# Fundamental solutions of a class of pseudo-differential operators with time-dependent negative definite symbols.

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# Abstract

In this thesis, we construct the fundamental solutions of pseudo-differential operators $q(x, t, D)$ with time-dependent negative definite symbols $q(x, t, \xi)$ which are discussed as generators of Feller- and $L^2$-sub-Markovian semigroups . The results are based on the Hille-Yosida theorem, the standard results of analytic semigroups of operators and the fundamental solutions of time-dependent parabolic equations.

# Contents

# Part I

# Introduction & Notation

## • Introduction

Pseudo-differential operators with negative definite symbols $q(x,\xi)$ have since the last ca. 20 years been investigated as generators of Feller- and $L^2$-sub-Markovian semigroups, hence they are also generators of Markov processes. Following first work of N.Jacob [9],[10] and [11], compare also the monograph [14]-[16], the most general results are due to W.Hoh [6]-[8]. In [2] B.Böttcher used a direct approach to construct a fundamental solution to operators of type $\frac{\partial}{\partial t} - q(x,D)$,$q(x,\xi)$ having a negative definite symbol. He extended in [3] his considerations to some time dependent operators $\frac{\partial}{\partial t} - q(x,t;D)$. These operators are related to symbols used by W.Hoh in [8], i.e. they allow a symbolic calculus and hence must be smooth with respect to the co-variable $\xi$. We want also to mention the monograph [5] by S. Eidelman et al. and the work of V. Kolokoltsov [17] and [18] where for more classical symbols analogous results are discussed. Note that recently A. Potrykus in [21] and [22] could improve the results obtained in [8] and [12], respectively.
In our approach we want to minimize in the time-dependent case regularity with respect to $\xi$ of the negative definite symbol $q(x,t;\xi)$, in fact we assume no differentiability at all. For this we carefully revise the estimates and results obtained in [11] where an approach by O. Oleinik and E.Radkevic, see [19], was adopted for the time-independent case. We have in mind more a case study, we do not aim to optimize conditions on $q(x,t;\xi)$.

There are six parts in my thesis.

**Part I** Introduction & Notation.

**Part II** Fourier Analysis and Semigroups.

**Part III** One Parameter semigroups.

**Part IV** Fundamental Solutions Of Time-Dependent Parabolic Equation.

**Part V** Some Properties Of Pseudo-Differential Operators with Time Dependent Negative Definite Symbols.

**Part VI** $-q(x,t_0,D)$ as Generator Of A Feller Semigroup & Fundamental Solution For $\frac{\partial}{\partial t} - q_\lambda(x,t,D)$

**Part II** is an introduction to those definitions and theorems, we collect from [14], which we have to use later on. We start with introducing the Fourier transform on the Schwartz space $S(\mathbb{R}^n)$ and the space of tempered distribution, $S'(\mathbb{R}^n)$. Then we discuss the convolution property of two functions such as

$$(u*v)\hat{}(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\cdot\hat{v}(\xi)\ .$$

Starting with Section 3, we study the Fourier transform of bounded Borel measure, and show that their Fourier transforms are positive definite functions by applying the fundamental theorem, Bochner's theorem. In the negative definite functions part, we show many lemmas and examples. Most important is a Peetre inequality

$$\frac{1+|\psi(\xi)|}{1+|\psi(\eta)|} \leq 2(1+|\psi(\xi-\eta)|) \ .$$

In **Part III**, we will introduce the theory of one-parameter operator semigroups on Banach spaces $(X, \|\cdot\|_X)$ and the standard results of analytic semigroups of operators. Our aim is to study the theorems and properties of Feller and sub-Markovian semigroups. The main results are the theorems around the Hille-Yosida theorem, Theorem 4.23 and Theorem 4.26.

In **Part IV**,we will discuss the fundamental solutions of the time-dependent parabolic equation :

$$\frac{du(t)}{dt} + A(t)u(t) = f(t), \ 0 \leq t \leq T.$$

$$u(0) = u_0.$$

After constructing the fundamental solution $U(t, s)$ by applying many important assumptions and inequalities, we finally arrive at the main results, Theorem 5.15 and Theorem 5.16, showing that there exists a fundamental solution $U(t, s)$ of the time-dependent parabolic equation with certain conditions.

In **Part V**, we will study a large class of these pseudo-differential operators with coefficients depending on time which may for frozen time dependence extend to a generator of a Feller semigroup or a sub-Markovian semigroup. Our special interest is proving estimates for the operator $q(x, t, D)$ in Sobolev spaces related to continuous negative definite functions with time-dependent negative definite symbols, $q(x, t, \xi)$, more precisely, in the Hilbert space $H^{\psi,s}(\mathbb{R}^n)$ . After decomposing the time dependent symbols into $q_1(t, \xi)$ and $q_2(x, t, \xi)$, i.e. $q(x, t, \xi) = q_1(t, \xi) + q_2(x, t, \xi)$, we discuss the assumptions and estimates of the operator $q_1(t, D)$ and $q_2(x, t, D)$ separately. In the remaining of the part V, we introduce the Friedrichs mollifier and discuss its properties . Then we focus on the main theorem, Theorem 6.27, i.e. there exists a unique variational solution $u(\cdot, t) \in H^{\psi,s_2+2}(\mathbb{R}^n)$ to the equation $q_\lambda(x, t, D)u(x, t) = q(x, t, D)u(x, t) + \lambda u(x, t) = f$.

**Part VI** discusses in detail the construction of Feller and $L^2$-sub-Markovian semigroups with a pseudo-differential operator (with time-dependent negative definite symbols) as pre-generator using the Hille-Yosida theorem, which we discuss in part III. After applying the standard results of analytic semigroups of operators in part III and Part IV, we construct the fundamental solution $U(t, s)$ in a $L^2$-context to the parabolic problem:

$$\frac{\partial u(x,t)}{\partial t} + q(x, t, D)u(x, t) = 0 \qquad and \quad u(x, 0) = f(x).$$

## • Notation

1. $\mathbb{N}$ natural numbers
2. $\mathbb{R}$ real numbers
3. $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$
4. $\mathbb{R}^n$ euclidean vector space
5. $\mathbb{C}$ complex numbers
6. $B(\Omega)$ Borel measurable function
7. $B_b(\Omega)$ bounded Borel measurable function
8. $C(G)$ continuous functions
9. $C_0(G)$ continuous functions with compact support
10. $C_\infty(G)$ continuous functions vanishing at infinity
11. $C_b(G)$ bounded continuous functions
12. $C^m(G)$ m-times continuously differentiable functions
13. $C^\infty(G) = \bigcap_{m\in\mathbb{N}} C^m(G)$
14. $H^{\psi,s}(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n); \|u\|_{\psi,s} < \infty\}$
15. $L^p(\Omega, \mu)$ usual Lebesgue space over $(\Omega, \mathcal{A}, \mu)$
16. $M_b^+(\Omega)$ bounded measures on $\Omega$
17. $S(\mathbb{R}^n)$ Schwartz space of tempered functions
18. $S'(\mathbb{R}^n)$ tempered distributions
19. $CN((\mathbb{R}^n)$ continuous negative definite functions
20. $CP((\mathbb{R}^n)$ continuous positive definite functions
21. $N((\mathbb{R}^n)$ negative definite functions
22. $P((\mathbb{R}^n)$ positive definite functions
23. $(X, \|\cdot\|_X)$ Banach space $X$ with norm $\|\cdot\|_X$
24. $X \hookrightarrow Y$ continuous embedding if $X$ into $Y$
25. $(A, D(A))$ linear operator with domain $D(A)$

26. $D(A)$ domain of an operator

27. $R(A)$ range of an operator

28. $\Gamma(A)$ graph of an operator

29. $\rho(A)$ resolvent of an operator

30. $(R_\lambda)_{\lambda>0}$ resolvent of an operator

31. $q(x,D)$ pseudo-differential operator with symbol $q(x,\xi)$

32. $\psi(D)$ pseudo-differential operator with symbol $\psi(\xi)$

33. $(T_t)_{t\geq 0}$ one parameter semigroup of operator

34. $(T_t^{(\infty)})_{t\geq 0}$ semigroup on $C_\infty(\mathbb{R}^n)$

35. $A^{(\infty)}$ generator of $(T_t^{(\infty)})_{t\geq 0}$

36. $(T_t^{(p)})_{t\geq 0}$ semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$

37. $A^{(p)}$ generator of $(T_t^{(p)})_{t\geq 0}$

38. $a \wedge b = min(a,b)$

39. $a \vee b = max(a,b)$

40. $\partial^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$

41. $\chi_A$ characteristic function of the set $A$

42. $Re\ f$ real part of a function

43. $f \circ g$ composition of functions

44. $f * g$ convolution of functions (or distributions)

45. $\mu_1 * \mu_2$ convolution of the measures $\mu_1$ and $\mu_2$

46. $supp\ u$ support of a function or distribution

47. $supp\ \mu$ support of a measures $\mu$

48. $(\mu_t)_{t\geq 0}$ convolution semigroup of probabilities

49. $\|u\|_X$ norm of $u$ in the space of $X$

50. $\|u\|_0$, $(u,u)_0$ norm and scalar product in $L^2(\Omega,\mu)$

51. $\|u\|_\infty = \sup|u(x)|$ or $ess \sup |u(x)|$

52. $\|u\|_{\psi,s}$ norm in the space $H^{\psi,s}(\mathbb{R}^n)$

# Part II

# Fourier Analysis and Semigroups

## 1 The Fourier Transform in $S(\mathbb{R}^n)$

This chapter is devoted to introduce those theories which are necessary to explain why certain pseudo-differential operators are generators of Feller and sub-Markovian semigroups. We will study many definitions and theorems which we collect from [14]. We will start by introducing the Fourier transform on the Schwartz space, and discuss its properties.

**Definition 1.1.** *The* ***Schwartz space*** $S(\mathbb{R}^n)$ *consists of all functions* $u \in C^\infty(\mathbb{R}^n)$ *such that for all* $m_1, m_2 \in \mathbb{N}_0$

$$p_{m_1,m_2}(u) := \sup_{x\in\mathbb{R}^n}((1+|x|^2)^{m_1/2} \sum_{|\alpha|\le m_2} |\partial^\alpha u(x)|) < \infty \tag{1.1}$$

*The family* $(p_{m_1,m_2})_{m_1,m_2\in\mathbb{N}}$ *forms a family of separating seminorms.*

**Definition 1.2.** *Let* $u \in S(\mathbb{R}^n)$. *The* ***Fourier transform*** *of* $u$ *is defined by*

$$\hat{u}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)dx. \tag{1.2}$$

*Sometimes we will write* $F_{x\mapsto\xi}(u)(\xi)$ *or* $F(u)(\xi)$ *for* $\hat{u}(\xi)$.

**Definition 1.3.** *On* $S(\mathbb{R}^n)$ *we define the* ***inverse Fourier transform*** *by*

$$(F^{-1}u)(\eta) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\eta\cdot y} u(y)dy \tag{1.3}$$

*We also will use* $F^{-1}_{y\mapsto\eta}(u)$ *for denoting* $(F^{-1}u)(y)$.

The following theorem summarizes some useful properties of the Fourier transform on $S(\mathbb{R}^n)$.

**Theorem 1.4.** *The Fourier transform* $F$ *is a linear bijective and continuous operator from* $S(\mathbb{R}^n)$ *into itself which has a continuous inverse given by* (1.3).
*Thus, on* $S(\mathbb{R}^n)$ *we have* $F \circ F^{-1} = F^{-1} \circ F = id$.

**Remark 1.5.** *Note that on* $S(\mathbb{R}^n)$ *we have* $F^4 = id$, *or* $F^{-1} = F^3$.

**Theorem 1.6.** *For all* $u \in S(\mathbb{R}^n)$,

$$\|\hat{u}\|_\infty \le (2\pi)^{-n/2}\|u\|_{L^1} \tag{1.4}$$

*and*

$$\|u\|_0 = \|\hat{u}\|_0 \tag{1.5}$$

*hold.*

**Remark 1.7.** *From* (1.5), *we get immediately, for all* $u, v \in S(\mathbb{R}^n)$,

$$(u, v)_0 = (\hat{u}, \hat{v})_0 \,. \tag{1.6}$$

*Estimate* (1.4) *entails that we can extend the Fourier transform from* $S(\mathbb{R}^n)$ *to a continuous linear mapping from* $\mathbb{L}^1(\mathbb{R}^n)$ *to* $C_\infty(\mathbb{R}^n)$, *whereas* (1.8) *allows us to extend the Fourier transform to an isometry on* $\mathbb{L}^2(\mathbb{R}^n)$.
*Thus, on* $\mathbb{L}^2(\mathbb{R}^n)$ *we have*

$$\|u\|_{\mathbb{L}^2} = \|\hat{u}\|_{\mathbb{L}^2} \tag{1.7}$$

*and*

$$(u, v)_{\mathbb{L}^2} = (\hat{u}, \hat{v})_{\mathbb{L}^2} \,. \tag{1.8}$$

Furthermore, estimate (1.4) leads to the **Lemma of Riemann-Lebesgue**:

**Theorem 1.8.** *The Fourier transform is a continuous linear operator from* $\mathbb{L}^1(\mathbb{R}^n)$ *into* $C_\infty(\mathbb{R}^n)$ *and*

$$\|\hat{u}\|_\infty \leq (2\pi)^{-n/2} \|u\|_{\mathbb{L}^1} \tag{1.9}$$

*holds for all* $u \in \mathbb{L}^1(\mathbb{R}^n)$.

**Definition 1.9.** *Let* $u, v \in S(\mathbb{R}^n)$, *their* ***convolution*** *is the function*

$$x \mapsto (u * v)(x) = \int_{\mathbb{R}^n} u(x - y) v(y) dy \tag{1.10}$$

*which is again an element in* $S(\mathbb{R}^n)$.

**Definition 1.10.** *Let* $\mu_j \in \mathbb{M}_b^+(\mathbb{R}^n)$,$1 \leq j \leq k$, *be measures. The image of* $\mu_1 \otimes \ldots \otimes \mu_k$ *under* $A_k$ *is called the convolution of these measures and is denoted by*

$$\mu_1 * \ldots * \mu_k := A_k(\mu_1 \otimes \ldots \otimes \mu_k). \tag{1.11}$$

*where* $A_k : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$, $(y^1, \ldots, y^k) \mapsto A_k(y^1, \ldots, y^k) = y^1 + \cdots + y^k$.

**Theorem 1.11.** *Let* $u, v \in S(\mathbb{R}^n)$. *Then we have*

$$(u \cdot v)\,\hat{}\,(\xi) = (2\pi)^{-n/2} (\hat{u} * \hat{v})(\xi) \tag{1.12}$$

*and*

$$(u * v)\,\hat{}\,(\xi) = (2\pi)^{n/2} \hat{u}(\xi) \cdot \hat{v}(\xi) \quad . \tag{1.13}$$

# 2 The Fourier Transform in $S'(\mathbb{R}^n)$

In this section ,we will extend the Fourier transform from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$ by duality. On $S'(\mathbb{R}^n)$, we will always consider the weak-∗-topology in the following.

**Definition 2.1.** *$S'(\mathbb{R}^n)$ is called the* ***space of tempered distributions****.*
*It consists of all distribution $u \in D'(\mathbb{R}^n)$ having a continuous extension to $S(\mathbb{R}^n)$, i.e. $S'(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ , and $D'(\mathbb{R}^n)$ is the space of distributions on $\mathbb{R}^n$.*

**Remark 2.2.** *The weakest topology on $X^*$ which makes all elements of $X$ continuous, i.e. for every $u \in X$ a continuous linear functional on $X^*$ is given by $u(x^*) = x^*(u)$, is called the weak-∗-topology on $X^*$.*
*For every neighbourhood $U$ of $0 \in S'(\mathbb{R}^n)$, there exists a neighbourhood $V$ of $0 \in S'(\mathbb{R}^n)$ such that*

$$V = \{u \in S'(\mathbb{R}^n) |\ |< u, \hat{\phi}_j >|< 1 \quad for\ \phi_1, \cdots, \phi_k \in S(\mathbb{R}^n)\}.$$

*Moreover, a sequence $(u_\nu)_{\nu\in\mathbb{N}}$, $u_\nu \in S'(\mathbb{R}^n)$ , converges in the weak-∗-topology to $u \in S'(\mathbb{R}^n)$ if and only if*

$$< u_\nu, \phi > \quad \rightarrow \quad < u, \phi >, \quad for\ all \quad \phi \in S(\mathbb{R}^n).$$

**Definition 2.3.** *Let $u \in S'(\mathbb{R}^n)$. The* ***Fourier transform*** *$\hat{u}$ of $u$ is defined by*

$$< \hat{u}, \phi > := < u, \hat{\phi} >, \quad for\ all \quad \phi \in S(\mathbb{R}^n). \tag{2.1}$$

*As usual, we use also the notation $Fu$ for $\hat{u}$.*

**Remark 2.4.** *For $g \in L^1(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$, and $\phi \in S(\mathbb{R}^n)$, we find*

$$< \hat{g}, \phi > = < g, \hat{\phi} > = \int_{\mathbb{R}^n} g(x)\hat{\phi}(x)dx = \int_{\mathbb{R}^n} \hat{g}(x)\phi(x)dx,$$

*which shows that $\hat{g}$ in the sense of $S'(\mathbb{R}^n)$ coincides with $\hat{g}$ as it is defined on $L^1(\mathbb{R}^n)$.*

Furthermore, since convergence in $\mathbb{L}^2(\mathbb{R}^n)$ implies weak-∗-convergence in $S'(\mathbb{R}^n)$, we may deduce that the Fourier transform as defined on $S'(\mathbb{R}^n)$ also extends the Fourier transform as defined on $\mathbb{L}^2(\mathbb{R}^n)$.

**Theorem 2.5.** *The Fourier transform is a continuous linear operator from $S'(\mathbb{R}^n)$ into itself which is bijective and has a continuous inverse $F^{-1}$.*

**Theorem 2.6.** *For $u \in S'(\mathbb{R}^n)$ and $\phi \in S(\mathbb{R}^n)$, the convolution $u * \phi$ is defined and we have*

$$(u * \phi)\,\hat{} = (2\pi)^{n/2}\hat{\phi} \cdot \hat{u} \tag{2.2}$$

*and*

$$(\phi \cdot u)\,\hat{} = (2\pi)^{-n/2}\hat{u} * \hat{\phi} \quad . \tag{2.3}$$

# 3 Positive Definite Functions and Negative Definite Functions

In this section, we want to study the Fourier transform of Borel measures $\mu \in \mathbb{M}_b^+(\mathbb{R}^n)$.

Since $\mathbb{M}_b^+(\mathbb{R}^n)$ is a subset of $S'(\mathbb{R}^n)$, the Fourier transform $\hat{\mu}$ of $\mu$ is well defined and for $\phi \in S(\mathbb{R}^n)$, we have

$$< \hat{\mu}, \phi > = < \mu, \hat{\phi} > = \int_{\mathbb{R}^n} \hat{\phi}(\xi)\mu(d\xi). \tag{3.1}$$

From the definition of $\hat{\phi}$, we get, using Fubini's theorem,

$$\begin{aligned} < \hat{\mu}, \phi > &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\xi} \phi(x) dx \mu(d\xi) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-ix\xi} \mu(d\xi) \right) \phi(x) dx \\ &= \left\langle (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i(\cdot,\xi)} \mu(d\xi), \phi \right\rangle, \end{aligned}$$

hence we have

$$\hat{\mu}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \mu(dx) \tag{3.2}$$

**Definition 3.1.** *A function $u : \mathbb{R}^n \to \mathbb{C}$ is called* ***positive definite*** *if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \cdots, \xi^k \in \mathbb{R}^n$ the matrix $(u(\xi^j - \xi^l))_{j,l=1,2,\cdots,k}$ is positive Hermitian, i.e. for all $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$, we have*

$$\sum_{j,l=1}^{k} u(\xi^j - \xi^l)\lambda_j \bar{\lambda}_l \geq 0. \tag{3.3}$$

**Theorem 3.2.** *Let $\mu \in \mathbb{M}_b^+(\mathbb{R}^n)$. Then $\hat{\mu}$ is a positive definite function.*

*Proof.* For $k \in \mathbb{N}$ and $\xi^1, \cdots, \xi^k \in \mathbb{R}^n$ we find with $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$

$$\begin{aligned} \sum_{j,l=1}^{k} \lambda_j \overline{\lambda_l} \hat{\mu}(\xi^j - \xi^l) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \sum_{j,l=1}^{k} \lambda_j \overline{\lambda_l} e^{-i(\xi^j - \xi^l)\cdot x} \mu(dx) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left( \sum_{j=1}^{k} \lambda_j e^{-i\xi^j \cdot x} \right) \cdot \left( \overline{\left( \sum_{l=1}^{k} \lambda_l e^{-i\xi^l \cdot x} \right)} \right) \mu(dx) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left| \sum_{j=1}^{k} \lambda_j e^{-i\xi^j \cdot x} \right|^2 \mu(dx) \geq 0. \end{aligned}$$

□

Of fundamental importance is **Bochner's theorem**:

**Theorem 3.3.** *A function $u : \mathbb{R}^n \to \mathbb{C}$ is the Fourier transform of a measure $\mu \in \mathrm{M}_b^+(\mathbb{R}^n)$ with total mass $\|\mu\|$, if and only if the following conditions are fulfilled*

1. *$u$ is continuous;*
2. *$u(0) = \hat{\mu}(0) = (2\pi)^{-n/2}\|\mu\|$;*
3. *$u$ is positive definite.*

**Definition 3.4.** *A family $(\mu_t)_{t\geq 0}$ of bounded Borel measures on $\mathbb{R}^n$ is called a* ***convolution semigroup*** *on $\mathbb{R}^n$, if the following conditions are fulfilled*

$$\mu_t(\mathbb{R}^n) \leq 1 \text{ for all } t \geq 0 \tag{3.4}$$

$$\mu_s * \mu_t = \mu_{t+s}, s, t \geq 0 \text{ and } \mu_0 = \varepsilon_0, \tag{3.5}$$

$$\mu_t \to \varepsilon_0 \text{ vaguely as } t \to 0 \tag{3.6}$$

(Note that for $a \in \Omega$ the *Dirac measure* at $a$ is denoted by $\varepsilon_0$. In our case, $\Omega = \mathbb{R}^n$ and $a = 0$, we write $\varepsilon_0$ instead of $\varepsilon_a$.)

**Recall 3.5.** *Let $(\mu_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $\mathrm{M}_b^+(\mathbb{R}^n)$ and $\mu_0 \in \mathrm{M}_b^+(\mathbb{R}^n)$. We say that $(\mu_\nu)_{\nu\in\mathbb{N}}$* ***converges vaguely*** *to $\mu_0$, if for all $u \in C_0(\mathbb{R}^n;\mathbb{R})$ we have*

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^n} u(x)\mu_\nu\ (dx) = \int_{\mathbb{R}^n} u(x)\mu_0\ (dx). \tag{3.7}$$

**Theorem 3.6.** *Suppose that $(\mu_\nu)_{\nu\in\mathbb{N}}$, $\mu_\nu \in \mathrm{M}_b^+(\mathbb{R}^n)$, converges vaguely to $\mu \in \mathrm{M}_b^+(\mathbb{R}^n)$ and that $\lim_{\nu\to\infty} \mu_\nu(\mathbb{R}^n) = \mu(\mathbb{R}^n)$. Then $(\mu_\nu)_{\nu\in\mathbb{N}}$ converges weakly to $\mu$.*

**Recall 3.7.** *Let $(\mu_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $\mathrm{M}_b^+(\mathbb{R}^n)$ and $\mu_0 \in \mathrm{M}_b^+(\mathbb{R}^n)$. We say that $(\mu_\nu)_{\nu\in\mathbb{N}}$* ***converges weakly*** *to $\mu_0$, if (3.7) holds for all $u \in C_b(\mathbb{R}^n;\mathbb{R})$.*

**Lemma 3.8.** *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$. Then the mapping $t \mapsto \mu_t$ is continuous at $t = 0$ with respect to the Bernoulli topology, i.e. the topology of weak convergence of measures.*

*Proof.* For $\phi \in C_0(\mathbb{R})$,$0 \leq \phi \leq 1$ and $\phi(0) = 1$, we find by (3.4) and (3.6)

$$1 = \phi(0) = \lim_{t\to 0} \int_{\mathbb{R}^n} \phi d\mu_t \leq \liminf_{t\to 0} \mu_t(\mathbb{R}^n) \leq \limsup_{t\to 0} \mu_t(\mathbb{R}^n) \leq 1,$$

hence

$$\lim_{t\to 0} \mu_t(\mathbb{R}^n) = 1 = \varepsilon_0(\mathbb{R}^n)$$

and by Theorem 3.6 the lemma is proved. □

**Lemma 3.9.** *For any convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$ the mapping $t \mapsto \mu_t$ is continuous from $[0,\infty)$ into $\mathrm{M}_b^+(\mathbb{R}^n)$ equipped with the Bernoulli topology.*

*Proof.* For $t, t_0 > 0$ and $\xi \in \mathbb{R}^n$ we get

$$|\hat{\mu}_t(\xi) - \hat{\mu}_{t_0}(\xi)| \leq (2\pi)^{-n/2}|\hat{\mu}_{t_0}(\xi)||\hat{\mu}_{|t-t_0|}(\xi) - (2\pi)^{-n/2}| \leq |\hat{\mu}_{|t-t_0|}(\xi) - (2\pi)^{-n/2}|,$$

but by Lemma 3.8 we conclude that $\hat{\mu}_{|t-t_0|}(\xi) \to (2\pi)^{-n/2}$ uniformly on compact sets as $t \to t_0$. Then

$$\lim_{t\to t_0} \mu_t = \mu_{t_0} \tag{3.8}$$

in the Bernoulli topology.

□

Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$. It follows that the family $(\hat{\mu}_t)_{t\geq 0}$ of the Fourier transforms of $\mu_t, t \geq 0$, consists of continuous positive definite functions on $\mathbb{R}^n$ satisfying $|\hat{\mu}_t(\xi)| \leq (2\pi)^{-n/2}$. Our aim is to show the existence of a unique function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that $\hat{\mu}_t(\xi) := (2\pi)^{-n/2}e^{-t\psi(\xi)}$ holds.

**Theorem 3.10.** *Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$. Then there exists a function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\hat{\mu}_t(\xi) := (2\pi)^{-n/2}e^{-t\psi(\xi)} \tag{3.9}$$

*holds for all $\xi \in \mathbb{R}^n$ and $t \geq 0$.*

*Proof.* For $\xi \in \mathbb{R}^n$ fixed we consider the mapping $\phi_\xi : [0,\infty) \to \mathbb{C}$ defined by

$$\phi_\xi(t) := (2\pi)^{n/2}\hat{\mu}_t(\xi), \quad t \geq 0. \tag{3.10}$$

By Lemma 3.9 this mapping is continuous and the convolution theorem gives

$$\phi_\xi(s+t) = \phi_\xi(t)\phi_\xi(s) \tag{3.11}$$

and

$$\lim_{t\to 0} \phi_\xi(t) = 1. \tag{3.12}$$

It follows the existence of a unique complex number $\psi(\xi)$ such that

$$\phi_\xi(t) = e^{-t\psi(\xi)}, \quad t \geq 0. \tag{3.13}$$

Note that the mapping $\xi \mapsto e^{-t\psi(\xi)}$ must be positive definite and $\psi(0) \geq 0$. □

**Definition 3.11.** *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called* ***negative definite*** *if*

$$\psi(0) \geq 0 \tag{3.14}$$

*and*

$$\xi \mapsto e^{-t\psi(\xi)} \textit{ is positive definite for } t \geq 0. \tag{3.15}$$

Next we will introduce the classes $N(\mathbb{R}^n)$and $CN(\mathbb{R}^n)$ to study negative definite functions more closely.

**Definition 3.12.** *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ belongs to the class $N(\mathbb{R}^n)$ if for any choice of $k \in \mathbb{N}$ and vectors $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ the matrix*

$$(\psi(\xi^j) + \overline{\psi(\xi^l)} - \psi(\xi^j - \xi^l))_{j,l=1,\ldots,k} \tag{3.16}$$

*is positive Hermitian. Further we set*

$$CN(\mathbb{R}^n) := N(\mathbb{R}^n) \cap C(\mathbb{R}^n).$$

*For $\psi \in N(\mathbb{R}^n)$ we have obviously*

$$\psi(0) \geq 0, \tag{3.17}$$

*and since for $\xi \in \mathbb{R}^n$ the matrix*

$$\begin{pmatrix} \psi(\xi) + \overline{\psi(\xi)} - \psi(0) & \psi(\xi) + \overline{\psi(0)} - \psi(\xi) \\ \psi(0) + \overline{\psi(\xi)} - \psi(-\xi) & \psi(0) + \overline{\psi(0)} - \psi(0) \end{pmatrix} \tag{3.18}$$

*is positive Hermitian we find*

$$\psi(\xi) + \psi(0) - \psi(\xi) = \overline{\psi(0)} + \psi(\xi) - \overline{\psi(-\xi)},$$

*i.e.*

$$\psi(\xi) = \overline{\psi(-\xi)} \quad or \quad \psi(\xi) = \tilde{\psi}(\xi), \tag{3.19}$$

*where we used the notation $\tilde{u}(\xi) = \overline{u(-\xi)}$. Furthermore the determinant of the matrix* (3.18) *must be non-negative, implying that*

$$Re\ \psi(\xi) \geq \psi(0) \quad for\ all \quad \xi \in \mathbb{R}^n. \tag{3.20}$$

**Lemma 3.13.** *A. The set $N(\mathbb{R}^n)$ is a convex cone which is closed under pointwise convergence. B. For $\psi \in N(\mathbb{R}^n)$ it follows that $\tilde{\psi}$ and Re $\psi$ belong to $N(\mathbb{R}^n)$ too. C. Any non-negative constant is an element of $N(\mathbb{R}^n)$. D. For $\psi \in N(\mathbb{R}^n)$ and $\lambda > 0$ the function $\xi \mapsto \psi(\lambda\xi)$ belongs to $N(\mathbb{R}^n)$. E. The set $CN(\mathbb{R}^n)$ is a convex cone which is closed with respect to uniform convergence on compact sets. F. For $\psi_j \in N(\mathbb{R}^{n_j}), j = 1, 2,$ it follows that $\psi(\xi, \eta) := \psi_1(\xi) + \psi_2(\eta)$ defines an element in $N(\mathbb{R}^{n_1+n_2})$.*

**Lemma 3.14.** *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is an element in $N(\mathbb{R}^n)$ if and only if the following three conditions are fulfilled*

$$\psi(0) \geq 0; \tag{3.21}$$

$$\psi = \tilde{\psi}; \tag{3.22}$$

*and for any $k \in \mathbb{N}$ and any choice of vectors $\xi^1, \ldots, \xi^k \in \mathbb{R}^n$ and complex numbers $c_1, \ldots, c_k$*

$$\sum_{j=1}^{k} c_j = 0 \text{ implies that} \sum_{j,l=1}^{k} \psi(\xi^j - \xi^l) c_j \overline{c_l} \leq 0. \tag{3.23}$$

**Corollary 3.15.** *Let $u : \mathbb{R}^n \to \mathbb{C}$ be a positive definite function. Then the function $\xi \mapsto u(0) - u(\xi)$ is in $N(\mathbb{R}^n)$.*

**Theorem 3.16.** *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is an element of $N(\mathbb{R}^n)$ if and only if it is negative definite.*

**Remark 3.17.** *From now on we will denote the set of all negative definite functions by $N(\mathbb{R}^n)$, and $CN(\mathbb{R}^n)$ is the set of all continuous negative functions.*

**Theorem 3.18.** *(Schoenberg's theorem) For any convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$ there exists a uniquely determined continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ such that*

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-t\psi(\xi)}, \; t \geq 0 \text{ and } \xi \in \mathbb{R}^n, \tag{3.24}$$

*holds. Conversely, given a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$, then there exists a unique convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$ such that (3.22) holds.*

**Corollary 3.19.** *A function $\psi : \mathbb{R}^n \to \mathbb{C}$ is continuous and negative definite if and only if*

$$\psi(0) \geq 0 \tag{3.25}$$

*and*

$$\xi \mapsto e^{-t\psi(\xi)}, \; t > 0 \text{ is continuous and positive definite} \tag{3.26}$$

**Lemma 3.20.** *For any $\psi \in N(\mathbb{R}^n)$ we have*

$$\sqrt{|\psi(\xi+\eta)|} \leq \sqrt{|\psi(\xi)|} + \sqrt{|\psi(\eta)|},$$

$$\tag{3.27}$$

$$\left| \sqrt{|\psi(\xi)|} - \sqrt{|\psi(\eta)|} \right| \leq \sqrt{|\psi(\xi-\eta)|},$$

*and*

$$|\psi(\xi) + \psi(\eta) - \psi(\xi+\eta)| \leq 2(Re\ \psi(\xi))^{1/2} \cdot (Re\ \psi(\eta))^{1/2}.$$

**Lemma 3.21.** *For any locally bounded negative definite function $\psi \in N(\mathbb{R}^n)$ there exists a constant $c_\psi > 0$ such that for all $\xi \in \mathbb{R}^n$*

$$|\psi(\xi)| \leq c_\psi(1 + |\xi|^2). \tag{3.28}$$

**Lemma 3.22.** ***(Peetre's inequality)*** *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a negative definite function. Then we have*

$$\frac{1 + |\psi(\xi)|}{1 + |\psi(\eta)|} \leq 2(1 + |\psi(\xi - \eta)|) \tag{3.29}$$

*Proof.* For $\eta, \xi' \in \mathbb{R}^n$, we find

$$\begin{aligned}
& 2\,(1+|\psi(\eta)|)\,(1+|\psi(\xi')|) \\
= \;& 2+2|\psi(\eta)|+2|\psi(\xi')|+2|\psi(\eta)||\psi(\xi')| \\
= \;& (1+|\psi(\eta)|+|\psi(\xi')|)+(|\psi(\eta)|+|\psi(\xi')|)+(1+2|\psi(\eta)\psi(\xi')|) \\
\geq \;& 1+|\psi(\eta)|+|\psi(\xi')|+2\sqrt{|\psi(\eta)\psi(\xi')|} \\
= \;& 1+\left(\sqrt{|\psi(\eta)|}+\sqrt{|\psi(\xi')|}\right)^2
\end{aligned}$$

where we used the estimate $2\sqrt{|b_1 b_2|} \leq |b_1|+|b_2|$.

Using the subadditivity of $\eta \to \sqrt{|\psi(\eta)|}$, it follows that

$$2\,(1+|\psi(\eta)|)\,(1+|\psi(\xi')|) \geq \left(1+\sqrt{|\psi(\eta)\psi(\xi')|}^2\right) = 1+|\psi(\eta+\xi')|$$

Taking $\xi' = \xi - \eta$, we finally get

$$2(1+|\psi(\xi-\eta)|) \geq \frac{1+|\psi(\xi)|}{1+|\psi(\eta)|}.$$

□

**Example 3.23.** *Any non-negative symmetric quadratic form $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function. Note , that we do not assume $q$ to have full rank. A convolution semigroup with $q$ as corresponding continuous negative definite function is called a Gaussian semigroup.*

**Example 3.24.** *Let $l : \mathbb{R}^n \to \mathbb{R}$ be a linear functional and define $\psi(\xi) := il(\xi)$. Then the function $\psi$ is a continuous negative definite function. Moreover, whenever $\psi(\xi) = il(\xi)$, $l : \mathbb{R}^n \to \mathbb{R}$ , is a continuous negative definite function, then $l$ must be linear.*

Combining the last two examples and Lemma 3.13 we find that for any $c \geq 0, h \in \mathbb{R}^n$ and symmetric non-negative definite quadratic form $q$ the function

$$\xi \mapsto q(\xi) + ih \cdot \xi + c \tag{3.30}$$

is an element of $CN(\mathbb{R}^n)$. For the corresponding convolution semigroup $(\mu_t)_{t\geq 0}$ on $\mathbb{R}^n$ we find

$$\hat{\mu}_t(\xi) = (2\pi)^{-n/2} e^{-tq(\xi)} e^{-ith\cdot\xi} e^{-tc}.$$

Denoting by $(\mu_t^q)_{t\geq 0}, (\mu_t^h)_{t\geq 0}$ and $(\mu_t^c)_{t\geq 0}$ the convolution semigroup associated with $q, h$ and $c$ , respectively, we find

$$\hat{\mu}_t(\xi) = (2\pi)^n \hat{\mu}_t^q(\xi) \hat{\mu}_t^h(\xi) \hat{\mu}_t^c(\xi),$$

and the convolution theorem yields

$$\mu_t = \mu_t^q * \mu_t^h * \mu_t^c.$$

**Example 3.25.** *Since for $h \in \mathbb{R}^n$ the function $\xi \mapsto e^{-ih\cdot\xi}$ is positive definite and $e^0 = 1$, it follows from Corollary 3.15 that $\xi \mapsto (1 - e^{-ih\cdot\xi})$ is a continuous negative definite function implying that $\xi \mapsto (1 - \cos(h \cdot \xi))$ is an element in $CN(\mathbb{R}^n)$ too. For $h \in \mathbb{R}, h \geq 0$, and $t \geq 0$ let us consider on $\mathbb{R}$ the measures*

$$\mu_t = \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} \varepsilon_{hk}. \tag{3.31}$$

*Taking the Fourier transform of $\mu_t$ we get*

$$\begin{aligned} \hat{\mu}_t(\xi) &= \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} \hat{\varepsilon}_{hk} = \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} (2\pi)^{-1/2} \langle \varepsilon_{hk}, e^{-i(\cdot,k)} \rangle \\ &= (2\pi)^{-1/2} e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} e^{-ikh\xi} \\ &= (2\pi)^{-1/2} e^{-t} \sum_{k=0}^{\infty} \frac{(t \cdot e^{-h\xi})^k}{k!} \\ &= (2\pi)^{-1/2} e^{-t(1-e^{-ih\xi})}, \end{aligned}$$

*implying that $\xi \mapsto 1 - e^{-ih\xi}$ is a continuous negative definite function and that $(\mu_t)_t \geq 0$ is a convolution semigroup on $\mathbb{R}$, called the **Poisson semigroup**.*

Next we will focus on the **Lévy-Khinchin formula** which states that every continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$ has the representation:

$$\psi(\xi) = c + i(d \cdot \xi) + q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-ix\cdot\xi} - \frac{ix \cdot \xi}{1 + |x|^2}\right) \frac{1 + |x|^2}{|x|^2} \mu(dx) \tag{3.32}$$

with a non-negative constant $c \geq 0$, a vector $d \in \mathbb{R}^n$, a symmetric positive semidefinite quadratic form $q$, and a finite Borel measure $\mu$ on $\mathbb{R}^n \setminus \{0\}$. The function $\psi$ is uniquely determined by $(c, d, q, \mu)$ and any such quadruple defines via (3.32) a continuous negative definite function. Note that by Example 3.24 and Example 3.25 for every $x \in \mathbb{R}^n$ the function

$$\xi \mapsto \left(1 - e^{-ix\cdot\xi} - \frac{ix \cdot \xi}{1 + |x|^2}\right) \tag{3.33}$$

is negative definite, and taking into (3.32),the Lévy-Khinchin formula has the interpretation that every continuous negative definite function is a superposition of elementary continuous negative definite functions.

**Definition 3.26.** *Let $\mu$ be the measure in the Lévy-Khinchin presentation of the continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{C}$. The measure*

$$\nu(dx) = \frac{1 + |x|^2}{|x|^2} \mu(dx) \tag{3.34}$$

*defines on $B(\mathbb{R}^n \setminus \{0\})$ is called the **Lévy measure** associated with $\psi$.*

**Remark 3.27.** *Note that the Lévy measure satisfies* $\int_{\mathbb{R}^n\setminus\{0\}}(|x|^2 \wedge 1)\nu(dx) < \infty$.
*A. Let* $\psi : \mathbb{R}^n \to \mathbb{C}$ *be a continuous negative definite function, then we have the following representation*

$$\psi(\xi) = c + i(d\cdot\xi) + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-ix\cdot\xi} - \frac{ix\cdot\xi}{1+|x|^2}\right)\nu(dx) \tag{3.35}$$

*where* $\nu$ *integrates* $x \mapsto (|x|^2 \wedge 1)$.
*B. Let* $\psi : \mathbb{R}^n \to \mathbb{R}$ *be a real-valued continuous negative definite function and denote by* $\nu$ *its Lévy measure. It follows that*

$$\int_{\mathbb{R}^n\setminus\{0\}} \frac{|x|^2}{1+|x|^2}\nu(dx) < \infty, \tag{3.36}$$

*and* $\psi$ *has the representation*

$$\psi(\xi) = c + q(\xi) + \int_{\mathbb{R}^n\setminus\{0\}} \Big(1 - \cos(x\cdot\xi)\Big)\nu(dx). \tag{3.37}$$

The following Table 3.28 is taken from the monograph [1] of Chr. Berg and G. Forst. It summaries some of the examples of continuous negative definite functions and the associated convolution semigroups.

**Table 3.28**

| convolution semigroup | negative definite function |
|---|---|
| Degenerate semigroup : $\mu_t = e^{-at}\varepsilon_0$ , $a \geq 0$ | $a$ |
| Translation semigroup with speed $b \in \mathbb{R}$ : $\mu_t = \varepsilon_{bt}$ | $ib\xi$ |
| Poisson semigroup with jumps of size $s \geq 0$ : $\mu_t = \sum_{k=0}^{\infty} e^{-t}\frac{t^k}{k!}\varepsilon_{sk}$ | $1 - e^{-is\xi}$ |
| One-sided stable semigroup of order $\alpha \in [0,1]$: $\mu_t = \sigma_t^\alpha$ | $(i\xi)^\alpha$ |
| $\Gamma$ semigroup : $\mu_t(dx) = \chi_{(0,\infty)}(x)\frac{1}{\Gamma(t)}x^{t-1}e^{-x}\lambda^{(1)}(dx)$ | $\log(1+\xi^2) + i\arctan\xi$ |
| Brownian semigroup : $\mu_t(dx) = \frac{1}{(4\pi t)^{1/2}}e^{-\frac{x^2}{4t}}\lambda^{(1)}(dx)$ | $\xi^2$ |
| Symmetric stable semigroup of order $\alpha \in (0,2)$ : $\mu_t = \mu_t^{\alpha/2}$ | $\|\xi\|^\alpha$ |
| Cauchy semigroup: $\mu_t(dx) = \mu_t^{1/2}(dx) = \frac{t}{\pi}(t^2+x^2)^{-1}\lambda^{(1)}(dx)$ | $\|\xi\|$ |

# Part III

# One Parameter Semigroup

## 4 One Parameter semigroup

We will introduce the theory of one-parameter operator semigroups on Banach spaces $(X, \|\cdot\|_X)$. Our aim is to study Feller semigroups and sub-Markovian semigroups which play an important role in Part V.
(Note that our standard reference in this section is [14].)

**Definition 4.1.** *A. A one parameter family $(T_t)_{t\geq 0}$ of bounded linear operators $T_t : X \to X$ is called a **(one parameter) semigroup** of operators, if $T_0 = id$ and $T_{s+t} = T_s \circ T_t$ holds for all $s, t \geq 0$.*

*B. We call $(T_t)_{t\geq 0}$ **strongly continuous** if*

$$\lim_{t\to 0} \|T_t u - u\|_X = 0 \tag{4.1}$$

*for all $u \in X$.*

*C. The semigroup $(T_t)_{t\geq 0}$ is called a **contraction semigroup**, if for all $t \geq 0$*

$$\|T_t\| \leq 1 \tag{4.2}$$

*holds, i.e. if each of the operators $T_t$ is a contraction. As usual, $\|T_t\|$ denotes the operator norm $\|T_t\|_{X,X}$.*

**Definition 4.2.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on $(C_\infty(\mathbb{R}^n, \mathbb{R}), \|\cdot\|_\infty)$ which is **positivity preserving**, i.e. $u \geq 0$ implies $T_t u \geq 0$ for all $t \geq 0$. Then $(T_t)_{t\geq 0}$ is called a **Feller semigroup**.*

**Remark 4.3.** *A linear bounded operator $S : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), 1 \leq p < \infty$ is called **positive preserving** if $0 \leq u$ almost everywhere implies $0 \leq Su$ almost everywhere.*

**Example 4.4.** *Let $A : X \to X$ be a bounded linear operator and define*

$$T_t u := e^{tA} := \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k, \quad t \geq 0. \tag{4.3}$$

*First we find for $t \geq 0$ that*

*and as in the finite dimensional case we find now*

$$\sum^{\infty} \frac{1}{k!} t^k \|A\|^k = e^{t\|A\|},$$

$$e^{(s+t)A} = e^{sA} e^{tA} \text{ and } e^{0A} = id.$$

*and as in the finite dimensional case we find now*

$$e^{(s+t)A} = e^{sA} e^{tA} \text{ and } e^{0A} = id.$$

*Furthermore, we have the uniform continuity of the family $(T_t)_{t\geq 0}$ as $t$ tends to zero, i.e.*

$$\lim_{t\to 0}\|e^{tA}-id\|=0,$$

*implying that* $\lim_{t\to 0}\|e^{tA}u-u\|_X=0$. *Hence,* $(e^{tA})_{t\geq 0}$ *is a strongly continuous one parameter semigroup on* $(X,\|\cdot\|_X)$.

**Example 4.5.** *Let* $(\mu_t)_{t\geq 0}$ *be a convolution semigroup on* $\mathbb{R}^n$. *On the Banach space* $(C_\infty(\mathbb{R}^n),\|\cdot\|_\infty)$ *we define the operator*

$$T_t u(x) := \int_{\mathbb{R}^n} u(x-y)\mu_t(dy). \tag{4.4}$$

*We claim that* $(T_t)_{t\geq 0}$ *is a strongly continuous contraction semigroup. First, since* $u\in C_\infty(\mathbb{R}^n)$ *is bounded we find*

$$|T_t(x)| \leq \int_{\mathbb{R}^n}|u(x-y)|\mu_t(dy) \leq \|u\|_\infty \mu_t(\mathbb{R}^n).$$

*But* $\mu_t(\mathbb{R}^n)\leq 1$. *which implies*

$$\sup_{x\in\mathbb{R}^n}|T_t u(s)| \leq \|u\|_\infty, \tag{4.5}$$

*i.e.* $T_t$ *is defined on* $C_\infty(\mathbb{R}^n)$ *and* $T_t u$ *is a bounded function. But now it is easy to see that* $T_t u\in C_\infty(\mathbb{R}^n)$. *In fact, for* $u\in S(\mathbb{R}^n)$ *we find using the convolution theorem and Theorem 3.17 that*

$$(T_t u)\hat{}(\xi) = (2\pi)^{n/2}\hat{u}(\xi)\hat{\mu}_t(\xi) = \hat{u}(\xi)e^{-t\psi(\xi)}, \tag{4.6}$$

*where* $\psi:\mathbb{R}^n\to\mathbb{C}$ *is a continuous negative definite function. But* (4.6) *implies that* $(T_t u)\hat{}\in L^1(\mathbb{R}^n)$ *for* $u\in S(\mathbb{R}^n)$, *and the Riemann-Lebesgue lemma, Theorem 1.8, implies* $T_t u\in C_\infty(\mathbb{R}^n)$. *Thus, by* (4.4) *we find using the density of* $S(\mathbb{R}^n)$ *in* $C_\infty(\mathbb{R}^n)$ *that* $T_t$ *is a contraction on* $C_\infty(\mathbb{R}^n)$. *From the definition of the convolution of measures, Definition 1.9, we find*

$$\begin{aligned}
T_s\circ T_t u(x) &= \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} u(x-z-y)\mu_t(dy)\right\}\mu_s(dz)\\
&= \int_{\mathbb{R}^n} u(x-z)(\mu_t*\mu_s)(dz)\\
&= \int_{\mathbb{R}^n} u(x-z)\mu_{t+s}(dz)\\
&= T_{t+s}u(x).
\end{aligned}$$

*Since* $\mu_0=\varepsilon_0$, *we have immediately* $T_0=id$. *Finally, we prove that* $(T_t)_{t\geq 0}$ *is strongly continuous for* $t\to 0$. *For this note that any function in* $C_\infty(\mathbb{R}^n)$ *is uniformly continuous. Hence, for* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$|u(x)-u(x-y)|<\varepsilon \text{ for } |y|<\delta.$$

*The continuity of $(\mu_t)_{t\geq 0}$ in the Bernoulli topology implies that*

$$\lim_{t\to 0}\mu_t(B_\delta(0)) = \varepsilon_0(B_\delta(0)) = 1,$$

*which gives*

$$\mu_t(B_\delta^c(0)) < \varepsilon \text{ and } 1-\mu_t(\mathbb{R}^n) < \varepsilon \tag{4.7}$$

*for $0 < t \leq t_0$. Now we find*

$$\begin{aligned}
|T_tu(x)-u(x)| &\leq \left|\int_{|dsR^n|}\{u(x-y)-u(x)\}\mu_t(dy)\right| + |u(x)|(1-\mu_t(\mathbb{R}^n)) \\
&\leq \int_{B_\delta(0)}|u(x-y)-u(x)|\mu_t(dy) + \int_{B_\delta^c(0)}|u(x-y)-u(x)|\mu_t(dy) + \|u\|_\infty(1-\mu_t(\mathbb{R}^n)) \\
&\leq \varepsilon + 2\varepsilon\|u\|_\infty + \varepsilon\|u\|_\infty = \varepsilon(1+3\|u\|_\infty),
\end{aligned}$$

*implying that $(T_t)_{t\geq 0}$ is strongly continuous as $t \to 0$. Note that $T_t$, $t \geq 0$, is positivity preserving, i.e. $u \geq 0$ yields $T_tu \geq 0$ .*

**Definition 4.6.** *A. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. We call $(T_t)_{t\geq 0}$ a* ***sub-Markovian semigroup*** *on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, if for $u \in L^p(\mathbb{R}^n)$ such that $0 \leq u \leq 1$ almost everywhere if follows that $0 \leq T_tu \leq 1$ almost everywhere.*

*B. Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^p(\mathbb{R}^n), 1 \leq p < \infty$ or on $C_\infty(\mathbb{R}^n)$. We call $(T_t)_{t\geq 0}$* ***symmetric*** *if for all $u, v \in L^p(\mathbb{R}) \cap L^2(\mathbb{R}^n)$, or $u, v \in C_\infty(\mathbb{R}) \cap L^2(\mathbb{R}^n)$, respectively, we have*

$$(T_tu, v)_0 = (u, T_tv)_0 \quad . \tag{4.8}$$

**Lemma 4.7.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on $(X, \|\cdot\|_X)$. Then there exist constants $w \geq 0$ and $M_w \geq 1$ such that*

$$\|T_t\| \leq M_w e^{wt} \tag{4.9}$$

*where once again $\|\cdot\|$ denotes the operator norm $\|\cdot\|_{X,X}$.*

**Definition 4.8.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup. The* ***type*** *of $(T_t)_{t\geq 0}$ is defined by*

$$\omega_0 := \omega_0\Big((T_t)_{t\geq 0}\Big) := \inf\left\{\omega \in \mathbb{R}\,\middle|\,\|T_t\| \leq M_\omega e^{\omega t}\right\},$$

*where $\|T_t\| \leq M_\omega e^{\omega t}$ is to hold for some $M_\omega \geq 1$ and all $t \geq 0$.*

**Corollary 4.9.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on $(X, \|\cdot\|_X)$. For any $u \in X$, the mapping $t \to T_tu$ is continuous from $[0, \infty)$ to $X$.*

*Proof.* For $u \in X, t \geq 0$ and $h \geq 0$, we find using Example 4.5 that

$$\|T_{t+h}u - T_t u\|_X = \|T_t(T_h u - u)\|_X \leq Me^{wt}\|T_h u - u\|_X$$

and for $0 \leq h \leq t$ it follow that

$$\|T_{t-h}u - T_t u\|_X = \|T_{t-h}(T_h u - u)\|_X \leq Me^{wt}\|T_h u - u\|_X$$

implying the corollary. □

**Definition 4.10.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of operators on Banach space $(X, \|\cdot\|_X)$. The* ***generator*** *$A$ of $(T_t)_{t\geq 0}$ is defined by*

$$Au := \lim_{t\to 0} \frac{T_t u - u}{t} \quad \textit{(strong limit)} \tag{4.10}$$

*with domain*

$$D(A) := \left\{ u \in X \,\middle|\, \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists as strong limit} \right\}. \tag{4.11}$$

*Obviously, $D(A)$ is a linear subspace of $X$.*

**Example 4.11.** *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function with corresponding convolution semigroup $(\mu_t)_{t\geq 0}$ , i.e. $\hat{\mu}_t(\xi) = (2\pi)^{-n/2}e^{-t\psi(\xi)}$. Moreover, let $(T_t)_{t\geq 0}$ be the (Feller) semigroup defined by (4.4). Let $u \in S(\mathbb{R}^n)$, note $S(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$. It follows that*

$$\frac{T_t u - u}{t} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{e^{-t\psi(\xi)-1}}{t} \hat{u}(\xi)\, d\xi.$$

*Since $\hat{u} \in S(\mathbb{R}^n)$ and $|\psi(\xi)| \leq c_\psi(1 + |\xi|^2)$ by Lemma 3.21, we can define the operator*

$$\psi(D)u(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)\, d\xi. \tag{4.12}$$

*We claim that $S(\mathbb{R}^n) \subset D(A)$ and $Au = -\psi(D)u$ for $u \in S(\mathbb{R}^n)$ , where $A$ is the generator of the (Feller) semigroup $(T_t)_{t\geq 0}$. We will use the estimates (4.13) and (4.14):*

$$\frac{at}{1+at} \leq 1 + e^{-at} \leq at, \qquad a \geq 0,\ t \geq 0, \tag{4.13}$$

*and*

$$\left|\frac{e^{-at} - 1 + at}{t}\right| \leq \frac{1}{2}a^2 t, \qquad a \geq 0,\ t \geq 0. \tag{4.14}$$

*Now we find*

$$\left|\frac{e^{-t\psi(\xi)} - 1 + t\psi(\xi)}{t}\right| \leq t|\psi(\xi)|^2 \leq tc_\psi(1 + |\xi|^2)^2,$$

*which implies for $u \in S(\mathbb{R}^n)$ that*

$$\lim_{t\to 0} \frac{T_t u - u}{t} = -\psi(D)u.$$

**Lemma 4.12.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on the Banach space $(X, \|\cdot\|_X)$ and denote by $A$ its generator with domain $D(A) \subset X$.*
*A. For any $u \in X$ and $t \geq 0$ it follows that $\int_0^t T_s u\, ds \in D(A)$ and*

$$T_t u - u = A \int_0^t T_s u \, ds. \tag{4.15}$$

*B. For $u \in D(A)$ and $t \geq 0$ we have $T_t u \in D(A)$, i.e. $D(A)$ is invariant under $T_t$, and*

$$\frac{d}{dt} T_t u = A T_t u = T_t A u. \tag{4.16}$$

*C. For $u \in D(A)$ and $t \geq 0$ we always get*

$$T_t u - u = \int_0^t A T_s u \, ds = \int_0^t T_s A u \, ds. \tag{4.17}$$

**Corollary 4.13.** *Let $A$ be the generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on the Banach space $(X, \|\cdot\|_X)$. Then $D(A) \subset X$ is a dense subspace and $A$ is a closed operator. Moreover, $(T_t)_{t\geq 0}$ is a strongly continuous semigroup on $D(A)$ when $D(A)$ is equipped with the graph norm $\|u\|_{A,X} = \|Au\|_X + \|u\|_X$.*

**Remark 4.14.** *Let $(A, D(A))$ be a linear operator from $X$ to $Y$, both being topological vector spaces. A. We call $A$ a* ***closed operator****, if $\Gamma(A)$, the graph of an operator, is closed in $X \times Y$. B. The operator $A$ is* ***closable*** *if it has a closed extension.*

**Definition 4.15.** *The* ***resolvent set*** *$\rho(A)$ of $A$ consists of all $\lambda \in \mathbb{C}$ such that $\lambda id - A$ is surjective and has a continuous inverse $(\lambda id - A)^{-1}$ defined on $R(\lambda id - A) = X$. The set $\sigma(A) := \rho(A)^c$ is called the* ***spectrum*** *of $A$.*

**Definition 4.16.** *Let $A$ be a closed operator on the Banach space $(X, \|\cdot\|_X)$ with domain $D(A) \subset X$. The* ***resolvent*** *of $A$ is the family $(R_\lambda)_{\lambda\in\rho(A)}, R_\lambda := (\lambda - A)^{-1}$. The operator $R_\lambda$ is called the resolvent of $A$ at $\lambda$.*

**Lemma 4.17.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on the Banach space $(X, \|\cdot\|_X)$ with generator $(A, D(A))$. Then $\{\lambda \in \mathbb{C} | Re\lambda > 0\} \subset \rho(A)$ and we have*

$$R_\lambda u = (\lambda - A)^{-1} u = \int_0^\infty e^{-\lambda t} T_t u \, dt \tag{4.18}$$

*for all $u \in X$ and $Re\lambda > 0$.*

**Remark 4.18.** *Note that it is possible to define the resolvent $(R_\lambda)_{\lambda>0}$ (or $\lambda \in \rho(A)$) for a strongly continuous contraction semigroup directly by*

$$R_\lambda u = \int_0^\infty e^{-\lambda t} T_t u \, dt, \tag{4.19}$$

*i.e. without using its generator. For $u \in D(A)$ we find now for $\lambda > 0$ that*

$$\begin{aligned}
\|\lambda u - \lambda^2 R_\lambda u + Au\|_X &= \left\| \int_0^\infty e^{-\lambda t} \lambda^2 (u - T_t u) \, dt + Au \right\|_X \\
&= \left\| \int_0^\infty e^{-s} s \Big(\frac{u - T_{\frac{s}{\lambda}} u}{\frac{s}{\lambda}}\Big) \, ds + Au \right\|_X \\
&= \left\| \int_0^\infty e^{-s} s \Big(Au + \frac{u - T_{\frac{s}{\lambda}} u}{\frac{s}{\lambda}}\Big) \, ds \right\|_X \\
&\le \int_0^\infty e^{-s} s \left\| Au + \frac{u - T_{\frac{s}{\lambda}} u}{\frac{s}{\lambda}} \right\|_X ds,
\end{aligned}$$

*which implies that in the sense of the strong limit we have for all $u \in D(A)$*

$$\lim_{\lambda \to \infty} (-\lambda u + \lambda^2 R_\lambda u) = Au.$$

**Lemma 4.19.** *Let $A$ be a closed operator. For $\lambda, \mu \in \rho(A)$ the resolvent equation*

$$R_\lambda R_\mu = R_\mu R_\lambda = (\lambda - \mu)^{-1} (R_\mu - R_\lambda) \tag{4.20}$$

*holds.*

*Proof.* Since $(\lambda - A)(\mu - A) = (\mu - A)(\lambda - A)$ we find for $\lambda, \mu \in \rho(A)$ that

$$(\mu - \lambda)^{-1} (\lambda - \mu)^{-1} = (\lambda - A)^{-1} (\mu - A)^{-1}.$$

Furthermore, we have

$$\begin{aligned}
R_\lambda - R_\mu &= R\lambda(\mu - A) R_\mu - R\lambda(\lambda - A) R_\mu \\
&= -R_\lambda A R_\mu + \mu R_\lambda R_\mu + R_\lambda A R_\mu - \lambda R_\lambda R_\mu \\
&= (\mu - \lambda) R_\lambda R_\mu,
\end{aligned}$$

which yields (4.20)

□

**Definition 4.20.** *A linear operator $A, D(A) \to X, D(A) \subset X$ is called* ***dissipative****, more precisely $X$-dissipative, if*

$$\|\lambda u - Au\|_X \ge \lambda \|u\|_X \tag{4.21}$$

*holds for all $\lambda > 0$ and $u \in D(A)$.*

**Lemma 4.21.** *Let $(A, D(A))$ be a dissipative operator on $X$ and $\lambda > 0$. The operator $A$ is closed if and only if the range $R(\lambda - A)$ is closed.*

**Theorem 4.22.** *Let $A$ be a closed and dissipative operator which is densely defined on a Banach space $(X, \|\cdot\|_X)$. We assume that $(0, \infty) \subset \rho(A)$. The* ***Yosida approximation*** *of $A$ is defined for $\lambda > 0$ by*

$$A_\lambda = \lambda A(\lambda - A)^{-1} = \lambda A R_\lambda. \tag{4.22}$$

*It has the following properties*

1. *For all $\lambda > 0$ the operator $A_\lambda$ is bounded on $X$ and the semigroup $(e^{tA_\lambda})_{t \geq 0}$ is a strongly continuous contraction semigroup.*

2. *For all $\lambda, \mu > 0$ we have*

$$A_\lambda A_\mu = A_\mu A_\lambda.$$

3. *For $u \in D(A)$ it follows that*

$$\lim_{\lambda \to \infty} \|A_\lambda u - Au\|_X = 0.$$

**Theorem 4.23** (Hille and Yosida). *A linear operator $(A, D(A))$ on a Banach space $(X, \|\cdot\|_X)$ is the generator of strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ if and only if the following three conditions hold.*

1. *$D(A) \subset X$ is dense;*
2. *$A$ is a dissipative operator;*
3. *$R(\lambda - A) = X$ for some $\lambda > 0$.*

**Theorem 4.24.** *A linear operator on a Banach space $(X, \|\cdot\|_X)$ is closable and its closure $\overline{A}$ is the generator of a strongly continuous semigroup on $X$ if and only if the following three conditions are satisfied*

1. *$D(A) \subset X$ is dense;*

2. *$A$ is a dissipative operator;*

3. *$R(\lambda - A)$ is dense in $X$ for some $\lambda \geq 0$.*

**Definition 4.25.** *Let $A : D(A) \to B(\mathbb{R}^n)$ be a linear operator $D(A) \subset B(\mathbb{R}^n)$. We say that $(A, D(A))$ satisfies the* ***positive maximum principle*** *if for any $u \in D(A)$ such that for some $x_0 \in \mathbb{R}^n$ the fact that $u(x_0) = \sup_{x \in \mathbb{R}^n} u(x) \geq 0$ implies that $Au(x_0) \leq 0$.*

For later purposes it is helpful to introduce the following version of the **Hille-Yosida Theorem**:

**Theorem 4.26 (Hille-Yosida-Ray Theorem).** *A linear operator $(A, D(A)), D(A) \subset C_\infty(\mathbb{R}^n)$ on $C_\infty(\mathbb{R}^n)$ is closable and its closure is the generator of a Feller semigroup if and only if the three following conditions hold:*

*(i) $D(A) \subset C_\infty(\mathbb{R}^n; \mathbb{R})$ is dense;*

*(ii) $(A, D(A))$ satisfies the positive maximum principle;*

*(iii) $R(\lambda - A)$ is dense in $C_\infty(\mathbb{R}^n; \mathbb{R})$ for some $\lambda > 0$.*

**Example 4.27.** *Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous negative definite function. On $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ we define the operator*

$$-\psi(D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)d\xi.$$

*From Example 4.5 combined with Example 4.11 we know that $(-\psi(D), C_0^\infty(\mathbb{R}^n, \mathbb{R})$ has an extension generating a Feller semigroup, hence on $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ the operator $-\psi(D)$ satisfies the positive maximum principle.*

We may extend Example 4.27 to a more general situation.

**Theorem 4.28.** *Let $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a locally bounded function such that for any $x \in \mathbb{R}^n$ the function $q(x, \cdot) : \mathbb{R}^n \to \mathbb{C}$ is continuous and negative definite. Define on $C_0^\infty(\mathbb{R}^n)$ the operator*

$$-q(x, D)u(x) := -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x, \xi)\hat{u}(\xi)\, d\xi. \tag{4.23}$$

*Then the operator $(-q(x, D), C_0^\infty(\mathbb{R}^n; \mathbb{R}))$ satisfies the positive maximum principle, where we consider $C_0^\infty(\mathbb{R}^n; \mathbb{R})$ as a subspace of $C_0^\infty(\mathbb{R}^n)$.*

*Proof.* First note that by Lemma 3.22 we have

$$|q(x, \xi)| \leq \tilde{c}(x)(1 + |\xi|^2)$$

for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, which implies that the operator $q(x, D)$ is well defined on $C_0^\infty(\mathbb{R}^n; \mathbb{R})$. Now let $u \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ and $x_0 \in \mathbb{R}^n$ such that $u(x_0) = \sup_{x\in\mathbb{R}^n} u(x) \geq 0$. We have to prove that $-[q(x, D)u](x_0) \leq 0$ holds. Next we will consider the function $\psi_{x_0} : \mathbb{R}^n \to \mathbb{C}, \psi_{x_0}(\xi) = q(x_0, \xi)$. By our assumptions $\xi \to \psi_{x_0}(D)$ defined on $C_0^\infty(\mathbb{R}^n; \mathbb{R})$ by

$$\begin{aligned} -\psi_{x_0}(D)v(x) &= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi_{x_0}(\xi)\hat{v}(\xi)\, d\xi \\ &= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x_0, \xi)\hat{v}(\xi)\, d\xi \end{aligned}$$

satisfies the positive maximum principle, thus we have

$$-\psi_{x_0}(D)v(x_0) \leq 0$$

for $v(x_0) = \sup_{x\in\mathbb{R}^n} v(x) \geq 0$. But for any $v \in C_0^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned} -[q(x,D)v](x_0) &= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix_0\cdot\xi} q(x_0,\xi)\hat{v}(\xi)\,d\xi \\ &= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix_0\cdot\xi} \psi_{x_0}(\xi)\hat{v}(\xi)\,d\xi \\ &= -\psi_{x_0}(D)v(x_0) \end{aligned}$$

which implies the theorem. □

**Remark 4.29.** *Our arguments in Example 4.27 and Theorem 4.28 do not depend on the fact that the operator is defined on $C_0^\infty(\mathbb{R}^n)$. In case of the operator $-\psi(D)$ denote by $(A, D(A))$ the extension of $-\psi(D)$ as a generator of a Feller semigroup. Then by Theorem 4.26, the* ***Hille-Yosida-Ray Theorem****, the positive maximum principle holds for all extensions $(\tilde{A}, D(\tilde{A}))$ of $(-\psi(D), C_0^\infty(\mathbb{R}^n))$ with the property that $D(\tilde{A}) \subset D(A)$.*

Next we will introduce the **analytic semigroups** which we collect from [14].

Let $(T_t)_{t\geq 0}$ be a strongly continuous contraction semigroup in the Banach space $(X, \|\cdot\|_X)$ with generator $(A, D(A))$ and resolvent $(R_\lambda)_{\lambda>0}$. The relation between $(T_t)_{t\geq0}$ and $(R_\lambda)_{\lambda>0}$ is given by

$$R_\lambda u = \int_0^\infty e^{-\lambda t} T_t u\, dt. \tag{4.24}$$

We may try to invert (4.24) in order to express $T_t u$ with the help of $(R_\lambda)_{\lambda>0}$. Thus, it is necessary to extend $\lambda \longrightarrow R_\lambda u$ to some sector in the complex plane. It will become important to discuss whether or whether not $t \longrightarrow T_t u$ has an analytic extension to some sector in $\mathbb{C}$. Let $\omega \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$. Then the sector $S_{\theta,\omega} \subset \mathbb{C}$ is defined by

$$S_{\theta,\omega} := \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda \neq \omega \text{ and } |\arg(\lambda - \omega)| < \theta \right\}, \tag{4.25}$$

where $\arg z \in (\pi, \pi]$ is the argument of the complex number $z$.

**Definition 4.30.** *Let $A : D(A) \longrightarrow X, D(A) \subset X$, be a densely defined linear operator in the complex Banach space $(X, \|\cdot\|_X)$. We call $A$* ***sectorial*** *if there exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$ and $M > 0$ such that*

$$S_{\theta,\omega} \subset \rho(A) \tag{4.26}$$

*and*

$$\|R_\lambda\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}, \tag{4.27}$$

*hold.*

**Remark 4.31.** *A. Any sectorial operator is closed. B. Let $(X, \|\cdot\|_X)$ be a Hilbert space $(H, (\cdot,\cdot)_H)$ and $(A, D(A))$ be a sectorial operator on $H$. Then $-A$ is form sectorial with corresponding sector $S_{\theta-\frac{\pi}{2},-\omega}$.*

Suppose that $(A, D(A))$ is sectorial with $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$ and $M > 0$ as in Definition 4.30. Then the operator $A^\omega := A - \omega$ is sectorial now with $\omega = 0$, but $\theta$ and $M$ the same as for $A$. For a sectorial operator $A$ with $\omega = 0$ and $M = 1$ it follows that $(0, \infty) \subset \rho(A)$, hence for all $\lambda > 0$ the operator $\lambda - A$ is surjective, i.e. for all $f \in X$ the equation $\lambda u - Au = f$ has a solution. Moreover, since $M = 1$, we find for $u \in D(A)$ and $\lambda > 0$ that

$$\|u\|_X = \|R_\lambda(\lambda - A)u\|_X \leq \frac{1}{\lambda}\|(\lambda - A)u\|_X,$$

i.e. $(A, D(A))$ is a dissipative operator. Hence, by Theorem 4.26 , we have

**Corollary 4.32.** *Let $(A, D(A))$ be a sectorial operator in $(X, \|\cdot\|_X)$ such that (4.27) holds with $M = 1$. Taking $\omega \in \mathbb{R}$ as in (4.26), it follows that the operator $A^\omega := A - \omega$ with domain $D(A)$ is the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $X$. Moreover, the operator $(A, D(A))$ is the generator of the strongly continuous semigroup $(e^{\omega t}T_t)_{t\geq 0}$.*

**Proposition 4.33.** *Suppose that $A$ is a sectorial operator with sector $S_{\theta,\omega}$ and denoted by $(T_t)_{t\geq 0}$ the strongly continuous semigroup generated by $A$. It follows that for any $u \in X$, $k \in \mathbb{N}$ and $t > 0$ we have $T_t u \in D(A^k)$, hence*

$$T_t u \in \bigcap_{k\in\mathbb{N}} D(A^k), \tag{4.28}$$

*and we have for $u \in D(A^k)$*

$$A^k T_t u = T_t A^k u, \quad t \geq 0, \tag{4.29}$$

*and for suitable constants $M_k$, $k \in \mathbb{N}$, we find for $t > 0$ that*

$$\|t^k (A - \omega id)^k T_t\| \leq M_k e^{\omega t}. \tag{4.30}$$

*Moreover, the function $t \mapsto T_t$ is arbitrarily often differentiable and satisfies*

$$\frac{d^k}{dt^k} T_t u = A^k T_t u \tag{4.31}$$

*for all $u \in X$. Finally, the mapping $t \mapsto T_t$ has an analytic extension to the sector $\omega + S$, where $S$ is given by $S := \left\{\lambda \in \mathbb{C} \middle| |\arg\lambda| < \theta - \frac{\pi}{2}\right\}$.*

**Definition 4.34.** *A strongly continuous semigroup $(T_t)_{t\geq 0}$ is called an* ***analytic semigroup*** *of angle $\theta$, if the mapping $t \mapsto T_t$ has an analytic extension to the sector $S := \left\{z \in \mathbb{C} \setminus \{0\} \middle| |\arg z| < \theta - \frac{\pi}{2}\right\}$.*

**Theorem 4.35.** *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $(X, \|\cdot\|_X)$ such that $\|T_t\| \leq M$ for all $t \geq 0$ and some $M > 0$. Further, let $(A.D(A))$ be its generator and assume that $0 \in \rho(A)$. Then the following conditions are equivalent:*

1. *The semigroup $(T_t)_{t\geq 0}$ has an analytic extension to a sector $S_{\theta-\frac{\pi}{2},0}$, $\theta \in (\frac{\pi}{2}, \pi)$ and $\|T_z\|$ is uniformly bounded in every closed subsector $\bar{S}_{\theta'-\frac{\pi}{2},0}$, $\theta' \in (\frac{\pi}{2}, \pi)$ and $\theta' < \theta$.*

2. *There exists a constant $c$ such that for every $\sigma > 0$ and $\tau \neq 0$*
$$\|R_{\sigma+i\tau}\| \leq \frac{c}{|\tau|} \tag{4.32}$$
*holds.*

3. *There exists $\delta \in (0, \frac{\pi}{2})$ and $M > 0$ such that*
$$\Sigma := \left\{\lambda \in \mathbb{C} \middle| |\arg z| < \frac{\pi}{2} + \delta\right\} \cup \{0\} \subset \rho(A) \tag{4.33}$$
*and*
$$\|R_\lambda\| \leq \frac{M}{|\lambda|} \quad for \lambda \in \Sigma \setminus \{0\}. \tag{4.34}$$

4. *The mapping $t \mapsto T_t$ is differentiable in $(0, \infty)$ and with some constant $c'$ we have*
$$\|AT_t\| \leq \frac{c'}{t}, \quad t > 0. \tag{4.35}$$

Part IV

# Fundamental Solutions Of Time-Dependent Parabolic Equations

## 5 Fundamental Solutions of Time-Dependent Parabolic Equations

In this part, we will discuss equations:

$$\frac{du(t)}{dt} + A(t)u(t) = f(t), \; 0 \leq t \leq T. \tag{5.1}$$

$$u(0) = u_0. \tag{5.2}$$

where the operator $A$ depends on $t$. Our standard references are [23] and [20].

**Definition 5.1.** *We call* (5.1) *a* ***parabolic equation*** *if for each* $t \in [0,T]$*, the operator* $-A(t)$ *generates an analytic semigroup on some Banach space* $(X, \|\cdot\|_X)$.

Throughout this part (5.1) is always assumed to be a parabolic equation and $D(A(t)) \subset X$ denotes the domain of $A(t)$. We will assume that $D(A(t))$ is independent of $t$ and write therefore only $D(A)$. Then we may try to construct an operator $U(t,s)$ with the following properties:
$U(t,s)$ is a strongly continuous function, defined on $0 \leq s \leq t \leq T$ with value in $X$ such that

$$U(t,r)U(r,s) = U(t,s) \;\; for \;\; 0 \leq s \leq r \leq t \leq T, \tag{5.3}$$

$$U(s,s) = I \;\; for\ each \;\; s \in [0,T] \;\; , \tag{5.4}$$

$$(\partial/\partial t)U(t,s) = A(t)U(t,s), \tag{5.5}$$

$$(\partial/\partial s)U(t,s) = -U(t,s)A(s). \tag{5.6}$$

Here in (5.4) the operator $I$ denotes the identity on $X$.

Since,in general, the equations (5.5) and (5.6) involve unbounded operators on both sides, we assume that they hold in a dense subspace which is to be determined for each equation. Such an operator-valued function $U(t,s)$ is called a **fundamental solution** of (5.1). If $A(t) = A$ is independent of $t$, then $U(t,s) = \exp\Big((s-t)A\Big)$ is the fundamental solution, and (5.5)and (5.6) hold on $D(A)$. When the fundamental solution exists, one expects that the solution of (5.1),(5.2) can be written as

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds. \tag{5.7}$$

**Assumption 5.2.** *$A(t)$ for each $t \in [0,T]$ is a closed operator defined densely in a Banach space $X$. Its resolvent set $\rho(A(t))$ contains the half-plane $Re\lambda \le 0$, and $(1+|\lambda|)(A(t)-\lambda)^{-1}$ is uniformly bounded in $0 \le t \le T$ and $Re\lambda \le 0$.*

Hence, there exists a certain number $M$ and an angle $\theta \in (0,\pi/2)$ such that $\rho(A(t))$ contains the closed sector $\Sigma = \{\lambda : |\arg\lambda| \ge \theta\} \cup \{0\}$ and the estimate

$$\|(A(t)-\lambda)^{-1}\| \le M/(1+|\lambda|) \tag{5.8}$$

holds for $0 \le t \le T$ and $\lambda \in \Sigma$.

**Assumption 5.3.** *The domain $D(A(t)) \equiv D$ of $A(t)$ is independent of $t$ and, accordingly, $A(t)A(0)^{-1}$, being a bounded operator, is a Hölder continuous function of $t$ in the norm of $B(X)$, the algebra of bounded operators on $X$ . In other words, there exist positive numbers $\alpha \le 1$ and $L$ such that*

$$\|A(t)A(0)^{-1} - A(s)A(0)^{-1}\| \le L|t-s|^\alpha \tag{5.9}$$

*is satisfied for $0 \le s \le T$ and $0 \le t \le T$.*

Assumption 5.3 means that $A(t)A(0)^{-1}$ is a norm continuous function of $t$ and that $A(0)A(t)^{-1} = (A(t)A(0)^{-1})^{-1}$. Thus, $A(0)A(t)^{-1}$ is uniformly bounded. Therefore, we may suppose that

$$\|A(t)A(r)^{-1} - A(s)A(r)^{-1}\| \le L|t-s|^\alpha \tag{5.10}$$

holds for all $t,s,r \in [0,T]$, if necessary, by substituting $L$ by another number. Under Assumptions 5.2 and 5.3, the fundamental solution $U(t,s)$ is constructed as follows. Set

$$U(t,s) = \exp\Big(-(t-s)A(s)\Big) + W(t,s), \tag{5.11}$$

$$W(t,s) = \int_s^t \exp\Big(-(t-\tau)A(\tau)\Big)R(\tau,s)\,d\tau. \tag{5.12}$$

where $R(\tau,s)$ remains to be determined. A formal calculation gives

$$\begin{aligned}(\partial/\partial t)U(t,s) \;&=\; -A(s)\exp\Big(-(t-s)A(s)\Big) + R(t,s)\\ &\quad - \int_s^t A(\tau)\exp\Big(-(t-\tau)A(\tau)\Big)R(\tau,s)\,d\tau,\end{aligned}$$

$$
\begin{aligned}
A(t)U(t,s) &= A(t)\exp\Big(-(t-s)A(s)\Big) \\
&\quad + \int_s^t A(t)\exp\Big(-(t-\tau)A(\tau)\Big)R(\tau,s)\,d\tau.
\end{aligned}
$$

Summing these two equalities ,we obtain

$$
(\partial/\partial t)U(t,s) + A(t)U(t,s) = -R_1(t,s) + R(t,s) - \int_s^t R_1(t,\tau)R(\tau,s)\,d\tau, \tag{5.13}
$$

where

$$
R_1(t,s) = -\Big(A(t) - A(s)\Big)\exp\Big(-(t-s)A(s)\Big). \tag{5.14}
$$

Since we assume that $A(s)$ generates an analytic semigroup on $X$, there exists a constant $C_0$ such that

$$
\|\exp(-tA(s))\| \le C_0, \tag{5.15}
$$

$$
\|A(s)\exp(-tA(s))\| \le C_0 t^{-1}. \tag{5.16}
$$

Thus, by(5.10) and (5.16) using the uniform boundedness of $A(s)A(0)^{-1}$, we can estimate the norm of $R_1(t,s)$ for $s < t$ as

$$
\begin{aligned}
\|R_1(t,s)\| &\le \|(A(t)-A(s))A(s)^{-1}\|\|A(s)\exp(-(t-s)A(s))\| \\
&\le LC_0(t-s)^{\alpha-1}.
\end{aligned} \tag{5.17}
$$

Since the right-hand side of (5.13) vanishes if $t > s$, we find $R(t,s)$ as a solution of the integral equation

$$
R(t,s) - \int_s^t R_1(t,\tau)R(\tau,s)\,d\tau = R_1(t,s). \tag{5.18}
$$

It is easy to ascertain that $\exp(-(t-s)A(s))$ and $R_1(t,s)$ for $0 \le s < t \le T$ are continuous in the norm of $B(X)$. Because of (5.17), the integral equation (5.18) can be solved by successive iteration:

$$
R(t,s) = \sum_{m=1}^{\infty} R_m(t,s), \tag{5.19}
$$

$$
R_m(t,s) = \int_s^t R_1(t,\tau)R_{m-1}(\tau,s)\,d\tau. \tag{5.20}
$$

By induction, it is not difficult to see that

$$
\|R_m(t,s)\| \le (LC_0\Gamma(\alpha))^m (t-s)^{m\alpha-1}/\Gamma(m\alpha).
$$

Thus, we have

$$
\begin{aligned}
\|R(t,s)\| &\le \sum_{m=1}^{\infty}(LC_0\Gamma(\alpha))^m(t-s)^{m\alpha-1}/\Gamma(m\alpha) \\
&\le \sum_{m=1}^{\infty}(LC_0\Gamma(\alpha))^m T^{(m-1)\alpha}\Gamma(m\alpha)^{-1}(t-s)^{\alpha-1} = C(t-s)^{\alpha-1},
\end{aligned} \tag{5.21}
$$

$$\|W(t,s)\| \leq C(t-s)^{\alpha}, \tag{5.22}$$

$$\|U(t,s)\| \leq C. \tag{5.23}$$

**Lemma 5.4.** *Let $\beta$ be an arbitrary positive number satisfying $0 < \beta < \alpha$. Then there exists a constant $C_\beta$ such that the inequality*

$$\|R(t,s) - R(\tau,s)\| \leq C_\beta (t-s)^{\beta} (\tau - s)^{\alpha-\beta-1} \tag{5.24}$$

*holds for $0 \leq s < \tau < t \leq T$.*

*Proof.* Let us begin with the estimation of

$$\begin{aligned} R_1(t,s) - R_1(\tau,s) &= -(A(t) - A(\tau)) \exp(-(t-s)A(s)) \\ &\quad -(A(\tau) - A(s))\{\exp(-(t-s)A(s)) \\ &\quad - \exp(-(\tau - s)A(s))\}. \end{aligned} \tag{5.25}$$

The norm of the first term on the right-hand side does not exceed

$$\begin{aligned} \|(A(t) - A(\tau))A(s)^{-1}\| \|A(s) \exp(-(t-s)A(s))\| &\leq C(t-\tau)^{\alpha}(t-s)^{-1} \\ &\leq C(t-\tau)^{\alpha}(\tau-s)^{-1}. \end{aligned}$$

The norm of the second term does not exceed

$$\begin{aligned} &\left\|(A(\tau) - A(s)) \int_\tau^t (d/dr) \exp(-(r-s)A(s)) dr\right\| \\ = &\left\|(A(\tau) - A(s))A(s)^{-1} \int_\tau^t A(s)^2 \exp(-(r-s)A(s)) dr\right\| \\ \leq &\ C(\tau - s)^{\alpha} \int_\tau^t (r-s)^{-2} dr = C(t-\tau)(t-s)^{-1}(\tau-s)^{\alpha-1} \\ \leq &\ C(t-\tau)(\tau-s)^{\alpha-2}, \end{aligned}$$

but it can also be estimated as

$$\begin{aligned} &\|(A(\tau) - A(s)) \exp(-(t-s)A(s))\| + \|(A(\tau) - A(s)) \exp(-(\tau-s)A(s))\| \\ \leq &\ C(\tau-s)^{\alpha}(t-s)^{-1} + C(\tau-s)^{\alpha-1} \leq C(\tau-s)^{\alpha-1}; \end{aligned}$$

thus, it does not exceed

$$C\{(t-\tau)(\tau-s)^{\alpha-2}\}^{\alpha}\{(\tau-s)^{\alpha-1}\}^{1-\alpha} \leq C(t-\tau)^{\alpha}(\tau-s)^{-1}.$$

In this way, we have

$$\|R_1(t,s) - R_1(\tau,s)\| \leq C(t-\tau)^{\alpha}(\tau-s)^{-1}. \tag{5.26}$$

On the other hand, it follows (5.17) that

$$\begin{aligned}\|R_1(t,s)-R_1(\tau,s)\| &\le \|R_1(t,s)\|+\|R_1(\tau,s)\| \\ &\le C(t-s)^{\alpha-1}+C(\tau-s)^{\alpha-1}\le C(\tau-s)^{\alpha-1}.\end{aligned} \tag{5.27}$$

The combination of (5.26) and (5.27) gives

$$\begin{aligned}\|R_1(t,s)-R_1(\tau,s)\| &\le C\{(t-\tau)^{\alpha}(\tau-s)^{-1}\}^{\beta/\alpha}\{(\tau-s)^{\alpha-1}\}^{(\alpha-\beta)/\alpha} \\ &= C(t-\tau)^{\beta}(\tau-s)^{\alpha-\beta-1}.\end{aligned} \tag{5.28}$$

By the help of the relations

$$\begin{aligned}R(t,s)-R(\tau-s) &= R_1(t,s)-R_1(\tau,s)+\int_{\tau}^{t}R_1(t,\sigma)R(\sigma,s)\,d\sigma \\ &\quad +\int_{s}^{\tau}(R_1(t,\sigma)-R_1(\tau,\sigma))R(\sigma,s)\,d\sigma,\end{aligned}$$

$$\begin{aligned}\left\|\int_{\tau}^{t}R_1(t,\sigma)R(\sigma,s)\,d\sigma\right\| &\le C\int_{\tau}^{t}(t-\sigma)^{\alpha-1}(\sigma-s)^{\alpha-1}\,d\sigma \\ &\le C\int_{\tau}^{t}(t-\sigma)^{\alpha-1}\,d\sigma(\tau-s)^{\alpha-1} \\ &= C(t-\tau)^{\alpha}(\tau-s)^{\alpha-1},\end{aligned}$$

which are obtained from (5.17) ,(5.18),(5.21) , the inequalities (5.21) and (5.28) lead immediately to the inequality (5.24) .

□

Next, we will introduce **quadratic forms** which will be used to construct the fundamental solutions. Let $X$ be a complex Hilbert space; its inner product and norm will be denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. Let $V$ be another Hilbert space with inner product and norm denoted by $((\cdot,\cdot))$ and $\|\cdot\|$, respectively. We assume that $V$ is (linearly) embedded in $X$ as a dense subspace and that $V$ has a stronger topology than $X$. Therefore, there exists an $M_0$ such that $|u|\le M_0\|u\|$ for all $u\in V$. Let $a(u,v)$ be a **quadratic form** defined on $V\times V$. That is, to each $u,v\in V$ there corresponds a complex number $a(u,v)$ and $a(u,v)$ is linear in $u$ and antilinear in $v$:

$$a(u_1+u_2,v)=a(u_1,v)+a(u_2,v),$$

$$a(u,v_1+v_2)=a(u,v_1)+a(u,v_2),$$

$$a(\lambda u,v)=\lambda a(u,v),\quad a(u,\lambda v)=\bar{\lambda}a(u,v).$$

In this section, the space of all continuous antilinear functionals defined on $V$ and $X$ are denoted by $V^*$ and $X^*$, respectively.

In our case, we will add time dependence for the quadratic form. We assume that $a(t; u, v)$ is bounded, i.e., there exists a constant $M$ such that

$$|a(t; u, v)| \leq M\|u\|\|v\| \tag{5.29}$$

holds for all $u, v \in V$ and $0 \leq t \leq T$. Assume also that there exists a positive number $\delta > 0$ and a real number $k$ such that

$$Re\ a(t; u, v) \geq \delta\|u\|^2 - k|u|^2 \tag{5.30}$$

is satisfied for all $t \in [0, T]$ and $u \in V$. Furthermore, $a(t; u, v)$ is assumed to be Hölder continuous in $t$ in the following sense: there exists a certain number $\alpha \in (0, 1]$ such that

$$|a(t; u, v) - a(s; u, v)| \leq K|t - s|^\alpha \|u\|\|v\| \tag{5.31}$$

for all $t \in [0, T]$ and $u, v \in V$. Denote by $A(t)$ the operator determined by $a(t; u, v)$, i.e., we set

$$a(t; u, v) = (A(t)u, v).$$

Then we have the following theorem.

**Theorem 5.5.** *For all $t \in [0, T]$ and $u, v \in V$, we have $a(t; u, v) = (A(t)u, v)$. Then $-A$ generates an analytic semigroup in both $X$ and $V^*$.*

By Theorem 5.5, $-A(t)$ generates an analytic semigroup both in $X$ and in $V^*$, and when it is regarded as an operator in $V^*$, its domain coincides with $V$ and, hence, is independent of $t$. Thus, if (5.1) and (5.2) are considered as equations in $V^*$, we can construct its fundamental solution $U(t, s)$. Next, if $\alpha > 1/2$, the restriction of $U(t, s)$ to $X$ is found to be the desired fundamental solution of (5.1) and (5.2). We first make (5.30) valid for $k = 0$. Then, by (5.29), the inequality

$$\delta\|u\| \leq \|A(t)u\|_* \leq M\|u\| \tag{5.32}$$

holds for all $t \in [0.T]$ and $u \in V$. Now (5.31) immediately implies that

$$\|A(t)u - A(s)u\|_* \leq K|t - s|^\alpha \|u\|. \tag{5.33}$$

From this and (5.32) it follows that

$$\|A(t)A(0)^{-1} - A(s)A(0)^{-1}\|_* \leq C|t - s|^\alpha. \tag{5.34}$$

Therefore, it is possible to construct the fundamental solution of (5.1), regarded as an equation in $V^*$, by using (5.11),(5.12),(5.14),(5.19) and (5.20). It follows that $U(t, s)$ is a bounded operator in $V^*$. Next, we will discuss its restriction to $X$.

Using these previous inequalities and Lemma 5.4, we can get

$$\|R_1(t, s)\|_* \leq C(t - s)^{\alpha - 1}, \quad 0 \leq s < t \leq T \tag{5.35}$$

and

$$\|R_1(t,s) - R_1(\tau,s)\|_* \le C_\beta (t-\tau)^\beta (\tau - s)^{\alpha-\beta-1}, \quad 0 \le s < \tau < t \le T \tag{5.36}$$

hold, where $\beta$ is an arbitrary positive number smaller than $\alpha$.

Next, we will discuss more inequalities with $f \in X$.

**Lemma 5.6.** *For Re $\lambda \le 0$ and $A(s)$ as above, the following estimates hold with the constant $C$ independent of $f$ :*

$$|(A(s)-\lambda)^{-1} f| \le C|\lambda|^{-1}|f|, \tag{5.37}$$

$$|(A(s)-\lambda)^{-1} f| \le C|\lambda|^{-1/2}\|f\|_*, \tag{5.38}$$

$$\|(A(s)-\lambda)^{-1} f\| \le C|\lambda|^{-1/2}|f|, \tag{5.39}$$

$$\|(A(s)-\lambda)^{-1} f\| \le C\|f\|_*, \tag{5.40}$$

$$\|(A(s)-\lambda)^{-1} f\|_* \le C|\lambda|^{-1}\|f\|_*, \tag{5.41}$$

**Lemma 5.7.** *There exists a constant $C$ such that the following inequalities hold for all $t > 0$ and $C$ independent of the function $f$:*

$$|\exp(-tA(s))| \le 1, \tag{5.42}$$

$$\|\exp(-tA(s))\|_* \le C, \tag{5.43}$$

$$|\exp(-tA(s))f| \le Ct^{-1/2}\|f\|_*, \tag{5.44}$$

$$\|\exp(-tA(s))f\| \le Ct^{-1/2}|f|, \tag{5.45}$$

$$\|\exp(-tA(s))f\| \le Ct^{-1}\|f\|_*, \tag{5.46}$$

$$|A(s)\exp(-tA(s))f| \le Ct^{-3/2}\|f\|_*, \tag{5.47}$$

$$\|A(s)\exp(-tA(s))f\| \le Ct^{-3/2}|f|. \tag{5.48}$$

Moreover, from (5.33) and (5.45), we have

$$\|R_1(t,s)f\|_* \le C(t-s)^{\alpha-1/2}|f|. \tag{5.49}$$

**Lemma 5.8.** *If $f \in X$ and $0 \le s < t \le T$, then*

$$\|R(t,s)f\|_* \le C(t-s)^{\alpha-1/2}|f|. \tag{5.50}$$

*Proof.* By induction, it can be shown that there exists some constants $C_0$ and $C_1$ such that the following inequalities holds for all $m$:

$$\|R_m(t,s)f\|_* \le C_0\Gamma(\rho+1/2)C_1^{m-1}\Gamma(\rho)^{m-1}(t-s)^{m\rho-1/2}|f|/\Gamma(m\rho+1/2),$$

from which (5.50) follows immediately.

□

**Lemma 5.9.** *For each $f \in X$, $0 \le s < \tau < t \le T$ and $0 < \beta < \alpha$, we have*

$$\|R(t,s)f - R(\tau,s)f\|_* \le C_\beta\Big\{(t-\tau)^\alpha(\tau-s)^{-1/2} + \int_\tau^t (t-\tau)^{\alpha-1}(\tau-s)^{\alpha-1/2}\,d\tau + (t-\tau)^\beta(\tau-s)^{2\alpha-\beta-1/2}\Big\}|f|. \tag{5.51}$$

*Proof.* By the definition (5.14) of $R_1(t,s)$, we have

$$\begin{aligned}\|R_1(t,s)f - R_1(\tau,s)f\|_* &\le \|(A(t)-A(\tau))\exp(-(t-s)A(s))f\|_* \\ &+\|(A(\tau)-A(s)) \\ &\times\{\exp(-(t-s)A(s)) - \exp((-\tau-s)A(s))\}f\|_*.\end{aligned} \tag{5.52}$$

Because of (5.45) and (5.33), the first term on the right-hand side does not exceed $C(t-\tau)^\alpha(t-s)^{-1/2}|f|$. The inequality (5.48) provides

$$\begin{aligned}&\|\{\exp(-(t-s)A(s)) - \exp(-(\tau-s)A(s))\}f\| \\ =\ & \left\|\int_\tau^t A(s)\exp(-(r-s)A(s))f\,dr\right\| \\ \le\ & C\int_\tau^t (r-s)^{-3/2}|f|\,dr \\ =\ & C\{(\tau-s)^{-1/2} - (t-s)^{-1/2}\}|f| \\ =\ & C(\tau-s)^{-1/2}\{1-(\tau-s)^{-1/2}(t-s)^{-1/2}\}|f| \\ \le\ & C(\tau-s)^{-1/2}\{1-(\tau-s)/(t-s)^{-1}\}|f| \\ =\ & C(t-\tau)(t-s)^{-1}(\tau-s)^{-1/2}|f| \le C(t-\tau)(\tau-s)^{-3/2}|f|,\end{aligned}$$

which,combining with (5.33),shows that the second term on the right-hand side does not exceed $C(t-\tau)(\tau-s)^{\alpha-3/2}|f|$. On the other hand, by (5.33) and (5.45), the same term does not exceed

$$\begin{aligned}&\|(A(\tau)-A(s))\exp(-(t-s)A(s))f\|_* \\ &+\|(A(\tau)-A(s))\exp(-(\tau-s)A(s))f\|_* \\ &\le C(t-s)^\alpha(t-s)^{-1/2}|f| + C(\tau-s)^{\alpha-1/2}|f| \le C(\tau-s)^{\alpha-1/2}|f|.\end{aligned}$$

Thus, it is bounded by

$$C\{(t-\tau)(\tau-s)^{\alpha-3/2}\}^{\alpha}\{(\tau-s)^{\alpha-1/2}\}^{1-\alpha}|f| = C(t-\tau)^{\alpha}(\tau-s)^{-1/2}|f|.$$

Therefore, we have

$$\|R_1(t,s)f - R_1(\tau,s)f\|_* \le C(t-\tau)^{\alpha}(\tau-s)^{-1/2}|f|. \tag{5.53}$$

The estimate (5.51) is easily obtained by the use of (5.18),(5.35),(5.36),(5.50) and (5.53).

□

**Lemma 5.10.** *Define*

$$Y(t,s) = \exp(-(t-s)A(s)) - \exp(-(t-s)A(t))$$

*for* $0 \le s < t \le T$. *It satisfies the estimate*

$$\|Y(t,s)f\| \le C(t-s)^{\alpha-1}\|f\|_*. \tag{5.54}$$

*Proof.* $Y(t,s)$ can be expressed as

$$Y(t,s) = \frac{2}{2\pi i}\int_{\Gamma} e^{-\lambda(t-s)}(A(s)-\lambda)^{-1}(A(t)-A(s))(A(t)-\lambda)^{-1}\,d\lambda.$$

By (5.33) and (5.40), we have

$$\|(A(s)-\lambda)^{-1}(A(s)-A(t))(A(t)-\lambda)^{-1}f\| \le C(t-s)^{\alpha}\|f\|_*,$$

from which (5.54) follows immediately. □

**Lemma 5.11.**

$$|W(t,s)f| \le C(t-s)^{\alpha}|f|, \tag{5.55}$$

$$\|W(t,s)f\| \le C(t-s)^{\alpha-1/2}|f|. \tag{5.56}$$

*for* $f \in X$ *and* $0 \le s < t \le T$.

*Proof.* (5.55) is a direct consequence of Lemma 5.8 and (5.44). Next, (5.56) is obtained by representing $W(t,s)$ as

$$\begin{aligned} W(t,s) \;&=\; \int_s^t Y(t,\tau)R(\tau,s)\,d\tau + \int_s^t \exp(-(t-\tau)A(t)) \\ &\quad \times(R(\tau,s)-R(t,s))\,d\tau \\ &\quad +A(t)^{-1}\{1-\exp(-(t-s)A(t))\}R(t,s) \end{aligned}$$

and by applying Lemma 5.8 and 5.10 to the first term on the right-hand side, (5.46) and Lemma 5.9 to the second term, and (5.32),(5.43) and Lemma 5.8 to the third term, respectively.

□

**Theorem 5.12.** *For $0 \leq s \leq t \leq T$ , the function $(s,t) \mapsto U(s,t)$ is a strongly continuous function with values in $B(X)$ and satisfies*

$$|U(t,s)f| \leq C|f| \tag{5.57}$$

*and*

$$\|U(t,s)f\| \leq C(t-s)^{-1/2}|f|, \tag{5.58}$$

*where $f \in X$ and $0 \leq s < t \leq T$. Moreover, for $f \in V^*$, it satisfies*

$$|U(t,s)f| \leq C(t-s)^{-1/2}\|f\|_*. \tag{5.59}$$

*Proof.* (5.57) and (5.58) are evident from Lemma 5.11 and (5.11). (5.59) follows from (5.35) and (5.44).

□

This theorem implies that

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\, ds \tag{5.60}$$

belongs to $C([0.T];X)$ if $u_0 \in X$ and $f \in C([0,T];X)$. Moreover, the following theorem ensures that (5.60) forms a solution of the equations (5.1) and (5.2) in $V^*$.

**Theorem 5.13.** *Under Assumptions 5.2 and 5.3, a fundamental solution of* (5.1) *and* (5.2) *exists. For $0 \leq s < t \leq T$, the range $R(U(t,s)) \subset D$, the operator $(\partial/\partial t)U(t,s)$ exists as an element of $B(X)$, and the following inequalities hold:*

$$\|(\partial/\partial t)U(t,s)\| = \|A(t)U(t,s)\| \leq C(t-s)^{-1},$$
$$\|A(t)U(t,s)A(s)^{-1}\| \leq C.$$

*$U(t,s)u$ for each $t \in (0,T)$ and each $u \in D$ is differentiable with respect to $s$ in $0 \leq s \leq t$ and satisfies*

$$(\partial/\partial s)U(t,s)u = U(t,s)A(s)u. \tag{5.61}$$

By this theorem if $u_0$ is an arbitrary element of $X$, then $u(t) = U(t,0)u_0$ is continuous in $0 \leq t \leq T$, differentiable in $0 < t \leq T$, and is a solution of the homogeneous equation $du(t)/dt + A(t)u(t) = 0$, which coincides with $u_0$ at $t = 0$.

**Theorem 5.14.** *Let $u_0 \in X$ and $f \in C([0,T];X)$. If $u$ is a solution of* (5.1) *and* (5.2), *then it is expressible as* (5.60), *so that the solution of* (5.1) *and* (5.2) *is unique.*

*Proof.* Assume $0 < \varepsilon < s < t$. Then, from (5.61), it follows that

$$(\partial/\partial s)(U(t,s)u(s)) = U(t,s)u'(s) + U(t,s)A(s)u(s) = U(t,s)f(s). \tag{5.62}$$

By integrating this equation from $\varepsilon$ to $t$ and letting $t \to 0$, we obtain (5.60) immediately.

□

**Theorem 5.15.** *Let $u_0$ be an arbitrary element of $X$ and $f$ an arbitrary function Hölder-continuous in $[0,T]$. Then the function $u$ defined by* (5.60) *is the unique solution of* (5.1) *and* (5.2).

*Proof.* Define

$$S(t,s) = A(t)\exp(-(t-s)A(t)) - A(s)\exp(-(t-s)A(s)) \tag{5.63}$$

for $0 \leq s < t \leq T$, and put

$$W_\varepsilon(t,s) = \int_s^{t-\varepsilon} \exp(-(t-\tau)A(\tau))R(\tau,s)\, d\tau$$

for $0 < \varepsilon < t - s. W_\varepsilon(t,s) \to W(t,s)$ as $\varepsilon \to 0$. By differentiation, we have

$$\begin{aligned}(\partial/\partial t)W_\varepsilon(t,s) &= \exp(-\varepsilon A(t-\varepsilon))R(t-\varepsilon,s) \\ &\quad - \int_s^{t-\varepsilon} A(\tau)\exp(-(t-\tau)A(\tau))R(\tau,s)\, d\tau.\end{aligned}$$

Upon observing the relation

$$A(t)\exp(-(t-\tau)A(t)) = (\partial/\partial\tau)\exp(-(t-\tau)A(t)),$$

the right-hand side can be rewritten as

$$\begin{aligned}(\partial/\partial t)W_\varepsilon(t,s) &= \exp(-\varepsilon A(t-\varepsilon))R(t-\varepsilon,s) + \int_s^{t-\varepsilon} S(t,\tau)R(\tau,s)\, d\tau \\ &\quad - \int_s^{t-\varepsilon} \exp(-(t-\tau)A(t))(R(\tau,s) - R(t,s))\, d\tau \\ &\quad - \{\exp(-\varepsilon A(t)) - \exp(-(t-s)A(t))\}R(t,s).\end{aligned} \tag{5.64}$$

By (5.15), (5.16), (5.21) , (5.63) and Lemma 5.4, it is found that the norm of $(\partial/\partial t)W_\varepsilon(t,s)$ satisfies

$$\|(\partial/\partial t)W_\varepsilon(t,s)\| \leq C(t-s-\varepsilon)^{\alpha-1}, \tag{5.65}$$

where $C$ is a constant independent of $\varepsilon$ as well. It is easy to see, each term on the right-hand side of (5.64) converges strongly as $\varepsilon \to 0$. Putting $W'(t,s) = \lim_{\varepsilon\to 0}(\partial/\partial t)W_\varepsilon(t,s)$, we obtain from (5.64) and (5.65) that

$$\begin{aligned} W'(t,s) &= \int_s^t S(t,\tau)R(\tau,s)\, d\tau \\ &\quad - \int_s^t A(t)\exp(-(t-\tau)A(t))(R(\tau,s)-R(t,s))\, d\tau \\ &\quad + \exp(-(t-s)A(t))R(t,s), \end{aligned} \tag{5.66}$$

$$\|W'(t,s)\| \le C(t-s)^{\alpha-1}. \tag{5.67}$$

Letting $\varepsilon \to 0$ in

$$W_\varepsilon(t',s) - W_\varepsilon(t,s) = \int_t^{t'} (\partial/\partial r)W_\varepsilon(r,s)\, dr$$

with $t' > t > s+\varepsilon$, we have, owing to (5.65),

$$W(t',s) - W(t,s) = \int_t^{t'} W'(r,s)\, dr.$$

Since $W'(t,s)$ is strongly continuous in $0 \le s < t \le T, W(t,s)$ is strongly continuous differentiable with respect to $t$, and, hence, it is found that

$$(\partial/\partial t)W(t,s) = W'(t,s). \tag{5.68}$$

Therefore, the derivative

$$(\partial/\partial t)U(t,s) = -A(s)\exp(-(t-s)A(s)) + (\partial/\partial t)W(t,s) \tag{5.69}$$

exists and satisfies

$$\|(\partial/\partial t)U(t,s\| \le C(t-s)^{-1}.$$

The relations (5.67) and (5.68)implies that

$$(\partial/\partial t)\int_0^t W(t,s)f(s)ds = \int_0^t W(t,s)f(s)ds. \tag{5.70}$$

Also, as in the proof of (5.66), we have

$$\begin{aligned} (\partial/\partial t)\int_0^t (-(t-s)A(s))f(s)ds &= \int_0^t S(t,s)f(s)ds \\ &\quad - \int_0^t A(t)\exp(-(t-s)A(s))(f(s)-f(t))ds \\ &\quad + \exp(-tA(t))f(t). \end{aligned} \tag{5.71}$$

From these equations, and noting that

$$(\partial/\partial t)\int_0^{t-\varepsilon} U(t,s)f(s)ds - A(t)\int_0^{t-\varepsilon} U(t,s)f(s)ds - f(t)$$
$$= U(t,t-\varepsilon)f(t-\varepsilon) - f(t) \to 0,$$

as $\varepsilon \to +0$,we obtains the conclusion of the theorem.

□

**Lemma 5.16.** *The following inequality holds for* $0 \le s < t \le T$*:*

$$|(\partial/\partial t)W(t,s)| \le C(t-s)^{\alpha-1}. \tag{5.72}$$

Lemma 5.16 implies that $U(t,s)$ is differentiable with respect to $t$ in $B(X)$ and that, for $0 \le s < t \le T$, the following results hold: $R(U(t,s)) \subset D(A(t))$, $(\partial/\partial t)U(t,s) + A(t)U(t,s) = 0$, and

$$|(\partial/\partial t)U(t,s)| = |A(t)U(t,s)| \le C(t-s)^{-1}. \tag{5.73}$$

A quadratic form adjoint to $a(t;u,v)$ is denoted by $a^*(t;u,v)$, that is , $a^*(t;u,v) = \overline{a(t;v,u)}$. Let $A^*(t)$ be the operator determined by $a^*(t;u,v)$. Then, as in the above, we can construct an operator-valued function $V(t,s)$ $(0 \le s < t \le T)$ satisfying

$$-(\partial/\partial s)V(t,s) + A^*(s)V(t,s) = 0, \quad 0 \le s < t \le T,$$

$$V(t,t) = I, \quad 0 \le t \le T.$$

For $f, g \in X$, and $s < r < t$, we have

$$\begin{aligned} & (\partial/\partial r)(U(r,s)f, V(t,r)g) \\ = & -(A(r)U(r,s)f, V(t,r)g) + (U(r,s)f, A^*(r)V(t,r)g) \\ = & -a(r;U(r,s)f, V(t,r)g) + a(r;U(r,s)f, V(t,r)g) = 0, \end{aligned}$$

so that $(U(r,s)f, V(t,r)g)$ is independent of $r$ in the open interval $(s,t)$. Therefore, by letting $r \to s$ and $r \to t$, we get

$$U(t,s) = V^*(t,s). \tag{5.74}$$

As in (5.73), $V(t,s)$ is differentiable with respect to $s$ and satisfies

$$|(\partial/\partial s)V(t,s)| = |A^*(s)V(t,s)| \le C(t-s)^{-1}, \tag{5.75}$$

which combining with (5.74), implies that $U(t,s)$ is also differentiable with respect to $s$ in the interval $0 \le s < t$ and that $(\partial/\partial s)U(t,s)$ is a bounded extension of $U(t,s)A(s)$ in $X$, and

$$|(\partial/\partial s)U(t,s)| \le C(t-s)^{-1}. \tag{5.76}$$

**Theorem 5.17.** *If the inequalities* (5.29),(5.30) *and* (5.31) *with* $\alpha > 1/2$ *are satisfied, there exists a fundamental solution* $U(t,s)$ *of the equation* (5.1) *in* $X$. $U(t,s)$ *is differentiable with respect to* $t, s$ *in* $0 \le s < t \le T$, *its range* $R(U(t,s)) \subset D(A(T))$, $(U(t,s)A(s))$ *has a bounded extension in* $X$, *and we have* (5.73) *and* (5.76).

**Theorem 5.18.** *Suppose that the assumptions of the above theorem hold. Let* $u_0 \in X$ *and let* $f$ *be a Hölder continuous function with values in* $X$. *Then* (5.60) *is the unique solution of* (5.1) *and* (5.2).

The proof is similar to that of Theorem 5.15.

# Part V

# Some Properties of Pseudo-Differential Operators with Time Dependent Negative Definite Symbols

## 6 Some Properties of Pseudo-Differential Operators with Time Dependent Negative Definite Symbols

In this part, we will study a large class of these pseudo-differential operators with coefficients depending on time which may for frozen time dependence extend to a generator of a Feller semigroup or a sub-Markovian semigroup. Our standard reference in this part is [15].

We will starting with the estimates for these operators which will be used in a later section.

Thus, we will consider operators of the following form

$$q(x,t,D)u(x,t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} q(x,t,\xi)\widehat{u}(\xi,t)d\xi \tag{6.1}$$

and we assume that $q : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$ is a locally bounded function such that for every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$, the function $q(x,t,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is a negative definite and continuous.

**Definition 6.1.** *We call a function $q : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$ a* ***time dependent continuous negative definite symbol*** *if $q$ is continuous and for each $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, the function $q(x,t,\cdot) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ is negative definite.*

*In the following we will always assume $q$ to be continuous and use the short hand time dependent negative definite symbol, taking the continuity for granted.*

Note that for a time dependent negative definite symbol $q$ for every compact set $K \subset \mathbb{R}^n$, there exists a constant $C_K(t)$ such that

$$|q(x,t,\xi)| \leqslant C_K(t)(1+|\xi|^2)$$

holds for all $x \in K$, and $\xi \in \mathbb{R}^n$. In fact, we may take

$$C_K(t) = 2 \sup_{\substack{x \in K \\ |\eta| \leqslant 1}} |q(x,t,\eta)|.$$

We will add the additional assumption that we can find a bound for $C_K(t)$ independent of $t$, i.e. we will require for all $\xi \in \mathbb{R}^n$

$$|q(x,t,\xi)| \leqslant C_K(1+|\xi|^2) \tag{6.2}$$

with $C_K$ independent of $t \geqslant 0$ and $x \in K$.

A pseudo-differential operator corresponding to a time dependent negative definite symbol is called a pseudo-differential operator with time dependent negative definite symbol.

A pseudo-differential operator with time dependent negative definite symbol has also the representation as integro-differential operator which we obtain by using the Lévy-Khinchin formula:

$$\begin{aligned} q(x,t,D)u(x,t) &= -\sum_{k,l=1}^{n} a_{kl}(x,t)\frac{\partial^2 u(x,t)}{\partial x_k \partial x_l} + \sum_{j=1}^{n} d_j(x,t)\frac{\partial u(x,t)}{\partial x_j} \\ &\quad + c(x,t)u(x,t) - \int_{\mathbb{R}^n\setminus\{0\}} \left( u(y,t) - u(x,t) + \sum_{j=1}^{n} \frac{y_i}{1+|y|^2}\frac{\partial u(x,t)}{\partial x_j} \right) N(x,t,dy). \end{aligned}$$

Here $c : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}, c(x,t) \geqslant 0$ , $d = (d_1,...,d_n) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ are continuous functions, and $a_{kl} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}, 1 \leqslant k,l \leqslant n$, are Borel functions such that $a_{kl}(x,t) = a_{lk}(x,t)$.

For all $x \in \mathbb{R}^n$ , $t \in \mathbb{R}_+$ and $\xi = (\xi_1,...,\xi_n) \in \mathbb{R}^n$, it holds $\sum a_{kl}(x,t)\xi_k\xi_l \geqslant 0$.

Furthermore, for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$ fixed $N(x,t,\cdot)$ is a measure integrating the function $y \mapsto (1 \wedge |y|^2)$, i.e. $\int (1 \wedge y^2) N(x,t,dy) < \infty$. We note that for $t \in \mathbb{R}^n$ fixed $q(x,t,D)$ maps function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in C_0^\infty(\mathbb{R}^n)$ to Borel measurable function, i.e. $(q(x,t,D)u)(\cdot,t)$ is Borel measurable.

Furthermore, for $t_0 \in \mathbb{R}_+$ fixed $q(x,t_0,D)$ satisfies the **positive maximum principle**, i.e. it holds

$$if \quad \sup_{x\in\mathbb{R}^n} u(x_0,t_0) \geq 0, \quad then \quad (q(x,t_0,D)u)(x_0,t_0) \leq 0.$$

We are interested in proving estimates for the operator $q(x,t,D)$ in Sobolev spaces related to continuous negative definite functions, more precisely, in the Hilbert space $H^{\psi,s}(\mathbb{R}^n)$ for a fixed continuous negative definite function $\psi$ and $s \in \mathbb{R}$.

The Hilbert space $H^{\psi,s}(\mathbb{R}^n)$ with a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$H^{\psi,s}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n),\ \|u\|_{\psi,s} < \infty\}, \quad s \geqslant 0$$

and

$$\|u\|^2_{\psi,s} = \int_{\mathbb{R}^n} (1+\psi(\xi))^s |\widehat{u}(\xi)|^2 d\xi, \quad s \geqslant 0. \tag{6.3}$$

We want to extend these spaces in order to handle t-dependent functions and operators. For the moment it is sufficient to note that for $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$, such that $u(\cdot, t) \in \mathbb{L}^2(\mathbb{R}^n)$, we can define

$$\|u(\cdot,t)\|^2_{\psi,s} := \int_{\mathbb{R}^n} (1+\psi(\xi))^s |\widehat{u}(\xi,t)|^2 d\xi < \infty \,.$$

In some estimates we need a more precise control on constants. For this, we note that

$$\int_{\mathbb{R}^n} (1+|x|^2)^{-k/2} dx = \frac{\pi^{n/2}\Gamma\left(\frac{k-n}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} =: \widetilde{c}_{n,k}, \quad k > n \tag{6.4}$$

and $\widetilde{\gamma}_{m,n}$ is the smallest constant such that

$$(1+|\xi|^2)^{m/2} \leqslant \widetilde{\gamma}_{m,n} \sum_{|\alpha| \leqslant m} |\xi^\alpha| \tag{6.5}$$

holds. An upper bound for $\widetilde{\gamma}_{m,n}$ is $(n+1)^{m/2}$. Finally, for a continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ the constant $C_\psi$ is defined as the smallest constant such that

$$\psi(\xi) \leqslant C_\psi (1+|\xi|^2) \tag{6.6}$$

holds. For a (negative definite) symbol $q(x,t,\xi)$, we set

$$\begin{aligned} \widetilde{q}(\eta,t,\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\eta} q(x,t,\xi) dx \\ &= F_{x\to\eta}(q(x,t,\xi))(\eta) \end{aligned} \tag{6.7}$$

whenever this Fourier transform exists.

**Lemma 6.2.** *Let $q : \mathbb{R}^n \times \mathbb{R}_+ \times R \to \mathbb{C}$ be a measurable function and further let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a continuous negative definite function. In addition assume that $q(\cdot, t, \xi) : \mathbb{R}^n \to \mathbb{C}$ is m-times continuously differentiable.*

*A. If for every multiindex $\alpha \in \mathbb{N}_0^n, |\alpha| \leqslant m$, the estimate*

$$|\partial_x^\alpha q(x,t,\xi)| \leqslant C_\alpha (1+\psi(\xi)) \tag{6.8}$$

*holds with $C_\alpha$ independent of $x$ and $t$ , then we find for all $\varphi \in C_0^\infty(\mathbb{R}^n)$*

$$|(\varphi q)\,\hat{}\,(\eta,t,\xi)| \leqslant C_\varphi (1+|\eta|^2)^{-m/2} (1+\psi(\xi)). \tag{6.9}$$

*B. Suppose that there are functions $\varphi_\alpha \in L^1(\mathbb{R}^n), |\alpha| \leqslant m$ such that*

$$|\partial_x^\alpha q(x,t,\xi)| \leqslant \varphi_\alpha(x)(1+\psi(\xi)) \tag{6.10}$$

*holds. Note that the right hand side is again independent of $t$. Then we have for all $k \in \mathbb{N}_0, |k| \leqslant m$, the estimate*

$$|\widetilde{q}(\eta,t,\xi)| \leqslant \widetilde{\gamma}_{k,n} \sum_{|\alpha| \leqslant k} \|\varphi_\alpha\|_{L^1} (1+|\eta|^2)^{-k/2} (1+\psi(\xi)) \tag{6.11}$$

*Proof.* Compare [15] for the time independent case.

A. For $\beta \in \mathbb{N}_0^n, |\beta| \leqslant m$, we have

$$\left|\eta^\beta \int_{\mathbb{R}^n} e^{-ix\eta}\varphi(x)q(x,t,\xi)dx\right| = \left|\int_{\mathbb{R}^n} (\partial_x^\beta(e^{-ix\eta}))\varphi(x)q(x,t,\xi)dx\right|$$
$$= \left|\int_{\mathbb{R}^n} e^{-ix\eta}(\partial_x^\beta(\varphi(x)q(x,t,\xi))dx\right|$$
$$\leqslant \sum_{\gamma\leqslant\beta}\binom{\beta}{\gamma}C_\gamma(1+\psi(\xi))\left\|\partial^{\beta-\gamma}\varphi\right\|_{L^1}$$
$$\leqslant C_{\beta,\varphi}(1+\psi(\xi))$$

which leads to

$$\left(1+|\eta|^2\right)^{m/2}|(\varphi q)^\wedge(\eta,t,\xi)| \leqslant (2\pi)^{-n/2}\widetilde{C}_\varphi(1+\psi(\xi))$$

i.e. (6.9) is proved.

B. We may use the calculation of part A to obtain

$$\left|\eta^\beta \int_{\mathbb{R}^n} e^{-ix\eta}q(x,t,\xi)dx\right| = \left|\int_{\mathbb{R}^n} e^{-ix\eta}\partial_x^\beta q(x,t,\xi)dx\right|$$
$$\leqslant \|\varphi_\beta\|_{L^1}(1+\psi(\xi))$$

which gives for $k \leqslant m$

$$(1+|\eta|^2)^{k/2}|\widehat{q}(\eta,t,\xi)| \leqslant \widetilde{\gamma}_{k,n}\sum_{|\beta|\leqslant k}\|\varphi_\beta\|_{L^1}(1+\psi(\xi))$$

and (6.11) is proved. □

In order to prove a first consequence of Lemma 6.2 we need

**Lemma 6.3.** *Let $k \in L^1(\mathbb{R}^n)$. Then we have for all $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t), v(\cdot,t) \in \mathbb{L}^2(\mathbb{R}^n)$*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(\xi-\eta)u(\eta,t)v(\xi,t)d\eta d\xi\right| \leqslant \|k\|_{L^1}\|u(\cdot,t)\|_0\|v(\cdot,t)\|_0 \tag{6.12}$$

*Proof.* Using the Cauchy-Schwarz inequality, we find

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(\xi-\eta)u(\eta,t)v(\xi,t)d\eta d\xi\right|$$
$$\leqslant \int_{\mathbb{R}^n}\left(\left(\int_{\mathbb{R}^n}|k(\xi-\eta)|d\eta\right)^{1/2}\left(\int_{\mathbb{R}^n}|k(\xi-\eta)||u(\eta,t)|^2d\eta\right)^{1/2}|v(\xi,t)|\right)d\xi$$
$$\leqslant \|k\|_{L^1}^{1/2}\|v\|_0\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|k(\xi-\eta)||u(\eta,t)|^2d\eta d\xi\right)^{1/2}$$
$$\leqslant \|k\|_{L^1}\|u(\cdot,t)\|_0\|v(\cdot,t)\|_0$$

where in the last step Young's inequality was used. □

**Proposition 6.4.** *Suppose that* $q : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{C}$ *satisfies the assumptions of Lemma 6.2.A for* $m \geqslant n+1$. *Then for every* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *and for all* $u : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{C}$ *such that* $u(\cdot,t) \in H^{\psi,2}(\mathbb{R}^n)$ *it holds*

$$\|\varphi(q(x,t,D))u(\cdot,t)\|_0 \leq \tilde{c}_\varphi \|u(\cdot,t)\|_{\psi,2} \tag{6.13}$$

*with* $\tilde{c}_\varphi$ *independent of* $t \geq 0$.

*Proof.* We will prove (6.13) for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and all $u : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{C}$ such that $u(\cdot,t) \in S(\mathbb{R}^n)$. Note that for $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$, such that $u(\cdot,t), v(\cdot,t) \in S(\mathbb{R}^n)$ we have by (6.9) and Lemma 6.3

$$\begin{aligned}
|((\varphi(q(x,t,D)u)(\cdot,t), v(\cdot,t))_0| &= \left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\varphi q)\hat{}\,(\xi-\eta,t,\eta)\hat{u}(\eta,t)\overline{\hat{v}(\xi,t)}d\eta d\xi \right| \\
&\leqslant c_\varphi \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-(n+1)/2}(1+\psi(\eta))|\hat{u}(\eta,t)||\hat{v}(\xi,t)|d\eta d\xi \\
&\leqslant c_\varphi \tilde{c}_{n,n+1}\|(1+\psi(\cdot))\hat{u}(\cdot,t)\|_0\|\hat{v}(\cdot,t)\|_0 \\
&\leqslant \tilde{c}_\varphi \|u(\cdot,t)\|_{\psi,2}\|v(\cdot,t)\|_0,
\end{aligned}$$

or

$$\sup_{v \in L^2(\mathbb{R}^n),\ v \neq 0} \frac{|(\varphi(q(x,t,D)u)(\cdot,t), v(\cdot,t))_0|}{\|v(\cdot,t)\|_0} \leqslant \tilde{c}_\varphi \|u(\cdot,t)\|_{\psi,2}$$

which implies the proposition by the density of the Schwartz in the space $H^{\psi,s}(\mathbb{R}^n)$.

□

The proof of Proposition 6.4 shows already a principal problem when estimating a pseudo-differential operator with negative definite symbol in some of the spaces $H^{\psi,s}(\mathbb{R}^n)$. In order to improve (6.13) to a global estimate. i.e. to

$$\|q(x,t,D)u(\cdot,t)\|_0 \leqslant c\|u(\cdot,t)\|_{\psi,2}$$

additional assumptions on $q(x,t,D)$ are necessary.

We will consider time dependent symbols which decompose as follows

$$q(x,t,\xi) = q_1(t,\xi) + q_2(x,t,\xi) \ . \tag{6.14}$$

Of course, we have still to assume that $q(x,t,\cdot) : \mathbb{R}^n \to \mathbb{C}$ is for all $x \in \mathbb{R}^n$ a continuous negative definite function. In addition, we assume that $q_1(\cdot,t) : \mathbb{R}^n \to \mathbb{C}$ is also a continuous negative definite function. The operator $q_1(t,D)$ and $q_2(x,t,D)$ will be estimated separately.

Note that decomposition (6.14) may arise when freezing the coefficients, i.e.

$$q(x,t,\xi) = q(x_0,t,\xi) + (q(x,t,\xi) - q(x_0,t,\xi)) \quad .$$

Next we will need some general assumptions on $q_1$ and $q_2$.

**Assumption 6.5.** *We assume that the function $q : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$ is a time dependent continuous negative symbol having the decomposition $q(x,t,\xi) = q_1(t,\xi) + q_2(x,t,\xi)$ into a continuous function $q_1 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$ such that $q_1(\cdot,t) : \mathbb{R}^n \to \mathbb{C}$ is negative definite, and a continuous function $q_2 : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}$ . Further let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a fixed negative definite function. A.1. The function $q_1$ satisfies for $t \geq 0$ fixed with $\gamma_0 > 0$ and $\gamma_1, \gamma_2 \geqslant 0$ all independent of $t$, the estimates*

$$\gamma_0 \psi(\xi) \leqslant \text{Re } q_1(t,\xi) \leqslant \gamma_1 \psi(\xi), \quad \textit{for all } |\xi| \geqslant 1, t \in \mathbb{R}_+, \tag{6.15}$$

*and*

$$|\text{Im } q_1(t,\xi)| \leqslant \gamma_2 \text{Re } q_1(t,\xi), \quad \textit{for all } \xi \in \mathbb{R}^n, t \in \mathbb{R}_+ \ . \tag{6.16}$$

*Note that* (6.15) *and* (6.16) *implies*

$$1 + \text{Re } q_1(t,\xi) \leqslant 1 + |q_1(t,\xi)| \leqslant \gamma_1(1 + \psi(\xi)), \tag{6.17}$$

*for all $\xi \in \mathbb{R}^n$, all $t \geq 0$ with some $\gamma_1 > 0$ independent of $t$.*

*A.2.m. For $m \in \mathbb{N}_0$, the function $x \mapsto q_2(x,t,\xi)$ belongs to $C^m(\mathbb{R}^n)$ and we have the estimate*

$$|\partial_x^\alpha q_2(x,t,\xi)| \leqslant \varphi_\alpha(x)(1 + \psi(\xi)) \tag{6.18}$$

*for all $\alpha \in \mathbb{N}_0^n, |\alpha| \leqslant m$ with function $\varphi_\alpha \in L^1(\mathbb{R}^n)$, i.e. the right hand side of* (6.18) *is independent of $t$.*

We start with estimates for the operator $q_1(t,D)$.

**Proposition 6.6.** *We assume A.1. For any $s \in \mathbb{R}$ the operator $q_1(t,D)$ satisfies the estimates*

$$\|q_1(t,D)u(\cdot,t)\|_{\psi,s-2} \leqslant \gamma_1 \|u(\cdot,t)\|_{\psi,s} \tag{6.19}$$

*and*

$$\|q_1(t,D)u(\cdot,t)\|_{\psi,s-2} \geqslant \gamma_0 \|u(\cdot,t)\|_{\psi,s} - \lambda_0 \|u(\cdot,t)\|_{\psi,s-2} \tag{6.20}$$

*with a suitable constant $\lambda_0$ independent of $t$.*

*Proof.* It is sufficient to prove (6.19) and (6.20) for all $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in S(\mathbb{R}^n)$. For $s \in \mathbb{R}$, we find using (6.17) i.e.

$$1 + \text{Re } q_1(t,\xi) \leqslant 1 + |q_1(t,\xi)| \leqslant \gamma_1(1+\psi(\xi))$$

that

$$\begin{aligned}
\|q_1(t,D)u(\cdot,t)\|^2_{\psi,s-2} &= \int_{\mathbb{R}^n} (1+\psi(\xi))^{s-2}|q_1(t,\xi)\widehat{u}(\xi,t)|^2 d\xi \\
&\leqslant \gamma_1^2 \int_{\mathbb{R}^n} (1+\psi(\xi))^{s-2}(1+\psi(\xi))^2|\widehat{u}(\xi,t)|^2 d\xi \\
&= \gamma_1^2\|u\|^2_{\psi,s}
\end{aligned}$$

proving (6.19). To prove (6.20) observe that (6.15) , i.e.

$$\gamma_0\psi(\xi) \leqslant \text{Re } q_1(t,\xi) \leqslant \gamma_1\psi(\xi) \quad ,$$

holds for $|\xi| \geq 1$ and therefore we find

$$\begin{aligned}
\|q_1(t,D)u(\cdot,t)\|^2_{\psi,s-2} &= \int_{B_1^c(0)} (1+\psi(\xi))^{s-2}|q_1(t,\xi)|^2|\widehat{u}(\xi,t)|^2 d\xi \\
&\quad + \int_{B_1(0)} (1+\psi(\xi))^{s-2}|q_1(t,\xi)|^2|\widehat{u}(\xi,t)|^2 d\xi \\
&\geqslant \int_{B_1^c(0)} (1+\psi(\xi))^{s-2}(\text{Re } q_1(t,\xi))^2|\widehat{u}(\xi,t)|^2 d\xi \\
&\geqslant \gamma_0^2 \int_{B_1^c(0)} (1+\psi(\xi))^{s-2}(\psi(\xi))^2|\widehat{u}(\xi,t)|^2 d\xi \\
&\geqslant \gamma_0^2 \int_{B_1^c(0)} (1+\psi(\xi))^{s}|\widehat{u}(\xi,t)|^2 d\xi - \gamma_0^2 \int_{B_1^c(0)} (1+\psi(\xi))^{s-2}|\widehat{u}(\xi,t)|^2 d\xi \\
&= \gamma_0^2\|u(\cdot,t)\|^2_{\psi,s} - \gamma_0^2 \int_{B_1^c(0)} (1+\psi(\xi))^{s-2}|\widehat{u}(\xi,t)|^2 d\xi \\
&\quad -\gamma_0^2 \int_{B_1(0)} (1+\psi(\xi))^{s}|\widehat{u}(\xi,t)|^2 d\xi \\
&\geqslant \gamma_0^2\|u(\cdot,t)\|^2_{\psi,s} - \gamma_0^2 \sup_{\xi\in B_1(0)} (1+\psi(\xi))^2\|u(\cdot,t)\|^2_{\psi,s-2}
\end{aligned}$$

which implies (6.20). □

**Corollary 6.7.** *Suppose that* $\lim_{|\xi|\to\infty} \psi(\xi) = \infty$. *Then we have in the situation of Proposition 6.6 for every* $\varepsilon > 0$ *and* $s \geqslant 0$

$$\|q_1(t,D)u(\cdot,t)\|_{\psi,s-2} \geqslant (\gamma_0 - \varepsilon)\|u(\cdot,t)\|_{\psi,s} - \lambda_\varepsilon\|u(\cdot,t)\|_0, \tag{6.21}$$

*for some* $\lambda_\varepsilon \geqslant 0$ *independent of* $t$.

*Proof.* Since $\lim_{|\xi|\to\infty} \psi(\xi) = \infty$, we find for $\varepsilon > 0$ a suitable constant $\rho_\varepsilon \geqslant 0$ such that

$$(1+\psi(\xi))^{s-2} \leqslant \frac{\varepsilon^2}{\lambda_0^2}(1+\psi(\xi))^s$$

holds for all $\xi \in \mathbb{R}^n$, $|\xi| \geqslant \rho_\varepsilon$. Note that the case $\lambda_0 = 0$ is trivial. It follows that

$$\begin{aligned}
\lambda_0^2\|u(\cdot,t)\|^2_{\psi,s-2} &= \lambda_0^2 \int_{\mathbb{R}^n} (1+\psi(\xi))^{s-2}|\widehat{u}(\xi,t)|^2 d\xi \\
&= \lambda_0^2 \int_{B^c_{\rho_\varepsilon}(0)} (1+\psi(\xi))^{s-2}|\widehat{u}(\xi,t)|^2 d\xi \\
&\quad +\lambda_0^2 \int_{B_{\rho_\varepsilon}(0)} (1+\psi(\xi))^{s-2}|\widehat{u}(\xi,t)|^2 d\xi \\
&\leqslant \varepsilon^2 \int_{B^c_{\rho_\varepsilon}(0)} (1+\psi(\xi))^{s}|\widehat{u}(\xi,t)|^2 d\xi \\
&\quad +\lambda_0^2 \left(\sup_{x\in B_{\rho_\varepsilon}(0)} (1+\psi(\xi))^{s-2}\right) \int_{B^c_{\rho_\varepsilon}(0)} |\widehat{u}(t,\xi)|^2 d\xi \\
&\leqslant \varepsilon^2\|u(\cdot,t)\|^2_{\psi,s} + \lambda_\varepsilon^2\|u(\cdot,t)\|_0^2 \quad .
\end{aligned}$$

Hence (6.20) implies

$$\|q_1(t,D)u(\cdot,t)\|_{\psi,s-2} \geqslant \gamma_0\|u(\cdot,t)\|_{\psi,s} - \lambda_0\|u(\cdot,t)\|_{\psi,s-2} \geqslant (\gamma_0-\varepsilon)\|u(\cdot,t)\|_{\psi,s} - \lambda_\varepsilon\|u(\cdot,t)\|_0 \quad .$$

□

**Remark 6.8.** *It is possible to get a smaller value for* $\lambda_\varepsilon$, *but we do not need it later on.*

Next we want to estimate the operator $q_2(x,t,D)$ assuming that $q_2(x,t,\xi)$ fullfills A.2.m with a suitable large $m$.

In the following [A,B] denotes as usual the commutator of two operators A and B, i.e.

$$[A,B] = AB - BA.$$

**Theorem 6.9.** *Let* $s \geq \frac{1}{2}$ *and assume that* $q_2(x,t,\xi)$ *satisfies A.2.m with* $m > n+2s$. *Then we have*

$$\|[(1+\psi(D))^s, q_2(x,t,D)]u(\cdot,t)\|_0 \leq k_{n,m,s,\psi} \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,2s+1} \tag{6.22}$$

*for all $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot, t) \in H^{\psi,2s+1}(\mathbb{R}^n)$, where*

$$k_{n,m,s,\psi} = (2\pi)^{-n/2} 2^{2s+3} s c_\psi \widetilde{\gamma}_{m,n} \widetilde{c}_{n,m-2s} \tag{6.23}$$

*with $\widetilde{c}_{n,m-2s}$, $\widetilde{\gamma}_{m,n}$ and $c_\psi$ as in (6.4)- (6.6), i.e.*

$$\int_{\mathbb{R}^n} (1+|x|^2)^{-(m-2s)/2} dx = \frac{\pi^{n/2}\Gamma\left(\frac{(m-2s)-n}{2}\right)}{\Gamma\left(\frac{m-2s}{2}\right)} =: \widetilde{c}_{n,m-2s}, \quad m-2s > n \, .$$

*Proof.* For $s \geq \frac{1}{2}$, by Lemma 6.2. B, we have

$$|\widehat{q}_2(\xi - \eta, t, \eta)| \leq \widetilde{\gamma}_{m,n} \sum_{|\alpha| \leq m} \|\varphi_\alpha\|_{L^1} \left(1 + |\xi - \eta|^2\right)^{-m/2} (1 + \psi(\eta)) \, . \tag{6.24}$$

Now for $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot, t), v(\cdot, t) \in C_0^\infty(\mathbb{R}^n)$, we find using (3.29) and (6.24)

$$\begin{aligned}
& |([(1+\psi(D))^s, q_2(x,t,D)]u(\cdot,t), v(\cdot,t))_0| \\
= \; & (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widehat{q}_2(\xi - \eta, t, \eta) \left((1+\psi(\xi))^s - (1+\psi(\eta))^s\right) \widehat{u}(\eta, t) \widehat{v}(\xi, t) d\eta d\xi \right| \\
\leq \; & (2\pi)^{-n/2} 2^{2s+3} s c_\psi \widetilde{\gamma}_{m,n} \sum_{|\alpha| \leq m} \|\varphi_\alpha\|_{L^1} \\
& \times \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(1 + |\xi - \eta|^2\right)^{-\frac{m}{2}+s} (1+\psi(\eta))^{\frac{2s+1}{2}} |\widehat{u}(\eta,t)||\widehat{v}(\xi,t)| d\eta d\xi \\
\leq \; & k_{n,m,s,\psi} \sum_{|\alpha| \leq m} \|\psi_\alpha\|_{L^1} \|u(\cdot,t)\|_{\psi,2s+1} \|v(\cdot,t)\|_0,
\end{aligned}$$

which implies (6.22) and (6.23). □

**Corollary 6.10.** *Let $s_1 \geq 0$, $s_2 \geq \frac{1}{2}$ and assume that $q_2(x,t,\xi)$ satisfies A.2.m. with $m > n + 2s_1 + 2s_2$. Then we have with a suitable constant $\widetilde{c}$,*

$$\|[(1+\psi(D))^{s_1}, q_2(x,t,D)]u(\cdot,t)\|_{\psi,2s_2} \leq \widetilde{c} \sum_{|\alpha| \leq m} \|\varphi_\alpha\|_{L^1} \|u(\cdot,t)\|_{\psi,2s_1+2s_2+1}. \tag{6.25}$$

*Proof.* Since

$$\begin{aligned}
& \|[(1+\psi(D))^{s_1}, q_2(x,t,D)]u(\cdot,t)\|_{\psi,2s_2} \\
= \; & \|(1+\psi(D))^{s_2}[(1+\psi(D))^{s_1}, q_2(x,t,D)]u(\cdot,t)\|_0
\end{aligned}$$

and

$$\begin{aligned}
& (1+\psi(D))^{s_2}[(1+\psi(D))^{s_1}, q_2(x,t,D)]u(x,t) \\
= \; & [(1+\psi(D))^{s_1+s_2}, q_2(x,t,D)]u(x,t) - [(1+\psi(D))^{s_2}, q_2(x,t,D)](1+\psi(D))^{s_1}u(x,t)
\end{aligned}$$

It follows from Theorem 6.9 that

$$
\begin{aligned}
&\|[(1+\psi(D))^{s_1}, q_2(x,t,D)]u(\cdot,t)\|_{\psi,2s_2} \\
\leq\ & \left\|[(1+\psi(D))^{s_1+s_2}, q_2(x,t,D)]u(\cdot,t)\right\|_0 \\
&+\|[(1+\psi(D))^{s_2}, q_2(x,t,D)](1+\psi(D))^{s_1}u(\cdot,t)\|_0 \\
\leq\ & c\sum_{|\alpha|\leq m}\|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,2s_1+2s_2+1}+c^1\sum_{|\alpha|\leq m}\|\varphi_\alpha\|_{L^1}\|(1+\psi(D))^{s_1}u(\cdot,t)\|_{\psi,2s_2} \\
\leq\ & \tilde{c}\sum_{|\alpha|\leq m}\|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,2s_1+2s_2+1}
\end{aligned}
$$

□

Now we can show

**Theorem 6.11.** *Let $s \geq 1$ and suppose that $q_2(x,t,\xi)$ satisfies A.2.m with $m \geq n+[s]+1$. Then we have for all $s_1 \geq 0$ and $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in H^{\psi,s_1+2}(\mathbb{R}^n)$, the estimate*

$$\|q_2(x,t,D)u(\cdot,t)\|_{\psi,s_1} \leq \widetilde{c}_{n,m,s_1,\psi}\|u(\cdot,t)\|_{\psi,s_1+2} \tag{6.26}$$

*where for $s_1 = s$, we have*

$$\widetilde{c}_{n,m,s,\psi} = \left(\widetilde{\gamma}_{n+1,n}\widetilde{c}_{n,n+1}+k_{n,m,\frac{s}{2},\psi}\right)\sum_{|\alpha|\leq m}\|\varphi_\alpha\|_{L^1} \tag{6.27}$$

*where $k_{n,m,\frac{s}{2},\psi}$ is defined as in* (6.23), *i.e.*

$$k_{n,m,s,\psi} = (2\pi)^{-n/2}2^{2s+3}sc_\psi\widetilde{\gamma}_{m,n}\widetilde{c}_{n,m-2s}.$$

*Proof.* A. First note that for $u(\cdot,t), v(\cdot,t) \in S(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} q_2(x,t,D)u(x,t)v(x,t)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\widehat{q}_2(\xi-\eta,t,\eta)\widehat{u}(\eta,t)\overline{\widehat{v}(\xi,t)}d\eta d\xi$$

since by Plancherel's theorem

$$
\begin{aligned}
&\int_{\mathbb{R}^n} q_2(x,t,D)u(x,t)\overline{v(x,t)}dx \\
=\ & (q_2(x,t,D)u(\cdot,t), v(\cdot,t)) \\
=\ & ((q_2(x,t,D)u)\hat{\ }(\cdot,t), \hat{v}(\cdot,t))),
\end{aligned}
$$

and

$$
\begin{aligned}
&(q_2(x,t,D)u)\hat{\ }(\xi,t) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\xi}\left(\int_{\mathbb{R}^n} e^{ix\eta}q_2(x,t,\eta)\widehat{u}(\eta,t)\right)d\eta d\xi \\
=\ & (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix(\xi-\eta)}q_2(x,t,\eta)\widehat{u}(\eta,t)d\eta d\xi \\
=\ & \int_{\mathbb{R}^n}\widehat{q}_2(\xi-\eta,t,\eta)\widehat{u}(\eta,t)d\xi.
\end{aligned}
$$

We find

$$\int_{\mathbb{R}^n} q_2(x,t,D)u(x,t)\overline{v(x,t)}dx$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{q}_2(\xi-\eta,t,\eta)\widehat{u}(\eta,t)\overline{\widehat{v}(\xi,t)}d\eta d\xi.$$

Applying Lemma 6.2 B and Lemma 6.3, it follows that

$$\begin{aligned}
&|(q_2(x,t,D)u(\cdot,t),v(\cdot,t))_0| \\
\leq\ & \widetilde{\gamma}_{n+1,n} \sum_{|\alpha|\leq n+1} \|\varphi_\alpha\|_{L^1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-\frac{n+1}{2}} \\
& \times(1+\psi(\eta))|\widehat{u}(\eta,t)||\widehat{v}(\xi,t)|d\eta d\xi \\
\leq\ & \widetilde{\gamma}_{n+1,n}\widetilde{c}_{n,n+1} \sum_{|\alpha|\leq n+1} \|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,2}\|v(\cdot,t)\|_0 \quad ,
\end{aligned}$$

which implies

$$\|q_2(x,t,D)u(\cdot,t)\|_0 \leq \widetilde{\gamma}_{n+1,n}\widetilde{c}_{n,n+1} \sum_{|\alpha|\leq n+1} \|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,2} \tag{6.28}$$

i.e. (6.26) for $s_1 = 0$,and all constants are independent of $t$.

B. Next let $s \geq 1$ and observe that

$$\begin{aligned}
&\|q_2(x,t,D)u(\cdot,t)\|_{\psi,s} = \left\|(1+\psi(D))^{s/2}\circ q_2(x,t,D)u(\cdot,t)\right\|_{L^2} \\
=\ & \Big\|\big((1+\psi(D))^{s/2}\circ q_2(x,t,D)u(\cdot,t)\big) + \big(q_2(x,t,D)\circ(1+\psi(D))^{s/2}u(\cdot,t)\big) \\
& - \big(q_2(x,t,D)\circ(1+\psi(D))^{s/2}u(\cdot,t)\big)\Big\|_{L^2} \\
\leq\ & \left\|q_2(x,t,D)(1+\psi(D))^{s/2}u(\cdot,t)\right\|_0 + \Big\|\big((1+\psi(D))^{s/2}\circ q_2(x,t,D)u(\cdot,t)\big) \\
& - \big(q_2(x,t,D)(1+\psi(D))^{s/2}u(\cdot,t)\big)\Big\|_0,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\|q_2(x,t,D)u(\cdot,t)\|_{\psi,s} \\
\leq\ & \|q_2(x,t,D)(1+\psi(D))^{s/2}u(\cdot,t)\|_0 + \|[(1+\psi(D))^{s/2},q_2(x,t,D)]u(\cdot,t)\|_0 \quad .
\end{aligned}$$

Since $(1+\psi(D))^{r/2}$ maps $H^{\psi,t}(\mathbb{R}^n)$ bijectively and bicontinuously $H^{\psi,t-r}(\mathbb{R}^n)$ with $t \geq r > 0$, then we find for $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in H^{\psi,s+2}(\mathbb{R}^n)$ that

$$(1+\psi(D))^{s/2}u(\cdot,t) := v(\cdot,t) \in H^{\psi,(s+2)-s}(\mathbb{R}^n) = H^{\psi,2}(\mathbb{R}^n),$$

and therefore

$$\begin{aligned}
&\left\|q_2(x,t,D)(1+\psi(D))^{s/2}u(\cdot,t)\right\|_0 \\
\leq\ & \widetilde{\gamma}_{n+1,n}\widetilde{c}_{n,n+1} \sum_{|\alpha|\leq n+1} \|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,s+2}.
\end{aligned} \tag{6.29}$$

On the other hand, we may apply Theorem 6.9 with $s$ replaced by $s/2$ to get

$$\left\|[(1+\psi(D))^{s/2}, q_2(x,t,D)]u(\cdot,t)\right\|_0 \leq k_{n,m,\frac{s}{2},\psi} \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1} \|u(\cdot,t)\|_{\psi,s+1} \tag{6.30}$$

which leads together with (6.29) to

$$\|q_2(x,t,D)u(\cdot,t)\|_{\psi,s} \leq \left(\widetilde{\gamma}_{n+1,n}\widetilde{c}_{n,n+1} + k_{n,m,\frac{s}{2},\psi}\right) \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1} \|u(\cdot,t)\|_{\psi,s+2}$$

and the theorem is proved for $s \geq 1$ and $m > [s] + n + 2$. □

**Remark 6.12.** *Note that our proof of Theorem 6.11 yields the estimate*

$$\|q_2(x,t,D)u(\cdot,t)\|_{\psi,s_2} \leq \widetilde{c}^{*}_{n,s_2,\psi,m} \sum_{|\alpha|\leq m} \|\psi_\alpha\|_{L^1} \|u(\cdot,t)\|_{\psi,s_2+2} \tag{6.31}$$

*for all $s_2 \geq 0$ with $m$ suitably large.*

Note that in the following theorem the relations of the constants in (6.21) with $s-2$ replaced by $s$, i.e.

$$\|q_1(t,D)u(\cdot,t)\|_{\psi,s_1-2} \geq (\gamma_0 - \varepsilon)\|u(\cdot,t)\|_{\psi,s_1} - \lambda_\varepsilon \|u(\cdot,t)\|_0$$

and the constant $c_{n,m,s_1,\psi}$ from (6.27), i.e.

$$\widetilde{c}_{n,m,s_1,\psi} = \left(\widetilde{\gamma}_{n+1,n}\widetilde{c}_{n,n+1} + k_{n,m,\frac{s_1}{2},\psi}\right) \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1}$$

will become important.

For simplicity, let us restate (6.21) as

$$\|q_1(t,D)u(\cdot,t)\|_{\psi,s_1} \geq \eta\gamma_0 \|u(\cdot,t)\|_{\psi,s_1+2} - \gamma_{\eta,s_1}\|u(\cdot,t)\|_0 \tag{6.32}$$

which holds for all $s_1$ and $\eta, 0 < \eta < 1$. Further recall the estimate (6.26)

$$\|q_2(x,t,D)u(\cdot,t)\|_{\psi,s_2} \leq \widetilde{c}_{n,m,s_2,\psi}\|u(\cdot,t)\|_{\psi,s_2+2}$$

and for $s_1 \geq 1$ and $m \geq n + [s_1] + 1$:

$$\|q_2(x,t,D)u(\cdot,t)\|_{\psi,s_1} \leq \widetilde{c}_{n,m,s_1,\psi}\|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,s_1+2} \tag{6.33}$$

where $\widetilde{c}_{n,m,s_1,\psi} = \left(\widetilde{\gamma}_{n+1,n}\widetilde{c}_{n,n+1} + k_{n,m,\frac{s_1}{2},\psi}\right)$. Combining (6.32) and (6.33), we get

$$\begin{aligned}
\|q(x,t,D)u(\cdot,t)\|_{\psi,s_1} &\geq \|q_1(t,D)u(\cdot,t)\|_{\psi,s_1} - \|q_2(x,t,D)u(\cdot,t)\|_{\psi,s_1} \\
&\geq \eta\gamma_0\|u(\cdot,t)\|_{\psi,s_1+2} - \gamma_{\eta,s_1}\|u(\cdot,t)\|_0 - \widetilde{c}_{n,m,s_1,\psi} \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1}\|u(\cdot,t)\|_{\psi,s_1+2} \\
&= \left(\eta\gamma_0 - \widetilde{c}_{n,m,s_1,\psi} \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1}\right) \|u(\cdot,t)\|_{\psi,s_1+2} - \gamma_{\eta,s_1}\|u(\cdot,t)\|_0 \quad .
\end{aligned}$$

Thus we have proved

**Theorem 6.13.** *Suppose that $q(x,t,\xi) = q_1(t,\xi) + q_2(x,t,\xi)$ satisfies A.1 and A.2.m with $m > n + [s] + 1$ for some $s \geq 1$. Further assume that for some $\eta \in (0,1)$, we have*

$$\widetilde{c}_{n,m,s,\psi} \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1} < \eta\gamma_0 \quad . \tag{6.34}$$

*Then we have for all $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in H^{\psi,s}(\mathbb{R}^n)$, the lower estimate*

$$\|q(x,t,D)u(\cdot,t)\|_{\psi,s} \geq \delta_0 \|u(\cdot,t)\|_{\psi,s+2} - \gamma_{\eta,s}\|u(\cdot,t)\|_0 \tag{6.35}$$

*with*

$$\delta_0 = \eta\gamma_0 - \widetilde{c}_{n,m,s,\psi} \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1} > 0 \quad . \tag{6.36}$$

**Remark 6.14.** *Later on, we will see that often*

$$(q(x,t,D)u(\cdot,t), u(\cdot,t))_0 \geq 0 \tag{6.37}$$

*holds for all $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that $u(\cdot,t) \in H^{\psi,2}(\mathbb{R}^n)$. Taking in (6.35) $s = 2$, we find then for all $\lambda \geq 0$,*

$$\begin{aligned}
&\|q(x,t,D)u(\cdot,t) + \lambda u(\cdot,t)\|_0^2 \\
= \ &\|q(x,t,D)u(\cdot,t)\|_0^2 + 2\lambda(q(x,t,D)u(\cdot,t), u(\cdot,t))_0 + \lambda^2\|u(\cdot,t)\|_0^2 \\
\geq \ &\|q(x,t,D)u(\cdot,t)\|_0^2 + \lambda^2\|u(\cdot,t)\|_0^2
\end{aligned}$$

*or*

$$\begin{aligned}
\|q(x,t,D)u(\cdot,t) + \lambda u(\cdot,t)\|_0 &\geq \|q(x,t,D)u(\cdot,t)\|_0 + \lambda \|u(\cdot,t)\|_0 \\
&\geq \delta_0 \|u(\cdot,t)\|_{\psi,2} - \gamma_{\eta,2}\|u(\cdot,t)\|_0 + \lambda\|u(\cdot,t)\|_0 \quad .
\end{aligned}$$

*Thus for $\lambda \geq \gamma_{\eta,2}$, we have under the assumption (6.37), i.e. $(q(x,t,D)u(\cdot,t), u(\cdot,t))_0 \geq 0$*

$$\|q(x,t,D)u(\cdot,t) + \lambda u(\cdot,t)\|_0 \geq \delta_0\|u(\cdot,t)\|_{\psi,2} \quad . \tag{6.38}$$

*In order to prove regularity results with respect to $x \in \mathbb{R}^n$ for solutions of the equation*

$$q(x,t,D)u(x,t) = f$$

*we have to introduce the Friedrichs mollifier : The operator*

$$J_\varepsilon(u)(x,t) := (j_\varepsilon * u)(x,t) = \int_{\mathbb{R}^n} j_\varepsilon(x-y)u(y,t)dy$$

*where $j : \mathbb{R}^n \to \mathbb{R}$ is the function*

$$j(x) := \begin{cases} c_0 \exp((|x|^2-1)^{-1}), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

*and $c_0^{-1} = \int_{|x|<1} \exp((|x|^2-1)^{-1})dx$, is called the* ***Friedrichs mollifier****.*

For $\varepsilon > 0$ set $j_\varepsilon(x) := \varepsilon^{-n} j\left(\frac{x}{\varepsilon}\right)$. It follows that $j_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $j_\varepsilon(x) \geq 0$, $\operatorname{sup} j_\varepsilon = \overline{B_\varepsilon(0)}$ and $\int_{\mathbb{R}^n} j_\varepsilon(x)dx = 1$.

**Proposition 6.15.** *Let $J_\varepsilon$ be defined as above, i.e.*

$$J_\varepsilon(u)(x,t) := \int_{\mathbb{R}^n} j_\varepsilon(x-y)u(x,t)dy = (j_\varepsilon * u)(x,t)$$

*For any $s_1 \geq 0$ and $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in H^{\psi,s_1}(\mathbb{R}^n)$, we have for $t \geq 0$ fixed*

$$J_\varepsilon(u)(\cdot,t) \in \bigcap_{s_2 \geq 0} H^{\psi,s_2}(\mathbb{R}^n) \bigcap C^\infty(\mathbb{R}^n) \tag{6.39}$$

*and*

$$\|J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} \leq \|u(\cdot,t)\|_{\psi,s_1}, \tag{6.40}$$

*as well as*

$$\lim_{\varepsilon\to 0} \|J_\varepsilon(u)(\cdot,t) - u(\cdot,t)\|_{\psi,s_1} = 0. \tag{6.41}$$

*In addition, if for $\varepsilon \in (0,\rho), \rho > 0$, we have for some $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in L^2(\mathbb{R}^n)$,*

$$\|J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} \leq c_{u,s_1,t} \tag{6.42}$$

*with a constant independent of $\varepsilon$, it follows that $u(\cdot,t) \in H^{\psi,s_1}(\mathbb{R}^n)$.*

*Proof.* A. For $s_2 \geq 0$, we find

$$\begin{aligned} \|J_\varepsilon(u)(\cdot,t)\|^2_{\psi,s_2} &= \int_{\mathbb{R}^n} (1+\psi(\xi))^{s_2} \left|(j_\varepsilon * u)\hat{}(\xi,t)\right|^2 d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} (1+\psi(\xi))^{s_2} \left|\widehat{j_\varepsilon}(\xi)\right|^2 |\widehat{u}(\xi,t)|^2 d\xi \quad . \end{aligned}$$

Since $j_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, it follows that $\widehat{j_\varepsilon} \in S(\mathbb{R}^n)$. Hence there is a constant $c_{s_2,s_1,\varepsilon}$ such that

$$(1+\psi(\xi))^{s_2}|\widehat{j_\varepsilon}(\xi)| \leq c_{s_2,s_1,\varepsilon}(1+\psi(\xi))^{s_1}$$

which implies $\|J_\varepsilon(u)(\cdot,t)\|^2_{\psi,s_2} \leq c_{s_2,s_1,\varepsilon}\|u(\cdot,t)\|^2_{\psi,s_1}$, hence we have $J_\varepsilon(u)(\cdot,t) \in \bigcap_{s_2\geq 0} H^{\psi,s_2}(\mathbb{R}^n)$ and (6.39) holds.

B. The lemma of Riemann-Lebesgue implies

$$\left|\widehat{j_\varepsilon}(\xi)\right| = \left|\widehat{j}(\varepsilon\xi)\right| \leq (2\pi)^{-n/2}\|j\|_{L^1} = (2\pi)^{-n/2}$$

which yields

$$\begin{aligned} \|J_\varepsilon(u)(\cdot,t)\|^2_{\psi,s_1} &= (2\pi)^n \int_{\mathbb{R}^n} (1+\psi(\xi))^{s_1} \left|\widehat{j}_\varepsilon(\xi)\right|^2 |\widehat{u}_\varepsilon(\xi,t)|^2 \, d\xi \\ &\leq \int_{\mathbb{R}^n} (1+\psi(\xi))^{s_1} |\widehat{u}_\varepsilon(\xi,t)|^2 \, d\xi \\ &= \|u(\cdot,t)\|^2_{\psi,s_1}. \end{aligned}$$

C. In order to show (6.41), observe that

$$\|J_\varepsilon(u)(\cdot,t) - u(\cdot,t)\|^2_{\psi,s_2} = \int_{\mathbb{R}^n} (1+\psi(\xi))^{s_2} |(j_\varepsilon * u)\hat{}\,(\xi,t) - \widehat{u}(\xi,t)|^2 \, d\xi$$

$$= (2\pi)^n \int_{\mathbb{R}^n} (1+\psi(\xi))^{s_2} \left|\widehat{j}_\varepsilon(\xi) - (2\pi)^{-n/2}\right|^2 |\widehat{u}(\xi,t)|^2 d\xi \quad .$$

Since $\widehat{j}_\varepsilon(\xi) = \widehat{j}(\varepsilon\xi) \to \widehat{j}(0) = (2\pi)^{-n/2}$ and $\left|\widehat{j}_\varepsilon(\xi)\right| \leq (2\pi)^{-n/2}$, for all $\xi \in \mathbb{R}^n$ and $\varepsilon > 0$ , the dominated converges theorem yields

$$\lim_{\varepsilon\to 0} \|J_\varepsilon(u)(\cdot,t) - u(\cdot,t)\|^2_{\psi,s_2} = 0 \quad .$$

It remains to prove that (6.42) implies $u(\cdot,t) \in H^{\psi,s_1}(\mathbb{R}^n)$. From (6.42) it follows that $(J_{1/n}(u)(\cdot,t))_{n\geq 1}$ converges weakly for $t \geq 0$ fixed in $H^{\psi,s_1}(\mathbb{R}^n)$ to some element $v(\cdot,t) \in H^{\psi,s_1}(\mathbb{R}^n)$. By the linearity of the continuous embedding of $H^{\psi,s_1}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, it follows that $(J_{1/n}(u)(\cdot,t))_{n\geq 1}$ converges also weakly in $L^2(\mathbb{R}^n)$ to $v(\cdot,t)$.

But by (6.41) above we know that $(J_{1/n}(u)(\cdot,t))_{n\geq 1}$ converges strongly in $L^2(\mathbb{R}^n)$ to $u(\cdot,t)$, hence $u(\cdot,t) = v(\cdot,t)$ and $u(\cdot,t) \in H^{\psi,s_1}(\mathbb{R}^n)$.

□

**Theorem 6.16.** *Suppose that $q_2(x,t,\xi)$ satisfies A.2.m. Further let $s_1 \geq 0$ such that $|s_1 - 1| + n + 1 \leq m$. Then we have for all $\varepsilon \in (0,1]$ and all $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in H^{\psi,s_1+1}(\mathbb{R}^n)$*

$$\|[J_\varepsilon, q_2(x,t,D)]u(\cdot,t)\|_{\psi,s_1} \leq c\|u(\cdot,t)\|_{\psi,s_1+1} \tag{6.43}$$

*with a constant $c$ independent of $\varepsilon \in (0,1]$ and $t \geq 0$ .*

*Proof.* First observe that

$$\begin{aligned} &([J_\varepsilon, q_2(x,t,D)]\, u(x,t))\,\hat{}\,(\xi) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{q}_2(\xi-\eta,t,\eta)(\widehat{j}(\varepsilon\xi) - \widehat{j}(\varepsilon\eta))\widehat{u}(\eta,t) d\eta \, . \end{aligned} \tag{6.44}$$

Further we claim

$$(\widehat{j}(\varepsilon\xi) - \widehat{j}(\varepsilon\eta))(1+|\xi|^2)^{1/2} \leq c^*(1+|\xi-\eta|^2)^{1/2} \tag{6.45}$$

with $c^*$ independent of $\varepsilon \in (0,1]$.

In fact, since $|\widehat{j}(\xi)| \leq (2\pi)^{-n/2}$ for all $\xi \in \mathbb{R}^n$, we have for $|\xi - \eta| \geq \frac{1}{2}|\xi|$,

$$
\begin{aligned}
& |\widehat{j}(\varepsilon\xi) - \widehat{j}(\varepsilon\eta)|(1+|\xi|^2)^{1/2} \\
\leq \; & 2(2\pi)^{-n/2}(1+|\xi|^2)^{\frac{1}{2}} \leq c^1(1+|\xi-\eta|^2)^{1/2} \quad .
\end{aligned}
$$

On the other hand, for $|\xi - \eta| \leq \frac{1}{2}|\xi|$, since $\widehat{j} \in S(\mathbb{R}^n)$ we have $\nabla\widehat{j}(\xi) \leq \widetilde{c}(1+|\xi|^2)^{-1/2}$, the mean-value theorem yields for $\varepsilon \leq \sqrt{2}$.

$$
\begin{aligned}
& \left|\widehat{j}(\varepsilon\xi) - \widehat{j}(\varepsilon\eta)\right| (1+|\xi|^2)^{1/2} \\
\leq \; & |\varepsilon\xi - \varepsilon\eta|\widetilde{c}\left(1+\left|\frac{\varepsilon\xi}{2}\right|\right)^{-1/2} \left(1+|\xi|^2\right)^{-1/2} \\
\leq \; & c''(1+|\xi-\eta|^2)^{1/2}\left(\frac{1}{\varepsilon^2}+\frac{|\xi|^2}{2}\right)^{-1/2}(1+|\xi|^2)^{1/2} \qquad (6.46)
\end{aligned}
$$

and (6.45) is proved.

Now we get using (6.43), Lemma 6.2 B, (6.45), Peetre's inequality for continuous negative definite functions , and the estimate

$$(1+\psi(\xi)) \leq c_\psi(1+|\xi|^2)$$

$$
\begin{aligned}
& \left|(1+\psi(\xi))^{s_1/2}([J_\varepsilon, q_2(x,t,D)]u(x,t))(\xi)\right| \\
= \; & (2\pi)^{-n/2}\left|\int_{\mathbb{R}^n} \widehat{q}_2(\xi-\eta,t,\eta)\left(\widehat{j}(\varepsilon\xi)-\widehat{j}(\varepsilon\eta)\right)(1+\psi(\xi))^{s_1/2}\widehat{u}(\eta,t)d\eta\right| \\
\leq \; & c'''\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{-m/2}(1+\psi(\eta))(1+|\xi-\eta|^2)^{1/2} \\
& \times\frac{(1+\psi(\xi))^{1/2}}{(1+|\xi|^2)^{1/2}}(1+\psi(\xi))^{\frac{s_1-1}{2}}|\widehat{u}(\xi,t)|d\xi \\
\leq \; & c^{**}\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{-\frac{m+1}{2}}(1+\psi(\xi-\eta))^{\frac{|s_1-1|}{2}}(1+\psi(\eta))^{\frac{s_1+1}{2}}|\widehat{u}(\eta,t)|d\eta \\
\leq \; & \bar{c}\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{-\frac{m+1+|s_1-1|}{2}}(1+\psi(\eta))^{\frac{s_1+1}{2}}|\widehat{u}(\eta,t)|d\eta
\end{aligned}
$$

with $\bar{c}$ independent of $\varepsilon \in (0,1]$ and $t \geq 0$ .

Now, we finally get by Young's inequality

$$
\begin{aligned}
\|[J_\varepsilon, q_2(x,t,D)]u(\cdot,t)\|_{\psi,s_1} \; & \leq \; \bar{c}\left\|(1+|\cdot|^2)^{-\frac{m+1+|s_1-1|}{2}} * (1+\psi(\cdot))^{-\frac{s_1+1}{2}}|\widehat{u}(\cdot,t)|\right\|_0 \\
& \leq \; c\|u(\cdot,t)\|_{\psi,s_1+1}
\end{aligned}
$$

which is independent of $\varepsilon \in (0,1]$. ☐

**Theorem 6.17.** *Suppose that $q(x,t,\xi) = q_1(t,\xi) + q_2(x,t,\xi)$ fullfills A.1 and A.2.m with $m \geq [s_1] + n + 2$ for some $s_1 \geq 0$ and (6.34)*

$$i.e. \quad \widetilde{c}_{n,m,s_1,\psi} \sum_{|\alpha|\leq m} \|\varphi_\alpha\|_{L^1} < \eta\gamma_0 \qquad for\ \gamma_0 > 0\ and\ \eta \in (0,1)\ .$$

*Further suppose that for some $\lambda \in \mathbb{R}$ and $t \geq 0$ fixed $f(\cdot,t) \in H^{\psi,s_1}(\mathbb{R}^n)$. For a solution $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t) \in H^{\psi,s_1+1}(\mathbb{R}^n)$ to the equation*

$$q_\lambda(x,t,D)u(x,t) = q(x,t,D)u(x,t) + \lambda u(x,t) = f \tag{6.47}$$

*Then it follows that $u(\cdot,t) \in H^{\psi,s_1+2}(\mathbb{R}^n)$.*

*Proof.* Using Theorem 6.13, we find

$$\begin{aligned}
&\delta_0 \|J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1+2} - \gamma_{\eta,s_1} \|J_\varepsilon(u)(\cdot,t)\|_0 - \|\lambda J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} \\
\leq\ & \|q(x,t,D)J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} - \|\lambda J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} \\
\leq\ & \|q(x,t,D)J_\varepsilon(u)(\cdot,t) + \lambda J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} \\
\leq\ & \|J_\varepsilon(q(x,t,D)u(\cdot,t) + \lambda u(\cdot,t))\|_{\psi,s_1} + \|[J_\varepsilon, q_2(x,t,D)](u)(\cdot,t)\|_{\psi,s_1} \\
=\ & \|J_\varepsilon(f)(\cdot,t)\|_{\psi,s_1} + \|[J_\varepsilon, q_2(x,t,D)](u)(\cdot,t)\|_{\psi,s_1} \quad .
\end{aligned}$$

Thus we get by Theorem 6.16

$$\begin{aligned}
\delta_0 \|J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1+2} \leq\ & \|J_\varepsilon(f)(\cdot,t)\|_{\psi,s_1} + \|[J_\varepsilon, q_2(x,t,D)](u)(\cdot,t)\|_{\psi,s_1} \\
& + |\lambda| \|J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} + \gamma_{\eta,s_1} \|J_\varepsilon(u)(\cdot,t)\|_{\psi,s_1} \\
\leq\ & \|f(\cdot,t)\|_{\psi,s_1} + \widetilde{c}\|u(\cdot,t)\|_{\psi,s_1} + c\|u(\cdot,t)\|_{\psi,s_1+1}
\end{aligned}$$

with $\widetilde{c}$ independent of $\varepsilon \in (0,1]$,and $t \geq 0$, therefore the theorem follows from Proposition 6.15 .

□

We introduce the t-dependent sesquilinear form

$$B(t;u,v) := (q(x,t,D)u(\cdot,t), v(\cdot,t)) \tag{6.48}$$

which is associated to $q(x,t,D)$ and defined for all $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t), v(\cdot,t) \in C_0^\infty(\mathbb{R}^n)$. Clearly we have the decomposition

$$B(t;u,v) = B^{q_1}(t;u,v) + B^{q_2}(t;u,v) \tag{6.49}$$

where $B^{q_1}(t;u,v) := (q_1(t,D)u(\cdot,t), v(\cdot,t))_0$ and $B^{q_2}(t;u,v) := (q_2(x,t,D)u(\cdot,t), v(\cdot,t))_0$.

**Proposition 6.18.** *Suppose that $q_1$ satisfies A.1. The sesquilinear form $B^{q_1}$ satisfies*

$$|B^{q_1}(t;u,v)| \le \gamma_1 \|u(\cdot,t)\|_{\psi,1} \|v(\cdot,t)\|_{\psi,1} \tag{6.50}$$

*for all $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t), v(\cdot,t) \in \mathbb{H}^{\psi,1}(\mathbb{R}^n)$ with $\gamma_1$ as in* (6.17).

*Furthermore we have with some $\widetilde{\lambda}_0 \ge 0$,*

$$|B^{q_1}(t;u,v)| \ge \text{Re}B^{q_1}(t;u,v) \ge \gamma_0 \|u(\cdot,t)\|^2_{\psi,1} - \widetilde{\lambda}_0 \|u(\cdot,t)\|^2_0 \quad . \tag{6.51}$$

*Proof.* It is sufficient to prove (6.50) and (6.51) for all $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t), v(\cdot,t) \in S(\mathbb{R}^n)$. From (6.17)

$$\text{i.e.} \quad 1 + \text{Re}q_1(t,\xi) \le 1 + |q_1(t,\xi)| \le \gamma_1(1+\psi(\xi))$$

we deduce

$$\begin{aligned} |B^{q_1}(t;u,v)| &= \left| \int_{\mathbb{R}^n} q_1(t,\xi)\widehat{u}(\xi,t)\overline{\widehat{v}(\xi,t)}dt \right| \\ &\le \gamma_1 \int_{\mathbb{R}^n} (1+\psi(\xi))|\widehat{u}(\xi,t)||\widehat{v}(\xi,t)|d\xi \\ &\le \gamma_1 \|u(\cdot,t)\|_{\psi,1} \|v(\cdot,t)\|_{\psi,1} \end{aligned}$$

which proves (6.50).

To see (6.51) observe that

$$\begin{aligned} |B(t;u,v)| &\ge \text{Re}B(t;u,v) \\ &= \int_{\mathbb{R}^n} \text{Re}q_1(t,\xi)|\widehat{u}(\xi,t)|^2 d\xi \\ &\ge \gamma_0 \int_{B_1^c(0)} \psi(\xi)|\widehat{u}(\xi,t)|^2 d\xi + \int_{B_1(0)} \text{Re}q_1(t,\xi)|\widehat{u}(\xi,t)|^2 d\xi \\ &\ge \gamma_0 \|u(\cdot,t)\|^2_{\psi,1} - \gamma_0 \|u(\cdot,t)\|^2_0 - \sup_{|\xi|\le 1} |\text{Re}q_1(t,\xi) - \gamma_0\psi(\xi)| \|u(\cdot,t)\|^2_0 \end{aligned}$$

and (6.51) is shown. □

**Remark 6.19.** *Since $q_1(t,D)$ maps real-valued functions onto real-valued functions, it follows that for real-valued $u(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$, we have*

$$B^{q_1}(t;u,v) \ge \gamma_0 \|u(\cdot,t)\|^2_{\psi,1} - \widetilde{\lambda}_0 \|u(\cdot,t)\|^2_0. \tag{6.52}$$

Next, we will estimate $B^{q_2}$

**Proposition 6.20.** *Suppose that A.2.m. holds for $m \geq n+2$. Then for all $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t), v(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$, the estimate*

$$B^{q_2}(t;u,v) \leq k_2 \sum_{|\alpha|\leq n+2} \|\varphi_\alpha\|_{L^1} \|u(\cdot,t)\|_{\psi,1} \|v(\cdot,t)\|_{\psi,1} \tag{6.53}$$

*holds.*

*Proof.* As in the proof of Theorem 6.11, we find for $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t), v(\cdot,t) \in S(\mathbb{R}^n)$,

$$\begin{aligned} |B^{q_2}(t;u,v)| &= \left| \int_{\mathbb{R}^n} q_2(x,t,D)u(x,t)\overline{v(x,t)}dx \right| \\ &\leq \widetilde{\gamma}_{n+2,n} \sum_{|\alpha|\leq n+2} \|\psi_\alpha\|_{L^1} \\ &\quad \times \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-\frac{(n+2)}{2}}(1+\psi(\eta))|\widehat{v}(\eta,t)||\widehat{u}(\xi,t)|d\eta d\xi \\ &\leq \widetilde{\gamma}_{n+2,n} \sum_{|\alpha|\leq n+2} \|\varphi_\alpha\|_{L^1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-\frac{(n+2)}{2}} \left(\frac{1+\psi(\eta)}{1+\psi(\xi)}\right)^{1/2} \\ &\quad \times (1+\psi(\eta))^{1/2} |\widehat{v}(\eta,t)| \, (1+\psi(\xi))^{1/2} |\widehat{u}(\xi,t)| d\eta d\xi \\ &\leq \sqrt{2}\widetilde{\gamma}_{n+2,n}(1\vee c_\psi)^{1/2} \sum_{|\alpha|\leq n+2} \|\varphi_\alpha\|_{L^1} \\ &\quad \times \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (1+|\xi-\eta|^2)^{-\frac{(n+1)}{2}}(1+\psi(\eta))^{1/2}(1+\psi(\xi))^2|\widehat{v}(\eta,t)||\widehat{u}(\xi,t)|d\eta d\xi \\ &\leq k_2 \sum_{|\alpha|\leq n+2} \|\varphi_\alpha\|_{L^1} \|v(\cdot,t)\|_{\psi,1} \|u(\cdot,t)\|_{\psi,1} \end{aligned} \tag{6.54}$$

with $k_2 = \sqrt{2}(1\vee c_\psi)^{1/2}\widetilde{\gamma}_{n+2,n}\widetilde{c}_{n,n+1}$.

□

Combining Proposition 6.18 and Proposition 6.20 , we obtain the following

**Theorem 6.21.** *Suppose that $q_1(t,\xi)$ and $q_2(x,t,\xi)$ satisfy A.1. and A.2.m. with $m \geq n+2$. Then we have for all $u, v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u(\cdot,t), v(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$,*

$$|B(t;u,v)| \leq c\|u(\cdot,t)\|_{\psi,1}\|v(\cdot,t)\|_{\psi,1} \quad . \tag{6.55}$$

Further we have

**Theorem 6.22.** *Suppose that $q_1(t,\xi)$ and $q_2(x,t,\xi)$ satisfy A.1. and A.2.m. with $m \geq n+2$. Assume further with $k_2$ from (6.54) that*

$$\delta_1 := \gamma_0 - k_2 \sum_{|\alpha|\leq n+2} \|\varphi_\alpha\|_{L^1} > 0 \tag{6.56}$$

*Then we have for all* $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ *such that* $u(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$

$$|B(t;u,u)| \geq \mathrm{Re}B(t;u,u) \geq \delta_1 \|u(\cdot,t)\|^2_{\psi,1} - \widetilde{\lambda}_0 \|u(\cdot,t)\|_0^2 \qquad (6.57)$$

*where* $\widetilde{\lambda}_0$ *is taken from* (6.51).

*Proof.* Using Proposition 6.20 and Proposition 6.22, we get

$$\begin{aligned} |B(t;u,u)| &\geq \mathrm{Re}\ B(t;u,u) \\ &\geq B^{q_1}(t;u,u) - |B^{q_2}(t;u,u)| \\ &\geq \gamma_0 \|u(\cdot,t)\|^2_{\psi,1} - \widetilde{\lambda}_0 \|u(\cdot,t)\|_0^2 - k_2 \sum_{|\alpha| \leq n+2} \|\varphi_\alpha\|_{L^1} \|u(\cdot,t)\|^2_{\psi,1} \\ &= \delta_0 \|u(\cdot,t)\|^2_{\psi,1} - \widetilde{\lambda}_0 \|u(\cdot,t)\|_0^2 \quad . \end{aligned}$$

□

**Corollary 6.23.** *In the situation of Theorem 6.22, we have for all* $\lambda \geq \widetilde{\lambda}_0$

$$\begin{aligned} |B_\lambda(t;u,u)| &\geq \mathrm{Re}B_\lambda(t;u,u) \\ &= \mathrm{Re}B(t;u,u) + \lambda \|u(\cdot,t)\|^2_{\psi,1} \\ &\geq \delta_1 \|u(\cdot,t)\|^2_{\psi,1} \quad . \end{aligned} \qquad (6.58)$$

Next, let us suppose that $q_1(t,\xi)$ and $q_2(x,t,\xi)$ satisfy A.1. and A.2.m. and $m \geq n+2$. Further let $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $f(\cdot,t) \in L^2(\mathbb{R}^n)$.

**Definition 6.24.** *We call* $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ *such that* $u(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$ *a* ***variational solution*** *to the equation*

$$q_\lambda(x,t,D)u(x,t) = q(x,t,D)u(x,t) + \lambda u(x,t) = f \qquad (6.59)$$

*if*

$$\begin{aligned} B_\lambda(t;u,\varphi) &= B(t;u,\varphi) + \lambda(u(\cdot,t),\varphi(\cdot,t))_0 \\ &= (f(\cdot,t),\varphi(\cdot,t))_0 \end{aligned} \qquad (6.60)$$

*holds for all* $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ *such that* $\varphi(\cdot,t) \in C_0^\infty(\mathbb{R}^n)$*, or equivalently for all* $\varphi(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$.

Note that by applying Theorem 6.11 with $s_1 = 0$ ,then we can see that a variational solution of (6.59) with $u(\cdot,t) \in H^{\psi,2}(\mathbb{R}^n)$ satisfies (6.59) already in the strong sense . i.e. $q_\lambda(x,t,D)u(\cdot,t) \in L^2(\mathbb{R}^n)$ and (6.59) is an equality in $L^2(\mathbb{R}^n), t \geq 0$ being fixed , compare Theorem 6.11 which provides the regularity estimates for variational solutions needed for this statement.

**Theorem 6.25.** *Let $q_1(t,\xi)$ and $q_2(x,t,\xi)$ satisfy A.1. and A.2.m. with $m \geq n+2$ and take $\widetilde{\lambda}_0$ from* (6.52). *Moreover assume* (6.56), *i.e. $\delta_1 > 0$. Then for $t \geq 0$ fixed and every $\lambda \geq \widetilde{\lambda}_0$, there exists a unique variational solution $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ to equation* (6.59), *i.e.*

$$B_\lambda(t; u, \varphi) = (\varphi(\cdot,t), f(\cdot,t))_0$$

*for all $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $\varphi(\cdot,t) \in C^\infty(\mathbb{R}^n)$.*

*Proof.* First note that by

$$|(\varphi(\cdot,t), f(\cdot,t))_0| \leq \|f(\cdot,t)\|_0 \|\varphi(\cdot,t)\|_0 \leq \|f(\cdot,t)\| \|\varphi(\cdot,t)\|_{\psi,1} \tag{6.61}$$

every $f(\cdot,t) \in L^2(\mathbb{R}^n)$, $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ defines a continuous linear functional on $H^{\psi,1}(\mathbb{R}^n)$,$t$ being fixed. Moreover, Theorem 6.21 and Theorem 6.22 implies

$$|B_\lambda(t; u, \varphi)| \leq \|u(\cdot,t)\|_{\psi,1} \|\varphi(\cdot,t)\|_{\psi,1}$$

and

$$|B_\lambda(t; u, u)| \geq \text{Re}B_\lambda(t; u, u)) \geq \delta_1 \|u(\cdot,t)\|^2_{\psi,1}, \quad \delta_1 > 0 \quad .$$

Thus, the Lax-Milgram theorem gives the existence of a unique element $u(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$ such that

$$B_\lambda(t; u, \varphi) = (\varphi(\cdot,t), f(\cdot,t))_0$$

holds for $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ with $\varphi(\cdot,t) \in C_0^\infty(\mathbb{R}^n)$, which proves the theorem.

□

We want to prove that the unique variational solution constructed in Theorem 6.25 has more regularity properties.

**Theorem 6.26.** *Let $q(x,t,D) = q_1(t,D) + q_2(x,t,D)$ and $u(\cdot,t) \in H^{\psi,1}(\mathbb{R}^n)$, $f(\cdot,t) \in L^2(\mathbb{R}^n)$ be as Theorem 6.25 and assume Theorem 6.17 with $s_1 = 0$. Then $u(\cdot,t)$ belongs to $H^{\psi,2}(\mathbb{R}^n)$.*

*Proof.* Denote by $J_\varepsilon, \varepsilon \in (0,1]$, the Friedrichs mollifier and fixed $t \geq 0$. Further let $(u_k)_{k\in\mathbb{N}}$ be a sequence of function $u_k : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{C}$ such that $u_k(\cdot,t) \in C_0^\infty(\mathbb{R}^n)$. Assume that $(u_k(\cdot,t))_k \in \mathbb{N}$ converges in $H^{\psi,1}(\mathbb{R}^n)$ to $u(\cdot,t)$. It follows that

$$\begin{aligned}
& B_\lambda(t; J_\varepsilon(u_k), \varphi) = (q(x,t,D)J_\varepsilon(u_k)(\cdot,t) + \lambda J_\varepsilon(u_k)(\cdot,t), \varphi(\cdot,t))_0 \\
= \; & (J_\varepsilon((q(x,t,D) + \lambda)u_k(\cdot,t)), \varphi(\cdot,t))_0 - ([J_\varepsilon, q(x,t,D)]u_k(\cdot,t), \varphi(\cdot,t))_0 \\
= \; & B_\lambda(t; u_k, J_\varepsilon(\varphi)) - ([J_\varepsilon, q_2(x,t,D)]u_k(\cdot,t), \varphi(\cdot,t))_0 \quad .
\end{aligned}$$

From Theorem 6.16, we obtain (for $k$ large)

$$\|[J_\varepsilon, q_2(x,t,D)]u_k(\cdot,t)\|_0 \leq c\|u_k(\cdot,t)\|_{\psi,1} \leq \widetilde{c}\|u(\cdot,t)\|_{\psi,1}$$

implying that $[J_\varepsilon, q_2(x,t,D)]u_k(\cdot,t) \to w_\varepsilon(\cdot,t)$ in $L^2(\mathbb{R}^n)$ for some $w_\varepsilon(\cdot,t) \in L^2(\mathbb{R}^n)$ and $\|w_\varepsilon(\cdot,t)\|_0 \leq \widetilde{c}\|u(\cdot,t)\|_{\psi,1}$ for $\varepsilon \in (0,1]$ with $\widetilde{c}$ independent of $\varepsilon$.

Thus for $k \to \infty$, we obtain

$$\begin{aligned} B_\lambda(t; J_\varepsilon(u), \varphi) &= B_\lambda(t; u, J_\varepsilon(\varphi)) - (w_\varepsilon(\cdot,t), \varphi)_0 \\ &= (J_\varepsilon(\varphi)(\cdot,t), f(\cdot,t))_0 - (w_\varepsilon(\cdot,t), \varphi(\cdot,t))_0 \\ &= (\varphi(\cdot,t), J_\varepsilon(f)(\cdot,t))_0 - (w_\varepsilon(\cdot,t), \varphi(\cdot,t))_0 \quad . \end{aligned}$$

It follows that

$$\|q_\lambda(x,t,D) J_\varepsilon(u)(\cdot,t)\|_0 \le \|J_\varepsilon(f)(\cdot,t)\|_0 + \|w_\varepsilon(\cdot,t)\|_0 \le \|f(\cdot,t)\|_0 + \widetilde{c}\|u(\cdot,t)\|_{\psi,1}$$

or

$$\|q_\lambda(x,t,D) J_\varepsilon(u)(\cdot,t)\|_0 \le |\lambda| \|u(\cdot,t)\|_0 + \|f(\cdot,t)\|_0 + \widetilde{c}\|u(\cdot,t)\|_{\psi,1}$$

which implies by Theorem 6.13 that $\|J_\varepsilon(u)(\cdot,t)\|_{\psi,2} \le c$ for all $\varepsilon \in (0,1]$ with $c$ independent of $\varepsilon \in (0,1]$. Thus, we have $u(\cdot,t) \in H^{\psi,2}(\mathbb{R}^n)$ by Proposition 6.15.

□

Combining the results for $B(t,\cdot,\cdot)$ with Theorem 6.17, we get

**Theorem 6.27.** *Suppose that $q_1(t,\xi)$ and $q_2(x,t,\xi)$ satisfy A.1. and A.2.m.with $m \ge [s_2] + n + 2$, $s_2 \ge 0$. Moreover assume* (6.34) *as well as* (6.56) *(with $s_2$ instead of $s$). If $\lambda \ge \widetilde{\lambda}_0$, $\widetilde{\lambda}_0$ taken from* (6.51), *and $f(\cdot,t) \in H^{\psi,s_2}(\mathbb{R}^n)$, then there exists a unique variational solution $u(\cdot,t) \in H^{\psi,s_2+2}(\mathbb{R}^n)$ to* (6.59).

Note that Theorem 6.27 tells that in particular every variational solution to (6.59) is already a strong solution.

# Part VI

# The Operator $-q(x, t_0, D)$ as Generator of a Feller Semigroup & Fundamental Solutions for $\frac{\partial}{\partial t} - q_\lambda(x, t, D)$

## 7 The Operator $-q(x, t_0, D)$ as Generator of a Feller Semigroup

Let us assume that $q(x, t, \xi)$ satisfies the conditions of Theorem 6.27 and **fix** $t = t_0 \geq 0$. In this case, It follows from conditions in [15] , see also [11], that $-q(x, t_0, D)$ extends to a generator of a Feller semigroup. Note that in the following we let $-q(x, t_0, D)$ act only on functions depending on $x$. We will summarize the arguments leading to the above mentioned result.

Our aim is to use the Hille-Yosida-Ray theorem to get the Feller semigroup generated by $-q(x, t_0, D)$, $t_0 \geq 0$ being fixed.

The three conditions of the Hille-Yosida-Ray Theorem are the following (compare Theorem 4.26) :

(i) The domain of $-q(x, t_0, D)$ is dense in $C_\infty(\mathbb{R}^n)$ when considered as operator in $C_\infty(\mathbb{R}^n)$;

(ii) $-q(x, t_0, D)$ satisfies the positive maximum principle on its domain;

(iii) $R(\lambda + q(x, t_0, D))$ is dense in $C_\infty(\mathbb{R}^n)$ .

As shown in [14] the operator $-q(x, t_0, D)$ satisfies on $C_0^\infty(\mathbb{R}^n)$ the positive maximum principle. However, solving $(\lambda + q(x, t_0, D))u = f$ for a dense set in $C_\infty(\mathbb{R}^n)$ such that $u \in C_0^\infty(\mathbb{R}^n)$ is hard to attack. In view of Theorem 6.27 we want to use as domain for $-q(x, t_0, D)$ same space $H^{\psi,s+2}(\mathbb{R}^n)$ with $s$ such that $H^{\psi,s}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$. This however requires us to show that $-q(x, t_0, D)$ satisfies the positive maximum principle also on this larger domain. Here we just quote the corresponding results from [15] or [11] which we can apply to $-q(x, t_0, D)$.

**Theorem 7.1.** *Let $D(A) \subset C_\infty(\mathbb{R}^n; \mathbb{R})$ and suppose that $A : D(A) \to C_\infty(\mathbb{R}^n)$ is a linear operator. In addition assume that $C_0^\infty(\mathbb{R}^n) \subset D(A)$ is an operator core of $A$ in the sense that to every $u \in D(A)$ there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}, \varphi_k \in C_0^\infty(\mathbb{R}^n)$, such that*

$$\lim_{k\to\infty} \|\varphi_k - u\|_\infty = \lim_{k\to\infty} \|A\varphi_k - Au\|_\infty = 0.$$

*If $A|_{C_0^\infty}$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$, then it satisfies the positive maximum principle also on $D(A)$.*

Now that let us come back to equation

$$q_\lambda(x,t_0,D)u(x,\cdot) = q(x,t_0,D)u(x,\cdot) + \lambda u(x,\cdot) = f \tag{7.1}$$

For solving this we will not use the positive maximum principle, hence we may consider complex-valued functions, i.e. we may work in spaces of complex-valued functions.

Let us suppose that we may extend $q_\lambda(x,t_0,D)$ to some space $H^{\psi,s_1}(\mathbb{R}^n)$ where $\psi$ is a fixed real-valued continuous negative definite function.

In addition, suppose that $\psi$ satisfies

$$\psi(\xi) \geq c_0|\xi|^{r_0} \tag{7.2}$$

for some $c_0 > 0$, $r_0 > 0$ and all $\xi$.

Then it follows from the Sobolev embedding result that for $s > \frac{n}{r_0}$

$$H^{\psi,s}(\mathbb{R}^n) \hookrightarrow C_\infty(\mathbb{R}^n) \tag{7.3}$$

and

$$\|u\|_\infty \leq c_{s,r_0,n}\|u\|_{\psi,s} \tag{7.4}$$

holds.

Thus assuming (7.2), i.e. $\psi(\xi) \geq c_0|\xi|^{r_0}$ and having in mind some properties of the operators considered in Part IV , we may consider operators $q_\lambda(x,t_0,D)$ for $t_0 \geq 0$ fixed on $D(q(x,t_0,D)) = H^{\psi,s+2}(\mathbb{R}^n)$

$$q_\lambda(x,t_0,D) : H^{\psi,s+2}(\mathbb{R}^n) \to H^{\psi,s}(\mathbb{R}^n), \tag{7.5}$$

and

$$\|q_\lambda(x,t_0,D)u(x,\cdot)\|_{\psi,s} \leq c\|u\|_{\psi,s+2} \tag{7.6}$$

Since now we have $H^{\psi,s+2}(\mathbb{R}^n) \subset H^{\psi,s}(\mathbb{R}^n) \subset C_\infty(\mathbb{R}^n)$, it follows that $q_\lambda(x,t_0,D)$, i.e. $-q_\lambda(x,t_0,D)$ with domain $H^{\psi,s+2}(\mathbb{R}^n)$ is a densely defined operator on $C_\infty(\mathbb{R}^n)$.

Moreover by Theorem 7.1 , the operator $-q_\lambda(x,t_0,D)$ satisfies on $H^{\psi,s+2}(\mathbb{R}^n)$ also the positive maximum principle.

Thus by now,we have reduced an application of the Hille-Yosida-Ray Theorem to solving the equation $q_\lambda(x,t_0,D)u(x,\cdot) = f$, in the space $H^{\psi,s+2}(\mathbb{R}^n)$ for (all) $f \in H^{\psi,s}(\mathbb{R}^n)$

We may now work in the scale of Hilbert space $H^{\psi,s}(\mathbb{R}^n)$ and may consider first variational solution.

Suppose that for some $\lambda \geq 0$ there exists for all $f \in L^2(\mathbb{R}^n)$, a variational solution $u \in H^{\psi,1}(\mathbb{R}^n)$ to (7.1) , i.e. we may extend the sesquilinear form

$$B_\lambda(t_0, u, \varphi) := (q(x, t_0, D)u(x, \cdot), \varphi(x, \cdot))_0 + \lambda(u(x, \cdot), \varphi(x, \cdot))_0 \tag{7.7}$$

to $H^{\psi,1}(\mathbb{R}^n)$ and that for any $f \in L^2(\mathbb{R}^n)$ there exists $u \in H^{\psi,1}(\mathbb{R}^n)$ such that

$$B_\lambda(t_0, u, \varphi) = (\varphi, f)_0 \tag{7.8}$$

holds for all $\varphi \in H^{\psi,1}(\mathbb{R}^n)$.

Thus in the scale $H^{\psi,s}(\mathbb{R}^n)$: we need to prove that $f \in H^{\psi,s}(\mathbb{R}^n)$ always implies for a variational solution $u \in H^{\psi,1}(\mathbb{R}^n)$ that $u \in H^{\psi,s+2}(\mathbb{R}^n)$.

We will consider the operator handled in Part IV . Let $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a fixed continuous negative definite function satisfying

$$\psi(\xi) \geq c_0|\xi|^{r_0} \tag{7.9}$$

for some $c_0 > 0$, $r_0 > 0$ and all $|\xi| \geq 1$.

Further recall the definition of $\widetilde{c}_{n,k}, \widetilde{\gamma}_{m,n}$ and $c_\psi$ given in (6.5)- (6.7).

We want to apply Theorem 6.27 with $s_2$ such that

$$H^{\psi,s}(\mathbb{R}^n) \longrightarrow C_\infty(\mathbb{R}^n)$$

holds.

Thus, we have to take $s > \dfrac{n}{r_0}$, say

$$s := \left[\frac{n}{r_0}\right] + 1. \tag{7.10}$$

**Assumption 7.2.** *We assume that the function $q : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{C}$ is a continuous negative symbol having the decomposition*

$$q(x, t, \xi) = q_1(t, \xi) + q_2(x, t, \xi) \tag{7.11}$$

*and fix $t \geq 0$ with $\psi$ as in (7.9), the following conditions are assumed: A.1. The function $q_1$ is assumed to be continuous and negative definite and to satisfy with $\gamma_0, \gamma_1 > 0$*

$$\gamma_0\psi(\xi) \leq \mathrm{Re} q_1(t, \xi) \leq \gamma_1\psi(\xi), \quad \textit{for all } |\xi| \geq 1 \tag{7.12}$$

*and $|\mathrm{Im} q_1(t, \xi)| \leq \gamma_0 \mathrm{Re} q_1(t, \xi)$, for all $\xi \in \mathbb{R}^n$.*

*A.2.$m_0$ Set* $m_0 = s + n + 2 = \left[\frac{n}{r_0}\right] + n + 3$, *note* $s \geq 1$. *We assume that* $x \mapsto q_2(x,t,\xi)$ *belongs to* $C^{m_0}(\mathbb{R}^n)$ *for all* $\xi \in \mathbb{R}^n$ *and we have the estimate*

$$|\partial_x^\alpha q_2(x,t,\xi)| \leq \varphi_\alpha(x)(1+\varphi(\xi)) \tag{7.13}$$

*for all* $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m_0$ *with function* $\varphi_\alpha \in L^1(\mathbb{R}^n)$.

*A.3.$m_0$. With* $k_2$ *from* (6.54) *and* $\widetilde{c}_{n,m_0,\frac{s}{2},\psi}$ *from* (6.27), *we require*

$$\sum_{|\alpha|\leq m_0} \|\varphi_\alpha\|_{L^1} \leq \frac{1}{4}\gamma_0 \left(\frac{1}{k_2} \bigwedge \frac{1}{\widetilde{c}_{n,m_0,\frac{s}{2},\psi}}\right) \tag{7.14}$$

*Note that A.3.$m_0$. implies that* (6.35) *holds with* $\delta_0 = \frac{1}{4}\gamma_0$ *and* (6.57) *holds with* $\delta_1 = \frac{1}{4}\gamma_0$, *too.*

**Theorem 7.3.** *Suppose that Assumption 7.2 holds with* $s = \left[\frac{n}{r_0}\right] + 1$. *Then* $-q(x,t_0,D)$, $t_0 \geq 0$ *fixed, extends to a generator of a Feller semigroup.*

*Proof.* Consider the operator $(A, D(A))$ on $C_\infty(\mathbb{R}^n)$ with domain $D(A) = H^{\psi,s+2}(\mathbb{R}^n;)$ and $A = -q(x,t_0,D)$. It follow by Proposition 6.6 and Theorem 6.11 that $Au \in H^{\psi,s}(\mathbb{R}^n)$ if $u \in D(A)$, and therefore $Au \in C_\infty(\mathbb{R}^n)$ a dense domain. Moreover, $(-q(x,t_0,D), H^{\psi,s+2}(\mathbb{R}^n))$ satisfies the positive maximum principle.

Further we may apply Theorem 6.27, thus for $\lambda \geq \widetilde{\lambda}_0$, $\widetilde{\lambda}_0$ taken from (6.43), we find for every $f \in H^{\psi,s}(\mathbb{R}^n)$ a unique $u \in H^{\psi,s+2}(\mathbb{R}^n)$ satisfying

$$(A-\lambda)u(x,\cdot) = -q_\lambda(x,t_0,D)u(x,\cdot) = f$$

The theorem follows now by the Hille-Yosida-Ray Theorem.

□

# 8 Fundamental Solution for $\frac{\partial}{\partial t} - q_\lambda(x,t,D)$.

In Part VI - Section 7 , we focus on pseudo-differential operators with time dependent negative definite symbols and finally constructing Feller semigroups by extending the operator $-q(x,t_0,D)$ to a generator of such a semigroup. The next step in this part is to use our estimates in Part VI - Section 7 in order to prove that $-q_\lambda(x,t_0,D)$, $\lambda$ is sufficiently large and $-q_\lambda(x,t_0,D) = A-\lambda = -q(x,t_0,D)-\lambda$, extends to a generator of an $L^2$-sub-Markovian semigroup. Then applying Theorem 5.15, 5.16 and the results of Part IV , we may try to construct a fundamental solution in a $L^2$-context to the following parabolic problem:

$$\frac{\partial u(x,t)}{\partial t} + q(x,t,D)u(x,t) = 0 \qquad \textit{and} \quad u(x,0) = f(x). \tag{8.1}$$

We will start by introducing the definitions and theorems of sub-Markovian semigroups on $L^p(\mathbb{R}^n)$ which we collect from [14] . By definition, see Definition 4.6, a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ is called sub-Markovian if for all $u \in L^p(\mathbb{R}^n)$ such that $0 \leq u \leq 1$ almost everywhere it follows that $0 \leq T_t u \leq 1$ almost everywhere. In general we give the following definition:

**Definition 8.1.** *A. A linear bounded operator $S : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is called* ***sub-Markovian****, whenever*

$$0 \leq u \leq 1 \qquad a.e. \quad implies \quad 0 \leq Su \leq 1 \quad a.e. \tag{8.2}$$

*B. A linear , bounded operator $S : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $a \leq p < \infty$, is called* ***positivity preserving****, if*

$$0 \leq u \qquad a.e. \quad implies \quad 0 \leq Su \quad a.e. \tag{8.3}$$

**Remark 8.2.** *A strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ is sub-Markovian when each of the operators $T_t$, $t \geq 0$, is sub-Markovian. Moreover, we call a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ positivity preserving if each of the operators $T_t$, $t \geq 0$ is positivity preserving.*

**Lemma 8.3.** *Let $S : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ be a sub-Markovian operator. Then $S$ is positivity preserving.*

**Corollary 8.4.** *Any sub-Markovian semigroup on $L^p(\mathbb{R}^n)$ is also a positivity preserving semigroup on $L^p(\mathbb{R}^n)$.*

Let $(T_t)_{t\geq 0}$ be a sub-Markovian semigroup on $L^p(\mathbb{R}^n)$ and consider its resolvent $(R_\lambda)_{\lambda>0}$, i.e., the family of operators

$$R_\lambda u = \int_0^\infty e^{-\lambda t} T_t u \, dt \ . \tag{8.4}$$

Suppose that $0 \leq u \leq 1$ a.e. The sub-Markovian character of $T_t$ gives

$$0 \leq R_\lambda u \leq \int_0^\infty e^{-\lambda t} \, dt \leq \frac{1}{\lambda} \ , \tag{8.5}$$

or

$$0 \leq u \leq 1 \quad a.e. \quad implies \quad 0 \leq \lambda R_\lambda u \leq 1 \quad a.e. \tag{8.6}$$

Moreover, (8.4) implies

$$\|R_\lambda u\|_{L^p} \leq \int_0^\infty e^{-\lambda t} \|T_t u\|_{L^p} \, dt \leq \frac{1}{\lambda} \|u\|_{L^p} \ ,$$

i.e. $\lambda R_\lambda$ is a contraction on $L^p(\mathbb{R}^n)$.

**Definition 8.5.** *A resolvent $(R_\lambda)_{\lambda>0}$ on $L^p(\mathbb{R}^n)$ corresponding to a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ is called a sub-Markovian resolvent whenever* (8.6) *holds. It is called positivity preserving whenever $R_\lambda$ is for all $\lambda > 0$ a positivity preserving operator.*

**Remark 8.6.** *The resolvent of a sub-Markovian semigroup is sub-Markovian and that of a positivity preserving semigroup is positivity preserving too, which follows immediately from* (8.4).

**Lemma 8.7.** *A strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ is sub-Markovian if and only if its resolvent is sub-Markovian, and $(T_t)_{t\geq 0}$ is positivity preserving if and only if its resolvent is positivity preserving.*

**Lemma 8.8.** *Let $(T_t)_{t\geq 0}$ be a sub-Markovian semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with generator $(A, D(A))$. Then for all $u \in D(A) \subset L^p(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} (Au)((u-1)^+)^{p-1} \, dx \ = \ \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^n} (T_t u - u)((u-1)^+)^{p-1} \, dx \leq 0. \tag{8.7}$$

**Definition 8.9.** *A closed, densely defined linear operator $A : D(A) \to L^p(\mathbb{R}^n)$, $1 < p < \infty$, $D(A) \subset L^p(\mathbb{R}^n)$, is called a* ***Dirichlet operator*** *if for all $u \in D(A)$ the relation* (8.7) *holds.*

**Remark 8.10.** *The notation of a Dirichlet operator was introduced by N.Bouleau and F.Hirsch in [3] for self-adjoint operators on* $L^2(\mathbb{R}^n)$.

**Proposition 8.11.** *Suppose that a Dirchlet operator* $(A, D(A))$ *on* $L^p(\mathbb{R}^n)$ *with* $1 < p < \infty$, *generates a strongly continuous contraction semigroup* $(T_t)_{t\geq 0}$ *on* $L^p(\mathbb{R}^n)$ *with corresponding resolvent* $(R_\lambda)_{\lambda>0}$. *Then* $(T_t)_{t\geq 0}$ *and* $(R_\lambda)_\lambda > 0$ *are sub-Markovian.*

**Theorem 8.12.** *Let* $A$ *be a Dirichlet operator on* $L^p(\mathbb{R}^n)$, $1 < p < \infty$, *with the property that* $R(\lambda id - A) = L^p(\mathbb{R}^n)$ *for some* $\lambda > 0$. *Then* $A$ *generates a sub-Markovian semigroup on* $L^p(\mathbb{R}^n)$.

**Theorem 8.13.** *Let* $(A, D(A))$ *be a densely defined operator on* $L^p(\mathbb{R}^n)$, $1 < p < \infty$, *such that* (8.7) *holds for all* $u \in D(A)$ *and assume that for some* $\lambda > 0$ *we have* $\overline{R(\lambda id - A)} = L^p(\mathbb{R}^n)$. *Then* $A$ *is closable and its closure generates a sub-Markovian semigroup on* $L^p(\mathbb{R}^n)$.

We want to investigate the relation between generators of Feller semigroups and generators of sub-Markovian semigroups. For this we note first that whenever $(T_t^{(\infty)})_{t\geq 0}$, namely the semigroups on $C_\infty(\mathbb{R}^n)$, is a Feller semigroup, then for $u \in C_\infty(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$ such that $0 \leq u(x) \leq 1$ it follows that $0 \leq T_t^{(\infty)} u(x) \leq 1$ for all $x \in \mathbb{R}^n$.

**Theorem 8.14.** *Let* $A^{(\infty)} : D(A^{(\infty)} \to C_\infty(\mathbb{R}^n)$, $D(A^{(\infty)} \subset C_\infty(\mathbb{R}^n)$, *be the generator of a Feller semigroup* $(T_t^{(\infty)})_{t\geq 0}$. *Moreover, suppose that* $U \subset D(A^{(\infty)})$ *is a dense subspace of* $L^p(\mathbb{R}^n)$, $1 < p < \infty$. *If* $A^{(\infty)}|_U$ *extends to a generator* $A^{(p)}$ *of a strongly continuous contraction semigroup* $(T_t^{(p)})_{t\geq 0}$ *on* $L^p(\mathbb{R}^n)$ *for which* $V := (\lambda - A^{(p)})^{-1} U$ *is an operator core, then* $(A^{(p)}, D(A^{(p)}))$ *is a Dirichlet operator on* $L^p(\mathbb{R}^n)$ *and the semigroup* $(T_t^{(p)})_{t\geq 0}$ *is sub-Markovian.*

The proof is given in [14].

**Remark 8.15.** *A. Theorem 8.14 says in particular that operators defined on the space* $C_0^\infty(\mathbb{R}^n)$ *satisfying the positive maximum principle are also candidates for pre-generators of sub-Markovian semigroups. B. In case that* $p = 2$, *Theorem 8.14 was proved in [12], the general case in [13].*

Now let us come back to $-q_\lambda(x, t_0, D)$. The main idea is to prove that $-q_\lambda(x, t_0, D)$ extends to a generator of an $L^2$-contraction semigroup and then to apply Theorem 8.14.

**Theorem 8.16.** *Under the assumption Theorem 6.25 and* (6.35) *with* $s = 2$ *the operator* $-q_\lambda(x, t_0, D)$ *defined on* $H^{\psi,2}(\mathbb{R}^n)$ *is for* $\lambda \geq \tilde{\lambda}_0$, $\tilde{\lambda}_0$ *from* (6.52), *a generator of a strongly continuous contraction semigroup on* $L^2(\mathbb{R}^n)$.

*Proof.* We will apply the Hille-Yosida theorem, Theorem 6.23. From Part IV we know that $(-q_\lambda(x,t_0,D), H^{\psi,2}(\mathbb{R}^2))$ is a densely defined closed operator on $L^2(\mathbb{R}^n)$ and for $\lambda \geq \tilde{\lambda}_0$ the equation $q_\lambda(x,t_0,D)u(x,\cdot) = f$ is uniquely solvable for all $f \in L^2(\mathbb{R}^n)$ with a solution $u \in H^{\psi,2}(\mathbb{R}^n)$ by Theorem 6.26. It remains to prove that $-q_\lambda(x,t_0,D)$, $\lambda \geq \tilde{\lambda}_0$ is dissipative. But for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$(q_\lambda(x,t_0,D)\varphi,\varphi)_0 \geq 0$$

which extends to all $u \in H^{\psi,2}(\mathbb{R}^n)$. Now the dissipativity of $-q_\lambda(x,t_0,D)$, $\lambda \geq \tilde{\lambda}_0$, follows from

$$\|\tau u + q_\lambda(x,t_0,D)u\|_0^2 = \tau^2\|u\|_0^2 + \|q_\lambda(x,t_0,D)\|_0^2 + 2\tau(u, q_\lambda(x,t_0,D)u)_0$$

and the theorem is proved.

□

**Corollary 8.17.** *Under the assumption of Theorem 8.16 and Assumption 7.2 with* $t_0 = [\frac{n}{r_0}] + 3$ *the operator* $(-q_\lambda(x,t_0,D), H^{\psi,2}(\mathbb{R}^n))$ *is for all* $\lambda \geq \tilde{\lambda}_0$ *a Dirichlet operator and generates an* $L^2$*-sub-Markovian semigroup.*

*Proof.* We only have to apply Theorem 8.14 for $p = 2$ with $U = H^{\psi,2}(\mathbb{R}^n) \subset D(A^{(\infty)})$ and to note that $[q_\lambda(x,t_0,D)]^{-1}(H^{\psi,t_0+2}(\mathbb{R}^2)) = H^{\psi,t_0+4}(\mathbb{R}^n)$.

□

In the remaining of this section let us come back to the parabolic equation (8.1).

1. By Definition 4.30, 4.34 , Proposition 4.33 and Theorem 4.34, we can say that $(T_t)_{t\geq 0}$ is analytic on $L^2(\mathbb{R}^n)$.
2. From Part V , we find the t-dependent sesquilinear form $B_\lambda(t;u,v)$ satisfies (5.29) and (5.30) .
3. $-q_\lambda(x,t_0,D) = A - \lambda = -q(x,t_0,D) - \lambda$ extends to a generator of an $L^2$-sub-Markovian semigroup.

**Theorem 8.18.** *Suppose that all assumptions on* $q_\lambda(x,t,\xi)$ *such that analogous inequalities to* (5.29) *and* (5.30) *hold. Add* (5.31) *as extra assumption ,i.e.*

$$|B_\lambda(t;u,v) - B_\lambda(s;u,v)| \leq K|t-s|^\alpha \|u\|\|v\| \qquad t \in [0,T],\ \alpha \in (\frac{1}{2},1]\ . \tag{8.8}$$

*Then there exists a fundamental solution* $U(t,s)$ *to* (8.1).

**Remark 8.19.** *If for* $q_1$ *and* $q_2$,

$$|q_1(t,\xi) - q_1(s,\xi)| \leq C|t-s|^\alpha(1+\psi(\xi)) \tag{8.9}$$

*and*

$$|\partial_x^\alpha q_2(x,t,\xi) - \partial_x^\alpha q_2(x,s,\xi)| \leq C|t-s|^\alpha \varphi_\alpha(x)(1+\psi(\xi)). \tag{8.10}$$

*Then extra assumption* (8.8) *holds, which is seen by look to the proof of Proposition 6.18 and Proposition 6.20.*

## Part VII

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