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Strong convergence of a tamed theta scheme for NSDDEs with one-sided Lipschitz drift*

Li Tan[a,b] and **Chenggui Yuan**[c†]

[a]School of Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, P. R. China

[b]Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, P. R. China

[c]Department of Mathematics
Swansea University, Swansea, SA2 8PP, U.K.

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Abstract

This paper is concerned with strong convergence of a tamed theta scheme for neutral stochastic differential delay equations with one-sided Lipschitz drift. Strong convergence rate is revealed under a global one-sided Lipschitz condition, while for a local one-sided Lipschitz condition, the tamed theta scheme is modified to ensure the well-posedness of implicit numerical schemes, then we show the convergence of the numerical solutions.

MSC 2010: 65C30, 65L20

Key Words : tamed theta scheme; neutral stochastic differential delay equations; one-sided Lipschitz; strong convergence

1 Introduction

Numerical analysis plays an important role in studying stochastic differential equations (SDEs) because most equations can not be solved explicitly. The most commonly used method for approximating SDEs is the explicit Euler-Maruyama (EM) method. There are a lot of literature concerning with the explicit EM scheme for all kinds of SDEs, e.g., Hairer et

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†Contact e-mail address: C.Yuan@swansea.ac.uk

al. [1], Maruyama [9], Milstein [10], and Kloeden and Platen [6]. Most of the early works on explicit EM scheme were about the SDEs with the globally Lipschitz continuous coefficients, since the explicit EM scheme solutions may not converge in the strong sense to the exact solutions with one-sided Lipschitz continuous and superlinearly growing drift coefficients. Moreover, Hutzenthaler et al. [3] pointed out that the absolute moments of the EM scheme at a finite time could diverge to infinity. In order to cope with these difficulties, Higham et.al [2] studied a split-step backward Euler method for nonlinear SDEs, they showed that the implicit EM scheme converged if the drift coefficient satisfied a one-sided Lipschitz condition and the diffusion coefficient was globally Lipschitz. Hutzenthaler et al. [4] proposed a tamed EM scheme in which the drift term is modified to guarantee the boundness of moments. Later, Sabanis [11, 12] studied the strong convergence of the tamed EM scheme and extend the tamed EM scheme to SDEs with superlinearly growing drift and diffusion coefficients, respectively. Zong et al. [16] proposed a semi-tamed Euler scheme to solve the SDEs with the drift coefficient equipped with the Lipschitz continuous part and non-Lipschitz continuous part. Although additional computational effort is needed for implicit analysis, the implicit EM schemes have been showed better than the explicit EM scheme which converges strongly to the exact solution of SDEs under non-globally Lipschitz conditions. The implicit EM methods including the backward EM scheme, the split-step backward EM scheme and the theta scheme have been extensively studied, for example, Mao and Szpruch [8] studied strong convergence and almost sure stability of the backward EM scheme and the theta scheme to SDEs with non-linear and non-Lipschitzian coefficients, to name a few.

Recently, numerical analysis for neutral stochastic differential delay equations (NSDDEs) has also received a great deal of attention, see e.g., Lan and Yuan [7], Wu and Mao [13], Zhou [15], Zong et al. [17], Zong and Wu [18], and the references therein. However, the existing literature are difficult to deal with one-sided Lipschitz and superlinearly drift. To fill the gap, in this paper, we are going to introduce a tamed theta scheme and discuss the strong convergence of this scheme for NSDDEs in which the drift coefficients are one-sided Lipschitz and superlinearly.

The content of our paper is organized as follows. In section 2, we consider NSDDEs with global one-sided Lipschitz drift, the tamed theta scheme is introduced and strong convergence is investigated. We reveal that the tamed theta solution converges to the exact solution with order α (see (B1) below)under the global one-sided Lipschitz and the superlinearly growth condition. In section 3, the global one-sided Lipschitiz drift is replaced by the local one-sided Lipschitz drift, under which we show the convergence of the numerical solutions. In order to guarantee the well-posedness of the implicit tamed scheme, we impose a modified tamed theta scheme with a truncated skill.

2 Global One-sided Lipschitz Drift

For a fixed positive integer n, let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$ be an n-dimensional Euclidean space. Denote $\mathbb{R}^n \otimes \mathbb{R}^d$ by the set of all $n \times d$ matrices endowed with Hilbert-Schmidt norm $\|A\| := \sqrt{\text{trace}(A^*A)}$ for every $A \in \mathbb{R}^n \otimes \mathbb{R}^d$, in which A^* is the transpose of A. For a fixed $\tau \in (0, \infty)$, which will be referred to as the delay or memory, let $\mathscr{C} = C([-\tau, 0]; \mathbb{R}^n)$ be all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$, equipped with the uniform norm $\|\zeta\|_\infty := \sup_{-\tau \le \theta \le 0} |\zeta(\theta)|$ for every $\zeta \in \mathscr{C}$. By a filtered probability space, we mean a quadruple $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$, where $\mathscr{F}$ is a σ-algebra on the outcome space Ω, $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathscr{F})$, and $\{\mathscr{F}_t\}_{t\ge 0}$ is a filtration of sub-σ-algebra of $\mathscr{F}$, where the usual conditions are satisfied, i.e., $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$ and $\mathscr{F}_{t_+} := \bigcap_{s>t} \mathscr{F}_s = \mathscr{F}_t$. Let $\{W(t)\}_{t\ge 0}$ be a d-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$.

In this paper, we consider the following NSDDE

$$\mathrm{d}[X(t) - D(X(t-\tau))] = b(X(t), X(t-\tau))\mathrm{d}t + \sigma(X(t), X(t-\tau))\mathrm{d}W(t), t \ge 0 \tag{2.1}$$

with initial data

$$X_0 = \xi = \{\xi(\theta) : -\tau \le \theta \le 0\} \in \mathcal{L}^p_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n), p \ge 2,$$

that is, ξ is an $\mathscr{F}_0$-measurable $\mathscr{C}$-valued random variable such that $\mathbb{E}\|\xi\|^p_\infty < \infty$ for $p \ge 2$. Here, $D : \mathbb{R}^n \to \mathbb{R}^n$, and $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d$ are continuous in x and y. Fix $T > \tau > 0$, assume that T and τ are rational numbers, and the step size $\Delta \in (0, 1)$ be fraction of T and τ, so that there exist two positive integers M, m such that $\Delta = T/M = \tau/m$. Throughout the paper, we shall denote C by a generic positive constant, whose value may change from line to line. Further, for any $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$, we shall assume that:

(A1) For any $s, t \in [-\tau, 0]$ and $q > 0$, there exists a positive constant K_1 such that

$$\mathbb{E}\|\xi(s) - \xi(t)\|^q_\infty \le K_1 |s - t|^q.$$

(A2) $D(0) = 0$, and there exists a positive constant $\kappa \in (0, 1/2)$ such that

$$|D(x) - D(\overline{x})| \le \kappa |x - \overline{x}|.$$

(A3) There exists a positive constant K_2 such that

$$\langle x - D(y), b(x, y)\rangle \vee \|\sigma(x, y)\|^2 \le K_2(1 + |x|^2 + |y|^2).$$

(A4) There exist positive constants l, K_3 and K_4 such that for $p \ge 2$

$$\begin{aligned}&2\langle x - D(y) - \overline{x} + D(\overline{y}), b(x, y) - b(\overline{x}, \overline{y})\rangle + (p-1)\|\sigma(x, y) - \sigma(\overline{x}, \overline{y})\|^2 \\ &\le K_3(|x - \overline{x}|^2 + |y - \overline{y}|^2),\end{aligned}$$

and

$$|b(x, y) - b(\overline{x}, \overline{y})| \le K_4(1 + |x|^l + |\overline{x}|^l + |y|^l + |\overline{y}|^l)(|x - \overline{x}| + |y - \overline{y}|).$$

Remark 2.1. Due to the existence of implicitness and the neutral term, scopes of Δ and κ in assumption (A2) are given in order to guarantee rationality.

Remark 2.2. If $b(x,y)$ satisfies (A4), then, for any $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}|b(x,y)| &\leq |b(x,y) - b(0,0)| + |b(0,0)| \leq K_4(1 + |x|^l + |y|^l)(|x| + |y|) + |b(0,0)| \\ &\leq C(1 + |x| + |x|^{l+1} + |y| + |y|^{l+1}),\end{aligned}$$

where $C = K_4 \vee |b(0,0)|$. If the coefficients satisfy (A2) and (A4), then one has

$$\begin{aligned}&(p-1)\|\sigma(x,y) - \sigma(\overline{x},\overline{y})\|^2 \\ \leq\ & K_3(|x-\overline{x}|^2 + |y-\overline{y}|^2) + 2|x - D(y) - \overline{x} + D(\overline{y})||b(x,y) - b(\overline{x},\overline{y})| \\ \leq\ & C(1 + |x|^l + |\overline{x}|^l + |y|^l + |\overline{y}|^l)(|x-\overline{x}|^2 + |y-\overline{y}|^2).\end{aligned}$$

Remark 2.3. There are many examples such that the assumptions can be verified. For example, let

$$D(y) = -ay, \quad b(x,y) = x - x^3 + ay - a^3y^3, \quad \sigma(x,y) = x + ay,$$

for $x, y \in \mathbb{R}$, where a is a constant such that $|a| < 1/2$. We can check that assumptions (A2)-(A4) are satisfied. Since (A2) and (A3) are obvious, we only check (A4) here. By computation, we have

$$\begin{aligned}&2\langle x - D(y) - \overline{x} + D(\overline{y}), b(x,y) - b(\overline{x},\overline{y})\rangle + (p-1)\|\sigma(x,y) - \sigma(\overline{x},\overline{y})\|^2 \\ =& 2(x + ay - \overline{x} - a\overline{y})(x - x^3 + ay - a^3y^3 - \overline{x} + \overline{x}^3 - a\overline{y} + a^3\overline{y}^3) + (p-1)(x + ay - \overline{x} - a\overline{y})^2 \\ =& 2[x - \overline{x} + a(y-\overline{y})][(x-\overline{x})(1 - x^2 - \overline{x}^2 - x\overline{x}) \\ &+ a(y-\overline{y})(1 - a^2y^2 - a^2\overline{y}^2 - a^2y\overline{y})] + (p-1)[x - \overline{x} + a(y-\overline{y})]^2 \\ \leq& 4|x-\overline{x}|^2 + 4a^2|y-\overline{y}|^2 + 2(p-1)[(x-\overline{x})^2 + a^2(y-\overline{y})^2] \\ \leq& 2(p+1)(|x-\overline{x}|^2 + |y-\overline{y}|^2),\end{aligned}$$

and

$$\begin{aligned}&|b(x,y) - b(x,\overline{y})| = |x - x^3 + ay - a^3y^3 - \overline{x} + \overline{x}^3 - a\overline{y} + a^3\overline{y}^3| \\ =& |(x-\overline{x})(1 - x^2 - \overline{x}^2 - x\overline{x}) + a(y-\overline{y})(1 - a^2y^2 - a^2\overline{y}^2 - a^2y\overline{y})| \\ \leq& (1 + |x| + |\overline{x}| + |y| + |\overline{y}|)(|x-\overline{x}| + |y-\overline{y}|).\end{aligned}$$

Lemma 2.1. Let (A1)-(A4) hold, the NSDDE (2.1) admits a unique strong global solution $X(t), t \in [0,T]$, and

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X(t)|^p\right) \leq C$$

for $p \geq 2$. One can consult [5] for more details.

2.1 The Tamed Theta Scheme

Now we introduce a tamed theta scheme for (2.1). For $k = -m, \cdots, 0$, set $y_{t_k} = \xi(k\Delta)$; For $k = 0, 1, \cdots, M-1$, we form

$$\begin{aligned}y_{t_{k+1}} - D(y_{t_{k+1-m}}) =\ & y_{t_k} - D(y_{t_{k-m}}) + \theta b_\Delta(y_{t_{k+1}}, y_{t_{k+1-m}})\Delta \\ &+ (1-\theta)b_\Delta(y_{t_k}, y_{t_{k-m}})\Delta + \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k},\end{aligned} \tag{2.2}$$

where $t_k = k\Delta$, and $\Delta W_{t_k} = W(t_{k+1}) - W(t_k)$. Here $b_\Delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and satisfy some conditions given below. Besides, $\theta \in [0,1]$ is an additional parameter that allows us to control the implicitness of the numerical scheme. Since it is convenient to work with a continuous extension of a numerical method, we now define the equivalent continuous form for (2.2). Let $Y_\Delta(t) = \xi(t), t \in [-\tau, 0]$. For $t \in [0,T]$, we define the corresponding continuous-time tamed theta scheme by

$$\begin{aligned} Y_\Delta(t) =& D(\overline{Y}_\Delta(t-\tau)) + \xi(0) - D(\xi(-\tau)) + \theta \int_0^t b_\Delta(\overline{Y}_{\Delta+}(s), \overline{Y}_{\Delta+}(s-\tau))\mathrm{d}s \\ &+ (1-\theta)\int_0^t b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}s + \int_0^t \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}W(s), \end{aligned}$$

here $\overline{Y}_\Delta(t)$ is defined by

$$\overline{Y}_\Delta(t) = y_{t_k} \text{ and } \overline{Y}_{\Delta+}(t) = y_{t_{k+1}} \quad \text{for} \quad t \in [t_k, t_{k+1}), \tag{2.3}$$

thus $\overline{Y}_\Delta(t-\tau) = y_{t_{k-m}}$, and $\overline{Y}_{\Delta+}(t-\tau) = y_{t_{k+1-m}}$. However, this $Y_\Delta(t)$ is not $\mathscr{F}_t$-adapted, it does not meet the fundamental requirement in the Itô stochastic analysis. To avoid Malliavin calculus, we use the discrete split-step theta scheme introduced by Zong et al. [17] as follows: For $k = -m, \cdots, -1$, set $z_{t_k} = \xi(k\Delta)$. For $k = 0, 1, \cdots, M-1$, we reformulate the scheme (2.2) as follows

$$\begin{cases} y_{t_k} = D(y_{t_{k-m}}) + z_{t_k} - D(z_{t_{k-m}}) + \theta b_\Delta(y_{t_k}, y_{t_{k-m}})\Delta, \\ z_{t_{k+1}} = D(z_{t_{k+1-m}}) + z_{t_k} - D(z_{t_{k-m}}) + b_\Delta(y_{t_k}, y_{t_{k-m}})\Delta + \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}. \end{cases} \tag{2.4}$$

This scheme can also be rewritten as

$$\begin{aligned} z_{t_{k+1}} - D(z_{t_{k+1-m}}) &= z_{t_0} - D(z_{t_{-m}}) + \sum_{i=0}^{k} b_\Delta(y_{t_i}, y_{t_{i-m}})\Delta + \sum_{i=0}^{k} \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i} \\ &= \xi(0) - D(\xi(-\tau)) - \theta b_\Delta(\xi(0), \xi(-\tau))\Delta + \sum_{i=0}^{k} b_\Delta(y_{t_i}, y_{t_{i-m}})\Delta + \sum_{i=0}^{k} \sigma(y_{t_i}, y_{t_{i-m}})\Delta W_{t_i}. \end{aligned}$$

In order to simplify the computation, we define the corresponding continuous-time split-step tamed theta solution $Z_\Delta(t)$ as follows: For any $t \in [-\tau, 0)$, $Z_\Delta(t) = \xi(t)$, $Z_\Delta(0) = \xi(0) - \theta b_\Delta(\xi(0), \xi(-\tau))\Delta$. For any $t \in [0,T]$,

$$\mathrm{d}[Z_\Delta(t) - D(Z_\Delta(t-\tau))] = b_\Delta(\overline{Y}_\Delta(t), \overline{Y}_\Delta(t-\tau))\mathrm{d}t + \sigma(\overline{Y}_\Delta(t), \overline{Y}_\Delta(t-\tau))\mathrm{d}W(t), \tag{2.5}$$

where $\overline{Y}_\Delta(t)$ is defined by (2.3). With the split-step tamed theta scheme (2.4), the continuous form of the split-step tamed theta solution $Z_\Delta(t)$ and the tamed theta solution $Y_\Delta(t)$ have the following relation:

$$Y_\Delta(t) - D(Y_\Delta(t-\tau)) - \theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta = Z_\Delta(t) - D(Z_\Delta(t-\tau)).$$

Denote $\widetilde{Y}_\Delta(t) = Y_\Delta(t) - D(Y_\Delta(t-\tau)) - \theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta$, we can rewrite (2.5) as

$$\text{(2.6)}\quad \widetilde{Y}_\Delta(t) = \widetilde{Y}_\Delta(0) + \int_0^t b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}s + \int_0^t \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}W(s), t \in [0,T],$$

where $\widetilde{Y}_\Delta(0) = \xi(0) - D(\xi(-\tau)) - \theta b_\Delta(\xi(0), \xi(-\tau))\Delta$. It is easy to see $\widetilde{Y}_\Delta(t)$ coincides with $\overline{Y}_\Delta(t) - D(\overline{Y}_\Delta(t-\tau)) - \theta b_\Delta(\overline{Y}_\Delta(t), \overline{Y}_\Delta(t-\tau))\Delta$ at grid points $t = k\Delta, k = 0, 1, \cdots, M-1$, this also means that the continuous-time tamed theta solution $Y_\Delta(t)$ coincides with the discrete-time tamed theta solution $\overline{Y}_\Delta(t)$ at grid points $t = k\Delta, k = 0, 1, \cdots, M-1$.

We need some assumptions on $b_\Delta(x,y)$. We assume that there exists an $\alpha \in (0, 1/2]$ such that for any $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$, the following conditions hold:

(B1) There exist a positive constant $K_5 \geq 1$ such that

$$|b_\Delta(x,y)| \leq \min(K_5\Delta^{-\alpha}(1 + |x| + |y|), |b(x,y)|).$$

(B2) There exists a positive constant $\widetilde{K}_3$ such that

$$\langle x - D(y) - \overline{x} + D(\overline{y}), b_\Delta(x,y) - b_\Delta(\overline{x}, \overline{y})\rangle \leq \widetilde{K}_3(|x - \overline{x}|^2 + |y - \overline{y}|^2).$$

(B3) There exist positive constants l, K_6 such that

$$|b(x,y) - b_\Delta(x,y)|^p \leq K_6\Delta^{\alpha p}[1 + |x|^{(2l+1)p} + |y|^{(2l+1)p}].$$

Remark 2.4. With assumptions (B1)-(B3), (2.2) is well defined under some constraints on time step Δ. Furthermore, by (A3) and (B2), we have

$$\text{(2.7)}\qquad \langle x - D(y), b_\Delta(x,y)\rangle \vee \|\sigma(x,y)\|^2 \leq \widetilde{K}_2(1 + |x|^2 + |y|^2),$$

where $\widetilde{K}_2$ is a positive constant.

In order to ensure the implicitness of scheme (2.2) is well defined, an additional restriction is required on time step, i.e. $\theta\Delta\widetilde{K}_3 < 1$, where $\widetilde{K}_3$ is defined in (B2) (see [14] for more details). For $\theta \in (0,1]$, denote $\Delta_1 = \frac{1}{\theta\widetilde{K}_3}$. Further, in order to guarantee the boundedness of the p-th moment of numerical solutions, the step size is also required to satisfy $\theta^p\Delta < 6^{1-p}(2^{-p} - \kappa^p)/K_5^p$ for $p \geq 2$ where κ and K_5 are defined in (A2) and (B1). Denote $\Delta_2 = 6^{1-p}(2^{-p} - \kappa^p)/(\theta^p K_5^p)$ for $\theta \in (0,1]$. Thus in this section, we set $\Delta^* \in (0, \Delta_1 \wedge \Delta_2)$, and let $0 < \Delta \leq \Delta^*$ for $\theta \in (0,1]$, while for $\theta = 0$, we may set $\Delta \in (0,1)$.

Remark 2.5. Under conditions (A2)-(A4), the set of sequences of functions which satisfy (B1)-(B3) are non-empty. For example, let $b(x,y), \sigma(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, define

$$b_\Delta(x,y) = \frac{b(x,y)}{1 + \Delta^\alpha |b(x,y)|}.$$

It is easy to see $|b_\Delta(x,y)| \le |b(x,y)|$, and on the other hand, we have

$$|b_\Delta(x,y)| = \Delta^{-\alpha}\frac{|b(x,y)|}{\Delta^{-\alpha}+|b(x,y)|} \le \Delta^{-\alpha} \le K_5\Delta^{-\alpha}(1+|x|+|y|).$$

That is, (B1) is verified. We are now going to check (B2), we divide it into two cases. For $b(x,y)\cdot b(\overline{x},\overline{y})<0$,

$$\begin{aligned}
&\langle x-D(y)-\overline{x}+D(\overline{y}), b_\Delta(x,y)-b_\Delta(\overline{x},\overline{y})\rangle\\
=&\left\langle x-D(y)-\overline{x}+D(\overline{y}), \frac{b(x,y)}{1+\Delta^\alpha|b(x,y)|}-\frac{b(\overline{x},\overline{y})}{1+\Delta^\alpha|b(\overline{x},\overline{y})|}\right\rangle\\
\le&\frac{1}{2}K_3(|x-\overline{x}|^2+|y-\overline{y}|^2).
\end{aligned}$$

For $b(x,y)\cdot b(\overline{x},\overline{y})>0$,

$$\begin{aligned}
&\langle x-D(y)-\overline{x}+D(\overline{y}), b_\Delta(x,y)-b_\Delta(\overline{x},\overline{y})\rangle\\
=&\left\langle x-D(y)-\overline{x}+D(\overline{y}), \frac{b(x,y)-b(\overline{x},\overline{y})}{(1+\Delta^\alpha|b(x,y)|)(1+\Delta^\alpha|b(\overline{x},\overline{y})|)}\right\rangle\\
&+\left\langle x-D(y)-\overline{x}+D(\overline{y}), \frac{\Delta^\alpha[b(x,y)|b(\overline{x},\overline{y})|-|b(x,y)|b(\overline{x},\overline{y})]}{(1+\Delta^\alpha|b(x,y)|)(1+\Delta^\alpha|b(\overline{x},\overline{y})|)}\right\rangle\\
\le&\frac{1}{2}K_3(|x-\overline{x}|^2+|y-\overline{y}|^2).
\end{aligned}$$

Due to Remark 2.2, we see that

$$|b(x,y)-b_\Delta(x,y)|^p \le \Delta^{\alpha p}|b(x,y)|^{2p} \le K_6\Delta^{\alpha p}\left[1+|x|^{2(l+1)p}+|y|^{2(l+1)p}\right].$$

That is to say, (B3) is also satisfied.

2.2 Moment Bounds

In order to prove the main results, we now give some estimates for the numerical solution $Y_\Delta(t)$.

Lemma 2.2. Let (A1)-(A3) and (B1)-(B2) hold. Then it holds that for $p\ge 2$,

$$\sup_{0\le t\le T}\mathbb{E}|Y_\Delta(t)|^p \le C,$$

where the positive constant C is independent of Δ.

Proof. For $a > 0$, let $\lfloor a \rfloor$ be the integer part of a. Applying the Itô formula to $[1+|\widetilde{Y}_\Delta(t)|^2]^{\frac{p}{2}}$, we obtain

$$\begin{aligned}
\mathbb{E}[1+|\widetilde{Y}_\Delta(t)|^2]^{\frac{p}{2}} \leq & \mathbb{E}[1+|\widetilde{Y}_\Delta(0)|^2]^{\frac{p}{2}} + \frac{p}{2}\mathbb{E}\int_0^t [1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} 2\langle \widetilde{Y}_\Delta(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
& + \frac{1}{2}p(p-1)\mathbb{E}\int_0^t [1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} \|\sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\|^2 \mathrm{d}s \\
\leq & \mathbb{E}[1+|\widetilde{Y}_\Delta(0)|^2]^{\frac{p}{2}} + \frac{p}{2}\mathbb{E}\int_0^t [1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} (p-1)\|\sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\|^2 \mathrm{d}s \\
& + \frac{p}{2}\mathbb{E}\int_0^t [1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} 2\langle \overline{Y}_\Delta(s) - D(\overline{Y}_\Delta(s-\tau)), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
& + p\mathbb{E}\int_0^t [1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} \langle \widetilde{Y}_\Delta(s) - \vec{Y}_\Delta(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
=: & \mathbb{E}[1+|\widetilde{Y}_\Delta(0)|^2]^{\frac{p}{2}} + E_1(t) + E_2(t) + E_3(t),
\end{aligned}$$

where $\vec{Y}_\Delta(t) = \overline{Y}_\Delta(t) - D(\overline{Y}_\Delta(t-\tau)) - \theta b_\Delta(\overline{Y}_\Delta(t), \overline{Y}_\Delta(t-\tau))\Delta$. With (A2), (B1) and (2.7), we have

$$\begin{aligned}
E_1(t) + E_2(t) \leq & C\mathbb{E}\int_0^t [1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} (1 + |\overline{Y}_\Delta(s)|^2 + |\overline{Y}_\Delta(s-\tau)|^2)\mathrm{d}s \\
\leq & C\mathbb{E}\int_0^t [[1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p}{2}} + |\overline{Y}_\Delta(s)|^p + |\overline{Y}_\Delta(s-\tau)|^p]\mathrm{d}s \\
\leq & C\mathbb{E}\int_0^t \big[|Y_\Delta(s)|^p + |Y_\Delta(s-\tau)|^p + |\theta b_\Delta(Y_\Delta(s), Y_\Delta(s-\tau))\Delta|^p \\
& \qquad + |\overline{Y}_\Delta(s)|^p + |\overline{Y}_\Delta(s-\tau)|^p\big]\mathrm{d}s \\
\leq & C\mathbb{E}\int_0^t (|Y_\Delta(s)|^p + |Y_\Delta(s-\tau)|^p + |\overline{Y}_\Delta(s)|^p + |\overline{Y}_\Delta(s-\tau)|^p)\mathrm{d}s \\
& + C\Delta^{(1-\alpha)p}\mathbb{E}\int_0^t (1 + |Y_\Delta(s)|^p + |Y_\Delta(s-\tau)|^p)\mathrm{d}s \\
\leq & C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s.
\end{aligned}$$

Furthermore, it is easy to observe that,

$$\begin{aligned}
E_3(t) = & p\mathbb{E}\int_0^t [1+|\vec{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} \langle \widetilde{Y}_\Delta(s) - \vec{Y}_\Delta(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
& + p\mathbb{E}\int_0^t \left\{[1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} - [1+|\vec{Y}_\Delta(s)|^2]^{\frac{p-2}{2}}\right\} \langle \widetilde{Y}_\Delta(s) - \vec{Y}_\Delta(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
=: & \, pE_{31}(t) + pE_{32}(t),
\end{aligned}$$

where

$$\widetilde{Y}_\Delta(s) - \vec{Y}_\Delta(s) = \int_{\lfloor \frac{s}{\Delta} \rfloor \Delta}^s b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}u + \int_{\lfloor \frac{s}{\Delta} \rfloor \Delta}^s \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}W(u).$$

Due to (B1) and the Young inequality,

$$E_{31}(t) = \mathbb{E}\int_0^t [1+|\vec{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} \Big\langle \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}u, b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) \Big\rangle \mathrm{d}s$$
$$+ \mathbb{E}\int_0^t [1+|\vec{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} \Big\langle \mathbb{E}\int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}W(u)\Big|_{\mathscr{F}_{\lfloor \frac{s}{\Delta}\rfloor\Delta}}, b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) \Big\rangle \mathrm{d}s$$
$$\le \mathbb{E}\int_0^t [1+|\vec{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s |b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))|\mathrm{d}u|b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))|\mathrm{d}s$$
$$\le \Delta\mathbb{E}\int_0^t [1+|\vec{Y}_\Delta(s)|^2]^{\frac{p-2}{2}} |b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))|^2\mathrm{d}s$$
$$\le C\Delta^{1-2\alpha}\mathbb{E}\int_0^t (1+|\overline{Y}_\Delta(s)|^p + |\overline{Y}_\Delta(s-\tau)|^p)\mathrm{d}s$$
$$+ C\Delta^{1-2\alpha}\Delta^{(1-\alpha)p}\mathbb{E}\int_0^t (1+|\overline{Y}_\Delta(s)|^p + |\overline{Y}_\Delta(s-\tau)|^p)\mathrm{d}s$$
$$\le C + C\int_0^t \sup_{0\le u\le s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s.$$

Applying the Itô formula again, we obtain

$$[1+|\widetilde{Y}_\Delta(s)|^2]^{\frac{p-2}{2}}$$
$$\le [1+|\widetilde{Y}_\Delta(0)|^2]^{\frac{p-2}{2}} + (p-2)\int_0^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}u$$
$$+ \frac{1}{2}(p-2)(p-3)\int_0^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \|\sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\|^2\mathrm{d}u$$
$$+ (p-2)\int_0^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}W(u).$$

Thus we have

$$[1+|\vec{Y}_\Delta(s)|^2]^{\frac{p-2}{2}}$$
$$\le [1+|\vec{Y}_\Delta(0)|^2]^{\frac{p-2}{2}} + (p-2)\int_0^{\lfloor \frac{s}{\Delta}\rfloor\Delta} [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}u$$
$$+ \frac{1}{2}(p-2)(p-3)\int_0^{\lfloor \frac{s}{\Delta}\rfloor\Delta} [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \|\sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\|^2\mathrm{d}u$$
$$+ (p-2)\int_0^{\lfloor \frac{s}{\Delta}\rfloor\Delta} [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}W(u).$$

Hence,

$$
\begin{aligned}
E_{32}(t) \leq & (p-2)\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta} \rfloor \Delta}^{s} [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}u \\
& \times \langle \widetilde{Y}_\Delta(s) - \vec{Y}_\Delta(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
& + \frac{1}{2}(p-2)(p-3)\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta} \rfloor \Delta}^{s} [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \|\sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\|^2 \mathrm{d}u \\
& \times \langle \widetilde{Y}_\Delta(s) - \vec{Y}_\Delta(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
& + (p-2)\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta} \rfloor \Delta}^{s} [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}W(u) \\
& \times \langle \widetilde{Y}_\Delta(s) - \vec{Y}_\Delta(s), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
=: & (p-2)E_{321} + \frac{1}{2}(p-2)(p-3)E_{322} + (p-2)E_{323}.
\end{aligned}
$$

Using (A3) and (B1), the Young inequality, the Hölder inequality and the Burkholder-Davis-

Gundy (BDG) inequality, we compute

$$
\begin{aligned}
E_{321}(t) \leq & \mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}u \\
& \times \Big\langle \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}u, b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) \Big\rangle \mathrm{d}s \\
& + \mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}u \\
& \times \Big\langle \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}W(u), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) \Big\rangle \mathrm{d}s \\
\leq & \Delta\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-3}{2}} |b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))|^3 \mathrm{d}u\mathrm{d}s \\
& + C\mathbb{E}\int_0^t \bigg[\bigg(\int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-3}{2}} |b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))|\mathrm{d}u |b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))|\bigg)^{\frac{p}{p-1}} \\
& + \bigg|\int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}W(u)\bigg|^p\bigg]\mathrm{d}s \\
\leq & C\Delta^{2-3\alpha}\mathbb{E}\int_0^t (|Y_\Delta(s)|^p + |Y_\Delta(s-\tau)|^p + |\overline{Y}_\Delta(s)|^p + |\overline{Y}_\Delta(s-\tau)|^p)\mathrm{d}s \\
& + C\Delta^{2-3\alpha}\Delta^{(1-\alpha)p}\mathbb{E}\int_0^t (|Y_\Delta(s)|^p + |Y_\Delta(s-\tau)|^p)\mathrm{d}s \\
& + C\mathbb{E}\int_0^t \bigg(\int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-3}{2}} |b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))|^2 \mathrm{d}u\bigg)^{\frac{p}{p-1}} \mathrm{d}s \\
& + C\mathbb{E}\int_0^t \bigg(\int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s \|\sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\|^2 \mathrm{d}u\bigg)^{\frac{p}{2}} \mathrm{d}s \\
\leq & C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s + C\Delta^{\frac{p}{2}}\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s \\
\leq & C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s.
\end{aligned}
$$

Using the same techniques in the way to estimate $E_{321}(t)$, we get

$$
E_{322}(t) \leq C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s.
$$

Furthermore, by (A3) and (B1), we have

$$\begin{aligned}
E_{323}(t) =&\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}W(u) \\
&\times \Big\langle \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s b_\Delta(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}u, b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\Big\rangle \mathrm{d}s \\
&+\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}W(u) \\
&\times \Big\langle \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}W(u), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\Big\rangle \mathrm{d}s \\
=&\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-4}{2}} \langle \widetilde{Y}_\Delta(u), \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\rangle \mathrm{d}W(u) \\
&\times \Big\langle \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s \sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\mathrm{d}W(u), b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\Big\rangle \mathrm{d}s \\
\leq&\mathbb{E}\int_0^t \int_{\lfloor \frac{s}{\Delta}\rfloor\Delta}^s [1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p-3}{2}} \|\sigma(\overline{Y}_\Delta(u), \overline{Y}_\Delta(u-\tau))\|^2 \mathrm{d}u |b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))|\mathrm{d}s \\
\leq&C\Delta^{1-\alpha}\mathbb{E}\int_0^t (|Y_\Delta(s)|^p + |Y_\Delta(s-\tau)|^p + |\overline{Y}_\Delta(s)|^p + |\overline{Y}_\Delta(s-\tau)|^p)\mathrm{d}s \\
&+C\Delta^{1-\alpha}\Delta^{(1-\alpha)p}\mathbb{E}\int_0^t (|Y_\Delta(s)|^p + |Y_\Delta(s-\tau)|^p)\mathrm{d}s \\
\leq&C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s.
\end{aligned}$$

By sorting these equations, we conclude that

$$E_3(t) \leq C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s.$$

Thus, the estimate of $E_1(t) - E_3(t)$ results in

$$\sup_{0\leq u\leq t} \mathbb{E}|\widetilde{Y}_\Delta(u)|^p \leq \sup_{0\leq u\leq t} \mathbb{E}[1+|\widetilde{Y}_\Delta(u)|^2]^{\frac{p}{2}} \leq C + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s. \tag{2.8}$$

Since $|x-y|^p \geq 2^{1-p}|x|^p - |y|^p$, we have

$$\begin{aligned}
|\widetilde{Y}_\Delta(t)|^p \geq&2^{1-p}|Y_\Delta(t) - D(Y_\Delta(t-\tau))|^p - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta|^p \\
\geq&2^{1-p}[2^{1-p}|Y_\Delta(t)|^p - |D(Y_\Delta(t-\tau))|^p] - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta|^p.
\end{aligned}$$

This, combining with (A2) and (B1), yields that

$$|\widetilde{Y}_\Delta(t)|^p \geq(2^{2-2p} - \widetilde{C}_\Delta)|Y_\Delta(t)|^p - (2^{2-p}\kappa^p + \widetilde{C}_\Delta)|Y_\Delta(t-\tau)|^p - \widetilde{C}_\Delta,$$

where $\widetilde{C}_\Delta = \theta^p K_5^p 3^{p-1}\Delta$. Consequently,

$$\sup_{0\le u\le t}\mathbb{E}|Y_\Delta(u)|^p \le (2^{2-2p} - 2^{2-p}\kappa^p - 2\widetilde{C}_\Delta)^{-1}\left[\sup_{0\le u\le t}\mathbb{E}|\widetilde{Y}_\Delta(u)|^p + \widetilde{C}_\Delta + (2^{2-p}\kappa^p + \widetilde{C}_\Delta)\mathbb{E}\|\xi\|_\infty^p\right].$$

This, together with (2.8), implies

$$\sup_{0\le u\le t}\mathbb{E}|Y_\Delta(u)|^p \le C + C\int_0^t \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)|^p \mathrm{d}s.$$

Finally, the desired result is obtained by the Gronwall inequality. □

Lemma 2.3. Let (A1)-(A3), (B1)-(B2) hold. Then, we have, for $p\ge 2$

$$\mathbb{E}\left[\sup_{0\le k\le M-1}\sup_{t_k\le t<t_{k+1}}|Y_\Delta(t) - Y_\Delta(t_k)|^p\right] \le C\Delta^{\frac{p}{2}},$$

where C is a positive constant independent of Δ.

Proof. From the definition of numerical scheme (2.6), one sees that for $t\in[t_k, t_{k+1})$,

$$\widetilde{Y}_\Delta(t) - \widetilde{Y}_\Delta(t_k) = \int_{t_k}^t b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}s + \int_{t_k}^t \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}W(s).$$

By the elementary inequality $|a+b|^p \le 2^{p-1}(|a|^p + |b|^p), p\ge 1$, we compute

$$\begin{aligned}\mathbb{E}\left[\sup_{t_k\le t<t_{k+1}}|\widetilde{Y}_\Delta(t) - \widetilde{Y}_\Delta(t_k)|^p\right] &\le 2^{p-1}\mathbb{E}\left[\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}s\right|^p\right]\\ &\quad + 2^{p-1}\mathbb{E}\left[\sup_{t_k\le t<t_{k+1}}\left|\int_{t_k}^t \sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\mathrm{d}W(s)\right|^p\right].\end{aligned}$$

With (A3), (B1), Lemma 2.2, the Hölder inequality and the BDG inequality, we derive

$$\begin{aligned}&\mathbb{E}\left[\sup_{t_k\le t<t_{k+1}}|\widetilde{Y}_\Delta(t) - \widetilde{Y}_\Delta(t_k)|^p\right] \le 2^{p-1}\Delta^{p-1}\mathbb{E}\int_{t_k}^{t_{k+1}}\left|b_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\right|^p\mathrm{d}s\\ &+ C\mathbb{E}\left[\int_{t_k}^{t_{k+1}}\left\|\sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\right\|^2\mathrm{d}s\right]^{\frac{p}{2}}\\ \le& C\Delta^{(1-\alpha)p} + C\mathbb{E}\left[\int_{t_k}^{t_{k+1}}(1 + |\overline{Y}_\Delta(s)|^2 + |\overline{Y}_\Delta(s-\tau)|^2)\mathrm{d}s\right]^{\frac{p}{2}}\\ \le& C\Delta^{(1-\alpha)p} + C\Delta^{\frac{p}{2}} \le C\Delta^{\frac{p}{2}}.\end{aligned}$$

Denoting by $\widetilde{D}(t,t_k) := D(Y_\Delta(t-\tau)) - D(Y_\Delta(t_k-\tau))$, and $\widetilde{b}_\Delta(t,t_k) := b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau)) - b_\Delta(Y_\Delta(t_k), Y_\Delta(t_k-\tau))$, with (A2), we arrive at,

$$\begin{aligned}&|\widetilde{Y}_\Delta(t) - \widetilde{Y}_\Delta(t_k)|^p \ge 2^{1-p}|Y_\Delta(t) - Y_\Delta(t_k) - \widetilde{D}(t,t_k)|^p - \theta^p\Delta^p|\widetilde{b}_\Delta(t,t_k)|^p\\ \ge& 2^{2-2p}|Y_\Delta(t) - Y_\Delta(t_k)|^p - 2^{1-p}\kappa^p|Y_\Delta(t-\tau) - Y_\Delta(t_k-\tau)|^p - \theta^p\Delta^p|\widetilde{b}_\Delta(t,t_k)|^p.\end{aligned} \tag{2.9}$$

Obviously, for $0 \le t < t_1 = \Delta$, we have $t-\tau < t_1 - \tau < 0$, then we see from (2.9) and Lemma 2.2 that

$$\mathbb{E}\left[\sup_{0\le t<t_1}|Y_\Delta(t)-Y_\Delta(t_0)|^p\right] \le C\mathbb{E}\left[\sup_{0\le t<t_1}|\widetilde{Y}_\Delta(t)-\widetilde{Y}_\Delta(t_0)|^p\right]+C\Delta^{(1-\alpha)p} \le C\Delta^{\frac{p}{2}}.$$

For $t_1 \le t < t_2$, (2.9) and Lemma 2.2 lead to

$$\begin{aligned}\mathbb{E}\left[\sup_{t_1\le t<t_2}|Y_\Delta(t)-Y_\Delta(t_1)|^p\right] \le& C\mathbb{E}\left[\sup_{t_1\le t<t_2}|\widetilde{Y}_\Delta(t)-\widetilde{Y}_\Delta(t_1)|^p\right]\\ &+\mathbb{E}\left[\sup_{0\le t<(t_{2-m})\vee 0}|Y_\Delta(t)-Y_\Delta(t_{1-m})|^p\right]+C\Delta^{(1-\alpha)p}\\ \le& C\Delta^{\frac{p}{2}}.\end{aligned}$$

Consequently, the induction method yields,

$$\begin{aligned}\mathbb{E}\left[\sup_{t_k\le t<t_{k+1}}|Y_\Delta(t)-Y_\Delta(t_k)|^p\right] \le& C\mathbb{E}\left[\sup_{t_k\le t<t_{k+1}}|\widetilde{Y}_\Delta(t)-\widetilde{Y}_\Delta(t_k)|^p\right]+C\Delta^{(1-\alpha)p}\\ \le& C\Delta^{\frac{p}{2}}.\end{aligned}$$

The proof is therefore complete. □

2.3 Strong Convergence Rate

The following theorem reveals that the continuous form $Y_\Delta(t)$ of the tamed theta scheme (2.2) converges strongly to the exact solution $X(t)$.

Theorem 2.4. Let (A1)-(A4) and (B1)-(B3) hold, then it holds that for $p \ge 2$,

$$\mathbb{E}\sup_{0\le t\le T}|X(t)-Y_\Delta(t)|^p \le C\Delta^{\alpha p},$$

where α is defined in (B1) and C is a positive constant independent of Δ. That is, the strong convergence rate of the tamed theta scheme (2.2) is α.

Proof. Denote $I(t) = Y_\Delta(t) - D(Y_\Delta(t-\tau)) - \theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta - X(t) + D(X(t-\tau))$, then

$$\begin{aligned}I(t) =& I(0)+\int_0^t[b_\Delta(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau))-b(X(s),X(s-\tau))]\mathrm{d}s\\ &+\int_0^t[\sigma(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau))-\sigma(X(s),X(s-\tau))]\mathrm{d}W(s),\end{aligned}$$

where $I(0)=-\theta b_\Delta(\xi(0),\xi(-\tau))\Delta$. An application of the Itô formula yields,

$$
\begin{aligned}
|I(t)|^p \le & |I(0)|^p + p\int_0^t |I(s)|^{p-2}\langle I(s), b_\Delta(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - b(X(s),X(s-\tau))\rangle \mathrm{d}s\\
& + \frac{1}{2}p(p-1)\int_0^t |I(s)|^{p-2}\|\sigma(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - \sigma(X(s),X(s-\tau))\|^2 \mathrm{d}s\\
& + p\int_0^t |I(s)|^{p-2}\langle I(s), \sigma(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - \sigma(X(s),X(s-\tau))\rangle \mathrm{d}W(s)\\
\le & |I(0)|^p + \sum_{i=1}^{6} H_i(t),
\end{aligned}
$$

where

$$
\begin{aligned}
H_1(t) &:= p\int_0^t |I(s)|^{p-2}\langle I(s), b_\Delta(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - b(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s,\\
H_2(t) &:= p\int_0^t |I(s)|^{p-2}\langle I(s), b(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - b(Y_\Delta(s),Y_\Delta(s-\tau))\rangle \mathrm{d}s,\\
H_3(t) &:= p\int_0^t |I(s)|^{p-2}\langle I(s), b(Y_\Delta(s),Y_\Delta(s-\tau)) - b(X(s),X(s-\tau))\rangle \mathrm{d}s,\\
H_4(t) &:= p(p-1)\int_0^t |I(s)|^{p-2}\|\sigma(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - \sigma(Y_\Delta(s),Y_\Delta(s-\tau))\|^2 \mathrm{d}s,\\
H_5(t) &:= p(p-1)\int_0^t |I(s)|^{p-2}\|\sigma(Y_\Delta(s),Y_\Delta(s-\tau)) - \sigma(X(s),X(s-\tau))\|^2 \mathrm{d}s,\\
H_6(t) &:= p\int_0^t |I(s)|^{p-2}\langle I(s), \sigma(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - \sigma(X(s),X(s-\tau))\rangle \mathrm{d}W(s).
\end{aligned}
$$

By (A2), (B1), (B3), Lemma 2.2, and the Hölder inequality,

$$
\begin{aligned}
&\sup_{0\le u\le t}\mathbb{E}H_1(u) \le C\mathbb{E}\int_0^t |I(s)|^p \mathrm{d}s + C\mathbb{E}\int_0^t |b_\Delta(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau)) - b(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau))|^p \mathrm{d}s\\
&\le C\mathbb{E}\int_0^t [|Y_\Delta(s)-X(s)|^p + |Y_\Delta(s-\tau)-X(s-\tau)|^p + \theta^p\Delta^p|b_\Delta(Y_\Delta(s),Y_\Delta(s-\tau))|^p]\mathrm{d}s\\
&+ C\Delta^{\alpha p}\mathbb{E}\int_0^t (1+|\overline{Y}_\Delta(s)|^{2(l+1)p} + |\overline{Y}_\Delta(s-\tau)|^{2(l+1)p})\mathrm{d}s\\
&\le C\int_0^t \sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)-X(u)|^p \mathrm{d}s + C\Delta^{(1-\alpha)p} + C\Delta^{\alpha p}.
\end{aligned}
$$

By (A2), (A4), (B1), Lemmas 2.2-2.3, and the Hölder inequality,

$$
\begin{aligned}
&\sup_{0\le u\le t}\mathbb{E}H_2(u)\le C\mathbb{E}\int_0^t|I(s)|^p\mathrm{d}s+C\mathbb{E}\int_0^t|b(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau))-b(Y_\Delta(s),Y_\Delta(s-\tau))|^p\mathrm{d}s\\
\le&C\mathbb{E}\int_0^t[|Y_\Delta(s)-X(s)|^p+|Y_\Delta(s-\tau)-X(s-\tau)|^p+\theta^p\Delta^p|b_\Delta(Y_\Delta(s),Y_\Delta(s-\tau))|^p]\mathrm{d}s\\
&+C\mathbb{E}\int_0^t(1+|\overline{Y}_\Delta(s)|^l+|\overline{Y}_\Delta(s-\tau)|^l+|Y_\Delta(s)|^l+|Y_\Delta(s-\tau)|^l)^p\\
&\times(|\overline{Y}_\Delta(s)-Y_\Delta(s)|+|\overline{Y}_\Delta(s-\tau)-Y_\Delta(s-\tau)|)^p\mathrm{d}s\\
\le&C\mathbb{E}\int_0^t(|Y_\Delta(s)-X(s)|^p+|Y_\Delta(s-\tau)-X(s-\tau)|^p)\mathrm{d}s+C\Delta^{(1-\alpha)p}\\
&+C\int_0^t[\mathbb{E}(1+|\overline{Y}_\Delta(s)|^l+|\overline{Y}_\Delta(s-\tau)|^l+|Y_\Delta(s)|^l+|Y_\Delta(s-\tau)|^l)^{2p}]^{\frac{1}{2}}\\
&\times[\mathbb{E}(|\overline{Y}_\Delta(s)-Y_\Delta(s)|+|\overline{Y}_\Delta(s-\tau)-Y_\Delta(s-\tau)|)^{2p}]^{\frac{1}{2}}\mathrm{d}s\\
\le&C\int_0^t\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)-X(u)|^p\mathrm{d}s+C\Delta^{(1-\alpha)p}+C\Delta^{\frac{p}{2}}.
\end{aligned}
$$

Due to (A2), (B1), (2.7), Lemma 2.2, and the Hölder inequality,

$$
\begin{aligned}
&\sup_{0\le u\le t}\mathbb{E}H_3(u)+\sup_{0\le u\le t}\mathbb{E}H_5(u)\\
\le&C\mathbb{E}\int_0^t|I(s)|^{p-2}[|Y_\Delta(s)-X(s)|^2+|Y_\Delta(s-\tau)-X(s-\tau)|^2]\mathrm{d}s\\
&+C\mathbb{E}\int_0^t|I(s)|^{p-2}|\theta b_\Delta(Y_\Delta(s),Y_\Delta(s-\tau))\Delta||b(Y_\Delta(s),Y_\Delta(s-\tau))-b(X(s),X(s-\tau))|\mathrm{d}s\\
\le&C\mathbb{E}\int_0^t[|Y_\Delta(s)-X(s)|^p+\theta^p\Delta^p|b_\Delta(Y_\Delta(s),Y_\Delta(s-\tau))|^p]\mathrm{d}s\\
&+C\Delta^{1-\alpha}\mathbb{E}\int_0^t|I(s)|^{p-2}(1+|Y_\Delta(s)|+|Y_\Delta(s-\tau)|)\times\\
&\qquad(1+|Y_\Delta(s)|^l+|Y_\Delta(s-\tau)|^l+|X(s)|^l+|X(s-\tau)|^l)\times\\
&\qquad(|Y_\Delta(s)-X(s)|+|Y_\Delta(s-\tau)-X(s-\tau)|)\mathrm{d}s\\
\le&C\int_0^t\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)-X(u)|^p\mathrm{d}s+C\Delta^{(1-\alpha)p}.
\end{aligned}
$$

In the same way as the estimate of $H_1(t)$ and $H_2(t)$, we arrive at

$$\sup_{0\le u\le t}\mathbb{E}H_4(u)\le C\int_0^t\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)-X(u)|^p\mathrm{d}s+C\Delta^{(1-\alpha)p}+C\Delta^{\frac{p}{2}}.$$

Since $\mathbb{E}H_6(t)=0$, by sorting $H_1(t)-H_6(t)$ together, we derive

$$\sup_{0\le u\le t}\mathbb{E}|I(u)|^p\le C\int_0^t\sup_{0\le u\le s}\mathbb{E}|Y_\Delta(u)-X(u)|^p\mathrm{d}s+C\Delta^{\alpha p}.$$

By the definition of $I(t)$, we have

$$
\begin{aligned}
|I(t)|^p \geq & 2^{1-p}|Y_\Delta(t) - X(t) - D(Y_\Delta(t-\tau)) + D(X(t-\tau))|^p - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta|^p \\
\geq & 2^{1-p}[2^{1-p}|Y_\Delta(t) - X(t)|^p - |D(Y_\Delta(t-\tau)) - D(X(t-\tau))|^p] - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta|^p,
\end{aligned}
$$

this, together with (A2), leads to

$$
|I(t)|^p \geq 2^{2-2p}|Y_\Delta(t) - X(t)|^p - 2^{1-p}\kappa^p|Y_\Delta(t-\tau) - X(t-\tau)|^p - |\theta b_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta|^p.
$$

Taking (B1) and Lemma 2.2 into consideration yields

$$
\begin{aligned}
\sup_{0\leq u\leq t} \mathbb{E}|Y_\Delta(u) - X(u)|^p \leq & C \sup_{0\leq u\leq t} \mathbb{E}|I(u)|^p + C\Delta^{(1-\alpha)p} \\
\leq & C\Delta^{\alpha p} + C\int_0^t \sup_{0\leq u\leq s} \mathbb{E}|Y_\Delta(u) - X(u)|^p \mathrm{d}s.
\end{aligned}
$$

The desired result follows from the Gronwall inequality. □

Remark 2.6. If we replace (A3) by the following (A3'):

(A3') There exists a positive constant K_2 such that

$$
2\langle x - D(y), b(x,y)\rangle + (p-1)\|\sigma(x,y)\|^2 \leq K_2(1 + |x|^2 + |y|^2),
$$

and add another assumption (B4):

(B4) There exists a positive constant $\widetilde{K}_2$ such that

$$
2\langle x - D(y), b_\Delta(x,y)\rangle + (p-1)\|\sigma(x,y)\|^2 \leq \widetilde{K}_2(1 + |x|^2 + |y|^2),
$$

we can also show that under assumptions (A1)-(A2), (A3'), (A4), (B1)-(B4), the tamed theta scheme $Y_\Delta(t)$ converges strongly to the exact solution $X(t)$ with order α.

3 Local One-sided Lipschitz Drift

In this section, instead of the global one-sided Lipschitz condition (A4), we impose the following local one-sided Lipschitz condition:

(A5) For every $R > 0$, there exists a positive constant L_R such that

$$
\begin{aligned}
& \langle x - D(y) - \overline{x} + D(\overline{y}), b(x,y) - b(\overline{x},\overline{y})\rangle \vee \|\sigma(x,y) - \sigma(\overline{x},\overline{y})\|^2 \\
\leq & L_R(|x - \overline{x}|^2 + |y - \overline{y}|^2)
\end{aligned}
$$

for all $|x| \vee |y| \vee |\overline{x}| \vee |\overline{y}| \leq R$.

Remark 3.1. The local one-sided Lipschitiz condition is a weaker than the classical local Lipschitz condition, since if b satisfies the local Lipschitz condition such that,

$$|b(x,y)-b(\overline{x},\overline{y})|^2 \le L_R(|x-\overline{x}|^2+|y-\overline{y}|^2), |x|\vee|\overline{x}|\vee|y|\vee|\overline{y}|\le R,$$

we then have

$$\begin{aligned}&\langle x-\overline{x}, b(x,y)-b(\overline{x},\overline{y})\rangle \le \frac{1}{2}|x-\overline{x}|^2+\frac{1}{2}|b(x,y)-b(\overline{x},\overline{y})|^2\\ &\le \frac{1}{2}|x-\overline{x}|^2+\frac{1}{2}L_R(|x-\overline{x}|^2+|y-\overline{y}|^2) \le \widetilde{L}_R(|x-\overline{x}|^2+|y-\overline{y}|^2),\end{aligned}$$

where $\widetilde{L}_R=\max(\frac{1}{2},\frac{1}{2}L_R)$. This implies that the local one-sided Lipschitz condition holds.

On the other hand, if b satisfies the local one-sided Lipschitiz condition, it need not satisfy the classical local Lipschitz condition. For example, let $b(x,y)=x^3-x^{\frac{1}{3}}+y$, because of $x^{\frac{1}{3}}$, b does not satisfy the classical local Lipschitz condition. However, noting that

$$(x-\overline{x})(x^{\frac{1}{3}}-\overline{x}^{\frac{1}{3}})\ge 0 \text{ for all } x,\overline{x},$$

we see that

$$\begin{aligned}&\langle x-\overline{x}, x^3-\sqrt{x}+y-\overline{x}^3+\sqrt{\overline{x}}-\overline{y}\rangle\\ \le&|x-\overline{x}|^2(x^2+x\overline{x}+\overline{x}^2)+\frac{1}{2}(|x-\overline{x}|^2+|y-\overline{y}|^2)-(x-\overline{x})(x^{\frac{1}{3}}-\overline{x}^{\frac{1}{3}})\\ \le&|x-\overline{x}|^2(x^2+x\overline{x}+\overline{x}^2)+\frac{1}{2}(|x-\overline{x}|^2+|y-\overline{y}|^2)\\ \le&\widehat{L}_R(|x-\overline{x}|^2+|y-\overline{y}|^2), |x|\vee|\overline{x}|\vee|y|\vee|\overline{y}|\le R,\end{aligned}$$

where $\widehat{L}_R=\max(\frac{1}{2},3R^2)$. This means that b satisfies the local one-sided Lipschitiz condition.

Remark 3.2. Due to the continuity of $b(x,y)$, for every $R>0$, there exists a positive constant $\overline{L}_R$ such that

$$\sup_{|x|\vee|y|\le R}|b(x,y)|\le \overline{L}_R.$$

Remark 3.3. There are many examples such that the assumptions can be verified. For example, if we set

$$D(y)=\frac{1}{4}\cos y, \quad b(x,y)=x-x^3+\cos y, \quad \sigma(x,y)=y\sin x+x\sin y,$$

then assumptions (A2)-(A3) and (A5) hold.

Consider the following tamed theta scheme imposed in Section 2:

$$\begin{aligned}y_{t_{k+1}}-D(y_{t_{k+1-m}})=&\,y_{t_k}-D(y_{t_{k-m}})+\theta b_\Delta(y_{t_{k+1}},y_{t_{k+1-m}})\Delta\\ &+(1-\theta)b_\Delta(y_{t_k},y_{t_{k-m}})\Delta+\sigma(y_{t_k},y_{t_{k-m}})\Delta W_{t_k}.\end{aligned}$$

Generally speaking, for a given y_{t_k}, to guarantee a unique solution $y_{t_{k+1}}$ is to assume that there exists a positive constant L such that

$$\langle x - D(y) - \overline{x} + D(\overline{y}), b_\Delta(x, y) - b_\Delta(\overline{x}, \overline{y})\rangle \le L(|x - \overline{x}|^2 + |y - \overline{y}|^2)$$

as in Section 2. Moreover, as shown in Mao and Szpruch [8], this condition is somehow hard to relax. While in our assumption (A5), the drift coefficient b is local one-sided Lipschitz, thus in this case, the tamed drift b_Δ is hardly to be global one-sided Lipschitz. That is, we do not know if the tamed theta scheme (2.2) is well defined under assumptions (A2)-(A3) and (A5). In the following, we will provide an improved tamed theta scheme to ensure the well-posedness of implicit equations.

3.1 The Improved Tamed Theta Scheme

For any $R > 0$, define a smooth, non-negative function such that

$$\zeta_R(x, y) = \begin{cases} 1, & \text{for } |x|, |y| \le R, \\ 0, & \text{for } |x| \text{ or } |y| > R + 1, \end{cases}$$

and $\zeta_R(x, y) \le 1$ for all $x, y \in \mathbb{R}^n$. It is obvious that $\zeta_R(x, y)$ is Lipschitz with some constant C_ζ. Now we introduce the improved tamed theta scheme for (2.1). For $k = -m, \cdots, 0$, set $y_{t_k} = \xi(k\Delta)$; For $k = 0, 1, \cdots, M - 1$, we form

$$\begin{aligned} y_{t_{k+1}} - D(y_{t_{k+1-m}}) = {} & y_{t_k} - D(y_{t_{k-m}}) + \theta \overline{b}_\Delta(y_{t_{k+1}}, y_{t_{k+1-m}})\zeta_R(y_{t_{k+1}}, y_{t_{k+1-m}})\Delta \\ & + (1-\theta)\overline{b}_\Delta(y_{t_k}, y_{t_{k-m}})\zeta_R(y_{t_k}, y_{t_{k-m}})\Delta + \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}, \end{aligned} \tag{3.1}$$

where $t_k = k\Delta$, and $\Delta W_{t_k} = W(t_{k+1}) - W(t_k)$. Here $\overline{b}_\Delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Besides, $\theta \in [0, 1]$ is an additional parameter that allows us to control the implicitness of the numerical scheme. Denote

$$\widetilde{b}_\Delta(x, y) = \overline{b}_\Delta(x, y)\zeta_R(x, y),$$

then, (3.1) can be rewritten as

$$\begin{aligned} y_{t_{k+1}} - D(y_{t_{k+1-m}}) = {} & y_{t_k} - D(y_{t_{k-m}}) + \theta \widetilde{b}_\Delta(y_{t_{k+1}}, y_{t_{k+1-m}})\Delta \\ & + (1-\theta)\widetilde{b}_\Delta(y_{t_k}, y_{t_{k-m}})\Delta + \sigma(y_{t_k}, y_{t_{k-m}})\Delta W_{t_k}, \end{aligned} \tag{3.2}$$

which is exactly the form of (2.2). According to (3.2) we define $\overline{Y}_\Delta(t)$, $Y_\Delta(t)$, $\widetilde{Y}_\Delta(t)$, $Z_\Delta(t)$ by using the same notation as in Section 2. Instead of constraints on $\widetilde{b}_\Delta(x, y)$, we impose some assumptions on $\overline{b}_\Delta(x, y)$. Assume that there exists an $\alpha \in (0, 1/2]$ such that for any $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$, the following conditions hold:

(C1) There exists a positive constant $K_5 \ge 1$ such that

$$|\overline{b}_\Delta(x, y)| \le \min(K_5\Delta^{-\alpha}(1 + |x| + |y|), |b(x, y)|).$$

(C2) There exists a positive constant $\widetilde{K}_2$ such that

$$\langle x - D(y), \overline{b}_\Delta(x,y)\rangle \le \widetilde{K}_2(1 + |x|^2 + |y|^2).$$

(C3) For any $R > 0$, there exists a positive constant M_R such that

$$\langle x - D(y) - \overline{x} + D(\overline{y}), \overline{b}_\Delta(x,y) - \overline{b}_\Delta(\overline{x},\overline{y})\rangle \le M_R(|x-\overline{x}|^2 + |y-\overline{y}|^2).$$

for all $|x| \vee |y| \vee |\overline{x}| \vee |\overline{y}| \le R$.

(C4) For any $R > 0$, there exists a positive constant N_R such that

$$\sup_{|x|\vee|y|\le R} |b(x,y) - \overline{b}_\Delta(x,y)|^p \le N_R \Delta^{\alpha p} \to 0 \quad \text{as} \quad \Delta \to 0.$$

Lemma 3.1. Let (A2), (C1)-(C4) hold, then $\widetilde{b}_\Delta$ satisfies (C1), (C2), (C4) and the following (C3'):

(C3') There exists an $\overline{M}_{R_0}$ such that for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$

$$\langle x - D(y) - \overline{x} + D(\overline{y}), \widetilde{b}_\Delta(x,y) - \widetilde{b}_\Delta(\overline{x},\overline{y})\rangle \le \overline{M}_{R_0}(|x-\overline{x}|^2 + |y-\overline{y}|^2),$$

where $\overline{M}_{R_0} = M_{R_0} + 2C_\zeta \overline{L}_{R_0}$.

Proof. By the relationship between $\widetilde{b}_\Delta$ and $\overline{b}_\Delta$, (C1) and (C2) can be verified easily. Noting that for $|x| \vee |y| \le R$, $\zeta_R(x,y) = 1$, thus we get

$$\begin{aligned}
&\sup_{|x|\vee|y|\le R}\Big[|b(x,y) - \widetilde{b}_\Delta(x,y)|^p\Big] = \sup_{|x|\vee|y|\le R}\big[|b(x,y) - \overline{b}_\Delta(x,y)\zeta_\Delta(x,y)|^p\big] \\
=& \sup_{|x|\vee|y|\le R}\big[|b(x,y) - \overline{b}_\Delta(x,y)|^p\big] \le N_R\Delta^{\alpha p} \to 0, \quad \text{as} \quad \Delta \to 0,
\end{aligned}$$

then (C4) holds for $\widetilde{b}_\Delta(x,y)$. Now we are going to check (C3'). Divide it into four cases.
Case a: None of $|x|, |y|, |\overline{x}|, |\overline{y}|$ bigger than $R+1$. In this case, we see $0 \le \zeta_R(x,y), \zeta_R(\overline{x},\overline{y}) \le 1$. Rewrite $\widetilde{b}_\Delta$ with $\overline{b}_\Delta$, we have

$$\begin{aligned}
&\langle x - D(y) - \overline{x} + D(\overline{y}), \widetilde{b}_\Delta(x,y) - \widetilde{b}_\Delta(\overline{x},\overline{y})\rangle \\
=&\langle x - D(y) - \overline{x} + D(\overline{y}), \overline{b}_\Delta(x,y) - \overline{b}_\Delta(\overline{x},\overline{y})\rangle\zeta_R(x,y) \\
&+ \langle x - D(y) - \overline{x} + D(\overline{y}), (\zeta_R(x,y) - \zeta_R(\overline{x},\overline{y}))\overline{b}_\Delta(\overline{x},\overline{y})\rangle \\
=:&q_1 + q_2.
\end{aligned}$$

Since $0 \le \zeta_R(x,y) \le 1$, thus by (C3),

$$q_1 \le M_{R+1}(|x-\overline{x}|^2 + |y-\overline{y}|^2).$$

Further, noting that ζ_R is Lipschitz with constant C_ζ and for $|\overline{x}| \vee |\overline{y}| \leq R+1$, we see from (C1) and Remark 3.2 that $|\overline{b}_\Delta(\overline{x},\overline{y})| \leq |b(\overline{x},\overline{y})| \leq \overline{L}_{R+1}$, then (A2) leads to

$$q_2 \leq 2C_\zeta \overline{L}_{R+1}(|x-\overline{x}|^2 + |y-\overline{y}|^2).$$

Combining the estimation of q_1 and q_2, we get

$$\langle x - D(y) - \overline{x} + D(\overline{y}), \widetilde{b}_\Delta(x,y) - \widetilde{b}_\Delta(\overline{x},\overline{y})\rangle \leq (M_{R+1} + 2C_\zeta \overline{L}_{R+1})(|x-\overline{x}|^2 + |y-\overline{y}|^2).$$

Case b: One of $|x|, |y|, |\overline{x}|, |\overline{y}|$ bigger than $R+1$. Assume $|x| > R+1$ and $|y|, |\overline{x}|, |\overline{y}| \leq R+1$. In this case, we have $\zeta_R(x,y) = 0$ and $0 \leq \zeta_R(\overline{x},\overline{y}) \leq 1$. Similar to Case a, we have

$$\langle x - D(y) - \overline{x} + D(\overline{y}), \widetilde{b}_\Delta(x,y) - \widetilde{b}_\Delta(\overline{x},\overline{y})\rangle \leq 2C_\zeta \overline{L}_{R+1}(|x-\overline{x}|^2 + |y-\overline{y}|^2).$$

Case c: Two of $|x|, |y|, |\overline{x}|, |\overline{y}|$ bigger than $R+1$. We divide it into two cases.
i): Both $|x|, |y|$ bigger than $R+1$ or both $|\overline{x}|, |\overline{y}|$ bigger than $R+1$. Consider one of the case $|x|, |y| > R+1$ while $|\overline{x}|, |\overline{y}| \leq R+1$. It is obvious that $\zeta_R(x,y) = 0$ and $0 \leq \zeta_R(\overline{x},\overline{y}) \leq 1$. By taking similar steps as Case a, we can get

$$\langle x - D(y) - \overline{x} + D(\overline{y}), \widetilde{b}_\Delta(x,y) - \widetilde{b}_\Delta(\overline{x},\overline{y})\rangle \leq 2C_\zeta \overline{L}_{R+1}(|x-\overline{x}|^2 + |y-\overline{y}|^2).$$

ii): One of $|x|, |y|$ bigger than $R+1$ and one of $|\overline{x}|, |\overline{y}|$ bigger than $R+1$. Consider the case of $|x| > R+1, |y| \leq R+1, |\overline{x}| > R+1, |\overline{y}| \leq R+1$. Then $\zeta_R(x,y) = \zeta_R(\overline{x},\overline{y}) = 0$ and $\langle x - D(y) - \overline{x} + D(\overline{y}), \widetilde{b}_\Delta(x,y) - \widetilde{b}_\Delta(\overline{x},\overline{y})\rangle = 0$.
Case d: Three or four of $|x|, |y|, |\overline{x}|, |\overline{y}|$ bigger than $R+1$. Since we have $\zeta_R(x,y) = \zeta_R(\overline{x},\overline{y}) = 0$, the result is obvious.
Taking Cases a-d into consideration, there exists an $\overline{M}_{R_0}$ such that (C3') satisfies for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$.

□

Remark 3.4. Lemma 3.1 shows that with assumptions (C1)-(C3), (3.2) is well defined under some constraints on time step Δ. It is worth mentioning that (C3) and (C3') are merely used to guarantee the uniqueness of numerical solutions.

Remark 3.5. Under assumptions (A2)-(A3), (A5), we now give an example such that the set of sequences of functions satisfy (C1)-(C4). Let $b(x,y), \sigma(x,y)$ be one-dimensional and define

$$\overline{b}_\Delta(x,y) = \frac{1}{1 + \Delta^\alpha |b(x,y)| + \Delta^{\alpha/2}\|\sigma(x,y)\|} b(x,y)$$

for any $x, y \in \mathbb{R}$. It is easy to see $|\overline{b}_\Delta(x,y)| \leq |b(x,y)|$, and on the other hand, we have

$$|\overline{b}_\Delta(x,y)| = \frac{\Delta^{-\alpha}|b(x,y)|}{\Delta^{-\alpha} + |b(x,y)| + \Delta^{-\alpha/2}\|\sigma(x,y)\|} \leq K_5 \Delta^{-\alpha} \leq K_5 \Delta^{-\alpha}(1 + |x| + |y|).$$

Furthermore, due to (A3),

$$\langle x - D(y), \overline{b}_\Delta(x,y)\rangle = \frac{\langle x - D(y), b(x,y)\rangle}{1 + \Delta^\alpha |b(x,y)| + \Delta^{\alpha/2}\|\sigma(x,y)\|} \leq K_2(1 + |x|^2 + |y|^2).$$

That is to say, (C2) is satisfied. In order to show (C3), we have to divide it into several cases. Denote by $\Gamma(x,y) = 1 + \Delta^{\alpha}|b(x,y)| + \Delta^{\alpha/2}\|\sigma(x,y)\|$.

Case a: $b(x,y)\cdot b(\overline{x},\overline{y}) < 0$. We divide this into four classes.

i): For $b(x,y) > 0, b(\overline{x},\overline{y}) < 0$ and $x - D(y) - \overline{x} + D(\overline{y}) \geq 0$,

$$\begin{aligned}&\langle x - D(y) - \overline{x} + D(\overline{y}), \overline{b}_{\Delta}(x,y) - \overline{b}_{\Delta}(\overline{x},\overline{y})\rangle = \left\langle x - D(y) - \overline{x} + D(\overline{y}), \frac{b(x,y)}{\Gamma(x,y)} - \frac{b(\overline{x},\overline{y})}{\Gamma(\overline{x},\overline{y})}\right\rangle \\ \leq&\langle x - D(y) - \overline{x} + D(\overline{y}), b(x,y) - b(\overline{x},\overline{y})\rangle \leq L_R(|x-\overline{x}|^2 + |y-\overline{y}|^2).\end{aligned}$$

ii): For $b(x,y) > 0, b(\overline{x},\overline{y}) < 0$ and $x - D(y) - \overline{x} + D(\overline{y}) < 0$, the result is obvious.

iii): For $b(x,y) < 0, b(\overline{x},\overline{y}) > 0$ and $x - D(y) - \overline{x} + D(\overline{y}) < 0$,

$$\begin{aligned}&\langle x - D(y) - \overline{x} + D(\overline{y}), \overline{b}_{\Delta}(x,y) - \overline{b}_{\Delta}(\overline{x},\overline{y})\rangle \\ \leq&\langle x - D(y) - \overline{x} + D(\overline{y}), b(x,y) - b(\overline{x},\overline{y})\rangle \leq L_R(|x-\overline{x}|^2 + |y-\overline{y}|^2).\end{aligned}$$

iv): For $b(x,y) < 0, b(\overline{x},\overline{y}) > 0$ and $x - D(y) - \overline{x} + D(\overline{y}) \geq 0$, the result is also obvious.

Case b: $b(x,y)\cdot b(\overline{x},\overline{y}) > 0$. We compute

$$\begin{aligned}&\langle x - D(y) - \overline{x} + D(\overline{y}), \overline{b}_{\Delta}(x,y) - \overline{b}_{\Delta}(\overline{x},\overline{y})\rangle = \left\langle x - D(y) - \overline{x} + D(\overline{y}), \frac{b(x,y) - b(\overline{x},\overline{y})}{\Gamma(x,y)\Gamma(\overline{x},\overline{y})}\right\rangle \\ &+ \left\langle x - D(y) - \overline{x} + D(\overline{y}), \frac{\Delta^{\alpha}[b(x,y)|b(\overline{x},\overline{y})| - b(\overline{x},\overline{y})|b(x,y)|]}{\Gamma(x,y)\Gamma(\overline{x},\overline{y})}\right\rangle \\ &+ \left\langle x - D(y) - \overline{x} + D(\overline{y}), \frac{\Delta^{\alpha/2}\|\sigma(\overline{x},\overline{y})\|[b(x,y) - b(\overline{x},\overline{y})]}{\Gamma(x,y)\Gamma(\overline{x},\overline{y})}\right\rangle \\ &+ \left\langle x - D(y) - \overline{x} + D(\overline{y}), \frac{\Delta^{\alpha/2}b(\overline{x},\overline{y})[\|\sigma(\overline{x},\overline{y})\| - \|\sigma(x,y)\|]}{\Gamma(x,y)\Gamma(\overline{x},\overline{y})}\right\rangle \\ :=&\overline{q}_1 + \overline{q}_2 + \overline{q}_3 + \overline{q}_4.\end{aligned}$$

Obviously, $\overline{q}_2 = 0$. Noticing that $\Gamma(x,y) \geq 1, \Gamma(\overline{x},\overline{y}) \geq 1$ and $0 < \frac{\Delta^{\alpha/2}\|\sigma(\overline{x},\overline{y})\|}{\Gamma(\overline{x},\overline{y})} \leq 1$, we then derive from (A2), (A5) and Remark 3.2 that

$$\overline{q}_1 + \overline{q}_3 \leq 2L_R(|x-\overline{x}|^2 + |y-\overline{y}|^2),$$

and

$$\begin{aligned}\overline{q}_4 \leq&\frac{1}{2}|x - D(y) - \overline{x} + D(\overline{y})|^2 + \frac{\Delta^{\alpha}|b(\overline{x},\overline{y})|^2[\|\sigma(\overline{x},\overline{y})\| - \|\sigma(x,y)\|]^2}{2\Gamma^2(x,y)\Gamma^2(\overline{x},\overline{y})} \\ \leq&|x-\overline{x}|^2 + |y-\overline{y}|^2 + |b(x,y)|\|\sigma(\overline{x},\overline{y}) - \sigma(x,y)\|^2 \\ \leq&(1 + L_R\overline{L}_R)(|x-\overline{x}|^2 + |y-\overline{y}|^2).\end{aligned}$$

This shows that (C3) is satisfied. Thanks to (A3) and Remark 3.2, we see that

$$\sup_{|x|\vee|y|\leq R}|b(x,y) - \overline{b}_{\Delta}(x,y)|^p \leq \Delta^{\alpha p}\sup_{|x|\vee|y|\leq R}\frac{(|b(x,y)| + \|\sigma(x,y)\|^2)^p|b(x,y)|^p}{(1 + \Delta^{\alpha}|b(x,y)| + \Delta^{\alpha}\|\sigma(x,y)\|^2)^p} \leq C\Delta^{\alpha p} \to 0,$$

Thus, (C4) is shown.

Condition (C3') shows that $\widetilde{b}_\Delta$ is global one-sided Lipschitz. According to the monotone operator, the implicit scheme (3.2) is well defined with $\theta \overline{M}_{R_0}\Delta < 1$. Define $\Delta_3 = \frac{1}{\theta \overline{M}_{R_0}}$ for $\theta \in (0, 1]$. Thus in the following section, we set $\Delta^\star \in (0, \Delta_3 \wedge \Delta_2)$, and let $0 < \Delta \le \Delta^\star$ for $\theta \in (0, 1]$ while for $\theta = 0$, we set $\Delta \in (0, 1)$.

3.2 Convergence of the Numerical Solutions

We need the following lemma.

Lemma 3.2. Let (A1)-(A3) and (A5) hold, then it holds that

$$\sup_{0\le t\le T} \mathbb{E}|X(t)|^p \vee \sup_{0\le t\le T} \mathbb{E}|Y_\Delta(t)|^p \le C,$$

for $p \ge 2$.

Remark 3.6. Through the derivation process of Lemmas 2.2 and 2.3, we see that what we really used are assumptions (A1)-(A3), (B1) and (2.7) in Lemmas 2.2 and 2.3. In this section, $\overline{b}_\Delta$ and σ satisfy (C1)-(C2), which implies that the corresponding $\widetilde{b}_\Delta$ and σ also satisfy (B1) and (2.7). Thus, Lemmas 2.2 and 2.3 proposed in Section 2 still hold in this section.

We now state the main result in this Section.

Theorem 3.3. Let (A1)-(A3), (A5) and (C1)-(C4) hold, then the continuous form $Y_\Delta(t)$ of the tamed theta scheme (3.2) converges strongly to the exact solution $X(t)$ of (2.1), that is,

$$\lim_{\Delta\to 0} \sup_{0\le t\le T} \mathbb{E}|X(t) - Y_\Delta(t)|^2 = 0.$$

Proof. Denote $e(t) = Y_\Delta(t) - X(t)$, and for any $R > 0$ define the following stopping time

$$\tau_R = \inf\{t \ge 0 : |Y_\Delta(t)| \ge R\}, \rho_R = \inf\{t \ge 0 : |X(t)| \ge R\}, \nu_R = \tau_R \wedge \rho_R.$$

For any $\eta > 0$, by the Young inequality,

$$\begin{aligned} \sup_{0\le u\le T} \mathbb{E}|e(u)|^2 &= \sup_{0\le u\le T} \mathbb{E}(|e(u)|^2 \mathbf{I}_{\{\tau_R > T, \rho_R > T\}}) + \sup_{0\le u\le T} \mathbb{E}(|e(u)|^2 \mathbf{I}_{\{\tau_R \le T \text{ or } \rho_R \le T\}}) \\ &\le \sup_{0\le u\le T} \mathbb{E}(|e(u\wedge \nu_R)|^2 \mathbf{I}_{\{\nu_R > T\}}) + \frac{2\eta}{p} \sup_{0\le u\le T} \mathbb{E}|e(u)|^p \\ &\quad + \frac{p-2}{p\eta^{\frac{2}{p-2}}} \mathbb{P}(\tau_R \le T \text{ or } \rho_R \le T). \end{aligned} \tag{3.3}$$

Due to Lemma 2.2,

$$\mathbb{P}(\tau_R \le T) = \mathbb{E}\left(\mathbf{I}_{\{\tau_R \le T\}} \frac{|Y_\Delta(\tau_R)|^p}{R^p}\right) \le \frac{1}{R^p} \sup_{0\le u\le T} \mathbb{E}|Y_\Delta(u)|^p \le \frac{C}{R^p},$$

where here and in the following, we emphasize that C is a positive constant independent of Δ, R and ε, while C_R will be a positive constant depending on R. Similarly, we derive from Lemma 3.2 that

$$\mathbb{P}(\tau_R \le T \text{ or } \rho_R \le T) \le \mathbb{P}(\tau_R \le T) + \mathbb{P}(\rho_R \le T) \le \frac{2C}{R^p}. \tag{3.4}$$

On the other hand, Lemma 2.2 and Lemma 3.2 yield

$$\sup_{0\le u\le T} \mathbb{E}|e(u)|^p \le 2^{p-1} \sup_{0\le u\le T} \mathbb{E}(|Y_\Delta(u)|^p + |X(u)|^p) \le C. \tag{3.5}$$

Denote by $I(t) = Y_\Delta(t) - D(Y_\Delta(t-\tau)) - \theta\widetilde{b}_\Delta(Y_\Delta(t), Y_\Delta(t-\tau))\Delta - X(t) + D(X(t-\tau))$. Applying the Itô formula,

$$\begin{aligned}
\mathbb{E}|I(T\wedge\nu_R)|^2 =& |I(0)|^2 + 2\mathbb{E}\int_0^{T\wedge\nu_R} \langle I(s), \widetilde{b}_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau))\rangle \mathrm{d}s \\
&+ \mathbb{E}\int_0^{T\wedge\nu_R} \|\sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\|^2 \mathrm{d}s \\
\le& |I(0)|^2 + 2\mathbb{E}\int_0^{T\wedge\nu_R} \langle \overline{Y}_\Delta(s) - D(\overline{Y}_\Delta(s-\tau)) - X(s) + D(X(s-\tau)), \\
&\qquad b(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau))\rangle \mathrm{d}s \\
&+ 2\mathbb{E}\int_0^{T\wedge\nu_R} \langle \overline{Y}_\Delta(s) - D(\overline{Y}_\Delta(s-\tau)) - X(s) + D(X(s-\tau)), \\
&\qquad \widetilde{b}_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau))\rangle \mathrm{d}s \\
&+ 2\mathbb{E}\int_0^{T\wedge\nu_R} \langle Y_\Delta(s) - D(Y_\Delta(s-\tau)) - \overline{Y}_\Delta(s) + D(\overline{Y}_\Delta(s-\tau)), \\
&\qquad \widetilde{b}_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau))\rangle \mathrm{d}s \\
&- 2\theta\Delta\mathbb{E}\int_0^{T\wedge\nu_R} \langle \widetilde{b}_\Delta(Y_\Delta(s), Y_\Delta(s-\tau)), \widetilde{b}_\Delta(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - b(X(s), X(s-\tau))\rangle \mathrm{d}s \\
&+ 2\mathbb{E}\int_0^{T\wedge\nu_R} \|\sigma(\overline{Y}_\Delta(s), \overline{Y}_\Delta(s-\tau)) - \sigma(X(s), X(s-\tau))\|^2 \mathrm{d}s \\
\le& |I(0)|^2 + \sum_{i=1}^5 I_i(T),
\end{aligned}$$

where $I(0) = -\theta b_\Delta(\xi(0), \xi(-\tau))\Delta$. By assumption (A5) and Lemma 2.3,

$$\begin{aligned}
\sup_{0\le u\le T}(|I_1(u) + I_5(u)|) \le& C_R\int_0^{T\wedge\nu_R} [|\overline{Y}_\Delta(s) - X(s)|^2 + |\overline{Y}_\Delta(s-\tau) - X(s-\tau)|^2]\mathrm{d}s \\
\le& C_R\Delta + \int_0^t \sup_{0\le u\le s} \mathbb{E}|e(u\wedge\nu_R)|^2 \mathrm{d}s.
\end{aligned} \tag{3.6}$$

Obviously, due to (C4), Lemma 2.1 and Lemma 2.2,

$$\sup_{0\le u\le T}|I_2(u)| \le C_R\Delta^{\alpha} + C_R\Delta^{2\alpha}. \tag{3.7}$$

By (A2) and Lemma 2.3, we obtain,

$$\begin{aligned}\sup_{0\le u\le T}|I_3(u)| \le & C\int_0^{T\wedge\nu_R}\left(\mathbb{E}[|Y_\Delta(s)-\overline{Y}_\Delta(s)|^2+|Y_\Delta(s-\tau)-\overline{Y}_\Delta(s-\tau)|^2]\right)^{\frac{1}{2}} \\ & \left(\mathbb{E}|\widetilde{b}_\Delta(\overline{Y}_\Delta(s),\overline{Y}_\Delta(s-\tau))-b(X(s),X(s-\tau))|^2\right)^{\frac{1}{2}}\mathrm{d}s \le C_R\Delta^{\frac{1}{2}}.\end{aligned} \tag{3.8}$$

Furthermore, by Remark 3.2 and (C1)

$$\sup_{0\le u\le T}|I_4(u)| \le C_R\Delta. \tag{3.9}$$

Due to (3.6)-(3.9), we see

$$\sup_{0\le u\le T}\mathbb{E}|I(u\wedge\nu_R)|^2 \le C_R\int_0^T\sup_{0\le u\le s}\mathbb{E}|e(u\wedge\nu_R)|^2\mathrm{d}s + C_R\Delta^{\alpha}. \tag{3.10}$$

With assumption (A2), one has

$$\sup_{0\le u\le T}\mathbb{E}|e(u\wedge\nu_R)|^2 \le C\sup_{0\le u\le T}\mathbb{E}|I(u\wedge\nu_R)|^2 + \kappa\sup_{0\le u\le T}\mathbb{E}|e(u\wedge\nu_R-\tau)|^2 + C_R\Delta^2.$$

This implies

$$\sup_{0\le u\le T}\mathbb{E}|e(u\wedge\nu_R)|^2 \le C\sup_{0\le u\le T}\mathbb{E}|I(u\wedge\nu_R)|^2 + C_R\Delta^2. \tag{3.11}$$

Applying the Gronwall inequality, we derive from (3.10) and (3.11) that

$$\sup_{0\le u\le T}\mathbb{E}|e(u\wedge\nu_R)|^2 \le C_R\Delta^{\alpha}. \tag{3.12}$$

Thus, combining (3.4), (3.5) and (3.12), we see from (3.3) that for any given $\epsilon > 0$, one can choose η small enough such that

$$\frac{2\eta}{p}C < \frac{\epsilon}{3},$$

and then R big enough such that

$$\frac{p-2}{p\eta^{\frac{2}{p-2}}}\frac{2C}{R^p} < \frac{\epsilon}{3},$$

finally Δ small enough to satisfy

$$C_R\Delta^{\alpha} < \frac{\epsilon}{3}.$$

Therefore, we arrive at

$$\sup_{0\le u\le T}\mathbb{E}|Y_\Delta(u)-X(u)|^2 \to 0 \text{ as } \Delta\to 0,$$

as required. □

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