$$-\frac{1}{2}(x+\sqrt{3}\,y)\Big((x+\sqrt{3}\,y)^2+4z^2\Big)^{-\frac{3}{2}} - \frac{1}{2}(x-\sqrt{3}\,y)\Big((x-\sqrt{3}\,y)^2+4z^2\Big)^{-\frac{3}{2}}\Big]$$
$$-6x\Big(x^2+y^2+z^2\Big)^{-\frac{3}{2}},$$

$$\ddot{y} = k_1\Big[\Big(-3y\Big)\Big(3(x^2+y^2)\Big)^{-\frac{3}{2}} - \frac{\sqrt{3}}{2}(x+\sqrt{3}\,y)\Big((x+\sqrt{3}\,y)^2+4z^2\Big)^{-\frac{3}{2}}$$
$$+\frac{\sqrt{3}}{2}(x-\sqrt{3}\,y)\Big((x-\sqrt{3}\,y)^2+4z^2\Big)^{-\frac{3}{2}}\Big] - 6y\Big(x^2+y^2+z^2\Big)^{-\frac{3}{2}},$$

$$\ddot{z} = k_1\Big[(-2z)\Big(4(x^2+z^2)\Big)^{-\frac{3}{2}} - (2z)\Big((x+\sqrt{3}\,y)^2+4z^2\Big)^{-\frac{3}{2}}$$
$$-(2z)\Big((x-\sqrt{3}\,y)^2+4z^2\Big)^{-\frac{3}{2}}\Big] - 6z\Big(x^2+y^2+z^2\Big)^{-\frac{3}{2}}.$$

Note that the energy equation is

$$E = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \Big(3(x^2+y^2)\Big)^{-\frac{1}{2}} + \frac{1}{2}\Big((x+\sqrt{3}\,y)^2+4z^2)\Big)^{-\frac{1}{2}}$$
$$+\frac{1}{2}\Big((x-\sqrt{3}\,y)^2+4z^2)\Big)^{-\frac{1}{2}} + \frac{1}{2}\Big(4(x^2+z^2)\Big)^{-\frac{1}{2}} - 6\Big(x^2+y^2+z^2\Big)^{-\frac{1}{2}}.$$

5.4.3 Six electron case $(2, 2, 2)$

Having simplified the system we now have the equation of the motion for one of the 6-electrons case $(2, 2, 2)$, given by

$$
\begin{aligned}
\ddot{x} = k_1 \Bigg[\Big(-2x \Big) \Big(4(x^2 + y^2 + z^2) \Big)^{-\frac{3}{2}} \\
+ \Big((-2x) + (y+z) \Big) \Big(2(x^2 + y^2 + z^2) - 2(x(y+z) + yz) \Big)^{-\frac{3}{2}} \\
+ \Big((-2x) - (y+z) \Big) \Big(2(x^2 + y^2 + z^2) + 2(x(y+z) + yz) \Big)^{-\frac{3}{2}} \Bigg] \\
- 6x \Big(x^2 + y^2 + z^2 \Big)^{-\frac{3}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\ddot{y} = k_1 \Bigg[\Big(-2y \Big) \Big(4(x^2 + y^2 + z^2) \Big)^{-\frac{3}{2}} \\
+ \Big((-2y) + (x+z) \Big) \Big(2(x^2 + y^2 + z^2) - 2(x(y+z) + yz) \Big)^{-\frac{3}{2}} \\
+ \Big((-2y) - (x+z) \Big) \Big(2(x^2 + y^2 + z^2) + 2(x(y+z) + yz) \Big)^{-\frac{3}{2}} \Bigg] \\
- 6y \Big(x^2 + y^2 + z^2 \Big)^{-\frac{3}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\ddot{z} = k_1 \Bigg[\Big(-2z \Big) \Big(4(x^2 + y^2 + z^2) \Big)^{-\frac{3}{2}} \\
+ \Big((-2z) + (x+y) \Big) \Big(2(x^2 + y^2 + z^2) - 2(x(y+z) + yz) \Big)^{-\frac{3}{2}} \\
+ \Big((-2z) - (x+y) \Big) \Big(2(x^2 + y^2 + z^2) + 2(x(y+z) + yz) \Big)^{-\frac{3}{2}} \Bigg] \\
- 6z \Big(x^2 + y^2 + z^2 \Big)^{-\frac{3}{2}}.
\end{aligned}
$$

Note that the energy equation is

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\Big(4(x^2+y^2+z^2)\Big)^{-\frac{1}{2}}$$
$$+ \Big(2(x^2+y^2+z^2) - 2\big(x(y+z)+yz\big)\Big)^{-\frac{1}{2}}$$
$$+ \Big(2(x^2+y^2+z^2) + 2\big(x(y+z)+yz\big)\Big)^{-\frac{1}{2}} - 6\Big(x^2+y^2+z^2\Big)^{-\frac{1}{2}}.$$

We are now in a position to show the new style, weaving style and chasing style of the new family of classical periodic solutions.

5.5 Numerical illustrations

5.5.1 The weaving solution of the four electron case (2, 2)

The weaving motion of (2, 2)-electron atom problem is characterized by the particles (2-polygon) moving towards each other in pairs and following one of two paths related by rotation and (or) reflection. The motion of the particle in the planar case is illustrated below.

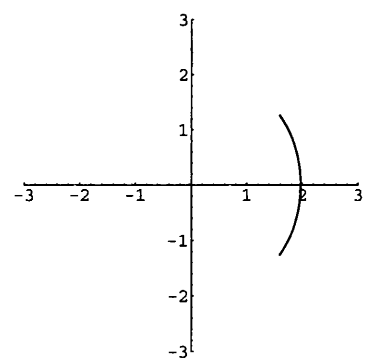


Figure 5.1:the planar motion of the particle.

In the case of the weaving motion of the first style, the basic initial data for the problem is $x(0) = 2$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0 =$

0.702824019, $\dot{z}(0) = \dot{z}_0 = 0.844139019$. the motion of one of the $(2,2)$-electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection that is the figure eight, of the path.

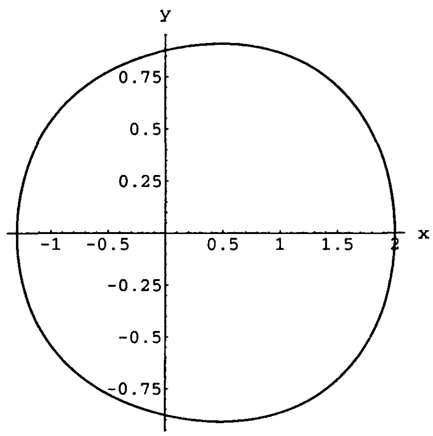


Figure 5.2:(x, y)-projection.

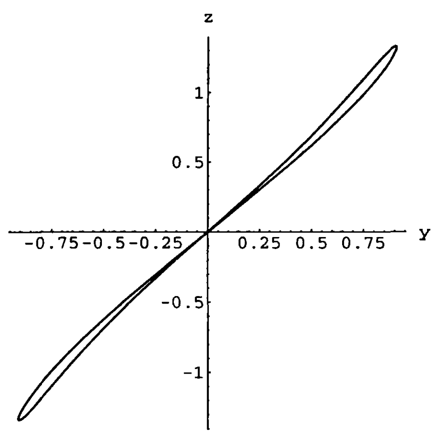


Figure 5.3:(y, z)-projection.

The weaving motion of the second style, the basic initial data for the problem is $x(0) = 2,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0 = 0.311175313,\ \dot{z}(0) = \dot{z}_0 = 1.247172813$. The motion of one of the $(2,2)$-electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

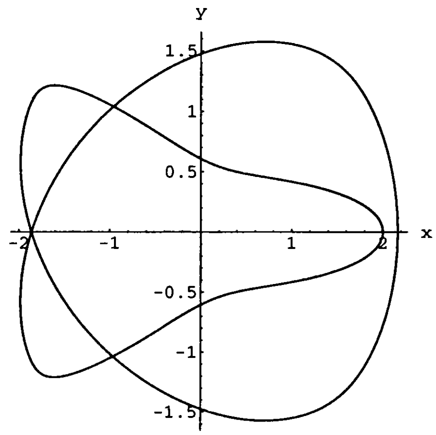


Figure 5.4:(x, y)-projection.

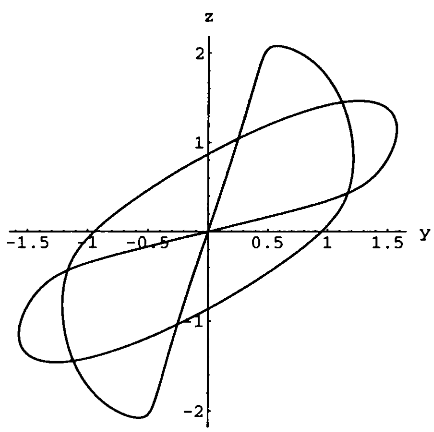


Figure 5.5:(y, z)-projection.

5.5.2 The weaving solution of the six electron case $(3, 3)$

The weaving motion of $(3, 3)$-electron atom problem is characterized by the polygon moving towards each other in pairs and following one of two paths related by rotation and (or) reflection. In the case of weaving motion of the first style, the basic initial data for the problem is $x(0) = 2$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0 = 1.052019592$, $\dot{z}(0) = \dot{z}_0 = 0.970048096$. The motion of one of the $(3, 3)$-electrons is illustrated below by plotting the

(x, y)-projection and the (y, z)-projection that is the figure eight but is very thin, of the path.

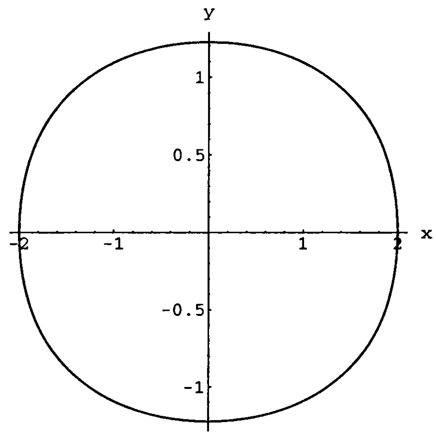


Figure 5.6:(x, y)-projection.

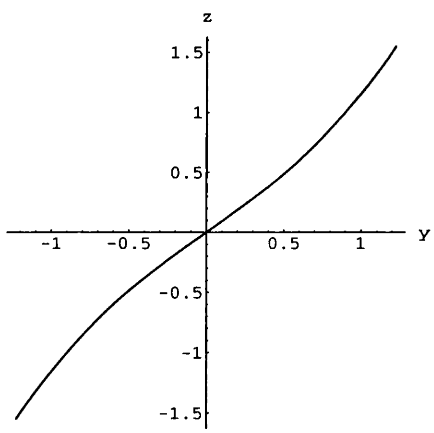


Figure 5.7:(y, z)-projection.

The weaving motion of the second style, the basic initial data for the problem is $x(0) = 2,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0 = 0.497197979,\ \dot{z}(0) = \dot{z}_0 = 0.804782126$. The motion of one of the $(3, 3)$-electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

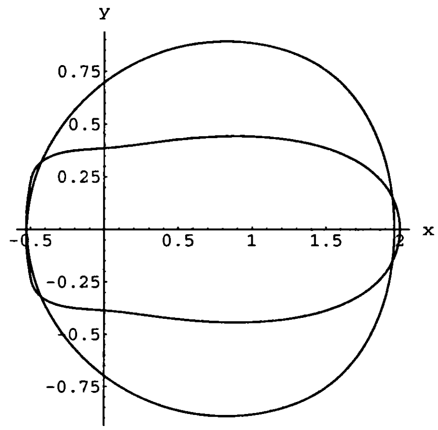


Figure 5.8:(x, y)-projection.

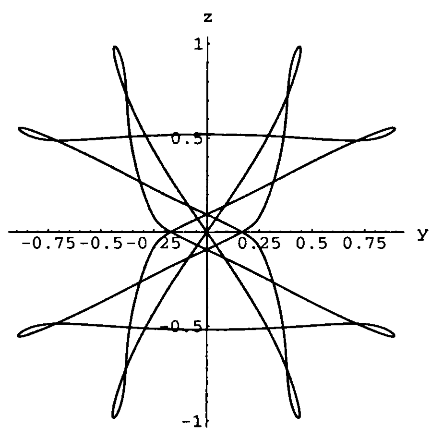


Figure 5.9:(y, z)-projection.

5.5.3 The chasing solution of the six electron case $(2, 2, 2)$

The chasing motion of $(2, 2, 2)$-electron atom problem are characterized by the particles pairwise chasing each other. In the case of chasing motion of the first style, the basic initial data for the problem is $x(0) = 2,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0 = 1.401844268,\ \dot{z}(0) = \dot{z}_0 = -0.477833459.$ the motion of one of the $(2, 2, 2)$-electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection that is the figure eight but is very thin, of the path.

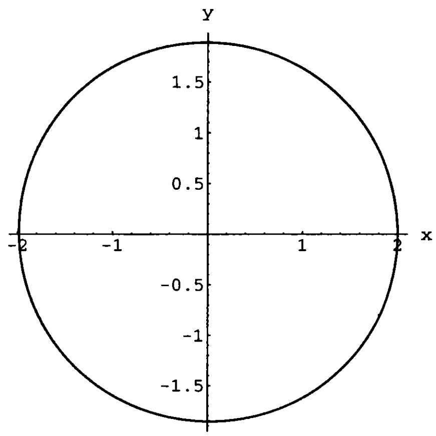


Figure 5.10:(x, y)-projection.

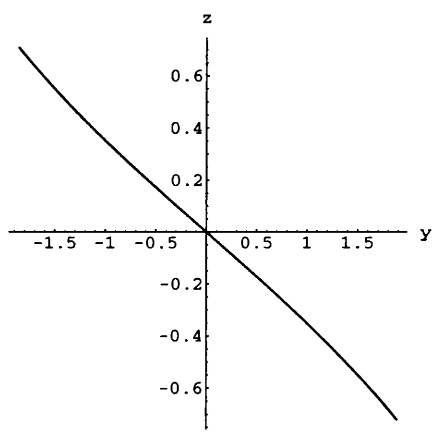


Figure 5.11:(y, z)-projcction.

The chasing motion of the second style, the basic initial data for the problem is $x(0) = 2$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0 = 0.82185$, $\dot{z}(0) = \dot{z}_0 = 1.28288$. The motion of one of the $(2, 2, 2)$-electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of an almost closed path.

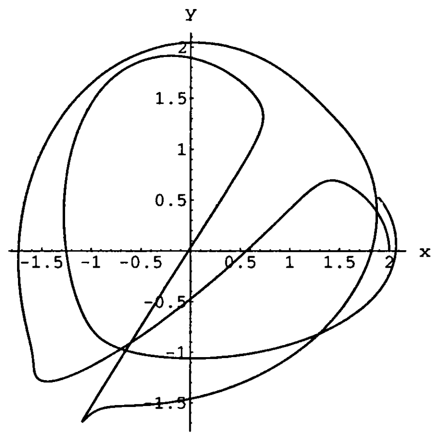


Figure 5.12: (x, y)-projection.

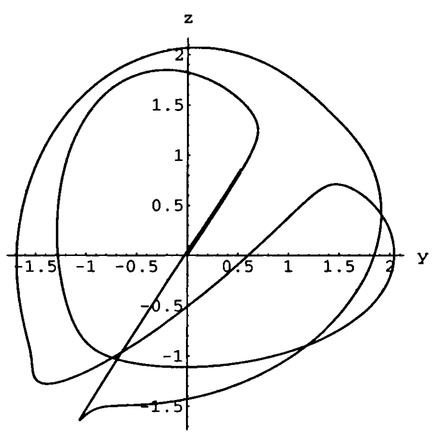


Figure 5.13: (y, z)-projection.

The interesting feature of the chasing solution is the replacement of the value of the initial velocities, $\dot{y}_0$, $\dot{z}_0$, to obtain the same periodic solution with different view.

5.6 The logarithmic pair potential

The purpose of this section is to study the motion of particles with logarithmic potential energy, to obtain the DTW-periodic solutions.

5.6.1 The one particle problem

Suppose two concentric cylindrical electrodes of radii a and b are maintained at potentials V_a and V_b as in figure 5.14. Suppose a charged particle with unit mass and with charge q is placed between these electrodes. Then, there will be a electric field between them so the force on the particle is

$$F(r) = -\frac{C}{r}, \quad r = |\mathbf{r}|, C = qK \text{ is a constant,}$$

where

$$K = \frac{V_a - V_b}{\log(b/a)}.$$

The constant K can be positive or negative, depending on whether the inner electrode is at a higher or lower potential than the outer electrode. Therefore, the force is attraction toward the inner electrode if $C > 0$, periodic motion occur, or repulsion from inner electrode if $C < 0$, periodic motion does not occur.

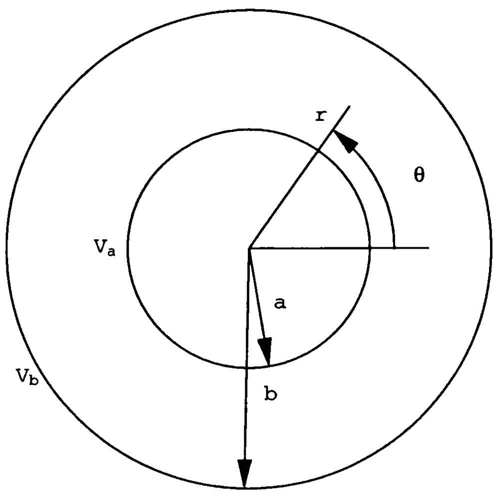


Figure 5.14:the cylindrical electrodes.

The equation of motion of the particle is in the form

$$\ddot{\mathbf{r}} = -\frac{C}{r^2}\,\mathbf{r}$$

This equation can not be solved in the same way as $\ddot{\mathbf{r}} = -\mu\,\mathbf{r}/r^3$, since in the plannar case one could reduce the equation of motion to the Duffing equation which shows the orbit is still almost an ellipse, but it does not quite close on itself so one could obtain one node, two node, three node, or more. For more details see [4], [56]. So in physical space in the case bounded state we will have one node, two node, three node or more. Some examples of n-node

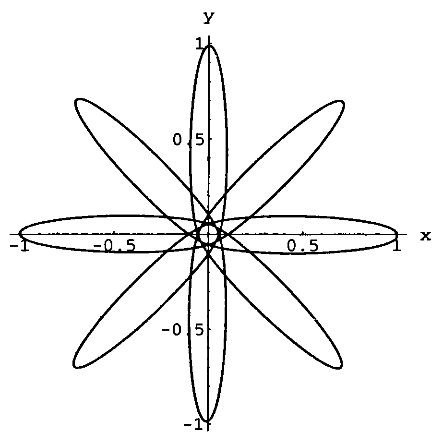


Figure 5.15:eight node.

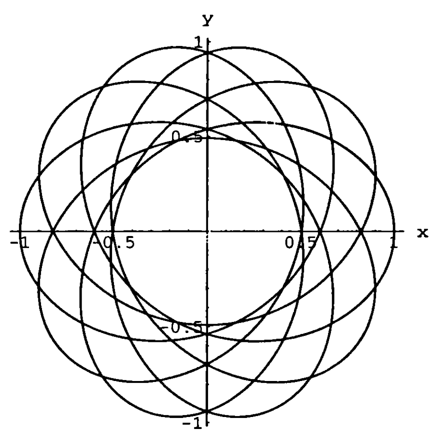


Figure 5.16:ten node.

5.6.2 $2n$ particle problem

In similar manner to chapter one, consider $2n$ particles (electrons, protons). The Hamiltonian $H(\mathbf{q},\mathbf{p})$, $H:\mathbb{R}^{12n}\longrightarrow\mathbb{R}$ is defined by

$$H(\mathbf{q},\mathbf{p})=\sum_{i=1}^{2n}\frac{1}{2}\mathbf{p}_i^2+k\sum_{i<j}^{2n}\log|\mathbf{q}_i-\mathbf{q}_j|-2n\alpha\sum_{i=1}^{2n}\log|\mathbf{q}_i|,$$

where $\mathbf{p}_i\in\mathbb{R}^3$, $\mathbf{q}_i\in\mathbb{R}^3$, $i=1,2,...,2n$, and k, α are constants. The equations of motion reduce to

$$\ddot{\mathbf{q}}_i=\sum_{\substack{j=1\\ j\neq i}}^{2n}k_1\frac{(\mathbf{q}_j-\mathbf{q}_i)}{|\mathbf{q}_j-\mathbf{q}_i|^2}-2nk_2\frac{\mathbf{q}_i}{|\mathbf{q}_i|^2}+k_3(\mathbf{B}\times\dot{\mathbf{q}}_i)\quad,i=1,2,...,2n.$$

5.6.3 Four electron, four proton case with constant magnetic field

Consider the motion of four electron, four proton problem with non-zero constant magnetic field. We have

$$\ddot{x}=2x(2x^2+2y^2+4z^2)^{-1}+2x(4x^2+4y^2)^{-1}-4x(x^2+y^2+z^2)^{-1}-B\dot{y},$$

$$\ddot{y}=2y(2x^2+2y^2+4z^2)^{-1}+2y(4x^2+4y^2)^{-1}-4y(x^2+y^2+z^2)^{-1}+B\dot{x},$$

$$\ddot{z}=4z(2x^2+2y^2+4z^2)^{-1}-4z(x^2+y^2+z^2)^{-1}.$$

The energy equation is

$$E=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{1}{2}\log(2x^2+2y^2+4z^2)+\frac{1}{4}\log(4x^2+4y^2)-2\log(x^2+y^2+z^2).$$

The basic initial data for the problem is $x(0)=1$, $y(0)=0$, $z(0)=0$, $\dot{x}(0)=0$, $\dot{y}(0)=\dot{y}_0$, $\dot{z}(0)=\dot{z}_0$. Four node solution for four electrons problem with non-zero magnetic field. Now let the initial velocity be as below

$$\dot{y}_0=0.158673861\quad\text{and}\quad\dot{z}_0=1.831370524\quad\text{and}\quad B=\frac{1}{2}.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}|=0.158673861,\quad E=\frac{1}{2}(\dot{y}_0^2+\dot{z}_0^2)+\log 2=2.382694875.$$

The motion of one of the electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

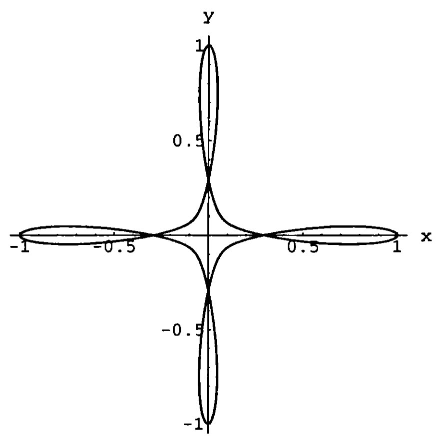


Figure 5.17:(x, y)-projection.

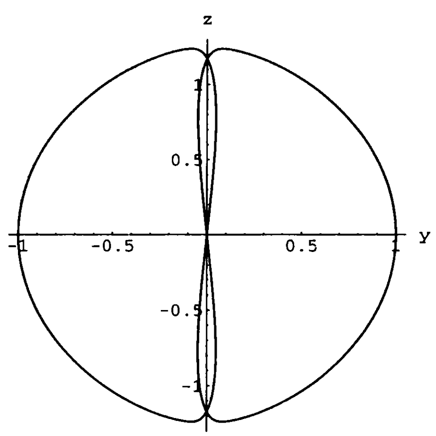


Figure 5.18:(y, z)-projection.

Some initial data for the four electron, four proton case with $B = 1/2$, we illustrated above the four node.

style	$\dot{y}_0$	$\dot{z}_0$
4-node	0.158673861	1.831370524
5-node	0.190906250	1.872131347
6-node	0.203952693	1.891726220
7-node	0.211099884	1.903146057
10-node	0.087936096	1.766124514
13-node	0.173666409	1.849080684
13-node	0.207975264	1.898108780
13-node	0.400657620	1.775244976
14-node	0.183710828	1.862125924
22-node	0.175981587	1.851996894

5.6.4 Four electron, four proton case

Consider the motion of four electron, four proton problem without constant magnetic field. We have

$$\ddot{x} = 2x(2x^2 + 2y^2 + 4z^2)^{-1} + 2x(4x^2 + 4y^2)^{-1} - 4x(x^2 + y^2 + z^2)^{-1},$$

$$\ddot{y} = 2y(2x^2 + 2y^2 + 4z^2)^{-1} + 2y(4x^2 + 4y^2)^{-1} - 4y(x^2 + y^2 + z^2)^{-1},$$

$$\ddot{z} = 4z(2x^2 + 2y^2 + 4z^2)^{-1} - 4z(x^2 + y^2 + z^2)^{-1}.$$

The energy equation is

$$E = \frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{1}{2}\log(2x^2+2y^2+4z^2)+\frac{1}{4}\log(4x^2+4y^2)-2\log(x^2+y^2+z^2).$$

The basic initial data for the problem is $x(0) = 1,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0,\ \dot{z}(0) = \dot{z}_0$. Five node solution for four electrons problem with zero magnetic field. Now let the initial velocity be as below

$$\dot{y}_0 = 0.232491359 \quad \text{and} \quad \dot{z}_0 = 1.737912948 \quad \text{and} \quad B = 0.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}| = 0.232491359, \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) + \log 2 = 2.230344004.$$

The motion of one of the electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

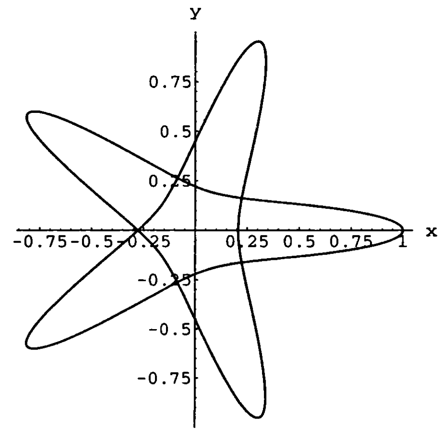


Figure 5.19:(x, y)-projection.

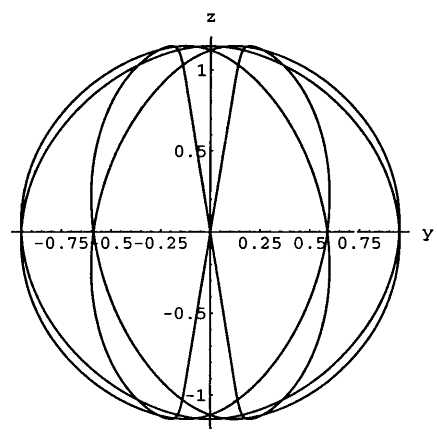


Figure 5.20:(y, z)-projection.

Some initial data for the four electron, proton problem with zero constant magnetic field, we illustrated above the five node.

style	$\dot{y}_0$	$\dot{z}_0$
2-node	1.179298267	1.006456761
5-node	0.232491359	1.737912948
7-node	0.383286960	1.642541768
9-node	0.517671389	1.561070041
11-node	0.139499899	1.810333983
12-node	0.308302908	1.688436452
13-node	0.184128906	1.77309125
14-node	0.129091780	1.820019069
15-node	0.771598659	1.390586697
16-node	0.453704842	1.600097593
17-node	0.123004394	1.825910703
23-node	0.203117381	1.758790886
25-node	0.133512571	1.815848282
28-node	0.207945839	1.755264923

5.6.5 Four ion case

Consider the motion of four ion problem. We have

$$\ddot{x} = -2x(2x^2 + 2y^2 + 4z^2)^{-1} + 2x(4x^2 + 4y^2)^{-1} - 4x(x^2 + y^2 + z^2)^{-1},$$

$$\ddot{y} = -2y(2x^2 + 2y^2 + 4z^2)^{-1} + 2y(4x^2 + 4y^2)^{-1} - 4y(x^2 + y^2 + z^2)^{-1},$$

$$\ddot{z} = -4z(2x^2 + 2y^2 + 4z^2)^{-1} - 4z(x^2 + y^2 + z^2)^{-1}.$$

The energy equation is

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\log(2x^2 + 2y^2 + 4z^2) - \frac{1}{4}\log(4x^2 + 4y^2).$$

The basic initial data for the problem is $x(0) = 1,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0,\ \dot{z}(0) = \dot{z}_0$. Eight node solution for four ions problem. Now let the initial velocity be as below

$$\dot{y}_0 = 0.211174259 \quad \text{and} \quad \dot{z}_0 = 1.002803401.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}| = 0.211174259, \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) = 0.525104614.$$

The motion of one of the ions is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

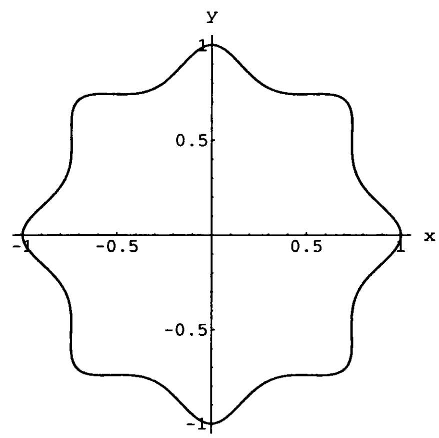


Figure 5.21:(x, y)-projection.

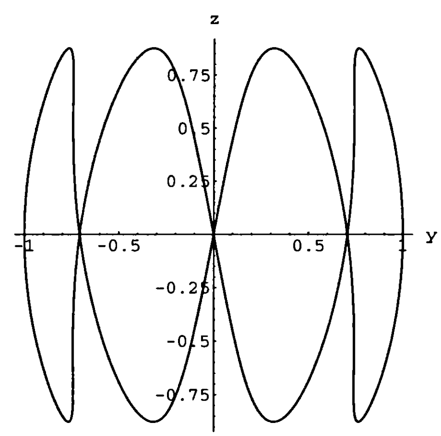


Figure 5.22:(y, z)-projection.
Some initial data for the four ion problem, we illustrated above the eight node.

style	$\dot{y}_0$	$\dot{z}_0$
5-node	0.413409154	0.813180543
8-node	0.211174259	1.002803401
9-node	0.183620195	1.020070847
10-node	0.162762366	1.031704395
11-node	0.146349422	1.039961938
12-node	0.133057406	1.046055914
13-node	0.122050845	1.050692850
17-node	0.196361809	1.012350808
35-node	0.137199918	1.044214351

5.6.6 Four body attractive case

Consider the motion of four body attractive problem. We have

$$\ddot{x} = -2x(2x^2 + 2y^2 + 4z^2)^{-1} - 2x(4x^2 + 4y^2)^{-1} - 4x(x^2 + y^2 + z^2)^{-1},$$

$$\ddot{y} = -2y(2x^2 + 2y^2 + 4z^2)^{-1} - 2y(4x^2 + 4y^2)^{-1} - 4y(x^2 + y^2 + z^2)^{-1},$$

$$\ddot{z} = -4z(2x^2+2y^2+4z^2)^{-1} - 4z(x^2+y^2+z^2)^{-1}.$$

The energy equation is

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}\log(2x^2 + 2y^2 + 4z^2) + \frac{1}{4}\log(4x^2 + 4y^2).$$

The basic initial data for the problem is $x(0) = 1,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0,\ \dot{z}(0) = \dot{z}_0$. 12 node solution for four body attractive problem with two different figures respectively. Now let the initial velocity be as below

$$\dot{y}_0 = 0.139931809 \quad \text{and} \quad \dot{z}_0 = 1.012789841.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}| = 0.139931809, \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) + \log 2 = 1.215809267.$$

$$\dot{y}_0 = 0.393976836 \quad \text{and} \quad \dot{z}_0 = 1.044904945.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}| = 0.393976836 = , \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) + \log 2 = 1.316669226.$$

The motion of one of the body attractive is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

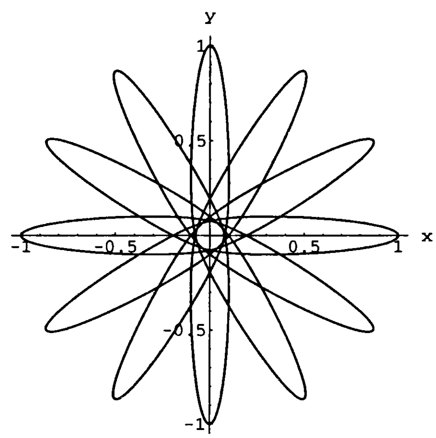


Figure 5.23:(x, y)-projection.

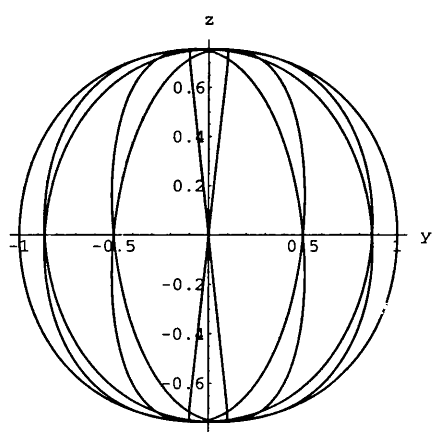


Figure 5.24:(y, z)-projection.

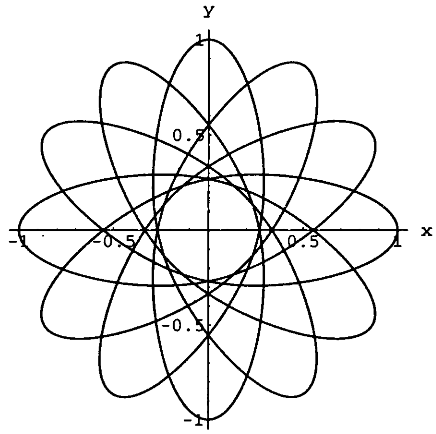


Figure 5.25:(x, y)-projection.

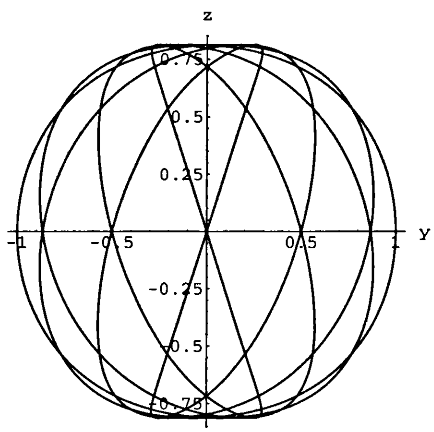


Figure 5.26:(y, z)-projection.

The basic initial data for the problem is $x(0) = 1,\ y(0) = 0,\ z(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = \dot{y}_0,\ \dot{z}(0) = \dot{z}_0$. 16 node solution for four body attractive problem with two different figures respectively. Now let the initial velocity be as below

$$\dot{y}_0 = 0.032086053 \quad \text{and} \quad \dot{z}_0 = 997224011.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}| = 0.032086053, \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) + \log 2 = 1.190889802.$$

$$\dot{y}_0 = 0.60338346 \quad \text{and} \quad \dot{z}_0 = 1.035290507.$$

This gives the angular momentum h and the energy as

$$|\mathbf{h}| = 0.60338346, \quad E = \frac{1}{2}(\dot{y}_0^2 + \dot{z}_0^2) + \log 2 = 1.411096197.$$

The motion of one of the body attractive is illustrated below by plotting the (x, y)-projection and the (y, z)-projection of the path.

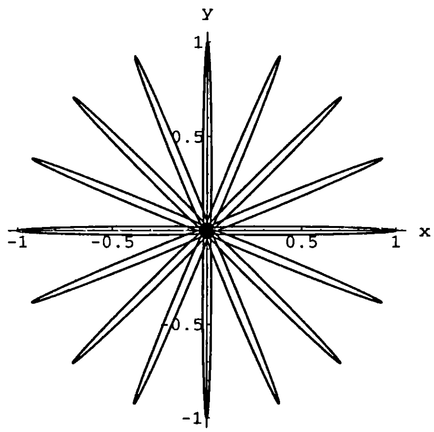


Figure 5.27:(x, y)-projection.

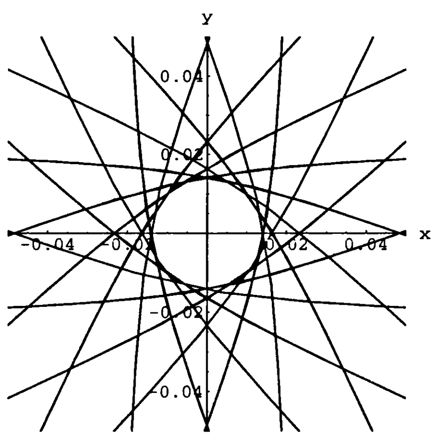


Figure 5.28:the central area.

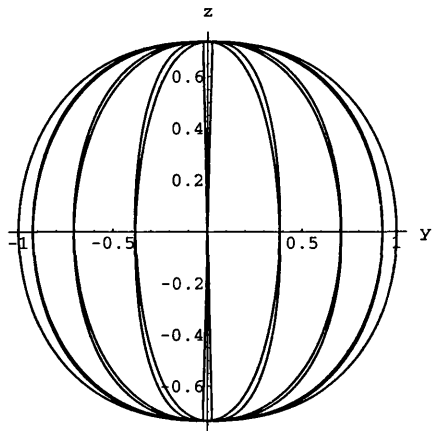


Figure 5.29:(y, z)-projection.

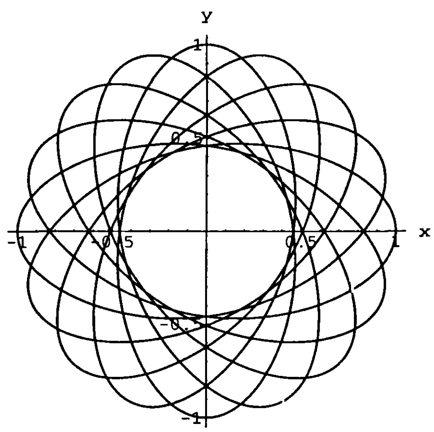


Figure 5.30:(x, y)-projection.

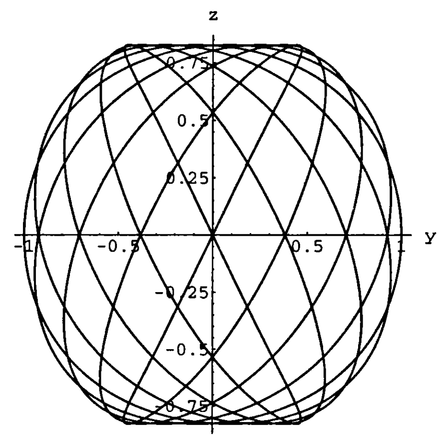


Figure 5.31:(y, z)-projection.

Some initial data for the four body attractive problem, we illustrated above the 12-node and 16-node.

style	$\dot{y}_0$	$\dot{z}_0$
7-node	0.531675084	1.044383593
9-node	0.650978515	1.024878906
11-node	0.712176249	1.005303124
12-node	0.139931809	1.012789841
12-node	0.393970836	1.044904945
15-node	0.777265557	0.975371421
16-node	0.032086053	0.997224011
16-node	0.603383460	1.035290507
19-node	0.454659968	1.046983750
19-node	0.100687214	1.006611136

5.7 Similar style to the Weaving style with logarithmic potential energy

In similar manner the motion of $(2, 2)$-electron atom problem with logarithmic potential energy, in the weaving motion, the basic initial data for the problem is $x(0) = 2$, $y(0) = 0$, $z(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = \dot{y}_0 = 1.004045317$, $\dot{z}(0) = \dot{z}_0 = 1.233972099$. The motion of one of the $(2, 2)$-electrons is illustrated below by plotting the (x, y)-projection and the (y, z)-

projection that is the figure eight but is very thin, of the path.

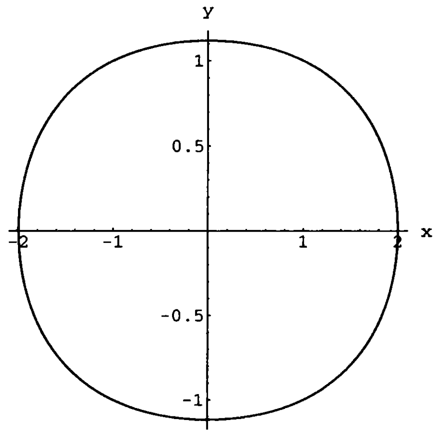


Figure 5.32:(x, y)-projection.

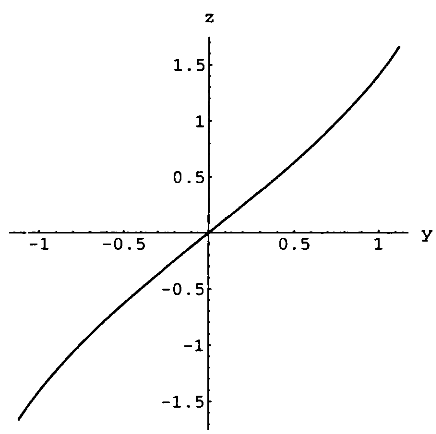


Figure 5.33:(y, z)-projection.

5.8 Conclusion

In this chapter we have obtained new style of periodic solutions with inverse-square law that mean with potential energy in general form $k/|\mathbf{q}_i - \mathbf{q}_j| - \alpha/|\mathbf{q}_i|$. The new types of periodic solution fall into two categories, weaving and chasing periodic solutions. The planar periodic solutions of CM [14] have a little more in common with our new style of periodic solution. We try to give approximate of new families of the $2n$ electron atom problem work as (n, n) where n is odd or even, the weaving periodic solutions are characterized by the n-polygon moving towards each other in pairs, such that every particle from the first polygon work with one particle from the second polygon in pairs. In general consider the n-polygon as particles move up and down in pairs. We also have obtained the DTW-periodic solutions with logarithmic potential energy and we could apply all the theory in chapters two and three and the numerical approch in chapter four to this system. We have also obtained the weaving style of the new style of the periodic solutions with the logarithmic potential energy. Unfortunately, we can not apply the f-equation for the new types of periodic solutions.

5.9 Further Research

One could continue to study the stability of the f-equation by considering non-linear stability criteria. A further avenue would be that of concentrating on the nature of f and ϵ at $r = r_m$ as opposed to $r = 1$. One could continue to solve the general ϵ-equation with respect to f insted of r. In some cases it might be appropriate to consider more general pure pair potentials, r^n, for both positive and negative n. Of more Physical interest might be combination of pure pair potentials as in the clasical Lennard-Jones potential. In all of these cases one could seek characterisations of the generic periodic solutions asuming that such systemes admit interesting styles of solutions. One could continue the investegation to the potential in the form $V(r) + W(r)$ and $V(r)W(r)$. For examples, $\mu/r + k/r^2$, $\mu/r + k_1 \log r$, $r(\log r) - r$, $(\log r)^2$, and where could obtain a precessing ellipse and where could apply (not apply) the f-equation. One could continue to study the linear (nonlinear) staiblity of the new styles of periodic solutions, since we can not use the f-equation with that periodic solutions. With the harmonic potential one could try to show that any periodic solutions are obtain with any number of particles that always be one node and two node some of them provided figure eight.

Chapter 6

MATHEMATICA CODE USED IN OBTAINING PERIODIC SOLUTIONS

Chapter 1

In sections 1.3.5, 1.3.6 and 1.3.7 of Chapter 1 we displayed some of the periodic solutions found by Davies, Truman and Williams in order to illustrate their basic nature. The system of equations to be solved in all cases is similar to that displayed below (those for the four body gravitational problem). The system below is a reduced system obtained after imposing the pseudo rotational symmetry on the motion. The full system would comprise twelve second order differential equations.

$$x''[t] = -2x[t]\left((4x[t]^2 + 2y[t]^2 + 2z[t]^2)^{-3/2} + (4x[t]^2 + 4y[t]^2)^{-3/2}\right)$$
$$y''[t] = -2y[t]\left((4x[t]^2 + 2y[t]^2 + 2z[t]^2)^{-3/2} + (4x[t]^2 + 4y[t]^2)^{-3/2}\right)$$
$$z''[t] = -4z[t]/(4x[t]^2 + 2y[t]^2 + 2z[t]^2)^{3/2}$$

The initial data for the system is fixed except for the values of the initial velocities in the y and z directions.

```
x[0] == 1.0, y[0] == 0.0, z[0] == 0.0,
x'[0] == 0, y'[0] == adot, z'[0] == gdot.
```

One of the early periodic solutions found, the four node solution, has the following values for the velocities.

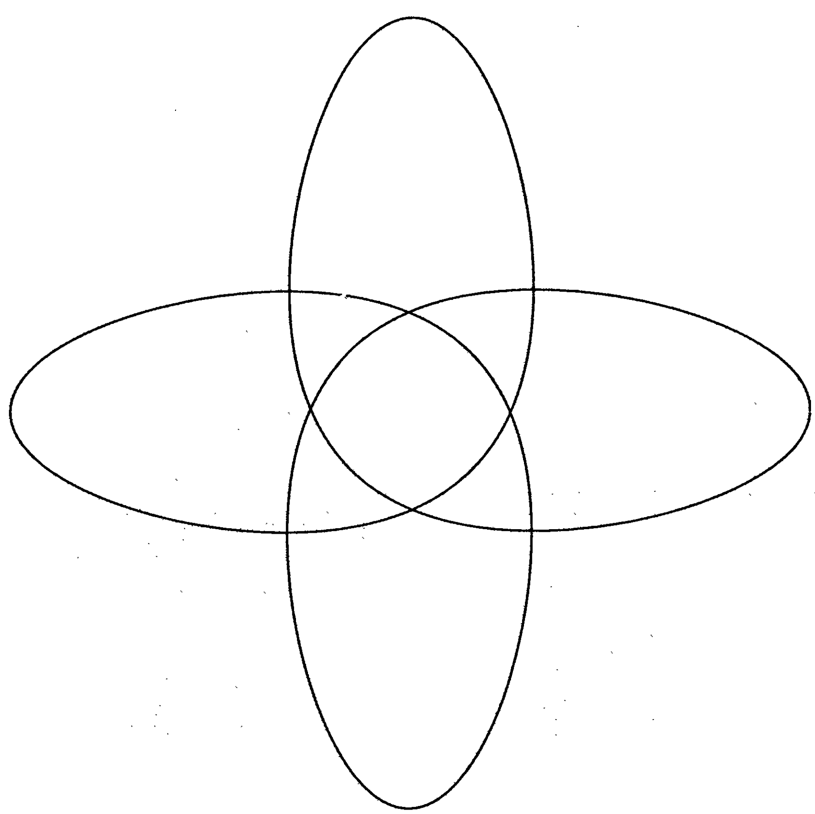
```
adot = 0.333250244;
gdot = 0.8417836914;
```

To compute the known periodic orbits we use native *Mathematica* code and assign the output to a named variable in order that we may manipulate it later for the purposes of viewing.

```
rsol = NDSolve[
   {x''[t] == -2 x[t] ((2 x[t]^2 + 2 y[t]^2 + 4 z[t]^2)^(-3/2) +
        (4 x[t]^2 + 4 y[t]^2)^(-3/2)),
    y''[t] == -2 y[t] ((2 x[t]^2 + 2 y[t]^2 + 4 z[t]^2)^(-3/2) +
        (4 x[t]^2 + 4 y[t]^2)^(-3/2)),
    z''[t] == -4 z[t] (2 x[t]^2 + 2 y[t]^2 + 4 z[t]^2)^(-3/2),
    x[0] == 1.0, y[0] == 0.0, z[0] == 0.0,
    x'[0] == 0.0, y'[0] == adot, z'[0] == gdot},
   {x, y, z}, {t, 0, 13}, MaxSteps → 2000];
```

Given the interpolating function representation of the solution we can either view the solution in plane sections by using **ParametricPlot** or in plane projection by use of **ParametricPlot3D**. We have one of the archetypal illustrations below.

```
ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. rsol],
 {t, 0, 13}, ViewPoint -> {0, 0, 50},
 Boxed -> False, Axes -> False, PlotPoints -> 2000]
```



```
- Graphics3D -
```

Chapter 2

In Chapter 2 the f equation was derived and discussed in some detail. The constants of the motion, angular momentum and energy, play a part in the statement of the f equation as do the functions F_1, F_2 and F_3. Once again we display these constants and functions in the case of the four body gravitational problem.

```
h = adot;
energy = 0.5 (adot adot + gdot gdot) - 1 / Sqrt[2] - 0.25;
```

```
Fone[f_, r_] := -4 / (4 f + 2 r^2) ^ (3 / 2);
Ftwo[f_, r_, h_] :=
  -2 r / (4 f + 2 r^2) ^ (3 / 2) - 1 / (4 r^2) + h^2 / r^3;
Fthree[f_, r_, h_] := -1 / (4 f + 2 r^2) ^ (1 / 2) -
   1 / (4 r) + h^2 / (2 r^2);
```

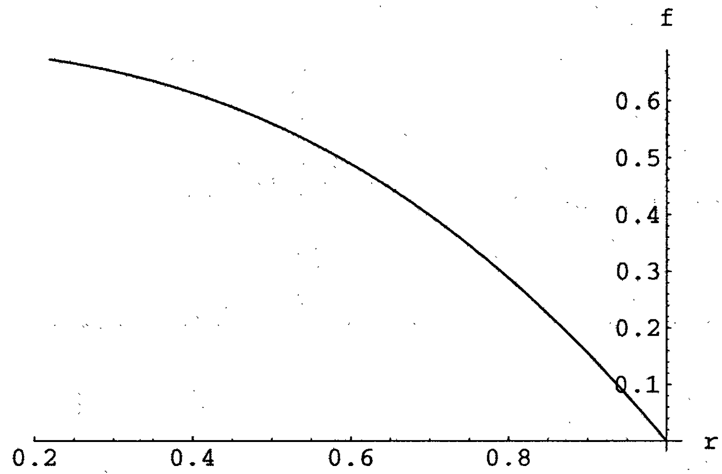
With everything stated and defined it would be simplest to employ **NDSolve** again to compute the numerical solution of the f equation. This was the earliest way we used but it no longer works in the current version of *Mathematica*. One would assign the output of **NDSolve** to a named variable and then display the computed function as desired. The original code is below.

```
fsol = NDSolve[{f[r] (f'[r] f'[r] / 2 - f[r] f''[r])
        (2 energy - 2 Fthree[f[r], r, h]) == f[r] (f'[r] f'[r] / 4 +
          f[r]) (f'[r] Ftwo[f[r], r, h] - 2 f[r] Fone[f[r], r]),
     f[1] == (10) ^ (-20), f'[1] == 2 gdot gdot / Ftwo[0.0, 1.0, h]},
   f, {r, 0.219, 1.0}];
```

The only way to force the above code to return a solution is to set f(1) equal to a very negative power of 10. We will demonstrate the current robust method for computing solutions of the f equation in the next section.

Given our InterpolatingFunction representation for f we typically have the behaviour as displayed below.

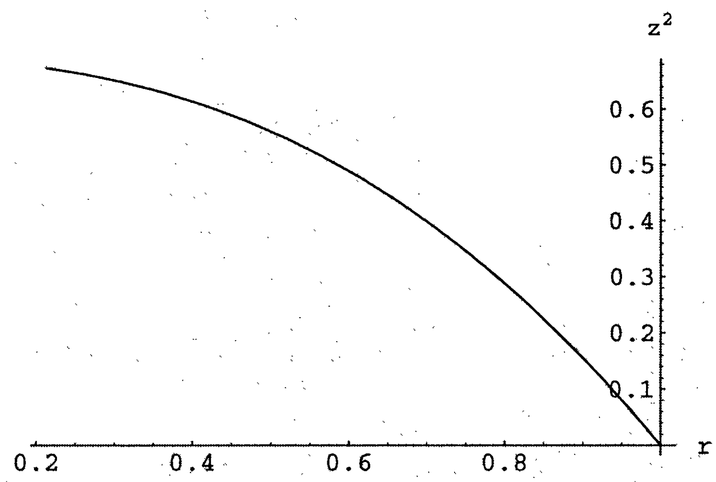
```
Plot[Evaluate[f[r] /. fsol],
 {r, 0.219, 1.0}, AxesLabel → {"r", "f"}]
```



- Graphics -

It is natural to compare this coerced solution of the f equation and the analogous values from the raw system of the previous section. The agreement appears to be good

```
ParametricPlot[Evaluate[{Sqrt[x[t]^2 + y[t]^2], z[t]^2} /. rsol],
 {t, 0, 1.62}, AxesLabel → {"r", "z²"}]
```



- Graphics -

although a better way to compare the solutions is to integrate the square of their difference. The first line below creates a function of r and the second line below extracts our computed f.

```
af = Interpolation[Flatten[
    Table[Evaluate[{Sqrt[y[t]^2 + x[t]^2], z[t]^2} /. rsol],
     {t, 0, 1.62, 0.01}], 1]];
```

```
df[x_] := First[f[x] /. fsol];
```

The result of our comparison is quite acceptable.

```
NIntegrate[ (af[x] - df[x] ) (af[x] - df[x]), {x, 0.219, 1.0}]

3.24247×10^-11
```

Chapter 4

The basic idea is to use a fourth order Runge Kutta scheme to compute the solution of the ϵ equation for a given system and solution of the f equation. There are several problems that arise almost immediately. First of all the f equation does not admit exact solutions except in the simplest of situations and so we must fall back on numerical representations of solutions of the f equation. Given that we must do this we can either compute particular solutions of the f equation in advance and store them to high accuracy and small mesh or we can enlarge the numerical scheme and compute the solution of the f equation alongside that of the ϵ equation. We choose to do the latter. The second problem is one seen already in the raw naive code used to solve the f equation namely the bad behaviour of some coefficients at r equals 1. We avoid this particular problem by using two forms of the Runge Kutta scheme. One, with the limiting values of the coefficients calculated algebraically, is used initially and then the standard scheme is applied repeatedly as usual. We make great use of *Mathematica* to compute the coefficients in what follows.

Let us put the gory bits at the end of this section.

We begin by choosing one of the standard problems and one of the four node periodic solutions. The initial velocities, energy and functions F_1, F_2 and F_3 are listed again for convenience.

```
adot = 0.333250244;
gdot = 0.8417836914;
```

```
h = adot;
energy = 0.5 (adot adot + gdot gdot) - 1 / Sqrt[2] - 0.25;
```

```
Fone[f_, r_] := -4 / (4 f + 2 r^2) ^ (3 / 2);
Ftwo[f_, r_, h_] :=
  -2 r / (4 f + 2 r^2) ^ (3 / 2) - 1 / (4 r^2) + h^2 / r^3;
Fthree[f_, r_, h_] := -1 / (4 f + 2 r^2) ^ (1 / 2) -
   1 / (4 r) + h^2 / (2 r^2);
```

The system of differential equations for f and ϵ are rewritten in first order vector form with the four vector representing the derivative as shown below. Note that it is a little complicated.

```
ff[{r_, {f_, fp_, e_, ep_}}] :=
 {fp, (fp^2 (2 energy - 2 Fthree[f, r, h]) / 2 -
      (fp^2 / 4 + f) (fp Ftwo[f, r, h] - 2 f Fone[f, r])) /
    (f (2 energy - 2 Fthree[f, r, h])), ep,
  -(ep ((f ( Ftwo[f, r, h] + 1 / 2 fp Fone[f, r]) -
          fp (2 energy - 2 Fthree[f, r, h]) +
          3/4 (fp^2 Ftwo[f, r, h] - 2 f fp Fone[f, r]) )) +
      e ((fp^2 (2 energy - 2 Fthree[f, r, h]) / 2 -
           (fp^2 / 4 + f) (fp Ftwo[f, r, h] - 2 f Fone[f, r]))
         ((2 energy - 2 Fthree[f, r, h]) + f Fone[f, r]) /
          (f (2 energy - 2 Fthree[f, r, h])) +
        (fp Ftwo[f, r, h] + (fp^2 / 4 + f)
           ((r fp / 2 - 2 f) Fone^(1,0) [f, r] - 4 Fone[f, r])))) /
    (f (2 energy - 2 Fthree[f, r, h]))}
```

The standard Runge Kutta scheme is implemented in the form below where we have made as much use of lists as possible. We have not bothered to compile these bits of code as they are not too demanding on current machines.

```
RungeKuttaFourC[{f_, h_, p0_}] := Module[{np1, k1, k2, k3, k4},
  k1 = f[p0];
  k2 = f[p0 + {h / 2, h k1 / 2}];
  k3 = f[p0 + {h / 2, h k2 / 2}];
  k4 = f[p0 + {h, h k3}];
  np1 = p0 + {h, h (k1 + 2 k2 + 2 k3 + k4) / 6};
  {f, h, np1}]
```

The initial Runge Kutta scheme differs only in that we have explicitly calculated the required value of $f''(1)$ and left it in terms of the partial derivatives of F_1, F_2 and F_3.

```
RungeKuttaFourCInitial[{f_, h_, p0_}] :=
 Module[{np1, k1, k2, k3, k4},
    k1 = {2 gdot gdot / Ftwo[0.0, 1.0, h], 1/(3 Ftwo[0, 1, adot]^3)
      (2 (4 gdot^2 Fone[0, 1] Ftwo[0, 1, adot] - gdot^2 Ftwo[0, 1, adot]
          Ftwo^(0,1,0)[0, 1, adot] - 2 gdot^4 Ftwo^(1,0,0)[0, 1, adot])),
    epsilonprime0, epsilondoubleprime0};
    k2 = f[p0 + {h / 2, h k1 / 2}];
    k3 = f[p0 + {h / 2, h k2 / 2}];
 k4 = f[p0 + {h, h k3}];
    np1 = p0 + {h, h (k1 + 2 k2 + 2 k3 + k4) / 6};
        {f, h, np1}]
```

The preceding code is almost all we require to solve for f and ϵ but we have to check on the values of the coefficients of ϵ and ϵ' to ensure that we have the correct dependence (or lack of it) between ϵ, ϵ' and ϵ'' at r equals 1. We simply have to extract the coefficients and substitute for several known quantities. This results in the numerical values for the coefficients of ϵ and ϵ' represented below as **nc0** and **nc1**.

```
c1 = f[r] (Ftwo[f[r], r, h] + 1 / 2 f'[r] Fone[f[r], r]) -
  f'[r] (2 energy - 2 Fthree[f[r], r, h]) +
  3/4 f'[r] ( f'[r] Ftwo[f[r], r, h] - 2 f[r] Fone[f[r], r]);
```

```
% /. r → 1;
```

```
% /. h → adot;
```

```
% /. f[1] → 0;
```

```
% /. f'[1] → 2 gdot gdot / Ftwo[0, 1, h];
```

```
nc1 = %

-0.593479
```

```
c0 = f''[r] ((2 energy - 2 Fthree[f[r], r, h]) + f[r] Fone[f[r], r]) +
  f'[r] Ftwo[f[r], r, h] -
  (f'[r] f'[r] + 4 f[r]) Fone[f[r], r] + (f'[r] f'[r] / 4 + f[r])
   (f'[r] Ftwo^(1,0,0)[f[r], r, h] - 2 f[r] Fone^(1,0)[f[r], r]);
```

```
% /. h → adot;
```

```
% /. r → 1;
```

```
% /. f[1] → 0;
```

```
% /. f'[1] → 2 gdot gdot / Ftwo[0, 1, h];
```

```
% /. f''[1] → 1 / (3 Ftwo[0, 1, adot]^3)
    (2 (4 gdot^2 Fone[0, 1] Ftwo[0, 1, adot] - gdot^2 Ftwo[0, 1, adot]
        Ftwo^(0,1,0)[0, 1, adot] - 2 gdot^4 Ftwo^(1,0,0)[0, 1, adot]));
```

```
nc0 = %

1.16967
```

Note the slightly messy form of $f''(1)$ shown above. This was obtained by repeated differentiation of $z^2 = f(r)$, a tedious task which was eased and certified by *Mathematica*. Given the values of **nc0** and **nc1** we can tie the initial values of ϵ and ϵ' together in the correct way and set up an initial data string for the Runge Kutta scheme. Note that we employ **RungeKuttaFourInitial** just the once and then fall back on the standard scheme. The first case.

```
epsilon0 = 1.0;
epsilondoubleprime0 = 0.0;
epsilonprime0 = - nc0 epsilon0 / nc1;
```

```
data0 = {1.0,
  {0.0, 2 gdot gdot / Ftwo[0.0, 1.0, h], epsilon0, epsilonprime0}}

{1., {0., -1.67508, 1., 1.97087}}
```

We begin our illustration below with a calculation of f and ϵ on [0.1,1.0].

```
RungeKuttaFourCInitial[{ff, -0.001, data0}]

{ff, -0.001, {0.999, {0.00164148, -1.67269, 0.998027, 2.00392}}}
```

```
pts = NestList[RungeKuttaFourC, %, 890];
```

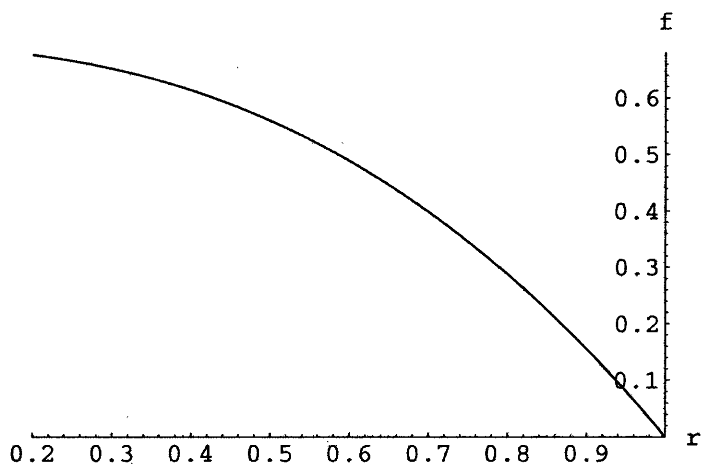
The raw data is stored as a list for future use even though it is quite bulky. We would only write out the data if it took a considerable time to compute. We extract the (r,ϵ) and (r,f) data pairs by a gratuitous use of list manipulation and after prepending the requisite initial data point we can display the results.

```
plotptseps = Transpose[Drop[
    Drop[Transpose[Map[Flatten[Last[#]] &, pts]], -1], {2, 3}]];
```

```
plotptsf =
  Transpose[Drop[Transpose[Map[Flatten[Last[#]] &, pts]], -3]];
```

```
plotpts = Prepend[plotptsf, {1.0, 0.0}];
```

```
ListPlot[plotpts, PlotJoined → True,
 PlotRange → {{0.2, 1.0}, {0.0, 0.68}}, AxesLabel → {"r", "f"}]
```

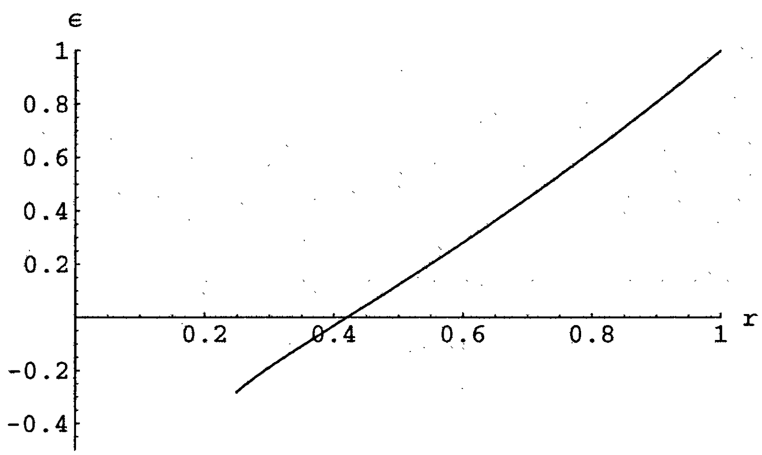



```
- Graphics -
```

The above is a plot of f against r and below is a plot of ϵ against r.

```
plotpts = Prepend[plotptseps, {1.0, epsilon0}];
```

```
ListPlot[Drop[plotpts, -140], PlotJoined → True,
 PlotRange → {{0.0, 1.0}, {-0.5, 1.0}},
 AxesOrigin → {0, 0}, AxesLabel → {"r", "ε"}]
```




```
- Graphics -
```

The second case.

```
epsilon0 = 0.0;
epsilondoubleprime0 = 1.0;
epsilonprime0 = -nc0 epsilon0 / nc1;
```

```
data0 = {1.0,
  {0.0, 2 gdot gdot / Ftwo[0.0, 1.0, h], epsilon0, epsilonprime0}}

{1., {0., -1.67508, 0., 0.}}
```

```
RungeKuttaFourCInitial[{ff, -0.001, data0}]

{ff, -0.001,
 {0.999, {0.00164148, -1.67269, 1.19518×10^-7, -0.0000489044}}}
```

```
pts = NestList[RungeKuttaFourC, %, 750];
```

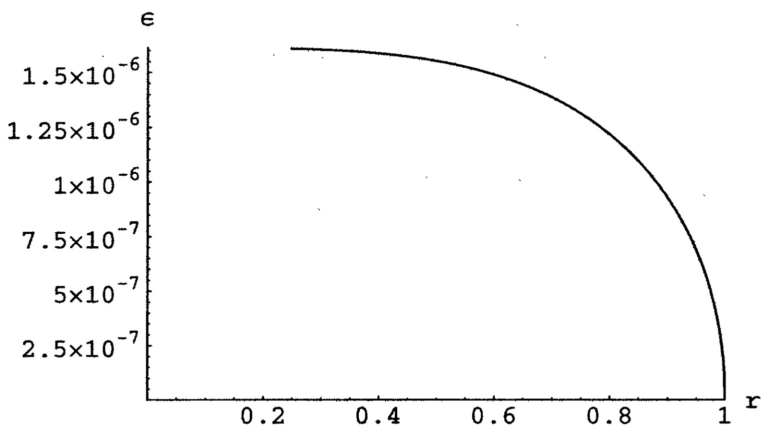
```
plotptseps = Transpose[Drop[
   Drop[Transpose[Map[Flatten[Last[#]] &, pts]], -1], {2, 3}]];
```

```
plotptsf =
  Transpose[Drop[Transpose[Map[Flatten[Last[#]] &, pts]], -3]];
```

Another plot below is for ϵ against r.

```
plotpts = Prepend[plotptseps, {1.0, epsilon0}];
```

```
ListPlot[plotpts, PlotJoined → True,
 PlotRange → {{0.0, 1.0}, {0.0, 0.000001615}},
 AxesOrigin → {0, 0}, AxesLabel → {"r", "ϵ"}]
```



- Graphics -

It must be emphasised that the manipulative power of *Mathematica* was most useful in calculating the coefficients employed above especially in their limiting cases. We have embedded the relevant cells below but do not list them in their open form herein.

■ Limiting value of f'' at r=1 (general form)

■ Limiting value of f''' at r=1 (general form)

■ Epsilon Equation Coefficients

Chapter 5

In this chapter we are concerned with obtaining new periodic solutions of our standard systems and periodic solutions of some new systems. As a starting point we sought planar solutions of a system with new symmetry. These solutions were illustrated in a very simple manner and the raw code has already been mentioned in an earlier section.

We start the motion off from a point with zero velocity and by doing so are able to characterise the motion by means of a critical angle, **tho**.

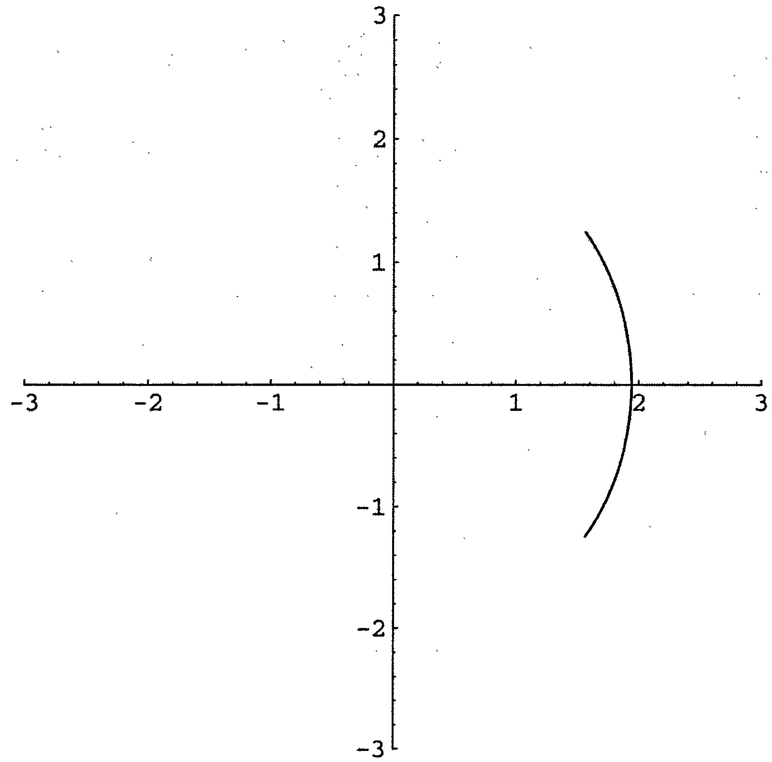
```
tho = N[π / 4.68];
```

We resort to using the native code to construct an interpolating function and assign the output to a named variable as before.

```
NDSolve[
 {x''[t] == (x[t] - y[t]) / (2 (x[t] - y[t])^2 + 4 z[t]^2)^(3/2) +
    (x[t] + y[t]) / (2 (x[t] + y[t])^2 + 4 z[t]^2)^(3/2) +
    2 x[t] / (4 x[t]^2 + 4 y[t]^2)^(3/2) - 4 x[t] / (x[t]^2 + y[t]^2 + z[t]^2)^(3/2),
  y''[t] == (y[t] - x[t]) / (2 (x[t] - y[t])^2 + 4 z[t]^2)^(3/2) +
    (x[t] + y[t]) / (2 (x[t] + y[t])^2 + 4 z[t]^2)^(3/2) +
    2 y[t] / (4 x[t]^2 + 4 y[t]^2)^(3/2) - 4 y[t] / (x[t]^2 + y[t]^2 + z[t]^2)^(3/2),
  z''[t] == 2 z[t] / (2 (x[t] - y[t])^2 + 4 z[t]^2)^(3/2) +
    2 z[t] / (2 (x[t] + y[t])^2 + 4 z[t]^2)^(3/2) -
    4 z[t] / (x[t]^2 + y[t]^2 + z[t]^2)^(3/2),
  x[0] == 2 Cos[tho], y[0] == 2 Sin[tho], z[0] == 0.0,
  x'[0] == 0.0, y'[0] == 0.0, z'[0] == 0.0},
 {x, y, z}, {t, 0, 4}, MaxSteps -> 5000 ]
```

We see the oscillatory nature of the motion for one particle in the diagram below.

```
ParametricPlot[Evaluate[{x[t], y[t]} /. rsol],
 {t, 0, 4}, PlotPoints → 2000,
 PlotRange → {{-3, 3}, {-3, 3}}, AspectRatio → 1]
```




```
- Graphics -
```

When we have to search for more involved periodic solutions we have initially mimiced the methods of Davies et al.

The basic idea is to numerically solve the reduced equations of motion and then at an appropriate time stop the solution and compare the position and velocity with the target position and velocity. Sometimes, searching for a multi-node solution for example, the target position and velocity will be related to the initial position and velocity by some fixed rotation with a possible additional reflection for the velocity. In some situations, however, the target position and velocity are just the initial position and velocity.

We take advantage of the native *Mathematica* code to find an approximate solution to the problem in hand and then use the more accurate Runge Kutta scheme to find a more accurate solution. After interpolating against the chosen coordinate, usually z (in this section y), we compute the six dimensional distance between the final position and velocity and the target position and velocity.

We start as always with some initial velocity data just in order to test some of the code as we proceed.

```
adot = 0.55;
gdot = 0.6;
```

The module **nsol** is used to construct a rough and ready solution to our problem. The bonus, however, is that it returns an interpolating function and so we can readily solve implicit equalities involving the solution.

```
nsol[adot_, gdot_] := NDSolve[
  {x''[t] == (x[t] - y[t]) / (2 (x[t] - y[t])^2 + 4 z[t]^2)^(3/2) +
     (x[t] + y[t]) / (2 (x[t] + y[t])^2 + 4 z[t]^2)^(3/2) +
     2 x[t] / (4 x[t]^2 + 4 y[t]^2)^(3/2) - 4 x[t] / (x[t]^2 + y[t]^2 + z[t]^2)^(3/2),
   y''[t] == (y[t] - x[t]) / (2 (x[t] - y[t])^2 + 4 z[t]^2)^(3/2) +
     (x[t] + y[t]) / (2 (x[t] + y[t])^2 + 4 z[t]^2)^(3/2) +
     2 y[t] / (4 x[t]^2 + 4 y[t]^2)^(3/2) - 4 y[t] / (x[t]^2 + y[t]^2 + z[t]^2)^(3/2),
   z''[t] == 2 z[t] / (2 (x[t] - y[t])^2 + 4 z[t]^2)^(3/2) +
     2 z[t] / (2 (x[t] + y[t])^2 + 4 z[t]^2)^(3/2) -
     4 z[t] / (x[t]^2 + y[t]^2 + z[t]^2)^(3/2), x[0] == 2.0, y[0] == 0.0,
   z[0] == 0.0, x'[0] == 0.0, y'[0] == adot, z'[0] == gdot},
  {x, y, z}, {t, 0, 20}, MaxSteps -> 5000 ]
```

The Runge Kutta scheme is the same as that displayed previously but this time we have a six vector for the representative of the derivative.

```
f[{t_, {x_, y_, z_, u_, v_, w_}}] :=
 {u, v, w, (x - y) / (2 (x - y)^2 + 4 z^2)^(3/2) + (x + y) / (2 (x + y)^2 + 4 z^2)^(3/2) +
   2 x / (4 x^2 + 4 y^2)^(3/2) - 4 x / (x^2 + y^2 + z^2)^(3/2),
  (y - x) / (2 (x - y)^2 + 4 z^2)^(3/2) + (x + y) / (2 (x + y)^2 + 4 z^2)^(3/2) +
   2 y / (4 x^2 + 4 y^2)^(3/2) - 4 y / (x^2 + y^2 + z^2)^(3/2),
  2 z / (2 (x - y)^2 + 4 z^2)^(3/2) + 2 z / (2 (x + y)^2 + 4 z^2)^(3/2) -
   4 z / (x^2 + y^2 + z^2)^(3/2)}
```

```
RungeKuttaFourC[{f_, h_, p0_}] := Module[{np1, k1, k2, k3, k4},
    k1 = f[p0];
    k2 = f[p0 + {h / 2, h k1 / 2}];
    k3 = f[p0 + {h / 2, h k2 / 2}];
  k4 = f[p0 + {h, h k3}];
    np1 = p0 + {h, h (k1 + 2 k2 + 2 k3 + k4) / 6};
       {f, h, np1}]
```

The initial data string, when used,, is again of a familiar form.

```
data0 = {0.0, {2.0, 0.0, 0.0, 0.0, adot, gdot}}

{0., {2., 0., 0., 0., 0.55, 0.6}}
```

The next module is the engine for the search. We find a rough and ready solution, **rsol**, starting with our initial velocity estimate and then find the first time at which we are nearest the target, in this case, **y=0**. Having found this time we run the Runge Kutta scheme to within 100 time steps of this target time and then construct a datastring of length 200 which takes the solution to approximately 100 time steps after the critical time. Identifying the first data point after the critical time, **y=0**, and the last data point after the critical time we obtain **gvec** and **lvec**. We then construct **cvec** by interpolation and it is **cvec** which we compare with our target position and velocity.

```
sixmet[adot_, gdot_, f_, h_] :=
 Module[{rsol, ru, nsteps, dataal, datast,
   vec, pos, gvec, lvec, yg, yl, cvec, size},
rsol = nsol[adot, gdot];
ru = FindRoot[(y /. First[rsol])[t] == 0, {t, 6}];
nsteps = Floor[(t /. ru) / h - 100];
dataal = Nest[RungeKuttaFourC,
    {f, h, {0.0, {2.0, 0.0, 0.0, 0.0, adot, gdot}}}, nsteps];
datast = NestList[RungeKuttaFourC, dataal, 200];
vec =
   First[Select[datast, (Part[Last[Last[#]], 2] >= 0.0) &, 1]];
pos = First[Flatten[Position[datast, vec]]];
gvec = Last[Last[vec]];
lvec = Last[Last[Part[datast, pos - 1]]];
yg = Part[gvec, 2];
yl = Part[lvec, 2];
cvec = (yg lvec - yl gvec) / (yg - yl);
size = Sqrt[(cvec - {2.0, 0.0, 0.0, 0.0, adot, gdot}).
     (cvec - {2.0, 0.0, 0.0, 0.0, adot, gdot})]]
```

```
sixmet[adot, gdot, f, 0.0001]

0.0180909
```

```
sixmet[0.7028240185972064, 0.8441390185972064, f, 0.0001]

0.00417142
```

In order that we can automate the search for periodic solutions we use the following module to run **sixmet** on an array of initial data points centred on some chosen initial velocity data. This module, **hunt**, either returns the data point with the smallest six-dimensional separation or if it is the initial data point it halves the working mesh. This method is crude or naive but it does seem to be effective. We may set the timestep for the Runge Kutta scheme outwith the control of the module **hunt** and further refine the size of the time mesh as required.

```
timestep = 0.0001;
```

```
hunt[{adot_, gdot_, mesh_}] := Module[{ar, rpos},
  ar = Table[sixmet[adot + mesh i, gdot + mesh j, f, timestep],
    {i, -1, 1, 1}, {j, -1, 1, 1}];
  rpos = First[Position[ar, Min[ar]]] - {2, 2};
  If[rpos == {0, 0}, {adot, gdot, mesh / 2},
   {adot + mesh First[rpos], gdot + mesh Last[rpos], mesh}]]
```

Given that we have some reasonably accurate initial data for a periodic solution of our chosen system we can store some spatial data for later use by means of the module **Wrapper**. This module simply sieves the computed data in order that we may represent the periodic solution in a more compact form.

```
adot = 0.7028240185972064;
gdot = 0.8441390185972064;
```

```
data0 = {0.0, {2.0, 0.0, 0.0, 0.0, adot, gdot}}

{0., {2., 0., 0., 0., 0.702824, 0.844139}}
```

```
Wrapper[u_] := Module[{te}, te = Nest[RungeKuttaFourC, u, 100];
  Write[stream, Drop[Last[Last[te]], -3]]; te]
```

A usual with *Mathematica* we may set the file input-output path to be something reasonable otherwise the files created can end up in some strange places.

```
SetDirectory["Ahmed:Thesis:Chapter(6):"]

Ahmed:Thesis:Chapter(6)
```

We assign the channel to the variable **stream** as this was quoted in **Wrapper** above.

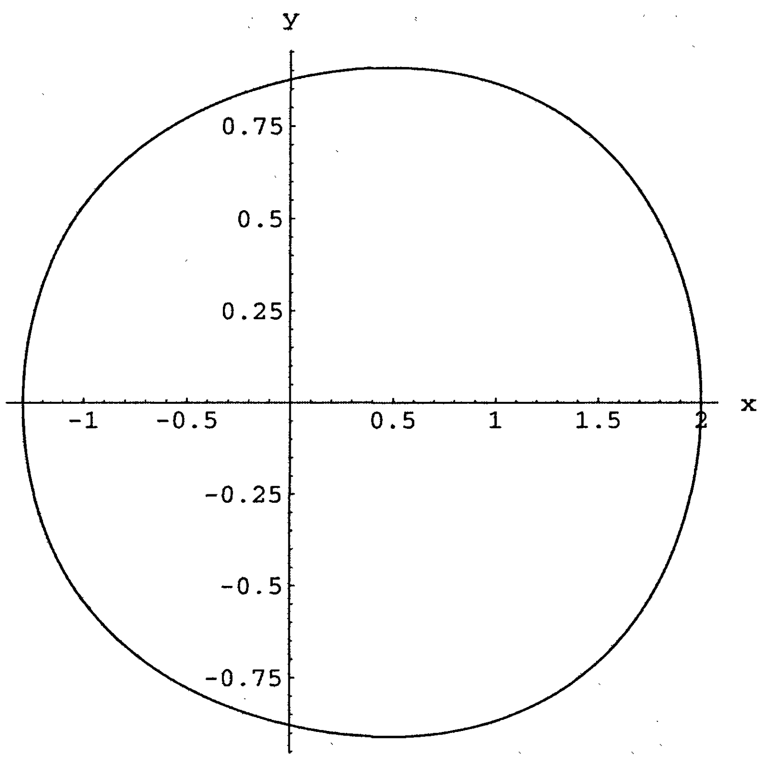
```
stream = OpenWrite["4=(2,2)eap_1"];
Nest[Wrapper, {f, 0.0001, data0}, 800];
Close[stream];
```

Once we have some data saved we simply read it in, choose which coordinates to plot and then employ **ListPlot** to do the work.

```
str = OpenRead["4=(2,2)eap_1"];
sarr = ReadList[str];
Close[str];
```

```
pointsxy = Transpose[Drop[Transpose[sarr], -1]];
```

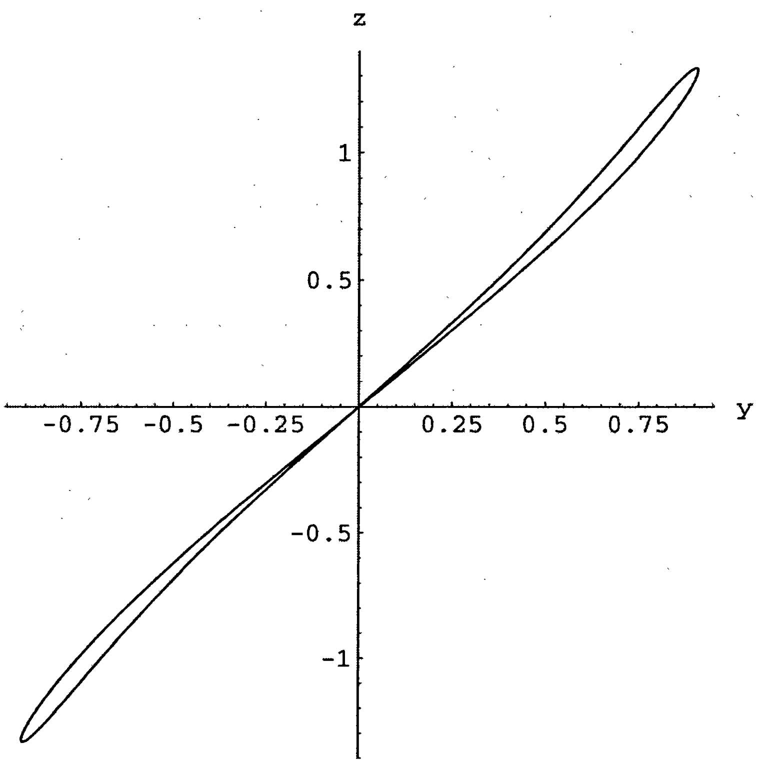
```
ListPlot[pointsxy, PlotJoined → True,
 PlotStyle → PointSize[0.001],
 AspectRatio → 1, AxesLabel → {"x", "y"}]
```



- Graphics -

```
pointsyz = Transpose[Drop[Transpose[sarr], 1]];
```

```
ListPlot[pointsyz, PlotJoined → True,
 AspectRatio → 1, AxesLabel → {"y", "z"}]
```



- Graphics -

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