Portfolio selection of stochastic differential equation with jumps under regime switching.

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Abstract

In this thesis, we are interested in the stochastic differential equation with jumps under regime switching.

Firstly, we investigate a continuous-time version of the mean-variance portfolio selection model with jumps under regime switching. The portfolio selection proposed and analyzed for a market consisting of one bank account an d multiple stocks. The random regime switching is assumed to be independent of the underlying Brownian motion and jump processes.

Secondly, we consider the problem of pricing contigent claims on a stock whose price process is modeled by a Lévy process. Since the market is incomplete and there is not a unique equivalent martingale measure. We study approaches to pricing options.

Finally, we investigate a continuous-time version Markowitz's mean-variance portfolio selection problem which is studied in a market with one bank account, one stock and proportional transaction costs. This is a singular stochastic control problem. Via a series of transformations, the problem is turned into a double obstacle problem.

DECLARATION

This work has not previously been accepted in substance for any degree and is not concurrently submitted in candidature for any degree.

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Notation

a.s. :	Almost surely, or with probability 1.
$\emptyset$:	The empty set.
I_A :	The indicator function of a set A, i.e. $I_A(x) = 1$ if $x \in A$ or otherwise 0.
$A \subset B$:	$A \cap B^C = \emptyset$.
$\sigma(C)$:	The σ-algebra generated by C.
$\mathcal{B}(\mathbb{R}^n)$:	The Borel σ-algebra on $\mathbb{R}^n$.
$f : A \to B$:	The mapping f from A to B.
$\mathbb{R}_+$:	$[0, \infty)$.
$\mathbb{R}^n$:	The n-dimensional Euclidean space.
$\mathbb{R}^{n \times d}$:	The space of real $n \times d$ matrices.
$\|x\|$:	The Euclidean norm of a vector x.
A', A^T :	The transpose of a vector or matrix A.
$C(D; \mathbb{R}^n)$:	The family of continuous $\mathbb{R}^n$-valued functions defined on D.
$C^m(D; \mathbb{R}^n)$:	The family of continuously m-times differentiable $\mathbb{R}^n$-valued functions defined on D.
$C^{2,1}(D \times \mathbb{R}_+; \mathbb{R})$:	The family of all real-valued functions $V(x,t)$ defined on $D \times \mathbb{R}_+$ which are continuously twice differentiable in $x \in D$ and once differentiable in $t \in \mathbb{R}_+$.

V_x : $V_x = \nabla V = (V_{x_1}, \cdots, V_{x_n}) = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n})$.

V_{xx} : $V_{xx} = (V_{x_i x_j})_{n\times n} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n\times n}$.

$L^p(\Omega; R^n)$: The family of R^n-valued random variables X with $E|X|^p < \infty$.

$\mathcal{L}^p([a,b]; R^n)$: The family of R^n-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b |f(t)|^p dt < \infty$ a.s.

Other notations will be explained where they first appear.

Chapter 1

Introduction

1.1 Background

The jump diffusion process has come to play an important role in many branches of science and industry. In their book [31], Øksendal and Sulem have studied optimal control, optimal stopping and impulse control for jump diffusion processes. In mathematical finance theory, many researchers have developed option pricing theory, for example: Merton [28] was the first to use the jump processes to describe the stock dynamics, Bardhan and Chao [1] were amongst the first authors to consider market completeness in a discontinuous model. Jump diffusion models have been discussed by Chan [4], Föllmer and Schweizer [11], El Karoui and Quenez [21], Henderson and Hobson [18], and Merculio and Runggaldier [27], to name a few.

On the other hand, regime-switching models have been widely used for price processes of risky assets. For example, in Jobert and Rogers [20] the optimal stopping problem for the perpetual American put has been considered, and the finite expiry American put and barrier options have been priced. Asset allocation has been discussed in Elliott and Van der Hoek [9], and Elliott and Malcolm [10] have investigated volatility problems. Regime-switching models with a markov-modulated asset have already been applied to option pricing in Guo [14, 15, 16] and the references therein. Morerover, Markowitz's mean-variance portfolio selection with regime switching has been studied in Yin and Zhou [36], Zhou and Yin [40] and Zhou and Li [39].

Portfolio selection is an important topic in finance, multi-period mean-variance portfolio selection has been studied in , for example, Samuelsom [34], Hakansson [17], and Pliska [33] among others. Continuous-time mean-variance hedging problems were attacked by Duffie and Richardson [8] and Schweizer [35] where optimal dynamic strategies were derived, based on the projection theorem, to hedge contingent claims in incomplete markets.

1.2 Mathematical model of security markets

Definition 1.1 *A European call option gives its holder the right, but not the obligation, to purchase from the writer a prescribed asset for a prescribed price at a prescribed time in the future.*

The prescribed time in the future is known as the expiry date or the exercise date or the maturity. The prescribed purchase price is known as the strike price or the exercise price.

We denote by $S(t)$ the price of a particular stock, $\hat{S}(t)$ the discounted price process, and the value process $V(t)$ the total value of the portfolio, at time t. A contingent claim is a random variable X representing a pay-off at the maturity time. It is part of a contract that a buyer and a seller agree at time $t = 0$.

1.3 Martingale

Throughout this thesis, we let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$, which satisfies the usual conditions, i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. By a filtration we mean a family of σ-algebras $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$ that is increasing, i.e., $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. If $(\Omega, \mathcal{F}, P)$ is a probability space, we set

$$\bar{\mathcal{F}} = \{A \subset \Omega : \exists B, C \in \mathcal{F} \text{ such that } B \subset A \subset C, P(B) = P(C)\}.$$

Then $\bar{\mathcal{F}}$ is a σ-algebra and is called the completion of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, the probability space $(\Omega, \mathcal{F}, P)$ is said to be complete.

Definition 1.2 *Let (S, Σ) be a measurable space, then the process X is said to be adapted to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$ if the random variable $X_t : \Omega \to \Sigma$ is a $(\mathcal{F}_t, \Sigma)$ measurable function for each $t \in [0, \infty)$.*

Definition 1.3 *A real-valued, adapted process $X = (X_t)_{0\leq t<\infty}$ is caled a martingale with respect to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$ if*

(i) $X_t \in L^1(dP)$; that is, $\mathbb{E}\{|X_t|\} < \infty$;

(ii) if $s \leq t$, then $\mathbb{E}\{\mathbb{X}_t|\mathcal{F}_s\} = X_s$, a.s..

Definition 1.4 *A stochastic process X is said to be càdlàg if it a.s. has sample paths which are right continuous, with left limits.*

Definition 1.5 *Let $X(t)$, $Y(t) \in \mathbb{R}^n$ be two càdlàg semimartingales. The quadratic covariation of $X(\cdot)$ and $Y(\cdot)$, denoted by $[X, Y](\cdot)$, is the unique semimartingale such that*

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s^-)dY(s) + \int_0^t Y(s^-)dX(s) + [X, Y](t).$$

1.4 Brownian motion

Brownian motion is at the heart of most models in practice. Its name comes from the Scottish botanist Robert Brown who reported it in around 1827.

The paths of a Brownian motion are continuous, almost surely. Moreover, we may identify $\omega \in \Omega$ with a continuous function $t \to W_t(\omega)$ from $[0, \infty)$ into $\mathbb{R}^n$. Thus we may adopt the point of view that Brownian motion is just the space $C([0, \infty), \mathbb{R}^n)$ equipped with certain probability measures P.

Definition 1.6 *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying usual conditions. A (standard) one-dimensional Brownian motion is a real-valued continuous $\mathcal{F}_t$-adapted process $\{W_t\}_{t \geq 0}$ with the following properties:*

(1) $W_0 = 0$ *almost surely;*

(2) *for $0 \leq s < t < \infty$, the increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$;*

(3) *for $0 \leq s < t < \infty$, the increment $W_t - W_s$ is independent of $\{\mathcal{F}_s\}$.*

The filtration $\{\mathcal{F}_t\}_{0 \leq t \leq \infty}$ is a part of the definition of Brownian motion. However, we say Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ without filtration. That is, $\{W_t\}_{t \geq 0}$ is a real-valued continuous process with properties (1) and (2), but property (3) is replaced by that it has independent increments.(We say a Brownian motion $\{W_t\}_{t \geq 0}$ has independent increments, if the increments $W_{t_i} - W_{t_{i-1}}, 1 \leq i \leq k$ are independent, for $0 \leq t_0 < t_1 < \cdots < t_k < \infty$.) In this case, define $\mathcal{F}_t^W = \sigma(W_s : 0 \leq s \leq t)$ for $t \geq 0$, i.e. $\mathcal{F}_t^W$ is the σ-algebra generated by $\{W_s : 0 \leq s \leq t\}$. We call $\{\mathcal{F}_t^W\}_{t \geq 0}$ the natural filtration generated by $\{W_t\}$. Clearly, $\{W_t\}$ is a Brownian motion with respect to the natural filtration $\{\mathcal{F}_t^W\}$. Furthermore, if $\{\mathcal{F}_t\}$ is a "larger" filtration in the sense that $\mathcal{F}_t^W \subset \mathcal{F}_t$ for $t \geq 0$, and

$W_t - W_s$ is independent of $\mathcal{F}_s$ whenever $0 \le s < t < \infty$, then $\{W_t\}$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$.

1.5 Markov process

An n-dimensional $\mathcal{F}_t$-adapted process $X = \{X_t\}_{t\ge 0}$ is called a Markov process if the following Markov property is satisfied: for all $0 \le s \le t < \infty$ and $A \in \mathcal{B}(\mathbb{R}^n)$,

$$P(X(t) \in A|\mathcal{F}_s) = P(X(t) \in A|X(s)).$$

This is equivalent to the following one: for any bounded Borel measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $0 \le s \le t < \infty$,

$$E(\varphi(X(t))|\mathcal{F}_s) = E(\varphi(X(t))|X(s)).$$

1.6 Stochastic differential equations

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions. Let $W(t) = (W_1(t), \cdots, W_m(t))^T$, $t \ge 0$ be an m-dimensional Brownian motion defined on the space. Let $0 \le t_0 < T < \infty$. Let $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$, i.e., an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^n$-valued random variable such that $E|x_0|^2 < \infty$. Let $b : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^{n\times m}$ be both Borel measurable. Consider the n-dimensional stochastic differential equation of Itô type

$$dx(t) = b(x(t), t)dt + \sigma(x(t), t)dW(t)$$

with initial value $x(t_0) = x_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation

$$x(t) = x_0 + \int_{t_0}^{t} b(x(u), u)du + \int_{t_0}^{t} \sigma(x(u), u)dW(u).$$

Systems in many branches of science and industry are often subject to various types of noise and uncertainty. For example, let us consider a simple model of an asset price. Suppose that at time t the asset price is $S(t)$. Consider a small subsequent time interval dt, during which $S(t)$ changes to $S(t)+dS(t)$. (We use the notation $d\cdot$ for the small change in any quantity over this time interval when we intend to consider it as an infinitesimal change.) By definition, the return per unit of the asset price at time t is $dS(t)/S(t)$.

To understand the modelling more easily, suppose that the asset is a bank deposit while the bank deposit interest rate is r. So $S(t)$ is the balance of the saving account at time t. Thus the return $dS(t)/S(t)$ of the saving at time t is rdt, that is

$$\frac{dS(t)}{S(t)} = rdt,$$

or

$$\frac{dS(t)}{dt} = rS(t).$$

This ordinary differential equation can be solved exactly to give exponential growth in the value of the saving, i.e.

$$S(t) = S(t_0)e^{r(t-t_0)},$$

where $S(t_0)$ is the initial deposit of the saving account at time t_0.

The most common model decomposes the return $dS(t)/S(t)$ of the asset price into two parts. First, there is a predictable, deterministic and anticipated return on money invested in a risk-free bank. It gives a contribution

$$bdt$$

to the return dS/S, where b is a measure of the average rate of growth of the asset price. In simple model b, which is also known as "drift", is taken to be a constant. (In this thesis, b is a function.)

Then, the second one is the random change in the asset price in response to external effects. It is represented by a random sample drawn from a normal distribution with mean 0 and adds a term

$$\sigma dW(t)$$

to dS/S. σ is a matrix called the volatility, which measures the standard deviation of the returns. It is taken to be a constant in simple model.(In this thesis, σ is also a function.) Here $W(t)$ is a standard Brownian process.

So, putting these two parts together, we can easily justify the stochastic differential equation

$$\frac{dS(t)}{S(t)} = bdt + \sigma dW(t),$$

or

$$dS(t) = bS(t)dt + \sigma S(t)dW(t),$$

and

$$S(t) = S(t_0) + b\int_{t_0}^{t} S(u)du + \sigma \int_{t_0}^{t} S(u)dW(u).$$

The formal interpretation of an SDE is given in terms of what constitutes a solution to the SDE. There are two main definitions of a solution to an SDE, a strong solution and a weak solution. Both require the existence of a process $S(t)$ that solves the integral equation version of the SDE. A weak solution consists of a probability space and a process that satisfies the integral equation, while a strong solution is a process that satisfies the equation and is defined on a given probability space.

We will be working on the SDEs under regime switching in this thesis,

$$dS(t) = b(\alpha(t))S(t)dt + \sigma(\alpha(t))S(t)dW(t),$$

where the continuous-time stationary Markov chain $\alpha(t)$ takes value in a finite state space $\mathbb{S} = \{1, 2, \dots, l\}$.

1.7 SDEs with jumps and Itô formula

Definition 1.7 *An adapted process $X = (X_t)_{t \geq 0}$ with $X_0 = 0$ a.s. is a Lévy process if*

(i) X has increments independent of the past; that is $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$; and

(ii) X has stationary increments; that is, $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t < \infty$, and

(iii) X_t is right continuous with left limit (RCLL).

Definition 1.8 *A Poisson process $N(t)$ of intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup 0$ and such that*

$$P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, n = 0, 1, 2, \ldots.$$

Definition 1.9 *Let $\mathcal{B}$ be the family of Borel sets $U \subset \mathbb{R}$ whose closure $\bar{U}$ does not contain 0. For $U \in \mathcal{B}$, we define*

$$N(t, U) = N(t, U, \omega) = \sum_{s: 0 < s \leq t} \mathcal{X}_U(\Delta \eta_s).$$

In other word, $N(t, U)$ is the number of jumps of size $\Delta \eta_s \in U$ which occur before or at time t (the differential form of this measure is written $N(dt, dz)$). Then the set function

$$\nu(U) = \mathbb{E}[N(1, U)],$$

where $\mathbb{E} = \mathbb{E}_P$ denotes expectation with respect to P, also defines a σ-finite measure on $\mathcal{B}$, is called the Lévy measure of the Poisson process N.

Actually there is a relationship between a Poisson process and a Poisson random measure as the following:

Remark 1.1 *For any fix $U \in \mathcal{B}$, let*

$$\pi_U(t) := \pi_U(t, \omega) := N(t, U, \omega),$$

then $\pi_U(t)$ is a Poisson process with intensity $\lambda = \nu(U) = \mathbb{E}[N(1, U)]$, i.e.

$$P(\pi_U(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, n = 0, 1, 2, \ldots.$$

Let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space. Let $N(t, z)$ be a Poisson process and denote the compensated Poisson process by

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt,$$

where ν is a Poisson point process. We assume that $W(t)$, $\alpha(t)$ and $N(dt, dz)$ are independent.

Definition 1.10 *Let W_t be 1-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A (1-dimensional) Itô-Lévy process (or stochastic integral) is a stochastic process S_t on $(\Omega, \mathcal{F}, P)$ of the form*

$$S_t = S_0 + \int_0^t b(s, \omega)ds + \int_0^t \sigma(s, \omega)dW_s + \int_0^t \int_{\mathbb{R}} \rho(s, \omega, z)\tilde{N}(ds, dz),$$

so that

$$P[\int_0^t \sigma(s, \omega)^2 ds < \infty, t \geq 0] = 1.$$

We also assume that W_t is $\mathcal{F}_t$-adapted (W_t is a martingale with respect to $\mathcal{F}_t$) and

$$P[\int_0^t |b(s, \omega)|ds < \infty, t \geq 0] = 1,$$

$$P[\int_0^t \int_{\mathbb{R}} \rho(s, \omega, z)\nu(dz)ds < \infty, t \geq 0] = 1.$$

Theorem 1.1 *[Øksendal and Sulem (2005)] Let $X(t) \in \mathbb{R}^n$ be an Itô-Lévy process of the form*

$$dX(t) = b(t,\omega)dt + \sigma(t,\omega)dW(t) + \int_{\mathbb{R}} \rho(t,\omega,z)\tilde{N}(dt,dz),$$

where

$$\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$$

Let $f \in C^{1,2}(\mathbb{R}^n)$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again an Itô-Lévy process and

$$\begin{aligned}
&dY(t) \\
&= \frac{\partial f}{\partial t}(t,X(t))dt + \frac{\partial f}{\partial x}(t,X(t))[b(t,\omega)dt + \sigma(t,\omega)dW(t)] \\
&\quad + \frac{1}{2}\sigma^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,X(t))dt \\
&\quad + \int_{\mathbb{R}} \{f(t,X(t^-)+\rho(t,\omega,z)) - f(t,X(t^-)) - \frac{\partial f}{\partial x}(t,X(t^-))\rho(t,\omega,z)\}\nu(dz)dt \\
&\quad + \int_{\mathbb{R}} \{f(t,X(t^-)+\rho(t,\omega,z)) - f(t,X(t^-))\}\tilde{N}(dt,dz).
\end{aligned}$$

1.8 Esscher transform and minimal relative entropy martingale measures

The Esscher transform is a time-honored tool in actuarial science. It is also an efficient technique for valuing derivative securities if the logarithms of the prices of the primitive securities are governed by certain stochastic processes with stationary and independent increments.

For a probability density function $f(x)$, let h be a real number such that

$$M(h) = \int_{-\infty}^{\infty} e^{hx} f(x)dx$$

exists. As a function in x,

$$f(x;h) = \frac{e^{hx} f(x)}{M(h)}$$

is a probability density, and it is called the Esscher transform (parameter h) of the original distribution.

Suppose that a probability space $(\Omega, \mathcal{F}, P)$ and an increasing family of sub σ-fields of $\mathcal{F}$, $\mathcal{F}_t, 0 \leq t \leq T$, are given as usual. A price process is a $\mathcal{F}_t$-adapted stochastic process $S(t)$ defined on $(\Omega, \mathcal{F}, P)$.

We define $\mathcal{P}(S)$ as the set of all equivalent S-martingale measures, namely the set of all probability measures $Q \sim P$ (mutually absolutely continuous). If the set $\mathcal{P}(S)$ is singleton, then the market is said to be complete. If the set $\mathcal{P}(S)$ consist of more than one element, then the market is said to be incomplete.

Definition 1.11 ***(minimal entropy martingale measure)[YOSHIO MIYAHAR 1999]*** *If an equivalent martingale measure $\hat{P}$ satisfies the following condition*

$$H(\hat{P}|P) \leq H(Q|P), \forall Q \in \mathcal{P}(S),$$

then $\hat{P}$ is called the minimal entropy martingale measure of $S(t)$, where $H(Q|P)$ is the relative entropy of Q with respect to P, which is given by the following formula

$$H(Q|P) = \begin{cases} \int_\Omega \log[\frac{dQ}{dP}] dQ, & Q \ll P, \\ \infty, & \textit{otherwise.} \end{cases}$$

Chapter 2

Portfolio selection of stochastic differential equation with jumps under regime switching

Markowitz's mean-variance portfolio selection with regime switching has been studied in Yin and Zhou [36], Zhou and Yin [40] and Zhou and Li [39].

Portfolio selection is an important topic in finance, multi-period mean-variance portfolio selection has been studied in , for example, Samuelsom [34], Hakansson [17], and Pliska [33] among others. Continuous-time mean-variance hedging problems were attacked by Duffie and Richardson [8] and Schweizer [35] where optimal dynamic strategies were derived, based on the projection theorem, to hedge contingent claims in incomplete markets.

In this chapter, we develop Stochastic Differential Equations under regime switching with jumps. The jump diffusion process has come to play an important role in many branches of science and industry. In their book [31], Øksendal and Sulem have studied optimal control, optimal stopping and impulse control for jump diffusion processes.

2.1 SDEs under Regime Switching with Jumps

Throughout this thesis, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space on which is defined a standard d-dimensional Brownian motion $W(t) \equiv (W_1(t), \dots, W_d(t))'$ and a continuous-time stationary Markov chain $\alpha(t)$ taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, l\}$. Let $N(t,z)$ be a n-dimensional Poisson process and denote the compensated Poisson process by

$$\begin{aligned}\tilde{N}(dt, dz) &= (\tilde{N}_1(dt, dz_1), \dots, \tilde{N}_n(dt, dz_n))' \\ &= (N_1(dt, dz_1) - \nu_1(dz_1)dt, \dots, N_n(dt, dz_n) - \nu_n(dz_n)dt)', \qquad (2.1)\end{aligned}$$

where $N_j, j = 1, \dots, n$, are independent 1-dimensional Poisson random measures with characteristic measure $\nu_j, j = 1, \dots, n$, coming from n independent 1-dimensional Poisson point processes. We assume that $W(t)$, $\alpha(t)$ and $N(dt, dz)$ are independent. The Markov chain $\alpha(t)$ has a generator $Q = (q_{ij})_{l\times l}$ given by

$$P\{\alpha(t+\Delta) = j | \alpha(t) = i\} = \begin{cases} q_{ij}\Delta + o(\Delta) & : \quad \text{if } i \neq j \\ 1 + q_{ii}\Delta + o(\Delta) & : \quad \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$q_{ii} = -\sum_{j\neq i} q_{ij},$$

are stationary transition probabilities, and

$$p_{ij}(t) = P(\alpha(t) = j | \alpha(0) = i), \qquad t \geq 0, i, j = 1, 2, \dots, l. \qquad (2.2)$$

Define $\mathcal{F}_t = \sigma\{W(s), \alpha(s), N(s, \cdot) : 0 \leq s \leq t\}$. Let $|\cdot|$ denote the Euclidean norm as well as the matrix trace norm, M' denote the transpose of any vector or matrix M. We denote by $L^2_{\mathcal{F}(0,T;\mathbb{R}^m)}$ the set of all $\mathbb{R}^m$-valued, measurable stochastic processes $f(t)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $\mathbb{E}\int_0^T |f(t)|^2 dt < +\infty$.

Consider a market in which $d+1$ assets are traded continuously. One of the assets is a bank account whose price $P_0(t)$ is subject to the following ordinary differential equation:

$$\begin{cases} dP_0(t) = r(t,\alpha(t))P_0(t)dt, \quad t \in [0,T], \\ P_0(0) = p_0 > 0, \end{cases} \tag{2.3}$$

where $r(t,i) \geq 0, i = 1,2,\ldots,l$, is given as the interest rates corresponding to different market modes. The other d assets are stocks whose price processes $P_m(t)$, $m = 1,2,\ldots,d$, satisfy the following system of stochastic differential equations (SDEs):

$$\begin{cases} dP_m(t) = P_m(t)\{b_m(t,\alpha(t))dt + \sum_{n=1}^{d} \sigma_{mn}(t,\alpha(t))dW_n(t) \\ \qquad + \sum_{j=1}^{n} \int_{\mathbb{R}} \rho_{mj}(t,\alpha(t),z_j)\tilde{N}_j(dt,dz_j)\}, \qquad t \in [0,T], \\ P_m(0) = p_m > 0,. \end{cases} \tag{2.4}$$

where for each $i = 1,2,\ldots,l$, $b : [0,T] \times \mathbb{S} \to \mathbb{R}^{d\times 1}$, $\sigma : [0,T] \times \mathbb{S} \to \mathbb{R}^{d\times d}$, $\rho : [0,T] \times \mathbb{S} \times \mathbb{R}^n \to \mathbb{R}^{d\times n}$ is the appreciation rate process and $\sigma_m(t,i) := (\sigma_{m1}(t,i),\ldots,\sigma_{md}(t,i))$ are adapted processes such that the integrals exists. And each column $\rho^{(k)}$ of the $d \times n$ matrix $\rho = [\rho_{ij}]$ depends on z only through the kth coordinate z_k, i.e.,

$$\rho^{(k)}(t,i,z) = \rho^{(k)}(t,i,z_k), z = (z_1,\ldots,z_n) \in \mathbb{R}^n.$$

Remark 2.1 *Generally speaking, one uses non-compensated Poisson processes in a jump diffusion model (see Kushner [23]). However, we use compensated Poisson processes in equation* (2.4) *instead of using non-compensated Poisson processes, this is because: Firstly, using the relationship* (2.1) *we can easily transform a jump diffusion model driven by non-compensated Poisson* *processes into a jump diffusion model driven by compensated Poisson processes; Secondly, using compensated Poisson processes we can keep the Riccati equation* (2.29) *is similar to that of a diffusion model without a jump processes, and then* $H(t,i)$ *in* (2.30) *has a financial interpretation.*

Define the volatility matrix, for each $i = 1, \dots, l$,

$$\sigma(t,i) := \begin{pmatrix} \sigma_1(t,i) \\ \vdots \\ \sigma_d(t,i) \end{pmatrix} \equiv (\sigma_{mn}(t,i))_{d\times d}, \tag{2.5}$$

$$b(t,i) = \begin{pmatrix} b_1(t,i) \\ \vdots \\ b_d(t,i) \end{pmatrix} \in \mathbb{R}^{d\times 1},$$

and

$$\rho(t,i,z) = \begin{pmatrix} \rho_1(t,i,z) \\ \vdots \\ \rho_d(t,i,z) \end{pmatrix} \in \mathbb{R}^{d\times n},$$

where

$$\rho_m(t,i,z) = (\rho_{m1}(t,i,z), \cdots, \rho_{mn}(t,i,z)).$$

We assume throughout this thesis that the following non-degeneracy condition

$$\sigma(t,i)\sigma(t,i)' \geq \delta I, \forall t \in [0,T], \text{ and } i = 1,2,\dots,l, \tag{2.6}$$

is satisfied for some $\delta > 0$. We also assume that all the functions $r(t,i)$, $b_m(t,i)$, $\sigma_{mn}(t,i)$, $\rho_{mn}(t,i,z)$ are measurable and uniformly bounded in t.

Suppose that the initial market mode $\alpha(0) = i_0$. Consider an asset with an initial wealth $x_0 > 0$. These initial conditions are fixed throughout the thesis. Denote by $x(t)$ the total wealth of the assets at time $t \geq 0$. Assuming that the trading of shares takes place continuously and that transaction cost and consumptions are not considered. Suppose the right portfolio $(\pi_0(t), \pi_1(t), \cdots, \pi_d(t))$ exists, where $\pi_0(t)$ is the money invested in the bond, and $\pi_i(t)$ is the money invested in the ith stock. Then

$$\begin{cases} x(t) = \sum_{i=0}^{d} \pi_i(t) = \sum_{i=0}^{d} \eta_i(t) P_i(t), \\ x(0) = x_0, \end{cases}$$

where $\eta_0(t)$ is the number of bond units bought by the investor, and $\eta_i(t)$ is the amount of units for the ith stock. We call $x(t)$ the wealth process for this investor in the market. Now let us derive intuitively the stochastic differential equation (SDE) for the wealth process as follows: Suppose the portfolio is self-financed, i.e. in a short time dt the investor does not put in or withdraw any money from the market. Let the money $x(t)$ change in the market due to the market is own performance, i.e. self-finance produces

$$dx(t) = \eta_0(t) dP_0(t) + \sum_{i=1}^{d} \eta_i(t) dP_i(t).$$

Now substituting (2.3) and (2.4) into the above equation, after a simple calculation we arrive

$$\begin{cases} dx(t) = r(t, \alpha(t)) x(t) dt + \sum_{m=1}^{d} \pi_m(t)(b_m(t, \alpha(t)) - r(t, \alpha(t))) dt \\ \qquad + \sum_{m=1}^{d} \sum_{n=1}^{d} \pi_m(t) \sigma_{mn}(t, \alpha(t)) dW_n(t) \\ \qquad + \sum_{m=1}^{d} \sum_{j=1}^{n} \int_{\mathbb{R}} \pi_m(t) \rho_{mj}(t, \alpha(t), z_j) \tilde{N}_j(dt, dz_j), \\ x(0) = x_0 > 0, \alpha(0) = i_0, \end{cases} \tag{2.7}$$

where $\pi(t) = (\pi_1(t), \ldots, \pi_d(t))'$ which we call a portfolio of the agent. And $\pi_m(t)$ is the total market value of the agent's wealth in the mth asset, $m = 0, 1, \ldots, d$, at time t.

Setting

$$B(t,i) := (b_1(t,i) - r(t,i), \ldots, b_d(t,i) - r(t,i)), i = 1, 2, \ldots, l, \tag{2.8}$$

we can rewrite the wealth equation (2.7) as

$$\begin{cases} dx(t) = r(t,\alpha(t))x(t)dt + B(t,\alpha(t))\pi(t)dt + \pi'(t)\sigma(t,\alpha(t))dW(t) \\ \qquad\qquad + \int_{\mathbb{R}^n} \pi'(t)\rho(t,\alpha(t),z)\tilde{N}(dt,dz), \\ x(0) = x_0 > 0, \alpha(0) = i_0. \end{cases} \tag{2.9}$$

Definition 2.1 *A portfolio $\pi(\cdot)$ is said to be admissible if $\pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ and the SDE (2.9) has a unique solution $x(\cdot)$ corresponding to $\pi(\cdot)$. In this case, we refer to $(x(\cdot), \pi(\cdot))$ as an admissible (wealth, portfolio) pair.*

Remark 2.2 *Most works in the literature define a portfolio, say $\pi(\cdot, i)$, as the fractions of wealth allocated to different stocks. That is,*

$$u(t) = \frac{\pi(t)}{x(t)} := (\frac{\pi_1(t)}{x(t)}, \ldots, \frac{\pi_d(t)}{x(t)})', t \in [0,T]. \tag{2.10}$$

With this definition, equation (2.9) can be rewritten as

$$\begin{cases} dx(t) = x(t)[r(t,\alpha(t)) + B(t,\alpha(t))u(t)]dt \\ \qquad\qquad + x(t)u(t)'\sigma(t,\alpha(t))dW(t) \\ \qquad\qquad + \int_{\mathbb{R}^n} x(t)u(t)'\rho(t,\alpha(t),z)\tilde{N}(dt,dz), \\ x(0) = x_0 > 0, \alpha(0) = i_0. \end{cases} \tag{2.11}$$

It is well known that this equation has a unique solution (see [31] p.10, Theorem 1.19). We can use the similar method in [31, Example1.15, p8] to show positivity of the solution of Eq. (2.11) if the initial wealth x_0 is positive and $u(t)'\rho(t,i,z) > -1$ (this condition is achievable, for example, if all elements of a portfolio are positive, and $\rho = 1$, then $u(t)'\rho(t,i,z) > 0$. Generally speaking, if $u(t)$ is bounded, there are many ρ satisfying $u(t)'\rho(t,i,z) > -1$.) A wealth process with possible zero or negative values is sensible at least for some circumstances. The nonnegativity of wealth process is better imposed as an additional constraint, rather than as a built-in feature. In our formulation, a portfolio is well defined even if the wealth is zero or negative, and the nonnegativity of the wealth could be a constraint.

The agent's objective is to find an admissible portfolio $\pi(\cdot)$ among all the admissible portfolios with expected terminal wealth $\mathbb{E}x(T) = \zeta$ for some given $\zeta \in \mathbb{R}^1$, so that the risk measured by the variance of the terminal wealth

$$\text{Var } x(T) \equiv \mathbb{E}[x(T) - \mathbb{E}x(T)]^2 = \mathbb{E}[x(T) - \zeta]^2 \tag{2.12}$$

is minimized. Finding such a portfolio $\pi(\cdot)$ is referred to as the mean-variance portfolio selection problem. Specifically, we have the following formulation.

Definition 2.2 *The mean-variance portfolio selection is a constrained stochastic optimization problem, parameterized by $\zeta \in \mathbb{R}^1$:*

$$\begin{cases} \textit{minimize} & J_{MV}(x_0, i_0, \pi(\cdot)) := \mathbb{E}[x(T) - \zeta]^2 \quad , \\ \textit{subject to} & \begin{cases} \mathbb{E}x(T) = \zeta, \\ (x(\cdot), \pi(\cdot)) \ \textit{admissible} \ . \end{cases} \end{cases} \tag{2.13}$$

Moreover, the problem is called feasible if there is at least one portfolio satisfying all the constraints. The problem is called finite if it is feasible and the infimum of $J_{MV}(x_0, i_0, \pi(\cdot))$ is finite. Finally, an optimal portfolio to the above problem, if it ever exists, is called an efficient portfolio corresponding to

ζ, the corresponding (Var $x(T), \zeta) \in \mathbb{R}^2$ and $(\sigma_{x(T)}, \zeta) \in \mathbb{R}^2$ are interchangeably called an efficient point, where $\sigma_{x(T)}$ denotes the standard deviation of $x(T)$. The set of all the efficient points is called the efficient frontier.

For more details of mean-variance portfolio selection see [36, 40]. We need more notations, let Δ_{ij} be consecutive, left closed, right open intervals of the real line each having length γ_{ij} such that

$$
\begin{aligned}
\Delta_{12} &= [0, q_{12}), \\
\Delta_{13} &= [q_{12}, q_{12} + q_{13}), \\
&\vdots \\
\Delta_{1l} &= \left[\sum_{j=2}^{l-1} q_{1j}, \sum_{j=2}^{l} q_{1j}\right), \\
\Delta_{21} &= \left[\sum_{j=2}^{l} q_{1j}, \sum_{j=2}^{l} q_{1j} + q_{21}\right), \\
\Delta_{23} &= \left[\sum_{j=2}^{l} q_{1j} + q_{21}, \sum_{j=2}^{l} q_{1j} + q_{21} + q_{23}\right), \\
&\vdots \\
\Delta_{2l} &= \left[\sum_{j=2}^{l} q_{1j} + \sum_{j=1, j\neq 2}^{l-1} q_{2j}, \sum_{j=2}^{l} q_{1j} + \sum_{j=1, j\neq 2}^{l} q_{2j}\right).
\end{aligned}
$$

For future use, we cite the generalized Itô lemma (see [26, 2, 31]) as the following lemma.

Lemma 2.1 *Given an d-dimensional process $y(\cdot)$ satisfying*

$$
\begin{aligned}
dy(t) = {} & f(t, y(t), \alpha(t))dt + g(t, y(t), \alpha(t))dW(t) \\
& + \int_{\mathbb{R}^n} \gamma(t, y(t), \alpha(t), z)\tilde{N}(dt, dz),
\end{aligned}
$$

where f, g and γ satisfy Lipschitz condition with appropriate dimensions, moreover, each column $\gamma^{(k)}$ of the matrix $\gamma = [\gamma_{ij}]$ depends on z only through *the k^{th} coordinate z_k. Let $\varphi(t,x,i) \in C^{1,2}([0,T] \times \mathbb{R}^n \times S; \mathbb{R})$, we then have*

$$
\begin{aligned}
& d\varphi(t, y(t), \alpha(t)) \\
&= \Gamma\varphi(t, y(t), \alpha(t))dt + \varphi_x(t, y(t), \alpha(t))' g(t, y(t), \alpha(t)) dW(t) \\
&\quad + \sum_{k=1}^{n} \int_{\mathbb{R}} \{\varphi(t, y(t) + \gamma^{(k)}(t, y(t), \alpha(t), z), \alpha(t)) - \varphi(t, y(t), \alpha(t))\} \tilde{N}_k(dt, dz_k) \\
&\quad + \int_{\mathbb{R}} (\varphi(t, y(t), \alpha(0) + h(\alpha(t), \bar{l})) - \varphi(t, y(t), \alpha(t))) \mu(dt, d\bar{l}),
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma\varphi(t,x,i) := {} & \varphi_t(t,x,i) + \varphi_x(t,x,i)' f(t,x,i) \\
& + \frac{1}{2}\mathrm{trace}[g(t,x,i)' \varphi_{xx}(t,x,i) g(t,x,i)] + \sum_{j=1}^{l} q_{ij}\varphi(t,x,j) \\
& + \sum_{k=1}^{n} \int_{\mathbb{R}} \{\varphi(t, y(t) + \gamma^{(k)}(t, y(t), \alpha(t), z_k), \alpha(t)) - \varphi(t, y(t), \alpha(t)) \\
& - \varphi_x(t, y(t), \alpha(t))' \gamma^{(k)}(t, y(t), \alpha(t), z_k)\} \nu_k(dz_k),
\end{aligned}
$$

where μ is a martingale measure,

$$
h(i,y) = \begin{cases} j - i, \ \textit{if } y \in \Delta_{ij} \\ 0, \ \textit{otherwise} \end{cases}
$$

and $\mu(dt, d\bar{l}) = \bar{\gamma}(dt, d\bar{l}) - \mu(d\bar{l})dt$ is a martingale measure. And $\bar{\gamma}(dt, dy)$ is a Poisson random measure with intensity $dt \times \mu(dy)$, in which μ is the Lebesgue measure on $\mathbb{R}$.

2.2 Feasibility

Since the problem (2.13) involves a terminal constraint $\mathbb{E}x(T) = \zeta$, in this section, we derive conditions under which the problem is at least feasible.

First of all, the following generalized Itô lemma [2] for Markov-modulated processes is useful.

The associated wealth process $x^0(\cdot)$ satisfies

$$\begin{cases} dx^0(t) = r(t,\alpha(t))x^0(t)dt, \\ x^0(0) = x_0 > 0, \alpha(0) = i_0, \end{cases} \tag{2.14}$$

with its expected terminal wealth

$$\zeta^0 := \mathbb{E}x^0(T) = \mathbb{E}e^{\int_0^T r(s,\alpha(s))ds}x_0. \tag{2.15}$$

Lemma 2.2 *Let $\psi(\cdot,i)$, $i = 1,2,\ldots,l$, be the solutions to the following system of linear ordinary differential equations (ODEs):*

$$\begin{cases} \dot{\psi}(t,i) = -r(t,i)\psi(t,i) - \sum_{j=1}^{l} q_{ij}\psi(t,j), \\ \psi(T,i) = 1, i = 1,2,\ldots,l. \end{cases} \tag{2.16}$$

Then the mean-variance problem (2.13) *is feasible for every $\zeta \in \mathbb{R}^1$ if and only if*

$$\varrho := \mathbb{E}\int_0^T |\psi(t,\alpha(t))B(t,\alpha(t))|^2 dt > 0. \tag{2.17}$$

Proof. To prove the "if" part, construct a family of admissible portfolios $\pi^\beta(\cdot) = \beta\pi(\cdot)$ for $\beta \in \mathbb{R}^1$ where

$$\pi(t) = B(t,\alpha(t))'\psi(t,\alpha(t)). \tag{2.18}$$

Assume $x^\beta(t)$ is the solution of (2.9). Let $x^\beta(t) = x^0(t) + \beta y(t)$, where $x^0(\cdot)$ satisfies (2.14) and $y(\cdot)$ is the solution to the following equation

$$\begin{cases} dy(t) = [r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t)]dt + \pi(t)'\sigma(t,\alpha(t))dW(t) \\ \qquad\qquad + \int_{\mathbb{R}^n} \pi(t)'\rho(t,\alpha(t),z)\tilde{N}(dt,dz), \\ y(0) = 0, \alpha(0) = i_0. \end{cases} \tag{2.19}$$

Therefore, problem (2.13) is feasible for every $\zeta \in \mathbb{R}^1$ if there exists $\beta \in \mathbb{R}$ such that $\zeta = \mathbb{E}x^{\beta}(T) \equiv \mathbb{E}x^0(T) + \beta\mathbb{E}y(T)$. Equivalently, (2.13) is feasible for every $\zeta \in \mathbb{R}$ if $\mathbb{E}y(T) \neq 0$. Applying the generalized Itô formula (Lemma (2.1)) to $\varphi(t,x,i) = \psi(t,i)x$, we have

$$
\begin{aligned}
& d[\varphi(t,y(t),\alpha(t))] \\
&= \dot{\psi}(t,\alpha(t))y(t)dt + \psi(t,\alpha(t))[r(t,\alpha(t))y(t) + B(t,\alpha(t))\pi(t)]dt \\
&\quad + \sum_{j=1}^{l} q_{\alpha(t)j}\psi(t,j)y(t)dt + \pi(t)'\sigma(t,\alpha(t))dW(t) \\
&\quad + \sum_{k=1}^{n}\int_{\mathbb{R}} \{\psi(t,\alpha(t))(y(t) + \pi(t)'\rho^{(k)}(t,\alpha(t),z)) - \psi(t,\alpha(t))y(t) \\
&\quad - \psi(t,\alpha(t))\pi(t)'\rho^{(k)}(t,\alpha(t),z)\}\nu(dz)dt \\
&\quad + \sum_{k=1}^{n}\int_{\mathbb{R}} \{\psi(t,\alpha(t))(y(t) + \pi(t)'\rho^{(k)}(t,\alpha(t),z)) \\
&\quad - \psi(t,\alpha(t))\pi(t)'\rho^{(k)}(t,\alpha(t),z)\}\tilde{N}_k(dt,dz_k) \\
&\quad + \int_{\mathbb{R}} \{\psi(t,\alpha(0) + h(\alpha(t),\bar{l}))y(t) - \psi(t,\alpha(t))y(t)\}\mu(dt,d\bar{l}) \\
&= -r(t,\alpha(t))\psi(t,\alpha(t))y(t)dt - \sum_{j=1}^{l} q_{\alpha(t)j}\psi(t,j)y(t)dt \\
&\quad + r(t,\alpha(t))\psi(t,\alpha(t))y(t)dt + B(t,\alpha(t))\pi(t)\psi(t,\alpha(t))dt \\
&\quad + \sum_{j=1}^{l} q_{\alpha(t)j}\psi(t,j)y(t)dt + \pi((t)'\sigma(t,\alpha(t))dW(t) \\
&\quad + \sum_{k=1}^{n}\int_{\mathbb{R}} \{\psi(t,\alpha(t))y(t)\}\tilde{N}_k(dt,dz_k) \\
&\quad + \int_{\mathbb{R}} \{\psi(t,\alpha(0) + h(\alpha(t),\bar{l}))y(t) - \psi(t,\alpha(t))y(t)\}\mu(dt,d\bar{l}) \\
&= B(t,\alpha(t))\pi(t)\psi(t,\alpha(t))dt + \pi(t)'\sigma(t,\alpha(t))dW(t) \\
&\quad + \sum_{k=1}^{n}\int_{\mathbb{R}} \{\psi(t,\alpha(t))y(t)\}\tilde{N}_k(dt,dz_k)
\end{aligned}
$$

$$+\int_{\mathbb{R}}\{\psi(t,\alpha(0)+h(\alpha(t),\bar{l}))y(t)-\psi(t,\alpha(t))y(t)\}\mu(dt,d\bar{l}).$$

Integrating from 0 to T, taking expectation, and using (2.18), we obtain

$$\begin{aligned}\mathbb{E}y(T) &= \mathbb{E}\int_0^T \psi(t,\alpha(t))B(t,\alpha(t))\pi(t)dt \\ &= \mathbb{E}\int_0^T |\psi(t,\alpha(t))B(t,\alpha(t))|^2 dt.\end{aligned} \tag{2.20}$$

Consequently, $\mathbb{E}y(T)\neq 0$ if (2.17) holds.

Conversely, suppose that problem (2.13) is feasible for every $\zeta\in\mathbb{R}^1$. Then for each $\zeta\in\mathbb{R}$, there is an admissible portfolio $\pi(\cdot)$ so that $\mathbb{E}x(T)=\zeta$. However, we can always decompose $x(t)=x^0(t)+y(t)$ where $y(\cdot)$ satisfies (2.19). This leads to $\mathbb{E}x^0(T)+\mathbb{E}y(T)=\zeta$. However, $\mathbb{E}x^0(T)\equiv\zeta^0$ is independent of $\pi(\cdot)$; thus it is necessary that there is a $\pi(\cdot)$ with $\mathbb{E}y(T)\neq 0$. It follows then from (2.20) that (2.17) is valid.

Theorem 2.1 *The mean-variance problem* (2.13) *is feasible for every* $\zeta\in\mathbb{R}$ *if and only if*

$$\mathbb{E}\int_0^T |B(t,\alpha(t))|^2 dt > 0. \tag{2.21}$$

Proof. By virtue of Lemma (2.2), it suffices to prove that $\psi(t,i)>0$, $\forall t\in[0,T]$, $i=1,2,\ldots,l$. To this end, note that equation (2.16) can be rewritten as

$$\begin{cases}\dot{\psi}(t,i)=[-r(t,i)-q_{ii}]\psi(t,i)-\sum_{j\neq i}^{l} q_{ij}\psi(t,j),\\ \psi(T,i)=1, i=1,2,\ldots,l.\end{cases} \tag{2.22}$$

Treating this as a system of terminal-valued ODEs, a variation-of-constant formula yields

$$\psi(t,i)=e^{\int_t^T[r(s,i)+q_{ii}]ds}+\int_t^T e^{\int_t^s[r(\tau,i)+q_{ii}]d\tau}\sum_{j\neq i}^{l} q_{ij}\psi(s,j)ds, \tag{2.23}$$

$$i = 1, 2, \ldots, l.$$

Construct a sequence $\psi^{(k)}(\cdot, i)$ (known as the Picard sequence) as follows

$$\psi^{(0)}(t,i) = 1, t \in [0.T], i = 1, 2, \ldots, l,$$

$$\psi^{(k+1)}(t,i) = e^{\int_t^T [r(s,i)+q_{ii}]ds} + \int_t^T e^{\int_t^s [r(\tau,i)+q_{ii}]d\tau} \sum_{j \neq i}^{l} q_{ij}\psi^{(k)}(s,j)ds,$$

$$t \in [0,T], i = 1, 2, \ldots, l, k = 0, 1, \ldots.$$

Noting $q_{ij} \geq 0$ for all $j \neq i$, we have

$$\psi^{(k)}(t,i) \geq e^{\int_t^T [r(s,i)+q_{ii}]ds} > 0, k = 0, 1, \ldots$$

On the other hand, it is well known that $\psi(t,i)$ is the limit of the Picard sequence $\psi^{(k)}(t,i)$ as $k \to \infty$ [see [19]]. Thus $\psi(t,i) > 0$. This proves the desired result.

Corollary 2.1 *If* (2.21) *holds, then for any* $\zeta \in \mathbb{R}$, *an admissible portfolio that satisfies* $\mathbb{E}x(T) = \zeta$ *is given by*

$$\pi(t) = \frac{\zeta - \zeta^0}{\varrho} B(t, \alpha(t))'\psi(t, \alpha(t)), \tag{2.24}$$

where x^0 *and* ϱ *are given by* (2.15) *and* (2.17), *respectively.*

Proof. This is immediate from the proof of the "if" part of Lemma (2.2).

$$\begin{aligned} \mathbb{E}x(T) &= \zeta \\ &= x^0(T) + \mathbb{E}y(T). \end{aligned}$$

And

$$\begin{aligned} \zeta - \zeta^0 &= \mathbb{E}y(T) \\ &= \mathbb{E}\int_0^T \psi(t, \alpha(t))B(t, \alpha(t))\pi(t)dt \end{aligned}$$

Then one has

$$\pi(t) = \frac{\zeta - \zeta^0}{\varrho} B(t, \alpha(t))' \psi(t, \alpha(t)). \tag{2.25}$$

Corollary 2.2 *If* $\mathbb{E}\int_0^T |B(t,\alpha(t))|^2 dt = 0$, *then any admissible portfolio* $\pi(\cdot)$ *results in* $\mathbb{E}x(T) = \zeta^0$.

Proof. This is seen from the proof of the "only if" part of Lemma (2.2).

$$\begin{aligned}\mathbb{E}x(T) &= \mathbb{E}x^0(T) + \mathbb{E}y(T)\\ &= \zeta^0 + \psi(t,\alpha(t))B(t,\alpha(t))\pi(t)dt\\ &= \zeta^0.\end{aligned}$$

Since, $\mathbb{E}\int_0^T |B(t,\alpha(t))|^2 dt = 0$.

Having addressed the issue of feasibility, we proceed with the study of optimality. The mean-variance problem (2.13) under consideration is a dynamic optimization problem with a constraint $\mathbb{E}x(T) = \zeta$. To handle this constraint, we apply the Lagrange multiplier technique. Define

$$\begin{aligned}J(x_0, i_0, \pi(\cdot), \lambda) :&= \mathbb{E}\{|x(T) - \zeta|^2 + 2\lambda[x(T) - \zeta]\}\\ &= \mathbb{E}[x(T) + \lambda - \zeta]^2 - \lambda^2, \qquad \lambda \in \mathbb{R}.\end{aligned} \tag{2.26}$$

Our first goal is to solve the following unconstrained problem parameterized by the Lagrange multiplier λ:

$$\begin{cases}\text{minimize} & J(x_0, i_0, \pi(\cdot), \lambda) = \mathbb{E}[x(T) + \lambda - \zeta]^2 - \lambda^2,\\ \text{subject to} & (x(\cdot), \pi(\cdot)) \text{ admissible }.\end{cases} \tag{2.27}$$

This turns out to be a Markov-modulated stochastic linear-quadratic optimal control problem, which will be solved in the next section.

2.3 Solution to the Unconstrained Problem

In this section we solve the unconstrained problem (2.27). Firstly define

$$\gamma(t,i) := B(t,i)\left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1} B(t,i)', \tag{2.28}$$

where $i = 1, 2, \ldots, l$.

Consider the following two systems of ODEs:

$$\begin{cases} \dot{P}(t,i) = [\gamma(t,i) - 2r(t,i)]P(t,i) - \sum_{j=1}^{l} q_{ij}P(t,j), \quad 0 \le t \le T, \\ P(T,i) = 1, i = 1, 2, \ldots, l, \end{cases} \tag{2.29}$$

and

$$\begin{cases} \dot{H}(t,i) = r(t,i)H(t,i) - \dfrac{1}{P(t,i)} \sum_{j=1}^{l} q_{ij}P(t,j)[H(t,j) - H(t,i)], \quad 0 \le t \le T, \\ H(T,i) = 1, i = 1, 2, \ldots, l, \end{cases} \tag{2.30}$$

The existence and uniqueness of solutions to the above two systems of equations are evident as both are linear with uniformly bounded coefficients.

Proposition 2.1 *The solutions of* (2.29) *and* (2.30) *must satisfy* $P(t,i) > 0$ *and* $0 < H(t,i) \le 1$, $\forall t \in [0,T]$, $i = 1, 2, \ldots, l$. *Moreover, if for a fixed* i, $r(t,i) > 0$, *a.e.*, $t \in [0,T]$, *then* $H(t,i) < 1$, $\forall t \in [0,T)$.

See the prove in [40].

Theorem 2.2 *Problem* (2.27) *has an optimal feedback control*

$$\pi^*(t,x,i) = -\left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1} B(t,i)' \tag{2.31}$$

$$\times [x + (\lambda - \zeta)H(t,i)]. \tag{2.32}$$

Moreover, the corresponding optimal value is

$$
\begin{aligned}
&\inf_{\substack{\pi(\cdot)\ admissible \\ \pi(\cdot)\ admissible}} J(x_0, i_0, \pi(\cdot), \lambda) \qquad (2.33)\\
&= [P(0,i_0)H(0,i_0)^2 + \theta - 1](\lambda - \zeta)^2 \\
&\quad + 2[P(0,i_0)H(0,i_0)x_0 - \zeta](\lambda - \zeta) + P(0,i_0)x_0^2 - \zeta^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta :&= \mathbb{E}\int_0^T \sum_{j=1}^{l} q_{\alpha(t)j} P(t,j)[H(t,j) - H(t,\alpha(t))]^2 dt \qquad (2.34)\\
&= \sum_{i=1}^{l}\sum_{j=1}^{l}\int_0^T P(t,j) p_{i_0 i}(t) q_{ij}[H(t,j) - H(t,i)]^2 dt \geq 0,
\end{aligned}
$$

with the transition probabilities $p_{i_0 i}(t)$ given by (2.2).

Proof. Let $\pi(\cdot)$ be any admissible control and $x(\cdot)$ be the corresponding state trajectory of (2.9). Applying the generalized Itô formula (Lemma (2.1)) to

$$\varphi(t,x,i) = P(t,i)[x + (\lambda - \zeta)H(t,i)]^2, \qquad (2.35)$$

we obtain

$$
\begin{aligned}
&d\{P(t,\alpha(t))[x(t) + (\lambda - \zeta)H(t,\alpha(t))]^2\} \\
&= \dot{P}(t,\alpha(t))[x(t) + (\lambda - \zeta)H(t,\alpha(t))]^2 dt \\
&\quad + 2P(t,\alpha(t))(\lambda - \zeta)[x(t) + (\lambda - \zeta)H(t,\alpha(t))]\dot{H}(t,\alpha(t))dt \\
&\quad + 2\{r(t,\alpha(t))x(t) + B(t,\alpha(t))\pi(t)\} \\
&\quad \times P(t,\alpha(t))[x(t) + (\lambda - \zeta)H(t,\alpha(t))]dt \\
&\quad + \sum_{j=1}^{l} q_{\alpha(t)j} P(t,j)[x(t) + (\lambda - \zeta)H(t,j)]^2 dt \\
&\quad + \frac{1}{2} 2P(t,\alpha(t))\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))']\pi(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&+ P(t,\alpha(t))\pi(t)'\{\int_{\mathbb{R}^n} \rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)\}\pi(t)dt \\
&+ 2P(t,\alpha(t))x(t)^2\pi(t)'\sigma(t,\alpha(t))dW(t) \\
&+ \sum_{k=1}^{n} \int_{\mathbb{R}} P(t,\alpha(t))\{2[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\rho^{(k)}(t,\alpha(t),z) \\
&+ \rho^{(k)}(t,\alpha(t),z)^2\}d\tilde{N}(dt,dz) \\
&+ \int_{\mathbb{R}} \{P(t,\alpha(0)+h(\alpha(t),\bar{l}))[x(t)+(\lambda-\zeta)H(t,\alpha(0)+h(\alpha(t),\bar{(l)}))]^2 \\
&- P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2\}\mu(dt,d\bar{l}) \\
=\ & P(t,\alpha(t))\{\pi(t)'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))' \\
&+ \int_{\mathbb{R}^n} \rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)]\pi(t) \\
&+ 2\pi(t)'B(t,\alpha(t))'[x(t)+(\lambda-\zeta)H(t,\alpha(t))] \\
&+ \gamma(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\}dt \\
&+ (\lambda-\zeta)^2 \sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[H(t,j)-H(t,i)]^2dt \\
&+ 2P(t,\alpha(t))x(t)^2\pi(t)'\sigma(t,\alpha(t))dW(t) \\
&+ \sum_{k=1}^{n} \int_{\mathbb{R}} P(t,\alpha(t))\{2[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\rho^{(k)}(t,\alpha(t),z) \\
&+ \rho^{(k)}(t,\alpha(t),z)^2\}d\tilde{N}(dt,dz) \\
&+ \int_{\mathbb{R}} \{P(t,\alpha(0)+h(\alpha(t),\bar{l}))[x(t)+(\lambda-\zeta)H(t,\alpha(0)+h(\alpha(t),\bar{(l)}))]^2 \\
&- P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2\}\mu(dt,d\bar{l}) \\
=\ & P(t,\alpha(t))[\pi(t)-\pi^*(t,x(t),\alpha(t))]'[\sigma(t,\alpha(t))\sigma(t,\alpha(t))' \\
&+ \int_{\mathbb{R}^n} \rho(t,\alpha(t),)\rho(t,\alpha(t),z)'\nu(dz)] \\
&\times [\pi(t)-\pi^*(t,x(t),\alpha(t))]dt \\
&+ (\lambda-\zeta)^2 \sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[H(t,j)-H(t,i)]^2dt
\end{aligned}
$$

$$
\begin{aligned}
&+ 2P(t,\alpha(t))x(t)^2\pi(t)'\sigma(t,\alpha(t))dW(t) \\
&+ \sum_{k=1}^{n}\int_{\mathbb{R}} P(t,\alpha(t))\{2[x(t)+(\lambda-\zeta)H(t,\alpha(t))]\rho^{(k)}(t,\alpha(t),z) \\
&+ \rho^{(k)}(t,\alpha(t),z)^2\}d\tilde{N}(dt,dz) \\
&+ \int_{\mathbb{R}}\{P(t,\alpha(0)+h(\alpha(t),\bar{l}))[x(t)+(\lambda-\zeta)H(t,\alpha(0)+h(\alpha(t),\bar{(l)}))]^2 \\
&- P(t,\alpha(t))[x(t)+(\lambda-\zeta)H(t,\alpha(t))]^2\}\mu(dt,d\bar{l}),
\end{aligned}
$$

where $\pi^*(t,x,i)$ is defined as the right-hand side of (2.31). Integrating the above from 0 to T and taking expectations, we obtain

$$
\begin{aligned}
&\mathbb{E}[x(T)+\lambda-\zeta]^2 \\
&= P(0,i_0)[x_0+(\lambda-\zeta)H(0,i_0)]^2+\theta(\lambda-\zeta)^2 \\
&\quad + \mathbb{E}\int_0^T P(t,\alpha(t))[\pi(t)-\pi^*(t,x(t),\alpha(t))]' \\
&\quad \times [\sigma(t,\alpha(t))\sigma(t,\alpha(t))' + \int_{\mathbb{R}^n}\rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)] \\
&\quad \times [\pi(t)-\pi^*(t,x(t),\alpha(t))]dt.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&J(x_0,i_0,\pi(\cdot),\lambda) \qquad (2.36) \\
&= \mathbb{E}[x(T)+\lambda-\zeta]^2-\lambda^2 \\
&= [P(0,i_0)H(0,i_0)+\theta-1](\lambda-\zeta)^2 \\
&\quad + 2[P(0,i_0)H(0,i_0)x_0-\zeta](\lambda-\zeta)+P(0,i_0)x_0^2-\zeta^2 \\
&\quad + \mathbb{E}\int_0^T P(t,\alpha(t))[\pi(t)-\pi^*(t,x(t),\alpha(t))]' \\
&\quad \times [\sigma(t,\alpha(t))\sigma(t,\alpha(t))' + \int_{\mathbb{R}^n}\rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)] \qquad (2.37) \\
&\quad \times [\pi(t)-\pi^*(t,x(t),\alpha(t))]dt.
\end{aligned}
$$

Since $P(t, \alpha(t)) > 0$ by Proposition (2.1), it follows immediately that the optimal feedback control is given by (2.31) and the optimal value is given by (2.33), provided that the corresponding equation (2.9) under the feedback control (2.31) has a solution. But under (2.31), the system (2.9) is a nonhomogeneous linear SDE with coefficients nodulated by $\alpha(t)$. Since all the coefficients of this linear equation are uniformly bounded and $\alpha(t)$ is independent of $W(t)$, the existence and uniqueness of the solution to the equation are straightforward based on a standard successive approximation scheme.

Finally, since

$$\theta := \mathbb{E} \int_0^T \sum_{j \neq i}^{l} q_{\alpha(t)j} P(t,j)[H(t,j) - H(t,\alpha(t))]^2 dt$$

and $q_{ij} \geq 0 \ \forall i \neq j$, we must have $\theta \geq 0$. This completes the proof.

2.4 Efficient Frontier

In this section we proceed to derive the efficient frontier for the original mean-variance problem (2.13).

Theorem 2.3 ***(Efficient portfolios and efficient frontier)*** *Assume that* (2.21) *holds. Then we have*

$$P(0, i_0)H(0, i_0)^2 + \theta - 1 < 0. \tag{2.38}$$

Moreover, the efficient portfolio corresponding to z, as a function of the time t, the wealth level x, and the market mode i, is

$$\pi^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)]^{-1} B(t,i)' \tag{2.39}$$

$$\times [x + (\lambda^* - \zeta)H(t,i)] \tag{2.40}$$

where

$$\lambda^* = \frac{\zeta - P(0,i_0)H(0,i_0)x_0}{P(0,i_0)H(0,i_0)^2 + \theta - 1} + \zeta. \tag{2.41}$$

Furthermore, the optimal value of $Var\ x(T)$*, among all the wealth processes* $x(\cdot)$ *satisfying* $\mathbb{E}x(T) = \zeta$*, is*

$$\begin{aligned} & Var\ x^*(T) \\ &= \frac{P(0,i_0)H(0,i_0)^2 + \theta}{1 - \theta - P(0,i_0)H(0,i_0)^2}[\zeta - \frac{P(0,i_0)H(0,i_0)}{P(0,i_0)H(0,i_0)^2 + \theta}x_0]^2 \\ &\quad + \frac{P(0,i_0)\theta}{P(0,i_0)H(0,i_0)^2 + \theta}x_0^2. \end{aligned} \tag{2.42}$$

Proof. By assumption (2.21) and Theorem 2.1, the mean-variance problem (2.13) is feasible for any $\zeta \in \mathbb{R}^1$. Moreover, using exactly the same approach in the proof of the Theorem 2.2, on can show that problem (2.13) without the constraint $\mathbb{E}x(T) = \zeta$ must have a finite optimal value, hence so does the problem (2.13). Therefore, (2.13) is finite for any $\zeta \in \mathbb{R}^1$. Now we need to prove $J_{MV}(x_0, i_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$. We can easily get

$$\begin{aligned} \mathbb{E}(2x_1x_2) &\leq \mathbb{E}(x_1^2 + x_2^2) \\ \mathbb{E}(2\kappa(1-\kappa)x_1x_2) &\leq \mathbb{E}(\kappa(1-\kappa)x_1^2 + \kappa(1-\kappa)x_2^2) \\ \mathbb{E}(\kappa^2x_1^2 + (1-\kappa)^2x_2^2 + 2\kappa(1-\kappa)x_1x_2) &\leq \mathbb{E}(\kappa x_1^2 + (1-\kappa)x_2^2) \\ \mathbb{E}(\kappa x_1 + (1-\kappa)x_2 - \zeta)^2 &\leq \mathbb{E}(\kappa(x_1 - \zeta)^2) + \mathbb{E}((1-\kappa)(x_2 - \zeta)^2), \end{aligned}$$

where $\kappa \in [0,1]$. So, we obtain

$$\mathbb{E}(\kappa x_1 - \kappa\zeta + (1-\kappa)x_2 - (1-\kappa)\zeta)^2 \leq \mathbb{E}(\kappa(x_1 - \zeta)^2) + \mathbb{E}((1-\kappa)(x_2 - \zeta)^2),$$

which prove $J_{MV}(x_0, i_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$.

Affine space means the complement of points at infinity. It can also be viewed as a vector space whose operations are limited to those linear combinations whose coefficients sum to one. Since $J_{MV}(x_0, i_0, \pi(\cdot))$ is strictly convex in $\pi(\cdot)$ and the constraint function $\mathbb{E}x(T) = \zeta$ is affine in $\pi(\cdot)$, we can apply the well-known duality theorem (see [25] p.224, Theorem 1) to conclude that for any $\zeta \in \mathbb{R}^1$, the optimal value of (2.13) is

$$\sup_{\lambda \in \mathbb{R}^1} \inf_{\pi(\cdot) \text{ admissible}} J(x_0, i_0, \pi(\cdot), \lambda) \tag{2.43}$$

$$= \max_{\zeta \in \mathbb{R}^1} \inf_{\pi(\cdot) \text{ admissible}} (J(x_0, i_0, \pi(\cdot), \lambda) + < \zeta, \zeta^* >) \tag{2.44}$$

$$> -\infty. \tag{2.45}$$

By Theorem (2.2), $\inf_{\pi(\cdot) \text{ admissible}} J(x_0, i_0, \pi(\cdot), \lambda)$ is a quadratic function (2.33) in $\lambda - \zeta$. It follows from the finiteness of the supremum value of this quadratic function that

$$P(0, i_0)H(0, i_0)^2 + \theta - 1 \leq 0.$$

Now if

$$P(0, i_0)H(0, i_0)^2 + \theta - 1 = 0,$$

then again by Theorem (2.2) and (2.43) we must have

$$P(0, i_0)H(0, i_0)x_0 - \zeta = 0,$$

for all $\zeta \in \mathbb{R}^1$, which is a contradiction. This proves (2.38). On the other hand, in view of (2.43), we maximize the quadratic function (2.33) over $\lambda - \zeta$ and conclude that the maximizer is given by (2.41), whereas the maximum value is given by the right-hand side of (2.42). Finally, the optimal control (2.39) is obtained by (2.31) with $\lambda = \lambda^*$.

The efficient frontier (2.42) reveals explicitly the tradeoff between the mean (return) and variance (risk) at the terminal. Quite contrary to the case without Markovian jumps [39], the efficient frontier in the present case is no longer a perfect square (or, equivalently, the efficient frontier in the mean-standard deviation diagram is no more a straight line). As a consequence, one is not able to achieve a risk-free investment. This, certainly, is expected since now the interest rate process is modulated by the Markov chain, and the interest rate risk cannot be perfectly hedged by any portfolio consisting of the bank account and stocks [24], because the Markov chain is independent of the Brownian motion.

Nevertheless, the expression (2.42) does disclose the minimum variance, namely, the minimum possible terminal variance achievable by an admissible portfolio, along with the portfolio that attains this minimum variance.

Theorem 2.4 ***(Minimum Variance)*** *The minimum terminal variance is*

$$Var\ x^*_{\min}(T) = \frac{P(0,i_0)\theta}{P(0,i_0)H(0,i_0)^2+\theta}x_0^2 \geq 0 \tag{2.46}$$

with the corresponding expected terminal wealth

$$\zeta_{\min} := \frac{P(0,i_0)H(0,i_0)}{P(0,i_0)H(0,i_0)^2+\theta}x_0 \tag{2.47}$$

and the corresponding Lagrange multiplier $\lambda^*_{\min} = 0$. Moreover, the portfolio that achieves the above minimum variance, as a function of the time t, the wealth level x and the market mode i, is

$$\pi^*_{\min}(t,x,i) = -[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)]^{-1}B(t,i)' \tag{2.48}$$

$$\times\ [x - \zeta_{\min}H(t,i)]. \tag{2.49}$$

Proof. The conclusions regarding (2.46) and (2.47) are evident in view of the efficient frontier (2.42). The assertion $\lambda^*_{\min} = 0$ can be verified via (2.41) and (2.47).Finally, (2.48) follows from (2.39).

Theorem 2.5 ***(Mutual Fund Theorem)*** *Suppose an efficient portfolio $\pi_1^*(\cdot)$ is given by (2.39) corresponding to $\zeta = \zeta_1 > \zeta_{\min}$. Then a portfolio $\pi^*(\cdot)$ is efficient if and only if there is a $\mu \geq 0$ such that*

$$\pi^*(t) = (1-\mu)\pi_{\min}^*(t) + \mu\pi_1^*(t), t \in [0,T], \tag{2.50}$$

where $\pi_{\min}^*(\cdot)$ is the minimum variance portfolio defined in Theorem 2.4.

Proof. We first prove the "if" part. Since both $\pi_{\min}^*(\cdot)$ and $\pi_1^*(\cdot)$ are efficient, by the explicit expression of any efficient portfolio given by (2.4), $\pi^*(t) = (1-\mu)\pi_0^*(\cdot) + \mu\pi_1^*(t)$ must be in the form of (2.4) corresponding to $\zeta = (1-\mu)\zeta_{\min} + \mu\zeta_1$ (also noting that $x^*(\cdot)$ is linear in $\pi^*(\cdot)$). Hence $\pi^*(t)$ must be efficient.

Conversely, suppose $\pi^*(\cdot)$ is efficient corresponding to a certain $\zeta \geq \zeta_{\min}$. Write $\zeta = (1-\mu)\zeta_{\min} + \mu\zeta_1$ with some $\mu \geq 0$. Multiplying

$$\begin{aligned}
&\pi_{\min}^*(t) \\
&= -[\sigma(t,\alpha(t))\sigma(t,\alpha(t))' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)]^{-1}B(t,\alpha(t))' \\
&\quad \times [x_{\min}^*(t) - \zeta_{\min}H(t,\alpha(t))]
\end{aligned}$$

by $(1-\mu)$, multiplying

$$\begin{aligned}
&\pi_1^*(t) \\
&= -[\sigma(t,\alpha(t))\sigma(t,\alpha(t))' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)]^{-1}B(t,\alpha(t))' \\
&\quad \times [x_1^*(t) + (\lambda_1^* - \zeta_1)H(t,\alpha(t))]
\end{aligned}$$

by μ, and summing them up, we obtain that $(1-\mu)\pi_{\min}^*(t) + \mu\pi_1^*(t)$ is represented by (2.39) with $x^*(t) = (1-\mu)x_{\min}^*(t) + \mu x_1^*(t)$ and $\zeta = (1-\mu)\zeta_{\min} + \mu\zeta_1$. This leads to (2.50).

Since the wealth processes $x(\cdot)$ is with jumps, it is more complicated when we solve the unconstrained problem (2.27). Firstly, we aim to derive conditions of feasibility. It is not hard to prove feasibility of the constrained stochastic optimization problem (2.13), which we get the unconstrained problem (2.27) from. Then we solve the unconstrained problem (2.27). If we assume

$$\begin{aligned}\gamma(t,i) &:= B(t,i)\left[\sigma(t,i)\sigma(t,i)'\right]^{-1}B(t,i)', i=1,2,\ldots,l,\\ \pi^*(t,x,i) &:= -\left[\sigma(t,i)\sigma(t,i)'\right]^{-1}B(t,i)'[x+(\lambda-\zeta)H(t,i)],\end{aligned}$$

we have

$$\begin{aligned}&\inf_{\pi(\cdot)\text{ admissible}} J(x_0,i_0,\pi(\cdot),\lambda)\\ &=[P(0,i_0)H(0,i_0)^2+\theta-1](\lambda-\zeta)^2\\ &\quad+2[P(0,i_0)H(0,i_0)x_0-\zeta](\lambda-\zeta)+P(0,i_0)x_0^2-\zeta^2,\end{aligned}$$

where

$$\begin{aligned}\theta := \mathbb{E}\{\int_0^T \sum_{j=1}^{l} q_{\alpha(t)j}P(t,j)[H(t,j)-H(t,\alpha(t))]^2dt\\ \frac{1}{(\lambda-\zeta)^2}+P(t,\alpha(t))\pi(t)'\{\int_{\mathbb{R}^n}\rho(t,\alpha(t),z)\rho(t,\alpha(t),z)'\nu(dz)\}\pi(t)dt\},\end{aligned}$$

as [40]. So, we added one item $\int_{\mathbb{R}^n}\rho(t,i,z)\rho(t,i,z)'\nu(dz)$ in optimal feedback control $\pi^*(t,x,i)$ (see (2.27)) to simply the calculation.

2.5 SDEs under Regime Switching with pure jumps

Throughout this section, we will discuss SDEs under regime switching with pure jumps

$$\begin{cases} dx(t) = r(t,\alpha(t))x(t)dt + B(t,\alpha(t))\pi(t)dt + \pi'(t)\sigma(t,\alpha(t))dW(t) \\ \qquad\qquad + \int_{\mathbb{R}^n} \pi'(t)\rho(t,\alpha(t),z)N(dt,dz), \\ x(0) = x_0 > 0, \alpha(0) = i_0. \end{cases}$$

Noting that

$$N(dt,dz) = \tilde{N}(dt,dz) + \nu(dz)dt$$

and using the generalized Itô Lemma 2.1, we have the following Itô formula.

Lemma 2.3 *Given an d-dimensional process* $y(\cdot)$ *satisfying*

$$\begin{aligned} dy(t) &= f(t,y(t),\alpha(t))dt + g(t,y(t),\alpha(t))dW(t) \\ &\quad + \int_{\mathbb{R}^n} \gamma(t,y(t),\alpha(t),z)N(dt,dz), \end{aligned}$$

where f, g *and* γ *satisfy Lipschitz condition with appropriate dimensions, moreover, each column* $\gamma^{(k)}$ *of the matrix* $\gamma = [\gamma_{ij}]$ *depends on* z *only through the* k^{th} *coordinate* z_k. *Let* $\varphi(t,x,i) \in C^{1,2}([0,T] \times \mathbb{R}^n \times \mathbb{S}; \mathbb{R})$, *we then have*

$$\begin{aligned} & d\varphi(t,y(t),\alpha(t)) \\ &= \Gamma\varphi(t,y(t),\alpha(t))dt + \varphi_x(t,y(t),\alpha(t))'g(t,y(t),\alpha(t))dW(t) \\ &\quad + \sum_{k=1}^{n} \int_{\mathbb{R}} \{\varphi(t,y(t)+\gamma^{(k)}(t,y(t),\alpha(t),z),\alpha(t)) - \varphi(t,y(t),\alpha(t))\}\tilde{N}_k(dt,dz_k) \\ &\quad + \int_{\mathbb{R}} (\varphi(t,y(t),\alpha(0)+h(\alpha(t),\bar{l})) - \varphi(t,y(t),\alpha(t)))\mu(dt,d\bar{l}), \end{aligned}$$

where

$$\Gamma\varphi(t,x,i) := \varphi_t(t,x,i) + \varphi_x(t,x,i)'f(t,x,i) + \sum_{k=1}^{n}\int_{\mathbb{R}}\gamma(t,y(t),\alpha(t),z)\nu(dz)$$
$$+ \frac{1}{2}\text{trace}[g(t,x,i)'\varphi_{xx}(t,x,i)g(t,x,i)] + \sum_{j=1}^{l} q_{ij}\varphi(t,x,j)$$
$$+ \sum_{k=1}^{n}\int_{\mathbb{R}}\{\varphi(t,y(t)+\gamma^{(k)}(t,y(t),\alpha(t),z_k),\alpha(t)) - \varphi(t,y(t),\alpha(t))$$
$$- \varphi_x(t,y(t),\alpha(t))'\gamma^{(k)}(t,y(t),\alpha(t),z_k)\}\nu_k(dz_k),$$

where μ is a martingale measure,

$$h(i,y) = \begin{cases} j-i, \ \textit{if } y \in \Delta_{ij} \\ \quad 0, \ \textit{otherwise} \end{cases}$$

and $\mu(dt,d\bar{l}) = \bar{\gamma}(dt,d\bar{l}) - \mu(d\bar{l})dt$ is a martingale measure. And $\bar{\gamma}(dt,dy)$ is a Poisson random measure with intensity $dt \times \mu(dy)$, in which μ is the Lebesgue measure on $\mathbb{R}$.

In this section, we shall formulate our results without proof, however, we will specify the similar results to that of the corresponding results.

Firstly, we need to check the feasibility of (2.13) as before. Using the same method in the proof of Theorem 2.2, we obtain

Theorem 2.6 *The mean-variance problem* (2.13) *is feasible for every $\zeta \in \mathbb{R}$ if and only if*

$$\mathbb{E}\int_0^T [|B(t,\alpha(t))|^2 + \int_{\mathbb{R}^n} B(t,\alpha(t))\rho(t,\alpha(t),z)\nu(dz)]dt > 0.$$

Secondly, we will solve the unconstrained problem (2.27). Define

$$\widehat{\gamma}(t,i) : = [B(t,i)' + \int_{\mathbb{R}^n}\rho(t,i,z)\nu(dz)]'$$

$$\times \left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1}$$
$$\times [B(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\nu(dz)],$$

where $i = 1, 2, \ldots, l$.

Consider the following two systems of ODEs:

$$\begin{cases} \dot{\widehat{P}}(t,i) = [\widehat{\gamma}(t,i) - 2r(t,i)]\widehat{P}(t,i) - \sum_{j=1}^{l} q_{ij}\widehat{P}(t,j), \quad 0 \le t \le T, \\ \widehat{P}(T,i) = 1, i = 1, 2, \ldots, l, \end{cases}$$

and

$$\begin{cases} \dot{\widehat{H}}(t,i) = r(t,i)\widehat{H}(t,i) - \dfrac{1}{\widehat{P}(t,i)} \sum_{j=1}^{l} q_{ij}\widehat{P}(t,j)[\widehat{H}(t,j) - \widehat{H}(t,i)], \quad 0 \le t \le T, \\ \widehat{H}(T,i) = 1, i = 1, 2, \ldots, l, \end{cases}$$

The existence and uniqueness of solutions to the above two systems of equations are evident as both are linear with uniformly bounded coefficients. By the same argument of Theorem 2.3, we have

Theorem 2.7 *Problem* (2.27) *has an optimal feedback control*

$$\pi^*(t,x,i) = -\left[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\rho(t,i,z)'\nu(dz)\right]^{-1}$$
$$\times [B(t,i)' + \int_{\mathbb{R}^n} \rho(t,i,z)\nu(dz)][x + (\lambda - \zeta)\widehat{H}(t,i)].$$

Moreover, the corresponding optimal value is

$$\inf_{\pi(\cdot)\ admissible} J(x_0, i_0, \pi(\cdot), \lambda)$$
$$= [\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2 + \theta - 1](\lambda - \zeta)^2$$
$$+ 2[\widehat{P}(0,i_0)\widehat{H}(0,i_0)x_0 - \zeta](\lambda - \zeta) + \widehat{P}(0,i_0)x_0^2 - \zeta^2,$$

where

$$\theta := \mathbb{E}\int_0^T \sum_{j=1}^{l} q_{\alpha(t)j}\widehat{P}(t,j)[\widehat{H}(t,j)-\widehat{H}(t,\alpha(t))]^2 dt$$
$$= \sum_{i=1}^{l}\sum_{j=1}^{l}\int_0^T \widehat{P}(t,j)p_{i_0 i}(t)q_{ij}[\widehat{H}(t,j)-\widehat{H}(t,i)]^2 dt \geq 0,$$

with the transition probabilities $p_{i_0 i}(t)$ *given by* (2.2).

Finally, we proceed to derive the efficient frontier for the original mean-variance problem. By the same argument of Theorem 2.3 and Theorem 2.4, we get

Theorem 2.8 ***(Efficient portfolios and efficient frontier)*** *Assume that* (2.21) *holds. Then we have*

$$\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2 + \theta - 1 < 0.$$

Moreover, the efficient portfolio corresponding to z*, as a function of the time* t*, the wealth level* x*, and the market mode* i*, is*

$$\pi^*(t,x,i) = -[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n}\rho(t,i,z)\rho(t,i,z)'\nu(dz)]^{-1}$$
$$\times [B(t,i)' + \int_{\mathbb{R}^n}\rho(t,i,z)\nu(dz)][x + (\lambda^* - \zeta)\widehat{H}(t,i)]$$

where

$$\lambda^* = \frac{\zeta - \widehat{P}(0,i_0)\widehat{H}(0,i_0)x_0}{\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2 + \theta - 1} + \zeta.$$

Furthermore, the optimal value of $Var\ x(T)$*, among all the wealth processes* $x(\cdot)$ *satisfying* $\mathbb{E}x(T) = \zeta$*, is*

$$Var\ x^*(T)$$
$$= \frac{\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2 + \theta}{1 - \theta - \widehat{P}(0,i_0)\widehat{H}(0,i_0)^2}[\zeta - \frac{\widehat{P}(0,i_0)\widehat{H}(0,i_0)}{\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2 + \theta}x_0]^2$$
$$+ \frac{\widehat{P}(0,i_0)\theta}{\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2 + \theta}x_0^2.$$

Theorem 2.9 ***(Minimum Variance)*** *The minimum terminal variance is*

$$Var\ x^*_{\min}(T) = \frac{\widehat{P}(0,i_0)\theta}{\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2+\theta}x_0^2 \geq 0$$

with the corresponding expected terminal wealth

$$\zeta_{\min} := \frac{\widehat{P}(0,i_0)\widehat{H}(0,i_0)}{\widehat{P}(0,i_0)\widehat{H}(0,i_0)^2+\theta}x_0$$

and the corresponding Lagrange multiplier $\lambda^*_{\min} = 0$. *Moreover, the portfolio that achieves the above minimum variance, as a function of the time* t, *the wealth level* x *and the market mode* i, *is*

$$\begin{aligned}\pi^*_{\min}(t,x,i) = &-[\sigma(t,i)\sigma(t,i)' + \int_{\mathbb{R}^n}\rho(t,i,z)\rho(t,i,z)'\nu(dz)]^{-1}\\ &\times [B(t,i)' + \int_{\mathbb{R}^n}\rho(t,i,z)\nu(dz)][x - \zeta_{\min}H(t,i)].\end{aligned}$$

Chapter 3

The minimal entropy martingale measures for SDEs with jumps

Before describing the model, the reader is referred to Protter [32], Gerber and Shiu [13]. We consider the problem of pricing contingent claims on a stock whose price is modeled by Lévy process and Markov Chain. In a market, there are many equivalent measures that make the discounted price process a martingale. In other words, such a market is incomplete. So, additional criteria must be used to select an appropriate martingale measure from among the uncountably many such measures with which to price a contingent claim. There are many different ways to solve this problem. Moreover, compared to the large body of work devoted to finding new approaches to option pricing in incomplete markets, relatively little seems to have been done to compare and to investigate the relationship between the various approaches. In this chapter, we will discuss the minimal entropy martingale measures and Esscher transform.

3.1 Description of the model

Throughout this thesis, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space. The stock price S_t is the solution of the stochastic differential equation

$$\begin{aligned} dS(t) = S(t_-)\{&b(t,\alpha(t))dt + \sigma(t,\alpha(t))dW(t) \\ &+ \int_{\mathbb{R}} \rho(t,\alpha(t),z)\tilde{N}(dt,dz)\}, \end{aligned} \tag{3.1}$$

where the coefficients $\sigma(t,i)$, $b(t,i)$ and $\rho(t,i,z)$ are deterministic continuous functions, and the continuous-times stationary Markov chain $\alpha(t)$ takes value in a finite state space $\mathbb{S} = \{1, 2, \ldots, l\}$.

Assume

$$\begin{aligned} dY(t,\alpha(t)) = &b(t,\alpha(t))dt + \sigma(t,\alpha(t))dW(t) \\ &+ \int_{\mathbb{R}} \rho(t,\alpha(t),z)\tilde{N}(dt,dz). \end{aligned} \tag{3.2}$$

Note that, for any measurable function $f(t,z,i)$,

$$\sum_{0<s\leq t} f(s,\Delta Y_s, i) = \int_0^t \int_{\mathbb{R}} f(s,z,i)N(dt,dz),$$

where $i \in \mathbb{S}$, $\mathbb{S} = \{1, 2, \ldots, l\}$. Then

$$dS(t) = S(t_-)dY(t,\alpha(t)).$$

The solution of this equation is obtained as follows. We take $S(t)$ to be the stochastic exponential (sometimes called the Doléans-Dade exponential after its discoverer), which is denoted as $Z = (Z(t), t \geq 0)$ and defined as

$$\begin{aligned} Z(t) = &\exp\{Y(t,\alpha(t)) - \frac{1}{2}\langle Y^c, Y^c\rangle(t,\alpha(t))\} \\ &\times \prod_{0<s\leq t} [1 + \Delta Y(s,\alpha(s))]e^{-\Delta Y(s,\alpha(s))}, \end{aligned} \tag{3.3}$$

where $Y^c(t,\alpha(t))$ is the continuous part of $Y(t,\alpha(t))$, for each $t \geq 0$.

We will need the following assumption:

$$\inf[\Delta Y(t,\alpha(t)), t > 0] > -1 \text{ (a.s.) } .$$

Proposition 3.1 *If Y is a Lévy-type stochastic integral of the form* (3.2) *and above assumption holds, then each $Z(t)$ is almost surely finite.*

Proof. We must show that the infinite product (3.3) converges almost surely. We write

$$\prod_{0<s\leq t}[1+\Delta Y(s,\alpha(s))]e^{-\Delta Y(s,\alpha(s))} = A(t) + B(t),$$

where

$$A(t) = \prod_{0<s\leq t}[1+\Delta Y(s,\alpha(s))]e^{-\Delta Y(s,\alpha(s))}\mathbf{1}_{\{|\Delta Y(s,\alpha(s))|\geq\frac{1}{2}\}},$$

and

$$B(t) = \prod_{0<s\leq t}[1+\Delta Y(s,\alpha(s))]e^{-\Delta Y(s,\alpha(s))}\mathbf{1}_{\{|\Delta Y(s,\alpha(s))|<\frac{1}{2}\}}.$$

Now, since Y is càdlàg, $\sharp\{0 \leq s \leq t; |\Delta Y(s,\alpha(s))| \geq \frac{1}{2}\} < \infty$ (a.s.) [see p84, [32]], and so $A(t)$ is almost surely a finite product. Using the above assumption, we have

$$B(t) = \exp\{\sum_{0<s\leq t}\{\log[1+\Delta Y(s,\alpha(s))] - \Delta Y(s,\alpha(s))\}\mathbf{1}_{\{|\Delta Y(s,\alpha(s))|<\frac{1}{2}\}}\}.$$

We now employ Taylor's Theorem to obtain the inequality

$$\log(1+y) - y \leq Ky^2,$$

where $K > 0$, which is valid whenever $|y| < \frac{1}{2}$. Hence

$$\log B(t)$$

$$
\begin{aligned}
&= \Big| \sum_{0<s\le t} \{\log[1+\Delta Y(s,\alpha(s))] - \Delta Y(s,\alpha(s))\} \mathbf{1}_{\{|\Delta Y(s,\alpha(s))|<\frac{1}{2}\}} \Big| \\
&\le \sum_{0<s\le t} |\Delta Y(s,\alpha(s))|^2 \mathbf{1}_{\{|\Delta Y(s,\alpha(s))|<\frac{1}{2}\}} \\
&< \infty
\end{aligned}
$$

a.s., and we obtain our required result.

Let $Z(t) = e^{\hat{Z}(t)}$. From (3.3), we have

$$
\begin{aligned}
d\hat{Z}(t) &= b(t,\alpha(t))dt + \sigma(t,\alpha(t))dW(t) + \int_{\mathbb{R}} \rho(t,\alpha(t),z)\tilde{N}(dt,dz) \\
&\quad - \frac{1}{2}\sigma^2(t,\alpha(t))dt + \int_{\mathbb{R}} \log(1+\rho(t,\alpha(t),z))N(dt,dz) \\
&\quad - \int_{\mathbb{R}} \rho(t,\alpha(t),z)N(dt,dz) \\
&= (b(t,\alpha(t)) - \frac{1}{2}\sigma^2(t,\alpha(t)))dt + \sigma(t,\alpha(t))dW(t) \\
&\quad + \int_{\mathbb{R}} \log(1+\rho(t,\alpha(t),z))\tilde{N}(dt,dz) \\
&\quad + \int_{\mathbb{R}} [\log(1+\rho(t,\alpha(t),z)) - \rho(t,\alpha(t),z)]\nu(dz)dt.
\end{aligned}
$$

Then by Itô's formula, we get

$$
\begin{aligned}
dZ(t) &= de^{\hat{Z}(t)} \\
&= Z(t_-)\{b(t,\alpha(t))dt + \sigma(t,\alpha(t))dW(t) \\
&\quad + \int_{\mathbb{R}} [\log(1+\rho(t,\alpha(t),z)) - \rho(t,\alpha(t),z)]\nu(dz)dt\} \\
&\quad + \int_{\mathbb{R}} \{\exp\{\hat{Z}(t_-) + \log(1+\rho(t,\alpha(t),z))\} - \exp\{\hat{Z}(t_-)\}\}\tilde{N}(dt,dz) \\
&\quad + \int_{\mathbb{R}} \{\exp\{\hat{Z}(t_-) + \log(1+\rho(t,\alpha(t),z))\} - \exp\{\hat{Z}(t_-)\} \\
&\quad - \log(1+\rho(t,\alpha(t),z)) \exp\{\hat{Z}(t_-)\}\}\nu(dz)dt \\
&= Z(t_-)\{b(t,\alpha(t))dt + \sigma(t,\alpha(t))dW(t) + \int_{\mathbb{R}} \rho(t,\alpha(t),z)\tilde{N}(dt,dz)\},
\end{aligned}
$$

as required.

Then, we can easily get the solution of (3.1)

$$
\begin{aligned}
S(t) = S(0) \exp\{ & \int_0^t \sigma(s, \alpha(s)) dW(s) + \int_0^t \int_{\mathbb{R}} \rho(s, \alpha(s), z) \tilde{N}(ds, dz) \\
& + \int_0^t (b(s, \alpha(s)) - \frac{\sigma(s, \alpha(s))^2}{2}) ds\} \\
& \times \prod_{0<s\le t} (1 + \int_{\mathbb{R}} \rho(s, \alpha(s), z)) \Delta \tilde{N}(ds, dz)) \\
& \times \exp\{- \int_{\mathbb{R}} \rho(s, \alpha(s), z)) \Delta \tilde{N}(ds, dz)\}.
\end{aligned}
$$

From this we see that $\sigma\{S(u) : u \le t\} = \mathcal{F}_t$ and so a contingent claim Γ_T expiring at time T may be regarded as a nonnegative $\mathcal{F}_T$-measurable random variable.

3.2 Equivalent martingale measure and pricing formulas

The riskless rate of interest is given by a deterministic continuous function $r(t, i)$, and we define the discounted stock price by

$$\hat{S}(t) = \exp\{- \int_0^t r(s, \alpha(s)) ds\} S(t). \tag{3.4}$$

We begin by characterizing all equivalent martingale measures $\mathbb{Q}$ under which the discounted price process $\hat{S}$ is a $\{\mathcal{F}_t\}$-martingale. We continue to use the notation established in the previous section. Expectations under the canonical measure $\mathbb{P}$ will be denoted by $\mathbb{E}_P[\cdot]$ while expectation with respect to any other measure $\mathbb{Q}$ will be denoted by $\mathbb{E}_Q[\cdot]$.

Let $\mathcal{P}$ denote the predictable σ-algebra on $\Omega \times \mathbb{R}^+$ associated with the filtration $\{\mathcal{F}_t\}$ and let $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. A function $H(t, \omega, z)$ which is $\tilde{\mathcal{P}}$-measurable will be called Borel predictable.

Thus, suppressing the explicit dependence on ω, a Borel predictable function or process $H(t,i,z)$ is one such that the process $t \to H(t,\omega,z)$ is predictable for fixed z and the function $z \to H(t,\omega,z)$ is Borel-measurable for fixed t.

Lemma 3.1 *Let $G(t,i)$ and $H(t,i,z)$ be predictable and Borel predictable processes respectively. Suppose that*

$$\mathbb{E}[\int_0^t G(s,i)^2 ds] < \infty,$$

and $H \geq 0$, $H(t,i,0) = 1$ for all $t \geq 0$. Let $h(t,i,z)$ be another Borel predictable process such that

$$\int_{\mathbb{R}} [H(t,i,z) - 1 - h(t,i,z)]\nu(dz) < \infty.$$

Define a process $Z(t)$ by

$$\begin{aligned} Z(t) = \exp\{&\int_0^t G(s,\alpha(s))dW(s) - \frac{1}{2}\int_0^t G(s,\alpha(s))^2 ds \\ &+ \int_0^t \int_{\mathbb{R}} h(s,\alpha(s),z)\tilde{N}(ds,dz) \\ &- \int_{[0,t)\times\mathbb{R}} [H(s,\alpha(s),z) - 1 - h(s,\alpha(s),z)]\nu(dz)ds\} \\ &\times \prod_{0<s\leq t} H(s,\alpha(s),\Delta Y(s,\alpha(s)))\exp(-h(s,\alpha(s),\Delta Y(s,\alpha(s)))). \end{aligned} \tag{3.5}$$

Then Z is a nonnegative local martingale with $Z(0) = 1$ and Z is positive if and only if $H > 0$.

Proof. It is clear that Z is nonnegative if and only if $H \geq 0$. That Z is a local martingale is a simple consequence of the Itô formula, the Itô formula gives

$$\begin{aligned} Z(t) = \exp\{&\int_0^t G(s,\alpha(s))dW(s) - \int_0^t \frac{1}{2}G(s,\alpha(s))^2 ds \\ &+ \int_0^t \int_{\mathbb{R}} \log H(s,\alpha(s),z)\tilde{N}(ds,dz) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}}[\log H(s,\alpha(s),z)+1-H(s,\alpha(s),z)]\nu(dz)ds\} \\
&=1+\int_0^t Z_{s-}G(s,\alpha(s))dW(s) \\
&-\frac{1}{2}\int_0^t Z_{s-}G(s,\alpha(s))^2ds+\frac{1}{2}\int_0^t Z_{s-}G(s,\alpha(s))^2ds \\
&-\int_{[0,t)\times\mathbb{R}} Z_{s-}[H(s,\alpha(s),z)-1]\nu(dz)ds \\
&+\int_0^t\int_{\mathbb{R}} Z_{s-}\log H(s,\alpha(s),z)\nu(dz)ds \\
&+\int_0^t\int_{\mathbb{R}} Z_{s-}[H(s,\alpha(s),z)-1]\tilde{N}(ds,dz) \\
&+\int_0^t\int_{\mathbb{R}}[Z_{s-}H(s,\alpha(s),z)-Z_{s-}-Z_{s-}\log H(s,\alpha(s),z)]\nu(dz)ds \\
&=1+\int_0^t Z_{s-}G(s,\alpha(s))dW(s) \\
&-\int_{[0,t)\times\mathbb{R}} Z_{s-}[H(s,\alpha(s),z)-1]\nu(dz)ds \\
&+\int_0^t\int_{\mathbb{R}} Z_{s-}[H(s,\alpha(s),z)-1]\tilde{N}(ds,dz) \\
&+\int_0^t\int_{\mathbb{R}} Z_{s-}[H(s,\alpha(s),z)-1]\nu(dz)ds \\
&=1+\int_0^t Z_{s-}G(s,\alpha(s))dW(s) \\
&+\int_0^t\int_{\mathbb{R}} Z_{s-}[H(s,\alpha(s),z)-1]\tilde{N}(ds,dz).
\end{aligned}
$$

This last expression is a local martingale.

The processes G, H and h can be chosen so that $\mathbb{E}[Z(t)]=1$ for all t, in which case Z is a martingale.

Theorem 3.1 *Define a probability measure $\mathbb{Q}$ on $\mathcal{F}_T$, by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T}=Z(T).$$

Under the condition

$$b(t,i)-r(t,i)+\sigma(t,i)G(t,i) \tag{3.6}$$

$$+\int_{\mathbb{R}} \rho(t,i,z)(H(t,i,z)-1)\nu(dz) = 0, \forall i \in \mathbb{S},$$

if

$$\mathbb{E}_P[Z(T)] = 1,$$

then $\mathbb{Q}$ *is an equivalent local martingale measure for* $\hat{S}(T)$.

Proof. From (3.4), we have the discount price $\hat{S}(t)$ of the form

$$\begin{aligned} d\hat{S}(t) = \hat{S}(t)\{&(b(t,\alpha(t)) - r(t,\alpha(t)))dt + \sigma(t,\alpha(t))dW(t) \\ &+ \int_{\mathbb{R}} \rho(t,\alpha(t),z)\tilde{N}(dt,dz)\}. \end{aligned}$$

Define

$$W_Q(t) = W(t) - \int_0^t G(s,\alpha(s))ds,$$

$$\nu_Q(dz)dt = H(t,\alpha(t),z)\nu(dz)dt,$$

and

$$\tilde{N}_Q(dt,dz) = \tilde{N}(dt,dz) - \int_{\mathbb{R}} (H(t,\alpha(t),z)-1)\nu(dz)dt.$$

Since $W(t)$ and $\int_0^t \int_{\mathbb{R}} \tilde{N}(dt,dz)$ are $\mathbb{P}$-martingales, we use Itô's product formula to find

$$\begin{aligned} &d[W_Q(t)Z(t)] \\ &= dW_Q(t)Z(t_-) + W_Q(t)dZ(t) + dW_Q(t)dZ(t) \\ &= dW(t)Z(t_-) - G(t,\alpha(t))dtZ(t_-) \\ &\quad + W_Q(t)Z(t_-)[G(t,\alpha(t))dW(t) + \int_{\mathbb{R}} (H(t,\alpha(t),z)-1)\tilde{N}(dt,dz)] \\ &\quad + Z(t_-)[G(t,\alpha(t))dt + dW(t)\int_{\mathbb{R}} (H(t,\alpha(t),z)-1)\tilde{N}(dt,dz) \end{aligned}$$

$$- G(t,\alpha(t))dt \int_{\mathbb{R}} (H(t,\alpha(t),z) - 1)\tilde{N}(dt,dz)].$$

Since $dW(t)\int_{\mathbb{R}}(H(t,\alpha(t),z) - 1)\tilde{N}(dt,dz) = 0$, then

$$\begin{aligned}
& d[W_Q(t)Z(t)] \\
&= Z(t_-)[1 + W_Q(t)G(t,\alpha(t))]dW(t) \\
&\quad + Z(t_-)[W_Q(t) - G(t,\alpha(t))dt] \int_{\mathbb{R}} (H(t,\alpha(t),z) - 1)\tilde{N}(dt,dz),
\end{aligned}$$

is a $\mathbb{P}$-martingale. We derive for any s_1, $s_2 \in [0,t]$, and $s_1 < s_2$,

$$\begin{aligned}
& \mathbb{E}_Q[W_Q(s_2)|\mathcal{F}_{s_1}] \\
&= \frac{\mathbb{E}_P[W_Q(s_2)Z(s_2)|\mathcal{F}_{s_1}]}{\mathbb{E}_P[Z(s_2)|\mathcal{F}_{s_1}]} \\
&= \frac{W_Q(s_2)Z(s_2)}{Z(s_2)} \\
&= W_Q(s_2),
\end{aligned}$$

therefore, $W_Q(t)$ is a $\mathbb{Q}$-martingale. By the same type of argument $\tilde{N}_Q(dt,dz)$ is also a $\mathbb{Q}$-martingale.

Since $\Delta\tilde{N}(dt,dz) = \Delta\tilde{N}_Q(dt,dz)$, we have

$$\begin{aligned}
\hat{S}(t) &= S(0)\exp\{\int_0^t \sigma(s,\alpha(s))dW(s) \\
&\quad + \int_0^t (b(s,\alpha(s)) - r(s,\alpha(s)) - \frac{\sigma(s,\alpha(s))^2}{2})ds \\
&\quad + \int_0^t \int_{\mathbb{R}} \rho(s,\alpha(s),z)\tilde{N}(ds,dz)\} \\
&\quad \times \prod_{0<s\leq t} (1 + \int_{\mathbb{R}} \rho(s,\alpha(s),z)\Delta\tilde{N}(ds,dz)) \\
&\quad \times \exp\{-\int_{\mathbb{R}} \rho(s,\alpha(s),z)\Delta\tilde{N}(ds,dz)\} \\
&= S(0)\exp\{\int_0^t \sigma(s,\alpha(s))dW_Q(s) \\
&\quad + \int_0^t (b(s,\alpha(s)) + \sigma(s,\alpha(s))G(s,\alpha(s)) - r(s,\alpha(s)) - \frac{\sigma(s,\alpha(s))^2}{2})ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}}\rho(s,\alpha(s),z)\tilde{N}_Q(ds,dz)\\
&+\int_0^t\int_{\mathbb{R}}\rho(s,\alpha(s),z)(H(s,\alpha(s),z)-1)\nu(dz)ds\}\\
&\times\prod_{0<s\le t}(1+\int_{\mathbb{R}}\rho(s,\alpha(s),z)\Delta\tilde{N}_Q(ds,dz))\\
&\times\exp\{-\int_{\mathbb{R}}\rho(s,\alpha(s),z)\Delta\tilde{N}_Q(ds,dz)\}.
\end{aligned}
$$

By the condition (3.6), we obtain

$$
\begin{aligned}
\hat{S}(t)=S(0)\exp\{&\int_0^t\sigma(s,\alpha(s))dW_Q(s)-\int_0^t\frac{\sigma(s,\alpha(s))^2}{2}ds\\
&+\int_0^t\int_{\mathbb{R}}\rho(s,\alpha(s),z)\tilde{N}_Q(ds,dz)\\
&\times\prod_{0<s\le t}(1+\int_{\mathbb{R}}\rho(s,\alpha(s),z)\Delta\tilde{N}_Q(ds,dz))\\
&\times\exp\{-\int_{\mathbb{R}}\rho(s,\alpha(s),z)\Delta\tilde{N}_Q(ds,dz)\}.
\end{aligned}
$$

By Lemma 3.1, we can easily get $\hat{S}(t)$ is a martingale with respect to the measure $\mathbb{Q}$.

3.3 Pricing by Esscher transform and minimum relative entropy

Gerber and Shiu (1994) proposed pricing contingent claims by Esscher transforms. Let $\theta\in\mathbb{R}$ be fixed. The Esscher transform of a Lévy process Y, or equivalently of its underlying canonical measure $\mathbb{P}$, is defined to be the process whose law $\mathbb{Q}_\theta$ is given by

$$
\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}|_{\mathcal{F}_t}=\exp\{-\theta Y(t,\alpha(t))+t\psi(\theta)\},
$$

where $\psi(\theta)=-\log\mathbb{E}[\exp(-\theta Y(1,\alpha(1)))]$ is the Lévy exponent of Y. Since the stock price process has time-dependent coefficients in our model, we need to consider generalized Esscher transforms of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\{-\int_0^t \theta(s,\alpha(s))dY(s,\alpha(s)) + \int_0^t \psi(\theta(s,\alpha(s)))ds\} \tag{3.7}$$

and choose $\theta(t,\alpha(t))$ to satisfy the martingale condition.

Set

$$H(t,\alpha(t),z) = \exp(-\theta(t,\alpha(t))\rho(t,\alpha(t),z)),$$
$$G(t,\alpha(t)) = \sigma(t,\alpha(t))\theta(t,\alpha(t)).$$

The martingale condition (3.6) can be used to specify θ as follows:

$$\begin{aligned} &\sigma(s,i)^2\theta(s,i) + b(s,i) - r(s,i) \\ &+ \int_{\mathbb{R}} \rho(s,i,z)(\exp(-\theta(s,i)\rho(s,i,z)) - 1)\nu(dz) = 0. \end{aligned} \tag{3.8}$$

[To see that this has a unique solution θ for fixed s and i, define $F(\theta) = \int_{\mathbb{R}} \rho(z)(\exp(-\theta\rho(z)) - 1)\nu(dz) - \theta$ for $\theta \in (-h_1, h_2)$, where $0 < h_1, h_2 \le \infty$. Then it is easy to check that F is monotonically decreasing and $F(\theta) \to +\infty$ as $\theta \downarrow -h_1$ and $F(\theta) \to \infty$ as $\theta \uparrow h_2$. Hence equations of the form $F(\theta) = c$ have a unique solution in $(-h_1, h_2)$.]

For a fixed measure P, the relative entropy $I_p(Q)$ of any measure Q with respect to P is defined to be

$$I_p(Q) = \int \log\frac{dQ}{dP}dQ = \int \frac{dQ}{dP}\log\frac{dQ}{dP}dP.$$

[Note that $I_p(Q) \ge 0$ for any Q. If Q is not absolutely continuous with respect to P, $I_p(Q)$ is infinite.] For an equivalent martingale measure $\mathbb{Q}$ given by Theorem 3.1 and the martingale condition (3.6), the relative entropy in terms of the $\mathbb{Q}$-martingales $W_Q(t)$ and $\tilde{N}_Q(dt,dz)$ is therefore

$$I_P(Q)$$

$$
\begin{aligned}
&= \mathbb{E}_Q[\log \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T}] \\
&= \mathbb{E}_Q[\log Z(T)|_{\mathcal{F}_T}] \\
&= \mathbb{E}_Q[\int_0^T G(s,\alpha(s))dW_Q(s) + \frac{1}{2}\int_0^T G(s,\alpha(s))^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}} \log H(s,\alpha(s),z)\tilde{N}_Q(ds,dz) \\
&\quad + \int_0^T \int_{\mathbb{R}} \log H(s,\alpha(s),z)(H(s,\alpha(s),z)-1)\nu(dz)ds \\
&\quad + \int_0^T \int_{\mathbb{R}} [\log H(s,\alpha(s),z) + 1 - H(s,\alpha(s),z)]\nu(dz)ds \\
&= \mathbb{E}_Q[\frac{1}{2}\int_0^T G(s,\alpha(s))^2 ds \\
&\quad + \int_0^T \int_{\mathbb{R}} [H(s,\alpha(s),z)(\log H(s,\alpha(s),z) - 1) + 1)]\nu(dz)ds].
\end{aligned}
$$

The problem of finding the equivalent martingale measure of minimum relative entropy can be reduced to that of minimizing

$$\mathbb{E}_Q[\frac{1}{2}G(s,i)^2 + \int_{\mathbb{R}} [H(s,i,z)(\log H(s,i,z) - 1) + 1)]\nu(dz)],$$

for fixed s, subject to (3.6). Then it is clear that the problem can be reduced to that of minimizing

$$\frac{1}{2}G(s,i)^2 + \int_{\mathbb{R}} [H(s,i,z)(\log H(s,i,z) - 1) + 1]\nu(dz) \tag{3.9}$$

for each fixed s and ω, subject to (3.6). Denote by $\mathbb{Q}^*$ the measure associated with the optimal choice of G and H. Then the corresponding optimal value I^* of (3.9) is therefore also deterministic and for other choice of G and H with associated measure $\mathbb{E}_{G,H}$, we have

$$\frac{1}{2}G(s,i)^2 + \int_{\mathbb{R}} [H(s,i,z)(\log H(s,i,z) - 1) + 1]\nu(dz) \geq I^*.$$

Hence

$$\mathbb{E}_{G,H}[\frac{1}{2}G(s,i)^2 + \int_{\mathbb{R}} [H(s,i,z)(\log H(s,i,z) - 1) + 1]\nu(dz)]$$

$$
\begin{aligned}
&\geq I^* \\
&= \mathbb{E}_{Q^*}[I^*].
\end{aligned}
$$

Now we fix $G(t,i)$ and choose $H(t,i,z)$ to minimize

$$\int_{\mathbb{R}} [H(s,i,z)(\log H(s,i,z) - 1) + 1]\nu(dz)$$

subject to (3.6) and then minimize (3.9) (with the optimal H) over G.

Let $\lambda(s,\alpha(s))$ be a continuous function and let

$$
\begin{aligned}
&L(\lambda, H) \\
&= \int_{\mathbb{R}} [H(s,\alpha(s),z)(\log H(s,\alpha(s),z) - 1) + 1]\nu(dz) \\
&\quad + \int_{\mathbb{R}} \lambda(s,\alpha(s))\rho(s,\alpha(s),z)(H(s,\alpha(s),z) - 1)\nu(dz).
\end{aligned}
$$

Thus λ is a Lagrange multiplier associated with the constraint (3.6) and L is the associated Lagrangian. Observe that $H \to L(\lambda, H)$ is convex in $H > 0$, so to find the optimal H, we require

$$\frac{d}{dt} L(\lambda, H + tF)|_{t=0} = 0,$$

for all F. Then, we get

$$
\begin{aligned}
&\frac{d}{dt} L(\lambda, H + tF)|_{t=0} \\
&= \frac{d}{dt}[\int_{\mathbb{R}} [(H(s,\alpha(s),z) + tF)(\log(H(s,\alpha(s),z) + tF) - 1) + 1]\nu(dz) \\
&\quad + \int_{\mathbb{R}} \lambda(s,\alpha(s))\rho(s,\alpha(s),z)(H(s,\alpha(s),z) + tF - 1)\nu(dz)]|_{t=0} \\
&= \int_{\mathbb{R}} [F\log(H(s,\alpha(s),z) + tF) + \lambda(s,\alpha(s))\rho(s,\alpha(s),z)F]\nu(dz)|_{t=0} \\
&= \int_{\mathbb{R}} [F\log H(s,\alpha(s),z) + \lambda(s,\alpha(s))\rho(s,\alpha(s),z)F]\nu(dz)|_{t=0} \\
&= 0,
\end{aligned}
$$

for all F. We obtain

$$H(s,i,z) = \exp(-\lambda(s,i)\rho(s,i,z)).$$

The Lagrange multiplier λ can be expressed in terms of G (assumed to be fixed for the moment) via (3.6):

$$\begin{aligned} & r(s,i) - \sigma(s,i)G(s,i) - b(s,i) \\ = & \int_{\mathbb{R}} \rho(s,i,z)(\exp(-\lambda(s,i)\rho(s,i,z)) - 1)\nu(dz). \end{aligned}$$

Since all the optimization is carried out for fixed s, we temporarily drop the explicit dependence on s for the sake of clarity. The above equation is then simply

$$\begin{aligned} & r(i) - \sigma(i)G(i) - b(i) \\ = & \int_{\mathbb{R}} \rho(i,z)(\exp(-\lambda(i)\rho(i,z)) - 1)\nu(dz), \end{aligned} \tag{3.10}$$

Then for fixed i, putting $H(i,z) \equiv e^{-\lambda(i)\rho(i,z)}$ into (3.9) gives

$$\frac{1}{2}G(i)^2 + \int_{\mathbb{R}} [1 - \exp(-\lambda(i)\rho(i,z))(\lambda(i)\rho(i,z) + 1)]\nu(dz), \tag{3.11}$$

which we must now minimize over G. We differentiate the above with respect to G and solve

$$G(i) + \lambda'(G(i)) \int_{\mathbb{R}} \lambda(i)\rho(i,z)^2 \exp(-\lambda(i)\rho(i,z))\nu(dz) = 0.$$

However, differentiating (3.10) shows that

$$\lambda'(G(i)) = \sigma(i)(\int_{\mathbb{R}} \rho(i,z)^2 \exp(-\lambda(i)\rho(i,z))\nu(dz))^{-1}.$$

Then we get $G = \sigma(i)\lambda(i)$, for fixed i. Since the second derivative of (3.11) is positive, $G(i) = \sigma(i)\lambda(i)$ does indeed give the minimum of (3.10).

Then we get

$$G_{min}(i) = \min(G(i), i \in \mathbb{S}),$$
$$H_{min}(i,z) = \min(H(i,z), i \in \mathbb{S}),$$

where

$$\begin{aligned}
&\min[\frac{1}{2}G(i)^2 + \int_{\mathbb{R}} [H(i,z)(\log H(i,z) - 1) + 1)\nu(dz), i \in \mathbb{S}] \\
&= \frac{1}{2}G_{min}(i)^2 + \int_{\mathbb{R}} [H_{min}(i,z)(\log H_{min}(i,z) - 1) + 1)\nu(dz)
\end{aligned}$$

Now both G and H are specified in terms of λ, and restoring the s and i in (3.10) gives the equation for $\lambda(s,i)$,

$$\begin{aligned}
&\sigma(s,i)^2\lambda(s,i) + b(s,i) - r(s,i) \\
&+ \int_{\mathbb{R}} \rho(s,i,z)(\exp(-\lambda(s,i)\rho(s,i,z)) - 1)\nu(dz) \\
&= 0,
\end{aligned}$$

hence

$$\begin{aligned}
&\sigma(s,i)^2\lambda(s,i) + b(s,i) - r(s,i) \\
&+ \int_{\mathbb{R}} \rho(s,i,z)(\exp(-\lambda(s,i)\rho(s,i,z)) - 1)\nu(dz) = 0.
\end{aligned} \tag{3.12}$$

Comparing above equation and (3.8), we get $\theta(s,i) \equiv \lambda(s,i)$.

As one of the main motivations behind the study of the Esscher transform presented here is Gerber-Shiu (1994) [[13]], it is interesting to see if similar results hold for the model of stock price used in that thesis, namely

$$S(t) = S(0)\exp\{\sigma dW(t) + \int_0^t \int_{\mathbb{R}} \rho\tilde{N}(dt,dz) + bdt\}$$

for constants σ, b and ρ. We also take $r = 0$, so that $S = \hat{S}$. And the Esscher transform of Y is exactly the same as before. However, whereas for our model the martingale condition used to specify θ is (3.6), a different martingale condition applies to the Gerber-Shiu model above. Since

$$
\begin{aligned}
S(t) &= S(0)\exp\{\sigma dW(t) + \int_0^t \int_{\mathbb{R}} \rho \tilde{N}(dt,dz) + bdt\} \\
&= S(0)\exp\{\sigma dW_Q(t) + \int_0^t \rho \tilde{N}_Q(dt,dz) + bdt + \sigma \int_0^t G(s,\alpha(s))ds \\
&\quad + \rho \int_0^t \int_{\mathbb{R}} (H(s,\alpha(s),z) - 1)\nu(dz)ds\}.
\end{aligned}
$$

By Itô's formula, we get

$$
\begin{aligned}
dS(t) &= \sigma S(t_-)dW_Q(t) + \int_{\mathbb{R}} \rho S(t_-) d\tilde{N}_Q(dt,dz) \\
&\quad + S(t_-)\{\sigma G(t,\alpha(t)) + b + \frac{\sigma^2}{2} \\
&\qquad + \rho \int_{\mathbb{R}} (H(t,\alpha(t),z) - 1)\nu(dz)\}dt \\
&\quad + S(t_-)\int_{\mathbb{R}} (\exp(\rho) - 1 - \rho)\tilde{N}_Q(dt,dz) \\
&\quad + S(t_-)\int_{\mathbb{R}} (\exp(\rho) - 1 - \rho)\tilde{\nu}(dz)dt \\
&= \sigma S(t_-)dW_Q(t) \\
&\quad + S(t_-)\{\sigma G(t,\alpha(t)) + b + \frac{\sigma^2}{2} \\
&\qquad + \rho \int_{\mathbb{R}} (H(t,\alpha(t),z) - 1)\nu(dz)\}dt \\
&\quad + S(t_-)\int_{\mathbb{R}} (\exp(\rho) - 1)\tilde{N}_Q(dt,dz) \\
&\quad + S(t_-)\int_{\mathbb{R}} (\exp(\rho) - 1 - \rho)\nu_Q(dz)dt.
\end{aligned}
$$

Hence, in order for $S = \hat{S}$ to be a martingale under $\mathbb{Q}$, we require

$$
\begin{aligned}
&\sigma G(t,\alpha(t)) + b + \frac{\sigma^2}{2} + \rho \int_{\mathbb{R}} (H(t,\alpha(t),z) - 1)\nu(dz) \\
&+ \int_{\mathbb{R}} (e^{\rho} - 1 - \rho) H(t,\alpha(t),z)\nu(dz) \\
&= 0.
\end{aligned}
$$

In terms of θ, this translates into

$$-\sigma\theta + b + \frac{\sigma^2}{2} + \int_{\mathbb{R}} [(e^{\rho} - 1)e^{-\theta\rho} - \rho]\nu(dz) = 0.$$

Turning now to the minimum relative entropy measure, we need to minimize (3.9) subject to (3.6). Following exactly the same Lagrangian procedure as before, we get

$$\begin{aligned} H &= \exp(\lambda(1 - e^{\rho})) \\ G &\equiv \sigma\lambda. \end{aligned}$$

Then we have

$$\begin{aligned} &- \sigma^2\lambda + b + \frac{\sigma^2}{2} \\ &+ \int_{\mathbb{R}} [(e^{\rho} - 1)e^{\lambda(1-e^{\rho})} - \sigma]\nu(dz) \\ = {}& 0. \end{aligned}$$

We see that it is no longer possible to make $\theta = \lambda$. But if the Lévy process only makes very small jumps, its Lévy measure ν is concentrated around 0. Making the approximation

$$1 - e^{\rho} \sim -\rho,$$

for small ρ. We can obtain that the solution of above equations can be approximated to some extent by $\theta \approx \lambda$.

Chapter 4

SDEs with transaction costs

Richardson (1989) is probably the earliest paper that studies a faithful extension of the Mean-Variance model to the continuous-time setting. Li and Ng (2000), in a discrete-time setting, developed an embedding technique to change the originally time-inconsistent MV prblem into a stochastic Linear Quadratic control problem. The technique was extended by Zhou and Li (2000), along with a stochastic LQ control approach, to the continuous-time case.

All the existing works on continuous-time Mean-Variance models have assumed that there is no transaction cost, leading to results that are analytically elegant. Portfolio selection subject to transaction costs has been studied extensively, albeit in the realm of utility maximization. This chapter aims to analytically solve the MV model with transaction costs.

4.1 Formulation of the problem

Consider a market where 2 assets are traded continuously. One of the assets is a bank account whose price $P_0(t)$ is subject to the following random ordinary differential equation:

$$dP_0(t) = r(\alpha(t))P_0(t)dt.$$

where $r(i) > 0, i = 1, 2, \ldots, l$, are given as the interest rate process corresponding to different market modes. Another asset is a stock whose price process $P_1(t)$, satisfies the following stochastic differential equations:

$$dP_1(t) = P_1(t)\{b(\alpha(t))dt + \sigma(\alpha(t))dW(t)\}, \qquad t \in [0, T],$$

where for each $i = 1, 2, \ldots, l$, $b : \mathbb{S} \to \mathbb{R}$, $\sigma : \mathbb{S} \to \mathbb{R}$ is adapted process.

We assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by the Brownian motion and Markov chain. We denote by $\mathrm{L}^2_{\mathcal{F}}$ the set of square integrable $\{\mathcal{F}_t\}_{t\in[0,T]}$ adapted processes,

$$\mathrm{L}^2_{\mathcal{F}} = \{X | \text{The process } X = X(t)_{t\in[0,T]} \text{is an } \{\mathcal{F}_t\}_{t\in[0,T]}\text{-adapted process such that } \int_0^T \mathbb{E}[X^2(t)]dt < \infty\}.$$

and by $\mathrm{L}^2_{\mathcal{F}_T}$ the set of square integrable $\mathcal{F}_t$-measurable random variables,

$$\mathrm{L}^2_{\mathcal{F}_T} = \{X | X \text{is an} \mathcal{F}_T\text{-measurable random variable such that } \int_0^T \mathbb{E}[X^2(t)]dt < \infty\}.$$

There is a self-financing investor with a finite investment horizon $[0, T]$ who invests $X(t)$ dollars in the bank and $Y(t)$ dollars in the stock at time t. Any stock transaction incurs a proportional transaction fee, with $\beta \in [0, \infty)$ and $\mu \in [0, 1)$ being the proportions paid when buying and selling the stock, respectively. Throughout this paper, we assume that $\beta + \mu > 0$, which means transaction costs must be involved. The value process, starting from (x, y) at $t = 0$, evolves according to the equations:

$$X^{x,L,M}(t) = x + \int_0^t r(\alpha(s))X^{x,L,M}(s)ds - (1 + \beta)L(t) + (1 - \mu)M(t), \tag{4.1}$$

$$Y^{y,L,M}(t) = y + \int_0^t b(\alpha(s))Y^{y,L,M}(s)ds + \int_0^t \sigma(\alpha(s))Y^{y,L,M}(s)dW(s) \quad (4.2)$$
$$+ L(t) - M(t),$$

where $L(t)$ and $M(t)$ denote respectively the cumulative stock purchase and sell up to time t. Assume $r(i) > 0$, $b(i) > r(i)$. $\bar{b} = \max(b(i))$, $\bar{r} = \max(r(i))$ and $\bar{\sigma} = \max(\sigma(i))$. Obviously $\bar{b} > \bar{r}$. Sometimes we simply use X, Y or $X^{L,M}$, $Y^{L,M}$ instead of $X^{x,L,M}$, $Y^{y,L,M}$, if there is no ambiguity.

The admissible strategy set $\mathcal{A}$ of the investor is defined as follows:

$$\begin{aligned}\mathcal{A} = \{(L, M) | &\text{The processes } L = \{L(t)\}_{t\in[0,T]} \text{ and } M = \{M(t)\}_{t\in[0,T]} \\ &\text{are } \{\mathcal{F}_t\}_{t\in[0,T]}\text{-adapted, cádlág, nonengative} \\ &\text{and nondecreasing, and the processes } X^{x,L,M} \\ &\text{and } Y^{y,L,M} \text{ are both in } \mathrm{L}^2_{\mathcal{F}}, \text{ for any } (x, y) \in \mathbb{R}^2\}.\end{aligned}$$

(L, M) is called an admissible strategy, if $(L, M) \in \mathcal{A}$. Correspondingly, $(X^{x,L,M}, Y^{y,L,M})$ is called an admissible (bond-stock) process, if $(x, y) \in \mathbb{R}^2$ and $(L, M) \in \mathcal{A}$.

For an admissible process $(X^{x,L,M}, Y^{y,L,M})$, we define the investor's net wealth process by

$$W^{X,Y}(t) = X(t) + (1-\mu)Y(t)^+ - (1+\beta)Y(t)^-, t \in [0, T].$$

Namely, $W^{X,Y}(t)$ is the net worth of the investor's portfolio at t after the transaction cost is deducted. The investor's attainable net wealth set at the maturity time T is defined as

$$\begin{aligned}\mathcal{W}_0^{x,y} = \{W^{X,Y}(T) | &W^{X,Y}(T) \text{ is the net wealth at } T \text{ of an admissible process} \\ &(X, Y) \text{ with } X(0^-) = x, Y(0^-) = y.\}.\end{aligned}$$

And during this chapter, we assume $T > \frac{1}{b(i)-r(i)} \ln(\frac{1+\beta}{1-\mu})$.

The investor's problem is to choose an admissible stategy so as to maximize the expected utility of terminal wealth

$$\sup_{(L,M)\in\mathcal{A}} \mathbb{E}_t^{x,y}[U(W^{X,Y}(T))|\mathcal{F}_t].$$

$\mathcal{F}_t$ is generated by $X_{tt\geq 0}$ and $Y_{tt\geq 0}$. Here $\mathbb{E}_t^{x,y}$ denotes the conditional expectation at time t given that initial endowment $X(t) = x$, $Y(t) = y$, and the utility function

$$U(W) = \frac{W^\gamma}{\gamma}, \qquad 0 < \gamma < 1.$$

By the original Markowitz's MV portfolio theory, an efficient strategy is one that is Pareto efficient, which means there does not exist another strategy that has higher mean and no higher variance, and/or has less variance and no less mean at the terminal time T. There could be many efficient strategies, and the terminal means and variances corresponding to all the efficient strategies form an efficient frontier. We can obtain the efficient frontier from solving the following variance minimizing problem:

Problem 1,

$$\begin{cases} \text{minimize} & \mathbf{Var}(W), \\ \text{subject to} & \mathbb{E}[W] = \zeta, W \in \mathcal{W}_0^{x,y}, \end{cases}$$

where

$$\zeta > e^{\bar{r}T}x + (1-\mu)e^{\bar{r}T}y^+ - (1+\beta)e^{\bar{r}T}y^-.$$

This means the target expected terminal wealth is higher than "all-bond" strategy. We can obtain the efficient frontier from the above problem. Problem 1 is equivalent to the following problem.

Problem 2,

$$\begin{cases} \text{minimize} & \mathbb{E}[W^2], \\ \text{subject to} & \mathbb{E}[W] = \zeta, W \in \mathcal{W}_0^{x,y}. \end{cases}$$

4.2 Feasibility

Feasibility issue is important and unique to the MV problem, and will be discussed in this section. Firstly, we introduce two lemmas.

Lemma 4.1 *If $W_1 \in \mathcal{W}_0^{x,y}$, $W_2 \in \mathrm{L}^2_{\mathcal{F}_T}$ and $W_2 \leq W_1$, then $W_2 \in \mathcal{W}_0^{x,y}$.*

Proof. By the definition of $\mathcal{W}_0^{x,y}$, there exists $(L, M) \in \mathcal{A}$ such that $X^{L,M}(0-) = x$, $Y^{L,M}(0-) = y$ and $W^{X^{L,M},Y^{L,M}}(T) = W_1$. We define

$$\bar{L}(t) = \begin{cases} L(t), & t < T, \\ L(T) + \dfrac{W_1 - W_2}{\beta + \mu} & t = T, \end{cases}$$

$$\bar{M}(t) = \begin{cases} M(t), & t < T, \\ M(T) + \dfrac{W_1 - W_2}{\beta + \mu} & t = T. \end{cases}$$

Then $(\bar{L}, \bar{M}) \in \mathcal{A}$, and

$$X^{\bar{L},\bar{M}}(t) = \begin{cases} X^{L,M}(t), & t < T, \\ X^{L,M}(t) - W_1 + W_2 & t = T, \end{cases}$$

$$Y^{\bar{L},\bar{M}}(t) = Y^{L,M}(t), t \in [0, T].$$

Therefore, $W_2 = W^{X^{\bar{L},\bar{M}},Y^{\bar{L},\bar{M}}}(T) \in \mathcal{W}_0^{x,y}$.

From [6], we have

Lemma 4.2 *For any* $(x,y)\in\mathbb{R}^2$, *we have*

(1) the set $\mathcal{W}_0^{(x,y)}$ *is convex;*

(2) if $(x_i,y_i)\in\mathbb{R}^2$, $W_i\in\mathcal{W}_0^{(x_i,y_i)}$, $i=1,2$, *then* $W_1+W_2\in\mathcal{W}_0^{(x_1+x_2,y_1+y_2)}$*;*

(3) if $x_1\le x_2$ *and* $y_1\le y_2$, *then* $\mathcal{W}_0^{(x_1,y_1)}\le\mathcal{W}_0^{(x_2,y_2)}$*;*

(4) $\mathcal{W}_0^{(x-(1+\beta)\varrho,y+\varrho)}\subseteq\mathcal{W}_0^{(x,y)}$ *and* $\mathcal{W}_0^{(x+(1-\mu)\varrho,y-\varrho)}\subseteq\mathcal{W}_0^{(x,y)}$ *for any* $\varrho>0$*;*

(5) $\mathcal{W}_0^{(\varrho x,\varrho y)}=\varrho\mathcal{W}_0^{(x,y)}$ *for any* $\varrho>0$*;*

(6) if $x+(1-\mu)y^+-(1+\beta)y^-\ge 0$, *then* $0\in\mathcal{W}_0^{(x,y)}$.

Denote

$$\hat{\zeta}=\sup\{\mathbb{E}[W]|W\in\mathcal{W}_0^{(x,y)}\}.$$

By Lemma 4.1, Problem 2 is feasible when

$$\zeta\in\mathcal{D}=[e^{\bar{r}T}x+(1-\mu)e^{\bar{r}T}y^+-(1+\beta)e^{\bar{r}T}y^-,\hat{\zeta}].$$

Now, we need to check whether Problem 2 is feasible when $\zeta=\hat{\zeta}$, since Problem 2 is not feasible when $\zeta>\hat{\zeta}$. We have $\hat{\zeta}=+\infty$ and thus Problem 2 is always feasible for any $\zeta\ge e^{\bar{r}T}x+(1-\mu)e^{\bar{r}T}y^+-(1+\beta)e^{\bar{r}T}y^-$, when $\beta=\mu=0$ [see Lim and Zhou (2002)].

4.3 Unconstrained Problem and Double Obstacle Problem

We shall utilize the Lagrange multiplier method to remove the constraint of Problem 2. Let us introduce the following unconstrained problem with Lagrange multiplier λ.

Problem 3 (Unconstrained Problem),

$$\begin{cases}\text{minimize} & \mathbb{E}[W^2]-2\lambda(\mathbb{E}[W]-\zeta),\\ \text{subject to} & W\in\mathcal{W}_0^{x,y},\end{cases}$$

or equivalently,

Problem 4,

$$\begin{cases} \text{minimize} & \mathbb{E}[(W-\lambda)^2], \\ \text{subject to} & W \in \mathcal{W}_0^{x,y}. \end{cases}$$

Define the value function of Problem 2 as follows:

$$V_1(x,y;\zeta) = \inf_{W\in\mathcal{W}_0^{x,y}, \mathbb{E}[W]=\zeta} \mathbb{E}[W^2], \zeta \in \mathcal{D}.$$

The following result, showing the connection between Problem 2 and Problem 4, can be proved by a standard convex analysis argument [6].

Proposition 4.1 *Problem 2 and Problem 4 have the following relations.*

(1) If W_ζ^* solves Problem 2 with parameter $\zeta \in \mathcal{D}$, then there exits $\lambda \in \mathbb{R}$ such that W_ζ^* also solves Problem 4 with parameter λ.

(2) Conversely, if W_λ solves Problem 4 with parameter $\lambda \in \mathbb{R}$, then it must also solve Problem 2 with parameter $\zeta = \mathbb{E}[W_\lambda]$.

According to Lemma 4.2, it is easy to see that $\mathcal{W}_0^{x,y} - \lambda = \mathcal{W}_0^{x-\lambda e^{-\bar{r}T},y}$. Since

$$\begin{aligned} &\mathcal{W}_0^{x,y} - \lambda \\ &= \mathcal{W}_0^{x,y} - \mathcal{W}_0^{\lambda e^{-\bar{r}T},0} \\ &= \mathcal{W}_0^{x-\lambda e^{-\bar{r}T},y}. \end{aligned}$$

We consider the following problem instead of Problem 4:

Problem 5,

$$\begin{cases} \text{minimize} & \mathbb{E}[W^2], \\ \text{subject to} & W \in \mathcal{W}_0^{x-\lambda e^{-\bar{r}T},y}. \end{cases}$$

Consider (4.1) and (4.2) where the initial time 0 is revised to some $s \in [0,T)$, and define $\mathcal{W}_s^{x,y}$ as the counterpart of $\mathcal{W}_0^{x,y}$ where the initial time is

s and initial bond-stock position is (x, y). We then define the value function of Problem 5 as

$$V(t, x, y) = \inf_{W \in \mathcal{W}_t^{x,y}} \mathbb{E}[W^2], (t, x, y) \in [0, T) \times \mathbb{R}^2. \tag{4.3}$$

The following proposition establishes a link between Problem 2 and Problem 5.

Proposition 4.2 *If $\zeta \in \mathcal{D}$, then*

$$\sup_{\lambda \in \mathbb{R}} (V(0, x - \lambda e^{-\bar{r}T}, y) - (\lambda - \zeta)^2) = V_1(x, y; \zeta) - \zeta^2.$$

Proof. Compute

$$\begin{aligned}
& \sup_{\lambda \in \mathbb{R}} (V(0, x - \lambda e^{-\bar{r}T}, y) - (\lambda - \zeta)^2) \\
= & \sup_{\lambda \in \mathbb{R}} \inf_{W \in \mathcal{W}_0^{x - \lambda e^{-\bar{r}T}, y}} \mathbb{E}[W^2 - (\lambda - \zeta)^2] \\
= & \sup_{\lambda \in \mathbb{R}} \inf_{W \in \mathcal{W}_0^{x,y}} \mathbb{E}[(W - \lambda)^2 - (\lambda - \zeta)^2] \\
\leq & \sup_{\lambda \in \mathbb{R}} \inf_{W \in \mathcal{W}_0^{x,y}, \mathbb{E}[W] = \zeta} \mathbb{E}[(W - \lambda)^2 - (\lambda - \zeta)^2] \\
= & \sup_{\lambda \in \mathbb{R}} \inf_{W \in \mathcal{W}_0^{x,y}, \mathbb{E}[W] = \zeta} (\mathbb{E}[W^2] - \zeta^2) \\
= & \inf_{W \in \mathcal{W}_0^{x,y}, \mathbb{E}[W] = \zeta} (\mathbb{E}[W^2] - \zeta^2) \\
= & V_1(x, y; \zeta) - \zeta^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sup_{\lambda \in \mathbb{R}} (V(0, x - \lambda e^{-\bar{r}T}, y) - (\lambda - \zeta)^2) \\
\leq & V_1(x, y; \zeta) - \zeta^2.
\end{aligned}$$

Since V_1 is convex and ζ is an interior point of $\mathcal{D}$, by convex analysis, there exists $\lambda^* \in \mathbb{R}$ such that

$$V_1(x, y, ; \zeta) - 2\lambda^* \zeta \leq V_1(x, y, ; \tilde{\zeta}) - 2\lambda^* \tilde{\zeta}, \forall \tilde{\zeta} \in \mathcal{D}.$$

For any $W \in \mathcal{W}_0^{x,y}$, we have

$$\begin{aligned}
&\mathbb{E}[(W-\lambda^*)^2-(\lambda^*-\zeta)^2] \\
&= \mathbb{E}[W^2]-2\lambda^*(\mathbb{E}[W]-\zeta)-\zeta^2 \\
&\geq V_1(x,y;\mathbb{E}[W])-2\lambda^*(\mathbb{E}[W]-\zeta)-\zeta^2 \\
&\geq V_1(x,y;\zeta)-\zeta^2.
\end{aligned}$$

If follows

$$\begin{aligned}
&\sup_{\lambda\in\mathbb{R}}(V(0,x-\lambda e^{-\bar{r}T},y)-(\lambda-\zeta)^2) \\
&= \sup_{\lambda\in\mathbb{R}}\inf_{W\in\mathcal{W}_0^{x,y}}\mathbb{E}[(W-\lambda)^2-(\lambda-\zeta)^2] \\
&\geq \inf_{W\in\mathcal{W}_0^{x,y}}\mathbb{E}[(W-\lambda^*)^2-(\lambda^*-\zeta)^2] \\
&\geq V_1(x,y;\zeta)-\zeta^2,
\end{aligned}$$

which yields the desired result.

Therefore, we need only to study the value function $V(t,x,y)$.

Lemma 4.3 *The value function V defined in 4.3 has the following properties.*

(1) For any $t\in[0,T)$, $V(t,\cdot,\cdot)$ is convex and continuous in $\mathbb{R}^2$.

(2) For any $t\in[0,T)$, $V(t,x,y)$ is nonincreasing in x and y.

(3) For any $\varrho>0$, $t\in[0,T)$, we have $V(t,x+(1-\mu)\varrho,y-\varrho)\geq V(t,x,y)$, $V(t,x-(1+\beta)\varrho,y-\varrho)\geq V(t,x,y)$.

(4) For any $\varrho>0$, $t\in[0,T)$, we have $V(t,\varrho x,\varrho y)=\varrho^2V(t,x,y)$.

(5) If $x+(1-\mu)y^+-(1+\beta)y^-\geq 0$, then $V(t,x,y)=0$.

Proof. All the results can be easily proved in term of the definition of V and Lemma 4.2.

We define

$$\bar{\varphi} = \{(x, y) \in \mathbb{R}^2 | x + (1-\mu)y^+ - (1-\beta)y^- > 0\}.$$

It is well known that the value function V is a viscosity solution (see Theorem 5.2, Chapter 4, Yong and Zhou) to the following HJB equation

$$\min\{-\varphi_t - \mathcal{L}_0\varphi, \varphi_y - (1-\mu)\varphi_x, (1+\beta)\varphi_x - \varphi_y\} = 0,$$
$$\forall (t, x, y, i) \in [0, T) \times \bar{\varphi} \times \mathbb{S},$$

with the terminal condition

$$\varphi(x, y, T, i) = \frac{(x + (1-\mu)y^+ - (1+\beta)y^-)^\gamma}{\gamma}, 0 < \gamma < 1,$$

where

$$\mathcal{L}_0\varphi(x, y, t, i) = \frac{1}{2}\sigma(i)^2 y^2 \varphi_{yy}(x, y, t, i) + b(i)y\varphi_y(x, y, t, i)$$
$$+ r(i)x\varphi_x(x, y, t, i) + \sum_{j=1}^{l} q_{ij}\varphi(x, y, t, j).$$

Due to the homotheticity of the utility function, we have for any positive constant ξ,

$$\varphi(\xi x, \xi y, t, i) = \xi^\gamma \varphi(x, y, t, i), 0 < \gamma < 1.$$

It is well known that under the assumption $\bar{b} > \bar{r}$, short selling is always suboptimal. Hence, we only need to consider $y > 0$. Then

$$\varphi(x, y, t, i) = y^\gamma \bar{V}(\frac{x}{y}, t, i), 0 < \gamma < 1.$$

So, above HJB equation is turned to

$$\begin{cases} \min\{-\bar{V}_t - \mathcal{L}_1\bar{V}, \gamma\bar{V} - (x+1-\mu)\bar{V}_x, (x+1+\beta)\bar{V}_x - \gamma\bar{V}\} = 0, & \text{in } \Omega, \\ \bar{V}(x, T, i) = \frac{1}{\gamma}(x+1-\mu)^\gamma, & \end{cases}$$

where $\Omega = (-(1+\mu), \infty) \times [0, T) \times \mathbb{S}, 0 < \gamma < 1$, and

$$\begin{aligned}
\mathcal{L}_1 \bar{V}(x,t,i) &= \frac{1}{2}\sigma^2(i) x^2 \bar{V}_{xx}(x,t,i) \\
&\quad - (b(i) - r(i) - \sigma^2(i)(1-\gamma)) x \bar{V}_x(x,t,i) \\
&\quad + \gamma(b(i) - \frac{1}{2}\sigma^2(i)(1-\gamma)) \bar{V}(x,t,i) \\
&\quad + \sum_{j=1}^{l} q_{ij} \bar{V}(x,t,j).
\end{aligned}$$

Further, let

$$w(x,t,i) = \frac{1}{\gamma} \ln(\gamma \bar{V}(x,t,i)).$$

Then $w(x,t,i)$ is governed by

$$\begin{cases}
\min\{-w_t - \mathcal{L}_2 w, (x+1-\mu) - \frac{1}{w_x}, \frac{1}{w_x} - (x+1+\beta)\} = 0, \quad \text{in } \Omega, \\
w(x,T,i) = \ln(x+1-\mu),
\end{cases} \tag{4.4}$$

where

$$\begin{aligned}
\mathcal{L}_2 w(x,t,i) &= \frac{1}{2}\sigma^2(i) x^2 (w_{xx}(x,t,i) \\
&\quad + \gamma w_x^2(x,t,i)) - (b(i) - r(i) - \sigma^2(i)(1-\gamma)) x w_x(x,t,i) \\
&\quad + b(i) - \frac{1}{2}\sigma^2(i)(1-\gamma) + \sum_{j=1}^{l} q_{ij} \frac{1}{\gamma} e^{\gamma(w(x,t,j) - w(x,t,i))}.
\end{aligned}$$

Now we relate this equation to a double obstacle problem. Let

$$v(x,t,i) = w_x(x,t,i).$$

Formally we have

$$\begin{aligned}
\frac{\partial}{\partial x}(\mathcal{L}_2 w(x,t,i)) &= \frac{1}{2}\sigma^2(i) x^2 v_{xx}(x,t,i) \\
&\quad - (b(i) - r(i) - (2-\gamma)\sigma^2(i)) x v_x(x,t,i)
\end{aligned}$$

$$
\begin{aligned}
&- (b(i) - r(i) - (1-\gamma)\sigma^2(i))v(x,t,i) \\
&+ \gamma\sigma^2(i)(x^2 v(x,t,i)v_x(x,t,i) + xv^2(x,t,i)) \\
&+ \sum_{j=1}^{l} q_{ij}(v(x,t,j) - v(x,t,i))v_0(i,j) \\
\triangleq\ & \mathcal{L}v,
\end{aligned}
$$

where $v_0(t,i,j) = \exp\{\gamma(w(x,t,j) - w(x,t,i))\}$.

This inspires us to consider the following double-obstacle problem:

$$
\begin{cases}
\max\{\min\{v_t + \mathcal{L}v, (x+1-\mu)v - 1\}, (x+1+\beta)v - 1\} = 0, \quad \text{in } \Omega, \\
v(x,T,i) = \dfrac{1}{x+1-\mu},
\end{cases}
\tag{4.5}
$$

equivalently,

$$
\begin{cases}
v_t + \mathcal{L}v = 0, & \dfrac{1}{x+1+\beta} < v < \dfrac{1}{x+1-\mu}, \\
v_t + \mathcal{L}v \leq 0, & v = \dfrac{1}{x+1+\beta}, \\
v_t + \mathcal{L}v \geq 0, & v = \dfrac{1}{x+1-\mu}, \\
v(x,T,i) = \dfrac{1}{x+1-\mu},
\end{cases}
$$

in Ω.

We define

$$
\begin{aligned}
\mathbf{SR} &= \{(x,t,i) \in \Omega : v(x,t,i) = \frac{1}{x+1-\mu}\}, \\
\mathbf{BR} &= \{(x,t,i) \in \Omega : v(x,t,i) = \frac{1}{x+1+\beta}\}, \\
\mathbf{NT} &= \{(x,t,i) \in \Omega : \frac{1}{x+1+\beta} < v < \frac{1}{x+1-\mu}\}.
\end{aligned}
$$

4.4 Existence and regularity of solution to problem (4.5)

We aim to prove the existence and regularity of solution to the double obstacle problem. One technical difficulty is that the upper obstacle is infinite on the boundary $x = -(1-\mu)$. To avoid the singularity , we confine ourselves within $\tilde{\Omega} = \{x > x^*, 0 < t < T\}$, where $x^* > -(1-\mu)$ is sufficiently close to $-(1-\mu)$, and the following boundary condition will be imposed on $x = x^*$:

$$v(x^*, t, i) = \frac{1}{x^* + 1 - \mu}, t \in [0, T). \tag{4.6}$$

We will see that (4.6) is indeed true because $x \leq x^*$ is contained in **SR** when $x^* \leq x_{s,\infty}$ defined in next section.

Proposition 4.3 *The double obstacle problem* (4.5) *has a unique solution* $v(x,t) \in W_p^{2,1}(\tilde{\Omega}_N \backslash |x| < \delta)$, *for any* $\delta > 0$, $1 < p < \infty$, *where* $\tilde{\Omega}_N$ *is any bounded set in* $\tilde{\Omega}$. *Moreover,*

$$v(x, t, i) \in C^\infty(\boldsymbol{NT}), \tag{4.7}$$

$$v_t(x, t, i) \geq 0, \tag{4.8}$$

and

$$v(0, t, i) = \frac{1}{1-\mu}, \text{ if } b(i) - r(i) - (1-\gamma)\sigma^2(i) \leq 0,$$

$$v(0, t, i) = \begin{cases} e^{-(b(i)-r(i)-(1-\gamma)\sigma^2(i))(T-t)} \dfrac{1}{1-\mu} & \text{for } t_1 < t \leq T, \\ \dfrac{1}{1+\beta} & \text{for } 0 \leq t \leq t_1, \end{cases}$$

if $b(i) - r(i) - (1-\gamma)\sigma^2(i) > 0$, *where*

$$t_1 = T - \frac{1}{b(i) - r(i) - (1-\gamma)\sigma^2(i)} \ln \frac{1+\beta}{1-\mu}.$$

The main difficulty of the proof lies in the degeneracy of operator $\mathcal{L}$ at $x = 0$. Before providing a proof, we would like to give its sketch. We can deal with the problem in $x^* < x < 0$ and in $x > 0$ independently in order to avoid the degeneracy of the operator $\mathcal{L}$ at $x = 0$, and no boundary value is required on $x = 0$. The standard penalty method can be adopted to show the $W_p^{2,1}$ regularity of solution, and (4.8) can be deduced from maximum principle. Regarding (4.7), we only need to show the smoothness on $\{x = 0\} \cap \mathbf{NT}$.

4.5 The proof of Proposition 4.3

We will only confine our attention to $x^* < x < 0$, and the case of $x > 0$ is similar. By transformation $x = -e^y$ and $u(y,t,i) = v(x,t,i)$, (4.5) and (4.6) become

$$\begin{cases} \min\{\max\{-u_t - \mathcal{L}_y u, u - \dfrac{1}{-e^y + 1 - \mu}\}, u - \dfrac{1}{-e^y + 1 + \beta}\} = 0, \\ u(y, T, i) = \dfrac{1}{-e^y + 1 - \mu}, -\infty < y < y^*, \\ u(y^*, t, i) = \dfrac{1}{-e^{y^*} + 1 - \mu}, 0 \le t < T, \end{cases} \tag{4.9}$$

where $y^* = \ln(-x^*)$, and

$$\begin{aligned} \mathcal{L}_y u(y,t,i) = & \frac{1}{2}\sigma^2 u_{yy}(y,t,i) - (b(i) - r(i) - (\frac{3}{2} - \gamma)\sigma^2(i))u_y(y,t,i) \\ & - (b(i) - r(i) - (1-\gamma)\sigma^2(i))u(y,t,i) \\ & - \gamma\sigma^2(i)e^y u(y,t,i)(u_y(y,t,i) + u(y,t,i)) \\ & + \sum_{j=1}^{l} q_{ij}(u(y,t,j) - u(y,t,i))v_0(i,j). \end{aligned}$$

Lemma 4.4 *Let u_i, $i = 1, 2$, satisfy*

$$\begin{cases} -u_{it} - \mathcal{L}_y u_i = 0, & \dfrac{1}{-e^y + 1 + \beta} < u_i < \dfrac{1}{-e^y + 1 - \mu} \\ -u_{it} - \mathcal{L}_y u_i \geq 0, & u_i = \dfrac{1}{-e^y + 1 + \beta} \\ -u_{it} - \mathcal{L}_y u_i \leq 0, & u_i = \dfrac{1}{-e^y + 1 - \mu} \\ u_i(y, T, i) = \psi_i(y), -\infty < y < y^*, \\ u_i(y^*, t, i) = \dfrac{1}{-e^{y^*} + 1 - \mu}, 0 \leq t < T, \end{cases} \tag{4.10}$$

Assume that

(a) $\psi_i(y^*) = \frac{1}{-e^{y^*}+1-\mu}$;

(b) $\psi_i(y)$ is bounded;

(c) for any $N > 0$, $u_i(y, t, i) \in W_p^{2,1}((-N, y^*) \times [0, T) \times \mathbb{S})$;

(d) $\partial_y u_1$ is bounded.

If $\psi_1(y) \geq \psi_2(y)$, then $u_1(y, t, i) \geq u_2(y, t, i)$.

Proof. Denote

$$\mathcal{N} = \{(y, t, i) : u_1(y, t, i) < u_2(y, t, i), -\infty < y < y^*, 0 \leq t < T, i \in \mathbb{S}\}.$$

Suppose not. Then $\mathcal{N}$ must be a nonempty open set, and

$$u_1 < \frac{1}{-e^y + 1 - \mu}, u_2 > \frac{1}{-e^y + 1 + \beta} \text{ in } \mathcal{N}.$$

It follows

$$-u_{1t} - \mathcal{L}_y u_1 \geq 0, -u_{2t} - \mathcal{L}_y u_2 \leq 0 \text{ in } \mathcal{N}.$$

Let $\bar{w} = u_1 - u_2$, which satisfies

$$\begin{cases} -\bar{w}_t - \bar{\mathcal{L}}_y \bar{w} \geq 0, & \text{in } \mathcal{N}, \\ \bar{w} = 0 & \text{on } \partial_p \mathcal{N}, \end{cases}$$

where $\partial_p\mathcal{N}$ is the parabolic boundary of $\mathcal{N}$. And

$$
\begin{aligned}
&\bar{\mathcal{L}}_y\bar{w}(y,t,i)\\
=&\frac{1}{2}\sigma^2(i)\bar{w}_{yy}(y,t,i)-(b(i)-r(i)-(\frac{3}{2}-\gamma)\sigma^2(i))\bar{w}_y(y,t,i)\\
&-(b(i)-r(i)-(1-\gamma)\sigma^2(i))\bar{w}(y,t,i)\\
&-\gamma\sigma^2(i)e^y[u_2(y,t,i)\bar{w}_y(y,t,i)\\
&+(u_{1y}(y,t,i)+u_1(y,t,i)+u_2(y,t,i))\bar{w}(y,t,i)]\\
&+\sum_{j=1}^{l}q_{ij}(\bar{w}(y,t,j)-\bar{w}(y,t,i))v_0(i,j).
\end{aligned}
$$

Since $\bar{w}$ is bounded and all coefficients in the above equation are bounded as well, applying the maximum principle, we have $\bar{w}\geq 0$ in $\mathcal{N}$, namely $u_1-u_2\geq 0$ in $\mathcal{N}$, which contradicts the definition of $\mathcal{N}$.

Since $(-\infty,y^*)$ is unbounded, we define a finite domain $(-N,y^*)\times[0,T)\times\mathbb{S}$ with $N>0$, namely,

$$
\begin{cases}
\min\{\max\{-u_t^N-\mathcal{L}_yu^N,u^N-\dfrac{1}{-e^y+1-\mu}\},u^N-\dfrac{1}{-e^y+1+\beta}\}=0,\\
u^N(y,T,i)=\dfrac{1}{-e^y+1-\mu},-N<y<y^*,\\
u^N(y^*,t,i)=\dfrac{1}{-e^{y^*}+1-\mu},u^N(-N,t,i)=\dfrac{1}{-e^{-N}+1-\mu},0\leq t<T,
\end{cases}
\tag{4.11}
$$

where a boundary condition on $y=-N$ is imposed.

Lemma 4.5 *For any $N>0$ given, the problem* (4.11) *has a solution $u^N(y,t,i)\in W_p^{2,1}((-N,y^*)\times[0,T)\times\mathbb{S})$, $1<p<+\infty$, and*

$$u_t^N\geq 0,$$

$$|u^N|_{W_p^{2,1}((-\tilde{N},y^*)\times[0,T)\times\mathbb{S})}\leq c,$$

where $\tilde{N}<N$ and c depends only on $\tilde{N}$ but is independent of N. Moreover,

$$|u_y^N|_{L^\infty((-N,y^*)\times[0,T)\times\mathbb{S})}\leq M, \tag{4.12}$$

where M is independent of N.

Proof. Following Friedman (1982), we consider a penalty approximation of the problem (4.11):

$$\begin{cases} -u_t^{N,\varepsilon} - \mathcal{L}_y u^{N,\varepsilon} + \beta_\varepsilon(u^{N,\varepsilon} - \dfrac{1}{-e^y+1+\beta}) + \gamma_\varepsilon(u^{N,\varepsilon} - \dfrac{1}{-e^y+1-\mu}) = 0, \\ u^{N,\varepsilon}(y,T,i) = \dfrac{1}{-e^y+1-\mu}, -N < y < y^*, \\ u^{N,\varepsilon}(y^*,t,i) = \dfrac{1}{-e^{y^*}+1-\mu}, u^{N,\varepsilon}(-N,t,i) = \dfrac{1}{-e^{-N}+1-\mu}, 0 \le t < T, \end{cases} \tag{4.13}$$

where

$$\begin{array}{ll} \beta_\varepsilon(\xi) \le 0, & \gamma_\varepsilon(\xi) \ge 0, \\ \beta_\varepsilon(\xi) = 0 \text{ if } \xi \ge \varepsilon, & \gamma_\varepsilon(\xi) = 0 \text{ if } \xi \le -\varepsilon, \\ \beta_\varepsilon(0) = -c_1, (c_1 > 0), & \gamma_\varepsilon(0) = c_2, (c_2 > 0), \\ \beta'_\varepsilon(\xi) \ge 0, & \gamma'_\varepsilon(\xi) \ge 0, \\ \beta''_\varepsilon(\xi) \le 0, & \gamma''_\varepsilon(\xi) \ge 0, \end{array}$$

with constants c_1 and c_2 to be chosen later. For any $\varepsilon > 0$ given, it is not hard to show by the fixed point theorem that the above semi-linear problem has a unique solution $u^{N,\varepsilon}(y,t,i) \in W_p^{2,1}((-N,y^*) \times [0,T) \times \mathbb{S})$.

Next we want to prove

$$\frac{1}{-e^y+1+\beta} \le u^{N,\varepsilon} \le \frac{1}{-e^y+1-\mu}, \text{ in } (-N,y^*) \times [0,T) \times \mathbb{S}. \tag{4.14}$$

Let

$$g(y) = \frac{1}{-e^y+1-\mu}.$$

We have

$$(-\frac{\partial}{\partial t} - \mathcal{L}_y)g(y) + \beta_\varepsilon(g(y) - \frac{1}{-e^y+1+\beta}) + \gamma_\varepsilon(g(y) - \frac{1}{-e^y+1-\mu})$$

$$
\begin{aligned}
&= \frac{1-\mu}{(-e^y+1-\mu)^3}[-(b-r)e^y + (b-r-(1-\gamma)\sigma^2)(1-\mu)] \\
&\quad + \beta_\varepsilon(\frac{\beta+\mu}{(-e^y+1+\beta)(-e^y+1-\mu)}) + \gamma_\varepsilon(0) \\
&\quad - \sum_{j=1}^{l} q_{ij}(\frac{1}{-e^y+1-\mu} - \frac{1}{-e^y+1-\mu}) \exp\{\gamma[w(x,t,j) - w(x,t,i)]\} \\
&= \frac{1-\mu}{(-e^y+1-\mu)^3}[-(b-r)e^y + (b-r-(1-\gamma)\sigma^2)(1-\mu)] \\
&\quad + \beta_\varepsilon(\frac{\beta+\mu}{(-e^y+1+\beta)(-e^y+1-\mu)}) + \gamma_\varepsilon(0) \\
&\geq -\frac{(1-\mu)^2(1-\gamma)\sigma^2}{(x^*+1-\mu)^3} + \beta_\varepsilon(\frac{\beta+\mu}{(-e^y+1+\beta)(-e^y+1-\mu)}) + \gamma_\varepsilon(0).
\end{aligned}
$$

When ε is sufficiently small, $\beta_\varepsilon(\frac{\beta+\mu}{(-e^y+1+\beta)(-e^y+1-\mu)}) = 0$. Take

$$
c_2 = \gamma_\varepsilon(0) = \frac{(1-\mu)^2(1-\gamma)\sigma^2}{(x^*+1-\mu)^3}.
$$

Then $\frac{1}{-e^y+1-\mu}$ is a supersolution, and thus

$$
u^{N,\varepsilon} \leq \frac{1}{-e^y+1-\mu}, \text{ in } (-N, y^*) \times [0,T) \times \mathbb{S}.
$$

In the same way, we can choose

$$
c_1 = -\beta_\varepsilon(0) = \frac{|b-r-(1-\gamma)\sigma^2|(1+\beta)^2}{(x^*+1+\beta)^3},
$$

such that $\frac{1}{-e^y+1+\beta}$ is a subsolution. So, we get (4.14).

Then, we obtain

$$
-c_1 \leq \beta_\varepsilon(u^{N,\varepsilon} - \frac{1}{-e^y+1+\beta}) \leq 0, \quad 0 \leq \gamma_\varepsilon(u^{N,\varepsilon} - \frac{1}{-e^y+1-\mu}) \leq c_2 \quad .
$$

We then deduce from (4.13) that $|u^{N,\varepsilon}|_{W_p^{2,1}((-N,y^*)\times[0,T)\times\mathbb{S})} \leq c$, where c is independent of ε because both c_1 and c_2 are independent of ε. Using $W_p^{2,1}$ interior estimate, we further have for any $\tilde{N} < N$,

$$
|u^{N,\varepsilon}|_{W_p^{2,1}((-\tilde{N},y^*)\times[0,T)\times\mathbb{S})} \leq c,
$$

where c depends on $\tilde{N}$ but is independent of ε and N.

Due to (4.14), we infer $u_t^{N,\varepsilon}|_{t=T} \geq 0$. Differentiating the equation in (4.13) w.r.t. t, we get an equation that $u_t^{N,\varepsilon}$ satisfies. Applying the maximum principle, we deduce

$$u_t^{N,\varepsilon} \geq 0,$$

which gives $u_t^N \geq 0$ by letting $\varepsilon \to 0$.

Since the bound of $u^{N,\varepsilon}$ and the C^2 norm of terminal value are independent of N and ε, we obtain by the $W_p^{2,1}$ interior estimate,

$$|u^{N,\varepsilon}|_{W_p^{2,1}((-y-1,y)\times[0,T)\times\mathbb{S})} \leq M,$$

for any $y \leq 0$, where M is independent of N and ε. Applying the imbedding theorem, we have

$$|u_y^{N,\varepsilon}|_{L^\infty((-y-1,y)\times[0,T)\times\mathbb{S})} \leq M.$$

Since y is arbitrary, it follows

$$|u_y^{N,\varepsilon}|_{L^\infty((-N,y^*)\times[0,T)\times\mathbb{S})} \leq M,$$

which yields (4.12) by letting $\varepsilon \to 0$. The proof is complete.

We can get following Lemma from Dai and Yi [7]

Lemma 4.6 *The problem* (4.9) *has a unique solution* $u(y,t,i)$,

$$u(y,t,i) \in W_p^{2,1}((-N,y^*)\times[0,T)\times\mathbb{S}), \text{ for any } N>0,$$
$$u_t \geq 0,$$
$$|u_y|_{L^\infty((-\infty,y^*)\times[0,T)\times\mathbb{S})} \leq M.$$

4.6 Characterization of free boundaries

A double obstacle problem usually gives rise to two free boundaries. Firstly, we will show that each free boundary can be expressed as a single-value function of time t. Then, we will examine the properties of the free boundaries.

Now, we introduce a lemma which will play a critical role in the existence proof of free boundaries.

Lemma 4.7 *Let $v(x,t,i)$ be the solution to the double obstacle problem* (4.5). *Then*

$$v_x + v^2 \leq 0, \ \ in\ \Omega.$$

Proof. It is clear that $v_x + v^2 = 0$ in **BR** and **SR**. So, the rest is to show $v_x + v^2 \leq 0$ in **NT**. Denote

$$p(x,t,i) = v_x(x,t,i), \qquad\qquad \bar{p}(x,t,i) = v^2(x,t,i).$$

Then

$$\begin{aligned}
&-p_t - \frac{1}{2}\sigma^2 x^2 p_{xx} + (b - r - (3-\gamma)\sigma^2)xp_x + (2b - 2r - (3-2\gamma)\sigma^2)p \\
&= \gamma\sigma^2(4xvv_x + x^2v_x^2 + x^2vv_{xx} + v^2) \\
&\quad + \sum_{j=1}^{l} q_{ij}(v_x(x,t,j) - v_x(x,t,i))v_0(i,j) \\
&\quad + \sum_{j=1}^{l} q_{ij}(v(x,t,j) - v(x,t,i))^2\gamma v_0(i,j),
\end{aligned}$$

and

$$\begin{aligned}
&-\bar{p}_t - \frac{1}{2}\sigma^2 x^2 \bar{p}_{xx} + (b - r - (2-\gamma)\sigma^2)x\bar{p}_x + (2b - 2r - (2-2\gamma)\sigma^2)\bar{p} \\
&= -\sigma^2x^2v_x^2 + \gamma\sigma^2(2x^2v^2v_x + 2xv^3)
\end{aligned}$$

$$+\sum_{j=1}^{l} q_{ij}2v(x,t,i)(v(x,t,j)-v(x,t,i))v_0(i,j),$$

in **NT**. Let $H(x,t,i) = v_x(x,t,i) + v^2(x,t,i) = p(x,t,i) + \bar{p}(x,t,i)$. Then

$$\begin{aligned}
& - H_t - \frac{1}{2}\sigma^2 x^2 H_{xx} + (b - r - (3-\gamma)\sigma^2 - \gamma\sigma^2 xv)xH_x \\
& + (2b - 2r - (3-2\gamma)\sigma^2 - 2\gamma\sigma^2 xv)H \\
= & -(1-\gamma)\sigma^2(xv_x + v)^2 \\
& + \sum_{j=1}^{l} q_{ij}(v_x(x,t,j) - v_x(x,t,i))v_0(i,j) \\
& + \sum_{j=1}^{l} q_{ij}(v(x,t,j) - v(x,t,i))^2\gamma v_0(i,j) \\
& + \sum_{j=1}^{l} q_{ij}2v(x,t,i)(v(x,t,j)-v(x,t,i))v_0(i,j).
\end{aligned}$$

We get $-(1-\gamma)\sigma^2(xv_x+v)^2 < 0$, since $0 < \gamma < 1$. Then we can find $v(x,t,i)$, which $q_{ij} \neq 0$ is small enough, to obtain

$$\begin{aligned}
& - H_t - \frac{1}{2}\sigma^2 x^2 H_{xx} + (b - r - (3-\gamma)\sigma^2 - \gamma\sigma^2 xv)xH_x \\
& + (2b - 2r - (3-2\gamma)\sigma^2 - 2\gamma\sigma^2 xv)H \\
\leq & \ 0.
\end{aligned}$$

Obviously $H(x,t,i) = 0$ on the parabolic boundary of **NT**. By the maximum principle, we have $v_x + v^2 \leq 0,$ in Ω.

Theorem 4.1 *There are two monotonically increasing functions $x_s(t,i)$: $[0,T] \times \mathbb{S} \to [-(1-\mu), +\infty)$ and $x_b(t,i) : [0,T] \times \mathbb{S} \to [-(1-\mu), +\infty)$, such that*

$$\boldsymbol{SR} = \{(x,t,i) \in \Omega : x \leq x_s(t), t \in [0,T)\}, i \in \mathbb{S},$$

and

$$\boldsymbol{BR} = \{(x,t,i) \in \Omega : x \geq x_b(t), t \in [0,T)\}, i \in \mathbb{S}.$$

Moreover,

$$x_s(t,i) < x_b(t,i) \textit{ for all } t \in [0,T).$$

Proof. Notice

$$\frac{\partial}{\partial x}(v - \frac{1}{x+1+\beta}) = v_x + \frac{1}{(x+1+\beta)^2} \leq v_x + v^2 \leq 0.$$

As a consequence, if $(x_1,t,i) \in \mathbf{BR}$, i.e. $v(x_1,t,i) = \frac{1}{x_1+1+\beta}$, then for any $x_2 > x_1$,

$$0 \leq v(x_2,t,i) - \frac{1}{x_2+1+\beta} \leq v(x_1,t,i) - \frac{1}{x_1+1+\beta} = 0,$$

from which we infer $v(x_2,t,i) = \frac{1}{x_2+1+\beta}$, i.e. $(x_2,t,i) \in \mathbf{BR}$. The existence of $x_b(t,i)$ (as a single-value function) follows.

We consider $\bar{v}(x,t,i) = (x+1-\mu)^2 v(x,t,i)$. Notice
$\mathbf{SR} = \{(x,t,i) \in \Omega : \bar{v}(x,t,i) = x+1-\mu\}$ and

$$\begin{aligned}
&\frac{\partial}{\partial x}(\bar{v} - (x+1-\mu)) \\
&= \frac{\partial}{\partial x}((x+1-\mu)^2(v - \frac{1}{x+1-\mu})) \\
&= -[(x+1-\mu)v - 1]^2 + (x+1-\mu)^2(v_x + v^2) \\
&\leq 0,
\end{aligned}$$

since $v_x + v^2 \leq 0$. If $(x_1,t,i) \in \mathbf{SR}$, $\bar{v}(x,t,i) = \frac{1}{x+1-\mu}$, for any $x_2 < x_1$, then

$$0 = v(x_1,t,i) - \frac{1}{x_1+1-\mu} \leq v(x_2,t,i) - \frac{2}{x_2+1-\mu} \leq 0.$$

Then the existence of $x_s(t,i)$ follows.

The monotonicity of $x_s(t,i)$ and $x_b(t,i)$ can be similarly deduced by virtue of

$$\frac{\partial}{\partial t}(v-(x+1+\lambda))=\frac{\partial}{\partial t}(v-(x+1-\mu))=v_t\geq 0.$$

Then we can get $x_s(t,i)<x_b(t,i)$, since $\mathbf{SR}\cap\mathbf{BR}=\emptyset$.

In finance, $x_s(t,i)$ and $x_b(t,i)$ stand for the optimal selling and buying boundaries, respectively. We can get their following behaviors from Dai and Yi [7].

Theorem 4.2 *Let $x_s(t,i)$ be the optimal selling boundary. Then*

(i)

$$x_s(t,i)\leq(1-\mu)x_M,$$

where $x_M(i)=-\frac{b(i)-r(i)-(1-\gamma)\sigma^2(i)}{b(i)-r(i)}$, *and*

$$x_s(T^{-},i)\triangleq\lim_{t\to T^-}x_s(t,i)=(1-\mu)x_{M,i};$$

(ii)

$$\begin{aligned}
&x_s(t,i)\equiv 0, && \text{when } b(i)-r(i)-(1-\gamma)\sigma^2(i)=0,\\
&x_s(t,i)>0, && \text{when } b(i)-r(i)-(1-\gamma)\sigma^2(i)<0,\\
&x_s(t,i)<0, && \text{when } b(i)-r(i)-(1-\gamma)\sigma^2(i)>0;
\end{aligned}$$

(iii) $x_s(t,i)$ is continuous. Moreover, $x_s(t,i)\in C^\infty([0,T)\times\mathbb{S})$.

We assume $T>\frac{1}{b(i)-r(i)}\ln(\frac{1+\beta}{1-\mu})$, then

Theorem 4.3 *Let $x_b(t,i)$ be the optimal buying boundary. Denote*

$$t_0=T-\frac{1}{b(i)-r(i)}\ln(\frac{1+\beta}{1-\mu})>0.$$

Then

(i)

$$x_b(t,i) \geq (1+\beta)x_M(i),$$

where $x_M(i) = -\frac{b(i)-r(i)-(1-\gamma)\sigma^2(i)}{b(i)-r(i)}$, *and*

$$x_b(t,i) = \infty, \text{ if and only if } t_0 \leq t \leq T;$$

(ii)

$$x_b(t,i) > 0, \text{ when } b(i) - r(i) - (1-\gamma)\sigma^2(i) \leq 0,$$

and

$$x_b(t,i) > 0 \text{ for } t \in (t_1, T), x_b(t_1, i) = 0, x_b(t,i) < 0 \text{ for } t \in (0, t_1)$$
$$\text{when } b(i) - r(i) - (1-\gamma)\sigma^2(i) > 0,$$

where $t_1 = T - \frac{1}{b(i)-r(i)-(1-\gamma)\sigma^2(i)} \ln \frac{1+\beta}{1-\mu}$.

(iii) $x_b(t,i)$ *is continuous.*

4.7 Equivalence

In this section, we will show the equivalence between the double obstacle problem (4.4) and the original problem (4.5).

Theorem 4.4 *Let* $v(x,t,i)$ *be the solution to the double-obstacle problem* (4.4). *Define*

$$w(x,t,i) = A(t,i) + \ln(x_s(t,i) + 1 - \mu) + \int_{x_s(t,i)}^{x} v(\xi, t, i) d\xi, \qquad (4.15)$$

where

$A(t,i)$

$$= \int_t^T \frac{rx^2 + (b(i)+r(i))(1-\mu)x + (b(i) - \frac{1}{2}\sigma^2(i)(1-\gamma))(1-\mu)^2}{(x+1-\mu)^2}|_{x=x_s(\tau),i} d\tau$$

$$+ \sum_{j=1}^{l} q_{ij} \frac{1}{\gamma^2(w_t(x,t,j) - w_t(x,t,i))} \exp\{\gamma(w(x,t,j) - w(x,t,i))\}.$$

Then $w(x,t,i)$ is the solution to the problem (4.5). Moreover,

$$w(x,t,i) \in C^{2,1}(\Omega \backslash F),$$

where F is the intersection of the free boundaries and the line $x = 0$, i.e., $F = \{(0,t,i)|x_s(t,i) = 0 \text{ or } x_b(t,i) = 0, t \in [0,T), i \in \mathbb{S}\}$.

Remark 4.1 *We exclude the set F on which some partial derivatives of $v(x,t,i)$ or $w(x,t,i)$ are discontinuous because of the degeneracy of the differential operator $\mathcal{L}$ or $\mathcal{L}_2$. Then*

$$F = \begin{cases} \emptyset, & \text{if } b(i) - r(i) - (1-\gamma)\sigma^2(i) < 0, \\ x = 0, & \text{if } b(i) - r(i) - (1-\gamma)\sigma^2(i) = 0, \\ (0, t_1) & \text{if } b(i) - r(i) - (1-\gamma)\sigma^2(i) > 0, \end{cases}$$

where t_1 is defined in Theorem (4.3).

Proof. Since $v(x,t,i) = \frac{1}{x+1-\mu}$ for $x \le x_s(t,i)$, we can get

$$w(x,t,i) = A(t,i) + \ln(x+1-\mu), x \le x_s(t,i). \tag{4.16}$$

So $w(x,t,i)$ satisfies the terminal condition. Therefore, to prove that $w(x,t,i)$ is the solution to the problem (4.5), it suffices to show

$$F = \begin{cases} -w_t - \mathcal{L}_2 w \ge 0, & \text{in } \mathbf{SR} \text{ and } \mathbf{BR}, \\ -w_t - \mathcal{L}_2 w = 0, & \text{in } \mathbf{NT}. \end{cases} \tag{4.17}$$

Observe

$$w_x(x,t,i) = v(x,t,i). \tag{4.18}$$

According to the definition of $A(t)$, we claim

$$-w_t - \mathcal{L}_2 w = 0, \text{ on } x = x_s(t,i).$$

Because of (4.18),

$$\begin{aligned}
&\mathcal{L}_2 w|_{x=x_s(t,i)} \\
&= \frac{1}{2}\sigma^2(i)x^2(w_{xx}(x,t,i) + \gamma w_x^2(x,t,i)) \\
&\quad - (b(i) - r(i) - \sigma^2(i)(1-\gamma))x w_x(x,t,i) \\
&\quad + b(i) - \frac{1}{2}\sigma^2(i)(1-\gamma) + \sum_{j=1}^{l} q_{ij}\frac{1}{\gamma}e^{\gamma(w(x,t,j)-w(x,t,i))}|_{x=x_s(t,i)} \\
&= \frac{1}{2}\sigma^2(i)x^2(v_x(x,t,i) + \gamma v^2(x,t,)) - (b(i) - r(i) - (1-\gamma)\sigma^2(i))x v(x,t,i) \\
&\quad - b(i) - \frac{1}{2}(1-\gamma)\sigma^2(i) + \sum_{j=1}^{l} q_{ij}\frac{1}{\gamma}e^{\gamma(w(0,t,j)-w(0,t,i))}|_{x=x_s(t,i)} \\
&= \frac{r(i)x_s^2(t,i) + (b(i)+r(i))(1-\mu)x_s(t,i) + (b(i) - \frac{1}{2}\sigma^2(i)(1-\gamma))(1-\mu)^2}{(x_s(t,i)+1-\mu)^2} \\
&\quad + \sum_{j=1}^{l} q_{ij}\frac{1}{\gamma}e^{\gamma(w(x_s(t,i),t,j)-w(x_s(t,i),t,i))} \\
&= -A'(t,i) = -w_t(x_s(t,i),t,i),
\end{aligned}$$

since (4.16).

So, we can easily get

$$\begin{aligned}
&\frac{\partial}{\partial x}(-w_t - \mathcal{L}_2 w) \le 0, \text{ in } \mathbf{SR}, \\
&\frac{\partial}{\partial x}(-w_t - \mathcal{L}_2 w) = 0, \text{ in } \mathbf{NT}, \\
&\frac{\partial}{\partial x}(-w_t - \mathcal{L}_2 w) \ge 0, \text{ in } \mathbf{BR}.
\end{aligned}$$

Then we deduce (4.17).

By Proposition 4.3, $v \in C^{1,0}(\Omega \backslash F)$ and then $w \in C^{2,0}(\Omega \backslash F)$. We will show $w_t \in C(\Omega \backslash F)$. According (4.15),

$$
\begin{aligned}
& w_t(x,t,i) \\
&= A'(t,i) + \frac{x_s'(t,i)}{x_s(t,i)+1-\mu} - v(x_s(t,i),t,i)x_s'(t,i) + \int_{x_s(t,i)}^{x} v_t(\xi,t,i)d\xi \\
&= A'(t,i) + \int_{x_s(t,i)}^{\max(\min(x,x_b(t,i)),x_s(t,i))} v_t(\xi,t,i)d\xi \\
&= A'(t,i) - \int_{x_s(t,i)}^{\max(\min(x,x_b(t,i)),x_s(t,i))} \mathcal{L}v(\xi,t,i)d\xi \\
&= A'(t,i) - \int_{x_s(t,i)}^{\max(\min(x,x_b(t,i)),x_s(t,i))} d\mathcal{L}_2 w \\
&= A'(t,i) + \mathcal{L}_2 w|_{x_s(t,i)} - \mathcal{L}_2 w|_{\max(\min(x,x_b(t,i)),x_s(t,i))} \\
&= -\mathcal{L}_2 w|_{\max(\min(x,x_b(t,i)),x_s(t,i))},
\end{aligned}
$$

which implies the continuity of $w_t(x,t,i)$. The proof is complete.

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