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Counting to one: reducibility of one- and two-loop amplitudes at the integrand level

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ABSTRACT: Calculation of amplitudes in perturbative quantum field theory involve large loop integrals. The complexity of those integrals, in combination with the large number of Feynman diagrams, make the calculations very difficult. Reduction methods proved to be very helpful, lowering the number of integrals that need to be actually calculated. Especially, the reduction at the integrand level technique, improves the speed and set-up of these calculations. In this article we demonstrate, by counting the numbers of tensor structures and independent coefficients, how to write such relations at the integrand level for one— and two—loop amplitudes. We clarify their connection to the so-called spurious terms at one loop and discuss their structure in the two—loop case. This method is also applicable to higher loops, and the results obtained apply to both planar and non-planar diagrams.

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1. Introduction

Modern colliders such as the Large Hadron Collider (LHC) (and the Tevatron before it was shut down) produce a large amount of experimental data. In order to understand the output of these experiments, comparison between very precise theoretical results and experimental results is needed. It is clear, from the theoretical point of view, that Next-to-Leading-Order (NLO) and Next-to-Next-to-Leading-Order (NNLO) calculations with many external legs have to be considered [1]. This implies that (very) large loop integrals have to be computed for very many Feynman diagrams, which has widely been considered the bottleneck of such calculations.

Reduction techniques form a way out. The idea of reducing Feynman integrals with a large number of denominators to a set of simpler integrals (*i.e.* with fewer denominators) at one loop goes surprisingly many years back [2, 3]. A typical integral with n such denominators, in d space-time dimensions, is given by

$$\int d^d q \frac{1}{D_1 D_2 ... D_n} \ ,$$

where $D_i = (q + p_i)^2 - m_i^2$ is the denominator of a generic propagator. In [2] the authors reduce a triangle (integral with 3 denominators) to bubbles (2 denominators) in 2 dimensions while in [3] a pentagon (5 denominators) is reduced to boxes (4 denominators) in 4 dimensions. We see that the result of the reduction depends on the dimension. However, the methods we will use can be applied to all dimensions. Our main interest is of course the case $d = 4$.

Since a few decades now [4], it is a very well known fact that a generic one-loop amplitude is decomposable in terms of scalar integrals, with one, two, three and four external legs (in $d = 4$). Passarino and Veltman [5] used Lorentz invariance to express tensor one-loop $n-$point integrals in terms of $m-$point scalar integrals ($m \leq n$). As a consequence, only the evaluation of scalar integrals (integrals with trivial numerators) is needed in order to perform a one-loop calculation.

In another attempt [6] a pentagon-to-boxes decomposition is performed in 4 dimensions. The importance of this paper is that it provides a basis in four-dimensional momentum space (the so called van Neerven-Vermaseren basis), which proved useful for understanding one-loop reduction. Another important fact about this paper is the use of what we call nowadays *spurious terms* to decompose a scalar pentagon to boxes. Spurious terms are terms that, by construction, vanish upon integration. Their rôle will be explained later when we consider reductions at the integrand level and we shall see why one cannot avoid them.

The next big step comes from unitarity methods [7–11]. Instead of working with specific Feynman diagrams these methods have a big advantage in that they try to decompose the whole one-loop amplitude in terms of the scalar integrals. By cutting propagators[1] the rational coefficients of loop integrals are given in terms of products of tree amplitudes. In generalized unitarity methods [12–17], the notion of *multiple cuts* is introduced. One can cut more than one propagator to find these coefficients. Note that, for $d = 4$, cutting four propagators essentially determines the loop momentum (there is, in general, more than a single solution since the D's are quadratic in the loop momentum).

The Ossola-Papadopoulos-Pittau (OPP) method [18–22] comes as a natural combination of all the above. Since every integral can be decomposed to scalar integrals with up to four denominators (for $d = 4$), every one-loop amplitude is written in terms of coefficients that multiply these scalar integrals. The OPP method works at the integrand level [23, 24], which means that for these decompositions to be possible one must also include spurious terms. Then one has to find a way to calculate the coefficients of the reduction and multiply them with the appropriate scalar integrals, using one of the packages available for the evaluation of them(i.e [25, 26]). Finding the coefficients is a purely algebraic problem. The method is suitable for a fully

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[1]'Cutting' a propagator means that loop momenta are chosen for which one or more of the D's vanish so that the integrand becomes singular. One also speaks of 'putting propagators on-shell'.

numerical implementation. The OPP method has been widely used so far in many one loop calculations (see for example [27–39]).

As we noticed before, in the case of one-loop calculations a basis for any integral is known in advance. Any one-loop integral can be written in terms of scalar boxes, triangles, bubbles and tadpoles. However, in the case of higher-loop integrals the situation is different. A basis is not known *a priori*. It is believed that unitarity methods can also be applied in that case and there are some recent papers in that direction, performing decomposition à la OPP [40–43], or using generalized unitarity [44–46] at two loops.

Two remarks are in order here. The first is that the basis of two-loop integrals does not include only scalar integrals. It includes integrals that also have irreducible scalar products (ISP) as numerators (to some power) that cannot be rewritten as existing denominators of the integral. In the one-loop case these ISP are always spurious and integrate to zero, but for higher loops this does no longer hold. The second remark is that if one is interested in constructing a unitarity-like basis, the set of integrals that ends up with is not necessarily a minimal one: the integrals are not by default Master Integrals (MI). There might be smaller sets of true MI and at two or more loops one can find them by using integration-by-parts (IBP) identities [47–49].

In this article we prove that any two-loop integral can be written as a linear combination of integrals with at most $2d$ denominators. From this set of integrals, one can in principle end up with true master integrals with numerators containing ISP 's. The layout of this paper is the following. We start with definitions and reduction at one loop. We see why one can have unitarity-like bases at the integrand level by writing first the numerator of our integrals (for scalar cases the number one) as a sum of coefficients times denominators. We investigate, using simple algebra, what is the form of these coefficients in order for our polynomial equation to have solutions. Then we repeat the same procedure in the case of two-loop integrands. Our method is not exactly the OPP one, but the connection of the two methods will become clear.

2. Reduction at one loop

2.1 Introduction : reduction with trivial coefficients

We start with integrands of any given one-loop amplitude. These integrands consist of the sum of integrands coming from the Feynman diagrams that contribute to a given process and share the same topology; the advantage is that we perform the decomposition once instead of reducing every single diagram separately. For that reason we deal with integrand-graphs, or iGraphs instead of Feynman diagrams. We

give an example of an iGraph of order 5 (pentagon) below, where $j = 1 \ldots 5$,

$$D_j \equiv D(q + p_j) = (q + p_j)^2 - m_j{}^2 = q^2 + 2(p_j \cdot q) + \mu_j \quad , \quad \mu_j \equiv p_j{}^2 - m_j{}^2 \quad , \text{(2.1)}$$
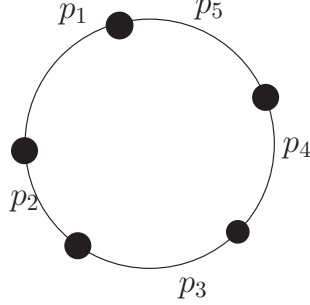
A one-loop pentagon iGraph. The dots serve to distinguish the
several denominators. The external momenta are indicated.

The loop momentum is denoted by q^μ, and $p_j{}^\mu$ is called the *external momentum*, where it must be realized that by this we do *not* mean a momentum related to a particle incoming or outgoing in a given amplitude: what we call external momenta are simply fixed momenta, given in some way by the configuration of incoming and outgoing momenta and the various diagram topologies.

Consider a one-loop iGraph of order n:

$$\frac{1}{D_1 D_2 D_3 D_4 ... D_n} \quad .$$

We say that we can decompose this iGraph *if* we can find funtions $T_{1,2,\ldots,n}(q)$ such that

$$D_1 T_1(q) + D_2 T_2(q) + \cdots + D_n T_n(q) = 1 \quad , \tag{2.2}$$

for then we have

$$\frac{1}{D_1 \cdots D_n} = \frac{T_1(q)}{D_2 D_3 D_4 ... D_n} + \frac{T_2(q)}{D_1 D_3 D_4 ... D_n} + \cdots + \frac{T_n(q)}{D_1 D_2 D_3 ... D_{n-1}} \quad , \tag{2.3}$$

and the original iGraph is decomposed into a sum of iGraphs of order $n - 1$ (or lower).

This immediately leads us to state the following theorem: *one-loop iGraphs of order d or smaller* **cannot** *be decomposed in the above manner*. The reason is simple: for $n \leq d$ there exist a cut through *all* propagators, so that $D_j = 0$ for $j = 1, \ldots, n$ and eq. (2.2) then would become $0 = 1$.

The simplest possibility for the functions $T_j(q)$ is to take them to be just numbers independent of q^μ ('trivial' coefficients):

$$T_j(q) = x_j \quad .$$

From eq. (2.2) then we have

$$q^2 \sum_{j=1}^{n} x_j + 2q_\mu \sum_{j=1}^{n} x_j p_j{}^\mu + \sum_{j=1}^{n} x_j \mu_j = 1 \ . \tag{2.4}$$

Since this has to hold for *any* value of q^μ we must have separately

$$\sum_{j=1}^{n} x_j = 0 \quad , \quad \sum_{j=1}^{n} x_j p_j{}^\mu = 0 \ , \tag{2.5}$$

and

$$\sum_{j=1}^{n} x_j \mu_j = 1 \ . \tag{2.6}$$

Note that if a nontrivial solution to the homogeneous equations (2.5) exists, then by suitable scaling we can always satisfy eq. (2.6). We see that, at one loop, for $d = 4$ any iGraph of order 6 or higher can be decomposed in this fairly trivial way. A pentagon in 4-dimensions thus cannot be decomposed that way. For general d, iGraphs of order $d + 2$ or higher are decomposable.

2.2 Reduction with coefficients linear in the loop momentum

For a one-loop iGraph of order 5 (or lower) no trivial decomposition exists in $d = 4$, assuming that four out of the five external momenta, p_j, in the loop can be considered as linearly independent. One way to see this is by shifting the loop momentum so that $\sum_j p_j{}^\mu = 0$. Then the only solution to the conditions in eq. (2.5) is $x_j = 0, j = 1 \ldots, 5$ which is unacceptable in eq. (2.6). We therefore turn to the next simplest possibility for the T's, with a linear q dependence :

$$T_j(q) = x_j + \sum_{k=1}^{4} x_{j,k}(q \cdot t_k) \ . \tag{2.7}$$

The single x_j is now replaced by 5 (or $d + 1$) variables to be determined in each T. Here, the four (or d) vectors $t_k{}^\mu$ must be linearly independent but are otherwise arbitrary. In analogy to eq. (2.5) and eq. (2.6), we now have more tensor structures in terms of the loop momentum : we can denote them by the shorthand

$$1 \ , \quad q^\mu \ , \quad q^\mu q^\nu \ , \quad q^2 q^\mu \ . \tag{2.8}$$

There are, for $d = 4$, therefore $1+4+10+4 = 19$ independent tensor structures. Note that the q^2 appearing in eq. (2.5) is no longer independent since it appears as the trace part of $q^\mu q^\nu$. This can be extended to the inclusion of higher-rank tensors and other dimensions : in d dimensions, and with the inclusion of tensor up to rank k, we find for the number $N(d, k)$ of independent tensor structures

$$N(d, k) = \binom{d-1+k}{k} + \sum_{p=0}^{k+1} \binom{d-1+p}{p} \ . \tag{2.9}$$

In the table below we give the results for various ranks and dimensionalities.

k	0	1	2	3	4
$d =1$	3	4	5	6	7
2	4	8	13	19	26
3	5	13	26	45	71
4	6	19	45	90	161
5	7	26	71	161	322
6	8	34	105	266	588

Values of $N(d,k)$

The number of coefficients x to be solved for is given by

$$X(n, d, k) = n \begin{pmatrix} d+k \\ k \end{pmatrix} . \tag{2.10}$$

Since for $d = 4$ and $k = 1$ we have $N(4, 1) = 19$, it would seem that iGraphs of order 5 and 4 are decomposable with linear terms. However, the situation is not so simple since it is not obvious that the 25 coefficients for $n = 5$ and the 20 coefficients for $n = 4$ allow us to actually build up the 19 required tensor structures. We now describe how we can ascertain the number of independent structures numerically, by an approach that may be dubbed *cancellation probing.*

We start by generating *random* values for the external momenta $p_j{}^\mu$ and m_j $(j = 1, \ldots, n)$. This avoids any possibility of us choosing, coincidentally, any special phase space point where degeneracy might occur. Then, we choose *random* values for q^μ precisely $\xi = X(n, d, k)$ times, and insert all this in eq. (2.2). We are left with a set of ξ linear equations for the ξ unknowns x :

$$\sum_{j=1}^{\xi} M^i{}_j x^j = 1 \quad , \quad j = 1, \ldots, \xi . \tag{2.11}$$

The $\xi \times \xi$ matrix M is purely numerical. We obtain it using the computer-algebra package `MAPLE`[2] which, although not numerically the fastest available, has the essential advantage that one can easily set the precision with which numerical operations are performed[3]. Now, if the number of independent tensor structures that can be formed with our T's is less than ξ, the determinant of M will vanish. In an ideal real-number model of computation, we would thus find $det(M) = 0$, but in our actual numerical computation there will be rounding errors. A cancellation of numbers to 'zero' will , in `MAPLE`, actually give a number of order 10^{-p}, where p is the number of digits specifies in the precision we tell `MAPLE` to use. If the matrix' determinant

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[2]http://www.maplesoft.com/
[3]The relevant variable is `Digits`.

is computed by Gaussian elimination[4], then a matrix with q zero eigenvalues will have a determinant of order 10^{-pq}. By letting p run down from 150 to 20 in steps of 10, we can obtain[5] a very accurate estimate of q, especially since q must be integer. We give two examples to demonstrate the use of this method for the calculation of the zero eigenvalues. In both examples we consider a decomposition of pentagon to boxes, with linear and quartic terms respectively.
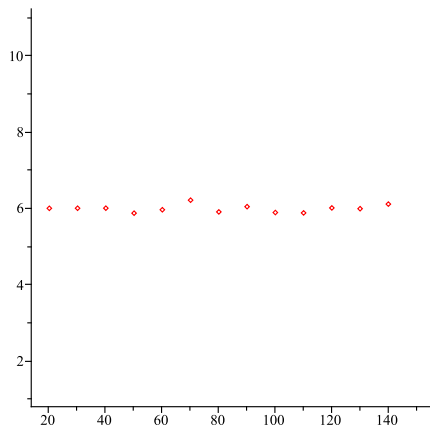
The number of zero eigenvalues using rounding errors for the case of a pentagon decomposition with coefficients linear in the loop momentum. The obtained value is $q = 5.988 \pm 0.098$; the (absolute values of the 25×25 determinants range from $1.6 \ 10^{-826}$ to $2.3 \ 10^{-108}$.
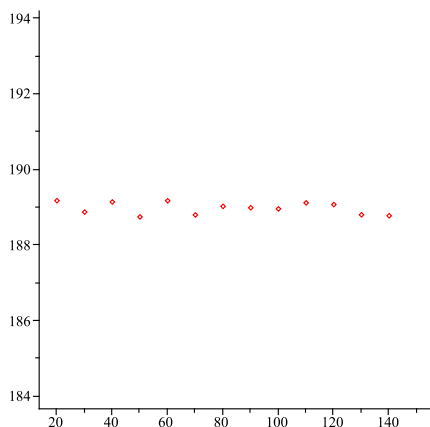
The number of zero eigenvalues using rounding errors for the case of a pentagon decomposition with up to quartic in the loop momentum coefficients. Here, $q = 188.98 \pm 0.15$, with the 350×350 determinants ranging from $2.7 \ 10^{-26000}$ to $2.3 \ 10^{-3320}$.

The difference $\xi - q$ then gives the rank of M, and *this* determines the decomposability of the iGraph : the rank must be equal to $N(d, k)$ for it to be decomposable. In the table below we give the results of cancellation probing for various n and d.

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[4]This is almost unavoidable since the matrix M is *not* sparse, and anyway we can choose Gaussian elimination as an option in any case.

[5]Surprisingly, the cancellation probing appears to fail for $p = 15$ and $p = 10$, possibly since MAPLE may have special ways to treat these accuracies ($p = 10$ is default in MAPLE).

n	$d = 6$	$d = 5$	$d = 4$	$d = 3$	$d = 2$	$d = 1$
2	14-0	12-0	10-0	8-0	6-0	4-0
3	21-1	18-1	15-1	12-1	9-1	6-2
4	28 -3	24-3	20-3	16-3	12-4	8-4
5	35-6	30-6	25-6	20-7	15-7	10-6
6	42-10	36-10	30-11	24-11	18-10	12-8
7	49-15	42-16	35-16	28-15	21-13	14-10
8	56-22	48-22	40-21	32-19	24-16	16-12

The rank of M for various d and n,
given as the difference $\xi - q$.

We have denoted the limit of decomposability with horizontal lines. We conclude that in four dimensions, $n = 5$ is precisely decomposable, but $n = 4$ is not. We now also see the deeper reason for this: in spite of there being 20 coefficients (one more than the minimum of 19), only 17 independent combinations can actually be formed. We also see that for sufficiently large n the number of independent combinations of coefficients saturates at $N(d, 1)$ as it ought to. We conclude that in d dimensions, an iGraph of order $d + 1$ is precisely decomposable with linear terms, but one of order d is of course not.

In the OPP method [18], the linear terms are precisely the spurious terms. We have to note that our general linear terms are not exactly those. The spurious terms have a specific property leading to fewer tensor structures, and we give an example in the appendix. Rewriting our general linear terms in terms of propagators and spurious terms, we see that we decompose a pentagon into boxes *and* triangles (like in [6]and [18] for example). It can be checked that the triangles always cancel, and therefore the decomposition is actually unique.

At this point it must be pointed out that, in all cases where a decomposition is possible in principle, we actually have obtained a solution for, the system (2.11). Once a would-be solution is found, it can easily be tested by evaluating eq. (2.2) for additional random values of the loop momentum[6] This 'global 1=1 test' then verifies this solution, where of course the equality is supposed to hold only up to the precision used.

This finishes the discussion for scalar one-loop iGraphs, that have unity for their numerator. Let us regard and iGraph of order n with a nontrivial numerator, for instance

$$\frac{q \cdot k}{D_1 D_2 \cdots D_N} \quad .$$

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[6]In a certain sense, eq. (2.2) implies an *infinite* number of linear equations, of which we take ξ and solve them.

Now, we can always arrange for $p_1{}^\mu = 0$ by a suitable shift of the loop momentum, and write the vector k^μ as

$$k^\mu = \omega^\mu + 2 \sum_{j=2}^{n} \zeta_j \, p_j{}^\mu \quad , \tag{2.12}$$

where the ζ's are fixed numbers and $\omega \cdot p_j = 0$ for all j (if $n \geq d+1$ then ω^μ simply vanishes). We can then write

$$(q \cdot k) = (q \cdot \omega) + \sum_{j=2}^{n} \zeta_j \, (D_j - D_1 - \mu_j + \mu_1) \tag{2.13}$$

so that this nonscalar iGraph decomposes into scalar iGraphs of order n and $n-1$, plus possibly a spurious term about which we do not worry since it integrates to zero. Our treatment of the scalar case is therefore sufficiently general.

## 3. Reduction of two-loop integrands

3.1 Preliminaries

We now turn to the problem of reducibility at two loops. Recently several attempts in this direction have appeared in the literature (see i.e. [44], [40], [41]). Let us assume that l_1 and l_2 are the two loop momenta. We consider three different kind of propagators for the three different loop lines of a generic two loop iGraph.

$$D(l_1 + p_i) \, , \; D(l_2 + p_j) \, , \; D(l_1 + l_2 + p_k) \tag{3.1}$$

where for instance $D(l_i + p_j) = (l_i + p_j)^2 - m_j^2$ and the p_j are the external momenta associated with the propagators of the diagram.

Such iGraphs can be denoted by the triplet (n_1, n_2, n_3) which indicates the number n_1 of propagators that contain only the one loop momentum l_1, the number n_3 of propagators containing only the other loop momentum l_2, and the n_2 propagators containing both. Obviously due to the symmetries of the iGraphs, for instance exchange $l_1 \leftrightarrow l_2$, we have relations of the form $(n_1, n_2, n_3) \leftrightarrow (n_3, n_2, n_1)$ or $(n_1, n_2, n_3) \leftrightarrow (n_1, n_3, n_2)$ provided we also exchange properly the external momenta. Predictably, we write the total order of the iGraph as $n = n_1 + n_2 + n_3$.
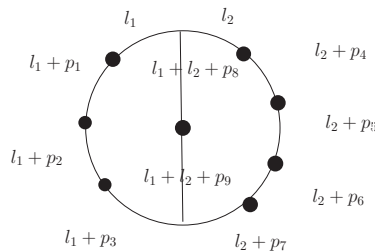
An example of the two-loop iGraph (4,2,5).
One can see the three different loop lines.

The propagators depending on both loop momenta are called *mixed propagators*. If these are absent the two integrals factor out and the problem becomes a double copy of one loop integrals. The same happens in case any other loop line is missing since, by shifting, one can always arrange the loop momenta such that they factor out. We consider these cases solved (by the one loop techniques) and will not discuss them further.

The general strategy consists in finding function $x_j \equiv x_j(l_1, l_2)$ satisfying the following equation

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_2 + p_j) = 1 \ . \quad (3.2)$$

As in the one-loop case, a generic graph of order $n \leq 2d$ with $n_{1,2,3} \leq 4$ cannot be decomposed in this way, since there are $l_{1,2}$ momenta for which all propagators appearing in the above equation can be simultaneously on-shell. This does *not* imply on the other hand that iGraphs of higher order must always be decomposable for any phase-space and mass configuration. A counterexample is the *Feynman* diagram of order 5 in two dimensions[7]: if all internal lines in this self-energy Feynman diagram are massless, it is possible to choose the two loop momenta components such that all five propagators are simultaneously on-shell.
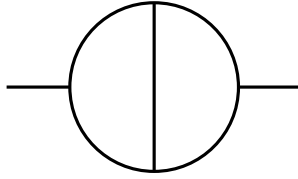
The requirement for trivial decomposition (with coefficients x_j that are constants with respect to the loop momenta) now reads

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_2 + p_j) = 1 \ . \quad (3.3)$$

Proceeding in analogy with our one-loop discussion, we find that we have to satisfy the following equations :

$$\sum_{j=1}^{n_1+n_2} x_j = \sum_{j=n_1+1}^{n} x_j = \sum_{j=n_1+1}^{n_1+n_2} x_j = 0 \ , \quad (3.4)$$

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[7]This diagram *can*, in fact, be decomposed, but not by the method described above: instead one has to use IBP techniques.

$$\sum_{j=1}^{n_1+n_2} x_j {p_j}^\mu = \sum_{j=n_1+1}^{n} x_j {p_j}^\mu = 0 \ , \tag{3.5}$$

and

$$\sum_{j=1}^{n} x_j \mu_j = 1 \ . \tag{3.6}$$

In total there are $2d+4$ conditions, so that the minimum size of a trivially decomposable iGraph is $2d+4$. In four dimensions, scalar iGraphs can therefore be decomposed down to $n=11$. Again in analogy, for $n=11$, since by shifting we can arrange $p_1 + \cdots + p_{n_1+n_2} = 0$ as well as $p_{n_1+1} + \cdots + p_n = 0$, the only solution to the $2d+3$ homogeneous equations is $x_j = 0$, $j = 1, \ldots, n$, and this fails the inhomogeneous equation. On the other hand, since any subset of an iGraph is itself an iGraph, any iGraph with $n_1 \geq 6$, $n_2 \geq 6$, or $n_3 \geq 6$ is trivially decomposable (for $d = 4$). Furthermore, with linear terms we see, from the one-loop discussion[8] that we only have to consider two-loop iGraphs with

$$n_{1,2,3} \leq 4 \ (= d) \ , \quad n_1 + n_2 + n_3 \leq 11 \ (= 2d + 3) \ . \tag{3.7}$$

A word of caution is in order here. We may have a case where $n_1 + n_2 \geq 6$ or $n_2 + n_3 \geq 6$ and then decide to perform a decomposition *à la* one-loop with trivial terms, taking l_1 as the loop momentum, for instance, and l_2 as one of the "external" momenta or vice-versa. Since the solution of the eq. (3.3) is nonlinear with respect to the external momenta, due to matrix inversion, the resultant decomposition will not have, in general, the simple form of iGraphs again, and the emerging integrals will belong to very different classes of functions.

The number of two-loop iGraphs that we have to consider is therefore not very large : 4 for $d = 2$, 10 for $d = 3$, 19 for $d = 4$.

3.2 Further reduction with linear terms

With trivial decomposition we see that we can always end up with an iGraph of order $2d+3$. Like in the one loop case, we now add coefficients linear in the loop momentum and hope for further reductions.

A note is in order here. In the one loop case the resulting integrals were always scalar. The reason is that any contraction of the loop momentum with any vector can either reconstruct denominators or be a spurious term. After integrating, in the

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[8]In case there at least 6 propagators in one loop line we can first reduce the propagators in this loop line with constant coefficients and then continue further if possible. In the case of 5 propagators we already know that adding linear terms that depend only on the loop momentum of this loop line and take all coefficients that depend on the other loop momentum to zero, we can again solve the problem à la one loop.

case a denominator is reconstructed the remaining integral is a scalar integral with fewer denominators. In case the term is spurious it vanishes after integration. In two loops this is not the case anymore. One can always use dot products of the loop momenta with the momenta of the integrals to write relations like

$$2l_1 \cdot p_j = [(l_1 + p_j)^2 - m_j^2] - l_1^2 - p_j^2 + m_j^2 \ . \tag{3.8}$$

The denominator $D(l_1 + p_j)$ may, however, not be present in the integral in case p_j appears in a propagator of another type such as $D(l_2 + p_j)^2$. Then, the product $l_1 \cdot p_j$ may be an *irreducible scalar product* (ISP) [40]. But not always; the ISP's of an integral are more complicated to write. For example, if there are enough propagators of the type $(l_1 + p_i)^2$ in the diagram such that the p_i's can form a basis, one can rewrite p_j as a linear combination of the p_i's and manage to reconstruct denominators. There is a specific number of ISP's in any diagram and one can have some freedom in how to write them. The integrals of the resulting basis can have numerators with ISP's in some power. It is not obvious that a scalar integral with a specific number of denominators is more difficult to calculate than a non-scalar integral with fewer denominators; however it is commonly accepted that this is the case.

Again, we want to write, if possible, the number 1 as in eq. (3.2). We use general linear terms in the sense that in every dimension we construct a basis t_i (possibly, but not necessarily, the external momenta in the iGraph) and we have

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j) \tag{3.9}$$

with the a_i, b_{ij}, and c_{ij} constants with respect to the loop momenta. Since in d dimensions, we need d vectors to construct such a basis, it is obvious that for an iGraph of order n we start with $(2d + 1)n$ coefficients. As in the one-loop case, we give a table that contains, for every dimension we worked with, the number of tensor structures (denoted by $T(d)$) and the number of independent coefficients that we have explicitly calculated by cancellation probing as described above.

$$T(d) = (4d^2 + 18d + 2)/2$$

n	$d=6$	$d=5$	$d=4$	$d=3$	$d=2$	$d=1$
3	39-0	33-0	27-0	21-0	15-0	9-0
4	52-0	44-0	36-0	28-0	20-0	12-2
5	65-1	55-1	45-1	35-1	25-1	15-5
6	78-3	66-3	54-3	42-3	30-3	
7	91-6	77-6	63-6	49-6	35-8	
8	104-10	88-10	72-10	56-10		
9	111-15	99-15	81-15	63-17		
10	130-21	110-21	90-21			
11	143-28	121-28	99-30			
12	156-36	132-36				
13	169-45	143-47				
14	182-55					
15	195-55					
$T(d)$	127	96	69	46	27	10

In the table a line distinguishes between reducible and non-reducible cases. Reducibility is explicitly checked by eq. (3.2). For all iGraphs defined by eq. (3.7), we see that the number of independent coefficients becomes equal to the number of independent tensor structures, when reducibility is attained. In all dimensions every $n = 2d + 2$ case is reducible with linear terms to a $n = 2d + 1$ iGraph. In four dimensions, we can decompose every integral down to integrals with 9 denominators. At this point, we are one step away from the limiting value of 8.

3.3 Comments on the number of independent coefficients

The most difficult part in our counting is always the number of independent coefficients. As shown, we do it numerically but we would like to understand more the reason why we have as many independent coefficients as we do find. We try to demonstrate here a way to estimate this number; for the case of linear terms we will give some examples.

We can rewrite the terms in eq. (3.9). As we mentioned, terms of the form $l_i \cdot p_j$ either reconstruct denominators either become ISP. Let us assume the $(4, 1, 4)$ iGraph in four dimensions. It has in principle two ISP that we call σ_1 and σ_2. To see this, we note that in any of the loop lines that consists of four denominators, there are three external momenta. We can always "borrow" a fourth one from the other line to have a complete basis and write any product $l_i \cdot p_j$ as a linear combination of the four propagators and an ISP. The ISP in that case would be the product of l_i with the momentum we borrowed. Repeating for the other loop line we get the two ISP's. Using the coefficients of eq. (3.9) in eq. (3.2) we can either get a denominator

times a constant or an ISP, or products of two denominators. We write this equation schematically as

$$1 = \sum_{i=1}^{9} D_i \{1, \sigma_1, \sigma_2\} + \sum_{i=1}^{9} \sum_{j,j \geq i}^{9} D_i D_j \{1\} \tag{3.10}$$

This means that by writing $\{1, \sigma_1, \sigma_2\} D_i$ there is a constant coefficient in front of every of these different terms:

$$\{1, \sigma_1, \sigma_2\} = a \cdot 1 + b_1 \sigma_1 + b_2 \sigma_2$$

where a, b_1, b_2 general numbers. In our particular example, we have $9 \times 3 + 45 \times 1 = 72$ coefficients. However, we started with a problem with up to cubic power in loop momenta and ended with up to quartic powers since we have these products of two denominators times some constants. These higher powers have to cancel, which means that we have to put extra constraints on our coefficients. We have 6 such constraints to cancel, namely the

$$l_1^4, \; l_1^2 l_2^2, \; l_2^4, \; (l_1 \cdot l_2) l_1^2, \; (l_1 \cdot l_2) l_2^2, \; (l_1 \cdot l_2)^2$$

terms. As a result we end up with $72 - 6 = 66$ independent coefficients, which is the number we get numerically as well. If we now try to decompose an iGraph of order 10 we can prove that eq. (3.10) becomes

$$1 = \sum_{i=1}^{10} D_i (1, \sigma_1, \sigma_2) + \sum_{i=1}^{9} \sum_{j,j \geq i}^{9} D_i D_j (1) \tag{3.11}$$

We don't need to go up to 10 in the product of 2 denominators since [9]

$$D_{10} \propto (D_1, ..., D_9, \sigma_1, \sigma_2)$$

We still need them, though, in the first term to produce terms of the type $(\sigma_i)^2$. In that case we have 69 independent coefficients and this graph is reducible. Adding more propagators we do not get more independent coefficients. In the same way one can count the independent coefficients in all dimensions although it is clear that it is safer to find their number numerically since there are a lot of overlaps in the tensor structures for higher cases. The way of rewriting the general linear terms as propagators and ISP's in the example above is still not the OPP method. In an extension of the OPP method to two loops, one would find the ISP's of every subdiagram and would avoid terms like D_i^2. We expect something similar to the one-loop case to happen then, rewriting σ_1 and σ_2 in the form of true ISP's of every subdiagram's contributions of the terms with the highest number of denominators to cancel. For that it is possible that special properties exist, as again in the one-loop case where spurious term solve a lot of equations by putting automatically tensor structures to zero.

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[9]This is actually the point where the number of dimensions plays a role in the counting.

3.4 Reduction with general quadratic terms

We try to reduce our iGraphs further with the use of general quadratic terms in the coefficients. In this case the coefficients become

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j) + \sum_{j \leq k} d_{ijk}(l_1 \cdot t_j)(l_1 \cdot t_k)$$
$$+ \sum_{j \leq k} e_{ijk}(l_2 \cdot t_j)(l_2 \cdot t_k) + \sum_{j,k} f_{ijk}(l_1 \cdot t_j)(l_2 \cdot t_k))$$

(3.12)

We give the number of tensor structures $T(d)$ and the original number of coefficients with general quadratic terms, $C_1(d)$. The coefficients depend on the number of propagators n. Notice that the expression for $T(d)$ is not valid for $d = 2$. In that case, there is more overlap between the highest tensor structures. More specifically, for this particular case one can completely reconstruct the $l_1^2 l_2^\mu l_2^\nu$ structure from $l_2^2 l_1^\mu l_1^\nu$ and $(l_1 \cdot l_2) l_1^\mu l_2^\nu$ allowing fewer independent structures to be constructed.

$$T(d) = 4d^3/3 + 10d^2 + 20d/3 - 2 \tag{3.13}$$
$$C_1(d) = (2d^2 + 3d + 1)n \tag{3.14}$$

With quadratic terms we start with $(2d^2 + 3d + 1) \times n$ coefficients in total, not all of them of course being independent. Using cancellation probing we are able to find the number of independent coefficients for different iGraphs in different dimensions. We put our findings in the following table :

n	$d = 4$	$d = 3$	$d = 2$
3	135-4	84-3	45-3
4	180-6	128-6	60-6
5	225-18	140-16	75-15
6	270-38	168-32	90-30
7	315-65	196-53	
8	360-98	224-80	
9	405-136	252-108	
10	450-180		
11	495-225		
$T(d)$	270	144	60

In the table above a line distinguishes again between reducible and non-reducible cases, according to eq. (3.2). As we see, we can decompose in 2 dimensions an iGraph of order 5 to lower order iGraphs as indicated by unitarity. However, there is no such solution in 3 and 4 dimensions and in this case we have to investigate what happens if we add cubic, quartic terms and so on. The two-dimensional case is exceptional here because of the extra properties that lower the number of tensor structures.

3.5 Reduction with general cubic terms

We focus now on 3 and 4 dimensions since we finished the reduction in 2 dimensions. We include general cubic terms and our coefficients become

$$
\begin{aligned}
x_i = a_i &+ \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j) + \sum_{j \le k} d_{ijk}(l_1 \cdot t_j)(l_1 \cdot t_k) \\
&+ \sum_{j \le k} e_{ijk}(l_2 \cdot t_j)(l_2 \cdot t_k) + \sum_{j,k} f_{ijk}(l_1 \cdot t_j)(l_2 \cdot t_k) \\
&+ \sum_{j \le k \le l} g_{ijkl}(l_1 \cdot t_j)(l_1 \cdot t_k)(l_1 \cdot t_l) + \sum_l \sum_{j \le k} h_{ijkl}(l_1 \cdot t_j)(l_1 \cdot t_k)(l_2 \cdot t_l) \\
&+ \sum_l \sum_{j \le k} p_{ijkl}(l_2 \cdot t_j)(l_2 \cdot t_k)(l_1 \cdot t_l) + \sum_{j \le k \le l} q_{ijkl}(l_2 \cdot t_j)(l_2 \cdot t_k)(l_2 \cdot t_l))
\end{aligned}
$$

$$(3.15)$$

We give the number of tensor structures $T(d)$ and the original number of coefficients with general cubic terms $C_1(d)$. The coefficients depend on the number of propagators n. The expression for $T(d)$ is valid for $d \ge 3$.

$$T(d) = 2d^4/3 + 22d^3/3 + 71d^2/6 + d/6 + 1 \tag{3.16}$$

$$C_1(d) = (4d^3/3 + 4d^2 + 11d/3 + 1)n \tag{3.17}$$

This means that in d dimensions we start with $C_1(d)$ coefficients, not all of them being independent. We run the MAPLE code in the case of the iGraph of order 7 in 3 dimensions and we find that out of 588 original coefficients, 360 are independent. This is the number of tensor structures as well. Using another PYTHON-based program[10], we can actually solve the system decomposing any iGraph of order 7 in 3 dimensions to lower iGraphs with general cubic terms, and perform the 1=1 test. This means that with cubic terms we are able to decompose any two-loop iGraph in 3 dimensions to up to a $2d$ iGraph as expected from unitarity. In the same way, we can investigate $d = 4$, and we get a valid decomposition: from our original 1485 coefficients, 831 are independent and all the tensor structures can be reconstructed. Actually we did the same in 5, 6, 7, and 8 dimensions, and managed to decompose every integral of generic order $2d + 1$ or higher, using cubic terms. We believe that this is a general result for any dimension, except of course for $d = 2$, where the same can be achieved with only quadratic terms.

4.   Conclusions

By introducing the notion of iGraphs, we have investigated the decomposability of Feynman amplitudes at one and two loops. In both cases we have demonstrated (in

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[10]http://codepad.org/zT4wUxCJ

the two-loop case, up to $d = 8$) that generic iGraphs can be decomposed down to the unitarity-based limit of $n = d$ and $n = 2d$, respectively. For one-loop graphs, the inclusion of linear terms is sufficient, and we have elucidated the relation between these linear terms and the OPP's spurious terms. At the two-loop level, ultimately cubic terms are needed (quadratic for $d = 2$), and the decomposition is seen to lead to non-scalar, non-vanishing integrals. If one wants to design a two-loop OPP method, it is clear that one has to take our general linear, quadratic, cubic terms and rewrite them in terms of propagators (this would lead to contributions with less denominators) and ISP's of each subdiagram seperately [11]. Our work is basically the starting point of an OPP method and a proof that reductions that were conjectured in [40, 41] are actually valid and survive the global $1 = 1$ test, in case one includes all relevant cuts.

We note once again that the resulting integral basis, obtained in this way, is clearly not a minimal one. We are aware of cases that could be further decomposed using IBP identities. Achieving a proper level of understanding of the interplay between OPP and IBP will certainly open the road for an efficient reduction of two-loop amplitudes at the integrand level.

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A. Appendix: Pentagons to Boxes with the OPP method

By defining

$$D_n(p_1, p_2, ..., p_n) = \frac{1}{D(q + p_1)D(q + p_2)...D(q + p_n)} \qquad \text{(A.1)}$$

we try to decompose this pentagon

$$D_5(p_1, p_2, p_3, p_4, p_5)$$

to the 5 following boxes

$$D_4(p_2, p_3, p_4, p_5), D_4(p_3, p_4, p_5, p_1), D_4(p_4, p_5, p_1, p_2)$$

$$D_4(p_5, p_1, p_2, p_3)D_4(p_1, p_2, p_3, p_4)$$

By shifting the loop momenta we can always arrange that the momenta p_i of the denominators sum up to zero:

$$\sum_{i=1}^{5} p_i = 0$$

---

[11]In the same way that one-loop OPP has different spurious terms for every subdiagram

We write the spurious terms in the following form

$$S_1 = \epsilon(q + p_2, q + p_3, q + p_4, q + p_5) = q_\mu(\epsilon^\mu(p_3, p_4, p_5) - \epsilon^\mu(p_2, p_4, p_5) +$$
$$\epsilon^\mu(p_2, p_3, p_5) - \epsilon^\mu(p_2, p_3, p_4)) + \epsilon(p_2, p_3, p_4, p_5) =$$
$$q_\mu A_1^\mu + B_1$$

$$(A.2)$$

We then have:

$$A_1^\mu = \epsilon^\mu(p_3, p_4, p_5) - \epsilon^\mu(p_2, p_4, p_5) + \epsilon^\mu(p_2, p_3, p_5) - \epsilon^\mu(p_2, p_3, p_4)$$
$$A_2^\mu = \epsilon^\mu(p_4, p_5, p_1) - \epsilon^\mu(p_3, p_5, p_1) + \epsilon^\mu(p_3, p_4, p_1) - \epsilon^\mu(p_3, p_4, p_5)$$
$$A_3^\mu = \epsilon^\mu(p_5, p_1, p_2) - \epsilon^\mu(p_4, p_1, p_2) + \epsilon^\mu(p_4, p_5, p_2) - \epsilon^\mu(p_4, p_5, p_1)$$
$$A_4^\mu = \epsilon^\mu(p_1, p_2, p_3) - \epsilon^\mu(p_5, p_2, p_3) + \epsilon^\mu(p_5, p_1, p_3) - \epsilon^\mu(p_5, p_1, p_2)$$
$$A_5^\mu = \epsilon^\mu(p_2, p_3, p_4) - \epsilon^\mu(p_1, p_3, p_4) + \epsilon^\mu(p_1, p_2, p_4) - \epsilon^\mu(p_1, p_2, p_3)$$

$$(A.3)$$

Using the OPP master formula [18] we get

$$1 = [d_1 + \tilde{d}_1 S_1]D(q + p_1) + [d_2 + \tilde{d}_2 S_2]D(q + p_2) + ... +$$
$$[d_5 + \tilde{d}_5 S_5]D(q + p_5)$$

$$(A.4)$$

We compare the polynomials of the two sides of eq. (A.4). The $q^2 q_\mu$ terms must vanish for any value of q which means

$$\sum_i (\tilde{d}_i A_i^\mu) = 0 \qquad (A.5)$$

Notice that from eq. (A.3)

$$\sum_{i=1}^{5} A_i = 0$$

which means that if all $\tilde{d}_i$ are equal eq. (A.5) is satisfied.
This is actually the only solution. To see that, consider the sum

$$\tilde{d}_1 A_1^\mu + \cdots + \tilde{d}_5 A_5^\mu$$

Substituting

$$p_5 = -p_1 - p_2 - p_3 - p_4$$

we get

$$\sum(\tilde{d}_i A_i^\mu) = (\tilde{d}_1 + \tilde{d}_2 - 4\tilde{d}_3 + \tilde{d}_4 + \tilde{d}_5)\epsilon^\mu(p_1, p_2, p_4)$$
$$+ (-\tilde{d}_1 + 4\tilde{d}_2 - \tilde{d}_3 - \tilde{d}_4 - \tilde{d}_5)\epsilon^\mu(p_1, p_3, p_4)$$
$$+ (-\tilde{d}_1 - \tilde{d}_2 - \tilde{d}_3 + 4\tilde{d}_4 - \tilde{d}_5)\epsilon^\mu(p_1, p_2, p_3)$$
$$+ (-4\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4 + \tilde{d}_5)\epsilon^\mu(p_2, p_3, p_4)$$

$$(A.6)$$

In 4 dimensions, for this sum to be zero, all coefficients in front of the vectors are zero (the vectors are linearly independent). This results in the system:

$$\tilde{d}_1 + \tilde{d}_2 - 4\tilde{d}_3 + \tilde{d}_4 + \tilde{d}_5 = 0$$
$$-\tilde{d}_1 + 4\tilde{d}_2 - \tilde{d}_3 - \tilde{d}_4 - \tilde{d}_5 = 0$$
$$-\tilde{d}_1 - \tilde{d}_2 - \tilde{d}_3 + 4\tilde{d}_4 - \tilde{d}_5 = 0$$
$$-4\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4 + \tilde{d}_5 = 0$$

$$(A.7)$$

One can solve the system and find

$$\tilde{d}_1 = \tilde{d}_2 = \tilde{d}_3 = \tilde{d}_4 = \tilde{d}_5 \tag{A.8}$$

The only solution is when all coefficients are equal and from now on we call them $\tilde{d}$. Then we take a look at the quadratic in q parts of the right-hand side of eq. (A.4). They cancel as well but one has to be careful since they come from two terms, the q^2 and the $q^\mu q^\nu$ term. We look at the latter term

$$\tilde{d}q_\mu \sum_{i=1}^{5}(q \cdot p_i)A_i^\mu$$

We have :

$$\tilde{d}q_\mu \sum_{i=1}^{5}(q \cdot p_i)A_i^\mu = \tilde{d}q_\mu((A_1 - A_5)(q \cdot p_1) +$$
$$(A_2 - A_5)(q \cdot p_2) + (A_3 - A_5)(q \cdot p_3) + (A_4 - A_5)(q \cdot p_4))$$

$$(A.9)$$

where $p_5 = -p_1 - p_2 - p_3 - p_4$ is again used. With this substitution we have:

$$
\begin{aligned}
A_1 - A_5 &= -5\epsilon^\mu(p_2, p_3, p_4) \\
A_2 - A_5 &= 5\epsilon^\mu(p_1, p_3, p_4) \\
A_3 - A_5 &= -5\epsilon^\mu(p_1, p_2, p_4) \\
A_4 - A_5 &= 5\epsilon^\mu(p_1, p_2, p_3)
\end{aligned}
$$

$$(A.10)$$

Inserting the Schouten Identity

$$
\epsilon(p_1, p_2, p_3, p_4)q^\mu = \epsilon^\mu(p_2, p_3, p_4)(q \cdot p_1) - \epsilon^\mu(p_1, p_3, p_4)(q \cdot p_2) + \\
\epsilon^\mu(p_1, p_2, p_4)(q \cdot p_3) - \epsilon^\mu(p_1, p_2, p_3)(q \cdot p_4)
$$

$$(A.11)$$

we get

$$
\tilde{d}q_\mu \sum_{i=1}^5 (q \cdot p_i) A_i^\mu = -5\tilde{d}q^2 \epsilon(p_1, p_2, p_3, p_4) \tag{A.12}
$$

That is exactly the property that the spurious terms have that makes the solution to the system possible. The $q^\mu q^\nu$ terms all vanish owing to the Schouten identity except for the trace part proportional to q^2 solving 9 out of 10 equations in one go. The total number of nontrivial equations is therefore 10 and not 19 in this case and the system has a solution. To complete the story, taking into account eq. (A.8) we are left with 5 d_i and one $\tilde{d}$, which are uniquely now determined from the remaining 6 equations, namely constant part, q^2 term and q_μ term.

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