Aspects of effective potentials in gauge theories.

Wild, Philip

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Philip Wild

Department of Physics, Swansea University
Singleton Park, Swansea, SA2 8PP, United Kingdom

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Abstract

In this thesis we present several results concerning effective potentials in gauge theories. Firstly we calculate the effective potential up to one loop of weakly coupled large N $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ on $S^3 \times S^1$, at low temperature and near-critical chemical potentials. We show that the theory contains a metastable phase for chemical potentials below a certain temperature-dependent value. We then give a review of Vilkovisky's formulation of the effective action, and use this method to calculate the effective potential of scalar electrodynamics with background VEVs for the scalar field and gauge field. Finally we describe the classical vacuum structure of a class of $\mathcal{N} = 1$ gauge theories, which may be used to calculate the exact effective potential using the Dijkgraaf-Vafa conjecture.

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Contents

1 Introduction

This work describes results in three areas, all concerning calculation of effective potentials in various different gauge theories. The largest part is Section 2, in which we give the calculation of the one-loop effective potential of weakly coupled large N $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$, at small non-zero temperature and with three chemical potentials associated to a maximal Abelian subalgebra of the $\mathfrak{su}(4)$ R-symmetry generators of the theory; this section reproduces results which have previously been given in [1]. Classically the theory is unstable whenever the size of any chemical potential exceeds a maximum value of β_0^{-1}, where β_0 is the radius of the three-sphere on which the theory lives. We therefore calculate the effective potential in the case where one, two or three chemical potentials are exactly at the critical value, with the remaining chemical potentials set to zero. We also consider the case where the chemical potentials are above their classically critical value, and show that there exists a range of values for which the theory is in a metastable phase whose lifetime diverges in the large N limit. This complements the work of Yamada and Yaffe [8] who showed that a similar metastable phase exists at high temperature and weak coupling.

While the work given in Section 2 was being carried out, it was believed at one point that it may be necessary to use the reparameterisation-invariant method of Vilkovisky to calculate the effective potential. This turned out not to be the case,[1] but this method is interesting in its own right. In Section 3 we therefore give a review of the method, in which emphasis is placed on the geometrical meaning of the quantities involved, followed by a calculation of the Vilkovisky effective potential of scalar electrodynamics with non-zero VEVs for both the scalar field and gauge field. The calculation of the unregularised potential is performed using both a generalised R_ξ gauge and unitary gauge, and the results are shown to agree. We then give an approximate expression for the dimensionally regularised effective potential which includes the exact divergent part, which is seen to be consistent with renormalisability. The work described in this section is merely the first step towards clarifying contradictory work in the literature, as well as considering scalar electrodynamics at finite temperature and chemical potential.

Finally, Section 4 derives the classical vacuum structure of certain $\mathcal{N} = 1$

[1] See footnote 5 of [1] for an explanation of why the effective potential calculated here is independent of ξ. This would no longer be the case if the chemical potentials were away from their critical values, so that it may be necessary to apply the method given in Section 3 in that case.

supersymmetric gauge theories, which, via the Dijkgraaf-Vafa conjecture, may be used to calculate the exact low energy effective potentials of certain generalisations of theories which were previously studied in [47].

2 The phase diagram of large N $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$

2.1 Introduction to Section 2

The large N limit of gauge theories [4] has been an area of interest in recent years for several reasons. Not only is it believed that they may share many qualitative properties with real $N = 3$ QCD [24], [25], but also that supersymmetric gauge theories may provide a window into the physics of string theory via the AdS/CFT correspondence [9], which we briefly review below. One principle difference between finite N gauge theory and the limit $N \to \infty$ is that the latter may exhibit phase transitions even in finite volume; this is because the large N limit acts as a kind of thermodynamic limit, as is seen in, for example, the Gross-Witten transition in the large N limit of two-dimensional $U(N)$ lattice gauge theory [5]. Therefore an interesting set of questions concerns the phase structure of such a theory. In this work, which reproduces and expands upon results first given in [1], we investigate a particular corner of the phase diagram of weakly coupled $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$, with gauge group $SU(N)$, in the limit of large N. In particular we will calculate the one-loop effective potential of the theory at low temperature, with chemical potentials near to their critical value (explained below), and it will be shown that there exists a metastable phase for a range of chemical potentials above the critical value and below a temperature-dependent limit. Our work complements that of Yamada and Yaffe [8], in which the weakly coupled theory was shown to have a metastable phase at high temperature, and it is reasonable to assume that the metastable phase found here is the continuation to low temperature of that found by Yamada and Yaffe; in fact we will further show that, in the case of a single non-zero chemical potential, the instability line varies continuously between the near-linear high temperature line described in [8] and that first described in [1]. The resulting phase diagram will be compared to that of the theory at strong coupling, using results which follow from the conjectured correspondence between the strongly coupled gauge theory and a supergravity theory on $AdS_5 \times S^5$, and will display many qualitative similarities.

The remainder of Section 2 is organised as follows: in Section 2.2 we give the action of the theory which will be considered, and define all relevant quantities. Section 2.3 introduces the three chemical potentials μ_p, $p = 1, 2, 3$, associated to a maximal Abelian $U(1)^3$ subgroup of the $SU(4)$ R-symmetry

of the theory, and explains how they are incorporated into the theory by an appropriate modification of the Lagrangian density. In Section 2.4 we describe the method which will be used to calculate the one-loop effective potential of the theory, and derive a number of formulae which will be used in the calculation. Section 2.5 presents the derivation of the effective potential at zero temperature, and shows that the theory at zero temperature has flat directions in its effective potential when at least one chemical potential is at its critical value. In Section 2.6 we then calculate the additional contribution to the effective potential at small, finite temperature, and show that a metastable phase exists when the chemical potentials are within a certain range. Finally, in Sections 2.7 and 2.8 we discuss the results and compare the known facts about the phase diagram at weak 't Hooft coupling to those at strong coupling, and suggest further questions which arise from these results.

We conclude this introduction by briefly reviewing some known facts about the phase diagram of $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$, both in the strongly coupled and weakly coupled cases, as well as that of real-world QCD.

2.1.1 Real QCD

Before turning our attention to the phase diagram of $\mathcal{N} = 4$ super Yang-Mills theory, we first describe some known features of the phase diagram of real QCD in the $T - \mu_B$ plane, where μ_B is the chemical potential associated with the Baryon number of the theory. The facts given here are explained in considerably more detail in [6], [7], and references therein.

At length scales larger than about 0.5fm the theory is confined. This is because the gluons' self-interaction causes the field lines emanating from a quark to form narrow "fluxtubes" whose energy per unit length is approximately constant. Each such fluxtube must begin and end on a quark or antiquark and therefore the only stable finite-energy combinations are those which are colour-neutral. As such it is difficult to make reliable predictions using perturbation theory, and it is instead necessary to use lattice techniques, or simplified models such as the bag model [6], in which quarks are assumed to move freely within a spherical hadron with a constant energy density of approximately $(200\text{MeV})^4$ (in natural units). On the other hand, at short distance scales the theory becomes asymptotically free, and is more accurately described as a theory of weakly interacting quarks and gluons. Thus at low temperatures the ground state of the theory is in a confining phase whose dominant degrees of freedom are hadrons, whereas at high temperatures the theory is in a "quark-gluon plasma" phase. In fact, lattice

calculations of order parameters such as the chiral condensate [6] show that, when the non-zero intrinsic masses of the quarks are taken into account, the phases are separated not by a singular phase transition but rather a smooth crossover.

At non-zero chemical potential lattice methods unfortunately become unreliable. This is because of the sign problem: as will be seen in Section 2.3, the Euclidean action in the presence of a chemical potential is no longer real, which has the effect that the Monte Carlo method may no longer give results that converge to the correct partition function in the limit of increasing lattice volume. Thus it is necessary to use calculations based on simplified models, such as the bag model. At zero temperature and small μ_B we expect that the ground state of the theory is just the vacuum, until μ_B reaches the nucleon rest mass minus the binding energy per nucleon of nuclear matter; at this point (around $\mu_B = 926\text{MeV}$) there is a first-order phase transition as the equilibrium baryon density n_B increases from zero to its average value of 0.15fm^{-3} in nuclear matter. By continuity this transition is expected to persist at non-zero temperature, and is indicated by the lowest of the three phase lines in Figure 1.

As μ_B increases further we expect another transition, to a deconfined phase where quarks are the dominant degrees of freedom, called quark matter; in the bag model this occurs when the volume of a single baryon is equal to the inverse of the mean baryon density, so that the individual baryons necessarily overlap. Different models predict varying values of μ_B at which this occurs, in the range $1100 - 1500\text{MeV}$. This phase line is expected to continue into the region of non-zero temperature, as indicated in Figure 1; however, since the deconfinement/confinement phase transition is given by a crossover rather than a well-defined line at zero temperature, it is expected that this line will end at a critical point, though due to the aforementioned difficulties the precise location of this point is not known. At still higher values of μ_B the ground state of the theory is a "colour-flavour-locked" (CFL) condensate, whose properties include colour superconductivity.

Although the $\mathcal{N} = 4$ theory differs from real QCD in many ways, we will see below that there are nonetheless some similarities between the phase diagrams of the two theories; in particular the deconfinement/confinement phase transition is still present in the supersymmetric theory at both strong and weak coupling. Note that, whereas real QCD has scale dependence due to both the non-zero intrinsic masses of the quarks and the β-function of the gauge coupling, the $\mathcal{N} = 4$ theory is conformal; however, the theory at finite volume may nonetheless depend non-trivially on temperature due to the scale-dependence introduced by the ratio between the inverse temperature

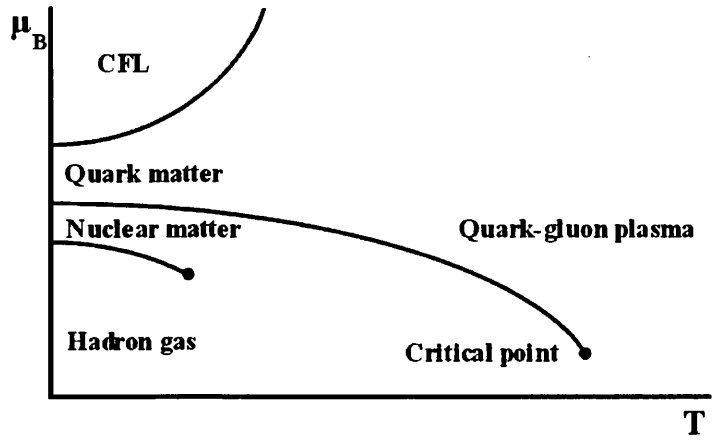

Figure 1: A schematic image of the proposed phase diagram of QCD.

and the radius of the spatial three-sphere. As mentioned above, it is also only in the large N limit that the theory with finite volume may exhibit a true phase transition; however it is argued in [25] that the 't Hooft expansion in powers of $1/N$ may already be a good approximation at $N = 3$.

2.1.2 Strong coupling and the AdS/CFT correspondence

The AdS/CFT conjecture [9] states that Type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super Yang-Mills theory, with gauge group $SU(N)$, on the boundary of AdS_5, with string coupling proportional to the Yang-Mills coupling g^2. In the limit $N \to \infty$ with the 't Hooft coupling $\lambda = g^2 N$ large but fixed the string theory is weakly coupled and therefore well approximated by classical supergravity. In [10] Witten gave a precise recipe which relates quantities in a boundary conformal theory to those in its supergravity dual, which we now outline. First note that any theory on $AdS_5 \times S^5$ can be considered a theory on AdS_5, by treating the Kaluza-Klein harmonics as distinct fields. Suppose now that we are given a compactification of AdS_5 including a boundary; for example, Euclidean AdS_5 can be described as the open unit ball B_5 in $\mathbb{R}^5$, with coordinates x^i, $i = 0 \ldots 4$, $x^2 \equiv \sum_i x^{i\,2} < 1$, with metric given by

$$\mathrm{d}s^2 = \frac{4 \sum_i \mathrm{d}x^{i\,2}}{(1 - x^2)^2};$$

an evident compactification is to include the boundary S^4. However, the metric given above is singular as $x^2 \to 1$, and so does not extend to a metric

on S^4. We may rectify this by choosing a function f, positive on B_5, which has a first order zero as $x \to 1$, and defining a new metric $\mathrm{d}s'^2$on B_5 by

$$\mathrm{d}s'^2 \equiv f^2 \mathrm{d}s^2.$$

This restricts to a metric on S^4. To each field ϕ in the theory on AdS_5 we may associate a local operator $\mathcal{O}$. Witten's proposal states that, for each classical field configuration ϕ_0 on the S^4 boundary of AdS_5, the generating functional of a theory on S^4 is given by

$$\left\langle \mathrm{e}^{\sum_{\text{fields}} \phi_0 \mathcal{O}} \right\rangle_{\text{CFT}} = \mathcal{Z}(\phi_0) \tag{1}$$

where the sum over fields in the left hand side includes an integration over S^4, and where $\mathcal{Z}(\phi_0)$ is the path integral of the theory on AdS_5 over all field configurations whose boundary values are given by ϕ_0. In the limit of classical supergravity the partition function $\mathcal{Z}(\phi_0)$ is simply given by

$$\mathcal{Z}(\phi_0) = \sum \mathrm{e}^{-S(\phi)} \tag{2}$$

where $S(\phi)$ is the classical supergravity action, and the sum is over all field configurations ϕ on B_5 which extend ϕ_0 and satisfy the classical equations of motion. Note that the theory on S^4 defined by the generating functionals (1) is a *conformal* field theory, since there is no natural choice of f used to define the metric on S^4; in particular, given any real function σ on the closure $\bar{B}_5$ of B_5, the replacement $f \mapsto \mathrm{e}^{\sigma} f$ also has a first order zero as $x \to 1$, and this changes the metric $\mathrm{d}s'^2$ by a conformal transformation.

In [10], [12], it was shown using an ansatz similar to (1) and the approximation (2) that the strongly coupled $\mathcal{N} = 4$ theory on $S^3 \times S^1$ exhibits a phase transformation in the large N limit: suppose that we can find Einstein manifolds X_i such that there exist compactifications $\bar{X}_i$, with boundaries $\bar{X}_i - X_i$ homeomorphic to $S^3 \times S^1$, that the metrics on X_i have a double pole on the boundary, and such that when multiplied by a function f^2 as before the metrics on $\bar{X}_i$ restrict to the conformal structure on $S^3 \times S^1$. We then use the ansatz

$$\mathcal{Z}_{\text{CFT}} = \sum_i \mathrm{e}^{-S(X_i)}. \tag{3}$$

In the limit of large N the supergravity action $S(X_i)$ is proportional to a positive power of N; define S_i by

$$S(X_i) \approx N^{\gamma} S_i \quad (N \to \infty)$$

From (3) we see that, in the limit of large N, the sum over i is dominated by that X_i for which S_i is smallest. Therefore if $S_i = S_j$ for some value of the parameters of the theory there may exist a "flop" from one phase to another. [10] gives two such X_i, first discovered by Hawking and Page [11], for which this flop exists. The first is a quotient of AdS_5 by a discrete subgroup isomorphic to $\mathbb{Z}$: AdS_5 may be defined the set of points in $\mathbb{R}^6$, coordinates u, v, x^i, $i = 1 \ldots 4$ satisfying

$$uv - \sum_i x^{i\,2} = b^2 \tag{4}$$

with $u, v > 0$, with metric $\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + \sum_i \mathrm{d}x^{i\,2}$. Let λ be a fixed positive real number and identify the point (u, v, x) with $(\lambda^n u, \lambda^{-n} v, x)$, $n \in \mathbb{Z}$; X_1 is the quotient of AdS_5 under this equivalence relation. By choosing v to lie in the interval $[1, \lambda]$ with ends identified, and solving u in terms of the remaining coordinates, it can be seen that X_1 is topologically $\mathbb{R}^4 \times S^1$. As $x \to \infty$ the right hand side of (4) becomes insignificant, and in order to describe the manifold at infinity we identify (u, v, x) satisfying $uv - \sum_i x^{i\,2} = 0$ with (ru, rv, rx) for each positive real number r. Choosing r such that $x^2 = 1$ we see that the coordinates x^i define a point in S^3, and v again defines a point in S^1, while we can uniquely solve for u in terms of v and x^i. So the boundary of X_1 can be identified with $S^3 \times S^1$, as required. The second spacetime X_2 given in [11] is the AdS black hole spacetime with mass m; it is homeomorphic to $S^3 \times \mathbb{R}^2$, with metric

$$\mathrm{d}s^2 = \left(\frac{r^2}{b^2} + 1 - \frac{wm}{r^2}\right)\mathrm{d}t^2 + \left(\frac{r^2}{b^2} + 1 - \frac{wm}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2 \tag{5}$$

where b is the radius of curvature of the AdS space, $\mathrm{d}\Omega_3^2$ is the volume form on S^3, and

$$w \equiv \frac{8G}{3\pi}$$

where G is Newton's constant. The spacetime is restricted to $r > r+$, where r_+ is the largest root of

$$\frac{r^2}{b^2} + 1 - \frac{wm}{r^2} = 0.$$

t is an angular variable, and the metric above is smooth and complete only if the period of t is

$$\beta = \frac{4\pi b^2 r_+}{4r_+^2 + 2b^2}.$$

The relevant action is the Einstein-Hilbert action:

$$S = -\frac{1}{8\pi G}\int \mathrm{d}^5x\,\sqrt{g}\left(\frac{1}{2}R + \Lambda\right) + \text{boundary term}$$

where R is the Ricci scalar; solving the equations of motion gives $\Lambda = -3R/10$, so the first term in the action is simply proportional to the scalar curvature multiplied by the volume of the spacetime. In fact the volumes of the two spacetimes described above are both infinite, but a finite answer for their difference may be obtained by regularising the action. To this end we first define $r^2 \equiv \sum_i x^{i\,2}$ and $t \equiv b\,(\log v - \log\,(r^2 + b^2))$ on X_1, so that the metric becomes

$$\mathrm{d}s^2 = \left(\frac{r^2}{b^2} + 1\right)\mathrm{d}t^2 + \left(\frac{r^2}{b^2} + 1\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2; \tag{6}$$

we then impose a finite cutoff r_c on the radial integrations. The period $\beta \equiv b\log\lambda$ of t is fixed by the requirement that the geometry of the two surfaces $r = r_c$ of X_1 and X_2 should coincide, i.e.

$$\beta\sqrt{\frac{r_c^2}{b^2} + 1} = \beta_0\sqrt{\frac{r_c^2}{b^2} + 1 - \frac{wm}{r_c^2}}.$$

The curvature of both spacetimes is given by $R = -20/b^2$, and from (6) and (5) it may be seen that $\sqrt{g}$ is simply the square root of the determinant of the metric on S^3, multiplied by r^3. In the limit $r_c \to \infty$ the boundary terms in the action vanish, and therefore the difference $S(X_2) - S(X_1)$ is given by

$$\begin{aligned}
&\frac{\mathrm{Vol}\,S^3}{2\pi G b^2}\lim_{r_c\to\infty}\left(\beta_0\int_{r_+}^{r_c}\mathrm{d}r\,r^3 - \beta_0\sqrt{1 - \frac{wmb^2}{r_c^2(r_c^2+1)}}\int_0^{r_c}\mathrm{d}r\,r^3\right)\\
&= \frac{\pi^2 r_+^3}{4G(b^2 + 2r_+^2)}\left(\frac{wmb^2}{r_+^2} - 2r_+^2\right) = \frac{\pi^2 r_+^3(b^2 - r_+^2)}{4G(b^2 + 2r_+^2)}
\end{aligned}$$

where wmb^2 has been eliminated in the last equality using the quadratic formula and the definition of r_+. This quantity is positive when β is sufficiently

large, and negative when β is sufficiently small, so that a "flop" phase transition indeed occurs. In [12] it is argued that the Hawking-Page transition described here is dual to the deconfinement/confinement phase transition of large N gauge theory, which will be briefly reviewed in the next subsection.

The above picture has been generalised to the case where the $\mathcal{N} = 4$ gauge theory on $S^3 \times S^1$ has chemical potentials, which are dual to angular velocities along the S^5 on the supergravity side [13], [14], [15]. For example, when all three chemical potentials are equal and fixed, the system undergoes a phase transition between the spinning thermal AdS geometry and the Reissner-Nordstrom-AdS black hole. This gives the transition line in the bottom left corner of Figure 2. In [14] it was also shown that the theory becomes unstable at a critical temperature above the Hawking-Page transition; this line is also shown in Figure 2. It is not known whether there exists a new phase of the theory above this line, or whether the line is instead a genuine boundary above which no theory exists. It has also been recently argued [16] that there exists a metastable phase above a fixed temperature, denoted by the dotted line in Figure 2.

On the other hand, directly analysing the strongly coupled gauge theory is difficult, and the deconfinement/confinement phase transition is therefore not well understood. However, it is possible to investigate the theory numerically using lattice techniques, which suggest that the transition exists and is first order for $N \geq 3$, see for example [26], [27].

2.1.3 Weak coupling

We now summarise previous results concerning the phase diagram of weakly coupled $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$ in the limit of large N. In [17], [18] it was shown that the theory with zero coupling and zero chemical potentials undergoes a first order deconfinement/confinement transition, much like the Hawking-Page transition seen at strong coupling. Hawking and Reall [19] considered the theory at non-zero chemical potential and found that the behaviour was qualitatively different from the case of strong coupling, however in that work they ignored the requirement that physical states should be gauge invariant. Yamada and Yaffe [8] found that when this requirement was taken into account the phase diagram of the free theory contained a deconfinement/confinement transition similar to that at strong coupling. There are two equivalent methods to evaluate the partition function: one is to take a path integral approach, as used in [8], and this is the approach which will be used in the later sections of this work. An alternative method, as used in [20], [21], involves exploiting the correspondence

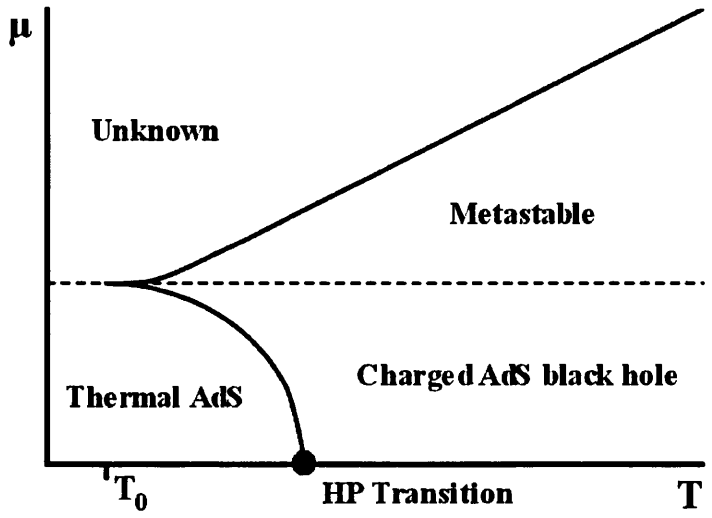

Figure 2: Known features of the phase diagram of strongly coupled $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$, which follow from the AdS/CFT conjecture. The dot at zero chemical potential is the Hawking-Page transition described above.

between states and operators in a conformal field theory and directly counting the gauge invariant multi-trace operators in the theory. We briefly outline this method and its application to studying the deconfinement/confinement phase transition. By decomposing each field into spherical harmonics on S^3 and treating each mode as a distinct particle species, the theory may be regarded as quantum mechanics with an infinite number of degrees of freedom, which in the free theory are simply decoupled harmonic oscillators. Under the state-operator correspondence the physical states correspond to products of traces of products of creation operators. Suppose that the oscillators are indexed by $i \in \mathbb{N}$, such that the i^{th} oscillator has energy E_i and charge q_{pi} corresponding to each chemical potential μ_p. The thermodynamics of the theory may be encoded in the "single-particle partition function" z: defining $x \equiv \mathrm{e}^{-\beta}$, where β is the inverse temperature, it is defined by

$$z(x) \equiv \sum_i x^{E_i - \sum_p \mu_p q_{pi}}.$$

The calculation of the complete partition function $\mathcal{Z}$ proceeds by first calculating the single trace partition function $\mathcal{Z}_{ST}$, which is defined to be the sum over all states corresponding to a single trace of a product of creation operators. In order to calculate $\mathcal{Z}_{ST}$ in terms of z it is necessary to carefully avoid overcounting; a naïve formula might read

$$\mathcal{Z}_{ST}(x) = \sum_{n=1}^{\infty} z(x)^n$$

but this would be incorrect, since traces which differ by cyclic permutation of their arguments are the same (for finite N there are also additional relations between the traces coming from Lie algebra identities, but this can be ignored in the large N limit). The method given in [20] to correctly account for the cyclicity of the trace uses Polya's theorem, and the result is

$$\mathcal{Z}_{ST} = -\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \log\,(1 - z(x^n))$$

where φ is Euler's totient function. From the above one may calculate the exact partition function in the limit of zero coupling, using the fact that single trace states act as identical Bosonic or Fermionic particles. The result is [20]

$$\log \mathcal{Z} = -\sum_{n=1}^{\infty} \log\,(1 - z_B(x^n) + (-)^n z_F(x^n))$$

where z_B and z_F are the Bosonic and Fermionic single-particle partition functions, defined in the obvious way.

In the case where all fields transform in the adjoint representation R of $SU(N)$, the partition function is given by

$$\mathcal{Z} = \prod_i \sum_{n_i=0}^{\infty} x^{\sum_i n_i E_i} \times \rho(n) \tag{7}$$

where $\rho(n)$ is the number of gauge singlets in the representation obtained by taking the product of the (anti-) symmetrised n_i^{th} products of the adjoint representation:

$$\rho(n) = \text{number of singlets in } \left(\bigotimes_{\text{Bosonic i}} \bigotimes_{S}^{n_i} R \otimes \bigotimes_{\text{Fermionic j}} \bigotimes_{A}^{n_j} R \right).$$

It is shown in [20] that (7) is equal to

$$\mathcal{Z} = \int_{SU(N)} \mathrm{d}U\, \mathrm{e}^{-S_{\text{eff}}}$$

where $\mathrm{d}U$ is the Haar measure on $SU(N)$, and the effective action S_{eff} is given by

$$S_{\text{eff}}(U) \equiv -\sum_n \frac{1}{n} \left(z_B(x^n) - (-)^n z_F(x^n) \right) \chi_R(U^n)$$

where $\chi_R(U)$ is the character of $U \in SU(N)$ in the representation R. Using the explicit basis given in Section 2.4 it can be seen that, when R is the adjoint representation of $SU(N)$,

$$\chi_R(U) = \operatorname{Tr} U \operatorname{Tr} U^\dagger - 1.$$

The partition function of the theory may therefore be described by a matrix model, involving the integral over a single matrix $U \in SU(N)$. Yamada and Yaffe, who derived the same expression for S_{eff} using path integral methods, give the following results for the single-particle partition functions[2]:

$$z_B(x) = z_S(x) + z_V(x)$$

where

$$\begin{aligned} z_S(x) &\equiv \frac{x + x^2}{(1-x)^3} \left(x^{\mu_1} + x^{-\mu_1} + x^{\mu_2} + x^{-\mu_2} + x^{\mu_3} + x^{-\mu_3} \right) \\ z_V(x) &\equiv \frac{6x^2 - 2x^3}{(1-x)^3} \end{aligned}$$

and

$$z_F(x) = \frac{2x^{3/2}}{(1-x)^3} \left(x^{\frac{\mu_1}{2}} + x^{-\frac{\mu_1}{2}} \right) \left(x^{\frac{\mu_2}{2}} + x^{-\frac{\mu_2}{2}} \right) \left(x^{\frac{\mu_3}{2}} + x^{-\frac{\mu_3}{2}} \right).$$

In order to determine the thermodynamics of the theory in the limit of large N, one may describe the matrix U by an eigenvalue density $\rho(\theta)$, defined such that the number of eigenvalues of U of the form $\mathrm{e}^{i\vartheta}$ with $\theta \leq \vartheta \leq \theta + \delta\theta$ is given by $N\rho(\theta)\delta\theta$ in the limit $\delta\theta \to 0$. After modifying the definition of S_{eff} to include the Van der Monde determinant arising from the Haar measure, one finds that the effective action in terms of ρ is

[2] In the relevant section of [8], Yamada and Yaffe took the theory to live on S^3 of radius $\beta_0 = 1$, whereas later we will leave β_0 arbitrary. Since the theory is conformal all physical quantities should only depend on the ratio β_0/β, so no generality is lost by the choice $\beta_0 = 1$.

$$S_{\text{eff}}(\rho) = N^2 \sum_{n=1}^{\infty} V_n(x)|\rho_n|^2$$

where ρ_n is the n^{th} Fourier coefficient of ρ:

$$\rho_n \equiv \int_0^{2\pi} \mathrm{d}\theta\, \rho(\theta) \mathrm{e}^{in\theta}$$

and V_n is defined by

$$V_n(x) \equiv \frac{1}{n}\left(1 - z_B(x^n) + (-)^n z_F(x^n)\right).$$

Each ρ_n behaves like an independent real variable under the influence of a potential V_n; if V_n is positive then the effective action is minimised by taking $\rho_n = 0$, and if this is true for each n then ρ is simply the uniform distribution, i.e. the theory is in a maximally disordered phase. This will be the case if, for each n,

$$z_B(x^n) - (-)^n z_F(x^n) < 1. \tag{8}$$

Note that, when the chemical potentials are smaller than their critical values of 1, the single particle partition functions z_B and z_F increase monotonically with x. Thus the statement that (8) holds for every n is equivalent to the statement that it holds for $n = 1$; moreover, since z_B and z_F approach zero at small temperature and are unbounded from above at high temperature, there exists a unique temperature at which

$$z_B(x) + z_F(x) = 1; \tag{9}$$

this equation therefore defines a phase line, below which the theory is in the confining phase with the eigenvalues of U distributed uniformly. On the other hand, at high temperatures V_n is negative for sufficiently small N, and so the distribution $\rho(\theta)$ approaches a delta function. The precise shape of the line (9) in $\mu - T$ space depends on the ratio of the chemical potentials μ_1, μ_2 and μ_3; some examples are given in Figure 2 of [8]. The phase line will be schematically shown in Figure 3 below. Note that the effective action is not unstable even if some of the V_n are negative, since the fact that ρ is non-negative means that each ρ_n is bounded. In the large N limit the Polyakov loop is simply given by ρ_1, and is therefore an order parameter for the deconfinement/confinement phase transition; at low temperature it is zero, whereas at high temperature it is non-zero, and since it transforms

non-trivially under the centre $\mathbb{Z}_N$ of the gauge group, this symmetry is spontaneously broken at high temperature.

Yamada and Yaffe also considered the theory at small, non-zero coupling and high temperature, in the region where $(\beta/\beta_0)^2$ and $\beta^2\mu^2$ are both of the order λ. At high temperature the non-static heavy modes decouple, and may be integrated out leaving a low-energy effective theory. In particular they considered the "electric" effective field theory, valid on length scales which are large compared to β, obtained by integrating out non-static fluctuations. This theory depends only on α and φ, the constant modes of the time component of the gauge field and the scalars respectively (the other fields, being vector- or spinor-valued on S^3, have no constant modes). They found that the theory at high temperature is in a metastable phase provided that the largest of the chemical potentials is less than a critical value μ_c, given by

$$\mu_c = \sqrt{\beta_0^{-2} + \lambda T^2}$$

where $T = 1/\beta$. Thus, like in the case of strong coupling, there is an instability line in the phase diagram which rises approximately linearly with temperature. This is also shown in Figure 3. In this work we will investigate the near critical region at low temperature, complementing the work of Yamada and Yaffe; in particular we will show that there exists a metastable phase at low temperatures, with an instability phase line which meets the deconfinement/confinement phase line at $T = 0$, $\mu = \beta_0^{-1}$, similar to the low temperature instability phase line in the strong-coupling diagram. We conjecture that this is the continuation to low temperature of the instability phase line found by Yamada and Yaffe. This region of the phase diagram was previously considered by Harmark and Orselli [23], who showed that in the limit $\beta \to \infty$, $\mu \to \beta_0^{-1}$ with $\beta(\mu - \beta_0^{-1})$ fixed, the theory reduces to various quantum mechanical sectors for well known systems such as the Heisenberg spin chain.

In [22], Basu and Wadia considered the partition function of the $\mathcal{N} = 4$ theory with fixed R-charges, instead of fixed chemical potentials. In Section 2.6 of [8] Yamada and Yaffe include a discussion of their results, and explain that they are not inconsistent with their own.

For more results along these lines, see also [33], [34].

2.2 The $\mathcal{N} = 4$ theory

We consider $\mathcal{N} = 4$ $SU(N)$ Super Yang-Mills theory on $S^3 \times S^1$, with radii β_0 and $\beta/2\pi$ respectively. The Euclidean action of the theory is (for background

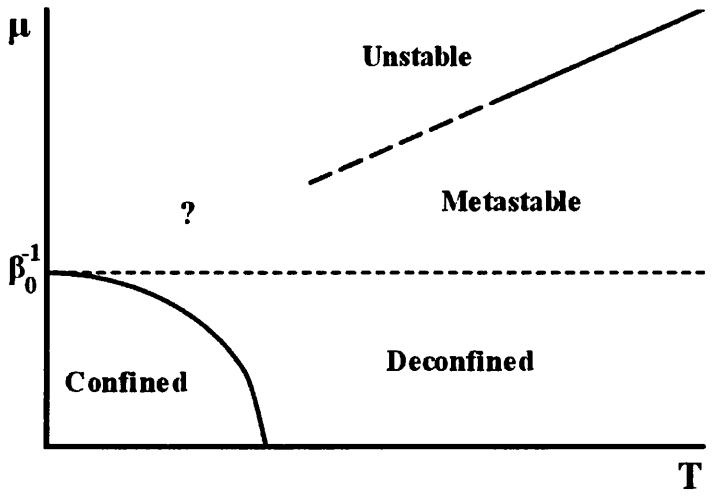

Figure 3: Previously known features of the phase diagram of weakly coupled $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$. In the remainder of Section 2 we will investigate the corner at low temperature near the critical chemical potential.

see e.g. [2], [3] and references therein)

$$\frac{1}{g^2}\int \sqrt{h}\,\mathrm{d}^4x\,\mathrm{Tr}\left(\frac{1}{2}F^{\mu\nu}F_{\mu\nu}+\sum_a D^\mu\phi_a D_\mu\phi_a+\frac{R}{6}\phi^2-\frac{1}{2}\sum_{ab}[\phi_a,\phi_b]^2\right.$$
$$\left.\frac{1}{g^2}\int \sqrt{h}\,\mathrm{d}^4x\,\mathrm{Tr}\left(\frac{1}{2}F^{\mu\nu}F_{\mu\nu}+\sum_{\substack{a\\A}} D^\mu\phi_a D_\mu\phi_a+\frac{R}{6}\phi^2-\frac{1}{2}\sum_{\substack{ab\\aAB}}[\phi_a,\phi_b]^2\right)\right. . \quad (10)$$

In the above ϕ_a, $a = 1\ldots6$ are real scalar fields, ψ_A, $A = 1\ldots4$ are four-component Majorana Fermions

$$\psi_A = (\lambda_{A\alpha}, \bar{\lambda}_A^{\dot{\alpha}})$$

with all matter fields transforming in the adjoint representation of the gauge group; Tr is the trace in the fundamental representation of the Lie algebra $\mathfrak{su}(N)$.

D_μ is the gauge-covariant and coordinate-covariant derivative:

$$\begin{aligned} D_\mu\phi_a &\equiv \partial_\mu\phi_a + [A_\mu, \phi_a] \\ D_\mu\psi_A &\equiv \partial_\mu\psi_A + [A_\mu, \psi_A] + \tilde{\Gamma}_\mu\psi_A. \end{aligned}$$

Here $\tilde{\Gamma}_\mu$ is the spin connection: given an orthonormal frame ($e_i = e_i^\mu \partial_\mu$), $h_{\mu\nu} e_i^\mu e_j^\nu = \eta_{ij}$, it is given by

$$\tilde{\Gamma}_\mu = \Omega_{\mu\, ij} \Gamma^{ij}$$

where Ω is the connection two-form, which is fixed by the requirement that the torsion tensor vanishes:

$$\begin{aligned} \Omega_{\mu\, ij} &= -\Omega_{\mu\, ji} \\ e_i^\mu \omega_\mu^j &= \delta_i^j \\ \mathrm{d}\omega^i + \Omega^i{}_j \wedge \omega^j &= 0 \end{aligned}$$

and the gamma matrices Γ^{ij} are given by

$$\begin{aligned} \Gamma^{ij} &\equiv \frac{1}{8}[\gamma^i, \gamma^j] \\ \{\gamma^i, \gamma^j\} &= 2\eta^{ij} \\ \gamma^\mu &= e_i^\mu \gamma^i. \end{aligned}$$

Note that S^3 is simply connected and parallelizable [28] and therefore possesses a unique spin structure. S^1 possesses two spin structures, corresponding to periodic or anti-periodic boundary conditions for the Fermions. In thermal field theory the path integral over fields with these boundary conditions gives the partition function $\mathrm{Tr}\, \mathrm{e}^{-\beta H}$ respectively with or without an additional factor of $(-)^F$, and so we are interested in the case with anti-periodic boundary conditions (this is also the only case which can be extended to the AdS Schwarzschild black hole spacetime, and therefore it is only with this choice of boundary condition that we expect the partition function to contain a phase transition dual to the Hawking-Page transition [10]).

The quantities Γ^{aAB} are defined by

$$\Gamma^{aAB} = \begin{cases} \alpha^p_{AB} & \text{for } a = 2p-1 \\ i\beta^p_{AB}\gamma^5 & \text{for } a = 2p \end{cases}$$

where α^p and β^p satisfy

$$\begin{aligned} \{\alpha^p, \alpha^q\} = \{\beta^p, \beta^q\} &= -2\delta^{pq} \\ [\alpha^p, \beta^q] &= 0 \end{aligned}$$

and may be given in terms of the Pauli sigma matrices as

$$\alpha^1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \quad \alpha^2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \quad \alpha^3 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}$$
$$\beta^1 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \quad \beta^2 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} \quad \beta^3 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}.$$

Finally, R denotes the Ricci scalar. On S^3 of radius β_0, with metric induced from that of $\mathbb{R}^4$, we have

$$R = \frac{6}{\beta_0^2}.$$

The theory contains an $SU(4)$ R-symmetry, under which the Fermions transform in the fundamental representation and the scalars transform in the $SO(6)$ representation, which will be explained shortly. It is also conformally invariant: the action is invariant under the Weyl rescaling

$$\begin{aligned} h_{\mu\nu} &\mapsto \mathrm{e}^{2\alpha} h_{\mu\nu} \\ A_\mu &\mapsto A_\mu \\ \phi_a &\mapsto \mathrm{e}^{-\alpha} \phi_a \\ \psi_A &\mapsto \mathrm{e}^{-\frac{3}{2}\alpha} \psi_A. \end{aligned}$$

Note that we can identify the cylinder $S^3 \times \mathbb{R}$ with Euclidean space with a point removed, $\mathbb{R}^4 - \{0\}$: if $(x,t) \in S^3 \times \mathbb{R}$ then (x,r) are polar coordinates on $\mathbb{R}^4$, where $r \equiv \mathrm{e}^t$. Under this identification, the metric on $\mathbb{R}^4$ is related to that of $S^3 \times \mathbb{R}$ by a Weyl rescaling:

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{R}^4} &= \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2 \\ &= \mathrm{e}^{2t}(\mathrm{d}t^2 + \mathrm{d}\Omega_3^2) = \mathrm{e}^{2t} \mathrm{d}s^2_{S^3 \times \mathbb{R}} \end{aligned}$$

where $\mathrm{d}\Omega_3^2$ is the volume element on S^3. Thus the time translation on $S^3 \times \mathbb{R}$ can be identified with the dilatation on $\mathbb{R}^4$, and quantization of the theory on $S^3 \times \mathbb{R}$ is equivalent to radial quantization of the theory in flat space.

2.3 R-charges and chemical potentials

As explained in [8], it is natural in quantum field theory to introduce chemical potentials associated to the generators of a maximal Abelian subgroup $H \subseteq G$ of a symmetry group; this is because any trace of the form

$$\mathcal{Z}(\beta, g) \equiv \mathrm{Tr}\,\left(\mathrm{e}^{-\beta H} U(g)\right),$$

with $g \in G$ and $U(g)$ the corresponding operator on Hilbert space, is equal to $\mathcal{Z}(\beta, h)$ for some $h \in H$. Therefore for $G = SU(4)$ we consider a subgroup isomorphic to $U(1)^3$, consisting of matrices of the form $\mathrm{diag}(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}, \mathrm{e}^{i\theta_3}, \mathrm{e}^{i\theta_4})$, with $\sum_a \theta_a = 2n\pi$, $n \in \mathbb{Z}$. In the fundamental representation this subgroup is generated by the space of diagonal, traceless, pure imaginary matrices. We therefore choose the three generators to be given by

$$\begin{aligned} Q_1 &= \frac{i}{2}\mathrm{diag}(1, 1, -1, -1) \\ Q_2 &= \frac{i}{2}\mathrm{diag}(1, -1, 1, -1) \\ Q_3 &= \frac{i}{2}\mathrm{diag}(1, -1, -1, 1). \end{aligned} \tag{11}$$

In addition to Fermions transforming in the fundamental representation of the R-symmetry there are also six real scalars transforming in the representation $SO(6)$ of $SU(4)$. This representation can be understood by first introducing six *complex* scalars $\Phi_{ij} = -\Phi_{ji}$, $i, j = 1 \ldots 4$, which transform in the antisymmetric representation **6** of $SU(4)$:

$$\Phi_{ij} \mapsto (g \Phi g^T)_{ij}.$$

We also define $\Phi^{ij} = \frac{1}{2}\epsilon^{ijkl}\Phi_{kl}$, which transforms like the complex conjugate of Φ_{ij}. We may therefore impose the reality condition $\Phi^{ij} = \Phi^*_{ij}$, which leaves three complex degrees of freedom; defining the real scalars ϕ_a by[3]

$$\begin{aligned} \Phi_{12} &\equiv \phi_1 + i\phi_2 \\ \Phi_{13} &\equiv \phi_3 + i\phi_4 \\ \Phi_{14} &\equiv \phi_5 + i\phi_6 \end{aligned}$$

gives

$$\Phi = \begin{pmatrix} 0 & \phi_1 + i\phi_2 & \phi_3 + i\phi_4 & \phi_5 + i\phi_6 \\ -\phi_1 - i\phi_2 & 0 & \phi_5 - i\phi_6 & -\phi_3 + i\phi_4 \\ -\phi_3 - i\phi_4 & -\phi_5 + i\phi_6 & 0 & \phi_1 - i\phi_2 \\ -\phi_5 - i\phi_6 & \phi_3 - i\phi_4 & -\phi_1 + i\phi_2 & 0 \end{pmatrix}.$$

[3]Since the fields are valued in the Lie algebra of $SU(N)$, for example, ϕ_1 and ϕ_2 should be understood to be the Hermitian and anti-Hermitian parts of Φ_{12}.

It follows that $\det \Phi = \phi^4 \equiv (\sum_a \phi_a \phi_a)^2$ is invariant, so that the action of g on ϕ is in $O(6)$; furthermore since this action is continuous it follows that the image of $SU(4)$ is connected and is therefore contained in $SO(6)$. It is easily verified that the generator $\sum_p \mu_p Q_p$ acts on the real vector $\phi = (\phi_1, \ldots, \phi_6)^T$ as follows:

$$\sum_p \mu_p Q_p^{(6)} = \begin{pmatrix} 0 & -\mu_1 & 0 & 0 & 0 & 0 \\ \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mu_2 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu_3 \\ 0 & 0 & 0 & 0 & \mu_3 & 0 \end{pmatrix}$$

so that the three charges $Q_p^{(6)}$ generate rotations in the (ϕ_{2p-1}, ϕ_{2p}) planes.

The quantity of interest is the grand canonical partition function, defined by

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H + \beta \mu_i Q_i} \tag{12}$$

where H is the Hamiltonian of the theory and Q_i are the conserved charges associated with each chemical potential. In general, when deriving a path integral formula for the grand canonical partition function, it is necessary to derive expressions for the charges Q^i in terms of the fields and their conjugate momenta. However, as pointed out by Yamada and Yaffe [8], when the charges are generators of a group which acts on the matter fields of a gauge theory this is unnecessary; instead it is sufficient to calculate the partition function of the canonical ensemble, with constant imaginary background field values for the time components of the gauge field. Here we make their argument explicit. Suppose that a subgroup of the symmetry group G is generated by an element X of the Lie algebra of G, and that the theory contains matter fields $\phi^{(i)a}$ transforming in representations $R^{(i)}$ of G; here a is an additional index, ranging over the dimension of $R^{(i)}$. Let us denote by $X^{(i)}$ the generator X in the representation $R^{(i)}$, so that $\delta\phi^{(i)} = \epsilon X^{(i)} \phi^{(i)}$ under an infinitesimal G transformation. The conserved charge associated with X is given by

$$Q = \int \mathrm{d}^3x \, \frac{\partial \mathcal{L}}{\partial \dot{\phi}^{(i)a}} X^{(i)a}{}_b \phi^{(i)b} = \int \mathrm{d}^3x \, p_{(i)a} X^{(i)a}{}_b \phi^{(i)b} \tag{13}$$

where $p_{(i)a}$ is the momentum conjugate to $\phi^{(i)a}$ and the integral is over spatial coordinates. In the absence of chemical potentials the partition function is

given by the path integral of the Euclidean theory, with time on a circle of circumference β:

$$\begin{aligned} \mathrm{Tr}\, \mathrm{e}^{-\beta H} &= \int \mathcal{D}[...]\, \mathrm{e}^{-S_E} \\ S_E &\equiv \int \mathrm{d}^4x\, \mathcal{L}_E. \end{aligned}$$

The key observation is that the time derivative of any matter field in a gauge theory only ever occurs as part of a covariant derivative. Let us therefore write

$$\mathcal{L} = F(\partial_0\phi^{(i)} + A_0^{(i)}\phi^{(i)}) = F(D_0\phi)$$

where the function F implicitly depends on all the fields and their spatial derivatives. The momentum conjugate to $\phi^{(i)a}$ is

$$p_{(i)a} = F_{,(i)a}(D_0\phi).$$

There are two distinct cases to consider: firstly, the case where the function $F_{,(i)a}$ can be inverted to give the $D_0\phi^{(i)a}$ in terms of the $p_{(i)a}$ (this is the case when $\phi^{(i)}$ is a scalar field). Then $\partial_0\phi^{(i)} = G^{(i)}(p_i) - A_0^{(i)}\phi^{(i)}$ for some function G and the Hamiltonian density of the theory is (ignoring terms arising from the time derivatives of fields other than $\phi^{(i)}$ which are irrelevant to this argument)

$$H = p_{(i)a}\left(G^{(i)a}(p_i) - A_0^{(i)a}{}_b\phi^{(i)b}\right) - F(G(p)) \tag{14}$$

Now consider the theory with a background value μX for A_0. The Lagrangian density is now

$$\mathcal{L} = F(D_0\phi^{(i)} + \mu X^{(i)}\phi^{(i)})$$

and the relation between the time derivative of a matter field and its conjugate momentum becomes $\partial_0\phi^{(i)} = G^{(i)}(p_i) - A_0^{(i)}\phi^{(i)} - \mu X^{(i)}\phi^{(i)}$. Thus the path integral of the Euclidean action gives the partition function of a theory with a Hamiltonian density given by

$$H' = p_{(i)a}\left(G^{(i)a}(p_i) - A_0^{(i)a}{}_b\phi^{(i)b} - \mu X^{(i)a}{}_b\phi^{(i)b}\right) - F(G(p)) = H - \mu q$$

with H given by (14) and q given by the integrand of (13); in other words it gives the grand canonical partition function (12), as required.

The second case is where F is linear in some $D_0\phi^{(i)}$, as is the case when $\phi^{(i)}$ is a spinor field. Write $F(D_0\phi) = B_{(i)a}D_0\phi^{(i)a} + C$, where B and C are again implicitly dependent on the fields and their spatial derivatives. $B_{(i)a}$ is the momentum conjugate to $\phi^{(i)a}$, and the Hamiltonian density (again ignoring irrelevant corrections) is

$$H = -p_{(i)a}A_0^{(i)a}{}_b\phi^{(i)b} - C. \tag{15}$$

Including a background value μX for A_0 as before leaves $p_{(i)a}$ unchanged, and gives rise to a modified Hamiltonian density

$$H' = -p_{(i)a}\left(A_0^{(i)a}{}_b\phi^{(i)b} + \mu X^{(i)a}{}_b\phi^{(i)b}\right) - C = H - \mu q$$

with H as in (15) and q as before. Note that in both cases the Euclidean action is derived from the Lorentzian action by the replacement $\partial_0\phi \mapsto -i\partial_0\phi$, so that the chemical potentials may be implemented by modifying the Euclidean theory by the replacement

$$D_0\phi^{(i)} \mapsto D_0\phi^{(i)} + i\mu X^{(i)}\phi^{(i)}.$$

This completes the argument.

Finally, then, the Euclidean Lagrangian density of the theory including the chemical potential terms is given by, using (10),

$$\begin{aligned}\mathcal{L}_E = \frac{1}{g^2}\mathrm{Tr}\Bigg(&\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \sum_p (D_\mu\phi_{2p-1} - i\delta_{\mu 0}\mu_p\phi_{2p})^2 \\ &+ \sum_p (D_\mu\phi_{2p} + i\delta_{\mu 0}\mu_p\phi_{2p-1})^2 + \frac{1}{\beta_0^2}\phi^2 - \frac{1}{2}\sum_{ab}[\phi_a,\phi_b]^2 \\ &+ i\sum_A \bar{\psi}_A(\gamma^\mu D_\mu - \tilde{\mu}_A\gamma^0\gamma^5)\psi_A - \sum_{pAB}\bar{\psi}_A[\alpha^p_{AB}\phi_{2p-1} + i\beta^p_{AB}\gamma^5\phi_{2p},\psi_B]\Bigg)\end{aligned} \tag{16}$$

where we have defined $\tilde{\mu_A}$ as follows (compare to (11)):

$$\begin{aligned}\tilde{\mu}_1 &= \frac{1}{2}(\mu_1 + \mu_2 + \mu_3) \\ \tilde{\mu}_2 &= \frac{1}{2}(\mu_1 - \mu_2 - \mu_3) \\ \tilde{\mu}_3 &= \frac{1}{2}(-\mu_1 + \mu_2 - \mu_3) \\ \tilde{\mu}_4 &= \frac{1}{2}(-\mu_1 - \mu_2 + \mu_3).\end{aligned}$$

Some comments on the properties of this action follow: firstly note that the Euclidean action is not real for non-zero chemical potentials. Note also that the terms quadratic in the scalar fields ϕ_{2p} and ϕ_{2p-1} are of the form

$$(\beta_0^{-2} - \mu_p^2)\phi^2$$

so that the theory has a classical instability when $|\mu_p| > \beta_0^{-1}$ for some p; in this case the grand canonical partition function becomes ill-defined. The term involving β_0 arises only in the case of finite volume, and so it is necessary to consider the theory quantized on S^3 if we wish to introduce a non-zero chemical potential.

In what follows we will consider the Wilsonian effective potential of the theory, that is, we will expand all fields in terms of Matsubara modes on S^1 and spherical harmonics on S^3 and then calculate the action density as a function of the light degrees of freedom after the heavy degrees of freedom have been integrated out. In the low temperature limit which we consider here, i.e. the limit $\beta \to \infty$, only the modes which are almost massless contribute to the partition function. The cases of interest are therefore those where some of the chemical potentials μ_p are near to their critical values of β_0^{-1}, so that the corresponding pairs of scalars ϕ_{2p} and ϕ_{2p-1} are massless at tree level. We therefore consider three cases:

$$\begin{array}{lll} \text{(i)} & \mu_1 \approx \beta_0^{-1}, & \mu_2 = \mu_3 = 0; \\ \text{(ii)} & \mu_1 \approx \mu_2 \approx \beta_0^{-1}, & \mu_3 = 0 \\ \text{(iii)} & \mu_1 \approx \mu_2 \approx \mu_3 \approx \beta_0^{-1}. & \end{array}$$

We introduce background fields corresponding to each of the light degrees of freedom as follows:

$$\begin{aligned} \varphi_a &\equiv \frac{1}{\mathrm{Vol}\, S^3 \times S^1} \int_{S^3 \times S^1} \phi_a \\ \alpha &\equiv \frac{1}{\mathrm{Vol}\, S^3 \times S^1} \int_{S^3 \times S^1} A_0; \end{aligned}$$

the volume of $S^3 \times S^1$ is given by $2\pi^2 \beta \beta_0^3$. Note that the light modes are necessarily constant on $S^3 \times S^1$, and the scalars and the time component of the gauge field are the only fields which admit non-zero constant background values.

The full quantum effective potential is given by an expansion in powers of the 't Hooft coupling $\lambda \equiv g^2 N$:

$$V_{\text{eff}} = V_0 + V_1 + \dots \tag{17}$$

with $V_n \propto \lambda^{n-1}$. The tree-level term is

$$V_0 = \frac{N}{\lambda}\text{Tr}\left(-\sum_a [A_0, \phi_a]^2 - \frac{1}{2}\sum_{ab}[\phi_a, \phi_b]^2 \right.$$
$$\left. +2\sum_p \mu_p \alpha[\phi_{2p-1}, \phi_{2p}] + \sum_p (\beta_0^{-2} - \mu_p^2)(\phi_{2p-1}^2 + \phi_{2p}^2) \right). \tag{18}$$

When any of the chemical potentials are significantly smaller than their critical values of β_0^{-1} the tree-level term forces the scalar VEVs to vanish. However, when the difference $\beta_0^{-2} - \mu_p^2$ is $\beta_0^{-2}\mathcal{O}(\lambda)$ the scalar mass terms actually contribute to the one-loop term in (17), and so should instead be considered at that order. We will therefore consider the one-loop correction to the effective action with background fields in the almost-flat directions, namely we assume that ϕ_{2p}, ϕ_{2p-1} and α are mutually commuting, with the former non-zero only when the chemical potential μ_p is close to its critical value.

Having chosen ϕ_a and α to be commuting, the remaining gauge freedom allows them to be simultaneously diagonalised. We therefore define

$$\begin{aligned} \varphi_a &= \text{diag}(\varphi_{a\,i}) \\ \alpha &= \beta^{-1}\text{diag}(\theta_i) \end{aligned} \tag{19}$$

with $\varphi_{a\,i}$, $\theta_i \in \mathbb{R}$, $i = 1 \dots N$, and $\sum_i \varphi_{a\,i} = \sum_i \theta_i = 0$; the factor of β^{-1} has been introduced to simplify some formulae which will arise later. The requirement that the background fields be constant and diagonal does not completely fix the gauge. In particular it leaves transformations g of the form

$$g(t) = \text{diag}\left(\exp(\tfrac{2\pi i n_i t}{\beta})\right) \quad n_i \in \mathbb{Z},\ \textstyle\sum_i n_i = 0$$

under which $\theta_i \mapsto \theta_i - 2\pi n_i$. Thus the θ_i should be considered angular variables.

Before calculating the effective potential it is necessary to fix the gauge. To this end we use the R_ξ gauge of spontaneously broken gauge theory; for convenience we will restrict our attention to Feynman gauge, i.e. $\xi = 1$,

throughout.[4] The resulting gauge-fixing addition to the Lagrangian is given by

$$\mathcal{L}_g = \frac{1}{g^2}\mathrm{Tr}\left(\nabla_i A^i + \tilde{D}_0 A^0 - i\sum_a [\varphi_a, \phi_a]\right)^2$$

where ∇_i is the coordinate covariant spatial derivative and $\tilde{D}_0$ denotes the time derivative including the background value for the time component of the gauge field:

$$\tilde{D}_0 \equiv \partial_0 + i[\alpha, \cdot].$$

When calculating the effective potential we will integrate out not only the nonconstant modes but also the constant modes of the off-diagonal components of the matter fields. From the Lagrangian density (16) together with the gauge-fixing term above it can be seen that the mass terms in the scalar fields are given by

$$\mathrm{Tr}\left(\sum_p (\beta_0^{-2} - \mu_p^2)(\phi_{2p-1}^2 + \phi_{2p}^2) + \sum_{ab} \phi_a[\varphi_b, [\varphi_b, \phi_a]]\right)$$
$$= \sum_{i,j=1}^{N} \sum_a |\phi_{a\,ij}|^2 \left(\beta_0^{-2} - \mu_{[(a+1)/2]}^2 + \sum_b \varphi_{b\,ij}^2\right)$$

where $[\cdot]$ denotes the floor function, and we have defined $\varphi_{b\,ij} \equiv \varphi_{b\,i} - \varphi_{b\,j}$. Thus the off-diagonal modes $\phi_{2p-1\,ij}$ and $\phi_{2p\,ij}$ have an effective mass given by

$$m_{ij} \sim \beta_0^{-2} - \mu_p^2 + \sum_b \varphi_{b\,ij}^2. \tag{20}$$

In order for the one-loop approximation to the Wilsonian effective potential to be valid it is necessary that these off-diagonal constant modes should have a mass which is large compared to that of the light degrees of freedom. This will be the case provided that no two $\varphi_{a\,i}$ are close together, or more precisely we demand that for each $i \neq j$

$$\sum_a \varphi_{a\,ij}^2 \gg |\beta_0^{-2} - \mu_p^2| \sim \beta_0^{-2}\mathcal{O}(\lambda)$$

whenever μ_p is close to its critical value.

[4]The issue of gauge independence is briefly discussed in [1]. Incidentally, in Section 3.3.2 we will show that under fairly general circumstances the gauge-dependence of the one-loop effective action can be eliminated if one takes seriously the geometry of configuration space.

2.4 Calculation of the effective potential

Having fixed the gauge and chosen appropriate values for the VEVs, the effective action is defined by

$$\mathrm{e}^{-S_{\mathrm{eff}}(\varphi,\alpha)} = \mathcal{N}\int \mathcal{D}[\text{massive modes}]\,\mathrm{e}^{-S_E}$$

where S_E is the Euclidean action including the gauge-fixing term and $\mathcal{N}$ is an irrelevant normalisation constant. Thus the one-loop contribution to the effective potential is found by performing the path integral over the massive degrees of freedom of the quadratic terms in the action, which will give a formal expression for V_1 as a sum of logarithms of the determinants of the kernels appearing in the quadratic part of the action, together with multipliers appropriate for the statistics of the fields in question ($-1/2$ for Bose fields and $+1$ for Fermi fields). In the near-critical region, when calculating to lowest order in the 't Hooft coupling, we may take $\mu_p = \beta_0^{-1}$ exactly for the large chemical potentials.

In the absence of chemical potentials our choice of gauge removes quadratic cross terms between the scalar fields and the gauge fields. However the presence of chemical potentials and background fields means that this is no longer the case. The Gaussian terms in the integer-spin part of the Lagrangian density are

$$\begin{aligned}\mathcal{L}_B = \frac{1}{g^2}\mathrm{Tr}\,\Big\{ & A^0(-\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2)A_0 + A^i(-\tilde{D}_0^2 - \Delta^{(v)} + \varphi^2)A_i \\ & + \sum_p \phi_{2p-1}(\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2 + \beta_0^{-2} - \mu_p^2)\phi_{2p-1} \\ & + \sum_p \phi_{2p}(\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2 + \beta_0^{-2} - \mu_p^2)\phi_{2p} \\ & + 2\sum_p i\mu_p(\phi_{2p-1}\tilde{D}_0\phi_{2p} - \phi_{2p}\tilde{D}_0\phi_{2p-1}) \\ & + 2\sum_p \mu_p(\phi_{2p-1}\varphi_{2p}A_0 - A_0\varphi_{2p}\phi_{2p-1} - \phi_{2p}\varphi_{2p-1}A_0 + A_0\varphi_{2p-1}\phi_{2p}) \\ & + \bar{c}(-\tilde{D}_0^2 - \Delta^{(s)} + \varphi^2)c\Big\} \end{aligned} \tag{21}$$

where $\varphi^2 \equiv \sum_a \varphi_a^2$, and $\Delta^{(s)}$ and $\Delta^{(v)}$ are the spatial Laplacian operators acting on scalar and vector fields respectively (we will review some properties of these operators below). In the above all background fields implicitly act to the right in the adjoint representation, so that, for example, $\alpha\phi = [\alpha, \phi]$

and $\alpha^2\phi = [\alpha,[\alpha,\phi]]$; the derivation of (21) makes use of the fact that Tr $([\alpha,X]Y) = -$Tr $(X[\alpha,Y])$. The Gaussian terms involving spin-1/2 fields are

$$\mathcal{L}_F = \frac{1}{g^2}\sum_{AB}\bar{\psi}_A\left(i\delta^{AB}(\gamma^\mu D_\mu - \tilde{\mu}_A\gamma^0\gamma^5)\right.$$
$$\left. - \sum_p(\alpha^p_{AB}\varphi_{2p-1} + i\beta^p_{AB}\gamma^5\varphi_{2p})\right)\psi_B. \tag{22}$$

In the above the term involving the ghost fields $\bar{c}$, c arises from the Faddeev-Popov determinant.

Therefore the one-loop effective potential is, up to an irrelevant constant, given by

$$V_1 = \frac{1}{\mathrm{Vol}\,S^3\times S^1}\left(\frac{1}{2}\log\det K_B - \frac{1}{2}\log\det K_F^2\right) \tag{23}$$

where K_B and K_F are the fluctuation operators appearing in (21) and (22) respectively. Note that implicit in the definition of the quantum theory is an adjoint-invariant integration measure over the Lie algebra of $SU(N)$; such a choice may be specified by a choice of adjoint-invariant metric on $\mathfrak{su}(N)$. We choose the metric given by $< X,Y >= \mathrm{Tr}\,(XY)$ (which is a metric since $\mathfrak{su}(N)$ is compact), so that the one-loop effective potential is indeed given by (23), although any other choice would merely add a physically irrelevant constant. In order to calculate the determinants in (23) we make use of the fact that the spatial Laplacian operators $\Delta^{(s)}$ and $\Delta^{(v)}$ as well as the background gauge-covariant time derivative $\tilde{D}_0$ commute with K_B and K_F, and thus the eigenspaces of the latter decompose into states which are eigenvalues of the former. The eigenvalues of the Laplacians are indexed by natural numbers l, with degeneracy d_l. Similarly the fluctuation operators commute with the adjoint action of the background fields α and φ_a, so that we may calculate their determinants as functions of *numbers* α and φ_a and then perform the product over eigenvalues of the corresponding matrices.

For fields in a given eigenstate of the Laplacian, $\tilde{D}_0$ and the background fields, it will turn out that the determinants of the fluctuation operators are polynomial functions of $\tilde{D}_0^2$, with order given by the number of particle species. Let the zeros of these polynomials, for a given l, be denoted by $\varepsilon_l^{n\,2}$, where n ranges over the number of particle species. Then the logarithm of the determinant of K_{Bl} or K_{Fl}^2 is

$$\sum_n \mathrm{Tr}\log\det\left(-\tilde{D}_0^2+\varepsilon_l^{n\,2}(\alpha,\varphi)\right) \tag{24}$$

where the trace is now over the *adjoint* representation of the gauge group, and the determinant is the product over the eigenvalues of $\tilde{D}_0$. In what follows we will make use of the following matrix identity: if $F(\varphi,\theta)$ is a power-series function of the diagonal matrices φ and θ then

$$\mathrm{Tr}\,F(\varphi,\theta)=\sum_{i,j=1}^{N}F(\varphi_{ij},\theta_{ij})-F(0) \tag{25}$$

where the trace is in the adjoint representation of $SU(N)$, and e.g. $\varphi_{ij}\equiv\varphi_i-\varphi_j$. This is proved by constructing an explicit basis for the Lie algebra of $SU(N)$, given in the fundamental representation by the space of $N\times N$ anti-Hermitian, traceless matrices. We choose the basis consisting of matrices $A^{(ij)}$, $B^{(ij)}$, with $i,j=1\ldots N$, $i<j$, and $C^{(i)}$, $i=1\ldots N-1$, where

$$\begin{aligned}
A^{(ij)}_{kl} &\equiv i(\delta^i_k\delta^j_l+\delta^i_l\delta^j_k)\\
B^{(ij)}_{kl} &\equiv (\delta^i_k\delta^j_l-\delta^i_l\delta^j_k)\\
C^{(i)}_{kl} &\equiv i(\delta^i_k\delta^i_l-\delta^N_k\delta^N_l).
\end{aligned}$$

In the adjoint representation, a matrix X acts on $T\in\mathfrak{su}(N)$ as $X_{\mathrm{ad}}T=[X,T]$. It is easy to check that, for $X=\mathrm{diag}(X_i)$, $X_{ij}\equiv X_i-X_j$,

$$\begin{aligned}
X_{\mathrm{ad}}A^{(ij)} &= iX_{ij}B^{(ij)}\\
X_{\mathrm{ad}}B^{(ij)} &= -iX_{ij}A^{(ij)}\\
X_{\mathrm{ad}}C^{(i)} &= 0
\end{aligned}$$

and therefore, for a product $XY\cdots Z$ of n diagonal matrices,

$$\left.\begin{aligned}
&X_{\mathrm{ad}}Y_{\mathrm{ad}}\cdots Z_{\mathrm{ad}}A^{(ij)}=iX_{ij}Y_{ij}\cdots Z_{ij}B^{(ij)}\\
&X_{\mathrm{ad}}Y_{\mathrm{ad}}\cdots Z_{\mathrm{ad}}B^{(ij)}=-iX_{ij}Y_{ij}\cdots Z_{ij}A^{(ij)}\\
&X_{\mathrm{ad}}Y_{\mathrm{ad}}\cdots Z_{\mathrm{ad}}C^{(i)}=0
\end{aligned}\right\}\ \text{if n is odd, or}$$

$$\left.\begin{aligned}
&X_{\mathrm{ad}}Y_{\mathrm{ad}}\cdots Z_{\mathrm{ad}}A^{(ij)}=X_{ij}Y_{ij}\cdots Z_{ij}A^{(ij)}\\
&X_{\mathrm{ad}}Y_{\mathrm{ad}}\cdots Z_{\mathrm{ad}}B^{(ij)}=X_{ij}Y_{ij}\cdots Z_{ij}B^{(ij)}\\
&X_{\mathrm{ad}}Y_{\mathrm{ad}}\cdots Z_{\mathrm{ad}}C^{(i)}=0
\end{aligned}\right\}\ \text{if n is even.}$$

Equation (25) follows. The reason for the term $-F(0)$ is that the trace of the constant term is given by $F(0)$ times the dimension N^2-1 of $SU(N)$,

whereas the sum over i and j has N^2 terms.

Therefore (23) becomes

$$V_1 = -\frac{T}{2\pi^2\beta_0^3}\frac{1}{2}\sum_n\sum_{ij}\sum_l d_l^n \log\det\left(-\tilde{D}_0^2 + \varepsilon_l^n(\alpha_{ij},\varphi_{ij})\right)$$

where d_l^n is the degeneracy of the eigenvalue of the Laplacian acting on particle species n with quantum number l, together with a multiplier of $+1$ if species n is Bosonic and -1 if it is Fermionic; the d_l corresponding to $c, \bar{c}$ also contains an additional factor of 2.

We now turn our attention to the possible eigenvalues of the Laplacian operators on S^3. The quadratic terms arising from the gauge sector of the action are

$$\frac{1}{g^2}\int \sqrt{h}\,\mathrm{d}^4x\,\left(-A^\mu(\tilde{D}_0^2 + \nabla^i\nabla_i)A_\mu - (\tilde{\nabla}^\mu A^\nu)(\tilde{\nabla}_\nu A_\mu)\right)$$

where $\tilde{\nabla}_\mu \equiv \nabla_\mu + i\delta^0_\mu\alpha$. In order to make the second term cancel with the term $(\tilde{D}_0 A_0 + \nabla^i A_i)^2$ arising from $\mathcal{L}_g$ it is necessary to reverse the order of the derivatives:

$$\begin{aligned}
&- \int \sqrt{h}\,\mathrm{d}^4x\,(\nabla^\mu A^\nu)(\nabla_\nu A_\mu) \\
&= \int \sqrt{h}\,\mathrm{d}^4x\,A^\nu h^{\mu\rho}(\nabla_\rho\nabla_\nu A_\mu) \\
&= \int \sqrt{h}\,\mathrm{d}^4x\,A^\nu h^{\mu\rho}\left(\nabla_\nu\nabla_\rho A_\mu + h_{\mu\eta}R^\eta_{\lambda\rho\nu}A^\lambda\right) \\
&= \int \sqrt{h}\,\mathrm{d}^4x\,\left(-(\nabla_\nu A^\nu)(\nabla^\mu A_\mu) + A^\nu R^\mu_\nu A_\mu\right)
\end{aligned}$$

where use has been made of the identity $[\nabla_\rho, \nabla_\nu]A^\eta = R^\eta_{\lambda\rho\nu}A^\lambda$, where R is the Riemann curvature tensor. We have also integrated by parts using the covariant divergence theorem, namely that on a manifold M of dimension n without boundary, for a covector field X,

$$\int_M \sqrt{h}\,\mathrm{d}^n x\,\nabla^\mu X_\mu = 0; \tag{26}$$

this is a consequence of Stokes's theorem applied to the $(n-1)$-form $*X$. Thus, we define

$$\begin{aligned}\Delta^{(s)}s &\equiv \nabla^j\nabla_j s \\ \Delta^{(v)}v_i &\equiv \nabla^j\nabla_j v_i - R_i^j v_j\end{aligned}$$

so that the quadratic terms in Bosonic fields are indeed given by (21). A straightforward calculation shows that the Ricci tensor of S^3 is given by

$$R_i^j = \frac{2}{\beta_0^2}\delta_i^j.$$

The scalar Laplacian has eigenvectors given by the spherical harmonics Y_l on S^3, with $l \in \mathbb{N}$. They satisfy

$$\Delta^{(s)}Y_l = -\beta_0^{-2}l(l+2)Y_l$$

and have degeneracy $(l+1)^2$. Meanwhile, consider any vector field A_i, considered as a one-form. By the Hodge decomposition theorem we may uniquely write A_i as $B_i + C_i$, where

$$\begin{aligned}C_i &= \mathrm{d}f_i = \nabla_i f \\ \mathrm{d}^\dagger B &= 0\end{aligned}$$

for some scalar function f. Since $\mathrm{d}^\dagger = \star\mathrm{d}\star$ acting on one-forms is proportional to the divergence, it is also the case that $\nabla^i B_i = 0$. If f is taken to be a spherical harmonic Y_l then it follows that

$$\Delta^{(v)}C_i = \nabla^j\nabla_j\nabla_i Y_l - R_i^j\nabla_j Y_l = \nabla_i\Delta^{(s)}Y_l = -\beta_0^{-2}l(l+2)C_i$$

where we have again used the facts that $[\nabla_j, \nabla_i]v^k = R^k_{lji}v^l$ and $[\nabla_j, \nabla_i]f = 0$, when v is a vector and f a scalar. These modes again have degeneracy $(l+1)^2$, with $l \in \mathbb{N} - \{0\}$ (the case $l = 0$ is excluded since the spherical harmonic Y_0 is constant and so its derivative is zero). A basis for modes B_i is given by vector fields V_l, which satisfy

$$\Delta^{(v)}V_{li} = -\beta_0^{-2}(l+1)^2 V_{li};$$

they have degeneracy $2(l+2)$, again with $l \in \mathbb{N} - \{0\}$. Note that, if we substitute $A_i = B_i + C_i$ into the integral of (21) with B_i and C_i as eigenstates of the Laplacian then the cross terms between B_i and C_i vanish by the divergence theorem (26), so that K_B is diagonal in B_i and C_i.

From (22), the Fermionic Laplacian is given by

$$\gamma^\mu D_\mu \gamma^\nu D_\nu.$$

It has eigenvalues $-\beta_0^{-2}(l+\frac{1}{2})^2$, $l \in \mathbb{N} - \{0\}$, with degeneracy $\frac{1}{2}l(l+1)$.

From now on we specialise to the case where $\mu_p = 0$ or $\mu_p = \beta_0^{-1}$, with $\varphi_{2p-1} = \varphi_{2p} = 0$ whenever $\mu_p = 0$. Consider first the integer-spin sector. From (21) it can be seen that K_{Fl} is diagonal in the fields B_i, C_i, $c, \bar{c}$ and ϕ_{2p-1}, ϕ_{2p} whenever $\mu_p = 0$. They contribute terms of the form (24) with energies ε_l which can be read off from (21), using the eigenvalues of the corresponding Laplacian operators given above:

$$\begin{aligned}
\varepsilon_l^{\phi_{2p-1}\,2} &= \beta_0^{-2}(l+1)^2 + \varphi^2 \\
\varepsilon_l^{\phi_{2p}\,2} &= \beta_0^{-2}(l+1)^2 + \varphi^2 \\
\varepsilon_l^{B_i\,2} &= \beta_0^{-2}(l+1)^2 + \varphi^2 \\
\varepsilon_l^{C_i\,2} &= \beta_0^{-2}l(l+2) + \varphi^2 \\
\varepsilon_l^{c,\bar{c}\,2} &= \beta_0^{-2}l(l+2) + \varphi^2.
\end{aligned}$$

On the other hand the fields A_0 and ϕ_{2p-1}, ϕ_{2p} with $\mu_p \neq 0$ have a mixed interaction. Suppose that $\mu_p = \beta_0^{-1}$ for $p = 1 \ldots n$. For a given l, φ_a and $\tilde{D}_0$ (21) gives rise to the following $(2n+1) \times (2n+1)$ matrix:

$$(-\tilde{D}_0^2 + \beta_0^{-2}l(l+2) + \varphi^2)\mathbb{1} + 2\beta_0^{-1}M \tag{27}$$

where M has the following block diagonal form:

$$\begin{aligned}
M &= \begin{pmatrix} 0 & X \\ -X^T & Y \end{pmatrix} \\
X &\equiv \begin{pmatrix} -\varphi_2 & \varphi_1 & -\varphi_4 & \varphi_3 & \cdots \end{pmatrix} \\
Y &\equiv \begin{pmatrix} 0 & i\tilde{D}_0 & 0 & 0 & \cdots \\ -i\tilde{D}_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & i\tilde{D}_0 & \cdots \\ 0 & 0 & -i\tilde{D}_0 & 0 & \cdots \\ \cdots & & & & \end{pmatrix}.
\end{aligned}$$

In order to calculate the determinant of (27) consider first the characteristic polynomial of M: using the matrix identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det \left(A - BD^{-1}C\right)$$

we find

$$\begin{aligned} \det (Y - \lambda \mathbb{1}) &= (\lambda^2 \tilde{D}_0^2)^n \\ (Y - \lambda \mathbb{1})^{-1} &= \frac{1}{\lambda^2 - \tilde{D}_0^2}(Y - \lambda \mathbb{1})^T \\ \det (M - \lambda \mathbb{1}) &= -\lambda(\lambda^2 - \tilde{D}_0^2)^{n-1}(\lambda^2 - \tilde{D}_0^2 + \varphi^2) \end{aligned}$$

so that its eigenvalues are 0, $\pm\tilde{D}_0$ with degeneracy $n-1$, and $\pm\sqrt{\tilde{D}_0^2 - \varphi^2}$. From this we find that the determinant of (27) is equal to

$$\begin{aligned} (-\tilde{D}_0^2 + \beta_0^{-2} l(l+2) + \varphi^2) \left[(-\tilde{D}_0^2 + \beta_0^{-2} l(l+2) + \varphi^2)^2 - 4\beta_0^{-2}\tilde{D}_0^2 + 4\beta_0^{-2}\varphi^2\right] \\ \times \left[(-\tilde{D}_0^2 + \beta_0^{-2} l(l+2) + \varphi^2)^2 - 4\beta_0^{-2}\tilde{D}_0^2\right]^{n-1}. \end{aligned}$$

The second term factors as

$$\left(-\tilde{D}_0^2 + \beta_0^{-2} l^2 + \varphi^2\right)\left(-\tilde{D}_0^2 + \beta_0^{-2}(l+2)^2 + \varphi^2\right)$$

while the third term factors as

$$\begin{aligned} &\left(-\tilde{D}_0^2 + \left(\beta_0^{-1} + \sqrt{\beta_0^{-2}(l+1)^2 + \varphi^2}\right)^2\right) \\ &\times \left(-\tilde{D}_0^2 + \left(\beta_0^{-1} - \sqrt{\beta_0^{-2}(l+1)^2 + \varphi^2}\right)^2\right) \end{aligned}$$

from which the corresponding values of ε_l may be read off.

We omit the details of the calculation of the Fermionic energies, and instead will simply state the results. With $\mu_1 = \beta_0^{-1}$, $\mu_2 = \mu_3 = 0$, they are

$$\begin{aligned} \varepsilon_l &= \sqrt{\beta_0^{-2} l^2 + \varphi^2} \\ \varepsilon_l &= \sqrt{\beta_0^{-2}(l+1)^2 + \varphi^2} \end{aligned}$$

both with degeneracy $4l(l+1)$. For $\mu_1 = \mu_2 = \beta_0^{-1}$, $\mu_3 = 0$ they are

$$\begin{aligned}
\varepsilon_l &= \sqrt{\beta_0^{-2} l^2 + \varphi^2} + \frac{1}{2\beta_0} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} l^2 + \varphi^2} - \frac{1}{2\beta_0} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} (l+1)^2 + \varphi^2} + \frac{1}{2\beta_0} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} (l+1)^2 + \varphi^2} - \frac{1}{2\beta_0}
\end{aligned}$$

each with degeneracy $2l(l+1)$. For $\mu_1 = \mu_2 = \mu_3 = \beta_0^{-1}$ they are

$$\begin{aligned}
\varepsilon_l &= \sqrt{\beta_0^{-2} l^2 + \varphi^2} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} (l+1)^2 + \varphi^2} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} l^2 + \varphi^2} + \frac{1}{\beta_0} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} l^2 + \varphi^2} - \frac{1}{\beta_0} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} (l+1)^2 + \varphi^2} + \frac{1}{\beta_0} \\
\varepsilon_l &= \sqrt{\beta_0^{-2} (l+1)^2 + \varphi^2} - \frac{1}{\beta_0}
\end{aligned}$$

with degeneracies $2l(l+1)$, $2l(l+1)$, $l(l+1)$, $l(l+1)$, $l(l+1)$ and $l(l+1)$ respectively.

The results may be compactly summarised as in Table 1; note that those energies given as functions of μ_p are only valid when $\mu_p = 0$ or $\mu_p = \beta_0^{-1}$, with $\mu_p \geq \mu_q$ when $p < q$. When an entry contains a $\pm$ symbol then both signs occur as distinct values in (24).

For a given term of the form

$$\log \det \left(-\tilde{D}_0^2 + \varepsilon^2 \right)$$

the logarithm of the determinant is calculated by summing over all eigenvalues of $\tilde{D}_0$. For an integer-spin field on $S^3 \times S^1$, with the S^1 of radius β, $\tilde{D}_0$ has eigenvalues $(2in\pi + i\theta)/\beta$, $n \in \mathbb{Z}$, where θ is some eigenvalue of the matrix $\beta\alpha$ (recall definition (19)). For a spin-1/2 field, which has anti-periodic

field	ε_l	d_l	l_0
B_i	$\sqrt{\beta_0^{-2}(l+1)^2+\varphi^2}$	$2l(l+2)$	1
C_i	$\sqrt{\beta_0^{-2}l(l+2)+\varphi^2}$	$(l+1)^2$	1
$(c,\bar{c})$	$\sqrt{\beta_0^{-2}l(l+2)+\varphi^2}$	$-2(l+1)^2$	0
(A_0,ϕ_a)	$\sqrt{\beta_0^{-2}l(l+2)+\varphi^2}$	$(l+1)^2$	0
(A_0,ϕ_a)	$\sqrt{\beta_0^{-2}(l+1\pm\beta_0\mu_1)^2+\varphi^2}$	$(l+1)^2$	0
(A_0,ϕ_a)	$\sqrt{\beta_0^{-2}(l+1)^2+\varphi^2}\pm\mu_2$	$(l+1)^2$	0
(A_0,ϕ_a)	$\sqrt{\beta_0^{-2}(l+1)^2+\varphi^2}\pm\mu_3$	$(l+1)^2$	0
$(\psi_A,\bar{\psi}_A)$	$\sqrt{\beta_0^{-2}(l+\frac{1}{2}\pm\frac{1}{2}\beta_0\mu_1)^2+\varphi^2}\pm\frac{1}{2}\mu_2\pm\frac{1}{2}\mu_3$	$-l(l+1)$	1

Table 1: Energies for $\mu_p = 0$ or $\mu_p = \beta_0^{-1}$, with $\mu_p \geq \mu_q$ when $p < q$.

boundary conditions on S^1, the eigenvalues of $\tilde{D}_0$ are $(2i(n+1/2)\pi + i\theta)/\beta$, $n \in \mathbb{Z}$. The product

$$\prod_{n\in\mathbb{Z}}\left(\varepsilon^2+\left(\frac{2\pi n+\theta}{\beta}\right)^2\right)$$

is clearly divergent, but it is standard practice in thermal field theory to regularise the product in such a way that it splits into a temperature-independent energy part together with a thermal part which vanishes in the limit $\beta \to \infty$. To this end, let us write the above product as

$$(\beta^2\varepsilon^2+\theta)\prod_{n\in\mathbb{Z}-\{0\}}\left(1+\frac{\beta^2\varepsilon^2+\theta^2+4\pi n\theta}{4\pi^2n^2}\right)\times\prod_{n\in\mathbb{Z}-\{0\}}\left(\frac{4\pi^2n^2}{\beta^2}\right).$$

We will treat the last factor as being an infinite constant, which will contribute an irrelevant addition to the effective potential after its logarithm is taken. Our regularisation scheme is therefore to ignore this factor completely. Multiplying together the contributions from n and $-n$ for $n \in \mathbb{N} - \{0\}$, the logarithm of the remaining terms may be written as

$$\log(\beta^2\varepsilon^2+\theta^2)+\sum_{n=1}^{\infty}\log\left(1+\frac{\beta^2\varepsilon^2-\theta^2}{2\pi^2n^2}+\frac{(\beta^2\varepsilon^2+\theta^2)^2}{16\pi^4n^4}\right). \qquad (28)$$

In order to evaluate this sum, rewrite the argument of the logarithm as

$$\left(1-\frac{((\theta+i\beta\varepsilon)/2\pi)^2}{n^2}\right)\left(1-\frac{((\theta-i\beta\varepsilon)/2\pi)^2}{n^2}\right)$$

and then use the identity [31]

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)=\frac{\sin(\pi z)}{\pi z}$$

with $z=(\theta\pm i\beta\varepsilon)/2\pi$: (28) is then equal to

$$\begin{aligned}&\log\left(4\sin\left(\frac{\theta+i\beta\varepsilon}{2}\right)\sin\left(\frac{\theta-i\beta\varepsilon}{2}\right)\right)\\&\quad=\beta\varepsilon+\log\left(1-2\mathrm{e}^{-\beta\varepsilon}\cos\theta+\mathrm{e}^{-2\beta\varepsilon}\right).\end{aligned}\tag{29}$$

Writing

$$\log\left(1-2\mathrm{e}^{-\beta\varepsilon}\cos\theta+\mathrm{e}^{-2\beta\varepsilon}\right)=\log\left(1-\mathrm{e}^{-\beta\varepsilon+i\theta}\right)+\log\left(1-\mathrm{e}^{-\beta\varepsilon-i\theta}\right)$$

and using the Taylor expansion for $\log(1-x)$, we find that this is equal to

$$\log\det\left(-\tilde{D}_0^2+\varepsilon^2\right)=\beta|\varepsilon|-2\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{e}^{-n\beta|\varepsilon|}\cos(n\theta).\tag{30}$$

For the Fermionic case, recall that the eigenvalues of $\tilde{D}_0$ are

$$(2\pi i(n+1/2)+i\theta)/\beta.$$

Therefore we may derive the regularised determinant of $-\tilde{D}_0^2+\varepsilon^2$ from (30) by the replacement $\theta\mapsto\theta+\pi$. The result is

$$\log\det\left(-\tilde{D}_0^2+\varepsilon^2\right)=\beta|\varepsilon|-2\sum_{n=1}^{\infty}\frac{(-)^n}{n}\mathrm{e}^{-n\beta|\varepsilon|}\cos(n\theta).\tag{31}$$

When calculating the effective potential the above formulae will be divided by $\mathrm{Vol}\,S^1=\beta$, so that all dependence on the temperature will be contained in the second terms.

2.5 The zero temperature effective potential

Before calculating the full one-loop effective potential using (23), (24), (30) and (31), let us calculate the Casimir energy $\mathcal{E}$ of the theory, i.e. the sum of the energies $|\varepsilon|$ in (30) and (31) without the terms which depend non-trivially on the temperature. Remarkably the result will turn out to be completely independent of the background fields. Before considering the $\mathcal{N}=4$ theory, we carry out the corresponding calculation in the case of a

part of the theory, namely a single conformally coupled free complex scalar on $S^3 \times S^1$, since it will highlight several conceptual issues which arise in the full $\mathcal{N} = 4$ case. The Lagrangian density of this theory includes a mass term $(\beta_0^{-2} + \varphi^2)\phi^2$, where we have labelled the variables thusly by analogy with the mass terms in (21). Without a chemical potential the energies ε_l come in pairs $\sqrt{\beta_0^{-2}(l+1)^2 + \varphi^2}$, with degeneracy $(l+1)^2$, $l \in \mathbb{N}$. The Casimir energy is therefore

$$\mathcal{E} = \sum_{l=0}^{\infty} (l+1)^2 \sqrt{\beta_0^{-2}(l+1)^2 + \varphi^2}.$$

This sum is divergent and needs to be regularised. A naïve regulator which gives a finite result is to simply subtract the value of $\mathcal{E}$ which arises in the limit of flat space, i.e. $\beta_0 \to \infty$, but as we will show below this method will remove some of the dependence of $\mathcal{E}$ on the mass φ, which is unacceptable when φ is interpreted as a background field. Instead we will employ an explicit regularisation, namely a cutoff on the energies of the modes.[5] Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $F(x) = 1$ for $x = 0$ and $F(x) = 0$ for $x > 1$. We define the regularised Casimir energy in terms of an arbitrary mass scale Λ by

$$\mathcal{E}_R \equiv \frac{1}{2} \sum_{l=0}^{\infty} d_l |\varepsilon_l| F(|\varepsilon_l|/\Lambda). \tag{32}$$

In order to perform the sum over l we make use of two versions of the Abel-Plana formula [29]. In fact in the case of the theory with at least one critical chemical potential it turns out that only the second of the following formulae are needed, but we include both here for completeness, and because the derivation of the first will serve as a warm-up: let $f(z)$ be a holomorphic function, real on the real axis. The form relevant to Fermionic sums is

$$\sum_{n=0}^{\infty} f(n + \frac{1}{2}) = \int_0^{\infty} f(x)\,\mathrm{d}x + 2\int_0^{\infty} \frac{\Im f(ix)}{\mathrm{e}^{2\pi x} + 1}\,\mathrm{d}x \tag{33}$$

while the formula used for Bosonic sums is

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} f(x)\,\mathrm{d}x + \frac{1}{2}f(0) - 2\int_0^{\infty} \frac{\Im f(ix)}{\mathrm{e}^{2\pi x} - 1}\,\mathrm{d}x. \tag{34}$$

To derive these equations we make use of the Cauchy integral formula. To derive (33) first note that $\mathrm{e}^{2\pi i(x - i\epsilon)} + 1$ and $\mathrm{e}^{-2\pi i(x - i\epsilon)} + 1$ both have zeros

[5] This issue is explained in more detail in footnote 30 of [20].

at $x = n + 1/2 + i\epsilon$, $n \in \mathbb{Z}$, with derivatives $-2\pi i$ and $2\pi i$ respectively, and that

$$\int_0^\infty f(x)\,\mathrm{d}x = \int_0^\infty \frac{f(x)}{\mathrm{e}^{2\pi i(x-i\epsilon)}+1}\,\mathrm{d}x + \int_0^\infty \frac{f(x)}{\mathrm{e}^{-2\pi i(x-i\epsilon)}+1}\,\mathrm{d}x. \tag{35}$$

We take ϵ to be small and positive. To evaluate the first integral in the right hand side of (35), integrate over a contour which runs up the negative imaginary axis, along the positive real axis, and closes along the bottom right quadrant of the complex plane with $|x| \to \infty$. Cauchy's formula, and taking $\epsilon \to 0$, gives

$$\int_0^\infty \frac{f(x)}{\mathrm{e}^{2\pi i x}+1}\,\mathrm{d}x = -i\int_0^\infty \frac{f(-ix)}{\mathrm{e}^{2\pi x}+1}\,\mathrm{d}x.$$

For the second integral it is necessary instead to close the contour in the upper right quadrant, thereby including poles at $x = 1/2 + i\epsilon, 3/2 + i\epsilon, \ldots$. We find

$$\int_0^\infty \frac{f(x)}{\mathrm{e}^{-2\pi i x}+1}\,\mathrm{d}x = i\int_0^\infty \frac{f(ix)}{\mathrm{e}^{2\pi x}+1}\,\mathrm{d}x + \sum_{n=0}^{\infty} f(n+\frac{1}{2}).$$

Adding these two results together, and using the fact that $f(-ix) = (f(ix))^*$ when $x \in \mathbb{R}$ and $f(x)$ is real on the real axis, gives (33). The proof of (34) proceeds similarly: write

$$\int_0^\infty f(x)\,\mathrm{d}x = -\int_0^\infty \frac{f(x)}{\mathrm{e}^{2\pi i(x-i\epsilon)}-1}\,\mathrm{d}x - \int_0^\infty \frac{f(x)}{\mathrm{e}^{-2\pi i(x-i\epsilon)}-1}\,\mathrm{d}x \tag{36}$$

where $\mathrm{e}^{2\pi i(x-i\epsilon)} - 1$ and $\mathrm{e}^{-2\pi i(x-i\epsilon)} - 1$ have zeros at $x = n + i\epsilon$, $n \in \mathbb{Z}$, with derivatives $2\pi i$ and $-2\pi i$. Choosing contours as before, the first term evaluates to

$$\int_0^\infty \frac{f(x)}{\mathrm{e}^{2\pi i x}-1}\,\mathrm{d}x = -i\int_0^\infty \frac{f(-ix)}{\mathrm{e}^{2\pi x}-1}\,\mathrm{d}x;$$

however, for the second term one must be more careful to avoid the pole at $x = i\epsilon$. We introduce two contours enclosing the upper right quadrant, one with a small semicircle around the pole at $i\epsilon$ to the left, the other with a small semicircle around the right of the pole. These contours include the poles at $x = 0 + i\epsilon, 1 + i\epsilon, \ldots$ and $x = 1 + i\epsilon, 2 + i\epsilon, \ldots$ respectively. When the integrals over these two contours are added together their contributions

from the vicinity of $x = i\epsilon$ cancel, leaving the correct integral from $i\epsilon$ to $i\infty$ along the positive imaginary axis. Therefore, again taking $\epsilon \to 0$,

$$2\int_0^\infty \frac{f(x)}{e^{-2\pi i x}-1}\,dx = 2i\int_0^\infty \frac{f(ix)}{e^{2\pi x}-1}\,dx - 2\sum_{n=1}^{\infty} f(n) - f(0).$$

Once again, substituting these results into (36) yields (34).

In the case at hand $f(x)$ is given by $x^2\sqrt{\beta_0^{-2}x^2+\varphi^2}F(\sqrt{\beta_0^{-2}x^2+\varphi^2}/\Lambda)$ (defining $n = l+1$, the sum from $n = 0$ to ∞ is equal to the sum from $l = 0$ to ∞ since the summand with $n = 0$ is zero). The last term in the Abel-Plana formula (34) is clearly convergent even in the limit $\Lambda \to \infty$, so in it we may replace $f(x)$ with $x^2\sqrt{\beta_0^{-2}x^2+\varphi^2}$. Then the imaginary part of $f(ix)$ is zero whenever $x \le \beta_0\varphi$, so this integral gives

$$\frac{2}{\beta_0}\int_{\beta_0\varphi}^\infty dx\, x^2 \frac{\sqrt{x^2-\beta_0^2\varphi^2}}{e^{2\pi x}-1}. \tag{37}$$

(Note that in deriving the Abel-Plana formula it was assumed that $f(x)$ was holomorphic in the half-plane $\Re x \ge 0$. The function $\sqrt{\beta_0^{-2}x^2+\varphi^2}$ has branch points at $x = \pm i\beta_0\varphi$; we may define the function to be branched along a curve running through the half-plane $\Re x < 0$ so that the Abel-Plana formula is applicable, and the above term has the correct sign). The first term can be calculated in the limit of a hard cutoff, $F(x) = \theta(1-x)$, where θ is the Heaviside function. The indefinite integral of $x^2\sqrt{\beta_0^{-2}x^2+\varphi^2}$ is given by

$$\int_0^X dx\, x^2\sqrt{\beta_0^{-2}x^2+\varphi^2} = \frac{\beta_0^3}{8}\left\{\beta_0^{-1}X\sqrt{\beta_0^{-2}X^2+\varphi^2}\left(2\beta_0^{-2}X^2+\varphi^2\right) - \varphi^4\log\left(\frac{\beta_0^{-2}X}{\varphi}+\sqrt{1+\frac{\beta_0^{-2}X^2}{\varphi^2}}\right)\right\}.$$

Expanding in powers of φ/Λ, we find

$$\int_0^{\beta_0\sqrt{\Lambda^2-\varphi^2}} dx\, x^2\sqrt{\beta_0^{-2}x^2+\varphi^2} = \frac{1}{4}\beta_0^3\left(\Lambda^4 - \Lambda^2\varphi^2 + \frac{1}{2}\varphi^4\log\frac{e^{1/2}\varphi}{2\Lambda}\right) + \mathcal{O}(\varphi/\Lambda) \tag{38}$$

where $\varphi \equiv \sqrt{\varphi^2}$. In the limit $\Lambda \to \infty$ it is only this latter term which is non-zero. Therefore the naïve renormalisation scheme subtracts this term. But it can be seen that it contains a part which depends non-trivially on φ. Although the result in (38) appears to be non-analytic as $\varphi \to 0$, in fact it can be shown that the logarithmic singularity precisely cancels with that arising from the integral (37).

In the case where the theory includes a chemical potential the fluctuation operator is modified as in (21), so that the mass term becomes $(\beta_0^{-2} + \varphi^2 - \mu^2)\phi^2$ and the Lagrangian density contains a cross term $2i\mu(\phi_1\partial_0\phi_2 - \phi_2\partial_0\phi_1)$. A simple calculation along the lines of the last section shows that the energies ε_l are now given by

$$\varepsilon_l = \sqrt{\beta_0^{-2}(l+1)^2 + \varphi^2} \pm \mu. \tag{39}$$

An obvious approach to regularising the theory with the chemical potential switched on is to repeat the above analysis with a cutoff function which depends on the modified energies ε_l, that is, to use (32) with ε_l given by (39). However, this would be incorrect. This is because altering the regularisation scheme to be dependent on the chemical potential will effectively change the path integral measure, but in order to calculate (12) we should only change the action of the theory. We therefore define instead

$$\mathcal{E}_R \equiv \frac{1}{2}\sum_{l=0}^{\infty} d_l |\varepsilon_l| F(|\varepsilon_l^{(0)}|/\Lambda) \tag{40}$$

where $\varepsilon_l^{(0)} = \sqrt{\beta_0^{-2}(l+1)^2 + \varphi^2}$ is the energy eigenvalue in the absence of any chemical potential. Therefore, provided that $\mu \leq \sqrt{\beta_0^2 + \varphi^2}$ (so that $|\varepsilon_l| = \varepsilon_l$ for all l), the pairs of values for ε_l sum to give $2\varepsilon_l^{(0)}$, and the presence of chemical potentials leaves the Casimir energy unchanged.

With the above borne in mind we now return to the $\mathcal{N} = 4$ theory. In the remainder of this section we leave the trace over the adjoint representation implicit, that is, we will write all results as functions of the variable φ, and it should be understood that the correct formula for the Casimir energy is given by the trace of the given functions (as in equation (25)). Firstly, from Table 1 it can immediately be seen that, for $l \geq 1$, the sum over energies from the fields C_i, $(c, \bar{c})$ and the first of the (A_0, ϕ_a) modes all cancel. Thus the only contribution from this sector comes from the $l = 0$ energies of $(c, \bar{c})$ and (A_0, ϕ_a), and the overall contribution to $\mathcal{E}$ is

$$-\frac{1}{2}\varphi. \tag{41}$$

The contribution from the remaining integer-spin modes can similarly be read off from Table 1. The result is (recall that the cutoff function takes as its argument the energies in the absence of chemical potentials)

$$\begin{aligned}
&\frac{1}{2}\sum_{l=1}^{\infty} 2l(l+2)\sqrt{\beta_0^{-2}(l+1)^2+\varphi^2}\,G(l+1)\\
+\;&\frac{4}{2}\sum_{l=0}^{\infty}(l+1)^2\sqrt{\beta_0^{-2}(l+1)^2+\varphi^2}\,G(l+1)\\
+\;&\frac{1}{2}\sum_{l=0}^{\infty}(l+1)^2\sqrt{\beta_0^{-2}(l+2)^2+\varphi^2}\,G(l+1)\\
+\;&\frac{1}{2}\sum_{l=0}^{\infty}(l+1)^2\sqrt{\beta_0^{-2}l^2+\varphi^2}\,G(l+1)
\end{aligned}$$

where we have defined $G(l) \equiv F(\sqrt{\beta_0^{-2}l^2+\varphi^2}/\Lambda)$ for brevity. This formula is valid when 1, 2 or 3 chemical potentials are at their critical value; from Table 1 it can be seen that the effect of turning on critical values for μ_2 or μ_3 is to split some of the energy levels into pairs of the form $\varepsilon^{(0)} \pm \mu_p$, so the difference cancels out. This can be written in a more convenient form by several appropriate changes of variable:

$$\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{\infty}\sqrt{\beta_0^{-2}n^2+\varphi^2}\\
\times\big\{&(6n^2-2)G(n)+(n+1)^2G(n+1)+(n-1)^2G(n-1)\big\}\\
&+\varphi G(0)-\frac{1}{2}\varphi G(1);
\end{aligned}$$

the last two terms arise from the difference in limits between the sums over l and those over n. In the limit $\Lambda \to \infty$, $G(0) = G(1) = 1$, and so these terms contribute $\varphi/2$ which cancels out with the contribution of C_i, $(c, \bar{c})$ and the first of the (A_0, ϕ_a) modes (41). We then apply the Abel-Plana formula to what remains. As before, the last term in (34) is convergent in the limit $\Lambda \to \infty$ so we may take $G(x) = 1$, and the result simplifies to give

$$\frac{8}{\beta_0}\int_{\beta_0\varphi}^{\infty} \mathrm{d}x\, x^2 \frac{\sqrt{x^2-\beta_0^2\varphi^2}}{\mathrm{e}^{2\pi x}-1}.$$

The first term, again evaluated in the limit $F(x) = \theta(1-x)$, is equal to

$$4\int_0^{\beta_0\sqrt{\Lambda^2-\varphi^2}} \mathrm{d}x\, x^2\sqrt{\beta_0^{-2}x^2+\varphi^2}$$
$$+\frac{1}{2}\int_{\beta_0\sqrt{\Lambda^2-\varphi^2}}^{\beta_0\sqrt{\Lambda^2-\varphi^2}+1} \mathrm{d}x\,\left\{(x-1)^2\sqrt{\beta_0^{-2}x^2+\varphi^2} - x^2\sqrt{\beta_0^{-2}(x-1)^2+\varphi^2}\right\} \quad (42)$$

where the last two terms arise from the differences $G(x+1) - G(x)$ and $G(x-1) - G(x)$. The first part of (42) was given earlier in (38). To evaluate the second part, note that, for large Λ, the limits of integration have $x \gg \varphi$, and so the square roots may be Taylor expanded to give

$$\begin{aligned}\frac{1}{2\beta_0}\int_{\beta_0\sqrt{\Lambda^2-\varphi^2}}^{\beta_0\sqrt{\Lambda^2-\varphi^2}+1} \mathrm{d}x\,&\left\{(x-1)^2x\left(1+\frac{\beta_0^2\varphi^2}{2x^2}\right)\right.\\ &\left.-x^2(x-1)\left(1+\frac{\beta_0^2\varphi^2}{2(x-1)^2}\right)\right\}+\ldots\\ &=\frac{1}{2}\left(-\beta_0\Lambda^2+\frac{1}{6\beta_0}-\frac{\beta_0\varphi^2}{2}\right)+\mathcal{O}(\varphi/\Lambda).\end{aligned}$$

Putting it all together, the total contribution from the integer-spin modes to the Casimir energy is

$$\mathcal{E}_B = \beta_0^3\Lambda^4 - \beta_0^3\Lambda^2\varphi^2 + \frac{1}{2}\beta_0^3\varphi^4\log\frac{\mathrm{e}^{1/2}\varphi}{2\Lambda} + \frac{8}{\beta_0}\int_{\beta_0\varphi}^{\infty}\mathrm{d}x\,x^2\frac{\sqrt{x^2-\beta_0^2\varphi^2}}{\mathrm{e}^{2\pi x}-1}$$
$$-\frac{1}{2}\beta_0\Lambda^2+\frac{1}{12\beta_0}-\frac{1}{4}\beta_0\varphi^2.$$

The spin-1/2 contribution to the Casimir energy is calculated similarly. Again it can be seen from Table 1 that critical values for μ_2 and μ_3 split the energy levels into pairs of the form $\varepsilon^{(0)} \pm \mu_p$, so that the difference $\varepsilon - \varepsilon^{(0)}$ cancels out. The result is

$$-\ 2\sum_{l=1}^{\infty} l(l+1)\sqrt{\beta_0^{-2}l^2+\varphi^2}\,G(l+1/2)$$
$$-\ 2\sum_{l=1}^{\infty} l(l+1)\sqrt{\beta_0^{-2}(l+1)^2+\varphi^2}\,G(l+1/2);$$

again we change variables to write this as

$$-2\sum_{n=0}^{\infty}\sqrt{\beta_0^{-2}n^2+\varphi^2}\left\{n(n+1)G(n+1/2)+n(n-1)G(n-1/2)\right\}.$$

As before, the second integral in (34) may be evaluated with $F(x)=1$, giving

$$-\frac{8}{\beta_0}\int_{\beta_0\varphi}^{\infty}\mathrm{d}x\,x^2\frac{\sqrt{x^2-\beta_0^2\varphi^2}}{\mathrm{e}^{2\pi x}-1}.$$

The first integral is evaluated with $F(x)=\theta(1-x)$, giving

$$\begin{aligned}&-4\int_0^{\beta_0\sqrt{\Lambda^2-\varphi^2}}\mathrm{d}x\,x^2\sqrt{\beta_0^{-2}x^2+\varphi^2}\\ +2\int_{\beta_0\sqrt{\Lambda^2-\varphi^2}}^{\beta_0\sqrt{\Lambda^2-\varphi^2}+1/2}&\mathrm{d}x\left\{(x-1/2)(x+1/2)\sqrt{\beta_0^{-2}(x-1/2)^2+\varphi^2}\right.\\ &\left.-x(x-1)\sqrt{\beta_0^{-2}x^2+\varphi^2}\right\}.\end{aligned}$$

The first term is as before. The last may again be evaluated by Taylor expanding the square roots. We find

$$\begin{aligned}\frac{2}{\beta_0}\int_{\beta_0\sqrt{\Lambda^2-\varphi^2}}^{\beta_0\sqrt{\Lambda^2-\varphi^2}+1/2}\mathrm{d}x&\left\{(x-1/2)^2(x+1/2)\left(1+\frac{\beta_0^2\varphi^2}{2(x-1/2)^2}\right)\right.\\ &\left.-x^2(x-1)\left(1+\frac{\beta_0^2\varphi^2}{2x^2}\right)\right\}+\ldots\\ &=\frac{1}{2}\beta_0\Lambda^2+\frac{1}{4}+\frac{5}{48\beta_0}+\mathcal{O}(\varphi/\Lambda).\end{aligned}$$

Adding these terms together, the spin-1/2 contribution to the Casimir energy is

$$\begin{aligned}\mathcal{E}_F=-\beta_0^3\Lambda^4+\beta_0^3\Lambda^2\varphi^2-\frac{1}{2}\beta_0^3\varphi^4\log\frac{\mathrm{e}^{1/2}\varphi}{2\Lambda}&-\frac{8}{\beta_0}\int_{\beta_0\varphi}^{\infty}\mathrm{d}x\,x^2\frac{\sqrt{x^2-\beta_0^2\varphi^2}}{\mathrm{e}^{2\pi x}-1}\\ &+\frac{1}{2}\beta_0\Lambda^2+\frac{1}{4}\beta_0\varphi^2+\frac{5}{48\beta_0}.\end{aligned}$$

The total one-loop contribution to the effective potential is therefore

$$\frac{1}{\mathrm{Vol}\, S^3}(\mathcal{E}_B + \mathcal{E}_F) = \frac{1}{\mathrm{Vol}\, S^3}\frac{3}{16\beta_0} = \frac{3}{32\pi^2\beta_0^4}.$$

Thus at critical chemical potential, and zero temperature, the effective potential is flat even after taking into account quantum effects at the one-loop level. Furthermore it contains no non-analyticity at the origin. In [1] an alternative explanation for this phenomenon is offered in terms of the BPS states of the theory becoming massless when the theory is deformed by introducing the critical chemical potentials; the ground states of the Hamiltonian with one, two or three critical chemical potentials are infinite sets of one half, one quarter or one eighth BPS states respectively, each of which has zero energy.

2.6 The effective potential at finite temperature

2.6.1 One critical chemical potential

Let us now turn on a small, non-zero temperature, and consider the effective potential for the case where $\mu_1 = \beta_0^{-1}$, $\mu_2 = \mu_3 = 0$; the second terms in (30) and (31) now give rise to additional terms in the one-loop effective potential. Those terms are

$$-\frac{1}{\beta \mathrm{Vol}\, S^3}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sum_{i,l} d_l^i \mathrm{e}^{-n\beta|\varepsilon_l^i|}\cos(n\theta) + (-)^n \sum_{j,l} d_l^i \mathrm{e}^{-n\beta|\varepsilon_l^j|}\cos(n\theta)\right\} \quad (43)$$

where the index i ranges over the integer-spin modes and j ranges over the spin-1/2 modes. Since the exponentials fall off faster than any power of l as $l \longrightarrow \infty$, there is no need to regularise the sums over l. Note that the factor of 2 in (30) and (31) has cancelled with the factor of 1/2 in (23). Also, we have again omitted the trace over the adjoint representation of the Lie algebra; it should be understood that the actual effective potential is given by the expression given above with φ replaced by $\varphi_{ij} \equiv \varphi_i - \varphi_j$, and similarly for θ, and the result summed over i and j.

Let us calculate the summand in (43) for fixed n. As in the calculation of the Casimir energy, from Table 1 it is again clear that the modes C_i, $(c, \bar{c})$ and the first (A_0, ϕ_a) cancel whenever $l \geq 1$, and so the overall contribution from these fields to (43) is

$$-\mathrm{e}^{-n\beta\varphi}\cos(n\theta),$$

arising from $l = 0$. It can be seen that the remaining energies are all of the form $\sqrt{\beta_0^2(l+m)^2+\varphi^2}$, for some $m \in \mathbb{N}$. Their contributions can therefore be written in the following form, by appropriate changes of variables $l \to l - m$:

$$\sum_{l=0}^{\infty} \mathrm{e}^{-n\sqrt{l^2+\varphi^2}} \cos(n\theta) \left\{ 2(l-1)(l+1) + (l+1)^2 + (l-1)^2 + 4l^2 \right.$$
$$\left. -(-)^n \left(4l(l+1) + 4l(l-1)\right) \right\} + \mathrm{e}^{-n\beta\varphi} \cos(n\theta).$$

The last term, which arises due to the change in ranges of summation over l, cancels with the contribution from the remaining modes. The coefficient occurring the in above sum is equal to

$$\begin{cases} 0 & \text{when } n \text{ is even} \\ 16l^2 & \text{when } n \text{ is odd.} \end{cases}$$

From (25) the one-loop effective potential V_1 is therefore, up to a constant,

$$\sum_{ij} \frac{1}{\mathrm{Vol}\, S^3} \left\{ \frac{3}{16\beta_0} - \frac{16}{\beta} \sum_{n,l=0}^{\infty} \frac{1}{2n+1} \cos\left((2n+1)\theta_{ij}\right) l^2 \mathrm{e}^{-(2n+1)\beta\sqrt{\beta_0^{-2}l^2+\varphi_{ij}^2}} \right\}. \tag{44}$$

It is interesting to note that this expression is analytic in φ even at the origin, at which point the off-diagonal modes become massless (recall equation (20)): The sum over n can be performed exactly, using

$$\sum_{n}^{\infty} \frac{1}{2n+1} \left(\mathrm{e}^{(2n+1)i\theta} + \mathrm{e}^{-(2n+1)i\theta}\right) \mathrm{e}^{-(2n+1)\beta\varepsilon}$$
$$= \tanh^{-1}\left(\mathrm{e}^{-\beta\varepsilon+i\theta}\right) + \tanh^{-1}\left(\mathrm{e}^{-\beta\varepsilon-i\theta}\right); \tag{45}$$

the $\tanh^{-1}$ functions only have a singularity as $\varepsilon, \theta \to 0$, but the only non-zero terms in (44) have $l \geq 1$, $\varepsilon \geq \beta_0^{-1}$. In particular it is straightforward to show that the sum over l and n is finite at $\theta = \varphi = 0$: define

$$x \equiv \mathrm{e}^{-\frac{\beta}{\beta_0}} < 1$$

and, performing the sum over l first,

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{l=0}^{\infty} l^2 x^{l(2n+1)} = \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{x^{2n+1}(1+x^{2n+1})}{(1-x^{2n+1})^3}.$$

Note that successive terms in the remaining sum over n satisfy

$$\frac{(2n+1)x^{2n+3}(1+x^{2n+3})/(1-x^{2n+3})^3}{(2n+3)x^{2n+1}(1+x^{2n+1})/(1-x^{2n+1})^3} < x^2$$

and so, by the ratio test [30], this sum converges absolutely. On a neighbourhood of the origin the sum over l is therefore a uniform limit of analytic functions and is therefore analytic. Thus the singularity in (29) as ε and θ approach zero has cancelled exactly.

We now wish to determine the minimum of the potential (44). For any given φ, it is minimised by taking $\theta_{ij} = 0$ for all i and j; the potential for θ is attractive . Therefore at finite temperature the VEVs for A_0 satisfy

$$\theta_1 = \theta_2 = \ldots \theta_N$$

and for large N the distribution of eigenvalues of α is a delta function. Setting $\theta_{ij} = 0$ in (44), we now show that quantum effects at one-loop contribute a small positive mass squared to φ_{ai}. The following identities are easily verified: if f is a smooth function, then

$$\begin{aligned}
\frac{\partial}{\partial\varphi_{ai}}\sum_{kl} f(\varphi_{kl}^2) &= 4\sum_k f'(\varphi_{ik}^2)\varphi_{ik} \\
\frac{\partial^2}{\partial\varphi_{ai}\partial\varphi_{aj}}\sum_{kl} f(\varphi_{kl}^2) &= 4\sum_k f'(\varphi_{ik}^2)\delta_{ij} - 4f'(\varphi_{ij}^2) \\
&+ 8\sum_k \varphi_{ik}^2 f''(\varphi_{ik}^2)\delta_{ij} - 8\varphi_{ij}^2 f''(\varphi_{ij}^2).
\end{aligned}$$

Applying this to (44), and using (45), we find that

$$\begin{aligned}
V_1|_{\varphi=\theta=0} &= \frac{N^2}{\mathrm{Vol}\,S^3}\left(\frac{3}{16\beta_0} - \frac{16}{\beta}\sum_{l=0}^{\infty} l^2 \tanh^{-1}\left(\mathrm{e}^{-l\frac{\beta}{\beta_0}}\right)\right) \\
\frac{\partial}{\partial\varphi_{ai}}V_1\bigg|_{\varphi=\theta=0} &= 0 \\
\frac{\partial^2}{\partial\varphi_{ai}\partial\varphi_{aj}}V_1\bigg|_{\varphi=\theta=0} &= \frac{32\beta_0}{\mathrm{Vol}\,S^3}\sum_{l=0}^{\infty} l\frac{\mathrm{e}^{-l\frac{\beta}{\beta_0}}}{1-\mathrm{e}^{-2l\frac{\beta}{\beta_0}}}\left(N\delta_{ij}-1\right). \qquad (46)
\end{aligned}$$

In the limit of small temperature (more precisely, the limit $\beta_0 T \ll 1$) the sums over l are dominated by their terms with $l = 1$. Keeping $\theta = 0$, and

expanding the effective potential up to second order in φ, we have for small temperature and one critical chemical potential,

$$V_0 + V_1 = \frac{N^2}{\pi^2\beta_0^3}\left(\frac{3}{32\beta_0} - \frac{8}{\beta}e^{-\frac{\beta}{\beta_0}}\right)$$
$$+\frac{N}{\lambda}(\beta_0^{-2} - \mu_1^2)\sum_i \varphi_i^2 + \frac{8}{\pi^2\beta_0^2}e^{-\frac{\beta}{\beta_0}}\sum_a \varphi_{ai}\varphi_{aj}M_{ij} + \mathcal{O}(\varphi^4) \qquad (47)$$

where M is the matrix with components $N\delta_{ij} - 1$. Its eigenvalues are 0, with degeneracy 1, and N, with degeneracy $N-1$. Although $\beta_0^{-2} - \mu_1^2$ is zero at the critical value for μ_1, we leave the tree-level mass term explicit since it will be used again shortly. It is interesting to note that the presence of a non-zero temperature has lowered the vacuum energy. It can also be seen that the non-zero temperature has caused the light scalars to gain a small mass: in order to find the scalar masses it is necessary to take into account the $\mathfrak{su}(N)$ condition that each φ_a is traceless. To this end we take φ_{ai}, $i = 1 \ldots N-1$ as independent variables and let

$$\varphi_{aN} = -\sum_{i=1}^{N-1}\varphi_{ai}.$$

For a given matrix (m_{ij}), $i, j = 1 \ldots N$, we the find that

$$\sum_{i,j=1}^{N}\varphi_i\varphi_j m_{ij} = \sum_{i,j=1}^{N-1}\varphi_i\varphi_j\left(m_{ij} - m_{iN} - m_{Nj} + m_{NN}\right). \qquad (48)$$

Therefore, with $\mu_1 = \beta_0^{-1}$ in (47), the mass terms for the scalars are

$$\frac{8}{\pi^2\beta_0^2}e^{-\frac{\beta}{\beta_0}}\sum_a\sum_{i,j=1}^{N-1}\varphi_{ai}\varphi_{aj}\tilde{M}_{ij}$$

where $\tilde{M}$ is the matrix with components $N\delta_{ij} + N$, whose eigenvalues are N with degeneracy $N-2$ and N^2 with degeneracy 1. Therefore the scalars have gained a small mass given by

$$m = \frac{4}{\pi\beta_0}\sqrt{N}e^{\frac{\beta}{2\beta_0}}$$

with degeneracy $N-2$ for each $a \in \{1, 2\}$, and

$$m = \frac{4}{\pi\beta_0}Ne^{\frac{\beta}{2\beta_0}}$$

with degeneracy 1 for each $a \in \{1, 2\}$.

We now show that, when the chemical potential is greater than but sufficiently near to its critical value, the theory is in a metastable phase. From the tree-level term (18) it appears that the theory is unstable whenever $\mu_1 > \beta_0^{-1}$. Recall, however, that the loop approximation to the effective potential is an expansion in the 't Hooft coupling λ. Therefore when the difference $\beta_0^{-2} - \mu_1^2$ is $\beta_0^{-2}\mathcal{O}(\lambda)$, which is what more precisely is meant by the near-critical region, the instability in the tree-level term V_0 is comparable to the V_1. Let $\mu_1 = \beta_0^{-1} + \delta\mu$, with $\delta\mu \in \beta_0^{-1}\mathcal{O}(\lambda)$. The change in V_1 arising from a non-zero $\delta\mu$ is $\mathcal{O}(\lambda)$, and so it only affects the two-loop correction to the effective potential. Thus the above analysis with $\mu_1 = \beta_0^{-1}$ is sufficient for our purposes, and we need only consider the change in tree-level term. From (44), at large (compared to β_0^{-1}) values for φ the φ-dependent part of the one-loop effective potential becomes insignificant, and is dominated by the negative mass term of the tree-level potential. However, for small φ the approximation (47) is appropriate. There, the mass terms in φ_{ai} are (neglecting $\delta\mu^2$, and using (48)):

$$\left(-\frac{2}{\lambda\beta_0}\delta\mu + \frac{8}{\pi^2\beta_0^2}\mathrm{e}^{-\frac{\beta}{\beta_0}}\right)\sum_a \sum_{i,j=1}^{N-1} \varphi_{ai}\varphi_{aj}\tilde{M}_{ij}$$

and therefore for small φ the potential has a positive mass squared provided

$$\delta\mu < \frac{4\lambda}{\pi^2\beta_0}\mathrm{e}^{-\frac{\beta}{\beta_0}}. \tag{49}$$

Figure 4 shows the shape of the potential $V_0 + V_1$ when μ_1 is in this range. One may reasonably conjecture that this metastable phase is the continuation to low temperature of the high-temperature metastable phase found by Yamada and Yaffe [8]. This situation will be illustrated in Section 2.7, Figure 6.

While in the metastable phase with small φ, the theory may become unstable either by increasing the temperature, or by quantum tunnelling into the unstable region. The probability per unit time for the latter to occur is proportional to e^{-h}, where h is the height of the barrier between the metastable and unstable regimes. In order to see how this probability scales with N, consider first the form of the potential when a single eigenvalue of φ grows large; more precisely, in order to satisfy the $\mathfrak{su}(N)$ constraint that $\mathrm{Tr}\,\varphi = 0$, consider the case where each φ_i is aligned along the $a = 1$ direction, and

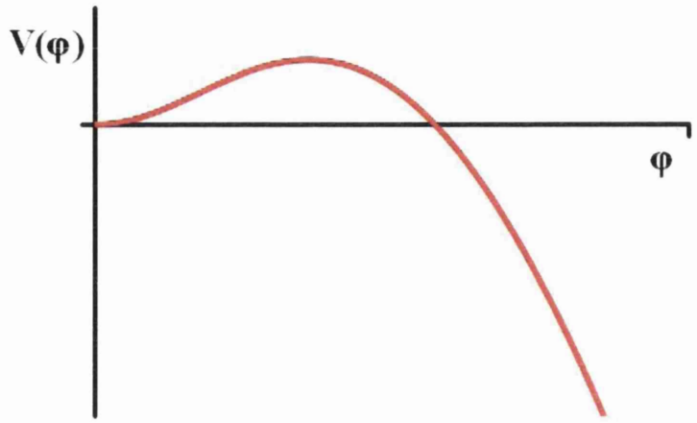

Figure 4: The effective potential up to one loop with $0 < \mu_1 - \beta_0 < 4\lambda e^{-\frac{\beta}{\beta_0}}/\pi^2\beta_0$, $\mu_2 = \mu_3 = 0$. The potential is shown as a function of φ_1 with $\varphi_i = 0$ for $i > 0$, and $\theta_i = 0$ for all i.

$$\varphi_{1\,i} = \begin{cases} x & i = 1 \\ -\frac{x}{N-1} & i = 2 \ldots N. \end{cases}$$

Then $V_0 + V_1$ is equal to

$$- \frac{2N\delta\mu}{\lambda\beta_0}\left(1 + \frac{N-1}{(N-1)^2}\right)x^2$$
$$- \frac{16}{\pi^2\beta_0^3\beta}(N-1)\sum_{l=0}^{\infty} l^2 \tanh^{-1}\left(e^{-\beta\sqrt{\beta_0^{-2}l^2 + \left(\frac{N-2}{N-1}\right)^2 x^2}}\right)$$

plus terms independent of x. For fixed x and large N this is approximately proportional to N, so that the probability of a single eigenvalue tunnelling into the unstable region scales like e^{-N}, and is therefore exponentially suppressed in the large N limit.

Alternatively, instead of considering the case where a single eigenvalue tunnels out of the metastable region, one may instead ask how the height of the barrier scales with N for coherent variations of the background scalar field. To this end we suppose that β, β_0 and N vary while the ratio $\delta\mu/\delta\mu_c \equiv k < 1$ is held fixed, where $\delta\mu_c$ is the quantity in the right hand side of (49). We also make a further approximation by considering only the term with $n = 0$, $l = 1$ in (44) and writing

$$e^{-\frac{\beta}{\beta_0}\sqrt{1+\beta_0^2\varphi^2}} \approx e^{-\frac{\beta}{\beta_0}} e^{-\frac{\beta\beta_0\varphi^2}{2}}$$

so that the effective potential $V_0 + V_1$ is approximately given by (ignoring field-independent terms)

$$-\frac{8N^2}{\pi^2\beta_0^3\beta} e^{-\frac{\beta}{\beta_0}} \left(\frac{k}{N} \sum_{i=1}^{N} \beta\beta_0\varphi_i^2 + \frac{1}{N^2} \sum_{i,j=1}^{N} e^{-\frac{\beta\beta_0\varphi_{ij}^2}{2}} \right).$$

In the large N limit we may introduce a density ρ of eigenvalues for the matrices $\sqrt{\beta\beta_0}\varphi_a$, and replace the sums over i and j with integrals over $\mathbb{R}^2$, together with a factor of N to account for the fact that $\int_{\mathbb{R}^2} dx^2 \rho(x) = N$; then the above expression for the effective potential becomes

$$-\frac{8N^2}{\pi^2\beta_0^3\beta} e^{-\frac{\beta}{\beta_0}} \left(k \int dx^2\, \rho(x)x^2 + \int dx^2 dy^2\, \rho(x)\rho(y) e^{-\frac{(x-y)^2}{2}} \right).$$

The quantity in parentheses contains no explicit dependence on β, β_0 or N and therefore all scaling of the height of the barrier, considered as a surface parameterised by the space of possible distributions ρ, is contained in the factor $\frac{8N^2}{\pi^2\beta_0^3\beta} e^{-\frac{\beta}{\beta_0}}$. Thus the probability per unit time for a coherent distribution of eigenvalues to tunnel into the unstable region scales like e^{-N^2} and so is even more suppressed in the large N limit than that for a single eigenvalue.

In fact one may attempt to calculate the effective mass at the origin at arbitrary (not necessarily small) temperature, using the exact expression given in (46), together with the tree-level mass term. One finds that for $\mu_1 > \beta_0^{-1}$ the theory is in a metastable phase provided that μ_1 is less than a critical value, given by

$$\mu_c^2 = \beta_0^{-2} + \frac{8\lambda}{\pi^2\beta_0^2} \sum_{n=1}^{\infty} n \frac{e^{-n\frac{\beta}{\beta_0}}}{1 - e^{-2n\frac{\beta}{\beta_0}}}. \tag{50}$$

Figure 5 shows an approximate plot of μ_c^2 against $T^2 = \beta^{-2}$, with $\beta_0 = 1$, $\lambda = \pi^2/8$. It can be seen to possess the qualitative features that we expect from comparison to the conjectured strong-coupling diagram (compare to Figure 8 of Section 2.7): at low temperature μ_c^2 shows exponential growth as in equation (49), while for high temperature the growth is linear as described in [8], where it was found that the theory at high temperature is in a metastable phase provided that

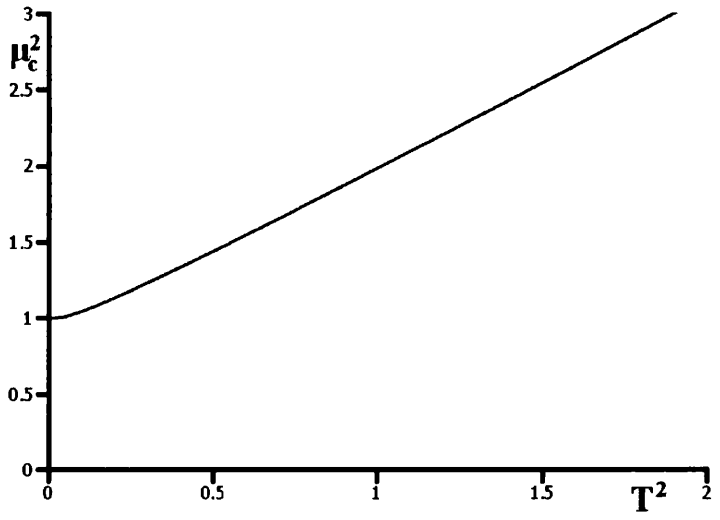

Figure 5: A plot of the critical chemical potential μ_c^2 as a function of temperature squared, as given by equation (50), with $\beta_0 = 8\lambda/\pi^2\beta_0^2 = 1$.

$$\beta_0^{-2} < \mu_1^2 < \beta_0^{-2} + \lambda T^2.$$

The emergence of linear behaviour at high temperature may be understood by the following heuristic argument. Consider a single summand

$$n\frac{\mathrm{e}^{-n\frac{\beta}{\beta_0}}}{1-\mathrm{e}^{-2n\frac{\beta}{\beta_0}}};$$

if $n\beta/\beta_0$ is $\mathcal{O}(1)$ then this term is also $\mathcal{O}(1)$, while for much larger n it is $\ll 1$. On the other hand, when $n\beta/\beta_0 \ll 1$, the exponentials are well approximated by the first and second terms in their Taylor expansions, so that

$$n\frac{\mathrm{e}^{-n\frac{\beta}{\beta_0}}}{1-\mathrm{e}^{-2n\frac{\beta}{\beta_0}}} \approx \frac{\beta_0}{2\beta}$$

which is $\gg 1$. It is therefore reasonable to assume that the sum over n is dominated by those terms for which

$$n\frac{\beta}{\beta_0} < \epsilon$$

for some small $\epsilon > 0$. Each such term contributes approximately $\beta_0/2\beta$ and there are approximately $\epsilon\beta_0/\beta$ such terms, so that the sum over n is approximately given by

$$\frac{\epsilon\beta_0^2}{2\beta^2} \propto T^2.$$

Assuming that the sum is indeed linear in T^2 up to terms which are $\mathcal{O}(1)$, it may further be seen from (50) that all dependence of μ_c on β is contained in the dimensionless quantity β/β_0. We therefore expect that

$$\lim_{\beta\to 0} \beta^2 \sum_{n=1}^{\infty} n \frac{\mathrm{e}^{-n\frac{\beta}{\beta_0}}}{1-\mathrm{e}^{-2n\frac{\beta}{\beta_0}}} = \beta_0^2 k$$

for some constant k. We now show[6] that $k = \pi^2/8$: write

$$\begin{aligned} x^2 \sum_{n=1}^{\infty} n \frac{\mathrm{e}^{-nx}}{1-\mathrm{e}^{-2nx}} &= x^2 \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} n\mathrm{e}^{-(2m+1)nx} \\ &= x^2 \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} n\mathrm{e}^{-(2m+1)nx} \\ &= \sum_{m=0}^{\infty} x^2 \frac{\mathrm{e}^{-(2m+1)x}}{\left(1-\mathrm{e}^{-(2m+1)x}\right)^2} \end{aligned}$$

where the absolute summability of the summand has allowed us to reverse the order of the sums over m and n. For $x \to 0$ the summand of the last line satisfies

$$\lim_{x\to 0} x^2 \frac{\mathrm{e}^{-(2m+1)x}}{\left(1-\mathrm{e}^{-(2m+1)x}\right)^2} = \frac{1}{(2m+1)^2}.$$

Moreover, defining $h \equiv -(2m+1)x$, it can be written as

$$\frac{1}{(2m+1)^2}\left[\frac{h^2\mathrm{e}^h}{(1-\mathrm{e}^h)^2}\right];$$

the term in brackets is easily seen to be bounded for $h \in \mathbb{R}$, and so the Weierstrass M-test [30] shows that the limit $x \to 0$ commutes with the sum over m. The result is therefore, using Euler's identity for $\sum_n 1/n^2$,

[6] I am very grateful to sci.math posters "Victor Meldrew" and "W^3" for providing this proof.

field	l	ε_l	d_l
(A_0, ϕ_a)	0	φ	1
(A_0, ϕ_a)	0	$\sqrt{\beta_0^{-2} + \varphi^2} - \beta_0^{-1}$	1
$(\psi_A, \bar{\psi}_A)$	1	$\sqrt{\beta_0^{-2} + \varphi^2} - \frac{1}{2}\beta_0^{-1}$	-4

Table 2: Light modes for $\mu_1 = \mu_2 = \beta_0^{-1}$, $\mu_3 = 0$.

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{6}\left(1 - \frac{1}{4}\right) = \frac{\pi^2}{8}$$

as required. Thus from (50) we expect that at high temperature the critical value μ_c^2 is close to

$$\mu_c^2 \approx \beta_0^{-2} + \lambda T^2.$$

in complete agreement with the high-temperature analysis of [8]. Note, however, that the above analysis rests on the assumption that $\mu_1 - \beta_0^{-1}$ is in $\beta_0^{-1}\mathcal{O}(\lambda)$, which breaks down at large μ_1. Therefore the above equation should only be considered valid in the limit of large T and small λ, with $\lambda T^2 \ll \beta_0^{-2}$.

2.6.2 Two critical chemical potentials

Let us now turn our attention to the contribution (43) in the case where $\mu_1 = \mu_2 = \beta_0^{-1}$, $\mu_3 = 0$. Again we consider first the summand for a given n. As before the modes C_i, $(c, \bar{c})$ and the first (A_0, ϕ_a) mode cancel whenever $l \geq 1$, and so the overall contribution from these fields to (43) is once again equal to

$$-\mathrm{e}^{-n\beta\varphi}\cos(n\theta); \tag{51}$$

for the remaining modes, however, the situation is more complicated than before. The energies are no longer all of the form $\sqrt{\beta_0^{-2}(l+m)^2 + \varphi^2}$, $m \in \mathbb{Z}$, so the different modes can no longer be reduced to the sum over l of a single term. Instead we consider the limit of low temperature, so that the sum is dominated by those modes of lowest energies. These may be read off from Table 1; the results are given in Table 2.

The first light (A_0, ϕ_a) mode cancels with the contribution from (51). The remaining modes give the following finite-temperature contribution to the effective potential:

$$-\frac{1}{\beta \mathrm{Vol}\, S^3}\sum_{n=1}^{\infty}\frac{1}{n}\cos(n\theta)\left\{\mathrm{e}^{-n\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\beta_0^{-1})}\right.$$
$$\left.-4(-)^n\mathrm{e}^{-n\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\frac{1}{2}\beta_0^{-1})}\right\}+\mathcal{O}(\mathrm{e}^{-\beta/\beta_0}).$$

The sum over n can be performed exactly, giving

$$\frac{1}{2\beta \mathrm{Vol}\, S^3}\left\{\log\left(1-2\mathrm{e}^{-\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\beta_0^{-1})}\cos(\theta)+\mathrm{e}^{-2\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\beta_0^{-1})}\right)\right.$$
$$\left.-4\log\left(1+2\mathrm{e}^{-\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\frac{1}{2}\beta_0^{-1})}\cos(\theta)+\mathrm{e}^{-2\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\frac{1}{2}\beta_0^{-1})}\right)\right\}. \quad (52)$$

For large values of φ it can be seen that the expression is exponentially suppressed. On the other hand, for small φ, θ we run into a problem: the effective potential has a logarithmic singularity at $\varphi = \theta = 0$, and is unbounded from below.

A resolution to this problem is offered in [1]. There it is explained that the singularity as θ and φ go to zero is a consequence of wrongly integrating out the light off-diagonal modes of ϕ, and that the correct approach is to compute the Feynman one-loop self-energy graphs at finite temperature. For fixed φ the potential for θ is attractive, and so the VEVs again satisfy

$$\theta_1 = \theta_2 = \ldots \theta_N.$$

The result given there for the effective potential computed by this alternative method is

$$\frac{N^2}{2\pi^2\beta_0^3}\left(\frac{3}{16\beta_0}-\frac{2}{\beta}\mathrm{e}^{-\frac{\beta}{2\beta_0}}+\frac{2\beta_0}{N}\mathrm{e}^{-\frac{\beta}{2\beta_0}}\mathrm{Tr}\,\varphi^2+\text{quartic}\right)$$

with the non-analyticity in the expression (52) coming from higher order interactions.

2.6.3 Three critical chemical potentials

The calculation of the effective potential at small but finite temperature when $\mu_1 = \mu_2 = \mu_3 = \beta_0^{-1}$ is similar to that of the previous subsection. The fields C_i, $(c, \bar{c})$ and the first (A_0, ϕ_a) mode again contribute a single term given by (51). Also, since the energies are no longer of the form $\sqrt{\beta_0^{-2}(l+m)^2+\varphi^2}$, $m \in \mathbb{Z}$, it is again necessary to consider only the lightest modes to make the potential analytically tractable. The light modes are given in Table 3.

field	l	ε_l	d_l
(A_0, ϕ_a)	0	φ	1
(A_0, ϕ_a)	0	$\sqrt{\beta_0^{-2} + \varphi^2} - \beta_0^{-1}$	2
$(\psi_A, \bar{\psi}_A)$	1	$\sqrt{\beta_0^{-2} + \varphi^2} - \beta_0^{-1}$	-2

Table 3: Light modes for $\mu_1 = \mu_2 = \mu_3 = \beta_0^{-1}$.

The resulting contribution to the effective potential is

$$-\frac{1}{\beta \mathrm{Vol}\, S^3} \sum_{n=0}^{\infty} \frac{1}{2n+1} \cos\left((2n+1)\theta\right) 4\mathrm{e}^{-(2n+1)\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\beta_0^{-1})} + \mathcal{O}(\mathrm{e}^{-\beta/\beta_0})$$
$$= -\frac{2}{\beta \mathrm{Vol}\, S^3} \left\{ \tanh^{-1}\left(\mathrm{e}^{-\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\beta_0^{-1})+i\theta}\right) + \tanh^{-1}\left(\mathrm{e}^{-\beta(\sqrt{\beta_0^{-2}+\varphi^2}-\beta_0^{-1})-i\theta}\right) \right\}.$$

Similar comments apply to the case of two critical chemical potentials: the potential for θ is attractive, so in the large N limit the eigenvalue distribution of θ is described by a delta function. Since the potential is singular as $\varphi, \theta \to 0$ it is not appropriate to integrate out the off-diagonal modes for ϕ, but instead we should use the method mentioned in the previous subsection. The result is [1]

$$\frac{N^2}{2\pi^2\beta_0^3}\left(\frac{3}{16\beta_0} - \frac{12}{\beta}\mathrm{e}^{-\frac{\beta}{\beta_0}} + \frac{8\beta_0}{N}\mathrm{e}^{-\frac{\beta}{\beta_0}}\mathrm{Tr}\,\varphi^2 + \text{quartic}\right).$$

Consider now the case where $\delta\mu_p \equiv \mu_p - \beta_0^{-1}$ is positive and $\mathcal{O}(\lambda)$, so that any change to V_1 from the case of critical chemical potential is $\mathcal{O}(\lambda)$ and therefore only affects the potential at the two-loop level. V_1 becomes constant and finite as $\varphi \to \infty$, so that the negative tree-level mass term makes the theory unstable in this limit. Close to $\varphi = 0$ the mass terms are, in the cases of two and three non-zero chemical potentials respectively,

$$\sum_p \left(\frac{N}{\lambda}(\beta_0^2 - \mu_p^2) + \frac{N}{\pi^2\beta_0^2}\mathrm{e}^{-\frac{\beta}{2\beta_0}}\right) \sum_i (\varphi_{2p-1\,i}^2 + \varphi_{2p\,i}^2)$$

$$\sum_p \left(\frac{N}{\lambda}(\beta_0^2 - \mu_p^2) + \frac{4N}{\pi^2\beta_0^2}\mathrm{e}^{-\frac{\beta}{\beta_0}}\right) \sum_i (\varphi_{2p-1\,i}^2 + \varphi_{2p\,i}^2).$$

Therefore the theory has a metastable phase if each $\delta\mu_p^2$ satisfies

$$\begin{cases} \delta\mu_p < \frac{\lambda}{2\pi^2\beta_0} e^{-\frac{\beta}{2\beta_0}} & \text{two non-zero chemical potentials} \\ \delta\mu_p < \frac{2\lambda}{\pi^2\beta_0} e^{-\frac{\beta}{\beta_0}} & \text{three non-zero chemical potentials.} \end{cases}$$

When in the metastable phase the theory may become unstable either by thermal activation or by quantum tunnelling through the barrier. From similar considerations to those of Section 2.6.1 the expected lifetime of the theory before the latter possibility takes place goes to infinity in the large N limit.

2.7 Summary of results

We now summarise the known facts about the phase diagram of weakly coupled, Euclidean $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$. As described by Yamada and Yaffe [8] the theory exhibits a first order deconfinement/confinement phase transition, with a phase line whose precise shape depends on the number of non-zero chemical potentials, but which schematically looks like that shown in Figure 6. It was also shown in [8] that at high temperature the effective potential at the origin of φ-space gives the scalar VEVs φ_{2p-1} and φ_{2p} an effective mass given by

$$m_p^2 = \beta_0^{-2} - \mu_p^2 + \lambda T^2,$$

while for large field amplitudes the quantum corrections are due to heavy states which are integrated out at low energies, so that the effective potential is dominated by its tree-level term. Therefore, for chemical potentials greater than their critical value of β_0^{-1} (so that the tree-level mass term is negative), there exists a metastable phase provided that the largest of the chemical potentials μ satisfies

$$\mu < \sqrt{\beta_0^{-2} + \lambda T^2}.$$

The lifetime of this state is determined by the probability per unit time for one eigenvalue of φ to tunnel out of the stable region, which for high temperature ($T \sim 1/\sqrt{\lambda}\beta_0$) is proportional to $\exp(-N\beta_0^3 T^3)$; in particular this probability is exponentially suppressed in the large N limit.

Meanwhile, in this work (which reproduces results first given in [1]) we have considered the theory at low temperature ($T \ll 1/\beta_0$), with one or more chemical potentials close to their critical value. We have shown that

the metastable phase line persists at low temperature, where the theory is in the metastable phase provided that the largest of the chemical potentials satisfies

$$\mu < \beta_0^{-1} + \frac{4\lambda}{\pi^2\beta_0} e^{-\frac{\beta}{\beta_0}},$$
$$\mu < \beta_0^{-1} + \frac{\lambda}{2\pi^2\beta_0} e^{-\frac{\beta}{2\beta_0}}$$

or

$$\mu < \beta_0^{-1} + \frac{2\lambda}{\pi^2\beta_0} e^{-\frac{\beta}{\beta_0}}$$

depending on whether one, two or three chemical potentials are near-critical. These results are summarised in Figure 6 (which should be merely considered schematic rather than numerically accurate). The figure also includes the conjectured shape of the instability phase line, if we assume that this line continues between the known high and low temperature limits. Figure 7 shows a plot of the instability phase line over a large temperature range, as given by equation (50) of Section 2.6.1. Interestingly it can be seen that its shape very closely matches the conjectured shape of Figure 6. Note however that equation (50) is only valid in the region where $\mu - \beta_0^{-1}$ is $\beta_0^{-1}\mathcal{O}(\lambda)$, and also only in the case where $\mu_2 = \mu_3 = 0$.

We now consider the corresponding phase diagram at strong coupling, where the theory is conjectured to be dual to Type IIB supergravity on $AdS_5 \times S^5$, with the three chemical potentials of the $\mathcal{N} = 4$ theory corresponding to three independent angular velocities along the S^5 direction. This is schematically shown in Figure 8. In the case corresponding to three equal chemical potentials the theory undergoes a phase transition similar to the Hagedorn/ deconfinement phase transition seen in the gauge theory. This is the Hawking-Page transition, which in the semiclassical supergravity approximation is explained by a "flop" between two possible bulk geometries: the spinning $AdS_5 \times S^5$ geometry and the Reissner-Nordstrom charged AdS black hole. Similar results apply to the case of more general chemical potentials.

When in the black hole phase, the supergravity theory becomes unstable for large values of μ_p, and at high temperature the instability line grows linearly. It has also recently been argued [16] that the line between stable and metastable phases shown in Figure 6 also exists in the supergravity case, and that the black hole geometry is metastable for $\beta_0^{-1} < \mu < \mu_c$, where

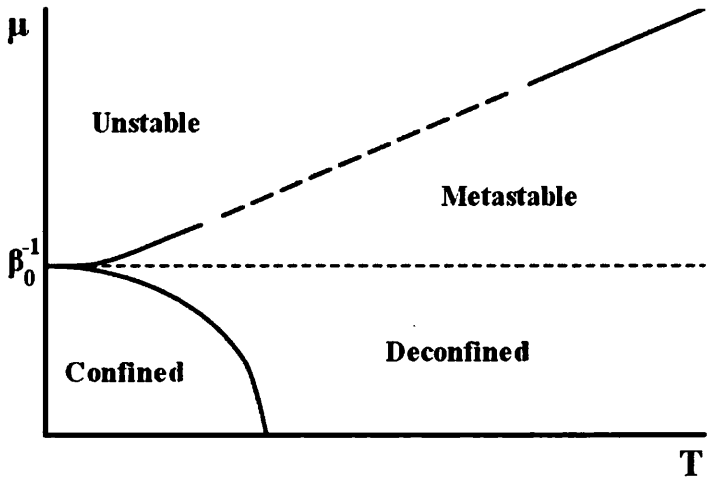

Figure 6: A schematic diagram including the known features of the phase diagram of weakly coupled $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$. The metastable phase at low temperature is that described here. At high temperature the almost-linear metastable phase line is that found by Yamada and Yaffe. The dashed line is to indicate that it is not known for certain whether this metastable phase line continues between the high and low temperature limits.

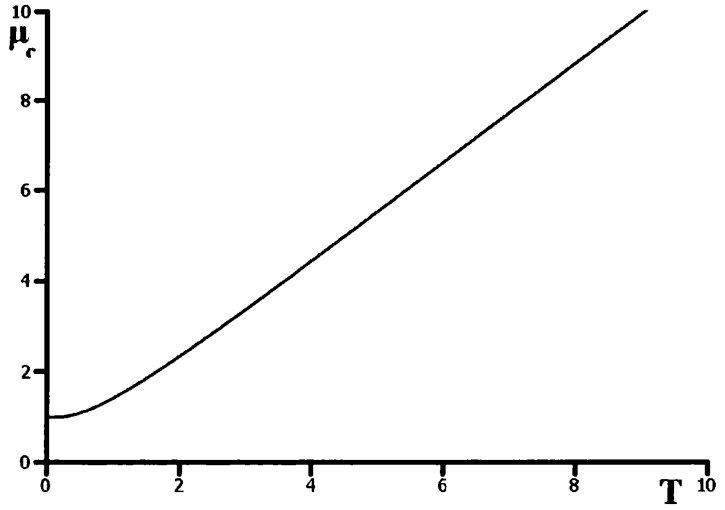

Figure 7: A plot of the critical chemical potential μ_c against temperature T, as given by equation (50), with $\beta_0 = 8\lambda/\pi^2\beta_0^2 = 1$; the sum over n was terminated when the summand dropped below 10^{-10}. Note the similarity of its shape to the conjectured instability phase line shown in Figure 6.

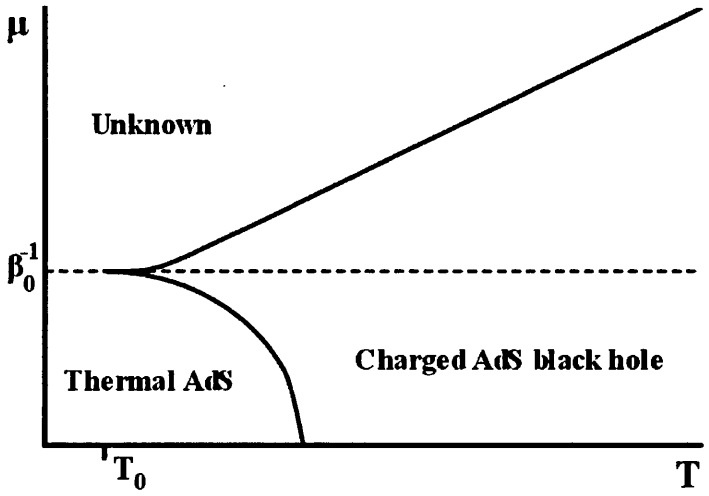

Figure 8: A schematic diagram showing features of the phase diagram of Type IIB supergravity on $AdS_5 \times S^5$, which is conjectured to be dual to strongly coupled $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$.

$\mu_c(T)$ defines the instability line. The similarities between the two diagrams 6 and 8 are obvious, and lend credence to the idea that the weak-coupling limit of gauge theories may be used to make predictions about their strong-coupling analogues, and also their conjectured duals. However there are some qualitative differences which should also be pointed out. In Figure 8, the instability line meets the Hawking-Page transition line at $T = T_0$, the temperature at which the black hole radius is zero. This depends on the relative size of the three chemical potentials: with $\mu_1 = \mu$, $\mu_2 = \mu_3 = 0$ we have $T_0 = 1/\pi\beta_0$, while for $\mu_1 = \mu_2 = \mu_3$, T_0 is zero. The behaviour of the theory at temperatures below T_0 is not well understood. Also, although the black hole instability line rises linearly at high temperature, unlike the high-temperature weak-coupling instability line its slope varies with the ratio of the three chemical potentials: with $\mu_1 = \mu, \mu_2 = \mu_3 = 0$ its slope is $\pi/\sqrt{2}$, while for $\mu_1 = \mu_2 = \mu_3$ it is 2π.

2.8 Further questions

In light of the results found here and elsewhere about the phase structure of $\mathcal{N} = 4$ super Yang-Mills theory on $S^3 \times S^1$, and in particular the similarity between that and the phase structure of the strongly coupled theory, it is reasonable to ask whether this similarity exists at all intermediate values

of the 't Hooft coupling. If not then this would imply that the theory has non-analyticities in the 't Hooft coupling. Otherwise it may be possible to understand how the phase diagram at weak coupling is mapped onto that at strong coupling, and in particular how and where the differences noted in the last section (namely the non-zero temperature T_0 and the change in slope of the instability line at high temperature) arise. One possible generalisation of the theory considered here is the β-deformed theory, which has three global $U(1)$ symmetries to which one may again associate three chemical potentials. Since [1] was completed this generalisation has been pursued by Elander [32], who found that the low-temperature metastable phase found in [1] exists also in the weakly coupled β-deformed theory using similar methods to those described here, and also showed that it exists in the strongly coupled theory by performing a probe-brane calculation in the dual gravitational background.

3 The Vilkovisky effective potential of scalar electrodynamics

3.1 Introduction to Section 3

Scalar electrodynamics, although not directly applicable to real-world physics, nonetheless provides an interesting toy model with which to study physical phenomena. One such phenomenon was first recognised by Coleman and Weinberg [35], namely that the theory may exhibit spontaneous symmetry breaking even when the classical action indicates no such breaking, due to one-loop radiative corrections. The natural tool for studying spontaneous symmetry breaking is the effective potential V; however it has long been recognised [36] that the ordinary effective action is gauge-dependent. This makes it difficult to assign any physical significance to the effective potential, and may also introduce false minima. It has also been claimed [37] that V cannot even be defined in the R_ξ gauges which are the usual setting for theories with spontaneously broken symmetries. In [38] Dolan and Jackiw calculated the effective potential of scalar electrodynamics using both the R_ξ gauge and the unitary gauge, which may either be obtained as the limit of the R_ξ gauge as some parameter goes to infinity, or alternatively considered to be a parameterisation of the physical configuration space of the theory in which gauge degrees of freedom have been removed. They found that V calculated in the R_ξ gauge is gauge-dependent, but argued that the non-renormalisability of the unitary Lagrangian prevented them from concluding that the unitary effective potential was always appropriate for practical calculations.

One possible resolution to this problem was given by Boyanovsky, Loinaz and Willey [39] who defined the physical effective potential as the expectation value of the Hamiltonian in physical states, and applied this to an Abelian Higgs model. Another approach, which we consider here, was put forward in a series of papers by Vilkovisky and DeWitt [40], [41]; this approach treats background fields as points in a configuration space which is not assumed to possess any affine structure. The idea behind their method is to replace the recurrence relation satisfied by the ordinary effective action Γ

$$\mathrm{e}^{-\Gamma[\varphi]} = \int \mathcal{D}\phi \exp\left(-S[\phi] + (\phi^i - \varphi^i)\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\right)$$

by a different recurrence relation in which the geometrically meaningless quantity $\phi^i - \varphi^i$ is replaced with a vector in the tangent space of the configuration space at φ. The effective action Γ is then defined iteratively using this

modified recurrence relation, which is manifestly invariant under coordinate transformations. For a review of Vilkovisky's method we recommend [42], [43]. Fradkin and Tseytlin [42] showed that the Vilkovisky effective action coincides with the ordinary effective action calculated in a particular gauge, namely Landau-DeWitt gauge, and demonstrated the calculation in the case of scalar electrodynamics using a generalised R_ξ gauge with two arbitrary real parameters. Russell and Toms [44] applied the method to the theory of a complex scalar in a background electromagnetic field, where coordinate invariance includes gauge invariance as a special case, as well as scalar electrodynamics in the unitary gauge; their result was in complete agreement with that of Fradkin and Tseytlin. More recently, Lin and Chiu [45] applied the method to both scalar electrodynamics and a simplified Standard Model, which was used to derive a gauge-invariant lower bound for the Higgs mass. In this work we attempt to generalise previous results on the effective action of scalar electrodynamics, by including a chemical potential associated to the $U(1)$ charge of the theory. This is implemented by giving the time-component (or 1-component in Euclidean space, which we consider here) a non-zero background field; recall from Section 2.3 that an imaginary background value for the gauge field is equivalent to a chemical potential associated to the electric charge of the theory.

The remainder of Section 3 is organised as follows. In Section 3.2 we will give a brief review of Vilkovisky's proposed modification of the effective action, in the context of a theory whose configuration space is described by an arbitrary Riemannian manifold $\mathcal{M}$, together with a scalar action S; the definition will be manifestly independent of a choice of coordinates on $\mathcal{M}$. In particular we show the Vilkovisky effective action may be understood as the conventional effective action of a theory parameterised by vectors in the tangent space of a point on $\mathcal{M}$. Section 3.3 explains how the definition of the previous section is applied to the case where $\mathcal{M}$ is the quotient of some manifold M under the action a gauge group G, by constructing a metric and associated connection on $\mathcal{M}$ in terms of a G-invariant metric on M. This section differs from other reviews that the author has seen in that coordinate-free definitions and proofs are presented whenever possible, since in the author's view the coordinate-based approach serves to obscure the underlying geometrical meaning of the concepts involved. In Section 3.3.1 we explain the relation between the integral over $\mathcal{M}$ with metric defined in the previous section and the Faddeev-Popov integral; in particular we show that the former is equal to a modified Faddeev-Popov integral which takes into account invariance under reparameterisations of the gauge group. In Section 3.3.2 we show that the one-loop effective potential of the gauge-fixed theory

calculated by Vilkovisky's method is, under fairly unrestrictive conditions, independent of the choice of gauge, and also has a simple geometrical interpretation in terms of the physical configuration space $\mathcal{M}$.

The remainder of Section 3 gives the calculation of the one-loop effective potential of scalar electrodynamics in Euclidean spacetime, with non-zero background fields for both the scalar and the gauge fields. In Section 3.4 we first derive unregularised expressions for the effective potential in both the generalised R_ξ gauge introduced by Fradkin and Tseytlin [42] and the unitary gauge, and show that they are equal (and in particular the former is gauge-independent), as predicted by the results of the previous section. In Section 3.5 we then attempt to explicitly calculate the regularised effective potential using dimensional regularisation; the presence of a non-zero background field for the gauge field presents some novel difficulties, and consequently we will only give an approximate expression valid when the gauge coupling g is small, together with a description of how terms of higher order in g may be systematically calculated.

3.2 The Vilkovisky Effective Action

In the conventional approach the effective action is defined by the Legendre transform (using the condensed notation of De Witt, in which the index i includes both position and discrete labels, and the sum over i includes integration over spacetime):

$$\Gamma[\varphi] \equiv W[J] - J_i\varphi^i \tag{53}$$

where $W[J]$ is the generating functional of connected Green's functions

$$\mathrm{e}^{-W[J]} \equiv \int \mathcal{D}\phi \exp\left(-S[\phi] - J_i\phi^i\right),$$

and J is given as a function of φ by solving the equation

$$\varphi^i = \frac{\delta W[J]}{\delta J_i}.$$

From this it follows that

$$\mathrm{e}^{-\Gamma[\varphi]} = \int \mathcal{D}\phi \exp\left(-S[\phi] + (\phi^i - \varphi^i)\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\right). \tag{54}$$

This equation can be used to solve for Γ iteratively, resulting in a loop expansion. The problem arises when one attempts to apply (54) to a theory

with a non-trivial configuration space $\mathcal{M}$, or to calculate the effective action after reparameterising the theory in a non-trivial way; geometrically the terms $\delta\Gamma[\varphi]/\delta\varphi^i$ are the components of a 1-form on the configuration space, and so should be contracted with a vector. However the quantities $(\phi^i - \varphi^i)$ are not the components of a vector, in fact in the absence of a natural affine structure on $\mathcal{M}$ this quantity makes no geometrical sense. The conventional effective action is therefore not invariant under coordinate transformations when φ is off-shell (i.e. when $\delta\Gamma[\varphi]/\delta\varphi^i \neq 0$).

The solution to this problem was given by Vilkovisky,[7] [40] who proposed an alternative to equation (54) which is manifestly invariant under diffeomorphisms of $\mathcal{M}$. The key step is to replace $(\phi^i - \varphi^i)$ with a quantity that transforms as a vector at φ, namely σ^i, defined as follows: given φ and ϕ on $\mathcal{M}$, let $c : [0,1] \to \mathcal{M}$ be a geodesic connecting φ and ϕ, with $c(0) = \varphi$ and $c(1) = \phi$, and define σ to be the vector tangent to $c(t)$ at $t = 0$. Then the new effective action is defined by

$$\mathrm{e}^{-\Gamma[\varphi]} = \int \mathcal{D}\phi \exp\left(-S[\phi] + \sigma^i \frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\right). \tag{55}$$

Let c now be given as a function of both t and σ, so that $c(\cdot, \sigma)$ is a geodesic through φ to which σ is tangent at $t = 0$. Let the coordinates of c be denoted c^i, and define $\eta^i = \phi^i - \varphi^i$. If we assume that in some neighbourhood of φ the components σ^i can be given as analytic functions of η, and vice-versa, then it is possible to give expressions for the coefficients of the relevant Taylor expansions. The defining equations for c^i are

$$\begin{aligned} \ddot{c}^i + \Gamma^i_{jk}\dot{c}^j\dot{c}^k &= 0 \\ \dot{c}^i(t=0) &= \sigma^i \\ c^i(t=1) &= \varphi^i + \eta^i. \end{aligned} \tag{56}$$

In the above Γ^i_{jk} is the Christoffel symbol of the metric on $\mathcal{M}$. Note that, as an immediate consequence of the above, we have that $c(t, a\sigma) = c(at, \sigma)$ for any real a. Suppose now that there exists a Taylor expansion for $\eta(\sigma)$:

$$\begin{aligned} \eta^i &= \sum_{n=1}^{\infty} \frac{1}{n!} b^i_{j_1\ldots j_n} \sigma^{j_1} \cdots \sigma^{j_n} \\ b^i_{j_1\ldots j_n} &= \frac{\partial^n \eta^i}{\partial\sigma^{j_1} \cdots \partial\sigma^{j_n}}. \end{aligned} \tag{57}$$

[7]The method was further refined by De Witt [41], however his definition of the effective action agrees with that of Vilkovisky in the one-loop approximation.

Using the fact that $c^i(t,\sigma) = \varphi^i + \eta^i(t\sigma)$, it follows that

$$\left.\frac{\partial^n c^i}{\partial t^n}\right|_{t=0} = \frac{\partial^n \eta^i}{\partial\sigma^{j_1}\cdots\partial\sigma^{j_n}}\sigma^{j_1}\cdots\sigma^{j_n}$$

so that the coefficients in (57) may be read off from the coefficients of $\sigma^{j_1}\cdots\sigma^{j_n}$ in the n^{th} time derivative of c^i at $t = 0$. (57) and (56) give

$$\begin{aligned} b^i_j &= \delta^i_j \\ b^i_{jk} &= -\Gamma^i_{jk} \end{aligned}$$

and for $n \geq 3$ induction on n gives

$$b^i_{j_1\ldots j_{n+1}} = b^i_{(j_1\ldots j_n, j_{n+1})} - n b^i_{k(j_1\ldots j_{n-1}}\Gamma^k_{j_n j_{n+1})}.$$

If we now suppose given a Taylor expansion for σ in terms of η

$$\sigma^i = \sum_{n=1}^{\infty}\frac{1}{n!}a^i_{j_1\ldots j_n}\eta^{j_1}\cdots\eta^{j_n} \tag{58}$$

then the coefficients $a^i_{j_1\ldots j_n}$ can be found by substituting (57) into (58). The first few coefficients are:

$$\begin{aligned} a^i_j &= \delta^i_j \\ a^i_{jk} &= \Gamma^i_{jk} \\ a^i_{jkl} &= \Gamma^i_{(jk,l)} + \Gamma^i_{m(j}\Gamma^m_{kl)}. \end{aligned}$$

As an aside, let f be an analytic function on $\mathcal{M}$, and let X be a vector field satisfying the geodesic equation

$$\nabla_X X = 0 \tag{59}$$

with $X_\varphi = \sigma$, so that $c(t) \equiv c(t,\sigma)$ is an integral curve of X. Assuming that c is analytic, the value of f at $\phi = c(1)$ is given by

$$f(\phi) = \sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{\partial^n(f\circ c)}{\partial t^n}\right|_{t=0}. \tag{60}$$

Note that, for any $(0,n)$-tensor ω on $\mathcal{M}$, we have

$$
\begin{aligned}
& \frac{\partial \omega_{c(t)}(X_{c(t)}, X_{c(t)}, \ldots X_{c(t)})}{\partial t} \\
= \; & X_{c(t)}\omega(X, \ldots) \\
= \; & (\nabla_X \omega)(X, \ldots) + \omega(\nabla_X X, \ldots) + \cdots \\
= \; & (\nabla_X \omega)(X, \ldots)
\end{aligned}
$$

where (59) has been used in the last equality; that is, the ordinary t-derivative of the scalar $\omega(X, \ldots)$ is equal to the covariant derivative of the tensor ω with all vector arguments set to X. Applying this inductively to successive t-derivatives of $f \circ c$ we find that

$$
\left. \frac{\partial^n (f \circ c)}{\partial t^n} \right|_{t=0} = \nabla_\sigma \cdots \nabla_\sigma f
$$

which, when substituted into (60) and written in component form, yields the elegant covariant Taylor expansion formula:

$$
f(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \nabla_{j_1} \cdots \nabla_{j_n} f(\varphi)\, \sigma^{j_1} \cdots \sigma^{j_n}. \tag{61}
$$

Therefore the Vilkovisky effective action (55) can be understood as the ordinary effective action of a theory reparameterised in terms of the linear variables σ^i, with a modified action given by (61) with $f = S$.

At this point the introduction of the quantity σ in (55) in order to make the effective action into a scalar may seem ad hoc; in particular it is unclear what the physical meaning of the newly-defined effective action should be. Recall that the conventional definition of the effective action (53) is defined to satisfy

$$
\frac{\partial \Gamma}{\partial \varphi^i} = -J_i
$$

and in particular, if the above equation is evaluated with φ^i set to the vacuum expectation value $< \phi^i >= \partial W[J]/\partial J_i|_{J=0}$ we have

$$
\left. \frac{\partial \Gamma}{\partial \varphi^i} \right|_{\varphi = <\phi>} = 0 \tag{62}
$$

so that the vacuum expectation value of the field components ϕ^i can be found by extremising the effective action. Since the Vilkovisky effective action is a scalar its stationary points are independent of the choice of parameterisation

of the configuration space. Thus, if we try to assume that an analogue of equation (62) holds for the Vilkovisky effective action, we would find that the VEV of a set of fields (considered as coordinates on $\mathcal{M}$) is covariant under diffeomorphisms h of $\mathbb{R}^m$, $m = \dim \mathcal{M}$, that is

$$< h(\phi) >= h(< \phi >).$$

But this equation is nonsensical; unless the measure on the configuration space is concentrated at a single point, expectation values will not behave this way. To put it another way, attempting to naïvely generalise (62) to the case where Γ is a scalar on $\mathcal{M}$ would mean treating the quantity $< \phi >$ as the expectation value of a point in $\mathcal{M}$ rather than of a vector-valued function on $\mathcal{M}$, but this is meaningless.

In the case where there exists an underlying flat structure on the configuration space the usefulness of the Vilkovisky effective action is clear: it allows one to calculate the effective action using a more convenient choice of coordinates (such as polar coordinates) and then use the stationary points of Γ to calculate the VEV of original fields; the fact that it is a scalar guarantees that the effective action calculated in the reparameterised fields will coincide with that calculated using the original parameterisation. On the other hand, in the case where there is no natural choice of coordinates (such as quantum gravity) its physical significance is not obvious.

When calculating the effective action for a gauge theory, in which the physical configuration space $\mathcal{M}$ is given by the quotient of some space M by the action of a group G of gauge transformations, it will be necessary to construct a gauge-invariant connection on $\mathcal{M}$ using the metric on M. The following section explains how to achieve this.

3.3 The Orbit Metric

Suppose that M is a differentiable manifold, G is a Lie group and $L : G \times M \to M$ is a smooth left action of G on M, which we also assume is free. For fixed $h \in G$ and $x \in M$ define $L_h : M \to M$ and $L_x : G \to M$ by $L_h(x) = L_x(h) = L(h, x)$; we also use the shorthand hx for $L(h, x)$. We assume that the quotient space $\mathcal{M} = M/G$ is also a differentiable manifold, and write q for the projection $M \to \mathcal{M}$. Suppose further that we are given a metric g on M which is G-invariant, that is, for any $h \in G$,

$$L_h^* g_{hx} = g_x. \tag{63}$$

The following is a procedure whereby the metric g is used to define a metric on the space of orbits $\mathcal{M}$. Firstly, for each point x in M let $V_{\|}(x)$ be the subspace of T_xM consisting of those vectors which are the push-forward under L_x of a vector in the Lie algebra T_eG of G; in other words $V_{\|}(x)$ contains those vectors which are generators of the action of G on the point $x \in M$. Similarly define $V_{\perp}(x)$ to be the orthogonal complement of $V_{\|}(x)$. Then the invariance of the metric g implies invariance of the subspaces $V_{\|}$ and $V_{\perp}$, i.e.

$$\begin{aligned} L_{h*}V_{\|}(x) &= V_{\|}(hx) \\ L_{h*}V_{\perp}(x) &= V_{\perp}(hx). \end{aligned}$$

To show the first of these, consider $R \in V_{\|}(x)$, so that $R = L_{x*}X$ for some $X \in T_eG$. Writing $\mathrm{Ad}_h(h') = hh'h^{-1}$, Ad_{h*} maps T_eG to itself, and since $L_{hx} \circ \mathrm{Ad}_h = L_h \circ L_x$ it follows that $L_{h*}R = (L_h \circ L_x)_*X = L_{hx*} \circ \mathrm{Ad}_{h*}X$ is in $L_{hx*}T_eG = V_{\|}(hx)$. Therefore $L_{h*}V_{\|}(x) \subseteq V_{\|}(hx)$. The same argument shows that $L_{h^{-1}*}V_{\|}(hx) \subseteq V_{\|}(x)$ and since $L_{h^{-1}*}$ is invertible we have that $L_{h*}V_{\|}(x) = V_{\|}(hx)$. To show the second of the above equalities, suppose that $X \in V_{\perp}(x)$, and let $R \in v_{\|}(hx)$. Then $R = L_{h*}Y$ for some $Y \in V_{\|}(x)$, and, using (63), $g_{hx}(L_{h*}X, R) = g_x(X, Y) = 0$ so that X is orthogonal to R; since R is arbitrary this shows that $L_{h*}X \in V_{\perp}(hx)$, so that $L_{h*}V_{\perp}(x) \subseteq V_{\perp}(hx)$. As before the same argument shows that this inclusion may be reversed.

The projection operator onto the subspace $V_{\perp}(x)$ of T_xM at each point $x \in M$ defines a $(1,1)$ tensor field Π on M. Since the decomposition $T_xM = V_{\|}(x) \oplus V_{\perp}(x)$ is G-invariant it follows that Π is also G-invariant. Therefore the degenerate $(0,2)$ tensor on M given by

$$\gamma(X, Y) \equiv g(\Pi X, \Pi Y) = g(\Pi X, Y)$$

is G-invariant; let us call this the "reduced metric". γ can be considered a metric on the space $\mathcal{M}$ of orbits in the following sense: given a "section" of M, meaning a differentiable map $p : \mathcal{M} \to M$ with the property that $q \circ p(x) = x$ for all $x \in \mathcal{M}$, a metric $\hat{g}$ on $\mathcal{M}$ is given by the pullback under p of γ. This definition makes sense since it will turn out to be independent of the choice of p; to show this, first note that the assumption that G acts freely on M means that, for each $x \in M$, the map L_{x*} is one-to-one. For suppose otherwise: then there exists non-zero X in T_eG such that $L_{x*}(X) = 0$. Let $c(t)$ be the one-parameter subgroup of G generated by X and consider the curve $c(t)x$ through M. At time t the vector tangent to this curve is given by $L_{x*} \circ L_{c(t)*}X = L_{c(t)*} \circ L_{x*}X = 0$ so that $c(t)x$ is constant, contrary to

the assumption that the group action is free. If a vector X is in the image of L_{x*} then it is tangent to a curve of the form $c(t)x$ with $c(t)$ a curve through $e \in G$; since q is constant on such a curve it follows that $q_*X = 0$, i.e.

$$\ker \Pi \subseteq \ker q_*$$

and furthermore the isomorphism $T_{q(x)}\mathcal{M} = \operatorname{im} q_* \simeq T_xM/\ker q_*$ shows that $\ker q_*$ has the same dimension as G,[8] so that

$$\ker q_* = \ker \Pi. \tag{64}$$

Now suppose that p' is another embedding of $\mathcal{M}$ into M, different from p, and suppose that, at a point $x \in \mathcal{M}$, $p(x) = hp'(x)$. Since $q \circ L_h \circ p' = q \circ p$ we have $p_*X - L_{h*} \circ p'_*X \in \ker q_*$ whenever $X \in T_x\mathcal{M}$, so $\Pi p_*X = \Pi L_{h*} \circ p'_*X$; therefore

$$\begin{aligned} p'^*\gamma_{p'(x)}(X,Y) &= \gamma_{p'(x)}(p'_*X, p'_*Y) \\ &= \gamma_{p(x)}(L_{h*} \circ p'_*X, L_{h*} \circ p'_*y) \\ &= \gamma_{p(x)}(p_*X, p_*Y) \\ &= p^*\gamma(X,Y) \end{aligned}$$

where the G-invariance of γ has been used in the second equality.

If $p^*(X,X) = 0$ then positivity of g implies that $\Pi(p_*X) = 0$ so that p_*X is in $\ker q_*$; therefore $X = q_* \circ p_*X = 0$, so that γ is non-degenerate.

In fact an explicit formula for Π can be given. Let $\{X_\alpha\}$ be a basis for the Lie algebra of G and define $R_\alpha(x) = L_{x*}X_\alpha$ for $x \in M$, so that $\{R_\alpha(x)\}$ is a basis for $V_\parallel(x)$. Omitting M-valued arguments for brevity:

$$\Pi^i_j = \delta^i_j - R^i_\alpha N^{\alpha\beta} R_{j\beta} \tag{65}$$

where $R_{j\beta} = g_{jk}R^k_\beta$ and $N^{\alpha\beta}$ is the inverse of the metric on T_eG induced by L_{x*}:

$$N_{\alpha\beta} = R^k_\alpha R_{k\beta}$$

(which is invertible since L_{x*} is injective). It is easy to verify that with this definition Π satisfies the defining equations for Π:

[8] Of course this argument breaks down when the dimension of G is infinite, a fact which we shall ignore.

$$\begin{aligned}\Pi^i_k \Pi^k_j &= \Pi^i_j \\ \Pi^i_k R^k_\alpha &= 0 \\ R_{k\alpha}\Pi^k_j &= 0.\end{aligned}$$

When calculating the Vilkovisky effective action of a theory it may be useful to parameterise the theory in terms of the full unphysical configuration space, that is in terms of the coordinates on M rather than those on $\mathcal{M}$ (this will be the case when we apply the Faddeev-Popov method to perform the integral over $\mathcal{M}$). Here we will outline how this can be achieved.

Suppose given a curve $c(t)$ through M, with tangent vectors $X(t)$, and a vector field $Y(t)$ defined on the image of $c(t)$ in M. We will be interested in the covariant derivative of q_*Y along $q \circ c(t)$ using the metric connection on $\mathcal{M}$. Suppose that we are given a connection ∇ on M. Let us try defining a connection on $\mathcal{M}$ by

$$\nabla_{q_*X} q_*Y = q_* \nabla_X Y. \tag{66}$$

It is easy to check that this does indeed define a connection on $\mathcal{M}$, that its torsion is given by

$$T(q_*Y, q_*Z) = q_* T(Y, Z)$$

and that it is compatible with $\hat{g}$ provided that ∇ is compatible with γ. This definition will therefore give the Levi-Civita connection on $\mathcal{M}$ if we can find a torsion-free connection on M which is compatible with γ. The Koszul formula shows that such a connection satisfies

$$\begin{aligned}2\gamma(\nabla_X Y, Z) &= X\gamma(Y,Z) + Y\gamma(X,Z) - Z\gamma(X,Y) \\ &+ \gamma([X,Y],Z) - \gamma([X,Z],Y) - \gamma([Y,Z],X). \end{aligned} \tag{67}$$

Since γ is degenerate equation (67) does not uniquely determine ∇, however given two such connections ∇ and ∇' we have, for all X, Y and Z,

$$\gamma(\nabla_X Y - \nabla'_X Y, Z) = 0$$

so that their difference is in $\ker \Pi = \ker q_* = V_\parallel$, meaning that they give rise to the same connection on $\mathcal{M}$.[9]

[9]When we calculate the 1-loop effective potential the difference between ∇ and ∇' will contribute terms proportional to $R^i_\alpha S_{,i}$ which are zero since the action is gauge invariant.

In order to find a connection satisfying (67), we first prove a useful fact. For fixed X and Y, denote the right hand side of (67) by $\omega(Z)$; it is easy to see that it is $\mathcal{F}(M)$-linear in Z. Suppose now that $\Pi Z = 0$. Then Z is a generator of a gauge transformation on M and so the Lie derivative of γ along Z is zero. Therefore

$$Z\gamma(X,Y) = \gamma([Z,X],Y) + \gamma(X,[Z,Y])$$

and the third, fifth and sixth terms in ω cancel; the remaining terms involve $\gamma(\cdot, Z)$ and consequently are zero. Thus $\omega(Z) = \omega(\Pi Z)$ for any Z. If we define a vector by $g^{kj}\omega_j$ it follows that

$$\gamma_{ik}g^{kj}\omega_j = \Pi_i^j\omega_j = \omega_i.$$

Therefore we may take the components of the connection to be given by

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\gamma_{li,j} + \gamma_{lj,i} - \gamma_{ij,l}\right) \tag{68}$$

and (67) will automatically be satisfied. A tedious calculation[10] shows that, up to terms of the form $R_\alpha^k K_{ij}^\alpha$ which can be ignored for the reason given above, we have

$$\begin{aligned} \Gamma_{ij}^k &= \left\{ \begin{array}{c} k \\ ij \end{array} \right\} + T_{ij}^k \\ T_{ij}^k &= -2R_{(i}^\alpha D_{j)}R_\alpha^k + R_{(i}^\alpha R_{j)}^\beta R_\alpha^l D_l R_\beta^k \end{aligned} \tag{69}$$

where $R_i^\alpha = g_{ij}N^{\alpha\beta}R_\beta^j$, D is the covariant derivative on M and the quantity in braces is the Christoffel symbol of the metric g. Note that the T_{jk}^i are manifestly the components of a tensor, and so Γ defined in (69) is the sum of a connection and a tensor and is therefore a connection on M; in particular it has the standard transformation properties of a connection under a change of coordinates.

[10]The easiest way to show this is to first rewrite the difference between the right hand side of (68) and the Christoffel symbol of g by replacing the ordinary derivatives with covariant derivatives and adding terms which cancel out the error introduced by this change; all of the new terms either cancel or are proportional to R_α^k. From there the calculation is straightforward. Killing's equation is needed.

3.3.1 Relation to Faddeev-Popov

Let μ be a G-invariant scalar function on M. The usual Faddeev-Popov formula for the integral of μ is given by

$$Z_{\mathrm{FP}} \equiv \int_M \mathrm{d}x^{m+n} \det{}^{1/2} g(x)\, \delta(\chi(x)) \det Q(x)\, \mu(x)$$

where m and n are the dimensions of $\mathcal{M}$ and G respectively, $\chi : M \to T_eG$ is a map from M to the Lie algebra of G such that the surface $\chi(x) = 0$ is the image of a section $p : \mathcal{M} \to M$, and for each $x \in M$, $Q(x)$ is the endomorphism of T_eG given by

$$Q(x)X = L_{x*} X\chi. \tag{70}$$

This definition, however, is not invariant under reparameterisations of the gauge group: under such a reparameterisation the delta function changes by a factor of the Jacobian of the corresponding change of basis for T_eG. This can be fixed by dividing the delta function by the square root of the determinant of some metric on T_eG; a natural choice is given by the metric $N(x)$ induced by the action of the gauge group at x. We therefore take the following definition of the Faddeev-Popov formula:

$$Z_{\mathrm{FP}} \equiv \int_M \mathrm{d}x^{m+n} \det{}^{1/2} g(x) \det{}^{-1/2} N(x)\, \delta(\chi(x)) \det Q(x)\, \mu(x). \tag{71}$$

It turns out that the integral (71) is exactly equal to the integral over $\mathcal{M}$ of μ, with volume form given by $\hat{g}$. To show this, let (z^i) be a coordinate chart on M, $i = 1 \cdots m+n$, and write $x^i = z^i$ for $i = 1 \cdots m$ and $y^\alpha = z^{m+\alpha}$ for $\alpha = 1 \cdots n$; without loss of generality we assume that the coordinates are ordered in such a way that the equation $\chi(x, y) = 0$ can be uniquely solved for y, and that the matrix J given by

$$J^\alpha{}_\beta \equiv \frac{\partial \chi^\alpha}{\partial y^\beta}$$

is invertible. Let y_x be the solution to $\chi(x, y) = 0$ for each x:

$$\chi(x, y_x) = 0. \tag{72}$$

Since the surface $\chi = 0$ is diffeomorphic to $\mathcal{M}$ the remaining coordinates (x^i) can be considered coordinates on $\mathcal{M}$. Integrating out the y coordinates in (71) (and suppressing function arguments for brevity) gives

$$Z_{FP} = \int \mathrm{d}x^m \det{}^{1/2} g \det{}^{-1/2} N \det{}^{-1} J \det Q \, \mu$$

and so the equivalence of the integral (71) to the integral over $\mathcal{M}$ will be proved if it can be shown that

$$\det g \, (\det J^{-1} Q)^2 = \det \hat{g} \det N \tag{73}$$

(here g and $\hat{g}$ should be understood to be the matrices of components of the metrics g and $\hat{g}$ in the coordinate systems given by z and x respectively). Define the following matrices:

$$\begin{aligned} S^i{}_\beta &\equiv R^i_\beta \\ T^\alpha{}_\beta &\equiv R^\alpha_\beta \\ W^\alpha{}_j &\equiv \frac{\partial y^\alpha_x}{\partial x^j}. \end{aligned}$$

The matrix $Q^\alpha{}_\beta$ is given by

$$Q^\alpha{}_\beta = J^\alpha{}_\gamma T^\gamma{}_\beta + \frac{\partial \chi^\alpha}{\partial x^i} S^i{}_\beta,$$

however differentiating (72) gives

$$\frac{\partial \chi^\alpha}{\partial x^i} + J^\alpha{}_\gamma W^\gamma{}_i = 0$$

so that

$$J^{-1} Q = T - WS.$$

The induced metric N is given by

$$N = \begin{pmatrix} S^T & T^T \end{pmatrix} g \begin{pmatrix} S \\ T \end{pmatrix} \tag{74}$$

while the metric $\hat{g}$ is given by (using (65))

$$\hat{g} = \begin{pmatrix} \mathbb{1} & W^T \end{pmatrix} \left\{ g - g \begin{pmatrix} S \\ T \end{pmatrix} N^{-1} \begin{pmatrix} S^T & T^T \end{pmatrix} g \right\} \begin{pmatrix} \mathbb{1} \\ W \end{pmatrix}. \tag{75}$$

Now define the matrix X by

$$X \equiv \begin{pmatrix} S^T & T^T \end{pmatrix} g \begin{pmatrix} \mathbb{1} \\ W \end{pmatrix};$$

then it is straightforward to show from the above that

$$\begin{pmatrix} \mathbb{1} & W^T \\ S^T & T^T \end{pmatrix} g \begin{pmatrix} \mathbb{1} & S \\ W & T \end{pmatrix} \begin{pmatrix} \mathbb{1} & \\ -N^{-1}X & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \hat{g} & X^T \\ & N \end{pmatrix}.$$

Taking the determinant of both sides, and using the fact[11] that

$$\det \begin{pmatrix} \mathbb{1} & S \\ W & T \end{pmatrix} = \det (T - WS) = \det J^{-1}Q$$

gives equation (73).

If we make a further assumption that, for each $\omega \in T_eG$, the zero set of the map $\chi - \omega$ defines a section of M, then we can further generalise (71) in the usual way: the above proof shows that the integral (71) is independent of the choice of gauge function χ, so that replacing χ by $\chi - \omega$ leaves the result unchanged. We may therefore integrate over ω, together with a weight function $B : T_eG \to \mathbb{R}$ such that the integral of B over T_eG is finite (at least after regularizing the theory), and the result will be the integral (71) multiplied by an irrelevant constant:

$$Z_{\text{FP}} \propto \int \mathrm{d}\omega^n \, B(\omega) \int_M \mathrm{d}x^{m+n} \det{}^{1/2} g(x) \det{}^{-1/2} N(x) \, \delta(\chi(x) - \omega) \det Q(x) \, \mu(x).$$

Reversing the order of integration, the delta functions render the integration over ω trivial, so we are finally left with

$$Z_{\text{FP}} \propto \int_M \mathrm{d}x^{m+n} \det{}^{1/2} g(x) \det{}^{-1/2} N(x) \, B(\chi(x)) \det Q(x) \, \mu(x). \qquad (76)$$

3.3.2 The 1-loop effective action

The one-loop effective action calculated from (55) using the Faddeev-Popov integral (76) is given in the usual manner by taking the quadratic terms of the integrand, evaluated at the background field x. The result is [45]

---

[11] This follows from the identity $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det (D - CA^{-1}B)$ when A is invertible.

$$\Gamma_{1-\text{loop}}(x) = S_0(x) + \frac{1}{2}\log\det(x)\,\Delta - \log\det Q(x) + \frac{1}{2}\log\det N(x) \qquad (77)$$

where

$$\Delta^i_j \equiv g^{ik}S_{;jk} = g^{ik}(S_{,jk} - \Gamma^l_{jk}S_{,l})$$

is the covariant fluctuation operator; here Γ is the connection on M whose projection onto $\mathcal{M}$ is the connection of the orbit space metric, as given by equation (69), S_0 is the original action of the theory and the action S includes the gauge-fixing term $-\log B(\chi)$. In general (77) is not independent of the choice of gauge function χ. However, under certain restrictions on the choice of gauge it will turn out that (77) is gauge independent, and moreover has a simple geometrical interpretation in terms of the physical configuration space $\mathcal{M}$. Suppose that χ is chosen such that the background field $x \in M$ lies on the surface $\chi = 0$, and such that Q is invertible, and suppose also that the function $B(\chi)$ is given by $\exp(-1/2N_0(\chi,\chi))$ for some fixed metric N_0 on the Lie algebra T_eG. Then (77) is given by

$$S_0 + \frac{1}{2}\log\det\nabla + \frac{1}{2}\log\det N_0 \qquad (78)$$

where we have defined ∇ to be the covariant fluctuation operator of the action S_0, considered as a function on $\mathcal{M}$:

$$\nabla^a_b \equiv \hat{g}^{ac}(S_{0,bc} - \Gamma^d_{bc}S_{0,d})$$

(throughout this subsection we will use lower case letters from the beginning of the alphabet to denote coordinate indices on $\mathcal{M}$). In order to prove (78) we need to prove the matrix identity

$$\det\Delta = \det\nabla\det Q^2\det N_0\det^{-1}N. \qquad (79)$$

In general this identity, when written out in terms of the various quantities defined in the previous section, is rather complicated. However it can be more easily proved by first noting that both sides are manifestly invariant under a change of coordinates on M, and choosing coordinates which split into coordinates on $\mathcal{M}$ and coordinates on G. Suppose that the surface $\chi = 0$ is a submanifold of M diffeomorphic to $\mathcal{M}$, and p is a section which takes $\mathcal{M}$ onto it. Consider the map $f : \mathcal{M} \times G \to M$ given by

$$f(x,h) = hp(x)$$

and let (x^a) and (y^α) be local charts on $\mathcal{M}$ and G, centred on the image $q(x)$ of $x \in M$ and the identity e of G respectively. p is an immersion by hypothesis, so that the images under f_* of $\partial/\partial x^a$ are linearly independent, and it was proved in Section 3.3 that the images under f_* of $\partial/\partial y^\alpha$ are also linearly independent. Furthermore, suppose that there is a non-zero vector X which lies in both of the above images. Let $c(t)$ be a curve in $\mathcal{M}$ through $q(x)$ such that X is tangent to $f(c(t), e)$, and consider the action of X on χ. Since $\chi = 0$ on $f(c(t), e)$ it follows that $X\chi = 0$, but since X is also in the image of L_{x*} we have from (70) that $\det Q = 0$, contrary to hypothesis. Thus f_* is an isomorphism, and so by the inverse function theorem there is a neighbourhood of x on which f^{-1} is defined and smooth. Therefore we may take (x^a, y^α) as local coordinates on M, centred at x.

Using these coordinates, the matrices defined in Section 3.3.1 take a particularly simple form. We have

$$\begin{aligned} S^i{}_\beta &= 0 \\ T^\alpha{}_\beta &= \delta^\alpha_\beta \\ W^\alpha{}_j &= 0 \\ \frac{\partial \chi^\alpha}{\partial y^\beta} &= Q^\alpha{}_\beta \\ \frac{\partial \chi^\alpha}{\partial x^i} &= 0. \end{aligned}$$

Note that only the first and third of these are valid over the entire domain of f^{-1}; the rest hold only on the surface $\chi = 0$. If we write the metric g in block form

$$g = \begin{pmatrix} G & F \\ F^T & M \end{pmatrix}$$

(where, from (74) and the expression given above for T, it follows that $M = N$ when $\chi = 0$) then the reduced metric and the configuration space metric are given by (75):

$$\begin{aligned} \gamma &= \begin{pmatrix} G - FM^{-1}F^T & 0 \\ 0 & 0 \end{pmatrix} \\ \hat{g} &= G - FN^{-1}F^T; \end{aligned}$$

note that an identity similar to that of footnote 11 gives

$$\det g = \det N \det \hat{g}. \tag{80}$$

Clearly the gauge invariance of S_0 implies that $\partial S_0/\partial y^\alpha = 0$. Moreover, the fact that the reduced metric γ is G-invariant, together with the fact that the Lie derivative of $\partial/\partial x^a$ along R_α is zero, shows that $\gamma_{ab,\alpha} = 0$, and therefore using (68) we have that

$$\Gamma^c_{a\beta} = \Gamma^c_{\alpha\beta} = 0$$

(ignoring terms proportional to R^α_β). Note further that $q_* \partial/\partial x^a = \partial/\partial x^a$, so that the components of the connection on $\mathcal{M}$ are simply the corresponding components Γ^c_{ab}. Using the fact that x lies on the surface $\chi = 0$, the components of the fluctuation operators with both indices lowered are:

$$\begin{aligned}
\nabla_{ab} &= \frac{\partial^2 S_0}{\partial x^a \partial x^b} - \Gamma^c_{ab}\frac{\partial S_0}{\partial x^c} \\
\Delta_{ab} &= \frac{\partial^2 S_0}{\partial x^a \partial x^b} - \Gamma^c_{ab}\frac{\partial S_0}{\partial x^c} = \nabla_{ab} \\
\Delta_{\alpha\beta} &= \frac{\partial \chi^\gamma}{\partial y^\alpha}\frac{\partial \chi^\delta}{\partial y^\beta} N_{0\gamma\delta} = (Q^T N_0 Q)_{\alpha\beta} \\
\Delta_{a\beta} &= \Delta_{\alpha b} = 0.
\end{aligned}$$

The above, together with (80), gives (79). It is important to note that, although this result shows that the one-loop effective action calculated from (77) is independent of the choice of gauge, it will not in general be gauge invariant, since in the above proof it was necessary to assume that $\chi(x) = 0$, i.e. that our background field is gauge-fixed. When we calculate the effective potential we will choose a gauge such that $\chi(\varphi)$ is zero whenever the field φ is constant, so that the effective potential should turn out to be invariant under constant elements of the gauge group.

3.4 The Vilkovisky effective potential for scalar electrodynamics

In this section we apply the method outlined in the previous sections to calculate, to one loop, the effective potential of scalar electrodynamics, with non-zero background scalar and gauge fields. We will calculate the effective potential using both the unitary gauge, which has been previously calculated using both the conventional definition of the effective action by Dolan and Jackiw[38] and the Vilkovisky method by Russell and Toms[44], and also the

generalised R_ξ gauge introduced by Fradkin and Tseytlin[42]. This theory contains a single complex scalar $\phi = \phi^1 + i\phi^2$ and a vector gauge field A_μ. Its bare Lagrangian density given by

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda(\phi^2)^2$$

where $D_\mu\phi^a = \partial_\mu\phi^a + gA_\mu\epsilon^{ab}\phi^b$ is the gauge-covariant derivative of ϕ, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature associated to the gauge field and $\phi^2 = \phi^a\phi^a$. Our sign conventions are those of Fradkin and Tseytlin; they differ from those of Russell and Toms by the replacement $e \mapsto -g$.

The infinitesimal form of the gauge transformation is

$$\begin{aligned} \delta A_\mu(x) &= -\partial_\mu\epsilon(x) \\ \delta\phi^a(x) &= g\epsilon^{ab}\phi^b(x)\epsilon(x). \end{aligned}$$

We take the following metric on the configuration space, which is easily seen to be gauge invariant and have vanishing Christoffel symbols:

$$\begin{aligned} g_{\phi^a(x)\phi^b(y)} &= \delta_{ab}\delta(x-y) \\ g_{A_\mu(x)A_\nu(y)} &= g^{\mu\nu}\delta(x-y) \end{aligned} \tag{81}$$

with all other components vanishing. Using (69) to find the coefficients of the connection on the space of gauge orbits, we find[12]:

$$\begin{aligned} N_{xy} &= (-\partial_x^2 + g^2\phi^2)\delta(x-y) \\ R_y^{\phi^a(x)} &= g\epsilon^{ab}\phi^b(x)\delta(x-y) \\ R_y^{A_\mu(x)} &= -\partial_{x_\mu}\delta(x-y) \\ \Gamma^{\phi^c(z)}_{A_\mu(x)A_\nu(y)} &= -g^2\partial_x^\mu N^{xz}\partial_y^\nu N^{yz}\phi^c(z) \\ \Gamma^{\phi^c(z)}_{A_\mu(x)\phi^b(y)} &= g\delta(z-y)\partial_x^\mu N^{xz}\epsilon^{cb} + g^3 N^{zy}\partial_x^\mu N^{xz}\epsilon^{bd}\phi^c(z)\phi^d(y) \\ \Gamma^{\phi^c(z)}_{\phi^a(x)\phi^b(y)} &= -g^2\delta(x-y)\phi^d\left(N^{xz}\epsilon^{cb}\epsilon^{ad} + N^{zy}\epsilon^{ca}\epsilon^{bd}\right) \\ &\quad -g^4 N^{xz}N^{zy}\epsilon^{ad}\epsilon^{be}\phi^c(z)\phi^d(x)\phi^e(y) \end{aligned}$$

with all other connection components equal to zero.

[12]Note that the gauge algebra is coordinatised by the values of $\epsilon(x)$ for all spacetime points x. For Lie algebra indices we use the shorthand x for the coordinate $\epsilon(x)$.

3.4.1 The generalised R_ξ gauge

Henceforth we work in Euclidean space (so that $g^{\mu\nu} = \delta^{\mu\nu}$) with the background fields for ϕ and A given by the fixed values φ and $\mathcal{B}$; we take $\mathcal{B}$ and φ to be aligned along the directions $\mu = 1$ and $a = 1$ respectively:

$$\begin{aligned} \mathcal{B}_\mu &= \delta_{\mu 1}\mathcal{B} \\ \varphi^a &= \delta^{a0}\rho. \end{aligned}$$

We will also temporarily set the number of spacetime dimensions to d, in anticipation of dimensional regularisation. The generalised R_ξ gauge introduced in [42] is defined by

$$\chi^x = \alpha^{-1/2}\left(\partial_\mu A^\mu(x) + \beta g \epsilon^{ab}\varphi^b\eta^a\right)$$

where $\eta^a \equiv \phi^a - \varphi^a$ is the difference between the field ϕ and its background value. As usual we take the function $B(\omega)$ in (76) to be given by $\exp(-1/2\int \mathrm{d}x\,\omega^{x\,2})$; this is of the form used in Section 3.3.2 with $N_0 = \mathbb{1}$ and gives an additional term to the Lagrangian density given by

$$\mathcal{L}_g = \frac{1}{2\alpha}\left(\partial_\mu A^\mu + \beta g \epsilon^{ab}\varphi^b\eta^a\right)^2. \tag{82}$$

The one-loop effective action calculated from (76) is given by

$$\Gamma(\varphi,\mathcal{B}) = S(\varphi,\mathcal{B}) + \frac{1}{2}\log\det\Delta - \log\det Q \tag{83}$$

where Δ is the covariant fluctuation operator evaluated at the background fields:

$$\Delta_{ij} \equiv S_{;ij} = S_{,ij} - \Gamma^k_{ij}S_{,k} \tag{84}$$

(with the action S including the gauge-fixing term (82)). Here Q is the Faddeev-Popov operator defined in Section 3.3.1. Note here that both indices on Δ are lower; since the metric (81) is field-independent in this parameterisation this makes no difference to the result. Also, we have omitted the term $-1/2\log\det N$ present in (77) in order to bring our results into line with previous results. The relevant derivatives of the action are given by (excluding terms involving derivatives of background fields which will be set to zero when calculating the effective potential)

$$
\begin{aligned}
S_{,\phi^a(x)} &= \left(g^2 A^\mu A_\mu + m^2 + \frac{1}{3!}\lambda\phi^2\right)\phi^a(x) \\
S_{,A_\mu(x)A_\nu(y)} &= \left(g^{\mu\nu}(-\partial_x^2 + g^2\phi^2) + (1-\frac{1}{\alpha})\partial_x^\mu\partial_x^\nu\right)\delta(x-y) \\
S_{,A_\mu(x)\phi^b(y)} &= \left(2g^2 A^\mu\phi^b + g(1-\frac{\beta}{\alpha})\epsilon^{bc}\phi^c\partial_x^\mu\right)\delta(x-y) \\
S_{,\phi^a(x)\phi^b(y)} &= \Big(\delta^{ab}(-\partial_x^2 + g^2 A^\mu A_\mu + m^2 + \frac{1}{3!}\lambda\phi^2) \\
&\quad - 2g\epsilon^{ab}A_\mu\partial_x^\mu + \frac{1}{3}\lambda\phi^a\phi^b + \frac{\beta^2}{\alpha}g^2\epsilon^{ac}\epsilon^{bd}\varphi^c\varphi^d\Big)\delta(x-y).
\end{aligned}
$$

Working in momentum space with a fixed value k for the momentum, (84) therefore gives the fluctuation operator:

$$
\Delta_k = \begin{pmatrix} A & C \\ C^\dagger & D \end{pmatrix}
$$

where A, C and D are the $d\times d$, $d\times 2$ and 2×2 matrices with components

$$
\begin{aligned}
A^{\mu\nu} &= g^{\mu\nu}\left(k^2 + g^2\rho^2\right) - k^\mu k^\nu\left(1 - \frac{1}{\alpha} - g^2\rho^2 L N_k^{-2}\right) \\
C^{\mu 1} &= 2\delta^{\mu 1} g^2\rho\mathcal{B} \\
C^{\mu 2} &= \mathrm{i}g\rho k^\mu\left(1 - \frac{\beta}{\alpha} + k^2 L N_k^{-2}\right) \\
D &= \begin{pmatrix} k^2 + L + \frac{1}{3}\lambda\rho^2 & 2\mathrm{i}g\mathcal{B}k_1 \\ -2\mathrm{i}g\mathcal{B}k_1 & k^2 + k^4 L N_k^{-2} + \frac{\beta^2}{\alpha}g^2\rho^2 \end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
N_k &\equiv k^2 + g^2\rho^2 \\
L &\equiv g^2\mathcal{B}^2 + m^2 + \frac{1}{3!}\lambda\rho^2.
\end{aligned}
$$

In order to calculate the determinant of Δ_k, consider first the matrix $A = a\mathbb{1} - bKK^T$, where $a \equiv N_k$, $b \equiv 1 - \frac{1}{\alpha} - g^2\rho^2 L N_k^{-2}$ and K is the column vector whose entries are the components of k. It is easy to see that KK^T has eigenvalues k^2 with multiplicity 1 and 0 with multiplicity $d-1$, whenever $k \neq 0$. Thus A has determinant $a^{d-1}(a - bk^2)$. In order to evaluate the

determinant of Δ_k we assume that neither a nor $a - bk^2$ is equal to zero, and use the identity

$$\det \begin{pmatrix} A & C \\ C^\dagger & D \end{pmatrix} = \det A \det (D - C^\dagger A^{-1} C); \tag{85}$$

the case where the determinant of A is zero will then follow from the non-singular case by continuity in a. A direct computation shows that

$$A^{-1} = \frac{1}{a}\left(\mathbb{1} + \frac{bKK^T}{a - bk^2}\right)$$

so that the above formula yields, after some simplification,

$$\begin{aligned} \det \Delta_k &= \frac{1}{3\alpha} N_k^{d-4} (k^2 + \beta g^2 \rho^2)^2 \left[\left((k^2 + g^2\rho^2)^2 + k^2 L \right) \right. \\ &\quad \left((k^2 + g^2\rho^2)(3k^2 + 3L + \lambda\rho^2) - 12 g^4 \mathcal{B}^2 \rho^2 \right) \\ &\quad \left. -12 g^2 \mathcal{B}^2 \left((k^2 + g^2\rho^2)^2 - g^2 L \rho^2 \right) k_1^2 \right]. \end{aligned} \tag{86}$$

Several facts about this formula are immediately apparent. Firstly, note that the d-dependence of the determinant is contained entirely in the term N_k^{d-4}, so that when calculating the regularised effective potential for non-integer d close to 4 the dependence of the determinant on d generalises in the obvious way. Secondly, the dependence on the gauge quantities α and β is contained in the factor $\alpha^{-1}(k^2 + \beta g^2 \rho^2)^2$. On the other hand, we have that

$$\begin{aligned} Q_y^x &= \int \mathrm{d}z \left(\frac{\partial \chi^x}{\partial \phi^a(z)} R_y^{\phi^a(x)} + \frac{\partial \chi^x}{\partial A_\mu(z)} R_y^{A_\mu(z)} \right) \\ &= \alpha^{-1/2} \left(\beta g^2 \rho^2 - \partial_x^2 \right) \delta(x - y) \end{aligned}$$

or, in momentum space

$$Q_k = \alpha^{-1/2} \left(k^2 + \beta g^2 \rho^2 \right).$$

From this we see that, when using equation (83) to calculate the effective potential, the term $\log Q_k$ arising from $\log \det Q$ will cancel exactly with the term $-1/2 \log \left(\alpha^{-1}(k^2 + \beta g^2 \rho^2)^2 \right)$ arising from $-1/2 \log \det \Delta$. Thus the result is completely independent of α and β, as expected from the results of Section 3.3.2.

3.4.2 Unitary gauge

In order to define unitary gauge, we first introduce the following field reparameterisation:

$$\begin{aligned} \phi^1 &= r\cos(\theta) \\ \phi^2 &= r\sin(\theta) \\ A_\mu &= B_\mu + \frac{1}{g}\partial_\mu\theta; \end{aligned}$$

note that this is a change of coordinates, not a gauge transformation, although the similarity of the definition of B to a gauge transformation shows that we may redefine $F_{\mu\nu}$ as $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and it will remain unchanged. A gauge transformation is now given by

$$\begin{aligned} r &\mapsto r \\ \theta &\mapsto \theta - g\epsilon \\ B_\mu &\mapsto B_\mu. \end{aligned} \tag{87}$$

In terms of these coordinates, the Lagrangian density becomes:

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu r\partial^\mu r + \frac{1}{2}g^2 B_\mu B^\mu r^2 + \frac{1}{2}m^2 r^2 + \frac{1}{4!}\lambda r^4$$

and it can be seen that θ is no longer a dynamical variable, but rather a variable whose only purpose is to label different points on a given gauge orbit. Therefore, as explained in more detail in [38], the parameterisation of the theory in terms of r and B_μ can be understood as describing the physical dynamics of the system with gauge degrees of freedom removed. As a gauge condition we take the surface $\theta = 0$. The integral (71) may be performed directly; Q is simply the identity map, and the integral over θ is trivial. Note that, with this choice of gauge, the coordinates (r, B_μ, θ) are closely related to those whose existence was proved in Section 3.3.2; from (87) they differ only in that the gauge parameter ϵ is related to θ by $\theta = -g\epsilon$. So this calculation should be equivalent to the direct calculation of the quantity in (78).

In terms of the new coordinates the non-zero components of the metric (81) are

$$g_{r(x)r(y)} = \delta(x-y)$$

$$
\begin{aligned}
g_{\theta(x)\theta(y)} &= (r^2 - \frac{1}{g^2}\partial_x^2)\delta(x-y) \\
g_{B_\mu(x)B_\nu(y)} &= g^{\mu\nu}\delta(x-y) \\
g_{B_\mu(x)\theta(y)} &= \frac{1}{g}\partial_x^\mu\delta(x-y)
\end{aligned} \tag{88}
$$

with inverse components

$$
\begin{aligned}
g^{r(x)r(y)} &= \delta(x-y) \\
g^{\theta(x)\theta(y)} &= \frac{1}{r^2}\delta(x-y) \\
g^{B_\mu(x)B_\nu(y)} &= (g_{\mu\nu} - \frac{1}{g^2r^2}\partial_\mu\partial_\nu)\delta(x-y) \\
g^{B_\mu(x)\theta(y)} &= -\frac{1}{gr^2}\partial_\mu\delta(x-y);
\end{aligned}
$$

its determinant is given by r. The direct calculation of the connection coefficients using this metric as well as (69) is lengthy, but having calculated the connection in terms of the original parameterisation of the theory we may instead use the general identity for a change of coordinates:

$$
\Gamma'^k_{ij} = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}\frac{\partial x'^k}{\partial x^l}\Gamma^l_{mn} + \frac{\partial^2 x^l}{\partial x'^i \partial x'^j}\frac{\partial x'^k}{\partial x^l}
$$

(recall the comment following equation (69)). Having integrated out the variables $\theta(x)$ the only relevant components of the connection are those involving $r(x)$ and $B_\mu(x)$; the non-zero components are

$$
\begin{aligned}
\Gamma^{r(z)}_{B_\mu(x)B_\nu(y)} &= -g^2\partial_x^\mu N^{xz}\partial_y^\nu N^{yz} r \\
\Gamma^{B_\mu(z)}_{B_\nu(x)r(y)} &= \partial_{z^\mu}\left(\delta(z-y)\partial_x^\nu N^{xz}\right)\frac{1}{r}.
\end{aligned}
$$

The relevant derivatives of the action are:

$$
\begin{aligned}
S_{,B_\mu(z)} &= g^2r^2B^\mu \\
S_{,r(z)} &= \left(g^2B^\mu B_\mu + m^2 + \frac{1}{3!}\lambda r^2\right)r \\
S_{,B_\mu(x)B_\nu(y)} &= \left(g^{\mu\nu}(-\partial_x^2 + g^2r^2) + \partial_x^\mu\partial_x^\nu\right)\delta(x-y) \\
S_{,B_\mu(x)r(y)} &= 2g^2B^\mu r\delta(x-y) \\
S_{,r(x)r(y)} &= \left(-\partial_x^2 + g^2B^\mu B_\mu + m^2 + \frac{1}{2}\lambda r^2\right)\delta(x-y)
\end{aligned}
$$

giving a fluctuation operator with components

$$\begin{aligned}
\nabla_{k\,B_\mu B_\nu} &= g^{\mu\nu}\left(k^2+g^2\rho^2\right) - k^\mu k^\nu\left(1-g^2\rho^2 L N_k^{-2}\right)\\
\nabla_{k\,B_\mu r} &= \nabla_{k\,r B_\mu} = 2\delta^{\mu 1} g^2 \rho \mathcal{B}\\
\nabla_{k\,rr} &= k^2 + L + \frac{1}{3}\lambda\rho^2.
\end{aligned}$$

When calculating the effective potential with this choice of coordinates it is necessary to raise one of the indices on ∇, or equivalently to multiply the determinant of the above operator by the inverse of the determinant of $\hat{g}$; using (75) and (88) we find that

$$\det \hat{g} = \frac{g^2 r^2}{\det N}.$$

Therefore, remembering that we omitted the term $1/2 \log\det N$ from (77), and also that our earlier choice of gauge-fixing term had $\det N_0 = 1$, from (78) we should find that

$$\frac{\det \nabla}{\det \hat{g}\det N} = \frac{\det \Delta}{\det Q^2}$$

or

$$\frac{1}{g^2\rho^2}\det \nabla_k = \frac{\det \Delta_k}{Q_k^2} \tag{89}$$

where $\det \Delta_k$ is given by (86). The left hand side of (89) can be calculated using the same method as before, namely using the determinant identity (85) with $A^{\mu\nu} = \nabla_{k\,B_\mu B_\nu}$, $C^\mu = \nabla_{k\,B_\mu r}$ and $D = \nabla_{k\,rr}$. A is again of the form $a\mathbb{1} - bKK^T$, so that the expressions given earlier for its determinant and inverse still apply, and it may readily be verified that (89) is satisfied.

3.5 Calculation of the determinant

We now calculate the 1-loop contribution to (83), using (86). Dimensional regularisation will be used, so that we introduce the cutoff parameter $\epsilon \equiv 4-d$ and an arbitrary mass scale μ. In anticipation of modified minimal subtraction we also introduce the mass scale M, where

$$M \equiv \frac{\mu}{\sqrt{4\pi}} \mathrm{e}^{\frac{\gamma}{2}}$$

where γ is the Euler-Mascheroni constant. The regularised, unrenormalised 1-loop contribution to the effective action is given by

$$V_1 = \frac{1}{2} M^{4-d} \int_{S^{d-1}} \frac{\mathrm{d}\Omega^d}{(2\pi)^d} \int_0^\infty \mathrm{d}k\, k^{d-1} \log\left(N_k^{d-4} D(k,\vartheta)\right) \tag{90}$$

where

$$\begin{aligned} D(k,\vartheta) &\equiv \left((k^2+g^2\rho^2)^2 + k^2 L\right) \\ &\quad \times \left((k^2+g^2\rho^2)(k^2+L+\lambda\rho^2/3) - 4g^4\mathcal{B}^2\rho^2\right) \\ &\quad -4g^2\mathcal{B}^2\left((k^2+g^2\rho^2)^2 - g^2 L\rho^2\right) k^2\cos^2\vartheta \\ N_k &\equiv k^2 + g^2\rho^2 \\ L &\equiv g^2\mathcal{B}^2 + m^2 + \frac{1}{3!}\lambda\rho^2 \end{aligned}$$

and

$$\mathrm{d}\Omega^d \propto \sin^{d-2}\vartheta \mathrm{d}\vartheta$$

is the integration measure on S^{d-1}. In order to reduce the integral to a single integral over k, we make use of the identity

$$\int_{S^{d-1}} \mathrm{d}\Omega^d\, f(\vartheta) = \mathrm{Vol}(S^{d-1}) \frac{\int_0^\pi \mathrm{d}\vartheta \sin^{d-2}\vartheta\, f(\vartheta)}{\int_0^\pi \mathrm{d}\vartheta \sin^{d-2}\vartheta} \tag{91}$$

as well as the general formula for the volume of S^{d-1},

$$\mathrm{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

(90) will be evaluated in the limit $\epsilon \to 0$, after renormalising the theory by discarding terms involving singularities as $\epsilon \to 0$. To this end we make repeated use of the following expansion:

$$x^{n-\epsilon} = x^n\left(1 - \epsilon \log x + \mathcal{O}(\epsilon^2)\right);$$

the terms singular as $\epsilon \to 0$ will turn out to be proportional to $1/\epsilon$, so when evaluating the individual quantities occurring in (90) it is only necessary to keep terms up to first order in ϵ, since higher order terms will not make any difference in the limit of zero ϵ. Consider first the denominator in (91). It is given by

$$\begin{aligned} &\int_0^\pi \mathrm{d}\vartheta\, \left(\sin^2\vartheta - \epsilon \sin^2\vartheta \log(\sin\vartheta) + \mathcal{O}(\epsilon^2)\right) \\ &\qquad = \frac{\pi}{2} + \epsilon\frac{\pi}{2}\left(\log 2 - \frac{1}{2}\right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{92}$$

In order to prove that the coefficient of ϵ given above is correct, use the identity given in footnote 8 of [8] to write

$$\int_0^\pi \mathrm{d}\vartheta \sin^2\vartheta \log(\sin\vartheta) = \int_0^\pi \mathrm{d}\vartheta \sin^2\vartheta \left(-\log 2 - \sum_{n=1}^{\infty} \frac{1}{n} \cos(2n\vartheta) \right).$$

It is easy to check that

$$\int_0^\pi \mathrm{d}\vartheta \sin^2\vartheta \cos(2n\vartheta) = \begin{cases} -\frac{\pi}{4} & n = 1 \\ 0 & n > 1 \end{cases}$$

from which (92) follows. The volume of S^{d-1} may be evaluated using the expansion

$$\Gamma\left(2 - \frac{\epsilon}{2}\right) = 1 + \left(\frac{\gamma}{2} - \frac{1}{2}\right)\epsilon + \mathcal{O}(\epsilon^2); \tag{93}$$

the result is

$$\mathrm{Vol}(S^{d-1}) = 2\pi^2 - \epsilon\pi^2 (\log\pi + \gamma - 1) + \mathcal{O}(\epsilon^2). \tag{94}$$

$M^{4-d}/(2\pi)^d$ is simply given by

$$\frac{1}{(2\pi)^4} + \epsilon \frac{1}{(2\pi)^4} \log(2\pi M) + \mathcal{O}(\epsilon^2). \tag{95}$$

To calculate the integral in (90), we define

$$\begin{aligned} P(k^2) &\equiv \left((k^2 + g^2\rho^2)^2 + k^2 L\right) \\ &\quad \times \left((k^2 + g^2\rho^2)(k^2 + L + \lambda\rho^2/3) - 4g^4\mathcal{B}^2\rho^2\right) \\ Q(k^2) &\equiv 4g^2\mathcal{B}^2 \left((k^2 + g^2\rho^2)^2 - g^2 L\rho^2\right) k^2; \end{aligned}$$

note that P is a monic polynomial in k^2. The integrand of (90) is then given by

$$k^{3-\epsilon} \left[-\epsilon \log N_k + \log P + \log\left(1 - \frac{Q}{P}\cos^2\vartheta\right) \right]. \tag{96}$$

For the first two terms the integral over ϑ is trivial, and the remaining integral over k is standard in evaluating effective potentials: let $-\varepsilon_1^2$, $-\varepsilon_2^2$, $-\varepsilon_3^2$ and $-\varepsilon_4^2$ be the four roots of P, considered a polynomial in k^2, and let $\varepsilon_0^2 \equiv g^2\rho^2$. The roots of P may be explicitly calculated: they give

$$\begin{aligned}
\varepsilon_{1,2} &= \frac{1}{2}\left(\sqrt{L+4g^2\rho^2} \pm \sqrt{L}\right) \\
\varepsilon_{3,4} &= \sqrt{\frac{1}{2}\left(L+\lambda\rho^2/3+g^2\rho^2 \pm \sqrt{16g^4B^2\rho^2+(L+\lambda\rho^2/3-g^2\rho^2)^2}\right)}.
\end{aligned}$$

For later use we note that

$$\begin{aligned}
\sum_{i=1}^{4}\varepsilon_i^2 &= 2L+3g^2\rho^2+\frac{1}{3}\lambda\rho^2 \\
\sum_{i=1}^{4}\varepsilon_i^4 &= 2L^2+8g^4B^2\rho^2+4g^2L\rho^2+\frac{2}{3}\lambda L\rho^2+3g^4\rho^4+\frac{1}{9}\lambda^2\rho^4. \qquad (97)
\end{aligned}$$

The integral over k of the first two terms above reduces to a sum of terms of the form

$$\begin{aligned}
&\int_0^\infty \mathrm{d}k\, k^{3-\epsilon}\log\left(k^2+\varepsilon_i^2\right) \\
&= -\left.\frac{\partial}{\partial t}\right|_{t=0}\int_0^\infty \mathrm{d}k\,\frac{k^{3-\epsilon}}{(k^2+\varepsilon_i^2)^t} \\
&= -\left.\frac{\partial}{\partial t}\right|_{t=0}\varepsilon_i^{4-\epsilon-2t}\frac{\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(t-2+\frac{\epsilon}{2}\right)}{2\Gamma(t)} \\
&= -\frac{1}{2}\varepsilon_i^{4-\epsilon}\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(-2+\frac{\epsilon}{2}\right) \qquad (98)
\end{aligned}$$

where the integral in the second line has been evaluated using an identity given in [46]. Using the expansion

$$\Gamma\left(-2+\frac{\epsilon}{2}\right) = \frac{1}{\epsilon}+\frac{3}{4}-\frac{\gamma}{2}+\mathcal{O}(\epsilon),$$

as well as (93), (98) is equal to

$$-\frac{1}{2\epsilon}\varepsilon_i^4+\frac{1}{2}\varepsilon_i^4\left(\log\varepsilon_i-\frac{1}{4}\right).$$

Putting this together with the expressions given in (94) and (95), the contribution to V_1 from the first two terms of (96) is

$$\frac{1}{32\pi^2}g^4\rho^4 + \frac{1}{32\pi^2}\sum_{i=1}^{4}\varepsilon_i^4\left(-\frac{1}{\epsilon} + \log\varepsilon_i - \log\left(2\sqrt{\pi}M\right) + \frac{\gamma}{2} - \frac{3}{4}\right) + \mathcal{O}(\epsilon)$$

$$= \frac{1}{32\pi^2}g^4\rho^4 + \frac{1}{32\pi^2}\sum_{i=1}^{4}\varepsilon_i^4\left(-\frac{1}{\epsilon} + \log\left(\varepsilon_i/\mu\right) - \frac{3}{4}\right) + \mathcal{O}(\epsilon). \tag{99}$$

For the final term in (96) the situation is more complicated. The exact value for the integral

$$\int_0^\pi \mathrm{d}\vartheta \sin^{d-2}\vartheta \log\left(1 - \frac{Q}{P}\cos^2\vartheta\right)$$

involves the generalised hypergeometric function evaluated with Q/P as an argument, and is therefore difficult to deal with analytically. However, an approximate expression may be found by noting that $Q/P \propto g^2$, and therefore for small g (or small $\mathcal{B}$) we may consider only the first few terms in the Taylor expansion of $\log\left(1 - Q/P\cos^2\vartheta\right)$. Some relevant integrals which arise are

$$\begin{aligned}
\int_0^\pi \mathrm{d}\vartheta \sin^2\vartheta\cos^2\vartheta &= \frac{\pi}{8} \\
\int_0^\pi \mathrm{d}\vartheta \sin^2\vartheta\cos^4\vartheta &= \frac{\pi}{16} \\
\int_0^\pi \mathrm{d}\vartheta \sin^2\vartheta\log\left(\sin\vartheta\right)\cos^2\vartheta &= \frac{\pi}{32}\left(-1 + 4\log 2\right) \\
\int_0^\pi \mathrm{d}\vartheta \sin^2\vartheta\log\left(\sin\vartheta\right)\cos^4\vartheta &= \frac{\pi}{192}\left(-1 + 12\log 2\right)
\end{aligned} \tag{100}$$

where the last two integrals may be evaluated using the same method that was used to prove equation (92); we omit the details. For the term proportional to $(Q/P)^1$, by expanding Q in powers of k^2 the remaining integral over k can be written as a sum of terms proportional to

$$\int_0^\infty \mathrm{d}k\, k^{3-\epsilon}\frac{k^{2n}}{P} = \int_0^\infty \mathrm{d}k \frac{k^{3+2n-\epsilon}}{(k^2+\varepsilon_1^2)(k^2+\varepsilon_2^2)(k^2+\varepsilon_3^2)(k^2+\varepsilon_4^2)}$$

for some $n \in \mathbb{N}$. In order to evaluate this we use the standard method of rewriting the denominator of the integrand [46]:

$$\frac{1}{(k^2+\varepsilon_1^2)(k^2+\varepsilon_2^2)(k^2+\varepsilon_3^2)(k^2+\varepsilon_4^2)}$$

$$= 3!\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_3 \times \left[(k^2+\varepsilon_4^2)x_3 + (k^2+\varepsilon_3^2)(x_2-x_3)\right.$$
$$\left. +(k^2+\varepsilon_2^2)(x_1-x_2) + (k^2+\varepsilon_1^2)(1-x_1)\right]^{-4}$$
$$= 3!\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_3 \frac{1}{(k^2+\mathcal{E}^2)^4}$$

where we have defined

$$\mathcal{E}^2(x_1,x_2,x_3) \equiv x_3(\varepsilon_4^2-\varepsilon_3^2) + x_2(\varepsilon_3^2-\varepsilon_2^2) + x_1(\varepsilon_2^2-\varepsilon_1^2) + \varepsilon_1^2.$$

Performing the integral over k first gives

$$3!\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_3 \int_0^\infty \mathrm{d}k \frac{k^{3+2n-\epsilon}}{(k^2+\mathcal{E}^2)^4}$$
$$= \frac{1}{2}\Gamma\left(2+n-\frac{\epsilon}{2}\right)\Gamma\left(2-n+\frac{\epsilon}{2}\right)\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_3\, \mathcal{E}^{2n-4-\epsilon}$$

where use has been made of the fact that $\Gamma(4) = 3!$. The integral over x_1, x_2 and x_3 is a special case of that considered in the appendix.[13] The result is

$$\int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_3\, \mathcal{E}^{2n-4-\epsilon}$$
$$= \frac{1}{(n-1-\epsilon/2)(n-\epsilon/2)(n+1-\epsilon/2)} \sum_{i=1}^4 A_i \varepsilon_i^{2n+2-\epsilon}$$

where

$$A_i \equiv \prod_{j\neq i} \frac{1}{\varepsilon_i^2-\varepsilon_j^2}.$$

Note that for $n = 1$ the factor $(n-1-\epsilon/2)(n-\epsilon/2)(n+1-\epsilon/2)$ is $\mathcal{O}(\epsilon)$, while for $n = 2$ or 3 it is $\mathcal{O}(1)$; on the other hand, the factor $\Gamma(2-n+\epsilon/2)$ is $\mathcal{O}(1)$ for $n = 1$ and $\mathcal{O}(1/\epsilon)$ for $n = 2$ or 3. Thus each power of k^2 in Q

[13]This assumes that the ε_i are distinct, which is false at $\rho = \mathcal{B} = 0$. However, when $\mathcal{B} = 0$ we have $Q = 0$ so that this problem doesn't arise.

contributes a pole of order at most 1 to V_1. In fact some simplification is possible: using the fact that

$$\Gamma(z+1) = z\Gamma(z)$$

it follows that

$$\begin{aligned} & \frac{\Gamma\left(2+n-\frac{\epsilon}{2}\right)\Gamma\left(2-n+\frac{\epsilon}{2}\right)}{2(n-1-\epsilon/2)(n-\epsilon/2)(n+1-\epsilon/2)} \\ = & -\frac{1}{2}\Gamma\left(n-\frac{\epsilon}{2}\right)\Gamma\left(1-n+\frac{\epsilon}{2}\right) \\ = & \frac{\pi}{2}\operatorname{cosec}\left(\frac{\pi\epsilon}{2}-\pi n\right) = (-)^n\frac{\pi}{2}\operatorname{cosec}\left(\frac{\pi\epsilon}{2}\right) \end{aligned}$$

where Euler's reflection formula [31] has been used in the last line. In particular, as $\epsilon \to 0$ this is equal to

$$(-)^n\frac{1}{\epsilon} + \mathcal{O}(\epsilon).$$

Consider now the sum

$$\sum_{i=1}^{4} \varepsilon_i^{2n+2-\epsilon} \prod_{j\neq i} \frac{1}{\varepsilon_i^2-\varepsilon_j^2} = \sum_{i=1}^{4} \varepsilon_i^{2n+2}(1-\epsilon\log\varepsilon_i)\prod_{j\neq i}\frac{1}{\varepsilon_i^2-\varepsilon_j^2} + \mathcal{O}(\epsilon^2).$$

The $\mathcal{O}(1)$ terms may be calculated using results given in the appendix. They are 0 for $n=1$, 1 for $n=2$ and $\sum_i \varepsilon_i^2$ for $n=3$. To the author's knowledge there is no simplification of those terms involving logarithms of ε_i, so we leave them explicit in what follows.
The factor $1/2\, M^{4-d}/(2\pi)^d \operatorname{Vol} S^{d-1} \int \mathrm{d}\vartheta \sin^{d-2}\vartheta\cos^2\vartheta / \int \mathrm{d}\vartheta \sin^{d-2}\vartheta$ is given by, using (92), (94), (95) and (100),

$$\frac{1}{64\pi^2} + \epsilon\frac{1}{64\pi^2}\left(\log\mu - \log 4 + \frac{5}{4}\right) + \mathcal{O}(\epsilon^2).$$

Therefore the overall contribution to V_1 from terms proportional to $\cos^2\vartheta$ is

$$\begin{aligned} & \frac{1}{16\pi^2}g^2\mathcal{B}^2\left\{\frac{1}{\epsilon}\sum_i\varepsilon_i^2 - \sum_i A_i\varepsilon_i^8\log\varepsilon_i + \left(\log\mu - \log 4 + \frac{5}{4}\right)\sum_i\varepsilon_i^2\right\} \\ & + \frac{1}{8\pi^2}g^4\mathcal{B}^2\rho^2\left\{-\frac{1}{\epsilon} + \sum_i A_i\varepsilon_i^6\log\varepsilon_i - \log\mu + \log 4 - \frac{5}{4}\right\} \\ & + \frac{1}{16\pi^2}\left(g^4L\mathcal{B}^2\rho^2 - g^6\mathcal{B}^2\rho^4\right)\sum_i A_i\varepsilon_i^4\log\varepsilon_i + \mathcal{O}(\epsilon). \quad (101) \end{aligned}$$

Consider now the term in (96) proportional to $(Q/P)^r$, $r \in N$. This gives rise to integrals of the form

$$\int_0^\infty \mathrm{d}k \, \frac{k^{3+2n-\epsilon}}{(k^2+\varepsilon_1^2)\cdots(k^2+\varepsilon_{4r}^2)}$$

where ε_i for $i > 4$ is defined so that $\varepsilon_i = \varepsilon_j$ whenever $i = j \bmod 4$; in fact we will temporarily assume that all the ε_i are distinct variables, and then take the limit as some of the ε become equal, as explained in the appendix. In the above n is an integer in $[r, 3r]$. Repeating the same steps as before we find that this is equal to

$$\int_0^1 \mathrm{d}x_1 \cdots \int_0^{x_{4r-2}} \mathrm{d}x_{4r-1} \int_0^\infty \mathrm{d}k \, \frac{k^{3+2n-\epsilon}}{(k^2+\mathcal{E}^2)^{4r}}$$

where the earlier definition of $\mathcal{E}$ has been generalised in the obvious way. This is equal to

$$\begin{aligned}
&\frac{\Gamma\left(2+n-\frac{\epsilon}{2}\right)\Gamma\left(4r-2-n+\frac{\epsilon}{2}\right)}{2(-4r+3+n-\epsilon/2)\cdots(n+1-\epsilon/2)} \sum_{i=1}^{4r} \varepsilon_i^{2n+2-\epsilon} \prod_{j\neq i} \frac{1}{\varepsilon_i^2-\varepsilon_j^2} \\
= \; & -\frac{1}{2}\Gamma\left(n-\frac{\epsilon}{2}\right)\Gamma\left(1-n+\frac{\epsilon}{2}\right) \sum_{i=1}^{4r} \varepsilon_i^{2n+2-\epsilon} \prod_{j\neq i} \frac{1}{\varepsilon_i^2-\varepsilon_j^2} \\
= \; & (-)^n \frac{\pi}{2} \operatorname{cosec}\left(\frac{\pi x}{2}\right) \sum_{i=1}^{4r} \varepsilon_i^{2n+2-\epsilon} \prod_{j\neq i} \frac{1}{\varepsilon_i^2-\varepsilon_j^2}.
\end{aligned}$$

Consider first the term of order $1/\epsilon$, namely

$$(-)^n \frac{1}{\epsilon} \sum_{i=1}^{4r} \varepsilon_i^{2n+2} \prod_{j\neq i} \frac{1}{\varepsilon_i^2-\varepsilon_j^2}. \tag{102}$$

Since $n \leq 3r$ we have $n+1 < 4r-1$ whenever $r > 2$, and $n+1 \leq 4r-1$ for $r = 2$. Therefore, using equation (106) of the appendix, it follows that this term is identically zero whenever $r > 2$, and the only non-zero contribution comes from $r = 2$, $n = 6$, in which case (102) is equal to $1/\epsilon$. Using (92), (94), (95) and (100), the factor $1/2\, M^{4-d}/(2\pi)^d \operatorname{Vol} S^{d-1} \int \mathrm{d}\vartheta \sin^{d-2}\vartheta \cos^4\vartheta / \int \mathrm{d}\vartheta \sin^{d-2}\vartheta$ is given by

$$\frac{1}{128\pi^2} + \epsilon \frac{1}{128\pi^2}\left(\log\mu - \log 4 + \frac{13}{12}\right) + \mathcal{O}(\epsilon^2)$$

and therefore the total contribution to V_1 from this term is equal to

$$\frac{1}{16\pi^2}g^4\mathcal{B}^4\left(-\frac{1}{\epsilon}-\log\mu+\log 4-\frac{13}{12}\right). \tag{103}$$

Note that the terms with $r \geq 2$ involving logarithms are $\mathcal{O}(1)$, and therefore the divergent part of the effective action is contained entirely in terms already calculated. Putting together the results in (99), (101), (103) and (97), the total divergent part of V_1 is equal to

$$\begin{aligned}&-\frac{1}{576\pi^2\epsilon}\left(36m^4+24m^2(3g^2+\lambda)\rho^2\right.\\&\left.+(54g^4+12g^2\lambda+5\lambda^2)\rho^4+180g^4B^2\rho^2\right).\end{aligned}$$

This is consistent with renormalisability, since the divergences proportional to ρ^2, ρ^4 and $B^2\rho^2$ may be absorbed into counterterms for m, λ and g respectively.

Defining Q_n^r to be the coefficient of k^{2n} in $Q(k^2)^r$, the remaining $\mathcal{O}(1)$ terms in the effective potential are

$$\begin{aligned}&\frac{1}{16\pi^3}\sum_{r=2}^{\infty}\frac{1}{r}\int_0^{\pi}\mathrm{d}\vartheta\sin^2\vartheta\cos^{2r}\vartheta\sum_{i=1}^{4r}\sum_{n=r}^{3r}(-)^nQ_n^r\varepsilon_i^{2n+2}\log\varepsilon_i^2\prod_{j\neq i}\frac{1}{\varepsilon_i^2-\varepsilon_j^2}\\=\;&\frac{1}{16\pi^3}\sum_{r=2}^{\infty}\frac{1}{r}\int_0^{\pi}\mathrm{d}\vartheta\sin^2\vartheta\cos^{2r}\vartheta\sum_{i=1}^{4r}\varepsilon_i^2Q(-\varepsilon_i^2)^r\log\varepsilon_i^2\prod_{j\neq i}\frac{1}{\varepsilon_i^2-\varepsilon_j^2}\end{aligned}$$

where the sum over i should be understood as the limit as $\varepsilon_5 \to \varepsilon_1$, $\varepsilon_6 \to \varepsilon_2$, and so on. This limit may be calculated using equation (110) of the appendix. For $r = 2$ the result is

$$\frac{1}{512\pi^2}\sum_{i=1}^{4}A_i^2\left\{-2B_i\varepsilon_i^2Q(-\varepsilon_i^2)^2\log\varepsilon_i^2+\left.\frac{\partial}{\partial x}\right|_{x=\varepsilon_i^2}\left(xQ(-x)^2\log x\right)\right\}$$

where A_i is defined as before, and

$$B_i \equiv \sum_{j\neq i}\frac{1}{\varepsilon_i^2-\varepsilon_j^2}.$$

This is equal to

$$\frac{1}{32\pi^2}g^4\mathcal{B}^4\sum_{i=1}^{4}A_i^2\left\{\left[B_i\left(-4\varepsilon_i^{14}+16g^2\rho^2\varepsilon_i^{12}+(8g^2L\rho^2-24g^4\rho^4)\varepsilon_i^{10}\right.\right.\right.$$
$$+(16g^6\rho^6-16g^4L\rho^4)\varepsilon_i^8+(8g^6L\rho^6-4g^8\rho^8-4g^4L^2\rho^4)\varepsilon_i^6\Big)$$
$$+14\varepsilon_i^{12}-48g^2\rho^2\varepsilon_i^{10}+(60g^4\rho^4-20g^2L\rho^2)\varepsilon_i^8+(32g^4L\rho^4-32g^6\rho^6)\varepsilon_i^6$$
$$\left.+(6g^8\rho^8-12g^6L\rho^6+6g^4L^2\rho^4)\varepsilon_i^4\right]\log\varepsilon_i+\varepsilon_i^{12}-4g^2\rho^2\varepsilon_i^{10}+(6g^4\rho^4$$
$$\left.-2g^2L\rho^2)\varepsilon_i^8+(4g^4L\rho^4-4g^6\rho^6)\varepsilon_i^6+(g^8\rho^8-2g^6L\rho^6+g^4L^2\rho^4)\varepsilon_i^4\right\}. \quad (104)$$

Note that (101), (103) and (104) appear as though they will behave badly under a change of units, since they involve terms such as $\log\varepsilon_i$ which are not balanced by $-\log\mu$. However this is an illusion which has been introduced by using (106) to simplify some of the sums over i. Had we left all such sums explicit the effective potential V_1 would contain terms of the form

$$\sum_{i=1}^{4}\left(A_i\varepsilon_i^p\log\varepsilon_i-A_i\varepsilon_i^p\log\mu\right);$$

for example, those terms in (104) proportional to $\varepsilon_i^p\log\varepsilon_i$, $p\leq 12$, may be replaced by $\varepsilon_i^p\log(\varepsilon_i/\mu)$ for *any* mass scale μ, since the overall coefficient of $\log\mu$ in the resulting expression is zero. Thus equations (101), (103) and (104) may be rewritten by removing all terms involving $\log\mu$ and replacing every occurrence of $\log\varepsilon_i$ with $\log(\varepsilon_i/\mu)$.

3.6 Discussion and further questions

In this section we have derived an expression for the unregularised one-loop Vilkovisky effective potential of scalar electrodynamics with non-zero background fields for both the scalar and gauge fields, using both the generalised R_ξ gauge and the unitary gauge, and have found that both answers are equal as predicted by the theory outlined in Sections 3.2 to 3.3. We have also attempted to calculate the effective potential using dimensional regularisation, but the expression given is approximate whenever the VEV for the gauge field is non-zero. We hope that it may be possible to give an exact closed-form expression for V_1 by extending the method given here (or perhaps by using some other method entirely). Having calculated the effective potential it would be interesting to examine the thermodynamic properties of the theory at non-zero chemical potential, but that is beyond the scope of the current work.

Since this work originally grew out of the work on Section 2 it would also be interesting to calculate the effective potential of $\mathcal{N} = 4$ super Yang-Mills theory using Vilkovisky's method, to compare the results to those of that section. However it appears that this would be a serious undertaking; in addition to the difficulties encountered in the relatively simple case of scalar electrodynamics, Fradkin and Tseytlin [42] explain that the problem of choosing a supergauge-invariant metric on the configuration space is non-trivial. In the case of the Wess-Zumino model, for example, no such local metric can be constructed in terms of the component fields, but it is instead necessary to start with the superspace formalism, in which the configuration space is treated as a supermanifold, coordinatised by both the component fields and the non-dynamical field F.

3.7 Appendix: an integral identity

In this section we prove an integral identity used in Section 3.5. Suppose that a_i, $i = 1, 2, \ldots N$ are distinct real numbers, $N \geq 2$, and $f(x)$ is a real function, differentiable at least $N - 1$ times. Then

$$\begin{aligned} & \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{N-2}} \mathrm{d}x_{N-1} \\ \times \quad & f^{(N-1)}\left(a_N x_{N-1} + a_{N-1}(x_{N-2} - x_{N-1}) + \ldots + a_1(1 - x_1)\right) \\ = \quad & \sum_{i=1}^{N} f(a_i) \prod_{i \neq j} \frac{1}{a_i - a_j} \end{aligned} \tag{105}$$

where $f^{(n)}$ denotes the n^{th} derivative of f. We prove this by induction on N. For $N = 2$ the result is immediate, so suppose that (105) holds for some $N = n$, and consider the case $N = n + 1$: we have

$$\begin{aligned} & \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-2}} \mathrm{d}x_{n-1} \int_0^{x_{n-1}} \mathrm{d}x_n \\ & \times f^{(n)}\left(a_{n+1} x_n + a_n(x_{n-1} - x_n) + \ldots + a_1(1 - x_1)\right) \\ = \quad & \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \cdots \int_0^{x_{n-2}} \mathrm{d}x_{n-1} \frac{1}{a_{n+1} - a_n} \\ \times \quad & \left\{ f^{(n-1)}\left(a_{n+1} x_{n-1} + a_{n-1}(x_{n-2} - x_{n-1}) + \ldots + a_1(1 - x_1)\right) \right. \\ - \quad & \left. f^{(n-1)}\left(a_n x_{n-1} + a_{n-1}(x_{n-2} - x_{n-1}) + \ldots + a_1(1 - x_1)\right) \right\}. \end{aligned}$$

Using the induction hypothesis, this is equal to

$$
\begin{aligned}
&\frac{1}{a_{n+1}-a_n}\left\{-\sum_{i=1}^{n} f(a_i)\prod_{j\neq i,n+1}\frac{1}{a_i-a_j}\right.\\
+\;&\left.\sum_{i=1}^{n-1} f(a_i)\prod_{j\neq i,n}\frac{1}{a_i-a_j}+f(a_{n+1})\prod_{j\neq n,n+1}\frac{1}{a_{n+1}-a_j}\right\}\\
=\;& f(a_{n+1})\prod_{j\neq n+1}\frac{1}{a_{n+1}-a_j}+f(a_n)\prod_{j\neq n}\frac{1}{a_n-a_j}\\
+\;&\frac{1}{a_{n+1}-a_n}\sum_{i=1}^{n-1} f(a_i)\left(\frac{1}{a_i-a_{n+1}}-\frac{1}{a_i-a_n}\right)\prod_{j\neq i,n,n+1}\frac{1}{a_i-a_j};
\end{aligned}
$$

finally, from

$$
\frac{1}{a_{n+1}-a_n}\left(\frac{1}{a_i-a_{n+1}}-\frac{1}{a_i-a_n}\right)=\left(\frac{1}{a_i-a_n}\right)\left(\frac{1}{a_i-a_{n+1}}\right)
$$

the result follows. Several special cases are used in Section 3.5, namely when $f(x)=x^p$ for some $p\in\mathbb{N}$. When $p<N-1$, note that the integral in the left hand side of (105) is zero since $f^{(N-1)}(x)=0$. For $p=N-1$, note that $f^{(N-1)}(x)=(N-1)!$, and it is easily verified that

$$
\int_0^1 \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{N-2}}\mathrm{d}x_{N-1}\,1=\frac{1}{(N-1)}!.
$$

Finally we need the case $p=N$, so that $f^{(N-1)}(x)=N!x$. Defining $x_0\equiv 1$ and $x_N\equiv 0$, it is easy to verify that

$$
\int_0^1 \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{N-2}}\mathrm{d}x_{N-1}\,x_i=\frac{N-i}{N!}.
$$

The coefficient of a_i in the integrand of (105) is $N!(x_{i-1}-x_i)$; we find that

$$
\begin{aligned}
&N!\int_0^1 \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\cdots\int_0^{x_{N-2}}\mathrm{d}x_{N-1}\,(x_{i-1}-x_i)\\
=\;&N!\left(\frac{N-i+1}{N!}-\frac{N-i}{N!}\right)=1.
\end{aligned}
$$

Therefore the special cases which are used in Section 3.5 are

$$\sum_{i=1}^{N} a_i^p \prod_{i\neq j} \frac{1}{a_i - a_j} = \begin{cases} 0 & p < N-1 \\ 1 & p = N-1 \\ \sum_i a_i & p = N. \end{cases} \tag{106}$$

Note that the case $f(x) = (-)^{N+1}x^{-1}$ gives the integral (11.A.1) of [46]. The result given there may be reproduced by noting that

$$\begin{aligned} (-)^{N+1} \sum_{i=1}^{N} \frac{1}{a_i} \prod_{j\neq i} \frac{1}{a_i - a_j} &= \frac{1}{\prod_{k=1}^{N} a_k} \sum_{i=1}^{N} \left(\frac{1}{a_i}\right)^{N-1} \prod_{j\neq i} \frac{1}{\frac{1}{a_i} - \frac{1}{a_j}} \\ &= \frac{1}{\prod_{k=1}^{N} a_k} \end{aligned} \tag{107}$$

where use has been made of (106) with $p = N - 1$.

We also need to consider the case where two or more a_i are equal. Rather than attempt to evaluate the integral of (105) directly, we instead suppose that they differ by some small set of distinct numbers, and take the limit of the right hand side of (105) as those numbers go to zero; the result will then follow by continuity. Without loss of generality suppose that a_i, $i = 1 \ldots r$ are close to some fixed a, and let

$$\delta_i = a_i - a, \quad i = 1 \ldots r.$$

The first r terms in the right hand side of (105) are given by

$$\sum_{i=1}^{r} f(a + \delta_i) \prod_{j=1\ldots r, j\neq i} \frac{1}{\delta_i - \delta_j} \prod_{k=r+1}^{N} \frac{1}{a - a_k + \delta_i}. \tag{108}$$

Let us define

$$\begin{aligned} \Pi &\equiv \prod_{k=r+1}^{N} (a - a_k) \\ G(\delta) &\equiv \prod_{k=r+1}^{N} \left(1 + \frac{\delta}{a - a_k}\right) \end{aligned}$$

so that (108) is equal to

$$\frac{1}{\Pi} \sum_{i=1}^{r} \frac{f(a + \delta_i)}{G(\delta_i)} \prod_{j\neq i} \frac{1}{\delta_i - \delta_j}. \tag{109}$$

We also assume that $f(x)$ is sufficiently smooth at a, so that

$$f(a+\delta_i) = \sum_{n=0}^{r-1} \frac{1}{n!} f^{(n)}(a)\delta_i^n + \mathcal{O}(\delta_i^r).$$

Provided that $|\delta_i| < |a - a_k|$ for $k = r+1, \ldots N$, $1/G(\delta_i)$ may be Taylor expanded around $\delta_i = 0$, so that (109) becomes a sum of terms of the form given in (105) with $N = r$, $a_i = \delta_i$, $f(x) = f^{(n)}(a)x^{n+p}$. When $n + p \geq r$ the sum in (109) is $\mathcal{O}(\delta)$. For $n + p < r - 1$ it is zero, using (106), so we need only consider the case where $n + p = r - 1$. Thus (109) is equal to

$$\frac{1}{\Pi} \sum_{n=0}^{r-1} \frac{1}{n!} G_{r-n-1} f^n(a) + \mathcal{O}(\delta) \tag{110}$$

where G_n is the coefficient of x^n in $1/G(x)$. Letting s_n denote the n^{th} elementary symmetric polynomial in the variables $1/(a - a_k)$, $k = r+1, \ldots N$, some of the G_n are given by

$$\begin{aligned} G_0 &= 1 \\ G_1 &= -s_1 \\ G_2 &= -s_2 + s_1^2 \\ G_3 &= s_3 - 2s_2 s_1 + s_1^3. \end{aligned}$$

By continuity, when two or more of the a_i are equal, those terms in the right hand side of (105) involving division by zero are replaced by terms of the form (110).

4 The classical vacuum structure of a deformation of $\mathcal{N} = 4$ super Yang-Mills theory

4.1 Introduction to Section 4

In this section we present the calculation of the classical vacuum structure of several generalisations of a theory which was previously investigated by Hollowood and Kumar [47]. This theory is of interest since under renormalisation group flow into the infrared it exhibits a duality cascade, a concept to be briefly summarised below. The principal tool which is used to investigate the infrared behaviour is the Dijkgraaf-Vafa conjecture [53] [54] [55], which relates the exact low-energy effective potential of the theory with a finite number N of colours to the partition function of a related matrix model in the limit of large N.

4.1.1 Seiberg duality and the duality cascade

In [48], Seiberg considered two $\mathcal{N} = 1$ supersymmetric theories, manifestly different at arbitrary distance scales, and argued that they are dual in the infrared. The first theory is supersymmetric quantum chromodynamics (henceforth referred to as SQCD) with gauge group $SU(N)$, and superpotential $W = 0$. It contains chiral multiplets Q^i, $i = 1 \ldots N_f$ transforming in the N representation of $SU(N)$, and $\widetilde{Q}_{\widetilde{i}}$, $\widetilde{i} = 1 \ldots N_f$ transforming in the $\bar{N}$ representation of $SU(N)$. i and $\widetilde{i}$ are flavour indices; gauge indices are not shown. The theory has a global symmetry group

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R;$$

the fields Q^i transform in the N_f representation of $SU(N_f)_L$ while transforming trivially under $SU(N_f)_R$, and $\widetilde{Q}_{\widetilde{i}}$ transform in the $\overline{N_f}$ representation of $SU(N_f)_R$ while transforming trivially under $SU(N_f)_L$. For the remaining $U(1)_B \times U(1)_R$ the fields Q^i and $\widetilde{Q}_{\widetilde{i}}$ have charges $(1, 1 - N/N_f)$ and $(-1, 1 - N/N_f)$ respectively. Some gauge-invariant operators of the theory are

$$\begin{aligned} M^i_{\widetilde{i}} &\equiv Q^{i\,a} Q_{\widetilde{i}\,a} \\ B^{i_1 \ldots i_N} &\equiv \epsilon_{a_1 \ldots a_N} Q^{i_1\,a_1} \ldots Q^{i_N\,a_N} \\ \widetilde{B}_{\widetilde{i}_1 \ldots \widetilde{i}_N} &\equiv \epsilon^{a_1 \ldots a_N} \widetilde{Q}_{\widetilde{i}_1\,a_1} \cdots \widetilde{Q}_{\widetilde{i}_N\,a_N} \end{aligned}$$

where a, a_r are gauge indices. For $N_f \geq N + 2$ the quantum moduli space is identical to the classical one, and the theory at the origin with $M = B =$

$\widetilde{B} = 0$ has been conjectured [49] to be in a non-Abelian Coulomb phase. The theory depends on a coupling constant g; by considering the beta function in the limit N, $N_f \to \infty$ with Ng^2 and $N_f/N = 3 - \epsilon$ fixed, Seiberg showed that the theory contains a non-trivial fixed point in this limit, and further claimed that this fixed point exists whenever $3N/2 < N_f < 3N$, and that the theory at this point is an interacting conformal field theory of quarks and gluons.

The second theory considered in [48] is SQCD with the same global symmetry group, but this time with gauge group $SU(N_f - N)$. It contains chiral fields $q_{\widetilde{i}}$, $\widetilde{i} = 1 \ldots N_f$, transforming in the $N_f - N$ representation of the gauge group, and $\widetilde{q}^i$, $i = 1 \ldots N_f$ transforming in the $\overline{N_f - N}$ representation. The theory has the same global symmetry group as before, but, as indicated by the indices on q, $\widetilde{q}$, the fields $q_{\widetilde{i}}$ transform in the $\overline{N_f}$ representation of $SU(N_f)_L$ while $\widetilde{q}^i$ transform in the N_f representation of $SU(N_f)_R$. For $U(1)_B \times U(1)_R$ they have charges $(N/(N_f - N), N/N_f)$ and $(-N/(N_f - N), N/N_f)$ respectively; note that with these definitions the gauge-invariant baryon fields defined by

$$\begin{aligned} B^{i_1 \ldots i_N} &\equiv \epsilon^{i_1 \ldots i_N \widetilde{j_1} \ldots \widetilde{j_{N_f - N}}} \epsilon^{a_1 \ldots a_{N_f - N}} q_{\widetilde{j_1} a_1} \cdots q_{\widetilde{j_{N_f - N}} a_{N_f - N}} \\ \widetilde{B}_{\widetilde{i_1} \ldots \widetilde{i_N}} &\equiv \epsilon_{\widetilde{i_1} \ldots \widetilde{i_N} j_1 \ldots j_{N_f - N}} \epsilon_{a_1 \ldots a_{N_f - N}} \widetilde{q}^{j_1 a_1} \cdots \widetilde{q}^{j_{N_f - N} a_{N_f - N}} \end{aligned}$$

have the same $U(1)_B \times U(1)_R$ charges as before, and also transform in the same way under the action of $SU(N_f)_L \times SU(N_f)_R$. The theory also contains independent meson fields $M^i_{\widetilde{i}}$ transforming in the representation $(N_f, \overline{N_f})$ of $SU(N_f)_L \times SU(N_f)_R$ with $U(1)_B \times U(1)_R$ charges $(0, 2 - 2N/N_f)$. In addition to the ordinary action of SQCD it has superpotential

$$W = y M^i_{\widetilde{i}} q_i \widetilde{q}^{\widetilde{i}}.$$

Note that $3N/2 < N_f < 3N$ if and only if $3(N_f - N)/2 < N_f < 3(N_f - N)$, so that, for the same range of values for N, N_f as before, this theory has an infrared fixed point. Seiberg proposed that at this fixed point the two theories are dual; more precisely, the gauge invariant operators B, $\widetilde{B}$ and M defined above have the same correlation functions, so that the two descriptions in terms of the elementary fields Q, $\widetilde{Q}$ or q, $\widetilde{q}$ are really two alternative descriptions of the same physics.

In [50] Klebanov and Strassler considered a supersymmetric gauge theory with gauge group $SU(N + M) \times SU(N)$, with matter fields A_1 and A_2 transforming in the bifundamental $(N + M, \overline{N})$ representation of the gauge group,

and also fields B_1 and B_2 transforming in the antibifundamental $(\overline{N+M}, N)$ representation. It has a global symmetry group

$$SU(2)_L \times SU(2)_R \times U(1)_B$$

under which (A_1, A_2) transforms in the fundamental representation of $SU(2)_L$ and (B_1, B_2) transforms in the fundamental representation of $SU(2)_R$; the fields A_i and B_i have baryon number $+1$ and -1 respectively. The superpotential is

$$W = h\mathrm{Tr}\left[A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right]$$

(where the trace is over gauge indices). They found that, under renormalisation group flow into the infrared, the theory underwent a sequence of Seiberg dualities, called a duality cascade, in which the number of colours in each gauge group factor repeatedly dropped by M. It has been speculated [50] that the standard model itself may lie at the base of a similar cascade, and that the elementary particles of the standard model may be dual to particles which are truly elementary in the ultraviolet.

For an introductory account of Seiberg duality and the duality cascade, see [51].

4.1.2 Matrix models and the Dijkgraaf-Vafa conjecture

It has long been known that the diagrammatic expansion of a matrix model may be understood as a sum over two-dimensional discretised surfaces (see e.g. [52]). For example, consider the matrix model with partition function

$$\mathcal{Z} = \mathrm{e}^{-F} = \int \mathrm{d}M \, \mathrm{e}^{N\mathrm{Tr}\left(-\frac{1}{2}M^2 + gM^3\right)}$$

where the integral is over $N \times N$ Hermitian matrices. Each Feynman diagram gives a triangulation of some surface, and the free energy F is a sum over connected surfaces. Simple power counting arguments [52] show that each diagram has an overall factor of $N^\chi = N^{2-2h}$ where χ is the Euler characteristic of the triangulation and h is its genus. Therefore in the large N limit we may expand F in inverse powers of N:

$$F = N^2 F_0 + F_1 + \ldots + F_h N^{2-2h} + \ldots$$

with F_h a sum over all connected surfaces with genus h. The relevance of matrix models to the theory investigated in [47] comes from the Dijkgraaf-Vafa conjecture [53], [54], [55]. Consider an $\mathcal{N} = 1$ $U(N)$ supersymmetric

gauge theory with a chiral superfield Φ, and superpotential $W(\Phi)$ which is the trace of a polynomial in Φ. The theory has a complex gauge coupling $\tau \equiv 4i\pi/g^2 + \theta/2\pi$. A supersymmetric vacuum may be obtained by diagonalising Φ and choosing each of its eigenvalues to be at an extremum of W; if W has extrema at w_i, $i = 1 \ldots p$, then such a vacuum is determined by the number N_i of eigenvalues of Φ which are equal to w_i. For a given vacuum the infrared limit of the theory involves a superpotential W_{eff} which is a function of some set of effective fields which characterise the low energy theory. Consider now the matrix model containing a single $\hat{N} \times \hat{N}$ Hermitian matrix Φ, whose partition function is given by

$$\mathcal{Z} = \int \mathcal{D}\Phi \, \mathrm{e}^{-\frac{1}{g_s} W(\Phi)};$$

in the above the integral over Φ should be understood as a multi-variable contour integral. In the limit $g_s \to 0$ the matrix model partition function may be calculated by the saddle-point approximation. As in the case of the field theory, a saddle point is determined by the number $\hat{N}_i$ of eigenvalues of Φ at each extremum of W. Consider now the zero-genus contribution F_0 to the free energy, evaluated in the limit $g_s \to 0$, $\hat{N} \to \infty$, with $S_i \equiv g_s \hat{N}_i$ fixed; we call this quantity $F_0(S_i)$. The Dijkgraaf-Vafa conjecture states that the S_i may be considered effective fields of the field theory, and the exact effective potential around the vacuum determined by the set of numbers N_i is given by

$$W_{\text{eff}}(S_i) = \sum_{i=1}^{p} \left(N_i \frac{\partial F_0(S_i)}{\partial S_i} - 2\pi i \tau S_i \right).$$

Thus in order to calculate the effective potential of a supersymmetric gauge theory it is first necessary to determine its classical vacuum structure. The remainder of Section 4 is devoted to this problem for some particular theories. For some tests and applications of the Dijkgraaf-Vafa conjecture, see also [56], [57], [58].

4.2 The β-deformation of $\mathcal{N} = 4$ super Yang-Mills theory

The original theory considered by Hollowood and Kumar[47] was the so-called "β-deformation" of $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(N)$. The theory contains three $N \times N$ matrix-valued chiral superfields in the adjoint representation of the gauge group, which we call X, Y and Z. The superpotential is given by

$$W = \mathrm{Tr}\,[XYZ - qXZY + V(X)]. \tag{111}$$

Here q is a primitive n^{th} root of unity, $q = e^{\frac{2\pi i}{n}}$, and $V(x)$ is a polynomial in x given by

$$V(x) = \sum_{j=1}^{l} \frac{a^j}{j} x^j \tag{112}$$

where we also demand that $a_j = 0$ whenever $j = 0 \bmod n$, for reasons that will become clear later. It will prove convenient to introduce the function $f(x)$ defined as follows:

$$f(x) - f(qx) \equiv xV'(x) \quad \Rightarrow \quad f(x) = \sum_{j=1}^{l} \frac{a_j x^j}{1 - q^j}. \tag{113}$$

In order to find the classical vacuum structure of this theory we in principle need to solve the F- and D-flatness conditions

$$\frac{\partial W}{\partial X^i} = \frac{\partial W}{\partial Y^i} = \frac{\partial W}{\partial Z^i} = 0 \qquad \text{and} \qquad D = 0$$

modulo gauge transformations; in fact, however, it turns out that this is equivalent to solving only the F-flatness conditions modulo *complexified* gauge transformations, that is we replace the gauge group $U(N)$ with $GL(N)$. The F-flatness equations for the superpotential (112) are:

$$XY - qYX = 0, \qquad ZX - qXZ = 0, \qquad YZ - qZY = -V'(X). \tag{114}$$

From (114) we find that the following four quantities commute with X, Y and Z:

$$u \equiv X^n, \quad v \equiv Y^n, \quad w \equiv Z^n \quad \text{and} \quad t \equiv XYZ + f(X). \tag{115}$$

These are not independent, but instead satisfy

$$uvw = (-)^{n-1} \prod_{j=1}^{n} (t - f(q^j X)). \tag{116}$$

Note that the right hand side of this equation is invariant under the replacement $X \mapsto qX$, and so may be written as a function of u. In order to prove (116), note first that, for any polynomial function $g(x)$, we have from the algebra (114)

$$g(X)Y = Yg(qX) \qquad g(X)Z = Zg(q^{-1}X) \quad \text{and} \quad YZ = qZY - V'(X)$$

from which it is straightforward to show that

$$\begin{aligned} X^{i+1}Y^{i+1}Z^{i+1} &= q^i X^i Y^i Z^i \{XYZ - \sum_{j=1}^{i} q^{-j} XV'(q^{-j}X)\} \\ &= q^i X^i Y^i Z^i (t - f(q^{-i}X)). \end{aligned}$$

Consequently

$$\begin{aligned} X^n Y^n Z^n &= q^{\sum_{j=0}^{n-1} j} \prod_{j=0}^{n-1} (t - f(q^{-j}X)) \\ &= q^{\sum_{j=0}^{n-1} j} \prod_{j=1}^{n} (t - f(q^{j}X)) \end{aligned}$$

where we have used the fact that $q^{-j} = q^{n-j}$ and the fact that the individual factors $(t - f(q^j X))$ commute with one another. Finally note that

$$q^{\sum_{j=0}^{n-1} j} = q^{\frac{1}{2}n(n-1)} = (-)^{n-1}.$$

To find solutions to the F-flatness equations we first use our gauge freedom to diagonalise the matrix X; any remaining elements of $GL(N)$ act on X as permutations. Writing $X_{ij} = \delta_{ij} X^{(i)}$ the first two of equations (114) become

$$\begin{aligned} X^{(i)} Y_{ij} &= q X^{(j)} Y_{ij} \\ X^{(j)} Z_{ij} &= q X^{(i)} Z_{ij} \end{aligned}$$

which simply states that Y_{ij} is zero whenever $X^{(i)} \neq qX^{(j)}$, and similarly for Z_{ji}. We may then use the remaining gauge freedom to divide X into blocks of the following form:

$$X = \mathrm{diag}(\overbrace{x, x, \cdots}^{n_0}, \overbrace{qx, qx, \cdots}^{n_1}, \cdots \overbrace{q^{p-1}x, q^{p-1}x, \cdots}^{n_{p-1}}) \tag{117}$$

with $p \leq n$. This choice reduces the gauge group to $U(n_0) \times U(n_1) \times \cdots \times U(n_{p-1})$. With this choice Y takes the form of off-diagonal blocks Y_i of size $n_{i+1} \times n_i$, which, under a gauge transformation $g_0 \times g_1 \times \cdots \times g_{p-1}$, transform as

$$Y_i \mapsto g_{i+1} Y_i g_i^{-1}. \tag{118}$$

Similarly Z consists of off-diagonal blocks Z_i of size $n_i \times n_{i+1}$, transforming under the bifundamental representation of $GL(n_i) \times GL(n_{i+1})$. For simplicity we restrict our attention to the case where every $n_i = 1$; in this case the group elements g_i are simply non-zero complex numbers. Consider first the case $p < n$. Assuming that all the Y_i are non-zero, it is possible to set them all to 1 by gauge transformations: setting $g_0 = 1$, for each i we let $g_{i+1} = Y_i^{-1} g_i$. Finally X, Y and Z take the form

$$X = x \begin{pmatrix} 1 & & & \\ & q & & \\ & & \ddots & \\ & & & q^{p-1} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix},$$
$$Z = \begin{pmatrix} 0 & z_0 & & \\ & \ddots & & \\ & & 0 & z_{p-2} \\ & & & 0 \end{pmatrix}. \tag{119}$$

The z_i are fixed by the third of equations (114); we find

$$\begin{aligned} -qz_0 &= -V'(x) \\ z_0 - qz_1 &= -V'(qx) \\ &\vdots \\ z_{p-2} &= -V'(q^{p-1}x). \end{aligned} \tag{120}$$

In addition, this implies that

$$\sum_{j=0}^{p-1} q^j V'(q^j x) = 0 \quad \Rightarrow \quad f(x) = f(q^p x). \tag{121}$$

Recall that it was assumed that all the Y_j were non-zero; if for example Y_i were zero then no constraints involving z_i would appear in equations (120) and the vacuum would be degenerate. However, note that in this case the equations (120) reduce to two sets of equations which simultaneously imply that $f(x) = f(q^{i+1}x) = f(q^p x)$, which can only occur in exceptional circumstances.

Now we move onto the case $p = n$. This differs in that now $X^{(0)} = q^n X^{(0)} = qX^{(p-1)}$, meaning that there are additional elements in the top right and bottom left corners of Y and Z respectively. Furthermore, having chosen group elements g_i, $i = 0 \cdots n-1$, we now no longer have the freedom to fix Y_{p-1}, meaning that these vacua are degenerate. We choose the following form for Y and Z:

$$Y = \begin{pmatrix} 0 & & & y \\ 1 & 0 & & \\ & & \ddots & \\ & & 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & z_1 & & \\ & \ddots & & \\ & & 0 & z_{p-2} \\ z_{p-1} & & & 0 \end{pmatrix}. \tag{122}$$

Substituting into the last of equations (114) gives

$$\begin{aligned} yz_{p-1} - qz_0 &= -V'(x) \\ z_0 - qz_1 &= -V'(qx) \\ &\vdots \\ z_{p-2} - qyz_{p-1} &= -V'(q^{n-1}x) \end{aligned}$$

which, for a given choice of, say, yz_{p-1}, may be solved provided that

$$\sum_{j=0}^{n-1} q^j V'(q^j x) = 0;$$

but this is true for any choice of x by the constraint $a_j = 0$ for $j = 0 \bmod n$. So we see that these vacua are truly degenerate, and are labelled by three

complex parameters. It should be pointed out that our choice of gauge (122) again assumed that all the Y_j were non-zero. If at most one of them is zero, then by permuting the elements of X we may send this Y_i to Y_{p-1}, and this reduces to a solution of the form (122), with $y = 0$; while if two or more are zero, then this reduces to a sum of solutions of the form (119), with some of the z_i as free parameters.

Below we consider two generalisations of the solutions considered above.

4.2.1 Generalisation to quiver theories

The first generalisation is a theory with the same field content and superpotential (111) considered above, but with a gauge group $U(N)^k$, $k \geq 2$. We also make the additional requirement that the fields, when expressed as $k \times k$ matrices, take the following form:

$$X = \mathrm{diag}(X_1, X_2, \cdots, X_k)$$

$$Y = \begin{pmatrix} 0 & & & Y_k \\ Y_1 & 0 & & \\ & \ddots & & \\ & & Y_{k-1} & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & Z_1 & & \\ & \ddots & & \\ & & 0 & Z_{k-1} \\ Z_k & & & 0 \end{pmatrix}. \tag{123}$$

In other words they look like the solutions (122), with $p = k$, although here the components are themselves $N \times N$ matrices. Note that, under a gauge transformation $g_1 \times g_2 \times \cdots g_k \in U(N)^k$, written as a matrix

$$\mathrm{diag}(g_1, g_2, \cdots, g_k)$$

the individual components transform as, for example

$$X_{ij} \mapsto g_i X_{ij} g_j^{-1}$$

so that the X_i transform in the adjoint representation of $U(N)_i$, the Y_i transform in the bifundamental representation of $U(N)_{i+1} \times U(N)_i$ (exactly like equation (118)), and the Z_i transform in that of $U(N)_i \times U(N)_{i+1}$.

In order to find solutions of the F-flatness equations (114) that resemble the solutions (119) and (122), it will prove useful to reorder the matrices X, Y and Z into the form of $N \times N$ matrices of $k \times k$ matrices, that is we send the $(i-1)N + j^{\text{th}}$ row to the $(j-1)k + i^{\text{th}}$, and similarly for columns. Note

that under this reordering the $i \times j$ component of a matrix is associated to the $(i \bmod k)^{\text{th}}$ $U(N)$ factor on the left and the $(j \bmod k)^{\text{th}}$ $U(N)$ factor on the right. Furthermore the non-zero elements of Y and Z^t are now located at pairs (i,j) where $i = j+1 \bmod k$.

As in Section 4.2 we may use our gauge freedom to diagonalise X, and again we make the simplifying assumption that the eigenvalues of X are non-degenerate. The gauge group does not allow arbitrary permutations of the elements of X, but rather it only allows permutations among those elements X_{ii}, X_{jj} with $i = j \bmod k$; however, if we have $X_{ii} = qX_{jj}$ with $i \neq j+1 \bmod k$, then these elements may not be "linked" with non-zero elements Y_{ij} or Z_{ji}; in other words we do not need to treat separately the case where, for example,

$$X = \mathrm{diag}(x, x', qx, \cdots)$$

because x and qx lie in different reducible blocks of the algebra spanned by X, Y and Z. And so, after reordering, the most general form for X we need consider may be reduced to blocks of the following form:

$$X = \mathrm{diag}(x, qx, \cdots, q^{p-1}x).$$

The case $p < n$ gives solutions that are identical to those of (119). As before, that part of the gauge group unbroken by our choice of X is $U(1)^p$, and may be used to set all non-zero elements of Y to 1. For the case $p = n$, on the other hand, a difference emerges. For the solutions

$$Y = \begin{pmatrix} 0 & & & y \\ 1 & 0 & & \\ & \ddots & & \\ & & 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & z_1 & & \\ & \ddots & & \\ & & 0 & z_{p-2} \\ z_{n-1} & & & 0 \end{pmatrix}$$

may only have non-zero y and z_{n-1} if $n = 0 \bmod k$, in other words if n is a multiple of k. So we see that the effect of generalising to a quiver theory is to constrain the allowed points in the moduli space which may be degenerate, and so reduce the number of massless degrees of freedom around a given classical vacuum.

4.2.2 Generalisation to $n_i = m$

In this section we return to the $U(N)$ gauge theory of Section 4.2, but consider a slightly more general class of solutions. Recall that following equation

(117) we assumed that all the n_i were equal to 1; we now partially relax this condition, and take $n_i = m$, $m \geq 2$, for all n_i. Explicitly, X is given by

$$X = \mathrm{diag}(\overbrace{x, x, \cdots}^{m}, \overbrace{qx, \cdots}^{m}, \cdots, \overbrace{q^{p-1}x, \cdots}^{m}).$$

Substituting into the first two of equations (114) tells us that Y and Z take the following block form, where Y_i and Z_i are $m \times m$ matrices:

$$Y = \begin{pmatrix} 0 & & & Y_{p-1} \\ Y_0 & 0 & & \\ & \ddots & & \\ & & Y_{p-2} & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & Z_0 & & \\ & \ddots & & \\ & & 0 & Z_{p-2} \\ Z_{p-1} & & & 0 \end{pmatrix} \tag{124}$$

where $Y_{p-1} = Z_{p-1} = 0$ if $p < n$. The complexified gauge group is broken to $GL(m)^p$. So in fact these solutions resemble the quiver theory studied in Section 4.2.1.

We may use our remaining gauge freedom to fix elements of Y. Elements of the gauge group are given by $g_0 \times g_1 \times \cdots \times g_{p-1}$, $g_i \in GL(m)$. We assume that all the Y_i are invertible; then letting $g_0 = g \in GL(m)$ we may set each Y_i, $i = 1 \ldots p-2$ to $\mathbb{1}_m$ by choosing $g_{i+1} = g_i Y_i^{-1}$. Under this transformation we find

$$Y_{p-1} \mapsto g Y_{p-1} Y_{p-2} \cdots Y_0 g^{-1}$$

so that we may also diagonalise Y_{p-1}. Considering first the case $p < n$, and substituting into the last of equations (114), we find that

$$\begin{aligned} -qZ_0 &= -V'(x)\mathbb{1}_m \\ Z_0 - qZ_1 &= -V'(qx)\mathbb{1}_m \\ &\vdots \\ Z_{p-2} &= -V'(q^{p-1}x)\mathbb{1}_m \end{aligned}$$

which can be solved provided that $f(x) = f(q^p x)$. Note that had one of the Y_j, say Y_i, been singular, then we could have fixed Y_j, $j < i$ to be the identity, and we would have

$$q^i Z_i Y_i = \sum_{j=0}^{i} q^j V'(q^j x)\mathbb{1}_m$$

which can only be satisfied if both sides are zero, so $f(x) = f(q^{i+1}x)$. Similarly we find $f(q^{i+1}x) = f(q^p x)$, so such solutions are unlikely.

In the case that $p = n$, we find that

$$\begin{aligned} Y_{p-1}Z_{p-1} - qZ_0 &= -V'(x)\mathbb{1}_m \\ Z_0 - qZ_1 &= -V'(qx)\mathbb{1}_m \\ &\vdots \\ Z_{p-2} - qY_{p-1}Z_{p-1} &= -V'(q^{n-1}x)\mathbb{1}_m; \end{aligned}$$

using the fact that $\sum_{j=0}^{n-1} q^j V'(q^j x) = 0$, these equations reduce to $Y_{p-1}Z_{p-1} - Z_{p-1}Y_{p-1} = 0$. Writing $Y_{p-1} = \mathrm{diag}(y_1, y_2, \cdots)$, this becomes

$$(y_i - y_j)Z_{p-1_{ij}} = 0$$

i.e. $Z_{p-1_{ij}}$ is zero whenever $y_i \neq y_j$; if any r of the y_i are equal, then our choice of g which was used to diagonalise Y_{p-1} contained a spare $GL(r)$ factor which may be used to diagonalise Z_{p-1}. Consequently the solutions with $p = n$ are degenerate with $2m+1$ free parameters, namely x and the diagonal components of Y_{p-1} and Z_{p-1}.

We now consider an example, which lies at the boundary between two types of solution considered in Sections 4.2 and 4.2.2. Let $n = 3$, $N = 6$ and

$$V(x) = -\epsilon(\frac{x^2}{2} - ax) \quad \Rightarrow \quad f(x) = -\epsilon(\frac{x^2}{1-q^2} - \frac{ax}{1-q}).$$

We will consider perturbations around a classical vacuum which is a sum of solutions of the form (119). Let

$$X_0 = \mathrm{diag}(x, qx, q^2x, x, qx, q^2x)$$

where x satisfies $f(x) = f(qx)$, that is $x = a$. Also let

$$Y_0 = \left(\begin{array}{c|cc|c|cc} 0 & & & & & \\ \hline & 0 & & & & \\ & 1 & 0 & & & \\ \hline & & & 0 & & \\ \hline & & & & 0 & \\ & & & & 1 & 0 \end{array}\right)$$

$$Z_0 = \left(\begin{array}{c|cc|c} 0 & & & \\ \hline & 0 & z & \\ & & 0 & \\ \hline & & & 0 \\ \hline & & & \begin{array}{cc} 0 & z \\ & 0 \end{array} \end{array}\right),$$

with $z = \epsilon a(q^2 - 1)$. This is a sum of solutions of the form (119) with $p = 1$ and $p = 2$, lying on the boundary of a solution of the form (124) with $p = 3$, $m = 2$; note, however, that X is ordered differently to equation (117), so as to resemble the vacua studied in Section 4.2. We consider the following perturbations about X, Y and Z:

$$X = X_0 + \mathrm{diag}(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$Y = Y_0 + \left(\begin{array}{c|cc|c} 0 & y_1 & & y_2 \\ \hline y_3 & 0 & y_4 & \\ & 0 & & y_5 \\ \hline & y_6 & 0 & y_7 \\ \hline y_8 & & y_9 & 0 \\ & y_{10} & & 0 \end{array}\right)$$

$$Z = Z_0 + \left(\begin{array}{c|cc|c} 0 & z_1 & & z_2 \\ \hline & 0 & & z_3 \\ z_4 & 0 & z_5 & \\ \hline & z_6 & 0 & z_7 \\ \hline & z_8 & & 0 \\ z_9 & & z_{10} & 0 \end{array}\right).$$

Expanding the superpotential (111) in terms of the variables x_i, y_i and z_i gives:

$$\begin{aligned} W = W_0 \; & - \; \frac{\epsilon}{2}\sum_{j=1}^{6} x_j^2 \\ & + \; x_2y_3z_1 + x_4y_7z_{10} + x_5y_8z_2 + x_6y_{10}z_3 + x_1y_1z_4 \\ & + \; x_4y_6z_5 + x_2y_4z_6 + x_5y_9z_7 + x_3y_5z_8 + x_1y_2z_9 \\ & - \; q\{x_1y_3z_1 + x_6y_7z_{10} + x_1y_8z_2 + x_2y_{10}z_3 + x_3y_1z_4 \\ & + \; x_3y_6z_5 + x_4y_4z_6 + x_4y_9z_7 + x_5y_5z_8 + x_6y_2z_9\}. \end{aligned} \tag{125}$$

From equation (125) we see that those perturbations around the diagonal elements of X have a mass $\sqrt{\epsilon}$, and that all other perturbations considered

are massless. Note that the choice of vacuum breaks the symmetry group to $U(1)^4$. In fact this theory has too many massless fields to be an interesting theory in the infrared.

Having found the classical vacuum structure of the theory one may proceed to calculate its low-energy effective potential by solving a corresponding matrix model; the details of this matrix model and its solution may be found in [47], [59]. However since the results given in Sections 4.2.1 and 4.2.2 exhibit no new phenomena compared to the theory studied in [47] this line of research has not been pursued.

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