Perturbations of an expanding brane.

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Perturbations of an Expanding Brane

By

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To my parents

Abstract

Motivated by the study of the cosmology of brane-worlds we begin by inducing vacuum decay in a 4+1 space-time assuming that the most likely configuration possesses spherical symmetry, and interpret the wall between the two vacua as a thick brane. We then set out to deduce the spectrum of perturbations about this solution with a view to making predictions about the CMB power spectrum of such a brane. However, we find that the instability of the spherically symmetric solution causes focusing of energy at the origin, which we suggest would result in black hole formation.

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First and foremost I would like to thank my parents for their continual support and understanding throughout this and all my endeavours – I'm sure I don't know how lucky I am.

Most of my friends heard about my struggles at some point or other, so I thank you all, but I will mention by name (in no particular order!) those that got the worst of it, and encouraged me to continue: Avtar, for fruitful discussions; Helen, you were my rock from the very beginning; Titti, for never giving up on my behalf; Steve, for keeping me sane in the office (but driving me nuts by leaving tunes in my head); Robin, whose thirst for knowledge put food on my table; and Daniel, for taking an interest!

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Apologies to anyone I have missed – it means only that I took you for granted, as I hope you will do me.

> I'll take your brain to another dimension.
> I'll take your brain to another dimension.
> I'll take your brain to another dimension.
> Pay close attention.
>
> *Out of Space*
> THE PRODIGY

Contents

Chapter 1

Introduction

1.1 High Energy Physics

One of the most ambitious aims of modern physics is to unify the fundamental forces into one theory. The Standard Model successfully unifies electromagnetism, and the strong and weak nuclear forces, but does not include gravity – although this is not a problem in practice since gravitational effects are small enough to be negligible, it should be possible to include gravity in principle. To unify gravity with the other forces, one needs not only a quantum theory of gravity, but to address the hierarchy problem too.

The hierarchy problem is basically that the range of physical values involved in the theory is very large. Although in quantum field theory particles interact by exchanging virtual particles, the low energy limit must be consistent with the classical expression of the forces, so taking gravity as the example, we may consider the Newtonian potential: the potential energy of two masses, m_1 and m_2 separated by a distance r is given by

$$V(r) = -\frac{G_{\mathrm{N}}\, m_1\, m_2}{r} \tag{1.1}$$

where G_{N} is Newton's constant, and is the physical value that determines the strength of the gravitational force. It is the extremely small value of G_{N},[1]

$$G_{\mathrm{N}} \sim (1.2 \times 10^{19}\,\mathrm{GeV})^{-2}$$

that makes gravity so weak.[2] In relation to this value is a mass, which characterizes the force, known as the Planck mass:

$$M_{\mathrm{Pl}} = G_{\mathrm{N}}^{-1/2} \sim 1.2 \times 10^{19}\,\mathrm{GeV}$$

The corresponding mass for the electroweak interaction is

$$M_{EW} \sim 10^{3}\,\mathrm{GeV}$$

and so we see that at these energies corrections due to gravity are negligible. It is this huge range of scales, $M_{\mathrm{Pl}}/M_{EW} \sim 10^{16}$ that comprises the hierarchy problem.

String Theory

String theory is a very popular (although not yet falsifiable) area of research seen as a possible route to a quantum theory of gravity. The basic idea is that instead of being point-like, particles are represented by (vibrational states of) small (possibly Planck length, $\sim 10^{-35}$m) one-dimensional 'strings', which can be open(-ended) or closed (loops). A promising feature of string theory is that one of the string states automatically corresponds to the graviton – the predicted mediator particle of the gravitational force. In fact, string theory developed beyond one-dimensional objects to general p-dimensional ones, called p-branes, so that the original strings are 1-branes, and 2-branes are 2-dimensional surfaces – membranes, where the name comes from.

Although string theory does not solve the hierarchy problem on its own – other mechanisms are needed for that such as supersymmetry (the proposal that fermions and bosons come in 'supersymmetric' pairs) resulting in 'superstring theory' where

[1] We work throughout in units of $c = \hbar = 1$.

[2] Of course, on the large scale mass accumulates, leading to large gravitational forces, but then quantum effects are negligible.

the Standard Model particles are revealed in some low energy limit – it has many interesting features, of which one is its requirement of extra dimensions.

There are five types of string theory, Types I, IIA, IIB, and the two heterotic types, $SO(32)$ and $E_8 \times E_8$, which despite their differences all have one thing in common: they require the existence of ten space-time dimensions to be consistent theories. This is a very novel feature of a theory, since most need to have the number of dimensions they act in entered 'by hand'. In fact, these different theories demonstrate mathematical relations, known as dualities, so much so that Edward Witten [1] proposed the concept of an 'M-theory' – a fundamental eleven-dimensional theory of which the five ten-dimensional theories are different limits.

Extra dimensions

That string theory requires ten dimensions of space-time is obviously in stark contrast to the four we are otherwise used to considering. This led to the revival of the idea of compactified dimensions such as in Kaluza-Klein theory. In Kaluza-Klein theory [2, 3] there is an extra dimension, which is compact and very small (Planck length), so small that it would not have been noticed, at least at the energy scales (or equivalently, distances) probed to date.

The possibility of extra dimensions has led to many ideas of how to reconcile the hierarchy problem. Arkani-Hamed et al [4] proposed that large extra dimensions (large compared to the Planck length, as in Kaluza-Klein scenarios) could eliminate the hierarchy problem by explaining the weakness of gravity by the fact that it is 'diluted' by the extra dimensions. The underlying premise is that gravity spreads out so that its strength is proportional to the 'area' covered at that distance, so that in d spatial dimensions (of infinite extent) the gravitational force is given by

$$F_d(r) \propto \frac{1}{M_{Pl(d)}^{d-1} r^{d-1}}$$

(where $M_{Pl(d)}$ is the d-dimensional Planck mass) and thus our usual three dimensions of space lead to the $1/r^2$ law. But what if there were in addition some extra compactified dimensions, i.e. dimensions whose extent is finite? Consider the case of $n+3$ spatial dimensions where the extra n dimensions have size R. At distances $r \ll R$ the force is spread over all $n+3$ dimensions so that we find

$$F_{n+3}(r) \propto \frac{1}{M_{Pl(n+3)}^{n+2} r^{n+2}} \quad (r \ll R) \tag{1.2}$$

but at distances $r \gg R$ the 'area' is proportional to $R^n r^2$ and so

$$F_{n+3}(r) \propto \frac{1}{M_{Pl(n+3)}^{n+2} R^n r^2} \quad (r \gg R) \tag{1.3}$$

which we perceive as

$$F_3(r) \propto \frac{1}{M_{Pl(3)}^2 r^2} \tag{1.4}$$

where

$$M_{Pl(3)}^2 = M_{Pl(n+3)}^{n+2} R^n \tag{1.5}$$

i.e. what we thought of as the Planck mass (in three spatial dimensions) is not a fundamental scale but rather a result of the higher dimensional Planck mass (which is fundamental) and the volume of the extra dimensions. For instance, if the fundamental Planck mass is actually of the same order of the electro-weak scale, and $n = 2$ (i.e. there are two extra dimensions) then those dimensions would have size of order a millimetre. This is interesting because unlike electromagnetism, the $1/r^2$ law for gravity has only been verified down to scales of millimetre-order [5,6], which means that deviations such as (1.2) could be consistent with experiments to date. So, if two such extra dimensions exist, then the previously large energy scale of gravity is brought down to that of the other forces, thus eliminating the hierarchy problem, and deviations from Newton's law can be expected at ranges of less than a millimetre.

Of course, the Standard Model interactions must not be allowed to experience the extra dimensions in the same way (since their behaviour at small distances is well-tested) – they must be constrained (in some way) to some small 'thickness' (which could at most be $1/M_{EW}$ since this is now the largest scale in the theory). In fact it has been shown [7] that this scenario can be explained using string theory.

In type II string theories, open strings must have one of their ends attached to a special type of p-brane, called a D-brane. These open strings describe gauge theories, so we say that gauge theories 'live' on the D-branes. Conversely, since gravitons are closed strings, they can exist throughout the 'bulk' (the space between branes). In fact, since the D-branes can emit and absorb closed strings, they are gravitating objects and thus massive. The essence of [7] is that our universe could be a three-dimensional D-brane on which all the particles and interactions of the Standard Model 'live', but with gravity acting throughout the bulk.

Brane-Worlds

So, treating our universe as a brane is an alternative way (to compactification) of explaining the lack of observational evidence of the extra dimensions predicted by string theory – the observable universe is constrained to some three-dimensional subspace (brane) in the higher dimensional bulk. Such a scenario became very popular in of itself, including as a way to deal with the hierarchy problem.

In contrast to the scenario mentioned already (of a large extra dimension bringing the Planck scale down to that of the electro-weak), a model by Randall and Sundrum used a small (compact) extra dimension to do the opposite, i.e. to bring the electro-weak scale up to the Planck scale [12]. In this model there are two branes at either end of the small dimension (specifically, they are at the fixed points of an S^1/Z_2 orbifold), the branes have tension of opposite sign and gravitationally repel each other, but are stabilized by a negative cosmological constant in the bulk. The bulk metric has a 'warp factor' so that distances on the negative tension brane (which is taken to be our universe) are exponentially smaller than on the other brane, thus explaining why $M_{\text{Pl}} \gg M_{EW}$.

In a variation to this model, Randall and Sundrum took our universe to be the positive tension brane [13]. Although this no longer solves the hierarchy problem, by taking the other brane to infinity and fine-tuning our brane's tension, a very surprising result was achieved: the linearized Einstein equations were found to hold on our brane (with small $1/r^2$ corrections to Newton's potential), i.e. the model could have a *non-compact* (infinite) extra dimension and not contradict observations of gravity. This is because the curved background supports a bound state of the higher dimensional graviton, effectively confining it to a small region within that dimension.

That our universe could contain large extra dimensions and still be consistent with the observed law of gravity sparked much interest in the cosmological properties of such scenarios – for these models to have any chance of being accurate descriptions of our universe, it is not enough just to have the same law of gravity – they must also be consistent with cosmological observations. So first let us review some of the basics of conventional cosmology in section 1.2 before turning to some of the related issues of 'brane cosmology' in section 1.3.

1.2 Standard Cosmology

1.2.1 Cosmological Evolution

The basis for a lot of cosmological theory is the 'Cosmological Principle', which (broadly) is the assumptions that the universe is homogeneous and isotropic. Although the universe is clearly not homogeneous on small scales – people, planets, stars, and so on are obvious density variations, and there appear to be clusters of galaxies and voids on scales up to 50 Mpc – on large scales (e.g. over 200 Mpc) the universe appears smooth. Homogeneity cannot be tested in the strictest sense, but it is a powerful assumption since our own observations are then enough to test the resulting cosmological models. The cosmic microwave background is very strong evidence for isotropy (see section 1.2.2) but even before that was discovered, radio surveys indicated that the distribution of radio sources on the sky is isotropic to a few per cent, which is compelling evidence since radio surveys reach out to great distances (due to the low galactic extinction of radio waves).

Perhaps the most important observation made in cosmology was that of Hubble [16] – that the spectral lines of distant galactic objects were red-shifted in a systematic way, so that by interpreting the red-shift as a Doppler effect, Hubble was able to deduce a relationship between an object's recession velocity and its distance from us – Hubble's law:

$$v = H_0\, d \tag{1.6}$$

where v is the object's velocity, d its distance, and the constant of proportionality H_0 is called Hubble's constant. This suggests that the universe is expanding (in a way consistent with homogeneity), and so the Cosmological Principle is refined to say that the universe is isotropic to comoving observers (i.e. comoving with the expansion). But what is the nature of this expansion?

Friedmann had already found solutions of Einstein's equations for a matter filled dynamically evolving universe, where the evolution was dependent on the spatial curvature of the universe. Later, Robertson and Walker came up with the most general form for the metric of a homogeneous and isotropic universe, now known as the FRW metric:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \left[\frac{\mathrm{d}r^2}{1 - k\,r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \tag{1.7}$$

where $a(t)$ is the scale factor of the universe, and k takes the characteristic values $(-1, 0, 1)$ denoting negative, zero, or positive spatial curvature respectively. In this coordinate system, comoving observers have constant space coordinates, and so the Hubble parameter is given by

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{1.8}$$

(where an overdot denotes a time-derivative), and the previously mentioned 'Hubble constant', H_0, is simply its current value.

By assuming the energy-momentum tensor has the perfect fluid form (consistent with the Cosmological Principle),

$$T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p) \tag{1.9}$$

(where ρ is the energy density and p the pressure) Einstein's equations take the form

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\,G\,\rho}{3} - \frac{k}{a^2} \tag{1.10}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi\,G}{3}\,(\rho + 3p) \tag{1.11}$$

(where H, a, ρ and p are all functions of time and any cosmological constant is included implicitly through ρ and p), the first of which is commonly known as Friedmann's equation. Eliminating $\ddot{a}$ from these equations yields

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \tag{1.12}$$

(sometimes known as the 'energy conservation equation'), so that in combination with the Friedmann equation we have two first order equations which (given an equation of state, i.e. a relation between ρ and p) can be solved for $a(t)$ and $\rho(t)$.

In cosmology, a linear equation of state is often assumed, in general:

$$p = (\gamma - 1)\rho \tag{1.13}$$

so that (1.12) gives

$$\rho \propto \frac{1}{a^{3\gamma}} \tag{1.14}$$

for instance, radiation has $\gamma = 4/3$, and dust has $\gamma = 1$. By considering mixtures of these possibilities, and more general ones altogether, cosmologists are able to model the evolution of the universe, and the various forms of matter it contains. A simple extrapolation of the expansion back in time suggests that in the past the universe was smaller and denser, and thus hotter. The logical conclusion is that the universe was once very small – the 'Big Squeeze' as Gamow called it, or as it later became known, the 'Hot Big Bang' model. This model is of our universe evolving from an initially hot dense state according to the laws of General Relativity (as related by Friedmann's equation etc.). Calculations of particle reactions in the early universe result in predictions of the relative abundance of light elements that we see in the universe today. The fact that the predictions match so well with observation is strong support for the Big Bang theory. Another prediction of the theory is a residual radiation from the early universe, which is now generally agreed to be the origin of the Cosmic Microwave Background.

1.2.2 Cosmic Microwave Background

In the early universe, it was so hot that the average photon energy was enough to ionise any atoms, and so all matter existed as ionised plasma in thermal equilibrium with the photons. But as the universe cooled (past about 3000 K, when the universe was approximately 300,000 years old) even the energy of the most energetic photons was less than the ionisation energy, and so nuclei and electrons could form atoms ('recombination'), and radiation (now 'decoupled') could propagate freely. Since photons are bosons, their characteristic distribution was that of a black-body spectrum. Due to the expansion of the universe, the photons are red-shifted, but one of the features of the black-body spectrum is that it is preserved by the expansion of the universe, so that today we can expect this background radiation to have maintained this characterisitc spectrum, but with a lower temperature.

Indeed, this is just what Penzias and Wilson observed in 1965 [17] – microwave background radiation with an extremely good fit to a black-body spectrum with characterisitc temperature 3 K, uniform in all directions of space, now known as the Cosmic Microwave Background (CMB). The high isotropy of the radiation lends strong support to the Cosmological Principle, but the small *anisotropies* are of just as much significance. The Cosmic Background Explorer satellite (COBE) measured the radiation to have a temperature of $2.726 \pm 0.010\,\mathrm{K}$ [20], and (after subtracting the effect of the Earth's motion through space, and the microwave emissions of our own galaxy) detected small fluctuations of order 1 part in 10^5, as can be seen in figure 1.1. Since the universe is inhomogeneous on the scale of (clusters of) galaxies, we can expect that at the time of decoupling there were small density fluctuations which could grow under the influence of their own gravity, into the structures we observe today. The CMB has these small fluctuations imprinted in it, so by analysing them we can learn about how the structures formed. For instance, the fluctuations are too small to lead to galaxy formation in a purely baryonic universe, which has led to models in which a high percentage of the matter is in some 'dark' (since it is not visible) non-baryonic form. The non-baryonic matter could have decoupled from the radiation before the baryonic matter, and thus fluctuations in its density would not be subject to the same constraints imposed by the CMB anisotropy data.

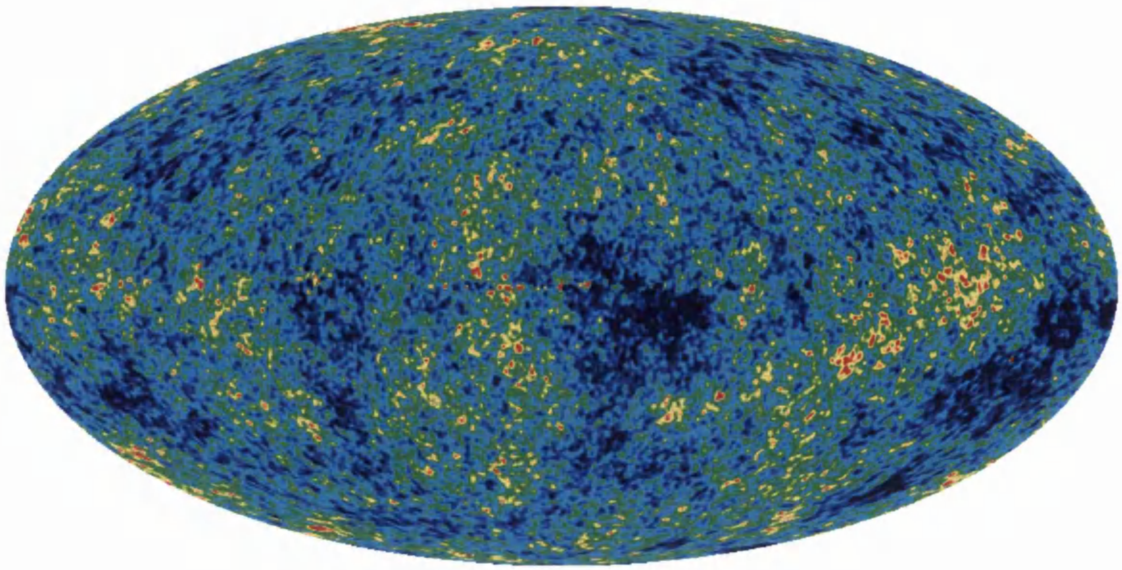

Figure 1.1: CMB map of the sky showing the anisotropies: red indicates warmer and dark blue indicates cooler areas, using a linear scale of $\pm 200\,\mu K$ [21]. Taken from the WMAP Science Team archive [22].

The anisotropies are usually analysed by decomposing them into spherical harmonics:

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi)$$

and defining the power spectrum by

$$C_l = \langle |a_{lm}|^2 \rangle$$

This data is conveyed by a plot of $l(l+1)C_l$ against l as in figure 1.2. From such plots it is possible to extract data, such as various cosmological parameters, or at least constrain relations among them (since there exists degeneracy among the parameters) – one of the most prominent being the matter density of the universe. We can see from (1.10) that a flat universe, $k = 0$, requires a particular value for the current matter density of the universe, known as the critical density:

$$\rho_c = \frac{3H^2}{8\pi G}$$

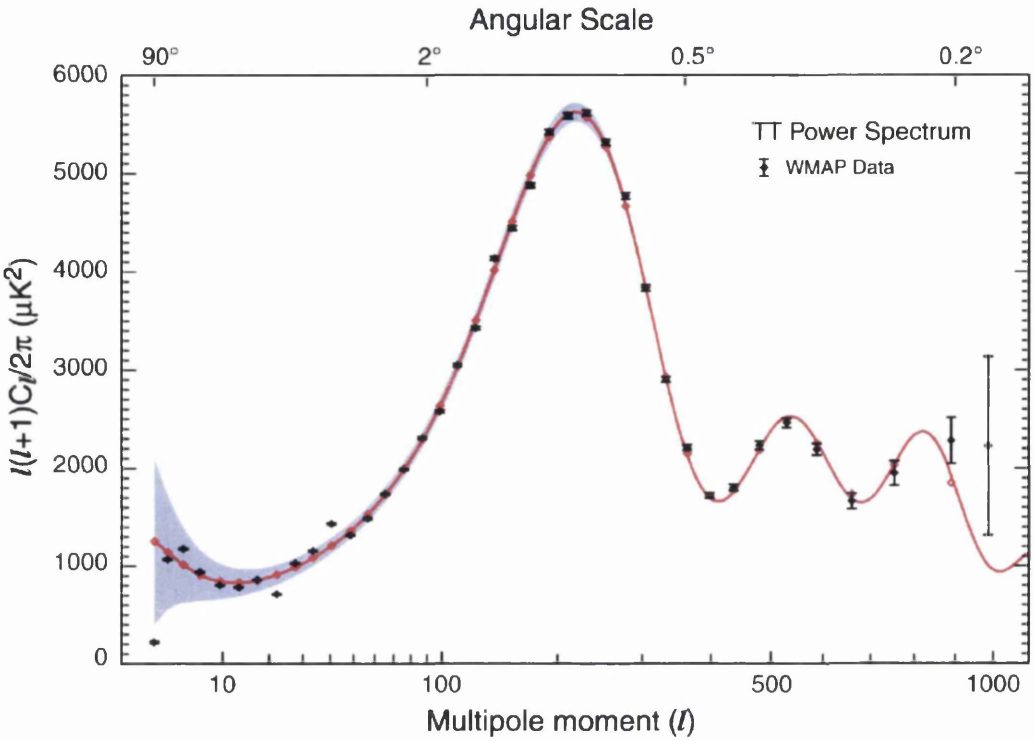

Figure 1.2: The CMB Power Spectrum. The data points are the WMAP 3-year data, the red curve is the best-fit ΛCDM model fit to the data [29], and the shaded region delineates cosmic variance about the model [21]. Taken from the WMAP Science Team archive [22].

In terms of this we define the density parameter:

$$\Omega = \frac{\rho_c}{\rho}$$

so that $\Omega = 1 \iff k = 0$. The first peak in the power spectrum is determined (mainly) from Ω_B (the baryonic contribution to the density parameter), according to $l_{\text{peak}} \propto {\Omega_B}^{-1/2}$. Since different models predict different power spectrums, this gives us a way to compare models with observational data.

1.3 Brane Cosmology

When creating models to study the cosmology of brane-worlds, the details of the underlying theory are often ignored, and instead just considered to motivate the new framework. The basic framework is to consider a higher-dimensional space (known as the bulk) throughout which gravity acts, and some sub-space of it (known as the brane) to which the Standard Model particles are confined (in some unspecified way). This sub-space is usually approximated to be infinitesimally 'thin' in the extra dimension, i.e. it is a hypersurface (whereas in the underlying theory it should have some thickness, as previously mentioned). So in brane cosmology when one refers to a 'brane' this may have nothing to do with the term as used in string theory, but rather refers to a submanifold of a space-time where ordinary matter resides.

The basic setup

To illustrate this 'metric-based' approach, we consider what is almost the canonical setup for a brane.[3] The metric of the bulk is given by[4]

$$\mathrm{d}s^2\big|_{\text{bulk}} = -n(t,w)^2\mathrm{d}t^2 + a(t,w)^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}w^2 \tag{1.15}$$

where γ_{ij} is the maximally symmetric three-dimensional metric and the extra dimension is spanned by the w-coordinate. Therefore, by specifying the brane to be at some particular w-coordinate, for instance $w = 0$, the metric on the brane is

$$\mathrm{d}s^2\big|_{\text{brane}} = -n(t,0)^2\mathrm{d}t^2 + a(t,0)^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{1.16}$$

where homogeneity has been automatically built into the cosmology of the brane. If t is chosen to be the proper time on the brane, then $n(t,0) = 1$. The next step is to introduce matter to the brane. The energy-momentum tensor can be split into contributions from the bulk and the brane, with the brane part containing a

[3] [30] is one of the earliest examples, or for a review see [31] (and references therein).

[4] We adopt the common notational convention that upper case Latin letters denote five-dimensional indices $(0,1,2,3,4)$, lower case Greek indices denote the four conventional dimensions $(0,1,2,3)$, and lower case Latin indices run over the ordinary spatial dimensions $(1,2,3)$.

Dirac-delta function, to ensure the matter is confined there:

$$T^A{}_B\big|_{\text{brane}} = \delta(w)\text{diag}(-\rho_b, p_b, p_b, p_b, 0) \tag{1.17}$$

where ρ_b and p_b are the energy-density and pressure of the brane respectively. Then, when it comes to solving Einstein's equations, one may consider the full energy-momentum tensor including the brane contribution (above), or, solve the equations in the bulk only, and utilise Israel's so called 'junction conditions' [32] on the brane.

The Israel junction conditions are effectively the integration of the Einstein equations across a surface separating two metrically different regions, and relate the energy-momentum tensor of the surface to the change in extrinsic curvature across it. Actually, in the Randall and Sundrum scenario the brane separates two metrically identical regions, but there is still a change in the extrinsic curvature across it. This arises via the Z_2 symmetry of the brane, i.e. the fact that the two bulk regions are mirror images, and thus have their normal vectors pointing in opposite directions (which in turn leads to equal but opposite extrinsic curvature). To remove the Z_2 symmetry but maintain a massive brane requires the two regions to be metrically different – the common case is two AdS_5 regions with different cosmological constants, in which case the brane is sometimes called a 'shell'. An explicit example of such a case is reviewed in section 2.1.

Pitfalls and Progress

Having used the Israel junction conditions, a common illness of brane-cosmology models was that the the resulting Friedmann-like equations were found to be significantly different from the standard ones, in that the Hubble parameter depends *quadratically* on the energy density (as opposed to linearly). For instance, in section 2.1.1 we will find that for a certain scenario the Friedmann-like equation is

$$H^2 = \left(\frac{\kappa_5\,\rho}{6}\right)^2 + \frac{\Lambda}{6} - \frac{k}{a^2} \tag{1.18}$$

(where Λ is an explicit cosmological constant and κ_5 is the Einstein constant of the five-dimensional space) which is in stark contrast to the equivalent expression from standard cosmology (1.10):

$$H^2 = \frac{\kappa_4 \, \rho}{3} + \Lambda - \frac{k}{a^2}$$

(where κ_4 is the Einstein constant of the four-dimensional space). This was a problem, since the cosmological evolution resulting from a relation like (1.18) is not in general compatible with nucleosynthesis constraints.

Fortunately, a way around this problem was soon established [33, 34]: it was shown that by fine-tuning the brane tension (interpreted as a cosmological constant on the brane) relative to the bulk cosmological constant, the equations could be modified to obtain behaviour more like the norm, and thus compatible with standard cosmology theory [36]. This will also be demonstrated in section 2.1.

Many studies have been made of perturbations of brane-worlds, with a view to calculating the CMB anisotropies they predict – see [37] (and references therein). One of the difficulties of this is connecting the brane perturbations with those of the bulk, since this involves solving differential equations across a surface where the metric is not smooth. Obviously this problem would not exist if one considered a brane across which the metric *is* smooth. One way to do this would be using a brane with a small but finite thickness (instead of infinitesimally thin) which we will call a 'thick brane'. There are also other reasons to consider such a scenario: the simplest models make use of a minimally coupled Lagrangian, i.e. one where the only curvature term is the Ricci scalar curvature (the Einstein-Hilbert action, which upon variation yields Einstein's equations – see section A.2), but if higher curvature terms are included (which terms 'should' be there ultimately depends on a successful quantum theory of gravity) such as in an Einstein-Gauss-Bonnet theory (which includes second order curvature terms) then the 'thin wall' approximation may no longer hold [41].

With these limitations in mind, in this work we propose a model in which the brane is generated dynamically, to be specific, by 'vacuum decay'. In a theory of a scalar field experiencing a potential that has two un-equal local minima (the global minimum being called the true vacuum, and the other the false vacuum), there exists the possibility of vacuum decay: starting with a space-time in which the scalar field is everywhere at the false vacuum (or very close to it), it is possible for a region to quantum mechanically tunnel to the true vacuum. This region would then expand to fill up the entire space. In between the two regions would be a domain wall, and it is this wall that we suggest could be interpreted as a thick brane. To proceed, we first review the theory of vacuum decay.

1.4 Vacuum Decay

In this section we review the work of [43, 44]. Consider the theory of a scalar field in five dimensional space-time subject to a potential of the form in figure 1.3, described by the action

$$S = \int \mathrm{d}^5 x \sqrt{g} \left[\frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi - U(\Phi) + \frac{R}{2\,\kappa} \right] \tag{1.19}$$

where the metric has signature $(+ - - - -)$. Classically, there are two stable minima, but quantum-mechanically the higher-energy minimum is only 'meta-stable' since it is possible to penetrate the barrier to reach the other minimum. The unstable minimum is termed the 'false vacuum' and transition from false vacuum to true is known as 'vacuum decay'. Qualitatively the process is well known – it is just the nucleation of a bubble: if the universe is filled by false vacuum, then

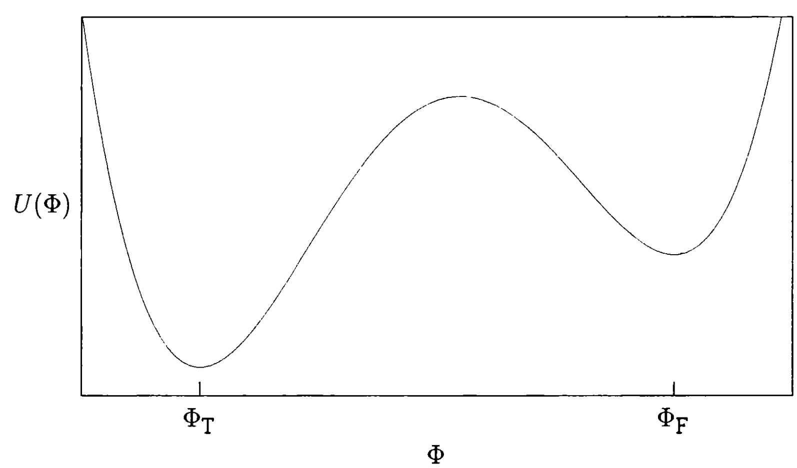

Figure 1.3: The potential $U(\Phi)$, with true and false vacuum indicated as Φ_T and Φ_F respectively.

quantum fluctuations mean that it is possible for a bubble of true vacuum to form, which can then grow to spread throughout the universe, converting false vacuum to true.

So, what form does the decay take? Simply (semi-classically), whichever is most probable. As discussed in [43], the probability of a decay per unit time per unit volume is given by

$$\frac{\Gamma}{V} \propto e^{-S_E/\hbar}(1+\mathcal{O}(\hbar))$$

where S_E is the Euclidean action of the decay, which is defined as i times the 'analytic continuation' of (1.19). To obtain this Euclidean action we first Wick-rotate the t-integral of (1.19) by $\frac{\pi}{2}$ into the complex plane so that it runs over purely imaginary values, and define $\tau = i\,t$ so that

$$\int_{-\infty}^{\infty} \mathrm{d}t \;\rightarrow\; \int_{-i\,\infty}^{i\,\infty} \mathrm{d}t \;\rightarrow\; i\int_{-\infty}^{\infty} \mathrm{d}\tau$$

Then we let $g_{\mu\nu} \longrightarrow -g_{\mu\nu}$ so that the metric in these coordinates has the signature $(+++++)$, which means that inner-products continue as $x\cdot y \longrightarrow -x\cdot y$, and we therefore find:

$$S_E = \int \mathrm{d}^5x\sqrt{g}\left[\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + U(\Phi) - \frac{R}{2\,\kappa}\right] \tag{1.20}$$

Finding the most probable decay has now been reduced to finding the configuration of Φ and $g_{\mu\nu}$ that minimises (1.20) subject to the conditions that $\Phi \longrightarrow \Phi_{\mathrm{F}}$ at Euclidean infinity and that Φ is not a constant (so that $\Phi \not\equiv \Phi_{\mathrm{F}}$). Such a solution is called the 'bounce' (in analogy with a similar solution in particle mechanics).

To solve for the bounce, we assume the solution has O(5) invariance.[5] If we also choose the invariance to be about the centre of the coordinate system then it amounts to specifying Φ is a function of the Euclidean-radius only (which we call ξ), and the form of the metric is simplified:

$$\begin{aligned} \Phi &= \Phi(\xi) \\ g_{\mu\nu} &= \mathrm{diag}\left(1, \rho(\xi)^2\gamma_{ij}\right) \end{aligned}$$

[5]For the case without gravity this was proved in [45]. With gravity it is an assumption, but not an unreasonable one, since the equations are $O(5)$ invariant.

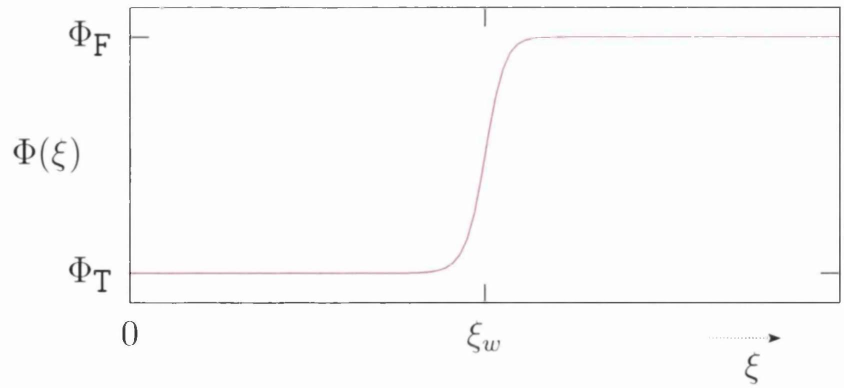

Figure 1.4: A tanh-like shape 'solution' with the wall between the two vacua at ξ_w.

where γ_{ij} is the metric of the 4-sphere. This reduces the equations of motion from partial- to ordinary-differential equations. As we will see in section 2.2 the solution is a tanh-like shape.[6]

So once again we ask: what does the decay look like? Let the solution be given by

$$\Phi(\xi) = f(\xi^2) \tag{1.21}$$

which is shown in figure 1.4. Since $\xi^2 = \tau^2 + r^2$ (where r is the usual Minkowski-space radius), all we need to do to get the solution in Minkowski-space is transform back $\tau = it$ so that $\xi^2 = -t^2 + r^2$, which is valid for $r^2 > t^2$, i.e. outside the lightcone. Thus the solution is

$$\Phi(t, r) = f(r^2 - t^2) \qquad (r^2 > t^2)$$

so we can see that at $t = 0$ the Φ-r profile looks like figure 1.4, and as time increases it looks like this profile is shifted in the positive-r direction, i.e. a bubble of true vacuum (of radius ξ_w) nucleates at the origin at $t = 0$ and grows with time. In fact, we can see from figure 1.4 that the 'wall' (the boundary between the two vacua)

[6]In [43] Coleman assumed $U(\Phi)$ to be nearly symmetrical, i.e. the energy difference between $U(\Phi_F)$ and $U(\Phi_T)$ was small, allowing him to solve for Φ explicitly in terms of a tanh.

has coordinates $\xi = \xi_w$ and thus traces out the hyperboloid

$$r^2 - t^2 = {\xi_w}^2$$

indicating that after nucleating, the bubble starts expanding at a velocity soon approaching that of light.

1.5 Plan for this Thesis

The plan for this work is to first create a model for a 'thick brane'. In the next chapter we will review a spherical shell-brane resulting from regions of differing cosmological constant in the bulk, including the use of the Israel junction conditions to find the relationship between the brane's evolution and its energy density, and fine-tuning the brane tension to recover Friedmann-like equations, as discussed in section 1.3. We will then find (numerically) the bounce for the vacuum decay of the theory defined by the action (1.19) and interpret the wall between the two vacua as a thick brane, in analogy with the aforementioned shell-brane.

The main part of this work will be to study some cosmological property of the thick brane, in particular, the fluctuations of the brane and the resulting anisotropies in the CMB. Although it was argued that the minimum energy (and thus most likely) configuration for vacuum decay will have spherical symmetry (in Euclidean space), there will be a large phase space of small fluctuations around this state, each of which has only slightly more energy (and is thus slightly less probable). Therefore if we consider the system to have some finite 'temperature', by appealing to a thermodynamic principle – that of energy versus entropy – we can determine how the energy (associated with that temperature) will distribute itself among the spectrum of energy states and thus their relative probability, which in turn will represent the weighting with which that configuration contributes to the CMB power spectrum. Therefore this method will lead to predictions of the *relative* heights of the peaks of the CMB spectrum, but not of the *absolute* heights, since that would depend on the amount of energy available, and thus the 'temperature' of the system.

In chapter 3 we introduce the formalism we will use to perturb the fields, and derive the equations governing the perturbations and the corresponding contribution to the action. In chapter 4 we solve the perturbation equations numerically, but find that there are no physically acceptable solutions. In chapter 5 we discuss this result, ultimately reasoning that due to radiation from the expanding bubble focusing at the origin, a different background needs to be used.

Chapter 2

Bubble Branes

Having discussed in broad terms the setup for a brane in chapter 1, we now look at an example in detail, ultimately using the Israel junction conditions to obtain the equation governing the evolution of the brane, and then introducing a brane tension to recover standard cosmology.

In section 2.2 we see how such a scenario can be generated, via vacuum decay; first setting up the system to solve, then solving it numerically, and finally comparing the metrics of the resulting brane with that of the preceding section.

2.1 Thin Branes

In this section we repeat the calculations of Deruelle and Doležel [42]. In their setup, the bulk consists of metrically different regions both described by the coordinates $x^A = \{t, r, \chi, \theta, \phi\}$, and separated by a shell, Σ. The two regions of the bulk are anti-de Sitter spacetimes – solutions of $G_{AB} + \Lambda_{\pm} g_{AB} = 0$ where $+/-$ denotes outside/inside the shell. The corresponding metrics are

$$\mathrm{d}s^2|_+ = -\Phi_+(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Phi_+(r)} + r^2\left[\mathrm{d}\chi^2 + f_k(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{2.1a}$$

$$\mathrm{d}s^2|_- = -\Phi_-(r)\alpha^2(t)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Phi_-(r)} + r^2\left[\mathrm{d}\chi^2 + f_k(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{2.1b}$$

where

$$\Phi_\pm(r) = k - L_\pm r^2 \qquad L_\pm = \frac{\Lambda_\pm}{6}$$
$$f_{-1}(\chi) = \sinh\chi \qquad f_0(\chi) = \chi \qquad f_1(\chi) = \sin\chi$$

and $\alpha(t)$ is known as the lapse function, introduced to join the metrics at the shell.

The evolution of the shell is specified parametrically by

$$r = a(\tau) \qquad t = t(\tau) \tag{2.2}$$

By choosing τ to be the proper time on Σ the metric on the shell takes the form:

$$\mathrm{d}s^2{}_\Sigma = -\mathrm{d}\tau^2 + a^2(\tau)\left[\mathrm{d}\chi^2 + f_k(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \tag{2.3}$$

Since we require the metric to be continuous across Σ, we 'join' this metric with those of (2.1). First, the metric outside the shell:

$$\begin{aligned}
& \mathrm{d}s^2|_\Sigma = \mathrm{d}s^2|_{+,\Sigma} \\
\Longrightarrow \quad & -\mathrm{d}\tau^2 = -\Phi_+(a)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Phi_+(a)} \\
\Longrightarrow \quad & -1 = -\Phi_+(a)\,\dot{t}^2 + \frac{\dot{a}^2}{\Phi_+(a)}
\end{aligned}$$

where an overdot denotes $\frac{\mathrm{d}}{\mathrm{d}\tau}$, and thus

$$\dot{t} = \frac{\sqrt{\Phi_+(a) + \dot{a}^2}}{\Phi_+(a)} \tag{2.4}$$

Joining the metric on the brane with that inside the shell we find:

$$\begin{aligned}
& \mathrm{d}s^2|_\Sigma = \mathrm{d}s^2|_{-,\Sigma} \\
\Longrightarrow \quad & -\mathrm{d}\tau^2 = -\Phi_-(a)\,\alpha^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\Phi_-(a)} \\
\Longrightarrow \quad & -1 = -\Phi_-(a)\,\alpha^2\,\dot{t}^2 + \frac{\dot{a}^2}{\Phi_-(a)}
\end{aligned}$$

After substituting for $\dot{t}$ using (2.4) we find:

$$\alpha(\tau) = \frac{\Phi_+(a)}{\Phi_-(a)}\sqrt{\frac{\dot{a}^2 + \Phi_-(a)}{\dot{a}^2 + \Phi_+(a)}} \tag{2.5}$$

The equation governing the evolution of the scale factor of this brane, $a(\tau)$, can be obtained by applying Einstein's equations in the form of Israel's junction conditions.

2.1.1 Israel's Junction Conditions

The Israel junction conditions relate the jump in extrinsic curvature across a surface to its energy-momentum tensor. Since extrinsic curvature can be expressed in terms of normal vectors, we first define these on the shell for the two regions.

Four independent tangent vectors on the shell are given by $||e_\tau|| = (\dot{t}, \dot{a}, 0, 0, 0)$ and ${e_i}^A = \delta_i^A$ (for $i = \chi, \theta, \phi$). We can use these to find the normal vectors to the shell since they satisfy ${e_\mu}^A n_A = 0$, giving

$$||n_A|| \propto \pm(-\dot{a}, \dot{t}, 0, 0, 0),$$

and defining the positive direction to be from inside to outside the shell causes the normal vectors in the positive direction to be $||n_A|| \propto (-\dot{a}, \dot{t}, 0, 0, 0)$. Normalising these vectors to unity: $n_A n^A = 1$, is where the difference in the two regions comes in, through the metric:

$$n_A^\pm n_B^\pm (g^{AB})^\pm = 1$$

so we find

$$\begin{aligned} ||n_A^+|| &= (-\dot{a}, \dot{t}, 0, 0, 0) \\ ||n_A^-|| &= \alpha(t)(-\dot{a}, \dot{t}, 0, 0, 0) \end{aligned}$$

Now that we have the normal vectors, we calculate the extrinsic curvature, $K_{\mu\nu}$, which is defined by

$$K_{\mu\nu}^\pm = -e_\mu^A e_\nu^B \nabla_A n_B \big|^\pm \tag{2.6}$$

where greek indices label tensors on the brane, i.e. with coordinates $(\tau, \chi, \theta, \phi)$ and induced metric (2.3), and $\nabla_A{}^{\pm}$ is the covariant derivative associated with the bulk metric (2.1). The non-zero components are

$$K^{\tau}{}_{\tau}{}^{\pm} = \frac{L_{\pm} - \frac{\ddot{a}}{a}}{\sqrt{H^2 + \frac{k}{a^2} - L_{\pm}}} \tag{2.7a}$$

$$K^{\chi}{}_{\chi}{}^{\pm} = K^{\theta}{}_{\theta}{}^{\pm} = K^{\phi}{}_{\phi}{}^{\pm} = -\sqrt{H^2 + \frac{k}{a^2} - L_{\pm}} \tag{2.7b}$$

where $H \equiv \frac{\dot{a}}{a}$.

Utilising the definition

$$\widehat{K}^{\mu}_{\nu} \equiv K^{\mu}{}_{\nu}{}^{+} - K^{\mu}{}_{\nu}{}^{-} \tag{2.8}$$

the Israel junction condition can be written

$$\kappa\, T^{\mu}_{\nu} = \widehat{K}^{\mu}_{\nu} - \delta^{\mu}_{\nu} \widehat{K} \tag{2.9}$$

where T^{μ}_{ν} is the energy-momentum tensor of matter on the brane. Since T^{μ}_{ν} has the perfect fluid form $T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p, p, p)$ – where ρ and p are the energy density and pressure of the shell – the τ-τ component of (2.9) becomes

$$\begin{aligned} \kappa\, T^{\tau}_{\tau} = -\kappa\, \rho &= \widehat{K}^{\tau}_{\tau} - \widehat{K} \\ &= \widehat{K}^{\tau}_{\tau} - (\widehat{K}^{\tau}_{\tau} + 3\, \widehat{K}^{\chi}_{\chi}) \\ &= -3\, \widehat{K}^{\chi}_{\chi} \\ &= -3\, (K^{\chi +}_{\chi} - K^{\chi -}_{\chi}) \end{aligned} \tag{2.10}$$

i.e.

$$\frac{\kappa\, \rho}{3} = \sqrt{H^2 + \frac{k}{a^2} - L_-} - \sqrt{H^2 + \frac{k}{a^2} - L_+} \tag{2.11}$$

This is our Friedmann-like equation, relating the scale factor to the energy density on the brane, the spatial curvature, and the cosmological constants in the bulk. As we anticipated (in section 1.3), in stark contrast to standard cosmology (1.10), we see $H^2 + \frac{k}{a^2}$ varies with ρ^2 rather than ρ.

As an aside, we can easily see here what the equivalent result would have been in (what Deruelle and Doležel call) a 'brane' rather than 'shell' scenario. In the brane case, there is only one cosmological constant, and thus $L_- = L_+ = L$, but the flip of the normal vector due to the Z_2-symmetry would change the sign of the second square root in (2.11), resulting in

$$\frac{\kappa\,\rho}{3} = 2\sqrt{H^2 + \frac{k}{a^2} - L}$$

Conversely, as noted in [42], in the Gauss-Bonnet theory of gravity, one needs neither two values for the cosmological constant nor the Z_2 symmetry to produce a massive shell, because the theory has two solutions in the bulk for a single cosmological constant: $L_\pm = \frac{1}{4\alpha}\left(-1 \pm \sqrt{1 + \frac{4\alpha\Lambda}{3}}\right)$ (where α is a coupling constant).

2.1.2 Recovering Standard Cosmology

To recover standard cosmology on the brane as per [33, 34], the energy density of the brane is decomposed into that of ordinary matter, ρ_m, and a constant brane 'tension', σ, which is really just a cosmological constant on the brane:

$$\rho = \rho_m + \sigma \qquad p = p_m - \sigma \tag{2.12}$$

By expanding (2.11) under the assumption $\rho_m \ll |\sigma|$ and fine tuning σ as

$$\sigma = \frac{3}{\kappa}\left(\sqrt{-L_-} - \sqrt{-L_+}\right) \tag{2.13}$$

(2.11) reduces to

$$H^2 + \frac{k}{a^2} = \frac{2\,\kappa\,\rho_m}{3}\frac{\sqrt{L_+ L_-}}{\sqrt{-L_+} - \sqrt{-L_-}} + \mathcal{O}\left({\rho_m}^2\right) \tag{2.14}$$

So, defining Newton's constant by

$$\frac{8\pi\,G}{3} = \frac{2\,\kappa}{3}\frac{\sqrt{L_+ L_-}}{\sqrt{-L_+} - \sqrt{-L_-}} \tag{2.15}$$

produces the Friedmann-like equation:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_m + \mathcal{O}(\rho_m{}^2) \tag{2.16}$$

Thus at late times (when ρ_m has become small enough for $\rho_m{}^2$ to be negligible) we recover the standard cosmological evolution.

From (2.15) we can see that for $G \geq 0$ we require $L_+ < L_- < 0$. Above it was implicitly assumed that $L_\pm \neq 0$; for the special case $L_+ = 0$ we would have fine-tuned $\sigma = \frac{3}{\kappa}\sqrt{-L_-}$ and defined

$$\frac{8\pi G}{3} = \frac{2\kappa}{3}\sqrt{-L_-} \tag{2.17}$$

to once again arrive at (2.16).

2.2 Thick Branes

In the previous section the differing cosmological constant of the two regions was key to producing a jump in the extrinsic curvature and thus a massive shell. One way to produce two such regions is via the process of vacuum decay; the two regions of differing cosmological constant correspond to a region of true vacuum and a region of false vacuum. If space is initially filled by false vacuum then there is the possibility of it decaying to true vacuum via a tunnelling event. The wall separating the two vacua then corresponds to the massive shell.

2.2.1 Vacuum Decay

We now consider the theory of a minimally coupled action for a scalar field, $\Phi(x)$, in a potential $U(\Phi(x))$, as in [44] but in five dimensions:

$$S = \int \mathrm{d}^5x\sqrt{|g|}\left[\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi - U(\Phi) + \frac{R}{2\,\kappa}\right] \tag{2.18}$$

where κ is the five dimensional Einstein constant and R the Ricci scalar curvature.

It was argued in section 1.4 that vacuum decay was possible for a potential with two un-equal minima, and that the configuration of such a scenario is obtained by minimising the Euclidean action of the theory, given by

$$S_E = \int \mathrm{d}^5x\sqrt{g}\left[\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + U(\Phi) - \frac{R}{2\,\kappa}\right] \tag{2.19}$$

To proceed we assume the bounce (the name given to the vacuum decay field-solution) is (hyper-)spherically symmetric. Working in spherical polar coordinates, this amounts to letting Φ be a function of Euclidean radius (denoted ξ) alone, i.e. $\Phi(x) = \Phi(\xi)$, and the line element taking the form

$$\mathrm{d}s^2 \equiv g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \mathrm{d}\xi^2 + \rho(\xi)^2\mathrm{d}{\Omega_4}^2 \tag{2.20}$$

where $\mathrm{d}{\Omega_4}^2$ is the line element of a unit 4-sphere.[1]

The assumption of symmetry greatly simplifies the equations of motion (derived in appendix A). The scalar field equation becomes

$$\Phi''(\xi) + \frac{4\,\rho'(\xi)}{\rho(\xi)}\Phi'(\xi) = U'\big(\Phi(\xi)\big) \tag{2.21}$$

while the radial and angular components of Einstein's equation are

$$-\frac{4\rho''(\xi)}{\rho(\xi)} = \kappa\left(\frac{2}{3}U(\Phi(\xi)) + \Phi'(\xi)^2\right) \tag{2.22}$$

and

$$3 - 3\rho'(\xi)^2 - \rho(\xi)\rho''(\xi) = \frac{2}{3}\,\kappa\,\rho(\xi)^2\,U(\Phi(\xi)) \tag{2.23}$$

respectively. Using (2.22) to substitute for $\rho''(\xi)$, (2.23) becomes

$$\rho'(\xi)^2 = 1 + \frac{\kappa\,\rho(\xi)^2}{12}\left(\Phi'(\xi)^2 - 2\,U(\Phi(\xi))\right) \tag{2.24}$$

Upon differentiating this and substituting for $\rho''(\xi)$ again, we arrive at the scalar

[1]Note that in this section $\rho(\xi)$ is the radius of curvature of the Euclidean space, not an energy density – which notation is meant will always be clear from the context.

field equation (2.21), thus showing that of the three equations of motion

$$\Phi''(\xi) = U'(\Phi(\xi)) - 4\frac{\rho'(\xi)}{\rho(\xi)}\Phi'(\xi) \tag{2.25a}$$

$$\rho''(\xi) = -\frac{\kappa\,\rho(\xi)}{12}\Big(2\,U(\Phi(\xi)) + 3\,\Phi'(\xi)^2\Big) \tag{2.25b}$$

$$\rho'(\xi)^2 = 1 + \frac{\kappa\,\rho(\xi)^2}{12}\Big(\Phi'(\xi)^2 - 2\,U(\Phi(\xi))\Big) \tag{2.25c}$$

only two are independent. We will use the first two as the differential equations to be integrated numerically, but the third will be important for determining the boundary conditions (particularly that $\rho'(\xi)^2 \to 1$ as $\xi \to 0$ and $\xi \to \infty$).

2.2.2 Numerical Solution

To solve equations (2.25) for $\Phi(\xi)$ and $\rho(\xi)$ we need to resort to numerical methods. This is done by taking the first two (second order equations) and transforming them into four first order equations[2], which can then be solved using a Runge-Kutta technique. We use Fehlberg's embedded Runge-Kutta formula (using the Cash-Karp parameters [15]) which adapts the step-size it uses based on an estimate of the relative error of the solutions compared to some specified tolerance. Once initial conditions are provided, in theory it is a simple case of calling a routine which integrates the equations out to a specified ξ-value. In practice it is not this simple, since the desired solution is not a stable one, and (as we will see) a little more work is required.

Any parameters of the system must also be specified before the numerical integration can take place. For this system these are $U(\Phi)$ and κ. We work in arbitrary units with $\kappa = 0.1$ and setup $U(\Phi)$ to have the required form (i.e. one conducive of vacuum decay). The values for these parameters can be placed in physical perspective by comparing with the work of section 2.1.2. There we found that to recover standard cosmology on the (thin) brane we had to fine-tune the brane tension, resulting in a relation between Newton's gravitational constant on

[2]An n-th order differential equation can always be transformed into a system of n coupled first order equations.

the brane and the Einstein and cosmological constants of the bulk, (2.17). The equivalent relation here would be:

$$\frac{8\pi G}{3} = \frac{\kappa}{3}\sqrt{\frac{-2\,U(\Phi_{\mathrm{T}})}{3}} \tag{2.26}$$

The basic form for $U(\Phi)$ is that of a quartic with three stationary points:

$$U'(\Phi) \propto (\Phi - \Phi_{\mathrm{F}})(\Phi - \Phi_{\mathrm{T}})(\Phi - \Phi_m)$$

where Φ_{F} and Φ_{T} will be local minima (the false and true vacua, respectively), and Φ_m a local maximum, and $\Phi_{\mathrm{T}} < \Phi_m < \Phi_{\mathrm{F}}$. The five degrees of freedom of the quartic are specified via the values of Φ_{T}, Φ_{F}, $U(\Phi_{\mathrm{T}})$, $U(\Phi_{\mathrm{F}})$, and $U(\Phi_m)$ – this completely determines $U(\Phi)$ and allows for the important features of the potential (e.g. the 'height' of the barrier between the minima) to be adjusted. Choosing these values to be[3]

$$\Phi_{\mathrm{T}} = -1 \qquad \Phi_{\mathrm{F}} = 1 \qquad U(\Phi_{\mathrm{T}}) = -7 \qquad U(\Phi_{\mathrm{F}}) = 0 \qquad U(\Phi_m) = 4 \tag{2.27}$$

produces the potential shown in figure 2.1. Now all that remains is to specify initial conditions for Φ and ρ, and then the numerical integration can be performed. To do this, we look at the behaviour of the fields as $\xi \to 0$. In this limit we can expand $\Phi(\xi)$ and $\rho(\xi)$ as power series (where we have used the facts that $\Phi'(0) = 0$ and $\rho(0) = 0$, and defined $\Phi_0 = \Phi(0)$):

$$\Phi(\xi) = \Phi_0 + \sum_{n=2}^{\infty} \frac{\Phi^{(n)}(0)}{n!}\xi^n \tag{2.28a}$$

$$\rho(\xi) = \sum_{n=1}^{\infty} \frac{\rho^{(n)}(0)}{n!}\xi^n \tag{2.28b}$$

[3]Choosing $U(\Phi_{\mathrm{F}}) = 0$ ensures that the action (2.19) is finite-valued for the bounce.

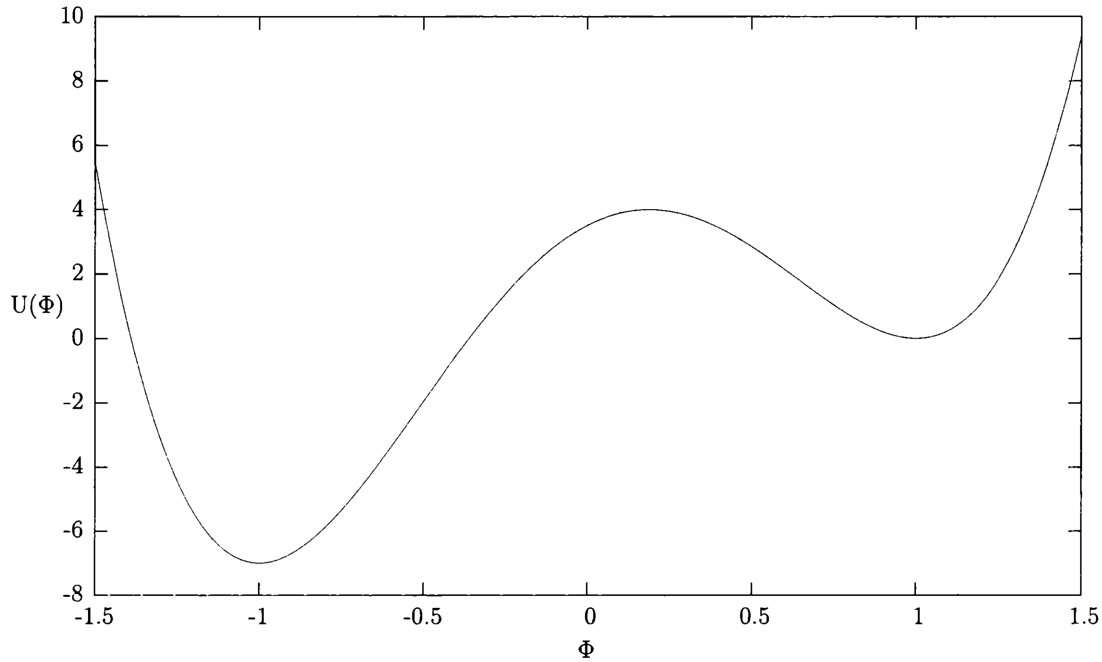

Figure 2.1: The potential, $U(\Phi)$, with the parameters specified in (2.27). Its salient features are that it has two local minima with different values (so that one is considered the 'true vacuum' and the other the 'false vacuum') separated by a barrier.

Substituting (2.28) into (2.25) and equating coefficients of ξ leads to

$$\Phi(\xi) = \Phi_0 + \frac{U'(\Phi_0)}{10}\,\xi^2 + \frac{U'(\Phi_0)}{2520}\Big(4\,\kappa\,U(\Phi_0) + 9\,U''(\Phi_0)\Big)\,\xi^4 + \mathcal{O}(\xi^6) \tag{2.29a}$$

$$\rho(\xi) = \xi + \kappa\left[-\frac{U(\Phi_0)}{36}\,\xi^3 + \frac{1}{108000}\Big(25\,\kappa\,U(\Phi_0)^2 - 144\,U'(\Phi_0)^2\Big)\,\xi^5 + \mathcal{O}(\xi^7)\right] \tag{2.29b}$$

so that the only parameter still un-specified is Φ_0. Before discussing how to determine Φ_0, we first look at the asymptotic forms of the solution as $\xi \longrightarrow \infty$.

In the limit $\xi \longrightarrow \infty$, a first approximation to (2.25) is $\Phi(\xi) = \Phi_F$ and $\rho(\xi) = \xi + b$, where b is an arbitrary constant.[4] To make a better approximation we first change

[4] This is in fact an *exact* solution, but a trivial one. It is an approximation in the sense that we know our desired solution has Φ_F as an asymptote as $\xi \longrightarrow \infty$.

variable:

$$z = \xi + b \tag{2.30}$$

so that equations (2.25a) and (2.25b) become

$$\Phi''(z) = U'(\Phi(z)) - \frac{4\,\rho'(z)\,\Phi'(z)}{\rho(z)} \tag{2.31a}$$

$$\rho''(z) = \frac{-\kappa\,\rho(z)\,\left(2\,U(\Phi(z)) + 3\,\Phi'(z)^2\right)}{12} \tag{2.31b}$$

let $\beta \equiv \sqrt{U''(\Phi_{\mathrm{F}})}$, and make the ansatz:

$$\Phi(z) = \Phi_{\mathrm{F}} + g(z)\,e^{-\beta\,z} + \mathcal{O}\left(e^{-2\,\beta\,z}\right) \tag{2.32a}$$

$$\rho(z) = z + \mathcal{O}\left(e^{-2\,\beta\,z}\right) \tag{2.32b}$$

Substituting this into (2.25) gives the following equation for $g(z)$:

$$0 = g''(z) + g'(z)\,\left(\frac{4}{z} - 2\,\beta\right) - \frac{4\,g(z)\,\beta}{z} \tag{2.33}$$

which has a solution

$$g(z) \propto \frac{1}{z^2} + \frac{1}{\beta\,z^3}$$

(there is another solution which diverges from Φ_{F}, in which we are not interested). So the asymptotic forms of $\Phi(z)$ and $\rho(z)$ are:

$$\Phi(z) = \Phi_{\mathrm{F}} + c\,e^{-\beta\,z}\left(\frac{1}{z^2} + \frac{1}{\beta\,z^3}\right) + \mathcal{O}\left(e^{-2\,\beta\,z}\right) \tag{2.34a}$$

$$\rho(z) = z + \mathcal{O}\left(e^{-2\,\beta\,z}\right) \tag{2.34b}$$

where c is an arbitrary constant.

We are now ready to integrate out. The key lies in finding the 'best' possible value for Φ_0; since the desired solution is unstable (separating divergent solutions from oscillating ones) Φ_0 is determined using a bisection algorithm, but it is still ultimately necessary to patch together the integrated solution and the known analytic form (2.34a). Full details are given in appendix B, the result being that an acceptable solution is found, which is shown in figure 2.2.

As discussed in section 1.4, this solution represents an expanding bubble of true vacuum, formed at $t = 0$ with radius ξ_b, where from the plot of $U(\Phi(\xi))$ versus ξ in figure 2.2 we can see that $\xi_b \approx 3$ (in the thin wall case this plot would be a step function).

As we integrate for $\Phi(\xi)$ and $\rho(\xi)$, we can also calculate S_E, which we do by recasting the integral expression for S_E as a differential equation. For a solution of the field equations obeying (2.20) and (2.22), the Euclidean action (2.19) becomes

$$S_E = \frac{16\,\pi^2}{9} \int_0^\infty \rho(\xi)^4\, U(\Phi(\xi))\, \mathrm{d}\xi \tag{2.35}$$

To recast this as a differential equation, consider

$$P(\xi) = \int_0^\xi f(\xi')\, \mathrm{d}\xi'$$

which implies that $S_E = P(\infty)$, and can be integrated numerically as the differential equation

$$\frac{\mathrm{d}P(\xi)}{\mathrm{d}\xi} = f(\xi)$$

When we come to perturb the bounce, we will calculate the perturbation's contribution to the action in the same way.

2.2.3 Brane Interpretation

In analogy with the theory of section 2.1, we interpret the two regions of vacuum as the bulk, and the wall separating them as a thick brane. But what of the metric on our brane?

The brane is located at some constant value of Euclidean radius, ξ_b, thus (2.20) reduces to:

$$\mathrm{d}s^2|_b = \rho(\xi_b)^2 \mathrm{d}\Omega_4{}^2 \tag{2.36}$$

To cast this in more familiar terms, we first expand $\mathrm{d}\Omega_4{}^2$:

$$\mathrm{d}\Omega_4{}^2 = \mathrm{d}\psi^2 + \sin^2\psi\, \mathrm{d}\Omega_3{}^2 \tag{2.37}$$

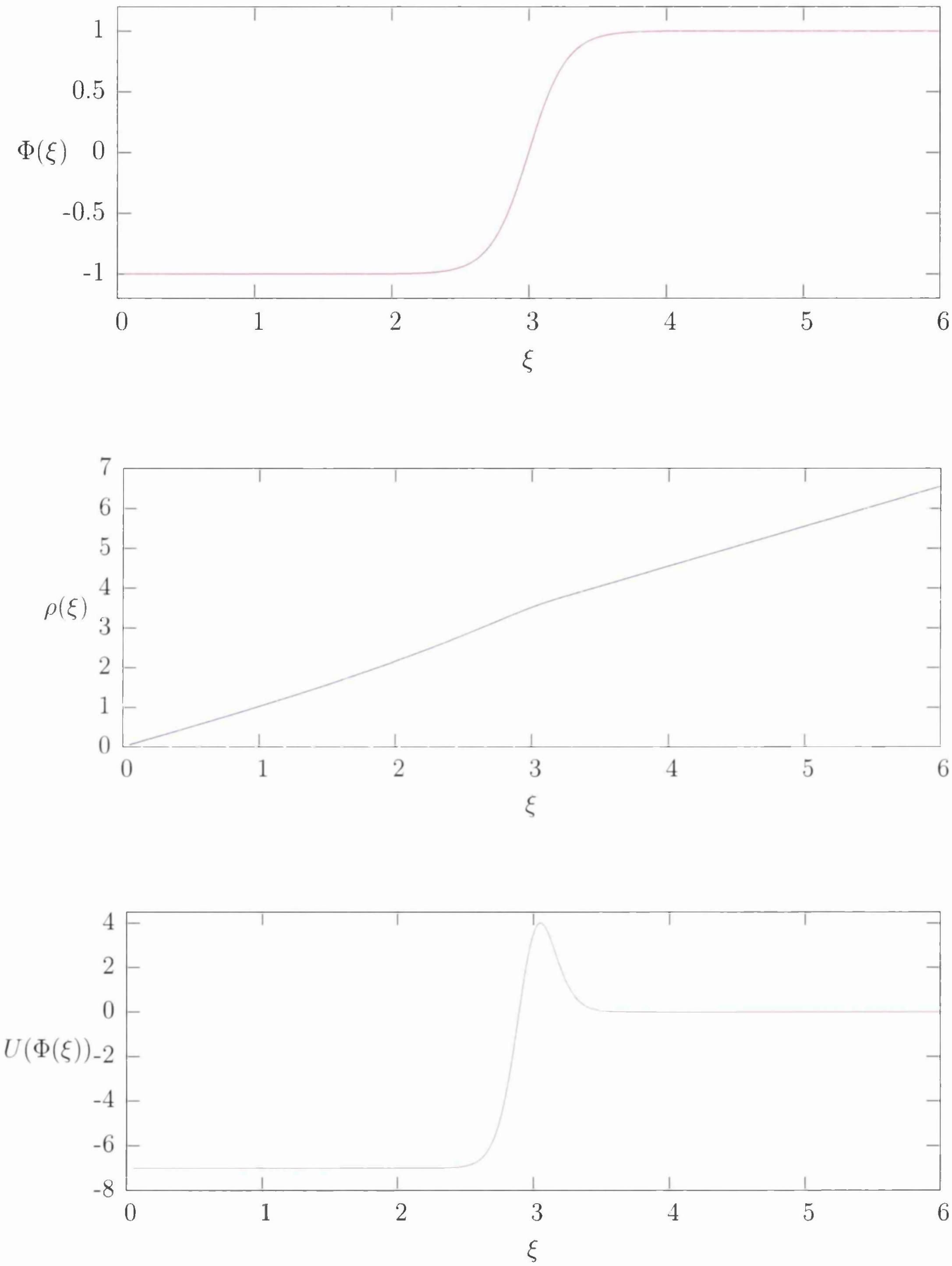

Figure 2.2: The bounce for the potential of figure 2.1.

which implicitly tells us the relation between ψ and the Euclidean 'time' coordinate τ (and thus the real time t), is:

$$\tau = i\,t = \xi \cos\psi$$

So, upon transforming from ψ to t, (2.37) becomes

$$\mathrm{d}\Omega_4{}^2 = \frac{-\mathrm{d}t^2}{\xi_b{}^2 + t^2} + \left(\frac{\xi_b{}^2 + t^2}{\xi_b{}^2}\right) \mathrm{d}\Omega_3{}^2$$

A further transformation,

$$t = \xi_b \sinh t_b$$

produces

$$\mathrm{d}\Omega_4{}^2 = -\mathrm{d}t_b{}^2 + \cosh^2 t_b\, \mathrm{d}\Omega_3{}^2$$

and so the complete expression for the metric on the brane (2.36) becomes

$$\mathrm{d}s^2|_b = \rho(\xi_b)^2 \left[-\mathrm{d}t_b{}^2 + \cosh^2 t_b \left(\frac{\mathrm{d}r^2}{1-r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\right)\right] \tag{2.38}$$

To compare this with the thin-brane of section 2.1, we return to the relation between $t(\tau)$ and $a(\tau)$ in (2.4) and now set $\Lambda_+ = 0$ (in keeping with $U(\Phi_{\mathrm{F}}) = 0$), and $k = 1$, resulting in

$$\dot{t}^2 - \dot{a}^2 = 1$$

which is satisfied by

$$t(\tau) = \rho(\xi_b) \sinh\left(\frac{\tau}{\rho(\xi_b)}\right) \qquad a(\tau) = \rho(\xi_b) \cosh\left(\frac{\tau}{\rho(\xi_b)}\right)$$

Thus the metric on the thin brane (2.3) becomes

$$\mathrm{d}s^2{}_\Sigma = -\mathrm{d}\tau^2 + \rho(\xi_b)^2 \cosh^2\left(\frac{\tau}{\rho(\xi_b)}\right) \left[\mathrm{d}\chi^2 + \sin(\chi)^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right] \tag{2.39}$$

or, upon transforming to $r = \sin\chi$,

$$\mathrm{d}s^2{}_\Sigma = -\mathrm{d}\tau^2 + \rho(\xi_b)^2 \cosh^2\left(\frac{\tau}{\rho(\xi_b)}\right)\left[\frac{\mathrm{d}r^2}{1-r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

which is (2.38) with the identification $\tau = \rho(\xi_b)\,t_b$.

An interesting feature of such a universe is that it starts with non-zero radius – in the case of (2.38) $\rho(\xi_b)$ gives the initial radius. Since $\rho(\xi_b)$ is related to ξ_b, there is a direct relationship between the form of the potential $U(\Phi)$ and the initial size of the brane-world – for instance the larger the difference in energy between the two minima, the smaller the initial size, since a smaller volume of negative energy density (the true vacuum) is needed to compensate the shell of positive energy density separating the two vacua. The affect of the form $U(\Phi(\xi))$ on the scale of the bounce is discussed more in section B.1 in the context of finding the numerical solution.

Chapter 3

Perturbations

In section 2.2 we solved for the bounce (vacuum decay configuration with spherical symmetry), having argued that this would minimise the Euclidean action, and thus be the most likely configuration to occur. If we interpret the expanding wall between the two vacua as our universe, then our space-time metric would inherit this symmetry, and thus be completely homogeneous and isotropic. In order to introduce anisotropies (as observed in the CMB) we now wish to relax this symmetry constraint, and obtain solutions which deviate slightly from the original solution, since these will have next-to-lowest Euclidean action and thus be the next most likely configurations to occur. To be able to compare the probability of the configurations, we will need to know the (relative) value of the Euclidean action for each one.

In theory we could try to solve the equations of motion without any loss of generality (i.e. without imposing any symmetries) by proposing an arbitrary metric and scalar field, but this would result in a large[1] set of coupled non-linear partial differential equations (in five independent variables) that would be very difficult to solve. So, to simplify the problem we use a perturbative expansion. In other

[1]There would be potentially sixteen equations for sixteen functions (fifteen from Einstein's equation, and one from the scalar field equation), less additional constraints imposed by gauge conditions.

words, we consider the perturbed variables (distinguished by a tilde)

$$\widetilde{\Phi}(x) = \Phi(\xi) + \varepsilon\,\Delta\Phi(x)$$
$$\widetilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \varepsilon\,h_{\mu\nu}(x)$$

where ε is some small parameter and $\Phi(\xi)$ and $g_{\mu\nu}(x)$ are the bounce solved for in chapter 2, i.e. $g_{\mu\nu} = \mathrm{diag}(1, \rho(\xi)^2\gamma_{ij})$ and $\Phi(\xi)$ and $\rho(\xi)$ are solutions of (2.25) with $\Phi(0) \approx \Phi_\mathrm{T}$ and $\Phi(\xi) \longrightarrow \Phi_\mathrm{F}$ as $\xi \longrightarrow \infty$. The resulting equations of motion can then be solved to first order in ε; the zeroth order part results in the original equations solved by the bounce, and the first order parts are our perturbation equations, which are linear in the perturbation variables ($\Delta\Phi$, $h_{\mu\nu}$) we want to solve for. The resulting perturbative expansion of the action can be written

$$\widetilde{S}_E = S_0 + \varepsilon\,S_1 + \varepsilon^2\,S_2 + \mathcal{O}(\varepsilon^3)$$

where S_0 is the action for the unperturbed solution, and since we have expanded about a solution which minimises the action, $S_1 = 0$. This leaves S_2 as the quantity we need to calculate for our solutions.

Although linear, the equations of motion from such an expansion are still very complicated. To simplify the problem further, we choose the form of $\Delta\Phi(x)$ and $h_{\mu\nu}(x)$ in a way that takes advantage of the symmetry of the zeroth-order solutions; to be specific, we use the cosmological perturbation theory of Kodama and Sasaki [49].

In this theory, perturbations are expanded in terms of harmonic functions on the invariant background – in this case the 4-sphere, which is invariant under spatial rotations (of the Euclidean 5-space). The power of this method is two-fold. First, the perturbations can be classified into three types based on their behaviour under transformations of the invariant background (i.e. rotations), which are called scalar-, vector-, and tensor-type accordingly. Equations of the type we will consider (linear second order differential) are then found to decompose into groups, each of which only contains perturbations of one type, thus allowing the three types of perturbation to be treated separately. Since we are interested in perturbations to the scalar field Φ, we will only consider scalar-type perturbations, because these are the only type of perturbation a scalar field can be subject to. The second powerful feature of this theory is that there is no coupling between different harmonic modes, which means that modes can be treated independently. Ultimately this means the equations of motion reduce to equations for the radial functions which are the coefficients of the harmonic expansion, i.e. a set of *ordinary* differential equations (in one independent variable) – a much less daunting system to solve. The formalism is reviewed in the current context (a five dimensional Euclidean space) in section 3.1.

Having reviewed the formalism we then employ it to deduce the equations of motion (section 3.2) and the action (section 3.3). Both calculations involve very long expressions which are dealt with using the Mathematica computer application, but the resulting expressions can be expressed relatively succinctly. In section 3.4 we discuss the issue of gauge invariance and ultimately introduce new *gauge invariant* variables.

3.1 Harmonic Expansion

We now review the formalism of [49] with a 4-sphere invariant background – first some definitions. The x-coordinate system covers the whole Euclidean space, and can be split into the radial and angular coordinates: $\|x^\mu\| = (\xi, \omega^i)$ where greek indices run from 0 to 4, and latin indices run from 1 to 4. Explicitly, the angular coordinates are $\|\omega^i\| = (\psi, \chi, \theta, \phi)$. Let γ_{ij} be the metric of the 4-sphere, so that the corresponding line element is given by

$$\mathrm{d}{\Omega_4}^2 = \gamma_{ij}\mathrm{d}\omega^i\mathrm{d}\omega^j$$

and $g_{ij} = \rho(\xi)^2\gamma_{ij}$. The covariant derivative corresponding to γ_{ij} of a tensor T with respect to ω^i is denoted $D_iT \equiv T_{|i}$, so that the Laplacian of the invariant background (4-sphere) is given by $\Delta = \gamma^{ij}D_iD_j$.

We are now ready to define the basis of harmonics in which we will expand the perturbations. Scalar quantities can be expanded in terms of a complete set of (scalar) harmonics on the 4-sphere, denoted Y^k and given by[2]:

$$(\Delta + k^2)Y^k = 0 \tag{3.1}$$

where $k^2 = m(m+3)$, $m \in \mathbb{Z}$ (full details of the 'hyper-spherical' harmonics, $Y^k(\omega)$, are given in section D.1). Scalar-type perturbations of vectors on the 4-sphere are expanded by

$$Y^k{}_i = \frac{-1}{k} Y^k{}_{|i} \tag{3.2}$$

and those of tensors by:

$$Y^k{}_{ij} = \frac{1}{k^2} Y^k{}_{|ij} + \frac{1}{4}\gamma_{ij} Y^k \quad \text{and} \quad \gamma_{ij} Y^k \tag{3.3}$$

Resuming the convention that perturbed quantities have a tilde over them (to distinguish them from corresponding background quantity) we now write our perturbative expansion as:

$$\widetilde{\Phi}(x) = \Phi(\xi) + \varepsilon \sum_k \Delta\Phi_k(\xi) Y^k(\omega) \tag{3.4a}$$

$$\widetilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \varepsilon \sum_k h_{\mu\nu}(x) \tag{3.4b}$$

Recalling that the background metric is given by

$$\begin{aligned} g_{00} &= 1 \\ g_{0j} &= 0 \\ g_{ij} &= \rho(\xi)^2 \gamma_{ij} \end{aligned}$$

and since $\widetilde{g}_{00}$, $\widetilde{g}_{0i}$, and $\widetilde{g}_{ij}$ transform (under a spatial rotation) as scalar, vector,

[2]N.B. the superscript k of Y^k labels the mode of the harmonic – it is not a tensor index.

and tensor respectively, $h_{\mu\nu}$ takes the form

$$h_{00} = 2\, A_k(\xi) Y^k(\omega) \tag{3.5a}$$

$$h_{0j} = \rho(\xi) B_k(\xi) Y^k_j(\omega) \tag{3.5b}$$

$$h_{ij} = 2\, \rho(\xi)^2 \Big[H_{\mathrm{L}k}(\xi) Y^k(\omega) \gamma_{ij} + H_{\mathrm{T}k}(\xi) Y^k_{ij}(\omega) \Big] \tag{3.5c}$$

Since there is no coupling between different modes they can be treated independently, and so the summation symbols (and corresponding labels) will usually be suppressed, i.e. (3.4) becomes

$$\widetilde{\Phi}(x) = \Phi(\xi) + \varepsilon\, \Delta\Phi(\xi) Y(\omega) \tag{3.6a}$$

$$\widetilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \varepsilon\, h_{\mu\nu}(x) \tag{3.6b}$$

3.2 Equations of motion

We wish to find the perturbed configurations of minimal Euclidean action, so we once again turn to the Euler-Lagrange equation for the scalar field and Einstein's equation (derived in appendix A, and re-stated here with tildes applied to perturbed quantities for emphasis):

$$\widetilde{\nabla}^2 \widetilde{\Phi} = U'\big(\widetilde{\Phi}\big) \tag{3.7}$$

$$\widetilde{R}_{\mu\nu} = \kappa \left(\frac{2}{3} \widetilde{g}_{\mu\nu}\, U\big(\widetilde{\Phi}\big) + \partial_\mu \widetilde{\Phi}\, \partial_\nu \widetilde{\Phi} \right) \tag{3.8}$$

(where $\widetilde{\nabla}^2$ is the Laplacian associated with $\widetilde{g}_{\mu\nu}$, i.e. of the perturbed 5-space). We wish to solve these equations to first order in ε, i.e. for the set of perturbation variables $\{\Delta\Phi(\xi), A(\xi), B(\xi), H_{\mathrm{L}}(\xi), H_{\mathrm{T}}(\xi)\}$. Einstein's equation (3.8) contributes four independent components coming from the four regions of the metric: $(\mu\nu) = (00), (0i), (ii), (ij)$. In 'raw' form these equations are very long, containing many angular-terms, such as various partial derivatives of the harmonic functions. However, after making use of the definition of the harmonics (see section D.1) and some algebraic manipulations, all the angular dependence separates out from the equations, reducing them to the following set of coupled ordinary differential equations:[3]

$$
\begin{aligned}
\Delta\Phi''(\xi) &= -\frac{4\,\Delta\Phi'(\xi)\,\rho'(\xi)}{\rho(\xi)} + \Delta\Phi(\xi)\left(\frac{k^2}{\rho(\xi)^2} + U''(\Phi(\xi))\right) + 2\,A(\xi)\,U'(\Phi(\xi)) \\
&\quad + A'(\xi)\,\Phi'(\xi) + \frac{k\,B(\xi)\,\Phi'(\xi)}{\rho(\xi)} - 4\,H_{\mathrm{L}}{}'(\xi)\,\Phi'(\xi) \qquad (3.9a) \\
0 &= -\frac{2\,\kappa\,\Delta\Phi(\xi)\,\rho(\xi)^2\,U'(\Phi(\xi))}{3} - 2\,\kappa\,\rho(\xi)^2\,\Delta\Phi'(\xi)\,\Phi'(\xi) \\
&\quad + A(\xi)\left(k^2 - \frac{4\,\kappa\,U(\Phi(\xi))\,\rho(\xi)^2}{3}\right) + 4\,\rho(\xi)\,A'(\xi)\,\rho'(\xi) + k\,B(\xi)\,\rho'(\xi) \\
&\quad + k\,\rho(\xi)\,B'(\xi) - 8\,\rho(\xi)\,H_{\mathrm{L}}{}'(\xi)\,\rho'(\xi) - 4\,\rho(\xi)^2\,H_{\mathrm{L}}{}''(\xi) \qquad (3.9b) \\
0 &= -\kappa\,\Delta\Phi(\xi)\,\Phi'(\xi) + \frac{3\,A(\xi)\,\rho'(\xi)}{\rho(\xi)} + \frac{3\,B(\xi)}{k\,\rho(\xi)} \\
&\quad - 3\,H_{\mathrm{L}}{}'(\xi) + 3\left(\frac{1}{k^2} - \frac{1}{4}\right) H_{\mathrm{T}}{}'(\xi) \qquad (3.9c) \\
0 &= -\frac{2\,\kappa\,\Delta\Phi(\xi)\,\rho(\xi)^2\,U'(\Phi(\xi))}{3} + A(\xi)\left(9 - \frac{k^2}{2} - \frac{4\,\kappa\,U(\Phi(\xi))\,\rho(\xi)^2}{3}\right) \\
&\quad + \rho(\xi)\,A'(\xi)\,\rho'(\xi) - \frac{(k^2-18)\;B(\xi)\,\rho'(\xi)}{2\,k} - \frac{(k^2-6)\;\rho(\xi)\,B'(\xi)}{2\,k} \\
&\quad - 8\,\rho(\xi)\,H_{\mathrm{L}}{}'(\xi)\,\rho'(\xi) - \rho(\xi)^2\,H_{\mathrm{L}}{}''(\xi) - \frac{3\;(k^2-4)\;\rho(\xi)\,H_{\mathrm{T}}{}'(\xi)\,\rho'(\xi)}{k^2} \\
&\quad - \frac{3\;(k^2-4)\;\rho(\xi)^2\,H_{\mathrm{T}}{}''(\xi)}{4\,k^2} \qquad (3.9d) \\
0 &= A(\xi) + \frac{3\,B(\xi)\,\rho'(\xi)}{k} + \frac{\rho(\xi)\,B'(\xi)}{k} + 2\,H_{\mathrm{L}}(\xi) \\
&\quad + \frac{H_{\mathrm{T}}(\xi)}{2} + \frac{4\,\rho(\xi)\,H_{\mathrm{T}}{}'(\xi)\,\rho'(\xi)}{k^2} + \frac{\rho(\xi)^2\,H_{\mathrm{T}}{}''(\xi)}{k^2} \qquad (3.9e)
\end{aligned}
$$

[3]The fact that (3.9) only involve the principal harmonic eigenvalue, k, justifies in (3.4) and (3.5) our implicit assumption that the radial functions are the same for the different modes with the same principal eigenvalue, i.e. that they only carry a k-label, rather than a full mode label, say $a = \{m_1, m_2, m_3, m_4\}$, as in

$$\widetilde{\Phi}(x) = \Phi(\xi) + \varepsilon \sum_a \Delta\Phi_a(x)\,Y_a(\omega)$$

3.3 The Action

In the previous section we derived differential equations governing the perturbation functions which in the next chapter we will attempt to solve numerically. As discussed in the beginning of this chapter, we expect the first-order perturbative contribution to the action, S_1, to be zero, since we have expanded about a stationary point – this is verified in appendix C. This leaves S_2, the second-order contribution to the action, as the quantity to calculate for each perturbation configuration. Since this term is second-order, it consists of a double-sum over the harmonic modes, and so we revert to the full form of the perturbative expansion, (3.4).

In this section we integrate-out the angular parts of S_2 leaving only a radial integral. This requires the decomposition of the harmonics in terms of the Gegenbauer functions, $\mathrm{C}[\lambda]_l^m$ (see section D.1):

$$Y_k(\omega) = \sum_{m_2=0}^{m_1} \sum_{m_3=0}^{m_2} \sum_{m_4=0}^{m_3} c_a \cos(m_4\omega_4 + \alpha_4) \prod_{i=1}^{3} \mathrm{C}\left[\tfrac{4-i}{2}\right]_{m_i}^{m_{i+1}}(\cos\omega_i) \tag{3.10}$$

where $k^2 = m_1(m_1+3)$, and c_a is the mode coefficient (a is shorthand for the mode eigenvalues, $a = \{m_1, m_2, m_3, m_4\}$).[4]

We will use the following shorthand for the five-dimensional integral and its limits:

$$\begin{aligned}\int \mathrm{d}^5x &= \int \mathrm{d}\xi \int \mathrm{d}^4\omega \\ &= \int_0^\infty \mathrm{d}\xi \int_0^\pi \mathrm{d}\psi \int_0^\pi \mathrm{d}\chi \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\end{aligned}$$

[4]To put this in familiar terms, on a 2-sphere we would have written

$$\begin{aligned}Y_k(\omega) &= \sum_{m_2=0}^{m_1} c_{m_1,m_2} \cos(m_2\omega_2 + \alpha_2)\, \mathrm{C}\left[\tfrac{1}{2}\right]_{m_1}^{m_2}(\cos\omega_1) \\ &= \sum_{m=0}^{l} c_{l,m} \cos(m\,\phi + \alpha)\, P_l^m(\cos\theta) \qquad \text{where } k^2 = l(l+1)\end{aligned}$$

So, using a prime to distinguish dummy indices, the first few terms of the expression resulting from substituting the perturbative expansion into the action and extracting the second-order part are:

$$
\begin{aligned}
S_2 = \sum_{\substack{m_1,m_2,\\ m_3,m_4}} \sum_{\substack{m'_1,m'_2,\\ m'_3,m'_4}} \int \mathrm{d}\xi \int \mathrm{d}^4\omega \Bigg\{ \\
&\frac{1}{2}\sin\theta\,\sin^2\chi\,\sin^3\psi\, Y_k(\psi,\chi,\theta,\phi) Y_{k'}(\psi,\chi,\theta,\phi)\times \\
&\qquad \rho(\xi)^4\, A_k(\xi)\, \Delta\Phi_{k'}(\xi) \left(U'(\Phi(\xi)) - \frac{\Phi'(\xi)^2}{\Phi(\xi)} \right) \\
&+ \frac{1}{2\,\kappa\, k^2} \rho(\xi)^4\, H_{\mathrm{T}k}{}''(\xi) A_{k'}(\xi)\, Y_{k'}(\psi,\chi,\theta,\phi) \Big(\\
&- k^2 \sin\theta\,\sin^2\chi\,\sin^3\psi\, Y_k(\psi,\chi,\theta,\phi) - \csc\theta\,\sin\psi\, Y_k^{(0,0,0,2)}(\psi,\chi,\theta,\phi) \\
&- \cos\theta\,\sin\psi\, Y_k^{(0,0,1,0)}(\psi,\chi,\theta,\phi) - \sin\theta\,\sin\psi\, Y_k^{(0,0,2,0)}(\psi,\chi,\theta,\phi) + \ldots \Big) + \ldots \Bigg\}
\end{aligned}
\tag{3.11}
$$

where the $Y_k(\omega)$ are now understood to be those of (3.10) with the summations moved outside.

To proceed, we separate out $Y_k(\omega)$ as per (3.10) and perform the resulting integration, one angle at a time. First we let

$$
Y_k(\psi,\chi,\theta,\phi) \longrightarrow \cos(m_4\,\phi + \alpha_{m_4})\, Y_k(\psi,\chi,\theta)
$$

where $Y_k(\psi,\chi,\theta)$ is understood to be the rest of the expression under the summation in (3.10), i.e.

$$
Y_k(\psi,\chi,\theta) = c_a \prod_{i=1}^{3} \mathrm{C}\left[\tfrac{4-i}{2}\right]_{m_i}^{m_{i+1}}(\cos\omega_i)
$$

and then perform all the ϕ-integrals using the orthogonality of cos and sin:

$$\int_0^{2\pi} \cos(m\,\phi + \alpha_m)\cos(m'\,\phi + \alpha_{m'})\,\mathrm{d}\phi = \pi\,\delta_{mm'}$$
$$\int_0^{2\pi} \sin(m\,\phi + \alpha_m)\sin(m'\,\phi + \alpha_{m'})\,\mathrm{d}\phi = \pi\,\delta_{mm'}$$
$$\int_0^{2\pi} \cos(m\,\phi + \alpha_m)\sin(m'\,\phi + \alpha_{m'})\,\mathrm{d}\phi = 0$$

Thus, (3.11) becomes

$$\begin{aligned}
S_2 = \sum_{\substack{m_1,m_2,\\ m_3,m_4}} \sum_{\substack{m_1',m_2',\\ m_3'}} \int \mathrm{d}\xi \int \mathrm{d}^3\omega \Bigg\{ \\
\frac{1}{2}\sin\theta\,\sin^2\chi\,\sin^3\psi\,Y_k(\psi,\chi,\theta)Y_{k'}(\psi,\chi,\theta)\times \\
\rho(\xi)^4\,A_k(\xi)\,\Delta\Phi_{k'}(\xi)\left(U'(\Phi(\xi)) - \frac{\Phi'(\xi)^2}{\Phi(\xi)}\right) \\
+ \frac{1}{2\,\kappa\,k^2}\rho(\xi)^4\,H_{\mathrm{T}k}{}''(\xi)A_{k'}(\xi)\,Y_{k'}(\psi,\chi,\theta)\Bigg(\\
- k^2\sin\theta\,\sin^2\chi\,\sin^3\psi\,Y_k(\psi,\chi,\theta) + {m_4}^2\csc\theta\,\sin\psi\,Y_k(\psi,\chi,\theta) \\
- \cos\theta\,\sin\psi\,Y_k^{(0,0,1)}(\psi,\chi,\theta) - \sin\theta\,\sin\psi\,Y_k^{(0,0,2)}(\psi,\chi,\theta) + \dots\Bigg) + \dots\Bigg\}
\end{aligned} \tag{3.12}$$

Next, we separate out θ in the same way:

$$Y_k(\psi,\chi,\theta) \to \mathrm{C}\left[\tfrac{1}{2}\right]_{m_3}^{m_4}(\cos\theta)\;Y_k(\psi,\chi)$$

The difference to the ϕ-case is that the θ-derivatives need extra treatment, so we let $u = \cos\theta$ and make use of the defining equation for $C\left[\frac{1}{2}\right]_{m_3}^{m_4}$ from section D.2, i.e.

$$0 = (1-u^2)C\left[\tfrac{1}{2}\right]_{m_3}^{m_4}{}''(u) - 2\,u\,C\left[\tfrac{1}{2}\right]_{m_3}^{m_4}{}'(u) + \left(m_3(m_3+1) - \frac{{m_4}^2}{1-u^2}\right)C\left[\tfrac{1}{2}\right]_{m_3}^{m_4}(u)$$

to eliminate all θ-derivatives, so that the θ-integral may be performed using the orthonormality relation[5] derived in section D.3:

$$\int_{-1}^{1} C\left[\tfrac{1}{2}\right]_{m_3}^{m_4}(u) C\left[\tfrac{1}{2}\right]_{m_3'}^{m_4}(u)\,\mathrm{d}u = \delta_{m_3 m_3'}$$

so that (3.12) becomes

$$\begin{aligned} S_2 = \sum_{\substack{m_1,m_2,\\ m_3,m_4}} \sum_{m_1',m_2'} \int \mathrm{d}\xi \int \mathrm{d}^2\omega \Bigg\{ \\ & -\frac{1}{2}\sin^2\chi\,\sin^3\psi\,Y_k(\psi,\chi)Y_{k'}(\psi,\chi)\times \\ & \qquad \rho(\xi)^4\,A_k(\xi)\,\Delta\Phi_{k'}(\xi)\left(U'(\Phi(\xi)) - \frac{\Phi'(\xi)^2}{\Phi(\xi)}\right) \\ & + \frac{1}{2\,\kappa\,k^2}\rho(\xi)^4\,H_{\mathrm{T}k}''(\xi)A_{k'}(\xi)\,Y_{k'}(\psi,\chi)\Bigg(\\ k^2\sin^2\chi\,\sin^3\psi\,Y_k(\psi,\chi) - m_3(m_3+1)\sin\psi\,Y_k(\psi,\chi) + \ldots\Bigg) + \ldots\Bigg\} \qquad (3.13) \end{aligned}$$

We then repeat the process for χ and ψ. Manipulating the Gegenbauer functions into the form of the orthogonality condition is not usually as easy as has been suggested here, due to the large variety of terms, for instance there are terms which are divergent or are otherwise difficult to integrate analytically. These terms must first be cancelled or combined with other terms before they can all be integrated. For more information about this see section D.4.

Once all the angular integrals have been performed, we are left with the ξ-integral, in which the only mode-dependence is of k (the principal mode), summed over all modes and weighted by the coefficient c_a:

$$S_2 = \pi \sum_{\substack{m_1,m_2,\\ m_3,m_4}} {c_a}^2\,S_{2,k}$$

---

[5]Since the $\mathrm{C}\left[\tfrac{1}{2}\right]$ are really just the associated Legendre functions in disguise, this may be more familiar as:

$$\int_{-1}^{1} P_l^m(u)\,P_{l'}^m(u)\,\mathrm{d}u \quad \propto \quad \delta_{ll'}$$

This means that we can calculate the action for each k-mode separately, and then trivially calculate the action for a general superposition of modes. Dropping the k-labels from the perturbation variables, we find $S_{2,k}$ is given by (3.14).

$$
\begin{aligned}
S_{2,k} = \int_0^\infty \mathrm{d}\xi \Bigg[& \\
& A(\xi)^2 \frac{\rho(\xi)^2}{\kappa}\Big(12 - 3\,\kappa\,\rho(\xi)^2 U(\Phi(\xi))\Big) + B(\xi)^2 \frac{\rho(\xi)^2}{\kappa}\left(\frac{9}{2} - \frac{4}{3}\kappa\,\rho(\xi)^2 U(\Phi(\xi))\right) \\
& + A(\xi) H_{\mathrm{L}}(\xi) \frac{\rho(\xi)^2}{3\,\kappa}\Big(32\,\kappa\,\rho(\xi)^2 U(\Phi(\xi)) - 9\,(12 + k^2)\Big) \\
& - H_{\mathrm{L}}(\xi)^2 \frac{\rho(\xi)^2}{3\,\kappa}\Big(8\,\kappa\,\rho(\xi)^2 U(\Phi(\xi)) + 9\,(k^2 - 8)\Big) \\
& - A(\xi) H_{\mathrm{T}}(\xi) \frac{3\,\rho(\xi)^2}{4\,\kappa}(k^2 - 4) - H_{\mathrm{L}}(\xi) H_{\mathrm{T}}(\xi) \frac{3\,\rho(\xi)^2}{2\,\kappa}(k^2 - 4) \\
& - H_{\mathrm{T}}(\xi)^2 \frac{\rho(\xi)^2}{16\,\kappa\,k^2}(k^2 - 4)\Big(36 + 3k^2 - 8\,\kappa\,\rho(\xi)^2 U(\Phi(\xi))\Big) \\
& + A(\xi)\Delta\Phi(\xi)\,\rho(\xi)^4 U'(\Phi(\xi)) + B(\xi)\Delta\Phi(\xi)\,k\,\rho(\xi)^3 \Phi'(\xi) \\
& + H_{\mathrm{L}}(\xi)\Delta\Phi(\xi)\,4\,\rho(\xi)^4 U'(\Phi(\xi)) + \Delta\Phi(\xi)^2 \frac{\rho(\xi)^2}{2}\Big(k^2 + \rho(\xi)^2 U''(\Phi(\xi))\Big) \\
& + A'(\xi) B(\xi) \frac{k\,\rho(\xi)^3}{\kappa} + A'(\xi) A(\xi) \frac{12\,\rho(\xi)^3 \rho'(\xi)}{\kappa} - A'(\xi) H_{\mathrm{L}}(\xi) \frac{16\,\rho(\xi)^3 \rho'(\xi)}{\kappa} \\
& + A(\xi) B'(\xi) \frac{k\,\rho(\xi)^3}{\kappa} + B(\xi) B'(\xi) \frac{4\,\rho(\xi)^3 \rho'(\xi)}{\kappa} + B(\xi) H_{\mathrm{L}}{}'(\xi) \frac{3\,k\,\rho(\xi)^3}{\kappa} \\
& - A(\xi) H_{\mathrm{L}}(\xi) \frac{20\,\rho(\xi)^3 \rho'(\xi)}{\kappa} + H_{\mathrm{L}}(\xi) H_{\mathrm{L}}{}'(\xi) \frac{40\,\rho(\xi)^3 \rho'(\xi)}{\kappa} - A'(\xi) H_{\mathrm{L}}{}'(\xi) \frac{4\,\rho(\xi)^4}{\kappa} \\
& + H_{\mathrm{L}}{}'(\xi)^2 \frac{2\,\rho(\xi)^4}{\kappa} + B(\xi) H_{\mathrm{T}}{}'(\xi) \frac{3\,\rho(\xi)^3}{4\,\kappa\,k}(k^2 - 4) \\
& - H_{\mathrm{T}}(\xi) H_{\mathrm{T}}{}'(\xi) \frac{15\,\rho(\xi)^3 \rho'(\xi)}{2\,\kappa\,k^2}(k^2 - 4) \\
& - H_{\mathrm{T}}{}'(\xi)^2 \frac{9\,\rho(\xi)^4}{8\,\kappa\,k^2}(k^2 - 4) - A(\xi)\Delta\Phi'(\xi)\,\rho(\xi)^4\,\Phi'(\xi) \\
& + H_{\mathrm{L}}(\xi)\Delta\Phi'(\xi)\,4\,\rho(\xi)^4\,\Phi'(\xi) + \Delta\Phi'(\xi)^2 \frac{\rho(\xi)^4}{2} - A(\xi) H_{\mathrm{L}}{}''(\xi) \frac{4\,\rho(\xi)^4}{\kappa} \\
& + H_{\mathrm{L}}(\xi) H_{\mathrm{L}}{}''(\xi) \frac{8\,\rho(\xi)^4}{\kappa} - H_{\mathrm{T}}(\xi) H_{\mathrm{T}}{}''(\xi) \frac{3\,\rho(\xi)^4}{2\,\kappa\,k^2}(k^2 - 4) \quad \Bigg]
\end{aligned}
\tag{3.14}
$$

3.4 Gauge Invariance

As they stand, the equations of motion for the perturbation variables (3.9) do not yield a unique solution because they contain gauge freedom from Einstein's equations – given a solution, others can be generated from it by a coordinate transformation (which we demonstrate below for a general field). To make sure we do not count a physical solution more than once, we will derive the gauge transformation under which (3.9) is invariant.

Consider some background vector field $F_\mu(x)$ and its perturbed counterpart, $\widetilde{F}_\mu(x)$ given by

$$\widetilde{F}_\mu(x) = F_\mu(x) + \varepsilon\,\Delta F_\mu(x) \tag{3.15}$$

Since we are only considering linear order perturbation theory, it is sufficient to consider only infinitesimal coordinate transformations. Doing this at $\mathcal{O}(\varepsilon)$, i.e.

$$x^\mu \longrightarrow \bar{x}^\mu = x^\mu + \varepsilon\,\delta x^\mu \tag{3.16}$$

(where an overbar indicates the new coordinate system) means that this coordinate transformation only acts at the perturbed level, leaving the background the same, i.e. $\bar{F}_\mu(x) = F_\mu(x)$. In the language of [49], this can be thought of as a change of correspondence between the unperturbed background and the perturbed world. Consider the transformation of $F_\mu(x)$ under the coordinate transformation (3.16):

$$\bar{\widetilde{F}}_\alpha(\bar{x}) = \widetilde{F}_\mu(x)\frac{\partial x^\mu}{\partial \bar{x}^\alpha} \tag{3.17}$$

The left hand side of this expression can be written explicitly in terms of the original coordinates as

$$\begin{aligned}\bar{\widetilde{F}}_\alpha(\bar{x}) &= \bar{\widetilde{F}}_\alpha(x + \varepsilon\,\delta x)\\ &\approx \bar{\widetilde{F}}_\alpha(x) + \varepsilon\,\widetilde{F}_{\alpha,\beta}(x)\,\delta x^\beta\end{aligned} \tag{3.18}$$

Thus (3.17) becomes

$$\bar{\widetilde{F}}_\alpha + \varepsilon\,\widetilde{F}_{\alpha,\beta}\,\delta x^\beta \approx \widetilde{F}_\mu\frac{\partial x^\mu}{\partial \bar{x}^\alpha} \tag{3.19}$$

Expanding this in terms of the background field (i.e. $\widetilde{F}_\alpha = F_\alpha + \varepsilon\,\Delta F_\alpha$) gives:

$$\begin{aligned}
\bar{F}_\alpha + \varepsilon\,\overline{\Delta F}_\alpha + \varepsilon\,F_{\alpha,\beta}\,\delta x^\beta &\approx (F_\mu + \varepsilon\,\Delta F_\mu)\,\frac{\partial x^\mu}{\partial \bar{x}^\alpha} \\
&\approx (F_\mu + \varepsilon\,\Delta F_\mu)\left(\delta^\mu_\alpha - \varepsilon\,\delta x^\mu{}_{,\alpha}\right) \\
&\approx F_\alpha + \varepsilon\,\Delta F_\alpha - \varepsilon\,F_\mu\,\delta x^\mu{}_{,\alpha}
\end{aligned} \tag{3.20}$$

which upon cancelling $\bar{F}_\alpha$ with F_α (since the background is unaffected by the coordinate transformation) yields the gauge transformation for ΔF_α:

$$\overline{\Delta F}_\alpha = \Delta F_\alpha - F_{\alpha,\beta}\,\delta x^\beta - F_\mu\,\delta x^\mu \tag{3.21}$$

We now wish to derive the gauge transformations for our perturbation fields. Once again the power of the theory presented in [49] is revealed: by expanding the coordinate transformation (3.16) in terms of the harmonics for scalars and vectors – (3.1) and (3.2) respectively – the different modes decouple and we are able to analyse the gauge transformations of the perturbation fields for each mode independently. Accordingly, we let the coordinate transformation (3.16) take the form

$$\xi \to \bar{\xi} = \xi + \varepsilon\,T(\xi)\,Y(\omega) \tag{3.22a}$$

$$\omega^i \to \bar{\omega}^i = \omega^i + \varepsilon\,L(\xi)\,Y^i(\omega) \tag{3.22b}$$

where T and L are arbitrary functions. Now we simply recast (3.21) for our scalar field and metric appropriately.

For the scalar field, we let $F_\mu(x) \to \Phi(\xi)$ and $\Delta F_\mu(x) \to \Delta\Phi(\xi) Y(\omega)$. Since Φ is a scalar (a rank-0 tensor) we drop the last term of (3.21) so that it becomes

$$\begin{aligned}
\overline{\Delta\Phi}(\xi)Y(\omega) &= \Delta\Phi(\xi)Y(\omega) - \Phi'(\xi)T(\xi)Y(\omega) \\
\Longrightarrow \qquad \overline{\Delta\Phi}(\xi) &= \Delta\Phi(\xi) - \Phi'(\xi)T(\xi)
\end{aligned} \tag{3.23}$$

Since the metric is a rank-2 tensor, we repeat the last term of (3.21) for the extra index, so that upon letting $F_\alpha \to g_{\alpha\beta}$ and $\Delta F_\alpha \to h_{\alpha\beta}$ we get

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - g_{\alpha\beta,\mu}\,\delta x^\mu - g_{\alpha\mu}\,\delta x^\mu{}_{,\beta} - g_{\mu\beta}\,\delta x^\mu{}_{,\alpha} \tag{3.24}$$

We now expand (3.24) for one region of the metric at a time, i.e. $(\alpha\beta) = (00), (0i), (ii)$. First, $(\alpha\beta) = (00)$:

$$\begin{aligned} \bar{h}_{00} &= h_{00} - g_{00,\mu}\,\delta x^\mu - g_{0\mu}\,\delta x^\mu{}_{,0} - g_{\mu 0}\,\delta x^\mu{}_{,0} \\ \Longrightarrow \quad 2\,\bar{A}(\xi)Y(\omega) &= 2\,A(\xi)Y(\omega) - 2\,T'(\xi)Y(\omega) \\ \Longrightarrow \quad \bar{A}(\xi) &= A(\xi) - T'(\xi) \end{aligned} \tag{3.25}$$

$(\alpha\beta) = (0i)$:

$$\begin{aligned} \bar{h}_{0i} &= h_{0i} - g_{0i,\mu}\,\delta x^\mu - g_{0\mu}\,\delta x^\mu{}_{,i} - g_{\mu i}\,\delta x^\mu{}_{,0} \\ \Longrightarrow \quad \rho(\xi)\bar{B}(\xi)Y_i(\omega) &= B(\xi)Y_i(\omega) - T(\xi)Y_{,i}(\omega) - \rho(\xi)^2\gamma_{ji}L'(\xi)Y^j(\omega) \\ &= B(\xi)Y_i(\omega) + k\,T(\xi)Y_i(\omega) - \rho(\xi)^2 L'(\xi)Y_i(\omega) \\ &= \rho(\xi)Y_i(\omega)\left(B(\xi) + k\frac{T(\xi)}{\rho(\xi)} - \rho(\xi)L'(\xi)\right) \\ \Longrightarrow \quad \bar{B}(\xi) &= B(\xi) + k\frac{T(\xi)}{\rho(\xi)} - \rho(\xi)L'(\xi) \end{aligned} \tag{3.26}$$

$(\alpha\beta) = (ii)$: (there is no implicit sum over the repeated i-index)

$$\begin{aligned} \bar{h}_{ii} &= h_{ii} - g_{ii,\mu}\,\delta x^\mu - g_{0\nu}\,\delta x^\nu{}_{,i} - g_{\mu i}\,\delta x^\mu{}_{,i} \\ &= h_{ii} - \left(\rho(\xi)^2\gamma_{ii}\right)_{,0} T(\xi)Y(\omega) - \rho(\xi)^2\gamma_{ii,m}L(\xi)Y^m(\omega) - 2\,\rho(\xi)^2\gamma_{im}L(\xi)Y^m{}_{,i}(\omega) \\ &= h_{ii} - 2\,\rho(\xi)\rho'(\xi)\gamma_{ii}T(\xi)Y(\omega) - \rho(\xi)^2\gamma_{ii,m}L(\xi)Y^m(\omega) - 2\rho(\xi)^2\gamma_{im}L(\xi)Y^m{}_{,i}(\omega) \\ &= h_{ii} - 2\rho(\xi)^2\left[\frac{\rho'(\xi)}{\rho(\xi)}\gamma_{ii}T(\xi)Y(\omega) + L(\xi)\left(\frac{1}{2}\gamma_{ii,m}Y^m(\omega) + \gamma_{im}Y^m{}_{,i}(\omega)\right)\right] \end{aligned} \tag{3.27}$$

To write the last term of this expression in the form of the tensor harmonics (3.3), consider expanding $Y^m_{|i}$:

$$
\begin{aligned}
& & Y^m{}_{|i} &= Y^m{}_{,i} + \Gamma^m{}_{ij}\, Y^j \\
&\implies & Y^m{}_{,i} &= Y^m{}_{|i} - \Gamma^m{}_{ij}\, Y^j \\
&\implies & \gamma_{im} Y^m{}_{,i} &= \gamma_{im} Y^m{}_{|i} - \gamma_{im} \Gamma^m{}_{ij}\, Y^j \\
& & &= Y_{i|i} - \Gamma_{iij}\, Y^j \\
& & &= Y_{i|i} - \frac{1}{2}\left(\gamma_{ii,j} + \gamma_{ij,i} - \gamma_{ij,i}\right) Y^j \\
& & &= Y_{i|i} - \frac{1}{2}\gamma_{ii,j} Y^j \\
&\implies & Y_{i|i} &= \frac{1}{2}\gamma_{ii,m} Y^m + \gamma_{im} Y^m{}_{,i}
\end{aligned}
$$

Thus (3.27) becomes

$$
\begin{aligned}
\bar{h}_{ii} &= h_{ii} - 2\rho(\xi)^2 \left[\frac{\rho'(\xi)}{\rho(\xi)}\gamma_{ii} T(\xi) Y(\omega) + L(\xi) Y_{i|i}(\omega)\right] \\
&= h_{ii} - 2\rho(\xi)^2 \left[\frac{\rho'(\xi)}{\rho(\xi)}\gamma_{ii} T(\xi) Y(\omega) - k\, L(\xi) Y_{ii}(\omega) + \frac{k}{4} L(\xi)\gamma_{ii} Y(\omega)\right] \\
&= 2\rho(\xi)^2 \Big[H_{\mathrm{L}}(\xi) Y(\omega)\gamma_{ii} + H_{\mathrm{T}}(\xi) Y_{ii}(\omega) - \frac{\rho'(\xi)}{\rho(\xi)}\gamma_{ii} T(\xi) Y(\omega) \\
&\qquad\qquad + k\, L(\xi) Y_{ii}(\omega) - \frac{k}{4} L(\xi)\gamma_{ii} Y(\omega)\Big] \\
&= 2\rho(\xi)^2 \left[\gamma_{ii} Y(\omega)\left(H_{\mathrm{L}}(\xi) - \frac{\rho'(\xi)}{\rho(\xi)} T(\xi) - \frac{k}{4} L(\xi)\right) + Y_{ii}(\omega)\left(H_{\mathrm{T}}(\xi) + k\, L(\xi)\right)\right]
\end{aligned} \tag{3.28}
$$

Expanding the left hand side of this expression:

$$
\bar{h}_{ii} = 2\,\rho(\xi)^2 \left[\overline{H}_{\mathrm{L}}(\xi) Y(\omega)\gamma_{ii} + \overline{H}_{\mathrm{T}}(\xi) Y_{ii}(\omega)\right]
$$

allows us to see that

$$
\overline{H}_{\mathrm{L}}(\xi) = H_{\mathrm{L}}(\xi) - \frac{\rho'(\xi)}{\rho(\xi)} T(\xi) - \frac{k}{4} L(\xi) \tag{3.29}
$$

$$
\overline{H}_{\mathrm{T}}(\xi) = H_{\mathrm{T}}(\xi) + k\, L(\xi) \tag{3.30}
$$

To recapitulate; we have found the perturbation fields gauge transform as:

$$\overline{\Delta\Phi}(\xi) = \Delta\Phi(\xi) - \Phi'(\xi)\,T(\xi) \tag{3.31a}$$

$$\bar{A}(\xi) = A(\xi) - T'(\xi) \tag{3.31b}$$

$$\bar{B}(\xi) = B(\xi) + k\frac{T(\xi)}{\rho(\xi)} - \rho(\xi)\,L'(\xi) \tag{3.31c}$$

$$\overline{H_{\mathrm{L}}}(\xi) = H_{\mathrm{L}}(\xi) - \frac{\rho'(\xi)}{\rho(\xi)}T(\xi) - \frac{k}{4}L(\xi) \tag{3.31d}$$

$$\overline{H_{\mathrm{T}}}(\xi) = H_{\mathrm{T}}(\xi) + k\,L(\xi) \tag{3.31e}$$

i.e. performing these transformations together leaves equations (3.9) unchanged.

By eliminating the gauge fields $L(\xi)$ and $T(\xi)$ from the transformation equations (3.31), we find combinations of the original fields which are gauge invariant. We call these $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z}$:

$$\mathcal{A}(\xi) = A(\xi) - \frac{\partial}{\partial\xi}\left[\frac{\rho(\xi)}{\rho'(\xi)}\left(H_{\mathrm{L}}(\xi) + \frac{H_{\mathrm{T}}(\xi)}{4}\right)\right] \tag{3.32a}$$

$$\mathcal{B}(\xi) = B(\xi) + \frac{\rho(\xi)}{k}H_{\mathrm{T}}'(\xi) + \frac{k}{\rho'(\xi)}\left(H_{\mathrm{L}}(\xi) + \frac{H_{\mathrm{T}}(\xi)}{4}\right) \tag{3.32b}$$

$$\mathcal{Z}(\xi) = \frac{\rho'(\xi)}{\rho(\xi)\,\Phi'(\xi)}\Delta\Phi(\xi) - \left(H_{\mathrm{L}}(\xi) + \frac{H_{\mathrm{T}}(\xi)}{4}\right) \tag{3.32c}$$

3.4.1 The Gauge Invariant Equations of Motion

Using (3.32) to substitute for A, B, and $\Delta\Phi$ in (3.9), the equations of motion become (3.33), which are expressed soley in terms of the gauge invariant variables (i.e. H_{L} and H_{T} have 'dropped out'). Thus we now have five equations in three variables ($\mathcal{A}$, $\mathcal{B}$, and $\mathcal{Z}$) which would appear to be over-constrained. We will return to them in the next chapter where we will solve them numerically.

To solve for S_2 it will be necessary to make the same transformation (to gauge independent variables), but since this is not so straight-forward (total divergences and thus boundary terms need to be considered), this is delayed until section 4.2.

The equations of motion expressed in gauge invariant variables are:

$$
\begin{aligned}
0 = \;& 2\,\mathcal{A}(\xi)\,\rho(\xi)^3\,U'(\Phi(\xi)) + \rho(\xi)^3\,\mathcal{A}'(\xi)\,\Phi'(\xi) + k\,\mathcal{B}(\xi)\,\rho(\xi)^2\,\Phi'(\xi) \\
&+ \frac{\mathcal{Z}(\xi)\,\rho(\xi)^2}{36\,\rho'(\xi)^4}\Bigg[-72\,\rho(\xi)\,U'(\Phi(\xi)) + 36\,k^2\,\rho'(\xi)\,\Phi'(\xi) \\
&\quad - 16\,\kappa^2\,U(\Phi(\xi))\,\rho(\xi)^4\,\rho'(\xi)\,\Phi'(\xi)^3 \\
&\quad + 6\,\kappa\,\rho(\xi)^3\,U'(\Phi(\xi))\left(2\,U(\Phi(\xi)) - 9\,\Phi'(\xi)^2\right) \\
&\quad - 4\,\kappa^2\,\rho(\xi)^5\,U'(\Phi(\xi))\,\Phi'(\xi)^2\left(-2\,U(\Phi(\xi)) + \Phi'(\xi)^2\right) \\
&\quad + 3\,\kappa\,\rho(\xi)^2\,\rho'(\xi)\,\Phi'(\xi)\left(-2\,\left(2+k^2\right)\,U(\Phi(\xi)) + \left(8+k^2\right)\,\Phi'(\xi)^2\right)\Bigg] \\
&- \frac{\rho(\xi)^3\,\mathcal{Z}'(\xi)}{3\,\rho'(\xi)^2}\Bigg[6\,\rho(\xi)\,U'(\Phi(\xi))\,\rho'(\xi) - 6\,\Phi'(\xi) \\
&\quad + \kappa\,\rho(\xi)^2\,\Phi'(\xi)\left(2\,U(\Phi(\xi)) + \Phi'(\xi)^2\right)\Bigg] - \frac{\rho(\xi)^4\,\Phi'(\xi)\,\mathcal{Z}''(\xi)}{\rho'(\xi)}
\end{aligned} \tag{3.33a}
$$

$$
\begin{aligned}
0 = \;& \mathcal{A}(\xi)\left(k^2 - \frac{4\,\kappa\,U(\Phi(\xi))\,\rho(\xi)^2}{3}\right) + 4\,\rho(\xi)\,\mathcal{A}'(\xi)\,\rho'(\xi) + k\,\mathcal{B}(\xi)\,\rho'(\xi) + k\,\rho(\xi)\,\mathcal{B}'(\xi) \\
&- \frac{2\,\kappa\,\mathcal{Z}(\xi)\,\rho(\xi)^2\,\Phi'(\xi)}{3\,\rho'(\xi)^2}\left(4\,\rho(\xi)\,U'(\Phi(\xi))\,\rho'(\xi) - 9\,\Phi'(\xi) + 2\,\kappa\,U(\Phi(\xi))\,\rho(\xi)^2\,\Phi'(\xi)\right) \\
&- \frac{2\,\kappa\,\rho(\xi)^3\,\mathcal{Z}'(\xi)\,\Phi'(\xi)^2}{\rho'(\xi)}
\end{aligned} \tag{3.33b}
$$

$$
0 = \frac{3\,\mathcal{A}(\xi)\,\rho'(\xi)}{\rho(\xi)} + \frac{3\,\mathcal{B}(\xi)}{k\,\rho(\xi)} - \frac{\kappa\,\mathcal{Z}(\xi)\,\rho(\xi)\,\Phi'(\xi)^2}{\rho'(\xi)} \tag{3.33c}
$$

$$
\begin{aligned}
0 = \;& \mathcal{A}(\xi)\left(9 + \frac{-k^2}{2} + \frac{-4\,\kappa\,U(\Phi(\xi))\,\rho(\xi)^2}{3}\right) + \rho(\xi)\,\mathcal{A}'(\xi)\,\rho'(\xi) \\
&+ \left(18 - k^2\right)\frac{\mathcal{B}(\xi)\,\rho'(\xi)}{2\,k} + \left(6 - k^2\right)\frac{\rho(\xi)\,\mathcal{B}'(\xi)}{2\,k} \\
&+ \frac{-2\,\kappa\,\mathcal{Z}(\xi)\,\rho(\xi)^3\,U'(\Phi(\xi))\,\Phi'(\xi)}{3\,\rho'(\xi)}
\end{aligned} \tag{3.33d}
$$

$$
0 = \mathcal{A}(\xi) + \frac{3\,\mathcal{B}(\xi)\,\rho'(\xi)}{k} + \frac{\rho(\xi)\,\mathcal{B}'(\xi)}{k} \tag{3.33e}
$$

Chapter 4

Seeking Solutions

In the previous chapter we setup perturbations of the bounce and derived the equations of motion for the gauge-invariant perturbation variables for each mode. In this chapter we will solve these equations numerically. As we do so, we will also want to calculate the action of each mode at the same time. To do this, we will need to express the action in terms of gauge-invariant variables alone, so that we can treat it as a differential equation that can be integrated numerically, just as we did for S_0 in section 2.2.2.

4.1 Reducing the Equations of Motion

The equations of motion for the gauge-invariant variables, (3.33), are five equations for three variables, $(\mathcal{A}(\xi), \mathcal{B}(\xi), \mathcal{Z}(\xi))$, which makes them over-constrained. To solve them numerically, we first solve them for the highest derivatives of the variables, i.e. $(\mathcal{A}'(\xi), \mathcal{B}'(\xi), \mathcal{Z}''(\xi))$. Attempting to do so reduces them to the following three equations, two of which are non-differential equations for $\mathcal{A}(\xi)$ and $\mathcal{B}(\xi)$ in terms of $\mathcal{Z}(\xi)$ and $\mathcal{Z}'(\xi)$, and the third of which is an uncoupled second order differential equation for $\mathcal{Z}(\xi)$:

$$\mathcal{A}(\xi) = \frac{\kappa\, \rho(\xi)^2 \Phi'(\xi)^2}{\rho'(\xi)\, s(\xi)} \left(\mathcal{Z}(\xi) \frac{s(\xi) - 3}{3\, \rho'(\xi)} - \rho(\xi) \mathcal{Z}'(\xi) \right) \tag{4.1a}$$

$$\mathcal{B}(\xi) = \frac{\kappa\, k\, \rho(\xi)^2 \Phi'(\xi)^2}{s(\xi)} \left(\frac{\mathcal{Z}(\xi)}{\rho'(\xi)} + \rho(\xi) \mathcal{Z}'(\xi) \right) \tag{4.1b}$$

$$
\begin{aligned}
\mathcal{Z}''(\xi) = \frac{\mathcal{Z}(\xi)}{s(\xi)} \Bigg[& 2\,k^2\,\kappa\,\Phi'(\xi)^2 + \frac{6\,(k^2-4)+s(\xi)\,(k^2+2)}{\rho(\xi)^2} \\
& + \frac{6\,(k^2-4)\,U'\left(\Phi(\xi)\right)\,\rho'(\xi)}{\rho(\xi)\,\Phi'(\xi)} \Bigg] \\
+ \frac{\mathcal{Z}'(\xi)}{s(\xi)} \Bigg[& 2\,k^2\,\kappa\,\rho(\xi)\,\rho'(\xi)\,\Phi'(\xi)^2 + \frac{2\,\rho'(\xi)}{\rho(\xi)}\left(3\,(k^2-4)+2\,s(\xi)\right) \\
& + \frac{6\,(k^2-4)\,U'\left(\Phi(\xi)\right)\,\rho'(\xi)^2}{\Phi'(\xi)} \Bigg]
\end{aligned}
\tag{4.1c}
$$

where $s(\xi)$ is the shorthand:

$$
s(\xi) \equiv 12 - 2\,\kappa\,\rho(\xi)^2\,U\left(\Phi(\xi)\right) - 3\,k^2\,\rho'(\xi)^2
$$

Thus the problem of solving for the perturbation variables has been reduced to solving a single differential equation, i.e. (4.1c).

To solve this equation we will need some boundary conditions, but before proceeding with this we first attempt to obtain an expression for the action in terms of the gauge invariant variables (so that it can be solved for at the same time as $\mathcal{Z}(\xi)$), which we will see also requires knowledge of boundary conditions.

4.2 The Gauge Invariant Action

As we solve equation (4.1c) for each value of k, we will also want to calculate $S_{2,k}$. But, since our current expression for $S_{2,k}$, (3.14), is in terms of the original variables, it is insufficient.

We first make the same substitutions from the original variables to the gauge-invariant ones (3.32), as we did for the equations of motion in section 3.4.1. However, unlike the equations of motion, after making the substitution the action still contains terms involving $H_{\mathrm{L}}(\xi)$ and $H_{\mathrm{T}}(\xi)$, so that the action takes the form:

$$
S_{2,k} = \int^{\infty} \mathrm{d}\xi \Big[\text{terms bi-linear in } (\mathcal{A},\mathcal{A}',\mathcal{B},\mathcal{B}',\mathcal{Z},\mathcal{Z}',H_{\mathrm{T}},H_{\mathrm{T}}',H_{\mathrm{T}}'',H_{\mathrm{L}},H_{\mathrm{L}}',H_{\mathrm{L}}'')\Big]
$$

$$
S_{2,k} = \int_0^{\infty} \mathrm{d}\xi \Big[\text{terms bi-linear in } (\mathcal{A},\mathcal{A}',\mathcal{B},\mathcal{B}',\mathcal{Z},\mathcal{Z}',H_{\mathrm{T}},H_{\mathrm{T}}',H_{\mathrm{T}}'',H_{\mathrm{L}},H_{\mathrm{L}}',H_{\mathrm{L}}'')
$$

We can use (4.1) to eliminate $\mathcal{A}(\xi)$ and $\mathcal{B}(\xi)$ from the terms paired with a $H_{\mathrm{L}}(\xi)$-

or $H_\mathrm{T}(\xi)$-term, but still $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$ remain:

$$\begin{aligned} S_{2,k} = \int_0^\infty \mathrm{d}\xi \Big[&\text{terms bi-linear in } (\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}', \mathcal{Z}, \mathcal{Z}') \\ &+ \text{terms bi-linear in } (\mathcal{Z}, \mathcal{Z}', H_\mathrm{T}, H_\mathrm{T}', H_\mathrm{T}'', H_\mathrm{L}, H_\mathrm{L}', H_\mathrm{L}'') \Big] \end{aligned} \tag{4.2}$$

If it were not for the remaining $H_\mathrm{L}(\xi)$- and $H_\mathrm{T}(\xi)$-terms, (4.2) would be ready for numerical evaluation alongside $\mathcal{Z}(\xi)$.

The integral of the unwanted terms can in fact proved to be zero, by virtue of them being equivalent to a total derivative. For instance, consider the following part of (4.2):

$$\frac{1}{\kappa}\int_0^\infty \mathrm{d}\xi \left[\frac{\rho(\xi)^5}{\rho'(\xi)}\Big(H_\mathrm{L}'(\xi)H_\mathrm{T}''(\xi) + H_\mathrm{L}''(\xi)H_\mathrm{T}'(\xi)\Big)\right]$$

The integrand of this expression can be expressed as part of a total derivative:

$$\begin{aligned} \frac{\partial}{\partial\xi}\left(H_\mathrm{L}'(\xi)H_\mathrm{T}'(\xi)\frac{\rho(\xi)^5}{\rho'(\xi)}\right) = {}& H_\mathrm{L}'(\xi)H_\mathrm{T}''(\xi)\frac{\rho(\xi)^5}{\rho'(\xi)} + H_\mathrm{L}''(\xi)H_\mathrm{T}'(\xi)\frac{\rho(\xi)^5}{\rho'(\xi)} \\ &+ H_\mathrm{L}'(\xi)H_\mathrm{T}'(\xi)\frac{\partial}{\partial\xi}\left(\frac{\rho(\xi)^5}{\rho'(\xi)}\right) \end{aligned}$$

Therefore, we can say that

$$\begin{aligned} &\int_0^\infty \mathrm{d}\xi \left[\frac{\rho(\xi)^5}{\rho'(\xi)}\Big(H_\mathrm{L}'(\xi)H_\mathrm{T}''(\xi) + H_\mathrm{L}''(\xi)H_\mathrm{T}'(\xi)\Big)\right] = \\ &\qquad - \int_0^\infty \mathrm{d}\xi \left[H_\mathrm{L}'(\xi)H_\mathrm{T}'(\xi)\frac{\partial}{\partial\xi}\left(\frac{\rho(\xi)^5}{\rho'(\xi)}\right)\right] + \left[H_\mathrm{L}'(\xi)H_\mathrm{T}'(\xi)\frac{\rho(\xi)^5}{\rho'(\xi)}\right]_0^\infty \end{aligned}$$

i.e. we have replaced terms in the integrand involving second derivatives of $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$ with terms involving only first derivatives, and introduced boundary terms. By systematically transforming all second derivatives and then first derivatives of $H_\mathrm{T}(\xi)$ and then $H_\mathrm{L}(\xi)$ in this way, eventually all of the $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$ terms are eliminated from the integrand, leaving only terms to be evaluated on the boundaries:

$$S_{2,k} = \int_0^\infty d\xi \left[\text{terms bi-linear in } (\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}', \mathcal{Z}, \mathcal{Z}')\right] + \left[\text{terms bi-linear in } (\mathcal{Z}, \mathcal{Z}', H_\mathrm{T}, H_\mathrm{T}', H_\mathrm{T}'', H_\mathrm{L}, H_\mathrm{L}', H_\mathrm{L}'')\right]_0^\infty \tag{4.3}$$

Thus all that remains is to show that the boundary terms are equal to zero. To do this we now turn our attention to the behaviour of $\mathcal{Z}(\xi)$, $H_\mathrm{L}(\xi)$, and $H_\mathrm{T}(\xi)$ in the limits $\xi \to 0$ and $\xi \to \infty$.

4.3 The Boundary Conditions

In order to solve (4.1c) numerically we will need to know how $\mathcal{Z}(\xi)$ behaves in the limit $\xi \to 0$ so that we can determine appropriate initial values. As we will see later, we will also want to know how $\mathcal{Z}(\xi)$ behaves as $\xi \to \infty$ to be able to check the validity of our numerical solution.

In the previous section we also saw that the action can only be expressed in an explicitly gauge-invariant way if it can be shown that certain terms evaluated on the boundaries of integration (i.e. zero and infinity) are equal to zero. To do this we need to know something about the behaviour of $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$ in these limits. Since we only need to set an upper limit on the boundary terms (and show that this is zero), it will suffice to know the largest possible form of $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$ consistent with their definition (rather than their exact form).

Asymptotic forms for $\mathcal{Z}(\xi)$

The limiting behaviour for $\Phi(\xi)$ and $\rho(\xi)$ as $\xi \to 0$ is given by (2.29). By substituting these approximations into (4.1c) and proposing suitable ansatz, we can deduce that in this limit two independent solutions are:

$$\mathcal{Z}(\xi) \propto \xi^{-(m+5)} \left(1 + \mathcal{O}(\xi^2)\right) \tag{4.4a}$$

$$\mathcal{Z}(\xi) \propto \xi^{m-2} \left(1 + \mathcal{O}(\xi^2)\right) \tag{4.4b}$$

where $k^2 = m(m+3)$. If we use the relation between $\Delta\Phi(\xi)$ and $\mathcal{Z}(\xi)$, (3.32c), neglecting the $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$ terms, we see this translates to the solutions:

$$\Delta\Phi(\xi) \propto \xi^{-(m+3)}\left(1+\mathcal{O}(\xi^2)\right) \tag{4.5a}$$
$$\Delta\Phi(\xi) \propto \xi^{m}\left(1+\mathcal{O}(\xi^2)\right) \tag{4.5b}$$

(where the relation between the solutions seems natural since m and $-(m+3)$ give the same value of k^2). Since $\Delta\Phi(\xi)$ must be everywhere finite, it is clear that the first of these solutions is inadmissible, and we will therefore take (4.4b) as our boundary condition.

Repeating this process using the approximations for $\Phi(\xi)$ and $\rho(\xi)$ as $\xi \to \infty$, i.e. (2.34), and recalling that $z = \xi + b$ and $\beta \equiv \sqrt{U''(\Phi_\mathrm{F})}$ (see section 2.2.2), we find the solutions for $\mathcal{Z}(\xi)$ in this limit are:

$$\mathcal{Z}(z) \propto \frac{1}{z}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{4.6a}$$
$$\mathcal{Z}(z) \propto \frac{e^{2\beta z}}{z}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{4.6b}$$

which translate to

$$\Delta\Phi(z) \propto \frac{e^{-\beta z}}{z^2}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{4.7a}$$
$$\Delta\Phi(z) \propto \frac{e^{\beta z}}{z^2}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{4.7b}$$

Once again the requirement that $\Delta\Phi(\xi)$ be finite renders the second of these solutions inadmissible.

Asymptotic limits for $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$

To set limits on the size of $H_\mathrm{L}(\xi)$ and $H_\mathrm{T}(\xi)$ we recall that they are effectively the perturbations to the ω^i-ω^j region of the metric and that they enter with factors of $\rho(\xi)^2$, i.e. we can say that

$$h_{ij} \sim \rho(\xi)^2 H_\mathrm{L}(\xi)\,,\ \rho(\xi)^2 H_\mathrm{T}(\xi)$$

Since $H_{\mathrm{L}}(\xi)$ and $H_{\mathrm{T}}(\xi)$ enter h_{ij} in the same way, they will have the same limiting behaviour, so while the following discussion is in terms of $H_{\mathrm{L}}(\xi)$, the same is true for $H_{\mathrm{T}}(\xi)$.

Since h_{ij} must be smooth everywhere, at the coordinate origin its ξ-derivative must be finite and therefore at most constant, i.e.

$$\lim_{\xi\to 0} \partial_\xi h_{ij} \lesssim \xi^0 \tag{4.8}$$

To translate this into a limit on $H_{\mathrm{L}}(\xi)$, let

$$\lim_{\xi\to 0} H_{\mathrm{L}}(\xi) \sim \xi^{q_0} \tag{4.9}$$

To find the limit for the exponent q_0 we substitute (4.9) into (4.8) and use the fact that $\lim\limits_{\xi\to 0}\rho(\xi) = \xi$:

$$\begin{aligned} & \lim_{\xi\to 0} \partial_\xi \left(\xi^{2+q_0}\right) \leq \xi^0 \\ \Longrightarrow \quad & \lim_{\xi\to 0} \xi^{1+q_0} \leq \xi^0 \\ \Longrightarrow \quad & 1+q_0 \geq 0 \\ \Longrightarrow \quad & q_0 \geq -1 \end{aligned}$$

In the limit $\xi \to z \to \infty$ the restriction on h_{ij} is that it must vanish:

$$\lim_{z\to\infty} h_{ij} = 0$$

We make a similar argument as above by letting

$$\lim_{z\to\infty} H_{\mathrm{L}}(z) \sim z^{q_\infty}$$

Thus, using the fact that $\lim\limits_{z\to\infty}\rho(z) = z$, we deduce:

$$q_\infty < -2$$

So now that we know the maximum size we can expect $H_{\mathrm{L}}(\xi)$ and $H_{\mathrm{T}}(\xi)$ to have $\left(\text{i.e. } \lim_{\xi\to 0} H_{\mathrm{L}}(\xi) = \xi^{-1} \text{ and } \lim_{z\to\infty} H_{\mathrm{L}}(z) = z^{-2}\right)$, we now return to where we left off in deriving the gauge-invariant action in section 4.2.

4.4 The Gauge Invariant Action (at last)

Now that we know how the fields behave in the limits $\xi \to 0$ and $\xi \to \infty$ we are ready to evaluate the boundary terms of (4.3). We return to the example term considered in section 4.2, i.e.

$$\left[H_{\mathrm{L}}'(\xi) H_{\mathrm{T}}'(\xi) \frac{\rho(\xi)^5}{\rho'(\xi)}\right]_0^\infty$$

First consider the limit at $\xi = 0$:

$$\begin{aligned} H_{\mathrm{L}}'(\xi) H_{\mathrm{T}}'(\xi) \frac{\rho(\xi)^5}{\rho'(\xi)}\bigg|_{\xi=0} &= \lim_{\xi\to 0} H_{\mathrm{L}}'(\xi) H_{\mathrm{T}}'(\xi) \frac{\rho(\xi)^5}{\rho'(\xi)} \\ &\lesssim \lim_{\xi\to 0} \xi^{-2} \times \xi^{-2} \times \xi^5 \\ &= \lim_{\xi\to 0} \xi \\ &= 0 \end{aligned}$$

Similarly for the limit at $\xi = \infty$:

$$\begin{aligned} H_{\mathrm{L}}'(\xi) H_{\mathrm{T}}'(\xi) \frac{\rho(\xi)^5}{\rho'(\xi)}\bigg|_{\xi=\infty} &= \lim_{z\to\infty} H_{\mathrm{L}}'(z) H_{\mathrm{T}}'(z) \frac{\rho(z)^5}{\rho'(z)} \\ &\lesssim \lim_{z\to\infty} z^{-3} \times z^{-3} \times z^5 \\ &= \lim_{z\to\infty} z^{-1} \\ &= 0 \end{aligned}$$

In this way we find that the entire boundary term of (4.3) is equal to zero and thus we now give the full form for the action expressed soley in terms of gauge-invariant variables:

$$
\begin{aligned}
S_{2,k} = \int_0^\infty \mathrm{d}\xi \Bigg[\\
&\mathcal{A}(\xi)^2 \frac{\rho(\xi)^2}{\kappa}\Big(12 - 3\,\kappa\,\rho(\xi)^2\,U(\Phi(\xi))\Big) + \mathcal{B}(\xi)^2\,\frac{\rho(\xi)^2}{\kappa}\left(\frac{9}{2} - \frac{4}{3}\,\kappa\,\rho(\xi)^2\,U(\Phi(\xi))\right) \\
&+ \mathcal{A}(\xi)\mathcal{Z}(\xi)\frac{\rho(\xi)^4\,\Phi'(\xi)^2}{3\,\rho'(\xi)^2}\Big(9 - 2\,\kappa\,\rho(\xi)^2\,U(\Phi(\xi))\Big) + \mathcal{B}(\xi)\mathcal{Z}(\xi)\frac{k\,\rho(\xi)^4\,\Phi'(\xi)^2}{\rho'(\xi)} \\
&+ \mathcal{Z}(\xi)^2\frac{\rho(\xi)^4}{72\,\rho'(\xi)^4}\Big[36\,\Phi'(\xi)^2\left(9 + k^2\,\rho'(\xi)^2\right) - 216\,\rho(\xi)\,\rho'(\xi)^3\,\Phi'(\xi)\,U'(\Phi(\xi)) \\
&\qquad + 36\,\rho(\xi)^2\Big(\rho'(\xi)^2 U'(\Phi(\xi))^2 + \Phi'(\xi)^2\left(-4\,\kappa\,U(\Phi(\xi)) + U''(\Phi(\xi))\right)\Big) \\
&\qquad + 6\,\kappa\,\rho(\xi)^3\,\rho'(\xi)\,\Phi'(\xi)\,U'(\Phi(\xi))\left(2\,U'(\Phi(\xi)) + 3\,\Phi'(\xi)^2\right) \\
&\qquad + \kappa\,\rho(\xi)^4\,\Phi'(\xi)^2\Big(16\,\kappa\,U(\Phi(\xi))^2 - 6\,U(\Phi(\xi))\,U''(\Phi(\xi)) + 3\,\Phi'(\xi)^2\,U''(\Phi(\xi))\Big)\Big] \\
&+ \mathcal{A}(\xi)\mathcal{A}'(\xi)\frac{12\,\rho(\xi)^3\,\rho'(\xi)}{\kappa} + \Big(\mathcal{A}(\xi)\mathcal{B}'(\xi) + \mathcal{B}(\xi)\mathcal{A}'(\xi)\Big)\frac{k\,\rho(\xi)^3}{\kappa} \\
&+ \mathcal{B}(\xi)\mathcal{B}'(\xi)\frac{4\,\rho(\xi)^3\,\rho'(\xi)}{\kappa} - \mathcal{A}(\xi)\mathcal{Z}'(\xi)\frac{\rho(\xi)^5\,\Phi'(\xi)^2}{\rho'(\xi)} + \mathcal{Z}'(\xi)^2\,\frac{\rho(\xi)^6\,\Phi'(\xi)^2}{2\,\rho'(\xi)^2} \\
&+ \mathcal{Z}(\xi)\mathcal{Z}'(\xi)\frac{\rho(\xi)^5\,\Phi'(\xi)}{3\,\rho'(\xi)^3}\Big(3\,\rho(\xi)\,\rho'(\xi)\,U'(\Phi(\xi)) - 9\,\Phi'(\xi) + 2\,\kappa\,\rho(\xi)^2\,\Phi'(\xi)\,U(\Phi(\xi))\Big)\Bigg]
\end{aligned}
\tag{4.10}
$$

(4.10) can be treated as a differential equation to be solved alongside $\mathcal{Z}(\xi)$, just as S_0 was.

4.5 The Numerical Integration

We have already obtained the form of $\mathcal{Z}(\xi)$ as $\xi \to 0$ and can thus generate appropriate initial conditions for $\mathcal{Z}(\xi)$ and $\mathcal{Z}'(\xi)$ which are needed to numerically integrate (4.1c). We solve this equation along-side those for $\Phi(\xi)$ and $\rho(\xi)$, which together with the equation for the action, (4.10) makes a seventh order set of ordinary differential equations.

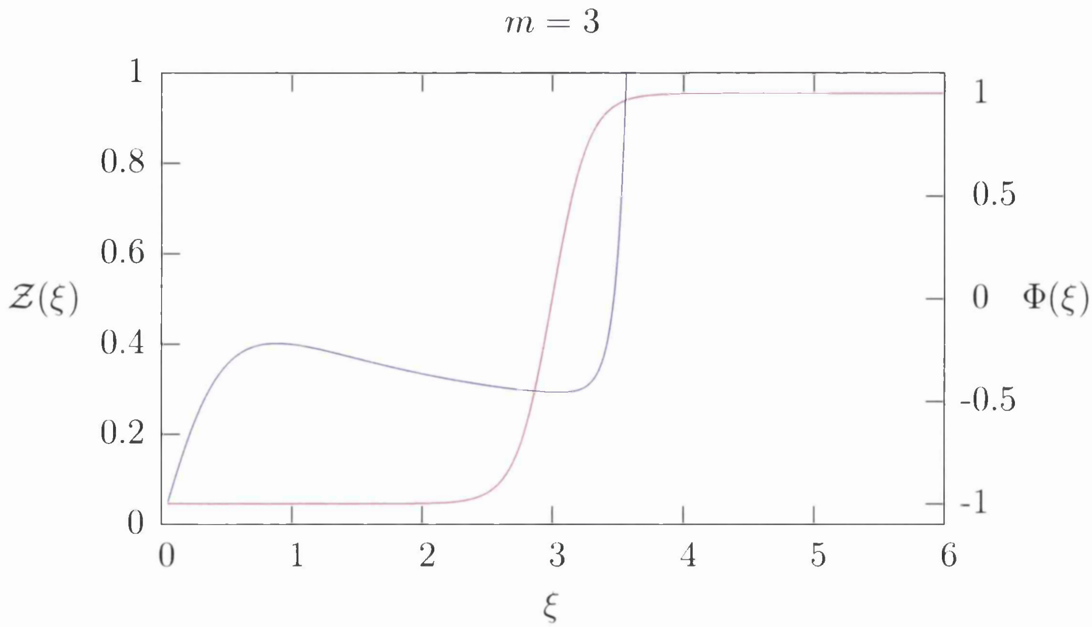

Figure 4.1: $\mathcal{Z}(\xi)$ is plotted in blue on the left-hand scale, and $\Phi(\xi)$ in red using the right-hand scale. Using the well-behaved solution at the origin results in $\mathcal{Z}(\xi)$ diverging when $\Phi(\xi)$ crosses from true vacuum to false.

Unfortunately, on integrating out, we find that soon after passing the wall $\mathcal{Z}(\xi)$ diverges, as shown in figure 4.1. From (4.6) we knew that it was possible for $\mathcal{Z}(\xi)$ to diverge exponentially as $\xi \to \infty$, so we need to consider our situation in more detail. Equation (4.1c) is a second order linear differential equation, so we expect its solutions to form a two-dimensional vector space. In section 4.3 we found a basis for this solution-space in the limits $\xi \to 0$ and $\xi \to \infty$. However, we found that in both limits only one of the two basis solutions was 'good' (i.e. physically acceptable), and therefore the only chance of there being a complete physical solution is if the good basis solution in each limit turn out to be limits of the same solution – mathematically we have no right to *expect* this, but the assumption that a physical solution exists might lead one to *hope* it. To put this in terms of the expansions (4.4) and (4.6) we let:

$$\lim_{\xi \to 0} \mathcal{Z}(\xi) = \alpha_{0,g}\, \mathcal{Z}_{0,g}(\xi) + \alpha_{0,b}\, \mathcal{Z}_{0,b}(\xi) \tag{4.11}$$

$$\lim_{\xi \to \infty} \mathcal{Z}(\xi) = \alpha_{\infty,g}\, \mathcal{Z}_{\infty,g}(\xi) + \alpha_{\infty,b}\, \mathcal{Z}_{\infty,b}(\xi) \tag{4.12}$$

where the α are arbitrary coefficients with the 'g' subscript indicating the 'good' (physical) solution, and the 'b' subscript indicating the 'bad' (divergent) solution.[1] Our 'hope' is that the solution $(\alpha_{0,g}, \alpha_{0,b}) = (1, 0)$ integrates out to the solution $(\alpha_{\infty,g}, \alpha_{\infty,b}) = (1, 0)$. As we integrate for $\mathcal{Z}(\xi)$, at each step there is some numerical error, so that even if at some point our solution is the one that is well-behaved at infinity, $(\alpha_{\infty,g}, \alpha_{\infty,b}) = (1, 0)$, eventually it will pick up some of the bad solution. Since the bad solution grows exponentially, even having a small amount of it present will over-shadow the good solution, so that numerically we are doomed to always find a divergent solution as $\xi \to \infty$. However, to check that a physical solution exits, in theory we need only show that $(\alpha_{0,g}, \alpha_{0,b}) = (1, 0)$ gives the best possible (i.e. smallest) solution as $\xi \to \infty$. To do this, we need only consider two ratios of $\alpha_{0,g} : \alpha_{\infty,g}$; integrate these solutions out to some value of ξ, say ξ_{f}, and from the values of $\mathcal{Z}(\xi_{\mathrm{f}})$ determine what ratio of $\alpha_{0,g} : \alpha_{\infty,g}$ would give $\mathcal{Z}(\xi_{\mathrm{f}}) = 0$ (which we are considering to be synonymous with $(\alpha_{\infty,g}, \alpha_{\infty,b}) = (1, 0)$). If we find this ratio to be $(\alpha_{0,g}, \alpha_{0,b}) = (1, 0)$ then a physical solution exists, and the problem becomes that of finding a best possible numerical fit to it, possibly as we did before for $\Phi(\xi)$ by patching solutions together (see section B.2).

Unfortunately we fall down at the first hurdle: considering two ratios of $(\alpha_{0,g}, \alpha_{0,b})$. This is because the omitted terms of the bad solution are bigger than the leading order term of the good solution, so an initial condition with $\alpha_{0,b} \neq 0$ cannot have $\alpha_{0,g}$ specified with any accuracy:

$$\begin{aligned}
\lim_{\xi \to 0} \mathcal{Z}(\xi) &= \alpha_{0,g}\, \mathcal{Z}_{0,g}(\xi) + \alpha_{0,b}\, \mathcal{Z}_{0,b}(\xi) \\
&= \alpha_{0,g}\, (\xi^{m-2} + \mathcal{O}(\xi^{m})) + \alpha_{0,b}\, (\xi^{-(m+5)} + \mathcal{O}\big(\xi^{-(m+3)}\big)) \\
&= \alpha_{0,b}\, \xi^{-(m+5)} + \mathcal{O}\big(\xi^{-(m+3)}\big)
\end{aligned}$$

Thus the only initial condition we can confidently use is $(\alpha_{0,g}, \alpha_{0,b}) = (1, 0)$.

[1] A solution is characterised by the ratio of $\alpha_g : \alpha_b$, so we consider solutions normalised by ${\alpha_g}^2 + {\alpha_b}^2 = 1$.

The antidote to this problem is to begin integration at some mid-point, say $\xi_{\rm m} = 2.5$; designate two solutions as our basis solutions, say $(\mathcal{Z}(\xi_{\rm m}), \mathcal{Z}'(\xi_{\rm m})) = (1, 0)$ and $(\mathcal{Z}(\xi_{\rm m}), \mathcal{Z}'(\xi_{\rm m})) = (0, 1)$, and once again find the ratio of these solutions that gives the best solution as $\xi \to \infty$. We then integrate $(\alpha_{0,g}, \alpha_{0,b}) = (1, 0)$ out to $\xi_{\rm m}$ to see how this solution is represented in terms of the basis states at $\xi_{\rm m}$, and hope it is consistent with the best solution for $\xi \to \infty$.

We find the best solution as $\xi \to \infty$ is given by $(\mathcal{Z}(\xi_{\rm m}), \mathcal{Z}'(\xi_{\rm m})) = (0.0707, -0.997)$, but that $(\alpha_{0,g}, \alpha_{0,b}) = (1, 0)$ corresponds to $(\mathcal{Z}(\xi_{\rm m}), \mathcal{Z}'(\xi_{\rm m})) = (0.991, -0.137)$. If we re-normalise these solutions to have $\mathcal{Z}(\xi_{\rm m}) = 1$ we find they are respectively $(\mathcal{Z}(\xi_{\rm m}), \mathcal{Z}'(\xi_{\rm m})) = (1, -14.1)$ and $(\mathcal{Z}(\xi_{\rm m}), \mathcal{Z}'(\xi_{\rm m})) = (1, -0.138)$, which are plotted in figure 4.2. We can clearly see that these solutions are very different, and repeating for different values of m reveals similar disparities, thus we must conclude that we cannot find any physically acceptable solutions. We discuss this in the final chapter.

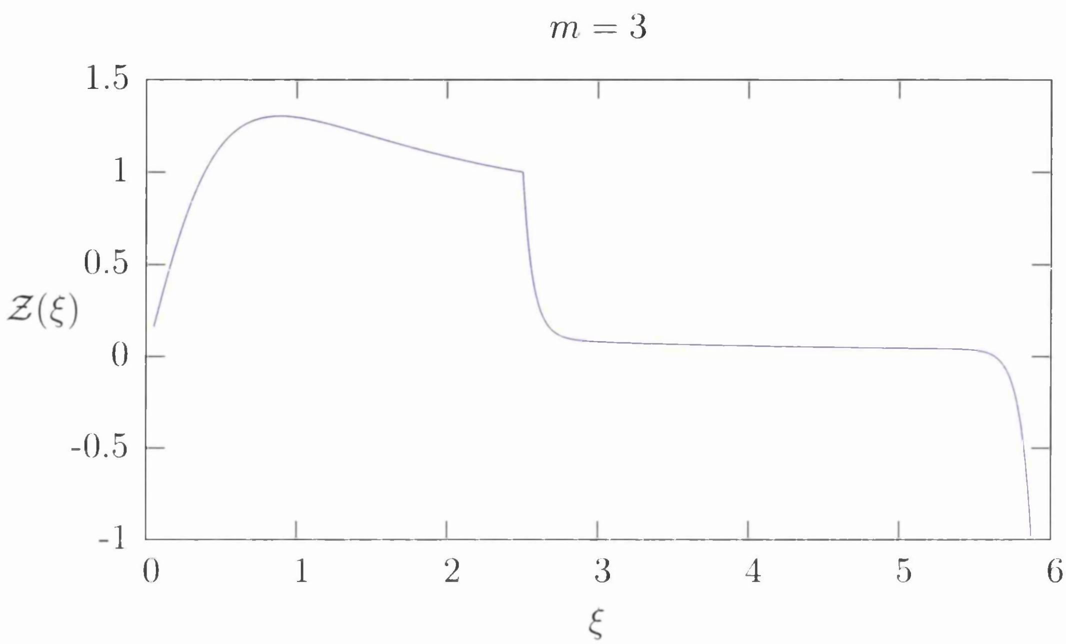

Figure 4.2: The non-diverging solution at the origin and the 'best' solution at infinity plotted together, both normalised to have $\mathcal{Z}(2.5) = 1$. The discontinuity of the gradient at this point indicates that these solutions are not the same, and thus no physically acceptable solution exists. The instability of the 'best' solution is revealed by its eventual divergence, seen here at $\xi \sim 5.5$

Chapter 5

Discussion & Conclusions

The Light Wall

In the previous chapter we discovered that there are no solutions to the perturbation field equations. Initially we find this counter-intuitive, since it seems a reasonable task to propose a solution (possessing symmetry) to a set of equations, and then to find perturbations to this solution which break the symmetry. Thus in this chapter we discuss possible explanations for this result.

When setting up a perturbative expansion, it is implicitly assumed that the perturbation fields are finite, so when (as in our case) we find a perturbation field diverging, we must admit that our perturbative expansion has broken down. A possible reason for this could be that the very act of introducing the expansion loses some important feature of the system. In this work we expanded to linear order, which is common even in the presence of gravity (which is an inherently non-linear theory), so we would not expect this to be a problem. Nevertheless, we may consider the case in which the 'mass' of the scalar field is 'light', and thus neglect the effects of gravity. Aside from this, removing gravity reduces the complexity of the problem, which is usually a good thing to do when trying to understand some result. To do this in practice, we merely let $\kappa = 0$. Returning to the equations of motion for the bounce, (2.25), we can immediately see that this gives $\rho(\xi) = \xi$, which we expect since in the absence of gravity space-time will be flat. Our only equation of motion is now the scalar field equation, (2.25a), which upon once again assuming a spherically symmetric solution, $\Phi(x) = \Phi(\xi)$ becomes:

$$\Phi''(\xi) = U'(\Phi(\xi)) - \frac{4}{\xi}\,\Phi'(\xi) \tag{5.1}$$

This equation can be solved numerically as before, producing a similar bounce-like solution. Upon once again introducing perturbations via

$$\widetilde{\Phi}(x) = \Phi(\xi) + \varepsilon\,\Delta\Phi(\xi)\,Y(\omega)$$

we find the order ε part of the scalar field equation is

$$\Delta\Phi''(\xi) = -\frac{4}{\xi}\,\Delta\Phi'(\xi) + \Delta\Phi(\xi)\left(\frac{k^2}{\xi^2} + U''\big(\Phi(\xi)\big)\right) \tag{5.2}$$

Since this is what we get by transforming (4.1c) back from $\mathcal{Z}(\xi)$ to $\Delta\Phi(\xi)$ with $\kappa = H_{\mathrm{L}}(\xi) = H_{\mathrm{T}}(\xi) = 0$ and $\rho(\xi) = \xi$, we instantly know that the solutions in the limits $\xi \to 0$ and $\xi \to \infty$ are the same as before, namely:

$$\lim_{\xi\to 0} \Delta\Phi(\xi) \propto \xi^{-(m+3)}\big(1 + \mathcal{O}(\xi^2)\big) \tag{5.3a}$$

$$\lim_{\xi\to 0} \Delta\Phi(\xi) \propto \xi^{m}\big(1 + \mathcal{O}(\xi^2)\big) \tag{5.3b}$$

and (recalling $z = \xi + b$)

$$\lim_{z\to\infty} \Delta\Phi(z) \propto \frac{e^{-\beta z}}{z^2}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \tag{5.4a}$$

$$\lim_{z\to\infty} \Delta\Phi(z) \propto \frac{e^{\beta z}}{z^2}\left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) \tag{5.4b}$$

Although we might once again hope that the good solution for $\xi \to 0$ would integrate out to the good solution for $\xi \to \infty$, we find that this is not the case.

Back to Lorentz space

To gain some physical insight we can consider the perturbation back in space-time, i.e. in Lorentz-signature 4+1 space (as opposed to Euclidean-signature space, where our physical intuition is not so easily applied). In Lorentz space as time increases from $-\infty$ the bounce is a shrinking spherical bubble of true vacuum which at $t=0$ reaches a minimum radius, 'bounces', and then expands for the rest of time. The perturbation field equation (5.2) is transformed to Lorentz space by merely flipping the sign of U, and so the solutions are obtained by the same transformation. Since $\beta \equiv \sqrt{U(\Phi_{\mathrm{F}})}$, this means the Lorentz space solutions are obtained by $\beta \longrightarrow i\,\beta$, and thus independent real-valued solutions are given by

$$\lim_{z\to\infty} \Delta\Phi(z) \propto \frac{\cos(\beta\, z)}{z^2}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{5.5a}$$

$$\lim_{z\to\infty} \Delta\Phi(z) \propto \frac{\sin(\beta\, z)}{z^2}\left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \tag{5.5b}$$

This looks promising – in Lorentz space both solutions are decaying oscillations for $\xi \longrightarrow \infty$. However, to obtain the solutions for $\xi \longrightarrow 0$, we must also consider the behaviour of the harmonics in Lorentz space.

Lorentz-space Harmonics

To obtain the Lorentz-space harmonics we will need the metric of the space in 'polar' coordinates. We started off in Lorentz space with coordinates $\{t, x, y, z, w\}$ where we would define a radial coordinate as

$$r^2 = x^2 + y^2 + z^2 + w^2$$

We then transformed to Euclidean-space with coordinates $\{\tau, x, y, z, w\}$, but worked in polar coordinates $\{\xi, \psi, \chi, \theta, \phi\}$ where ξ is the Euclidean-space 'radius':

$$\xi^2 = \tau^2 + x^2 + y^2 + z^2 + w^2$$

This transition from Cartesian coordinates to polars takes place via

$$r = \xi \sin\psi$$

$$\tau = \xi \cos\psi$$

where at each point (ξ, ψ) there is a 3-sphere coordinatised by (χ, θ, ϕ). The range of ψ is $[0, \pi]$ and the line element takes the form

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\xi^2 + \rho(\xi)^2 \mathrm{d}{\Omega_4}^2 \\ &= \mathrm{d}\xi^2 + \rho(\xi)^2 \big(\mathrm{d}\psi^2 + \sin^2\psi\, \mathrm{d}{\Omega_3}^2\big) \end{aligned}$$

where $\mathrm{d}{\Omega_n}^2$ is the line element of an n-sphere.

In Lorentz space (outside the light cone) the transformation from Cartesian to polar coordinates takes the form

$$\begin{aligned} r &= \xi \cosh\psi \\ t &= \xi \sinh\psi \end{aligned}$$

where there is once again a 3-sphere at each point, but now the range of ψ is $(-\infty, \infty)$ and the line element takes the form[1]

$$\mathrm{d}s^2 = \mathrm{d}\xi^2 + \rho(\xi)^2 \big(-\mathrm{d}\psi^2 + \cosh^2\psi\, \mathrm{d}{\Omega_3}^2\big)$$

Thus in this case when we come to solving for the harmonics, $Y(\omega)$, we separate the variables as before (see section D.1):

$$Y(\omega) = \prod_{i=1}^{4} f_i(\omega_i)$$

but find the equation for the ψ-function is:

$$0 = f''(\psi) + 3\tanh\psi\, f'(\psi) + \left(\frac{{k_2}^2}{\cosh^2\psi} - k^2\right) f(\psi) \tag{5.6}$$

(where k_2 is the second separation constant – see section D.1). We do not need to solve this equation in full for our current purpose, but just determine the allowed

[1] These coordinates are not those one would usually use in Lorentz space, but we do so here to see the connection between the harmonics in Euclidean and Lorentz space. In fact, these coordinates are quite natural for our situation, since (as we saw at the end of chapter 2) ψ relates to the proper time on the brane.

values of the separation constant (k^2), since these enter the expressions for $\Delta\Phi(\xi)$ in the limit $\xi \to 0$, (5.3), (recall $k^2 = m(m+3)$). Consider the limit $\psi \to \infty$, for which (5.6) becomes

$$0 = f''(\psi) + 3\,f'(\psi) - k^2\,f(\psi)$$

which has solutions:

$$\lim_{\psi\to\infty} f(\psi) \propto e^{\nu}$$
$$\lim_{\psi\to\infty} f(\psi) \propto e^{-\nu(\nu+3)}$$

where $k^2 = \nu(\nu+3)$. Since $f(\psi)$ must be finite for all values of ψ these solutions must have $\mathrm{Re}(\nu) \leq 0$ and $\mathrm{Re}(\nu) \geq -3$ respectively.

A similar analysis for $\psi \to -\infty$ reveals the same solutions, and thus in general we must combine the limits on ν to $-3 \leq \mathrm{Re}(\nu) \leq 0$. Clearly the range $\nu \in [-3, 0]$ satisfies this, but we can also consider complex values of ν and keep k^2 real-valued, by letting

$$\nu = -\frac{3}{2} + i\,\lambda \qquad (\lambda \in \mathbb{R})$$

which gives

$$k^2 = -\left(\frac{9}{4} + \lambda^2\right)$$

So whereas in Euclidean space we find $k^2 = m(m+3)$ with $m \in \mathbb{Z}^+$, in Lorentz space we find $k^2 = \nu(\nu+3)$ where either $\nu \in [0, -\frac{3}{2}]$ or $\nu = -\frac{3}{2} + i\,\lambda$ with $\lambda \in \mathbb{R}^+$.

Returning to (5.3), we can now determine the behaviour of $\Delta\Phi(\xi)$ as $\xi \to 0$ in Lorentz space by letting $m \to \nu$, which for real-valued ν ($\nu \in [0, -\frac{3}{3}]$) gives

$$\lim_{\xi\to 0} \Delta\Phi(\xi) \propto \xi^{-(\nu+3)}\big(1 + \mathcal{O}(\xi^2)\big) \tag{5.7a}$$

$$\lim_{\xi\to 0} \Delta\Phi(\xi) \propto \xi^{\nu}\big(1 + \mathcal{O}(\xi^2)\big) \tag{5.7b}$$

or for complex-valued ν ($\nu = -\frac{3}{2} + i\,\lambda$):

$$\lim_{\xi \to 0} \Delta\Phi(\xi) \propto \xi^{-3/2} \cos(\lambda \ln \xi)\big(1 + \mathcal{O}(\xi^2)\big) \tag{5.8a}$$

$$\lim_{\xi \to 0} \Delta\Phi(\xi) \propto \xi^{-3/2} \sin(\lambda \ln \xi)\big(1 + \mathcal{O}(\xi^2)\big) \tag{5.8b}$$

In either case (real or complex ν), we find both solutions are divergent at the origin, and thus not acceptable.

So, with gravity neglected, in Lorentz space we find the perturbations to the scalar field are all well behaved as $\xi \to \infty$, but divergent as $\xi \to 0$.

Incoming Radiation?

So what is the explanation for this lack of physical solutions? The factors of $\xi^{-3/2}$ in (5.8) indicate 'focusing' of the field towards the origin. Indeed, we can look at the equivalent result in a general number of dimensions, d. In that case, the Lorentz-space light-wall perturbation equation would have been

$$\Delta\Phi''(\xi) = -\frac{d-1}{\xi}\Delta\Phi'(\xi) + \Delta\Phi(\xi)\left(\frac{\nu(\nu+d-2)}{\xi^2} - U''\big(\Phi(\xi)\big)\right) \tag{5.9}$$

(so that for $d = 5$ we get back (5.2) with $k^2 = \nu(\nu+3)$). If we once again considered complex-values for ν (whilst keeping k^2 real):

$$\nu = 1 - \frac{d}{2} + i\,\lambda$$

we find solutions like those of (5.8) but with the factor $\xi^{-3/2}$ generalised to $\xi^{1-d/2}$.

The occurrence of radial functions diverging at the origin (and of factors like $\xi^{1-d/2}$) is reminiscent of scattering theory, in which one might expand a state with well defined linear momentum (i.e. a plane wave) in terms of states of well defined angular momentum. In doing so, the relevant radial functions are the spherical Bessel functions, which in d dimensions are related to the normal Bessel functions by

$$j_{n-1+d/2}(r) \sim r^{1-d/2}\, J_{n-1+d/2}(r)$$

There are two kinds of Bessel function, both of which are decaying oscillations at infinity, but while those of the first kind, $J_n(r)$, are finite at the origin, those of the second kind, $N_n(r)$ (also called Neumann functions) are divergent. Since a plane wave is well behaved at the origin, its expansion is in terms of Bessel functions of the first kind. But, in considering a scattering process (e.g. off of some potential localised around the origin), then the solution is no longer just a plane wave, and Neumann functions must also be included. In this case it is often convenient to work in a different basis of Bessel functions known as the Hankel functions, defined by

$$\begin{aligned} H_n^{(1)}(r) &= J_n(r) + i\,N_n(r) \\ H_n^{(2)}(r) &= J_n(r) - i\,N_n(r) \end{aligned}$$

The advantage of the Hankel functions is that at large distances they behave like:

$$\begin{aligned} \lim_{r\to\infty} H_n^{(1)}(r) &\sim \frac{1}{\sqrt{r}}\,e^{i\,r} \\ \lim_{r\to\infty} H_n^{(2)}(r) &\sim \frac{1}{\sqrt{r}}\,e^{-i\,r} \end{aligned}$$

allowing their interpretation as outward and inward propagating radiation. Whilst Hankel functions are decaying oscillations at infinity, they are divergent at the origin, just as we found $\Delta\Phi(\xi)$ to be earlier.

So perhaps the fact the solutions to our perturbation equation are divergent at the coordinate origin is a result of incoming radiation. One possibility is that this is an artefact of the mode expansion, i.e. that we have just chosen a bad basis. But if this is the case then presumably we could expect there to be some combination of different modes for which the divergences cancel, but this seems unlikely given the linear independence of the harmonic basis. Nevertheless, certainly one way to proceed would be to analyse the perturbations without performing the mode expansion. Given the complexity of the problem, the practical way to do this would seem to be a numerical simulation, i.e. to consider a large 'grid' of points in space-time on which the perturbation to the scalar field is defined.[2] The form of the bounce is known numerically, so there is no need to solve for this again – it could be entered as a background field. The simulation can begin at time $t = 0$ (when the bubble of true vacuum is at its smallest) with some small non-zero value to the perturbation fields, and run forward to see how this evolves – if there appears to be no problem at the origin, then this is indicative of a problem with the mode expansion.

This leaves the possibility that the incoming radiation is a truly physical effect (as opposed to an artefact of the mathematical formulation). This is certainly more appealing: the (perturbed) bounce is an expanding wall of higher energy density, and is thus able to radiate. Since this wall is in fact a spherical shell, it seems natural that energy radiated from it inwards would be focused towards the origin, which shows up as a divergence in the scalar field (just as in scattering theory). In the presence of gravity there is a further complication: a large amount of energy contained within a small region (in this case, a small radius about the origin) can give rise to a black hole. This is similar to the work of [46] where in $3+1$ dimensions and in the 'thin-wall' context it was concluded that vacuum decay could result in 'gravitational collapse', but since we have considered the light-wall case too, we conclude that the phenomenon is not a purely gravitational effect, but rather a result of the bubble's motion. On the one hand this seems like a problem since a black hole is rarely considered as a perturbative effect, since inside the event horizon the role of time and radial coordinate is reversed. On the other hand, outside the horizon there is no problem in considering the effects of the black hole perturbatively, and since the horizon acts as a new boundary, this is where we would now apply boundary conditions (i.e. that only incoming radiation be permitted) – which could be our saving since we could then allow solutions which diverge at the origin (in much the same way that in scattering theory one can allow incoming radiation if the region of interest does not include the origin).

In the presence of a black hole, a further complication could be if the new boundary condition (on the black hole's event horizon) is not Lorentz-invariant,

[2] To reduce the complexity this could be done in $2 + 1$ dimensions (at least to begin with), with no loss of generality, at least to the problem in hand, since we have seen that even for $d = 3$ our theory predicts divergences like $\xi^{-1/2}$.

since then it will not be rotationally invariant in Euclidean space. This means it may not be possible to apply the same perturbation formalism as we did here, in which case it would once again be prudent to perform a full numerical simulation as suggested above.

Summary

We now summarize this thesis. Motivated by interest in brane scenarios in which an extra dimension can be large, we considered a $4+1$ space-time occupied by a scalar field which experiences a potential that has two (un-equal) local minima. By assuming spherical symmetry, we solved the field equations for the 'bounce' – the configuration in which space is initially filled by false vacuum, but a bubble of true vacuum expands overcoming the false vacuum – commonly known as vacuum decay.

By comparing with an exact solution, we interpret the wall between the two vacua as a 'toy model' brane-world with some small thickness. The advantage of considering a 'thick' brane was meant to be the relative ease with which we could introduce perturbations to it, from which we could go on to construct a CMB spectrum. We setup perturbations to the bounce as an expansion in a harmonic basis with five radial field coefficients. The equations of motion for these perturbation fields were reduced to a single ordinary differential equation in one gauge invariant variable, which we could solve numerically. We also obtained an expression for the action as a one-dimensional integral in terms of the gauge invariant variable, so that the action of each mode could be calculated.

Surprisingly we found that there are no physically acceptable solutions to the perturbation field equations, and interpret this as either a deficiency of the mathematical formulation (i.e. the mode expansion), or an interesting physical effect; that of the focusing of radiation. For any configuration other than the perfect bounce, the wall radiates energy, which when focused at the origin can give rise to a black hole, which would require the anti-de Sitter space inside the bubble to be generalised to Schwarzschild-anti-de Sitter. In either case the suggested way to proceed is via a full numerical simulation on a discretized space-time grid.

Appendix A

Extremising the Action

In classical mechanics, Hamilton's principle of least action states that a system evolves so as to extremise its action. In quantum mechanics the system does not evolve so as to extremise the action, but rather the system evolves in every possible way with a probability related to the corresponding value of the action. However, to study some processes, such as tunnelling, which involves the use of instantons, one may want to know which path extremises the action, since this will give the most likely evolution of the system.

In this work we are interested in extremising the action of a scalar field and gravitational field, to find the configuration corresponding to vacuum decay. To extremise the action means to find the field solutions such that when subject to a small variation, the corresponding variation in the action is zero to first order, i.e. to find the fields Φ and $g_{\mu\nu}$ such that

$$S[\Phi + \varepsilon\,\delta\Phi, g_{\mu\nu} + \varepsilon\,\delta g_{\mu\nu}] = S[\Phi, g_{\mu\nu}] + \mathcal{O}(\varepsilon^2)$$

for arbitrary perturbations $\delta\Phi$ and $\delta g_{\mu\nu}$. We will see how to do this for our action in the following sections, where for the scalar field we find the Euler-Lagrange equation, and for the gravitational field, Einstein's field equation.

A.1 The Scalar Field Equation

Consider an action dependent on a scalar field and its derivatives (in d dimensions):

$$S = \int \mathcal{L}\left(\Phi(x), \partial_\mu \Phi(x)\right) \mathrm{d}^d x$$

If we subject the field to a small arbitrary variation, $\delta\Phi$, we find the resulting variation of the action (to first order in $\delta\Phi$) is

$$\begin{aligned}
\delta S &= \int \left(\delta\Phi \frac{\partial \mathcal{L}}{\partial \Phi} + \delta(\partial_\mu \Phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \mathrm{d}^d x \\
&= \int \left(\delta\Phi \frac{\partial \mathcal{L}}{\partial \Phi} + \partial_\mu \delta\Phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \mathrm{d}^d x
\end{aligned}$$

If we integrate this by parts and only consider variations such that $\delta\Phi(x) \to 0$ as $\boldsymbol{x} \to \infty$, we find

$$\delta S = \int \delta\Phi \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \mathrm{d}^d x$$

For $\Phi(x)$ to extremise the action, we must have $\delta S = 0$ for arbitrary $\delta\Phi$, which we can see implies

$$\frac{\partial \mathcal{L}}{\partial \Phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \tag{A.1}$$

Equation (A.1) is the Euler-Lagrange equation for a scalar field.

For the Euclidean action (2.19) we have

$$\mathcal{L} = \sqrt{g} \left[\frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi + U(\Phi) - \frac{R}{2\,\kappa} \right]$$

and thus (A.1) becomes

$$\sqrt{g}\, \frac{\mathrm{d}U}{\mathrm{d}\Phi} = \partial_\mu (\sqrt{g}\, g^{\mu\nu} \partial_\nu \Phi) \tag{A.2}$$

i.e.

$$\nabla^2 \Phi = \frac{\mathrm{d}U}{\mathrm{d}\Phi} \tag{A.3}$$

which is the equation of motion we will use for the scalar field.

A.2 Einstein's Equation

In this section we follow the outline given in [47]. To extremise the action with respect to the gravitational field, we consider it split into two parts:

$$S = S_M + S_G$$

where S_G is the action of the gravitational field, and S_M is the action of all other fields (treating the gravitational field as an external one). We now consider the variations to S_M and S_G resulting from the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$.

The variation to S_M is used to define an energy-momentum tensor for the system, via:

$$\delta S_M = \frac{1}{2}\int \mathrm{d}^d x \sqrt{|g|}\, T^{\mu\nu}\, \delta g_{\mu\nu} \tag{A.4}$$

i.e. $T^{\mu\nu}$ is related to the 'functional derivative' of S_M with respect to $g_{\mu\nu}$.

The choice of S_G is of great significance – it determines the theory of gravity that will be used. In this work we use an Einstein-Hilbert action since (as we shall see) its extremum gives Einstein's field equation of General Relativity.[1] The gravitational part of the Einstein-Hilbert action is

$$S_G = \frac{1}{2\,\kappa}\int \mathrm{d}^d x \sqrt{|g|}\, R$$

where R is the Ricci scalar curvature. To evaluate δS_G we must determine the variation in the integrand resulting from a variation in the metric, $\delta g_{\mu\nu}$:

$$\begin{aligned}
\delta(\sqrt{|g|}\,R) &= R\,\delta\sqrt{|g|} + \sqrt{|g|}\,\delta R \\
&= R\,\delta\sqrt{|g|} + \sqrt{|g|}\,\delta(g^{\mu\nu}R_{\mu\nu}) \\
&= R\,\delta\sqrt{|g|} + \sqrt{|g|}\,R_{\mu\nu}\,\delta g^{\mu\nu} + \sqrt{|g|}\,g^{\mu\nu}\,\delta R_{\mu\nu}
\end{aligned}$$

The last term of this expression can be re-written using the Palatini identity:

$$\sqrt{|g|}\,g^{\mu\nu}\,\delta R_{\mu\nu} = \sqrt{|g|}\,\left(g^{\mu\nu}\,\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} - \sqrt{|g|}\,\left(g^{\mu\nu}\,\delta\Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda}$$

[1] To work with a different or modified theory of gravity compared to Einstein's, extra terms would be included in S_G.

The advantage of this is that it is now evidently a covariant divergence, and thus when integrated over all space is equivalent to a surface term (by Gauss' theorem) which we take to be zero for a metric variation that vanishes at infinity, i.e. $\delta g_{\mu\nu} \to 0$ as $\boldsymbol{x} \to \infty$. Thus we can now say that

$$\delta S_G = \frac{1}{2\,\kappa} \int \mathrm{d}^d x \left(R\,\delta\sqrt{|g|} + \sqrt{|g|}\,R_{\mu\nu}\,\delta g^{\mu\nu} \right) \tag{A.5}$$

We wish to express this in terms $\delta g_{\mu\nu}$, and thus need expressions for $\delta g^{\mu\nu}$ and $\delta\sqrt{|g|}$ in terms of $\delta g_{\mu\nu}$, which we now derive:

$$\begin{aligned}
& & g_{\mu\nu}\,g^{\nu\rho} &= \delta^\rho_\mu \\
&\Longrightarrow & \delta(g_{\mu\nu}\,g^{\nu\rho}) &= 0 \\
&\Longrightarrow & \delta g_{\mu\nu}\,g^{\nu\rho} + g_{\mu\nu}\,\delta g^{\nu\rho} &= 0 \\
&\Longrightarrow & \delta g^{\sigma\rho} &= -g^{\mu\sigma}\,g^{\nu\rho}\,\delta g_{\mu\nu}
\end{aligned} \tag{A.6}$$

For $\delta\sqrt{|g|}$ first consider δg:[2]

$$\begin{aligned}
\delta g &= \delta(\det(g_{\mu\nu})) \\
&= \det(g_{\mu\nu} + \delta g_{\mu\nu}) - \det(g_{\mu\nu}) \\
&= \det(g_{\mu\sigma}(\delta^\sigma_\nu - g_{\nu\rho}\delta g^{\sigma\rho})) - \det(g_{\mu\nu}) \\
&= g\,(\det(\delta^\sigma_\nu - g_{\nu\rho}\delta g^{\sigma\rho}) - 1) \\
&= g\,(1 - \mathrm{Tr}\,(g_{\nu\rho}\delta g^{\sigma\rho}) - 1) \\
&= -g\,g_{\mu\nu}\delta g^{\mu\nu} \\
&= g\,g^{\mu\nu}\delta g_{\mu\nu}
\end{aligned}$$

[2] This uses the fact that for a matrix $\mathbf{A}$ and a small parameter ε:

$$\det(\mathbb{1} + \varepsilon\,\mathbf{A}) = 1 + \varepsilon\,\mathrm{Tr}\mathbf{A} + \mathcal{O}(\varepsilon^2)$$

which allows us to see that

$$
\begin{aligned}
\delta\sqrt{|g|} &= \frac{1}{2}\ |g|^{-1/2}\ \delta\,|g| \\
&= \frac{1}{2}\ |g|^{-1/2}\ |g|\ g^{\mu\nu}\delta g_{\mu\nu} \\
&= \frac{1}{2}\sqrt{|g|}\,g^{\mu\nu}\delta g_{\mu\nu}
\end{aligned} \tag{A.7}
$$

Thus (A.5) becomes

$$
\begin{aligned}
\delta S_G &= \frac{1}{2\,\kappa}\int \mathrm{d}^d x\left(R\,\frac{1}{2}\sqrt{|g|}\,g^{\mu\nu}\delta g_{\mu\nu} + \sqrt{|g|}\,R_{\mu\nu}\,(-g^{\mu\sigma}\,g^{\nu\rho}\,\delta g_{\sigma\rho})\right) \\
&= \frac{-1}{2\,\kappa}\int \mathrm{d}^d x\sqrt{|g|}\left(R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu}\right)\delta g_{\mu\nu}
\end{aligned}
$$

Recombining this with (A.4):

$$
\begin{aligned}
\delta S &= \delta S_M + \delta S_G \\
&= \int \mathrm{d}^d x\sqrt{|g|}\left[\frac{1}{2}\,T^{\mu\nu} - \frac{1}{2\,\kappa}\left(R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu}\right)\right]\delta g_{\mu\nu}
\end{aligned}
$$

we can see that requiring $\delta S = 0$ for arbitrary $\delta g_{\mu\nu}$ amounts to

$$
R^{\mu\nu} - \frac{1}{2}\,R\,g^{\mu\nu} = \kappa\,T^{\mu\nu} \tag{A.8}
$$

which is Einstein's field equation.

In this work we consider the Euclidean-space action (2.19):

$$
S_E = \int \mathrm{d}^5 x\sqrt{g}\left[\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + U(\Phi) - \frac{R}{2\,\kappa}\right]
$$

where the change in sign of the scalar curvature R means that the relevant matter-action needed to determine the energy momentum tensor appearing in (A.8) is

$$
S_M = -\int \mathrm{d}^5 x\sqrt{g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi + U(\Phi)\right]
$$

Thus using (A.4) with identities (A.6) and (A.7) we find

$$T_{\mu\nu} = -\frac{1}{2}\, g_{\mu\nu}\partial^{\lambda}\Phi\, \partial_{\lambda}\Phi - g_{\mu\nu}U(\Phi) + \partial_{\mu}\Phi\, \partial_{\nu}\Phi \tag{A.9}$$

It will be more convenient to use Einstein's equation in the form where the trace is taken of the energy-momentum tensor ($T = T^{\mu}_{\mu}$), rather than the Ricci tensor:

$$\begin{aligned} & & R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}R &= \kappa T_{\mu\nu} \\ &\implies & R - \frac{d}{2}R &= \kappa T \\ &\implies & R &= \frac{2}{2-d}\,\kappa T \end{aligned}$$

so that Einstein's equation becomes

$$R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{g_{\mu\nu}}{d-2}\, T \right)$$

Thus in the present case of $d = 5$ and (A.9), Einstein's equation becomes

$$R_{\mu\nu} = \kappa \left(\frac{2}{3}\, g_{\mu\nu}\, U(\Phi) + \partial_{\mu}\Phi\, \partial_{\nu}\Phi \right) \tag{A.10}$$

which is what we use as our equation of motion for the gravitational field $g_{\mu\nu}$.

Appendix B

Fine Tuning

In chapter 2 we solved the equations of motion, (2.25) numerically to obtain the bounce. It was hinted that doing so was more intricate than was explained in the main text – in this appendix we go into more detail. In the next section we see that the desired numerical solution is unstable, and how we get as close to it as possible. In section B.2 we see how the 'best possible solution' is improved by patching it with the known analytic behaviour in the long distance limit.

B.1 Finding Φ_0

We can gain useful physical insight by considering (2.25a) in a different light.

$$\Phi''(\xi) = U'(\Phi(\xi)) - 4\frac{\rho'(\xi)}{\rho(\xi)}\Phi'(\xi) \tag{2.25a}$$

If we imagine Φ to be the x-coordinate of a classical particle (with unit mass), and ξ to be the time-coordinate, then Φ'' is the acceleration of the particle, and (2.25a) tells us that the particle is experiencing a force resulting from the potential energy $-U(\Phi)$, and a (variable) damping force proportional to its velocity given by $4\frac{\rho'}{\rho}\Phi'$. The potential experienced by the particle, $-U(\Phi)$ is shown in figure B.1. In this picture, the solution of $\Phi(\xi)$ corresponding to vacuum decay is that of the particle starting at the top of the peak ($\Phi_0 = \Phi_\mathrm{T}$), rolling down it and then up the other peak to come to rest at Φ_F at some time in the infinite future ($\Phi \rightarrow \Phi_\mathrm{F}$ as $\xi \rightarrow \infty$).

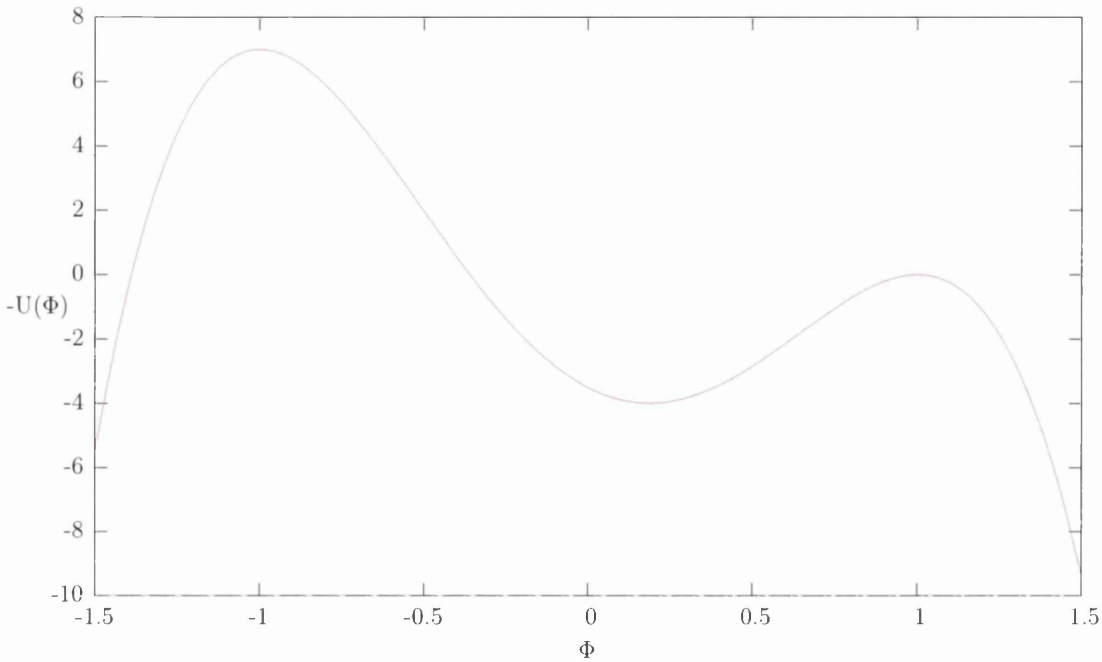

Figure B.1: The upside-down potential, $-U$, used to discuss the analogy between vacuum decay and the motion of a particle with position Φ at time ξ, as described by (2.25a)

Of course, this is a very unstable scenario, since the initial energy of the particle must be just right – not enough and it will not have enough to reach Φ_F, too much and it will reach Φ_T with a non-zero velocity and thus overshoot it (once $\Phi > \Phi_F$, the particle will roll off to infinity, i.e. $\Phi \to \infty$ as $\xi \to \infty$).

Our control of the initial energy given to the particle is through its starting position (Φ_0) alone, since we know it must start with zero velocity ($\Phi'(0) = 0$ is required for Φ to be smooth at the origin). Thus we now consider the possible scenarios resulting from differing values of Φ_0, so that we will be able to automate the process of finding the optimal value of Φ_0.

If we were to start with $\Phi_0 = \Phi_T$ then from (2.25a) we can see that the solution is $\Phi(\xi) \equiv \Phi_T$, i.e. the particle sits atop the peak forever – a state of unstable equilibrium. This is equivalent to the scalar field being at true vacuum for all space and time – obviously vacuum decay is not a possibility in such a case. Thus, we must start Φ off slightly past the peak, $\Phi_0 \gtrsim \Phi_T$, which means that the scalar

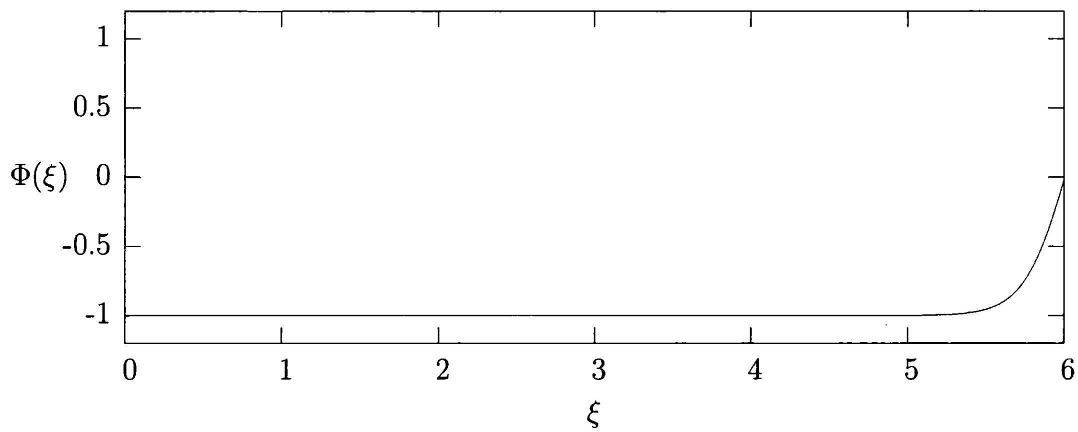

Figure B.2: Φ starts too close to Φ_T, and does not deviate much within the integration range considered.

field never reaches true vacuum *exactly.*

If Φ_0 is too close to Φ_T then the force (from the potential) acting on the particle is very small, and it will roll down the slope very slowly. Since in practice we only integrate out to a finite value of ξ, it is possible that by the time we get to this value Φ is still very close to Φ_T, as shown in figure B.2, which means our solution has not reached Φ_F and therefore does not correspond to vacuum decay. Thus, we must make sure that our starting value is far enough away from Φ_T so that Φ can reach Φ_F within our integration range – i.e. the particle starts rolling soon enough that it can reach Φ_F in the time range we consider.

Once we are sure Φ is starting far enough from Φ_T that it has enough 'time' to get to Φ_F, we should find a range of values for Φ_0 for which Φ overshoots Φ_F, never to return, as in figure B.3. Obviously for this range of values the particle has too much energy and we need to increase Φ_0 further. We say we *should* find a range of values, because it is possible that we will not, depending on the form of $U(\Phi)$; if $U(\Phi_m)$ is too big (i.e. the well between the peaks of figure B.1 is too deep) or $|\Phi_T|$ is too small (i.e. the peak at Φ_T in figure B.1 is not much higher than that at Φ_F) then the particle will lose too much energy due to the damping so that no matter what the value of Φ_0 it can never reach the top of the peak at Φ_F (in which case the output will look similar to figure B.4). Thus we must make sure the values chosen to define $U(\Phi)$ in (2.27) are such that this is not the case. In terms of the

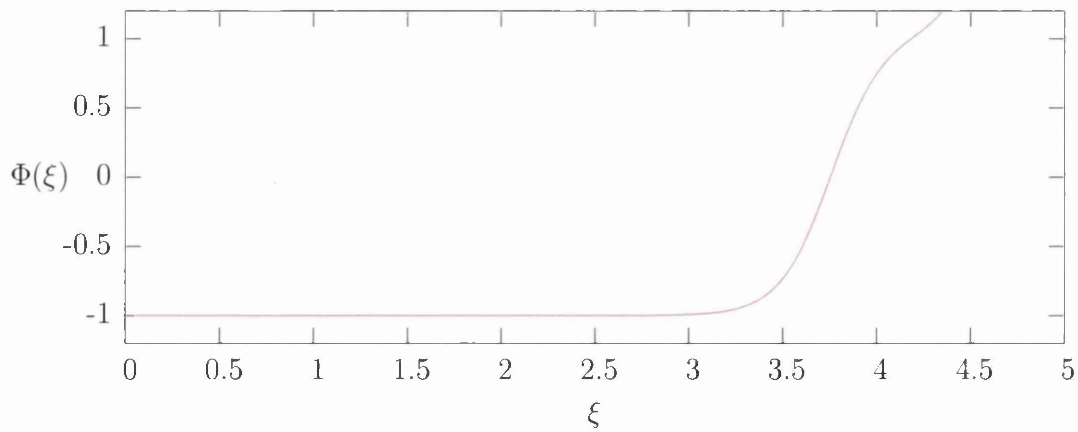

Figure B.3: Φ overshoots Φ_F. This indicates that Φ_0 needs to be increased.

scalar field, this means that there are possible forms for $U(\Phi)$ for which vacuum decay is not possible. We can understand this in terms of energy balance – for a bubble of true vacuum to form, the energy 'cost' of creating the wall between the vacua (where the potential has a maximum) must be compensated by the negative energy of the true vacuum relative to the false vacuum. While for a flat background there is always some radius for which this balance occurs (although it may be outside our range of integration), when gravity is included this is no longer true. As is explained (and proved in the limit $U(\Phi_T) \approx U(\Phi_F)$) in [44], the addition of gravitational potential energy and the modification to the volume and surface area of the bubble due to the curved background mean that there are configurations of $U(\Phi)$ for which no value of the bubble radius will balance the energy differences, and thus no bubble can form – gravity can stabilize the false vacuum.

Once we have adjusted our potential to ensure it permits the particle to reach Φ_F within our integration range, we must concern ourselves with what happens when Φ_0 is too far from Φ_T i.e. when the particle has not got enough energy. If the particle has not got enough energy to reach Φ_F then it will reach some maximum value, then fall back into the well and oscillate back and forth, losing energy due to the damping, as in figure B.4. This possibility is complicated further by the behaviour of $\rho(\xi)$. From (2.25b) we can see that as the contribution from $\Phi'(\xi)^2$

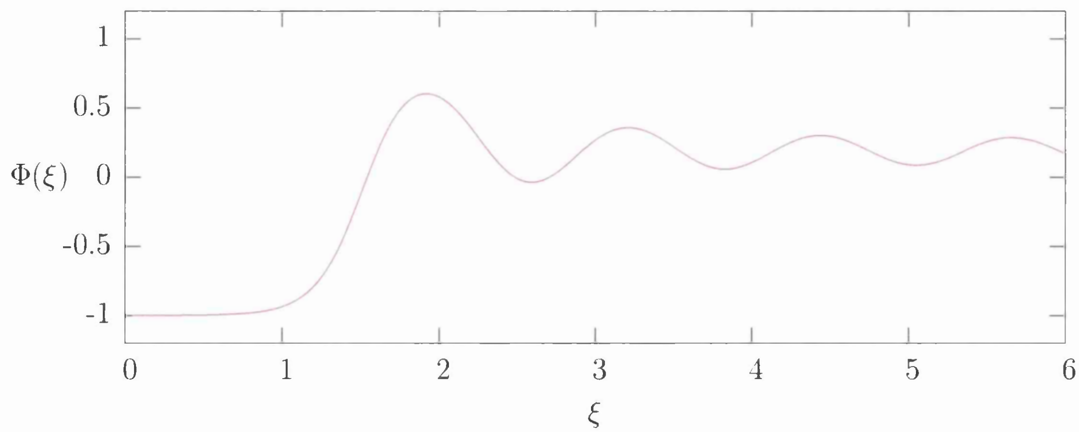

Figure B.4: Φ_0 is too far from Φ_T – the particle does not have enough energy to climb out of the well and falls back in.

becomes significant (due to the oscillations) $\rho''(\xi)$ can become negative enough for $\rho'(\xi)$ to become negative.

$$\rho''(\xi) = -\frac{\kappa\,\rho(\xi)}{12}\Big(2\,U(\Phi(\xi)) + 3\,\Phi'(\xi)^2\Big) \tag{2.25b}$$

If $\rho'(\xi)$ becomes negative, then the 'damping' term in (2.25a) switches sign and becomes an 'anti-damping' term, increasing the velocity of the particle, which ultimately might see the particle have enough energy to overshoot Φ_F (or Φ_T), as in figure B.5. In practice this means that if Φ exceeds Φ_F then this may in fact mean that Φ_0 was too far away from Φ_T rather than too close (as in figure B.3) – it is sufficient only to check the behaviour of $\rho(\xi)$ to determine which is the case.

The analysis thus far gets us quite close to a good solution, but to get even closer we make use of the known analytic form of our desired solution, (2.34):

$$\Phi(z) = \Phi_F + c\,e^{-\beta z}\left(\frac{1}{z^2} + \frac{1}{\beta\,z^3}\right) + \mathcal{O}\big(e^{-2\beta z}\big) \tag{2.34a}$$

$$\rho(z) = z + \mathcal{O}\big(e^{-2\beta z}\big) \tag{2.34b}$$

Using (2.34a) and the numerical data for $\Phi(\xi)$ and $\Phi'(\xi)$ we can calculate two values for the arbitrary constant c. By varying Φ_0 we search for the solution for

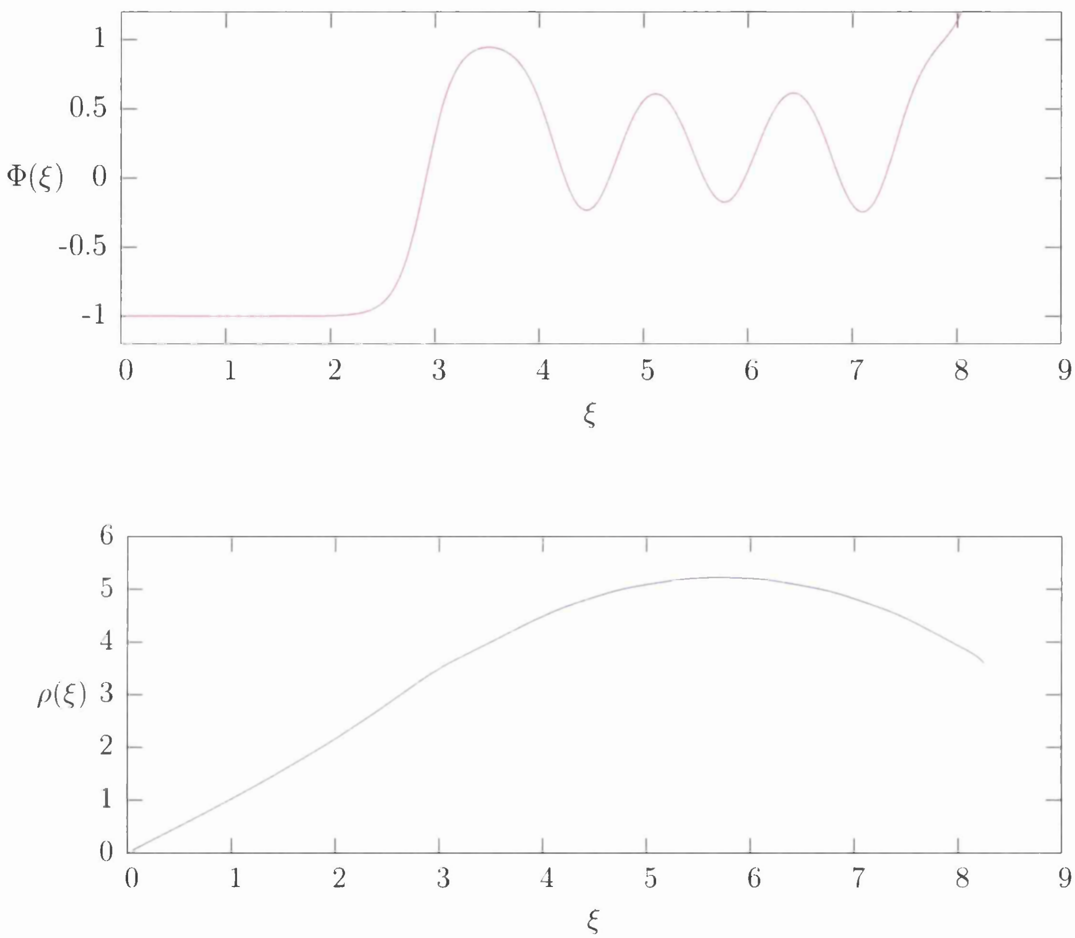

Figure B.5: Oscillations in Φ cause $\rho'(\xi)$ to becoming negative, leading to 'anti-damping' resulting in Φ shooting off to ∞.

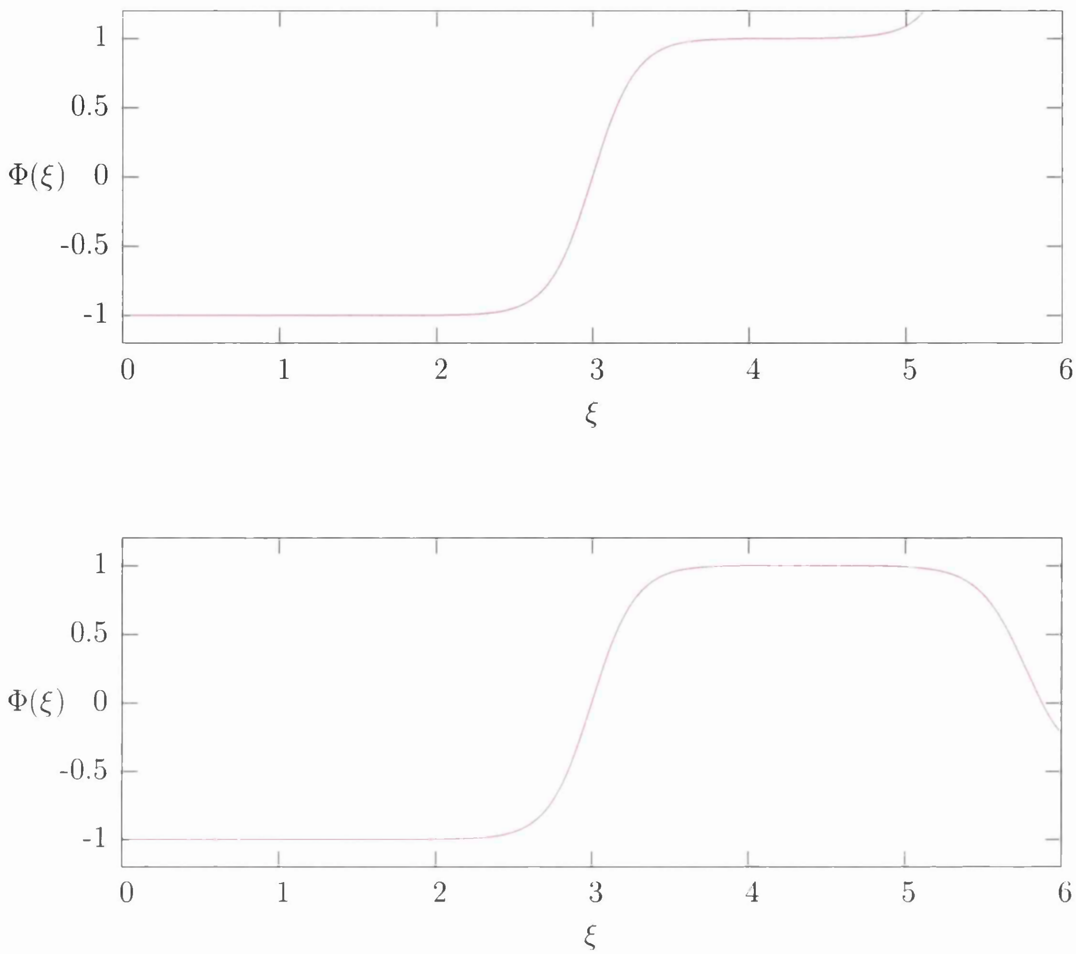

Figure B.6: Solutions which are very close to the desired one, but are ultimately unstable.

which these values of c coincide as best as possible. In this way we may find solutions which are very close to our desired one, but ultimately they are unstable, and diverge, as in figure B.6. To find our perfect solution we take one of these and 'patch' it with the exact solution, as described in the next section.

B.2 Patching Solutions

In the previous section we found the best possible solution, given numerical imprecision. Now we wish to improve on this by taking a 'best possible' solution, and from some point where it closely matches the analytic form (2.34) re-integrate using equations designed to enforce the asymptotic form. In this way, when we come to solve the perturbation field equations, we can use the same algorithms, and at some point merely switch from one set of equations to another.

So first, how do we determine at which point we will switch equations? Consider the solution for $\Phi(z)$ in the limit $z \longrightarrow \infty$, (2.34a), and its derivative, but with the constant c different in each case, the neglected terms of the expansion omitted, and using the definition $g(z) \equiv \dfrac{1}{z^2} + \dfrac{1}{\beta\, z^3}$;

$$\Phi(z) = \Phi_{\mathrm{F}} + c_1\, g(z)\, e^{-\beta\, z} \tag{B.2}$$

$$\Phi'(z) = c_2\, \left(g'(z) - \beta\, g(z)\right)\, e^{-\beta\, z} \tag{B.3}$$

These equations can be re-arranged to give expressions for c_1 and c_2 in terms of $\Phi(z)$ and $\Phi'(z)$ respectively, so that using the numerical data we can obtain values for c_1 and c_2. If our numerical solution was a perfect fit to (2.34a) then we would find $c_1 = c_2$, so to find the point where out solution best fits (2.34a) we look to make the ratio c_1/c_2 as close to 1 as possible. Typically we find a point where $\left|\dfrac{c_1}{c_2} - 1\right| \sim 10^{-3}$.

Once we have found this point, we integrate from there on using equations which enforce (2.34a). Consider again the expression for $\Phi'(z)$:

$$\Phi'(z) = c\, (g'(z) - \beta\, g(z))\, e^{-\beta\, z}$$

Substituting to express the right hand side in terms of $\Phi(z)$ gives:

$$\Phi'(z) = \left(\Phi(z) - \Phi_{\mathrm{F}}\right) \left(\frac{g'(z)}{g(z)} - \beta\right) \tag{B.4}$$

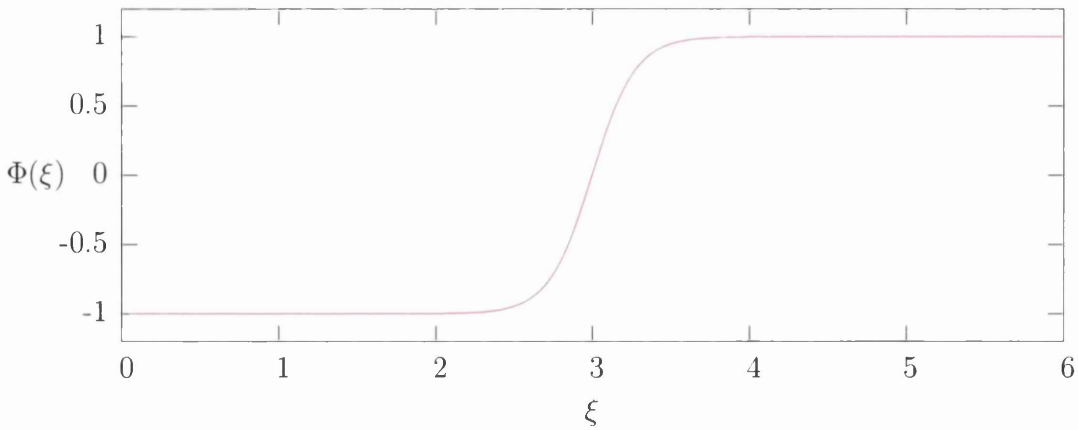

Figure B.7: Patched solution – at $\xi = 4.4$ we find $\frac{c_1}{c_2} = 1.003$ and from this point on integrate out using equations (B.4) and (B.5)

The solution to (B.4) is

$$\Phi(z) = \Phi_{\mathrm{F}} + d_1\, e^{-\beta\, z}$$

where d_1 is an arbitrary constant. Thus, integrating using (B.4) automatically yields a result of the desired form. Similarly, for $\Phi'(z)$ we use

$$\Phi''(z) = \frac{\Phi'(z)}{g'(z) - \beta\, g(z)} \big(g''(z) - 2\,\beta\, g'(z) + \beta^2\, g(z)\big) \tag{B.5}$$

because it has solution

$$\Phi'(z) = d_2\, \big(g'(z) - \beta\, g(z)\big) e^{-\beta\, z}$$

which is also of the desired form.

By doing all this we ultimately produce a solution like that in figure B.7 where Φ converges to Φ_{F} as $\xi \to \infty$, so we take this as our vacuum decay solution.

Appendix C

The Vanishing of S_1

In section 3.3 we claimed that having performed a perturbative expansion about a solution which extremises the action, the first order perturbative contribution to the action should be zero. In this appendix we verify that this is indeed the case for an arbitrary perturbation that vanishes at infinity (as was assumed when deriving the equations of motion in appendix A).

Whereas specifying an arbitrary perturbation to the scalar field is straightforward:

$$\widetilde{\Phi}(x) = \Phi(\xi) + \varepsilon\, \delta\Phi(x)$$

it is not quite so simple for the metric. To perturb the metric in the polar coordinate basis by an arbitrary amount would in general lead to singularities in the scalar curvature, so instead we perturb the metric in a Cartesian coordinate basis. First we change basis, using x as the polar coordinates, and c as the Cartesians:

$$g^c_{\mu\nu} = \frac{\partial x^\sigma}{\partial c^\mu} g^x_{\sigma\lambda} \frac{\partial x^\lambda}{\partial c^\nu}$$

so that $\mathbf{g^c}$ is the metric in the Cartesian basis and $\mathbf{g^x}$ in the polar basis. To perturb the Cartesian metric is trivial:

$$\widetilde{g}^c_{\mu\nu} = g^c_{\mu\nu} + \varepsilon\, \delta g_{\mu\nu}$$

(where the elements of the metric perturbation, $\delta g_{\mu\nu}$, are arbitrary functions of the Cartesian coordinates). We then transform this back into the polar basis:[1]

$$\widetilde{g}^{x}_{\mu\nu} = \frac{\partial c^{\sigma}}{\partial x^{\mu}} \widetilde{g}^{c}_{\sigma\lambda} \frac{\partial c^{\lambda}}{\partial x^{\nu}}$$

We are now in a position to calculate the first-order contribution to the action resulting from these perturbations. Expressing the action integral in cartesian coordinates, we may write:

$$\begin{aligned}
\widetilde{S}_E &= \int \widetilde{L}\, \mathrm{d}^5\mathbf{c} \\
\widetilde{L} &= \sqrt{\widetilde{g}^c}\left[\frac{1}{2}\,\partial^{\mu}\widetilde{\Phi}\,\partial_{\mu}\widetilde{\Phi} + U\big(\widetilde{\Phi}\big) - \frac{\widetilde{R}}{2\,\kappa}\right] \\
&= L_0 + \varepsilon\, L_1 + \mathcal{O}\big(\varepsilon^2\big)
\end{aligned}$$

The first few terms of L_1 are:

$$\begin{aligned}
L_1 = \delta\Phi(c)\frac{\rho(\xi)^4\, U'(\Phi(\xi))}{\xi^4} &+ \frac{\partial}{\partial c_1}\delta\Phi(c)\frac{\cos\psi\,\rho(\xi)^4\Phi'(\xi)^4}{\xi^4} + \cdots \\
+ \delta g_{11}(c)\Bigg[\frac{1}{4\,\kappa\,\xi^2}\Big(&-24\cos^2\psi + \sin^2\psi\big(\kappa\,\rho(\xi)^2\,\big(2\,U(\Phi(\xi)) + \Phi'(\xi)^2\big) \\
&+ 4\rho'(\xi)^2 + 4\rho(\xi)\rho''(\xi)\big)\Big) \\
&+ \frac{(5+7\cos(2\psi))\,\rho(\xi)\rho'(\xi)}{\kappa\,\xi^3} + \cdots\Bigg] + \cdots
\end{aligned} \tag{C.1}$$

We now eliminate derivatives of our scalar field and metric perturbations by integrating by parts, e.g. for some arbitrary term $f(x)$:

$$\int_{-\infty}^{\infty} f(x)\,\frac{\partial}{\partial c_1}\delta\Phi(c)\,\mathrm{d}c_1 = \Big[f(x)\,\delta\Phi(c)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial}{\partial c_1} f(x(c))\,\delta\Phi(c)\,\mathrm{d}c_1$$

where the boundary term is taken to be zero. Thus L_1 is now expressed in terms

[1] It is noted that $\widetilde{g}^{x}_{00}$ is multivalued at the origin, but that the scalar curvature is everywhere finite (and single-valued).

of $\delta\Phi$ and the $\delta g_{\mu\nu}$ only, i.e. (C.1) becomes:

$$\begin{aligned}L_1 = \delta\Phi(c)\frac{\rho(\xi)^3}{\xi^4}\Big[\rho(\xi)\big(U'(\Phi(\xi)) - \Phi''(\xi)\big) - 4\rho'(\xi)\Big] \\ + \delta g_{11}(c)\Bigg[\frac{\sin^2\psi}{4\,\kappa\,\xi^2}\left(6\,(\rho'(\xi)^2 - 1) + \kappa\,\rho(\xi)^2\left(2\,U(\Phi(\xi)) + \Phi'(\xi)^2\right) + 6\,\kappa\,\rho(\xi)\,\rho''(\xi)\right) \\ + \frac{\cos^2\psi\,\rho(\xi)^2}{4\,\kappa\,\xi^4}\left(12\,(\rho'(\xi)^2 - 1) + \kappa\,\rho(\xi)^2\left(2\,U(\Phi(\xi)) - \Phi'(\xi)^2\right)\right)\Bigg] + \cdots\end{aligned} \tag{C.2}$$

As soon as we insist that $\Phi(\xi)$ and $\rho(\xi)$ obey the equations of motion (2.25) we find that $L_1 \equiv 0$. Thus first order perturbations to the bounce are distinguished by their second order contribution to the action.

Appendix D

Hyper-Spherical Harmonics

In chapter 3 we performed a perturbative expansion in terms of harmonics of the invariant background, i.e. the 4-sphere, and then made use of their various properties to integrate over them. In this appendix we explore these hyper-spherical harmonics in more detail, so that we can see how this integration was performed.

D.1 General Solution

In this section we mirror the usual derivation of spherical harmonics, but generalised to n dimensions. Consider an n-sphere with coordinates ω_i $(i = 1, \ldots, n)$, and metric tensor γ_{ij} which has the canonical form:

$$\mathrm{d}\Omega_n{}^2 = \gamma_{ij}\,\mathrm{d}\omega^i \mathrm{d}\omega^j$$

$$\gamma_{ij} = \begin{cases} \prod_{m=1}^{i-1} \sin^2 \omega_m & i = j \\ 0 & i \neq j \end{cases}$$

Denoting the Laplacian with respect to γ_{ij} as Δ, spherical harmonics, $Y(\omega)$, are defined by

$$\Delta Y(\omega) = -k^2\, Y(\omega) \tag{D.1}$$

By separating the variables:

$$Y(\omega) = \prod_{i=1}^{n} f_i(\omega_i) \tag{D.2}$$

the partial differential equation (D.1) reduces to a set of (uncoupled) ordinary differential equations (using θ as a dummy variable):

$$0 = f_i''(\theta) + (n-i)\cot\theta\, f_i'(\theta) + f_i(\theta)\left({k_i}^2 - \frac{{k_{i+1}}^2}{\sin^2\theta}\right) \tag{D.3}$$

with separation constants k_i, where $k_1 \equiv k$ (the principal eigenvalue appearing in (D.1)) and $k_{n+1} \equiv 0$.

For $i = n$ we find $f_n''(\theta) = -{k_n}^2 f_n(\theta)$ which has solutions

$$f_n(\theta) \propto \cos(m_n\theta + \alpha_n)$$

where ${k_n}^2 = {m_n}^2$ and $m_n \in \mathbb{Z}$ (since f_n must be a periodic function with period 2π).

For $i \neq n$ we let $u = \cos\theta$ so that (D.3) becomes

$$0 = (1-u^2)f_i''(u) + (i-1-n)\,u\,f_i'(u) + f_i(u)\left({k_i}^2 - \frac{{k_{i+1}}^2}{1-u^2}\right) \tag{D.4}$$

From section D.2 we can see that the solutions of this equation are the associated Gegenbauer functions:

$$f_i(u) \propto \mathrm{C}\left[\tfrac{n-i}{2}\right]_{m_i}^{m_{i+1}}(u)$$

where ${k_i}^2 = m_i(m_i + n - i)$, $m_i \in \mathbb{Z}$ and $m_i \geq m_{i+1}$.

Thus (D.2) becomes

$$Y_k(\omega) \propto \cos(m_n\omega_n + \alpha_n)\prod_{i=1}^{n-1}\mathrm{C}\left[\tfrac{n-i}{2}\right]_{m_i}^{m_{i+1}}(\cos\omega_i)$$

where $k^2 = m_1(m_1 + n - 1)$ and $m_i \geq m_{i+1}$. But since (D.1) is linear in $Y(\omega)$, its general solution is a sum of such terms:

$$Y_k(\omega) = \sum_{m_2=0}^{m_1} \cdots \sum_{m_n=0}^{m_{n-1}} c_{m_1,\ldots,m_n} \cos(m_n\omega_n + \alpha_n) \prod_{i=1}^{n-1} \mathrm{C}\left[\tfrac{n-i}{2}\right]_{m_i}^{m_{i+1}} (\cos\omega_i)$$

where the $c_{m_1,\ldots,m_n}$ are arbitrary mode coefficients.

D.2 Gegenbauer Differential Equation

One way of writing the Gegenbauer differential equation is [50]:

$$0 = (1-x^2)\, y''(x) - (2\lambda+1)\, x\, y'(x) + \nu\,(\nu+2\lambda)\, y(x) \tag{D.5}$$

of which we can easily see that Legendre's equation is a special case with $\lambda = \frac{1}{2}$. It is possible to express the solution to (D.5) in terms of the Legendre functions of the first and second kind, but if the solution is required to be finite at $x = \pm 1$ then ν is required to be an integer, and the solutions are the Gegenbauer *polynomials*, $y(x) = \mathrm{C}[\lambda]_\nu(x)$.

In analogy with Legendre's equation, one can consider the 'associated' Gegenbauer differential equation:

$$0 = (1-x^2)\, y''(x) - (2\lambda+1)\, x\, y'(x) + \left({k_\lambda}^2 - \frac{{k_{\lambda-\frac{1}{2}}}^2}{1-x^2} \right) y(x) \tag{D.6}$$

where ${k_\lambda}^2 = m_\lambda(m_\lambda + 2\lambda)$ and $m_\lambda \in \mathbb{Z}$. The analogy continues since the solutions, which we may call 'associated Gegenbauer functions', $\mathrm{C}[\lambda]_{m_\lambda}^{m_{\lambda-1/2}}(x)$, can be expressed in terms of the Gegenbauer polynomials as:

$$\mathrm{C}[\lambda]_l^m(x) = (1-x^2)^{m/2} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m \mathrm{C}[\lambda]_l(x)$$

Since $\mathrm{C}[\lambda]_l(x)$ is an l-th degree polynomial in x, we see that $\mathrm{C}[\lambda]_l^m$ is zero for $m > l$.

D.3 Orthogonality

To deduce the orthogonality relation of the Gegenbauer functions, we first consider the (more general) Sturm-Liouville problem:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) + \beta\, w(x)\right) y(x) = 0 \tag{D.7}$$

Let $y_1(x)$ and $y_2(x)$ be two eigenfunctions of (D.7) with corresponding eigenvalues β_1 and β_2:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) + \beta_1\, w(x)\right) y_1(x) = 0 \tag{SL1}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) + \beta_1\, w(x)\right) y_2(x) = 0 \tag{SL2}$$

Now take the linear combination of these equations $y_2(x) \times (\mathrm{SL1}) - y_1(x) \times (\mathrm{SL2})$:

$$y_2(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x){y_1}'(x)\right] - y_1(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x){y_2}'(x)\right] + (\beta_1 - \beta_2)\, y_1(x)\, y_2(x)\, w(x) = 0$$

which upon collecting the derivatives together gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\left(y_2(x)\,{y_1}'(x) - y_1(x)\,{y_2}'(x)\right)\right] + (\beta_1 - \beta_2)\, y_1(x)\, y_2(x)\, w(x) = 0 \tag{D.8}$$

We now integrate over some interval $[a, b]$ (in which $p(x)$ and the eigenfunctions are continuous) to give

$$p(x)\left(y_2(x)\,{y_1}'(x) - y_1(x)\,{y_2}'(x)\right)\Big|_a^b + (\beta_1 - \beta_2)\int_a^b w(x)\, y_1(x)\, y_2(x)\, \mathrm{d}x = 0$$

If we now consider 'singular' boundary conditions, i.e. $p(a) = p(b) = 0$, we see that eigenfunctions with different eigenvalues are orthogonal with respect to the weight function $w(x)$, i.e.

$$\int_a^b w(x)\, y_1(x)\, y_2(x)\, \mathrm{d}x = 0 \qquad \text{for} \quad \beta_1 \neq \beta_2$$

By writing the associated Gegenbauer equation (D.6) as

$$(1-x^2)^{\frac{1}{2}-\lambda}\left(\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^{\lambda+\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}+\left({k_\lambda}^2(1-x^2)^{\lambda-\frac{1}{2}}+{k_{\lambda-\frac{1}{2}}}^2(1-x^2)^{\lambda-\frac{3}{2}}\right)\right)y(x)=0$$

we can see that this in Sturm-Liouville form with the identifications

$$\begin{aligned} p(x) &= (1-x^2)^{\lambda+\frac{1}{2}} \\ \beta &= {k_\lambda}^2 \\ w(x) &= (1-x^2)^{\lambda-1/2} \\ q(x) &= {k_{\lambda-1/2}}^2(1-x^2)^{\lambda-3/2} \end{aligned}$$

Therefore, since $p(\pm 1) = 0$ (for $\lambda > -\frac{1}{2}$), we can see that the associated Gegenbauer functions are orthogonal over the interval $[-1, 1]$ with respect to the weight function $(1-x^2)^{\lambda-\frac{1}{2}}$ i.e.

$$\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\,\mathrm{C}[\lambda]_l^m(x)\,\mathrm{C}[\lambda]_{l'}^m(x)\,\mathrm{d}x \quad = \quad 0 \qquad \text{for} \quad l \neq l' \tag{D.9}$$

In this work we choose to always work with normalised Gegenbauer functions, i.e. ones obeying the *orthonormality* relation:

$$\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\,\mathrm{C}[\lambda]_l^m(x)\,\mathrm{C}[\lambda]_{l'}^m(x)\,\mathrm{d}x \quad = \quad \delta_{ll'} \tag{D.10}$$

D.4 Less Obvious Integrals

In section 3.3 it was mentioned that to do all the θ-, χ- and ψ-integrals of S_2 was not as simple as to just employ the Gegenbauer equation and orthogonality relation. This is because there are terms with multiple derivatives, terms bilinear in derivatives, and terms with coefficients different from those in the Gegenbauer equation, and so on. Ultimately though, all these terms should reduce to a 'simple' (analytically integrable) term, since they all originated from a scalar (the action). To manipulate terms into an integrable form we systematically eliminate terms for which some kind of relation can be found. In the next two subsections we see two such relations and how they are derived. For brevity, the $[\lambda]$ and (u) of $\mathrm{C}[\lambda]_l^m(u)$ are dropped throughout.

D.4.1 Zero-boundary terms

Here we construct a term, $f(u)$, to be zero on the boundaries, $f(\pm 1) = 0$ so that we can say $\int_{-1}^{1} f'(u)\, du \equiv 0$. By expanding the integrand we then have an identity which can be used to eliminate a chosen term.

Consider the term $\frac{u}{(1-u^2)^n}\mathrm{C}_a^m \mathrm{C}_b^m$. We can expand the associated Gegenbauer functions in terms of the Gegenbauer polynomials to count the powers of $(1-u^2)$ contained in this term:

$$\frac{u}{(1-u^2)^n}\mathrm{C}_a^m \mathrm{C}_b^m = u\,(1-u^2)^{m-n}\mathrm{C}_a^{(m)}\,\mathrm{C}_b^{(m)}$$

Evaluating this at $u = \pm 1$:

$$\left.\frac{u}{(1-u^2)^n}\mathrm{C}_a^m \mathrm{C}_b^m\right|_{u=\pm 1} = 0 \quad \text{for} \quad m > n$$

But to be able to take advantage of this using Mathematica it is convenient to be able to make a substitution that is valid for general values of m and n. Therefore we require a term that is zero for $m \leq n$. For this we consider the quantity $\mathcal{M}_n(m)$ defined as

$$\mathcal{M}_n(m) = \prod_{i=0}^{n}(m-i)$$

where in the case that n is not an integer the product is understood to be up to the greatest integer less than or equal to n, so that

$$\mathcal{M}_n(m) = 0 \quad \text{for} \quad m \leq n$$

Thus (since m only takes integer values)

$$\left.\mathcal{M}_n(m)\frac{u}{(1-u^2)^n}\,\mathrm{C}_a^m\,\mathrm{C}_b^m\right|_{u=\pm 1} = 0 \quad \forall n$$

Since this is of the form specified at the beginning of this section, we may declare:

$$\mathcal{M}_n(m)\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u}{(1-u^2)^n}\,\mathrm{C}_a^m\,\mathrm{C}_b^m\right)\mathrm{d}u = 0 \tag{D.11}$$

For $n \neq 0$ this can be expanded to:

$$\begin{aligned} m^{n+1}\int_{-1}^{1}\frac{\mathrm{C}_a^m\,\mathrm{C}_b^m}{(1-u^2)^{n+1}}\,\mathrm{d}u &= \left(m^{n+1}-\mathcal{M}_n(m)\right)\int_{-1}^{1}\frac{\mathrm{C}_a^m\,\mathrm{C}_b^m}{(1-u^2)^{n+1}}\,\mathrm{d}u \\ &+ \frac{1}{2n}\mathcal{M}_n(m)\int_{-1}^{1}\left[(1-2n)\frac{\mathrm{C}_a^m\,\mathrm{C}_b^m}{(1-u^2)^n}+\frac{u}{(1-u^2)^n}\left(\mathrm{C}_a^{m\prime}\,\mathrm{C}_b^m+\mathrm{C}_a^m\,\mathrm{C}_b^{m\prime}\right)\right]\mathrm{d}u \end{aligned}$$

and thus used to eliminate terms of the form $m^p\int_{-1}^{1}\frac{\mathrm{C}_a^m\,\mathrm{C}_b^m}{(1-u^2)^{n+1}}\,\mathrm{d}u$ when p is greater than or equal to the greatest integer less than or equal to $n+1$ and $n+1 \geq 0$. By doing this for the largest value of $n+1$ first, and then working our way down, we eventually eliminate terms with large powers of $(1-u^2)^{-1}$ that do not fit into the orthogonality relation.

D.4.2 Covariant divergences

Another way to construct an identity is by considering covariant divergences. For instance, consider some function defined on the 2-sphere, $f(\theta,\phi)$. If we take the covariant divergence of this, and integrate over the 2-sphere, then by Gauss's theorem this must be zero:

$$\int_0^{\pi}\int_0^{2\pi}\sqrt{\gamma}\,\gamma^{ij}\,f(\theta,\phi)_{;ij}\,\mathrm{d}\phi\,\mathrm{d}\theta \quad = \quad 0$$

By expanding the integrand and doing any integrals we can explicitly, we then have an identity which can be used to eliminate a chosen term. The following table gives terms whose covariant divergence on the 2-sphere can be taken to form an identity, and a term which we eliminate using that identity.

covariant divergence of	used to eliminate
$(1-u^2)\mathrm{C}[\frac{1}{2}]_a^{m\prime}(u)\mathrm{C}[\frac{1}{2}]_b^{m\prime}(u)$	$m\dfrac{\mathrm{C}[\frac{1}{2}]_a^m(u)\mathrm{C}[\frac{1}{2}]_b^m(u)}{(1-u^2)^2}$
$m\dfrac{\mathrm{C}[\frac{1}{2}]_a^m(u)\mathrm{C}[\frac{1}{2}]_b^m(u)}{1-u^2}$	$m^2\dfrac{\mathrm{C}[\frac{1}{2}]_a^m(u)\mathrm{C}[\frac{1}{2}]_b^m(u)}{1-u^2}$
$m(m-1)\mathrm{C}[\frac{1}{2}]_a^{m\prime}(u)\mathrm{C}[\frac{1}{2}]_b^{m\prime}(u)$	$m^2\dfrac{\mathrm{C}[\frac{1}{2}]_a^m(u)\mathrm{C}[\frac{1}{2}]_b^m(u)}{(1-u^2)^3}$
$(1-u^2)\mathrm{C}[\frac{1}{2}]_a^{m\prime\prime}(u)\mathrm{C}[\frac{1}{2}]_b^m(u)$	$m\dfrac{\mathrm{C}[\frac{1}{2}]_a^m(u)\mathrm{C}[\frac{1}{2}]_b^m(u)}{1-u^2}$

In summary, when performing the angular integrals of S_2 we mostly just rely on the orthonormality of the Gegenbauer functions, but there are also some 'less obvious integrals'. For these we construct identities which are used to systematically eliminate the miscellaneous terms, so that ultimately each angular integral reduces down to its corresponding orthonormality relation.

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