Logical Methods in Computer Science
Vol. 14(2:14)2018, pp. 1–20
https://lmcs.episciences.org/

Submitted Mar. 28, 2017
Published Jun. 19, 2018

COMPUTABILITY IN BASIC QUANTUM MECHANICS

EIKE NEUMANN, MARTIN PAPE, AND THOMAS STREICHER

Aston University, School of Engineering and Applied Science, Aston Triangle, Birmingham B4 7ET, UK
e-mail address: neumaef1@aston.ac.uk

Kallinchener Str. 2B, Berlin D-15749, Germany
e-mail address: martin.pape@gmx.de

Fachbereich 4 Mathematik TU Darmstadt, Schloßgartenstr. 7, D-64289, Germany
e-mail address: streicher@mathematik.tu-darmstadt.de

Dedicated to Jiří Adámek on the occasion of his retirement

Abstract. The basic notions of quantum mechanics are formulated in terms of separable infinite dimensional Hilbert space $\mathcal{H}$. In terms of the Hilbert lattice $\mathcal{L}$ of closed linear subspaces of $\mathcal{H}$ the notions of *state* and *observable* can be formulated as kinds of measures as in [21]. The aim of this paper is to show that there is a good notion of computability for these data structures in the sense of Weihrauch's Type Two Effectivity (TTE) [26].

Instead of explicitly exhibiting admissible representations for the data types under consideration we show that they do live within the category $\mathbf{QCB}_0$ which is equivalent to the category **AdmRep** of admissible representations and continuously realizable maps between them. For this purpose in case of observables we have to replace measures by *valuations* which allows us to prove an effective version of von Neumann's Spectral Theorem.

1. Introduction

In his legendary book [17] from 1932 J. von Neumann gave a mathematical formulation of basic quantum mechanics based on separable Hilbert space $\mathcal{H}$ which may manifest itself as ℓ^2 as in Heisenberg's *matrix mechanics* or L^2 as in Schrödinger's *wave mechanics*. In this setting *observables* appear as *self adjoint operators* on $\mathcal{H}$ and *states* as particular observables, namely so-called *density operators* which are self adjoint operators $D \geq 0$ with $\mathsf{tr}(D) = 1$. The latter are closed under countable convex combinations. Those states which cannot be obtained as non-trivial countable convex combinations are called *pure* and correspond to 1-dimensional subspaces of $\mathcal{H}$.

It is *a priori* not clear why observables should be understood as self adjoint operators on Hilbert space. But this mystery is explained by von Neumann's famous *Spectral Theorem* already proved in [17] which establishes a 1-1-correspondence between self adjoint operators on $\mathcal{H}$ and *projector valued measures* on $\mathcal{H}$, i.e. measures on $\mathbb{R}$ taking values not in the unit interval $\mathbb{I} = [0, 1]$ but in the so-called *Hilbert lattice* $\mathcal{L}$ of closed linear subspaces of $\mathcal{H}$ which classically correspond to projectors, i.e. self adjoint operators $P = P^2$ on $\mathcal{H}$. A projector valued measure $o : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}$ together with a pure state as given by a unit vectors x in $\mathcal{H}$

gives rise to an ordinary probability measure $s_x \circ o$ where $s_x(P) = \langle x|Px\rangle$. This is explained in detail in the book [21] where it is also shown that states may be understood as measures on the Hilbert lattice $\mathcal{L}$. More details will be given subsequently in subsections 2.3 and 2.4, respectively.

In most physics textbooks one finds only the functional analytical account where observables are self adjoint operators because it is based on more traditional mathematics and more useful for (symbolic) computation (done by hand). There is a vast literature on the so-called "logico-algebraic" account based on the Hilbert lattice $\mathcal{L}$. It goes back to old work of Birkhoff and von Neumann where they proposed to consider $\mathcal{L}$ as a kind of "quantum logic". But since the lattice $\mathcal{L}$ is not distributive it interprets neither intuitionistic nor classical logic. In our paper we will not misuse $\mathcal{L}$ for logical purposes but rather as a tool for presenting a more algebraic and conceptual account of basic quantum mechanics as in [21] (despite its title).

The functional analytic formulation has already been studied in the framework of Type Two Effectivity [6, 25]. To our knowledge computability for the "logico-algebraic" approach has not been considered so far in the literature. At first sight this seems to be impossible since constructively they are not equivalent because projectors on separable infinite dimensional Hilbert space correspond to *located* closed linear subspaces and these are not even closed under binary intersection, see [9]. Nevertheless, we will show that the basic notions of the "logico-algebraic approach" can be endowed with an appropriate notion of computability in the sense of [26]. The key idea is to identify $\mathcal{L}$ not as a $\neg\neg$-subobject of $\mathfrak{B}(\mathcal{H})$ but as a $\neg\neg$-subobject of $\Sigma^{\mathcal{H}'}$ where Σ is the Sierpiński space and $\mathcal{H}'$ is the dual space of $\mathcal{H}$. Classically, the space $\mathcal{H}'$ is anti-isomorphic to $\mathcal{H}$ which, however, is not the case computationally. Rather it turns out that the topology on $\mathcal{H}'$ induced by computability is the sequentialization of the weak* topology.

In K. Weihrauch's book [26] one finds a theory of computability for classical spaces based on Turing machines with infinite input and output tapes. Based on [24], Bauer and Lietz have shown in [3, 4, 15] that computable analysis can be rephrased in terms of constructive analysis inside the *function realizability topos* $\mathbf{RT}(\mathcal{K}_2)$ or rather its restriction to effective morphisms, the so called *Kleene-Vesley topos* $\mathcal{KV}$, as described in [19]. In [2] it has been shown how to characterize abstractly within $\mathbf{RT}(\mathcal{K}_2)$ the category $\mathbf{AdmRep}$ of *admissible representations of spaces and continuous(ly realizable) maps* between them which forms the backbone of K. Weihrauch's account in [26]. Analogously, by restricting to $\mathcal{KV}$ one obtains the category $\mathbf{AdmRep}_{\mathsf{eff}}$ of *admissible representations of spaces and effectively realizable maps* between them since effectively realizable is equivalent to the existence of a Turing machine with infinite tapes performing the respective transformation of infinite sequences of natural numbers.

As shown in [2] the category $\mathbf{AdmRep}$ is equivalent to a (fairly) small full subcategory $\mathbf{QCB}_0$ of the category $\mathbf{Sp}$ of topological spaces and continuous maps, namely the one on T_0 quotients of countably based T_0 spaces. The equivalence of $\mathbf{QCB}_0$ and $\mathbf{AdmRep}$ is essentially due to the fact that all countably based T_0 spaces appear as quotients of subspaces of Baire space whose elements are used in [26] for representing elements of more abstract spaces.

This category $\mathbf{QCB}_0$ and thus also $\mathbf{AdmRep}$ has excellent categorical closure properties. In particular, it is *cartesian closed* and closed under regular, i.e. classical, subobjects. Within $\mathbf{AdmRep} \simeq \mathbf{QCB}_0$ one finds all complete separable metric spaces and, accordingly, it is a natural place for the Hilbert space approach to Quantum Mechanics as introduced by von Neumann in [17].

In our account, however, we provide a notion of computability for the more algebraic approach based on the Hilbert lattice $\mathcal{L}$ as described in [21]. Due to the closure properties of $\mathbf{AdmRep} \simeq \mathbf{QCB}_0$ and the fact that it hosts the Sierpiński space Σ and Hilbert space $\mathcal{H}$ it also hosts its dual $\mathcal{H}'$ and the classical subobject $\mathcal{L}$ of $\Sigma^{\mathcal{H}'}$ on closed subspaces of $\mathcal{H}'$. Notice that a closed linear subspace P of $\mathcal{H}'$ is represented by the continuous map $p \in \Sigma^{\mathcal{H}'}$ with $P = p^{-1}(\perp)$, i.e. somewhat surprisingly $\perp \in \Sigma$ plays the role of "true". As a consequence the natural order induced by Σ on $\mathcal{L}$ is **opposite** to subset inclusion as considered usually.

Let $\mathcal{I}$ be the unit interval $[0,1]$ with the *upper topology*, i.e. the Scott topology on the continuous lattice $([0,1], \geq)$. In $\mathbf{AdmRep} \simeq \mathbf{QCB}_0$ we will identify the space St of quantum **states** as the $\neg\neg$-subobject of $\mathcal{I}^{\mathcal{L}}$ consisting of those s which validate the conditions

(S1) $s(0) = 0$ and $s(\mathcal{H}) = 1$

(S2) $s(P \vee Q) = s(P) + s(Q)$ whenever $P \perp Q$

since s is continuous and thus preserves infima of decreasing ω-chains.[1]

By the spectral theorem for self-adjoint operators on $\mathcal{H}$ quantum **observables** correspond to *projection valued measures* on $\mathbb{R}$, i.e. certain maps from the set $\mathfrak{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ to $\mathcal{L}$. But since $\mathfrak{B}(\mathbb{R})$ does not live within $\mathbf{AdmRep} \simeq \mathbf{QCB}_0$ we have to restrict to a generating subcollection. It turns out that the object $\mathcal{C}(\mathbb{R})$ of closed subsets of $\mathbb{R}$ is a good choice for this purpose since observables can be characterized as those $\nu \in \mathcal{L}^{\mathcal{C}(\mathbb{R})}$ for which the map $\lambda C \in \mathcal{C}(\mathbb{R}).\langle x|\nu(C)x\rangle$ is a probability valuation on $\mathbb{R}$ for all unit vectors $x \in \mathcal{H}$.

Based on this reformulation of observables we will prove that the Spectral Theorem for bounded observables is effective in the sense that it holds in $\mathcal{KV}$. It will turn out that the induced topology on the operator side is the sequentialization of the strong operator topology and that a sequence (ν_n) of observables converges to ν_∞ w.r.t. the induced topology iff for all unit vectors the associated measures converge in the sense usually considered in the respective literature [5].

2. Basic Quantum Mechanics

We briefly recall how basic quantum mechanics can be formulated in terms of separable infinite dimensional complex Hilbert space $\mathcal{H}$ (see e.g. [16, 20] for background information) as pioneered in J. von Neumann's book from 1932 [17].

2.1. **Basics Facts about Hilbert Space.** Up to isomorphism there is just one separable infinite dimensional Hilbert space $\mathcal{H}$ over the field $\mathbb{C}$ of complex numbers, namely the space ℓ^2 of sequences x of complex numbers such that $\sum |x_n|^2$ converges. Addition and scalar multiplication is pointwise and the scalar product is given by $\langle x|y\rangle = \sum x_n^* y_n$ with x_n^* the complex conjugate of x_n. It is known to be a Banach space w.r.t. the norm $\|x\| = \sqrt{\langle x|x\rangle}$. There is a canonical countable orthonormal basis (e_n) for ℓ^2 where the n-th component of e_n is 1 and all other components are 0. We have $\langle e_n|e_n\rangle = 1$ and $\langle e_n|e_m\rangle = 0$ for $n \neq m$ and $x = \sum\limits_{n=0}^{\infty} \langle e_n|x\rangle e_n$ for all $x \in \mathcal{H}$.

[1] As usual $P \perp Q$ stands for $\forall x \in P. \forall y \in Q.\, \langle x|y\rangle = 0$.

We recall for later use that the *weak topology* on $\mathcal{H}$ is the coarsest topology for which every linear functional of the form $y \mapsto \langle x|y\rangle$ is continuous. The weak topology on $\mathcal{H}$ is known to be Hausdorff and the unit ball $B(\mathcal{H}) := \{x \in \mathcal{H} \mid \|x\| \leq 1\}$ is compact w.r.t. the weak topology but not w.r.t. the norm topology. Moreover, subspaces of $\mathcal{H}$ are closed w.r.t. the norm topology iff they are closed w.r.t. the weak topology on $\mathcal{H}$. Notice, moreover, that $\mathcal{H}$ with the weak topology is isomorphic to the dual space $\mathcal{H}'$ with the weak* topology, i.e. the coarsest topology rendering continuous all evaluation maps $\mathcal{H}' \to \mathbb{C} : f \mapsto f(x)$ for $x \in \mathcal{H}$.

We write $\mathcal{B}(\mathcal{H})$ for the space of *bounded linear operators* on $\mathcal{H}$. An $A \in \mathcal{B}(\mathcal{H})$ is called *self-adjoint* iff $\langle Ax|y\rangle = \langle x|Ay\rangle$ for all $x, y \in \mathcal{H}$. Such an A is called *positive* iff $\langle x|Ax\rangle \geq 0$ for all $x \in \mathcal{H}$ and it is called an *effect* iff $0 \leq \langle x|Ax\rangle \leq 1$ for all $x \in \mathcal{H}$ with $\|x\| = 1$. A *projector* is a self-adjoint $P \in \mathcal{B}(\mathcal{H})$ which, moreover, is idempotent, i.e. $PP = P$. As is well known projectors correspond to closed linear subspaces of $\mathcal{H}$ where a projector P maps $x \in \mathcal{H}$ to the best approximating element $P(x)$ of the closed subspace of $\mathcal{H}$ as given by fixpoints of P.

The trace $\mathsf{tr}(A)$ of a positive self-adjoint $A \in \mathcal{B}(\mathcal{H})$ is $\sum_n \langle e_n|Ae_n\rangle$ which exists iff this sum is bounded. Notice that $\mathsf{tr}(A)$ is independent from the choice of the orthonormal basis. A positive self-adjoint operator with trace 1 is called a *density operator*. Positive self-adjoint operators with trace ≤ 1 are often called *partial states*.

2.2. **Hilbert Lattice.** The *Hilbert lattice* $\mathcal{L}$ consists of the closed linear subspaces of $\mathcal{H}$ ordered by subset inclusion. The poset $\mathcal{L}$ is a lattice where meets are given by intersections and joins are given by closures of linear spans of unions. The bottom element of $\mathcal{L}$ is the zero subspace 0 of $\mathcal{H}$ whereas the top element of $\mathcal{L}$ is $\mathcal{H}$. Classically, we may identify a closed linear subspace of $\mathcal{H}$ with the corresponding projector of $\mathcal{H}$ on this subspace.

Notably, the Hilbert lattice $\mathcal{L}$ is *not* distributive and thus neither boolean nor a complete Heyting algebra. Nevertheless, for every $P \in \mathcal{L}$ we may consider its *orthocomplement*

$$P^\perp = \{x \in \mathcal{H} \mid \forall y \in P.\, \langle x|y\rangle = 0\}$$

which again is an element of $\mathcal{L}$. Notice that orthocomplementation $(\cdot)^\perp : \mathcal{L} \to \mathcal{L}$ reverses the order, i.e. $Q^\perp \subseteq P^\perp$ whenever $P \subseteq Q$, and is involutory in the sense that $P^{\perp\perp} = P$. We write $P \perp Q$ for $P \subseteq Q^\perp$ stating that all vectors in P are orthogonal to all vectors in Q. Notice that we always have $P \vee P^\perp = \mathcal{H}$ and $P \wedge P^\perp = 0$ though typically there will be many different $Q \in \mathcal{L}$ with $P \vee Q = \mathcal{H}$ and $P \wedge Q = 0$ in contrast to boolean algebras where such complements are unique. A distinguishing property of $\mathcal{L}$ is the law of *orthomodularity* stating that

$$Q = P \vee (Q \wedge P^\perp)$$

for $P \subseteq Q$.

It is well known, see e.g. [13], that subspaces of $\mathcal{H}$ are closed w.r.t. the norm topology iff they are closed w.r.t. the weak topology. Thus, we may identify $\mathcal{L}$ with closed linear subspaces of $\mathcal{H}'$ endowed with the weak* topology which is homeomorphic to $\mathcal{H}$ with the weak topology.

2.3. **Quantum States.** We recall the basic notions of quantum mechanics as can be found in the classical text [16] though we essentially follow the equivalent presentation of [21].

A (quantum) *state* is a function s from $\mathcal{L}$ to the unit interval $\mathbb{I} = [0, 1]$ satisfying the conditions

(s1) $s(0) = 0$ and $s(\mathcal{H}) = 1$
(s2) $s(\bigvee_n P_n) = \sum_n s(P_n)$ whenever $P_n \perp P_m$ for $n \neq m$.

Thus, one may think of a state as a kind of probability measure on $\mathcal{L}$ where disjointness is replaced by orthogonality.

Every density operator D on $\mathcal{H}$ induces a state

$$s(P) = \mathsf{tr}(DP)$$

where P on the right hand side refers to the corresponding projector on $\mathcal{H}$. By the famous *Gleason's Theorem* (see e.g. [11, 21]) this establishes a 1-1-correspondence between states and density operators.

A state s on $\mathcal{L}$ is called *pure* iff there is a unit vector $x \in \mathcal{H}$ with

$$s(P) = s_x(P) = \langle x|Px\rangle$$

for all $P \in \mathcal{L}$. One can show that

Proposition 2.1. *Every state s can be written as $\sum\limits_{n=0}^{\infty} \lambda_n s_{b_n}$ where the $\lambda_n \in \mathbb{I}$ with $\sum \lambda_n = 1$ and (b_n) is an orthonormal basis for $\mathcal{H}$.*

2.4. **Quantum Observables.** A (quantum) *observable* is a function o from the set $\mathfrak{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ to $\mathcal{L}$ such that

(o1) $o(\emptyset) = 0$ and $o(\mathbb{R}) = \mathcal{H}$
(o2) $o(\bigcup_n X_n) = \bigvee_n o(X_n)$
(o3) $o(X) \perp o(Y)$ whenever $X \cap Y = \emptyset$

i.e. o is a *projector valued measure* (when identifying elements of $\mathcal{L}$ with projectors).

By the famous von Neumann Spectral Theorem [17, 20] bounded self-adjoint operators A on $\mathcal{H}$ correspond to observables o which are bounded in the sense that $o([x, y]) = \mathcal{H}$ for some $x \leq y$ in $\mathbb{R}$ via

$$\langle x|Ay\rangle = \int_{\mathbb{R}} \lambda \; d\langle x|o((-\infty, \lambda))y\rangle$$

making use of the fact that $X \mapsto \langle x|o(X)(y)\rangle$ is a $\mathbb{C}$-valued measure on $\mathbb{R}$ (see e.g. [20].

Notice that an observable $o : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}$ composed with state $s : \mathcal{L} \to \mathbb{I}$ gives rise to a probability measure $s \circ o : \mathfrak{B}(\mathbb{R}) \to \mathbb{I}$ on $\mathbb{R}$.

2.5. **Alternative Characterizations of States and Observables.** For later use in section 4 it is useful to consider the following alternative characterizations of states and observables.

Proposition 2.2. *A map $s : \mathcal{L} \to \mathbb{I}$ is a state iff it satisfies the conditions*

(S1) $s(0) = 0$ *and* $s(\mathcal{H}) = 1$
(S2) *s preserves infima of decreasing ω-chains*
(S3) $s(P \vee Q) = s(P) + s(Q)$ *whenever* $P \perp Q$.

Proof. It is well known that states validate the condition (S1)–(S3) as shown e.g. in [21].

For the reverse direction suppose $s : \mathcal{L} \to \mathbb{I}$ validates conditions (S1)–(S3). Condition (s1) holds since it is the same as (S1). From this together with (S3) and $P \vee P^\perp = \mathcal{H}$ for all $P \in \mathcal{L}$, it is immediate that $s(P^\perp) = 1 - s(P)$. For this reason from (S2) it follows that s preserves suprema of increasing ω-chains. For showing $(s2)$ suppose (P_k) is a pairwise orthogonal sequence in $\mathcal{L}$. Let $Q_n = \bigvee_{k=0}^n P_k$. Then (Q_n) is an increasing ω-chain in $\mathcal{L}$ with $\bigvee P_k = \bigvee Q_n$. Thus, we have

$$\begin{aligned} s(\textstyle\bigvee_k P_k) &= s(\textstyle\bigvee_n Q_n) \\ &= \sup_n s(Q_n) \\ &= \sup_n s(\textstyle\bigvee_{k=0}^n P_k) \quad \text{(S3)} \\ &= \sup_n \textstyle\sum_{k=0}^n s(P_k) \\ &= \textstyle\sum_k s(P_k) \end{aligned}$$

showing that s satisfies (s2) and thus is a state. □

For well behaved spaces X like $\mathbb{R}$ probability measures on X are uniquely determined by their restrictions to $\mathcal{C}(X)$, the set of closed subsets of X, which forms a(n ω-)cpo w.r.t. $\supseteq$, see [10]. These restrictions of probability measures on X can be characterized as *valuations*, i.e. Scott continuous maps ν from $\mathcal{C}(X)$ to $\mathcal{I}$, the unit interval $\mathbb{I}$ ordered by $\geq$, satisfying $\nu(\emptyset) = 0$, $\nu(\mathbb{R}) = 1$ and

$$\nu(A) + \nu(B) = \nu(A \cup B) + \nu(A \cap B)$$

for $A, B \in \mathcal{C}(X)$. See [10] for more information on valuations though formulated there in terms of open instead of closed subsets of X.

We will now characterize quantum observables in terms of valuations.

Proposition 2.3. *Quantum observables correspond by restriction to $\mathcal{C}(\mathbb{R})$ to* quantum valuations, *i.e. maps $\nu : \mathcal{C}(\mathbb{R}) \to \mathcal{L}$ such that*

(O1) $\nu(\emptyset) = 0$ *and* $\nu(\mathbb{R}) = \mathcal{H}$
(O2) ν *preserves infima of decreasing ω-chains*
(O3) *for every unit vector x in $\mathcal{H}$ and $A, B \in \mathcal{C}(\mathbb{R})$*

$$\nu_x(A) + \nu_x(B) = \nu_x(A \cup B) + \nu_x(A \cap B)$$

where $\nu_x(C) = \langle x|\nu(C)(x)\rangle$ for $C \in \mathcal{C}(\mathbb{R})$.

Proof. Let $o : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}$ be an observable and ν its restriction to $\mathcal{C}(\mathbb{R})$. Condition (O1) for ν is immediate from condition (o1) for o. As shown in [21] o preserves infima of decreasing ω-chains from which (O2) is immediate. For every unit vector x in $\mathcal{H}$ the function $o_x = s_x \circ o$ is a measure from which (O3) is immediate.

Suppose $\nu : \mathcal{C}(\mathbb{R}) \to \mathcal{L}$ validates conditions (O1)-(O3). Then $E : \mathbb{R} \to \mathcal{L} : \lambda \mapsto 1 - \nu([\lambda, \infty))$ is a spectral family in the sense of [20] which as shown in *loc.cit.* uniquely extends to an observable $o : \mathfrak{B}(\mathbb{R}) \to \mathcal{L}$ with $o((-\infty, \lambda)) = E(\lambda)$. □

3. Topological Domain Theory

The Hilbert lattice $\mathcal{L}$ is complete and thus in particular a directed complete poset as studied in denotational semantics (see e.g. [10, 23]). However, we want to arrive at a notion of computability for the Hilbert lattice and derived notions such as states and observables and this is not possible for arbitrary directed complete posets or complete lattices. The first idea would be to exhibit $\mathcal{L}$ as an *effectively given domain* as described e.g. in [23]. But for this purpose $\mathcal{L}$ would have to be at least a *continuous* lattice in the sense of [10] which, alas, is not the case as has been pointed out to us by K. Keimel.

Proposition 3.1. *The Hilbert lattice $\mathcal{L}$ is not continuous.*

Proof. Suppose $\mathcal{L}$ were a continuous lattice. Then every atom a of $\mathcal{L}$ were compact. But there is an atom a in $\mathcal{L}$ such that for no n we have $a \leq \bigvee_{i=0}^{n} \langle e_n \rangle$ (where $\langle e_n \rangle$ is the one dimensional subspace spanned by e_n). But $a \leq \mathcal{H} = \bigvee_{n=0}^{\infty} \langle e_n \rangle$ and thus a is not compact. $\square$

Due to this shortcoming we will instead work in the framework of *topological domain theory* as described in [2] which subsumes both countably based continuous domains and complete separable metric spaces.

The basic idea of topological domain theory is to identify an appropriate full subcategory of the *function realizability topos* $\mathbf{RT}(\mathcal{K}_2)$ (as described e.g. in [19]) which is equivalent to the category **AdmRep** of *admissible representations* and *continuously realizable* maps between them.

3.1. **Admissible Representations.** Admissible representations are the basic structures underlying Weihrauch's *Type Two Effectivity* (TTE) as described in [26]. We briefly recall some basic notions.

The set of all functions from $\mathbb{N}$ to $\mathbb{N}$ endowed with the *initial segment* topology is commonly called *Baire space* for which we write $\mathbb{B}$. A *representation* of a topological T_0 space X is a *quotient* map ρ from a subspace B of $\mathbb{B}$ to X. For representations $\rho : B \to X$ and $\rho' : B' \to X'$ a function $f : X \to X'$ is called *continuously realizable* iff there exists a continuous function $\phi : B \to B'$ making the diagram

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "\rho"'] & B' \arrow[d, "\rho'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

commute. A representation $\rho : B \to X$ is called *admissible* iff for every continuous map f from a subspace B' of $\mathbb{B}$ to X there is a continuous map $\phi : B' \to B$ rendering the triangle

$$\begin{tikzcd} B' \arrow[r, "\phi"] \arrow[dr, "f"'] & B \arrow[d, "\rho"] \\ & X \end{tikzcd}$$

commutative. It is easy to see that for admissible representations $\rho : B \to X$ and $\rho' : B' \to X'$ a map $f : X \to X'$ is continuous iff it is continuously realizable as a map from ρ to ρ'. We write **AdmRep** for the ensuing category of admissible representations and continuous(ly realizable) maps between them.

We recall from [2] that complete separable metric spaces and countably based continuous domains form full subcategories of **AdmRep**.

3.2. **$\mathbf{QCB}_0$ Spaces.** As discussed in [2] the category **AdmRep** is equivalent to the following subcategory of the category of topological spaces and continuous maps.

Definition 3.2. A $\mathbf{QCB}_0$ *space* is a T_0-quotient of a countably based topological space. We write $\mathbf{QCB}_0$ for the ensuing category of $\mathbf{QCB}_0$ spaces and continuous maps between them.

The $\mathbf{QCB}_0$ spaces are precisely those topological T_0 spaces which admit an admissible representation. Moreover, as shown in [2](4.10) the category $\mathbf{QCB}_0$ is cartesian closed, countably complete and countably cocomplete.

Let Σ be the Sierpiński space $\{\bot, \top\}$ whose only nontrivial open set is $\{\top\}$. Obviously, continuous maps from X to Σ correspond to open subsets of X. For further reference we recall the following useful fact from [1].

Proposition 3.3. *For every $\mathbf{QCB}_0$ space X the exponential Σ^X in $\mathbf{QCB}_0$ is isomorphic to the space $\mathcal{O}(X)$ of open subsets of X endowed with the Scott topology arising from the subset ordering $\subseteq$.*

Proof. First of all the elements of Σ^X are the open subsets of X and the information ordering corresponds to $\subseteq$. Thus it suffices to show that every Scott closed subset C of Σ^X is closed w.r.t. the topology of Σ^X.

For this purpose we recall the following result. Let N_∞ be the one point compactification of $\mathbb{N}$ which is countably based and thus in $\mathbf{QCB}_0$. One can show that the map $\prod : \Sigma^{N_\infty} \to \Sigma : p \mapsto \bigwedge_{n\in\mathbb{N}} p_n$ is continuous.

Now suppose $p : N_\infty \to \Sigma^X$ with $p_n \in C$ for all $n \in \mathbb{N}$. Consider $q_n : X \to \Sigma : x \mapsto \bigwedge_{k\in\mathbb{N}} p_{n+k}(x)$ which is continuous since $\prod : \Sigma^{N_\infty} \to \Sigma$ is continuous. Obviously, we have $q_n \sqsubseteq p_n$ and thus $q_n \in C$ and $p_\infty = \bigsqcup_{n\in\mathbb{N}} q_n$. Thus $p_\infty \in C$ since C is Scott closed. □

On every topological space X we may consider the *specialization order*

$$x \sqsubseteq_X y \;\equiv\; \forall O \in \mathcal{O}(X).\, x \in O \implies y \in O$$

which allows one to define the following notions.

Definition 3.4. A *topological predomain* is a $\mathbf{QCB}_0$ space X where every ascending ω-chain (x_n) (w.r.t. $\sqsubseteq_X$) has a least upper bound x_∞. We write **TP** for the *category of topological predomains* which is a full subcategory of $\mathbf{QCB}_0$.

A *topological domain* is a topological predomain X which has a least element $\bot_X$ w.r.t. $\sqsubseteq_X$. We write **TD** for the ensuing *category of topological domains.*

One can show that

Proposition 3.5. *Every continuous function between topological predomains preserves suprema of ascending ω-chains.*

Furthermore as shown in [2] it holds that

Proposition 3.6. *The category* **TP** *is a full reflective exponential ideal of* $\mathbf{QCB}_0$.

Proposition 3.7. *The category* **TD** *is an exponential ideal of* $\mathbf{QCB}_0$ *and is closed under countable products in* $\mathbf{QCB}_0$.

3.3. **AdmRep within $\mathbf{RT}(\mathcal{K}_2)$.** Another important aspect of **AdmRep** is that it appears as a full reflective subcategory of the function realizability topos $\mathbf{RT}(\mathcal{K}_2)$ as described e.g. in [19].

The underlying set of the "second Kleene algebra" $\mathcal{K}_2$ is Baire space. For $\alpha, \beta \in \mathbb{B}$ we define $\alpha|\beta \simeq n$ iff $\alpha(\bar{\beta}(k)) = n + 1$ and $\alpha(\bar{\beta}(\ell)) = 0$ for $\ell < k$.[2] The partial application operation of $\mathcal{K}_2$ is defined as

$$\alpha\beta \simeq \gamma \iff \forall n \in \mathbb{N}.\ \alpha|(\langle n\rangle * \beta) = \gamma_n$$

where $*$ stands for concatenation of finite sequences with arbitrary sequences.

See [19] for the definition of the category $\mathbf{Asm}(\mathcal{K}_2)$ of *assemblies* which is equivalent to the full subcategory of $\mathbf{RT}(\mathcal{K}_2)$ on $\neg\neg$-separated objects and its full subcategory $\mathbf{Mod}(\mathcal{K}_2)$ of *modest sets*. Recall that modest sets are quotients of N^N in $\mathbf{RT}(\mathcal{K}_2)$ w.r.t. $\neg\neg$-closed partial equivalence where N is the natural numbers object of $\mathbf{RT}(\mathcal{K}_2)$.

Definition 3.8. An object X in $\mathbf{RT}(\mathcal{K}_2)$ is called Σ-*extensional* if the map

$$\eta_X : X \to \Sigma^{\Sigma^X} : x \mapsto \lambda p.p(x)$$

is a regular, i.e. $\neg\neg$-closed, monomorphism. We write $\mathbf{Mod}_\Sigma(\mathcal{K}_2)$ for the full subcategory of $\mathbf{RT}(\mathcal{K}_2)$ on Σ-extensional objects.

Theorem 6.1.9 of [1] guarantees that

Proposition 3.9.
(1) *The category* $\mathbf{Mod}_\Sigma(\mathcal{K}_2)$ *is equivalent to the category* $\mathbf{QCB}_0$.
(2) $\mathbf{Mod}_\Sigma(\mathcal{K}_2)$ *is an exponential ideal in* $\mathbf{RT}(\mathcal{K}_2)$.
(3) *Up to isomorphism the objects of* $\mathbf{Mod}_\Sigma(\mathcal{K}_2)$ *are the* $\neg\neg$*-subobjects of powers of* Σ.

4. Computable Basic Quantum Mechanics

The aim of this main section is to identify the Hilbert lattice $\mathcal{L}$, the type St of quantum states and the type Obs of quantum observables as objects of $\mathbf{AdmRep} \simeq \mathbf{QCB}_0$. This will induce a notion of computability on $\mathcal{L}$, St and Obs and suggest topologies on the respective sets which to our knowledge have not been considered so far in the literature on mathematical foundations of basic quantum mechanics. Of course, the sets $\mathcal{L}$, St and Obs can all be identified with particular subsets of $\mathfrak{B}(\mathcal{H})$ which itself can be endowed with the various different topologies as considered in (linear) functional analysis. We will discuss how these topologies relate to the ones induced by admissible representations.

4.1. **Separable Bananch Spaces within AdmRep.** As is well known from e.g. [26] all complete separable metric spaces can be endowed with admissible representations. This applies in particular to $\mathbb{R}$, $\mathbb{C}$ and separable Banach spaces over these fields such as $\mathcal{H}$.

[2] We write $\bar{\alpha}(n)$ for the code of the sequence $\langle \alpha_0, \ldots, \alpha_{n-1}\rangle$.

4.2. **Spaces of Bounded Linear Operators within AdmRep.** For separable Banach spaces E and F there arises the question what is the natural topology on the set $\mathfrak{B}(E,F)$ of bounded linear operators from E to F. The norm topology endows $\mathfrak{B}(E,F)$ with the structure of a Banach space which, however, in general is not separable. This holds in particular for $\mathfrak{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, which can be seen as follows. Consider the linear operator $T : \ell^\infty \to \mathfrak{B}(\ell^2)$ sending $x \in \ell^\infty$ to the linear operator $T(x) : \mathcal{H} \to \mathcal{H} : (y_n)_{n\in\mathbb{N}} \mapsto (x_n y_n)_{n\in\mathbb{N}}$ which has the same norm as x. Since ℓ^∞ is not separable the Banach space $\mathfrak{B}(\mathcal{H})$ is not separable w.r.t. the norm topology.[3]

However, since E and F are $\mathbf{QCB}_0$ spaces we may consider their exponential F^E in $\mathbf{QCB}_0$. Following the description of exponentials in $\mathbf{QCB}_0$ as given in e.g. [2] the underlying set of F^E is the set of all continuous functions from E to F where (f_n) converges to f_∞ in F^E iff for all sequences (x_n) in E converging to x_∞ the sequence $(f_n(x_n))$ converges to $f_\infty(x_\infty)$ in F. Since being linear is a $\neg\neg$-closed predicate on F^E we consider $\mathfrak{B}(E,F)$ as the corresponding $\neg\neg$-closed subobject of F^E which again is a $\mathbf{QCB}_0$ space. As shown in the next theorem the $\mathbf{QCB}_0$ topology on $\mathfrak{B}(E,F)$ is the sequentialization of a "traditional" topology on $\mathfrak{B}(E,F)$, namely the strong operator topology.

Theorem 4.1. *A sequence (T_n) converges to T in $\mathfrak{B}(E,F)$ w.r.t. its $\mathbf{QCB}_0$ topology iff (T_n) converges to T in the* strong operator topology*.*

Thus, the $\mathbf{QCB}_0$ topology of $\mathfrak{B}(E,F)$ is the sequentialization of the strong operator topology on $\mathfrak{B}(E,F)$.

Proof. The forward direction is obvious.

For the reverse direction suppose that (T_n) converges to T in the strong operator topology, i.e. $\lim\limits_{n\to\infty} T_n x = Tx$ for all $x \in E$. Thus, for all $x \in E$ the set $\{T_n x \mid n \in \mathbb{N}\} \cup \{Tx\}$ is bounded from which it follows by the Banach-Steinhaus theorem that $\{\|T_n\| \mid n \in \mathbb{N}\} \cup \{\|T\|\}$ is bounded by some $c > 0$. For showing that (T_n) converges to T in F^E suppose that (x_n) converges to x in E. We have

$$\begin{aligned} \|Tx - T_n x_n\| &\le \|Tx - T_n x\| + \|T_n x - T_n x_n\| \\ &\le \|Tx - T_n x\| + \|T_n\| \cdot \|x - x_n\| \\ &\le \|Tx - T_n x\| + c\|x_n - x\| \end{aligned}$$

for which reason $\lim\limits_{n\to\infty} \|Tx - T_n x_n\| = 0$ since $\lim\limits_{n\to\infty} \|Tx - T_n x\| = 0$ and $\lim\limits_{n\to\infty} \|x_n - x\| = 0$. Thus $\lim\limits_{n\to\infty} T_n x_n = Tx$ as desired. □

If E is separable Hilbert space $\mathcal{H}$ (e.g. ℓ^2) and F is $\mathbb{C}$ or $\mathcal{H}$ then the strong operator topology on $\mathfrak{B}(E,F)$ is *not sequential* (see solution of Problem 21 on p.185 of Halmos's *Hilbert Space Problem Book* [13]) for which reason one has to take its sequentialization to obtain the natural topology of $\mathfrak{B}(E,F)$ in $\mathbf{QCB}_0$. Thus, in particular, the natural topology on $E' = \mathfrak{B}(E,\mathbb{C})$ is the sequentialization of the weak* topology on E'.

Accordingly, in the following we will consider $\mathcal{H}'$ as endowed with the sequentialization of the weak* topology. We write $i : \mathcal{H} \to \mathcal{H}'$ for the map with $i(x)(y) = \langle x|y\rangle$ which is continuous but not a homeomorphism unless $\mathcal{H}$ is endowed with the sequentialization of the weak topology.

[3] We thank V. Brattka for drawing our attention to this counterexample.

4.3. **Hilbert Lattice within AdmRep.** Obviously, the Sierpiński space Σ also lives within **AdmRep**. Thus, also $\Sigma^{\mathcal{H}'}$ lives within **AdmRep**. From Proposition 3.3 we know that $\Sigma^{\mathcal{H}'}$ carries the Scott topology. Thus, when identifying $p \in \Sigma^{\mathcal{H}'}$ with the closed subset $p^{-1}(\bot)$ of $\mathcal{H}'$ the set $\mathcal{C}(\mathcal{H}')$ of closed subsets of $\mathcal{H}'$ gets endowed with the Scott topology induced by the partial order $\supseteq$ for which we write $\sqsubseteq_{\mathcal{C}(\mathcal{H}')}$ or simply $\sqsubseteq$ as is common for the specialization order.

However, for later use it is useful to make explicit what it means that a sequence (p_n) converges to p_∞ in $\Sigma^{\mathcal{H}'}$, namely that $(p_n(x_n))$ converges to $p_\infty(x_\infty)$ in Σ whenever (x_n) converges to x_∞ in $\mathcal{H}'$. Thus, the sequence (p_n) converges to p_∞ in $\Sigma^{\mathcal{H}'}$ iff for all (x_n) converging to x_∞ from $p_\infty(x_\infty) = \top$ it follows that $\exists n \forall k \geq n\ p_k(x_k) = \top$ iff for all (x_n) converging to x_∞ from $\forall n \exists k {\geq} n\ p_k(x_k) = \bot$ it follows that $p_\infty(x_\infty) = \bot$.

Since by Proposition 3.9 the category **AdmRep** is closed under $\neg\neg$-subobjects the collection of closed linear subspaces of $\mathcal{H}'$ gives rise to a $\neg\neg$-closed subobject of $\mathcal{C}(\mathcal{H}') \cong \Sigma^{\mathcal{H}'}$.

Definition 4.2. The *Hilbert lattice* $\mathcal{L}$ in $\mathbf{RT}(\mathcal{K}_2)$ is the subobject of $\Sigma^{\mathcal{H}'}$ consisting of all p satisfying the conditions

(1) $\forall x, y.\, p(x) = \bot \wedge p(y) = \bot \implies p(x + y) = \bot$
(2) $\forall x.\forall \lambda.\, p(x) = \bot \implies p(\lambda x) = \bot$.

Since conditions (1) and (2) are $\neg\neg$-closed $\mathcal{L}$ appears as a $\neg\neg$-subobject of $\Sigma^{\mathcal{H}'}$ and thus is an element of **AdmRep**. As follows from $[1, 2]$ the topology on $\mathcal{L}$ is the *sequentialization* of the subspace topology induced by the inclusion of $\mathcal{L}$ into $\Sigma^{\mathcal{H}'}$ which itself carries the Scott topology.

Nevertheless $\mathcal{L}$ inherits its specialization order from $\Sigma^{\mathcal{H}'}$ as follows from

Lemma 4.3. *Let $A \subseteq_{\neg\neg} \Sigma^X$ in $\mathbf{Mod}(\mathcal{K}_2)$. Then A inherits its information ordering from Σ^X, i.e. for $p, q \in A$ we have $p \sqsubseteq q$ iff $p(x) \sqsubseteq q(x)$ for all $x \in X$.*

Proof. By Proposition 3.3 the claim holds for Σ^X. But for $p, q \in \Sigma^X$ we have $p \sqsubseteq q$ iff there exists a morphism $f : \Sigma \to \Sigma^X$ in $\mathbf{Mod}(\mathcal{K}_2)$ with $f(\bot) = p$ and $f(\top) = q$.

Suppose $p, q \in A$. If $p \sqsubseteq_A q$ then $p \sqsubseteq_{\Sigma^X} q$ since the inclusion of A into Σ^X is a morphism in $\mathbf{Mod}(\mathcal{K}_2)$. On the other hand if $p \sqsubseteq_{\Sigma^X} q$ then by the observation above there is a morphism $f : \Sigma \to \Sigma^X$ in $\mathbf{Mod}(\mathcal{K}_2)$ with $f(\bot) = p$ and $f(\top) = q$. Since f factors through A it follows that $p \sqsubseteq_A q$. $\square$

Now we can show that

Proposition 4.4. *$\mathcal{L}$ is a topological domain.*

Proof. That $\mathcal{L}$ is a topological predomain is immediate from the fact that it appears as an equalizer of maps between topological predomains corresponding to conditions (1) and (2) of Def. 4.2.

Since the specialization order on $\mathcal{L}$ is inherited from $\Sigma^{\mathcal{H}'}$ the least element of $\mathcal{L}$ is given by the map $\bot_{\mathcal{L}} : \mathcal{H}' \to \Sigma$ with $\bot_{\mathcal{L}}(x) = \bot$ iff $x = 0$. $\square$

Let Prj be the $\neg\neg$-subobject of $\mathfrak{B}(\mathcal{H})$ consisting of projectors. Thus Prj is an object of **AdmRep** inheriting convergence from $\mathfrak{B}(\mathcal{H}) \subseteq_{\neg\neg} \mathcal{H}^{\mathcal{H}}$. Classically, every $p \in \mathcal{L} \subseteq \Sigma^{\mathcal{H}'}$ can be identified with the corresponding projector $P_p \in \mathfrak{B}(\mathcal{H})$ tacitly using that $\mathcal{H}'$ with the weak* topology is homeomorphic to $\mathcal{H}$ with the weak topology. The bijective map from Prj to $\mathcal{L}$ sending P to $\{x \in \mathcal{H} \mid Px \neq x\}$ is continuous since definable in the internal language of $\mathbf{RT}(\mathcal{K})$ but its inverse is not since it does not respect the specialization order. More generally, it holds that

Proposition 4.5. *The $\mathbf{QCB}_0$ spaces $\mathcal{L}$ and Prj are not isomorphic.*

Proof. If $\mathcal{L}$ and Prj were isomorphic then their specialization orders would be isomorphic, too, which, however, is not the case since on Prj it is flat whereas $\mathcal{L}$ has a least element w.r.t. its information ordering as follows from Lemma 4.3. □

Since $\mathbf{RT}(\mathcal{K}_2)$ is a model of Brouwerian intuitionistic mathematics (see [14,19]) it follows that one cannot prove constructively that $\mathcal{L}$ and Prj are in 1-1-correspondence. Moreover, Prop. 4.5 seems to show that von Neumann's Spectral Theorem does not hold in $\mathbf{RT}(\mathcal{K}_2)$ since from a classical point of view it entails a 1-1-correspondence between closed linear subspaces of $\mathcal{H}$ and projectors on $\mathcal{H}$. But, as we will see later in subsection 4.5 this is not the case for an appropriate formulation of the Spectral Theorem since $\mathcal{L}$ corresponds to spectral measures/valuations on Σ rather than on the discrete space $2 = \{0, 1\}$.

In view of Prop. 4.5 the subsequent Th. 4.6 might seem surprising. But in any case it will be crucial for proving our variant of the Spectral Theorem. For formulating Th. 4.6 we have to introduce a few conventions.

Let $S(\mathcal{H})$ be *projective Hilbert space*, i.e. unit vectors of $\mathcal{H}$ modulo the equivalence relation $x \sim y \equiv \forall p \in \mathcal{L}.\, p(x) = p(y)$. Notice that $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Recall that $\mathcal{I}$ is the unit interval $[0, 1]$ with the *upper* topology whose open sets are those downward closed subsets of $[0, 1]$ which are open in the usual Euclidean topology on $[0, 1]$. Notice that $x \sqsubseteq_{\mathcal{I}} y$ iff $x \geq y$. Moreover, one may characterize $\mathcal{I}$ as the Scott topology on the continuous lattice (in the sense of [10]) $[0, 1]$ ordered by $\geq$. For this reason $\mathcal{I}$ may be called the *upper interval*. We will write $\mathbb{I}_{\leq}$ for the dual but isomorphic notion, namely $\mathbb{I}$ endowed with the *lower* topology, i.e. the Scott topology induced by $\leq$ on $\mathbb{I}$.

Theorem 4.6. *The map $s : \mathcal{L} \to \mathcal{I}^{S(\mathcal{H})} : P \mapsto x \mapsto \langle x|Px\rangle$ is a morphism in $\mathbf{AdmRep}$, i.e. continuous w.r.t. the induced topologies. Moreover, the map s is a $\neg\neg$-mono, i.e. s is an iso when corestricted to its $\neg\neg$-image $\mathcal{L}_p$. Moreover, both s and $s^{-1} : \mathcal{L}_p \to \mathcal{L}$ have effective realizers and thus live in $\mathbf{AdmRep}_{\mathsf{eff}}$.*

Proof. We give a constructive proof which is partly inspired by the proof of Th. 4.15 in [18]. The basic idea is to consider the isomorphic copy $\mathcal{L}_q$ of $\mathcal{L}_p$ induced by the isomorphism between $\mathcal{I} = \mathbb{I}_{\geq}$ and $\mathbb{I}_{\leq}$ sending x to $\sqrt{1 - x^2}$, i.e. $\mathcal{L}_q$ consists of all functions from $S(\mathcal{H})$ to $\mathbb{I}_{\leq}$ which assign to $x \in S(\mathcal{H})$ the distance $d(x, L)$ to a closed linear subspace L of $\mathcal{H}'$. We will use some notation and facts as formulated and proven in Appendix B.

Let $L \in \mathcal{L}$ be given as a Σ-predicate on $\mathcal{H}'$. Then $B(\mathcal{H}) \cap L$ is also given by a Σ-predicate on $\mathcal{H}'$. For $c \in S(\mathcal{H})$ and $0 \leq r < 1$ by Theorem B.5 we have $r < d(c, L)$ iff $B(\mathcal{H}) \cap L \subseteq H_{c,r}$ which is in Σ since $B(\mathcal{H}) \cap L$ is a compact subset of $\mathcal{H}'$ and $H_{c,r}$ is open. Thus we have established the existence of s as a map from $\mathcal{L}$ to $\mathcal{I}_{\leq}^{S(\mathcal{H})}$ sending $L \in \mathcal{L}$ to $s(L) = \lambda r.\, r < d(x, L)$. Obviously $\mathcal{L}_q$ is the $\neg\neg$-image of this $s : \mathcal{L} \to \mathcal{I}_{\leq}^{S(\mathcal{H})}$.

For showing that the inverse s^{-1} of $s : \mathcal{L} \to \mathcal{I}_{\leq}^{S(\mathcal{H})}$ is computable we first discuss appropriate admissible representations of $\mathcal{L}$ and $\mathcal{L}_q$.[4] First notice that $[0, 1[\cap\mathbb{Q}$ and algebraic numbers can be coded effectively by natural numbers. We call elements of $S(\mathcal{H})$ "rational"

[4]The coding of $\mathcal{L}$ is a restriction of a coding of closed convex subsets of $B(\mathcal{L}')$ as can be found in [18].

iff all items are algebraic complex numbers and almost all of them vanish. Thus rational elements of $S(\mathcal{H})$ can be coded effectively by natural numbers. Elements $d \in \mathcal{L}_q$ are coded by sequences which enumerate all (codes of) pairs (c, r) of rational elements $c \in S(\mathcal{H})$ and $r \in [0, 1[$ such that $r < d(c)$. Elements $L \in \mathcal{L}$ are coded by sequences which enumerate all (codes of) pairs (c, r) of rational elements $c \in S(\mathcal{H})$ and $r \in [0, 1[$ such that $B(\mathcal{H}) \cap L \subseteq H_{c,r}$.[5] By Theorem B.5 an element of Baire space codes $L \in \mathcal{L}$ iff it codes $s(L) \in \mathcal{L}_q$. Thus both s^{-1} and s are coded by any code for the identity map on Baire space. Since there are effective codes for the latter both s and s^{-1} are computable. □

There arises the question to which extent the operations on $\mathcal{L}$ usually considered in the "logico-algebraic" approach do live within **AdmRep**. Of course, the antitonic operation $(-)^\perp : \mathcal{L} \to \mathcal{L}$ of orthocomplementation is not continuously realizable since otherwise it would be monotonic which is impossible since the specialization order on $\mathcal{L}$ is not discrete. The operation $\wedge : \mathcal{L} \times \mathcal{L} \to \mathcal{L} : (P, Q) \mapsto P \cap Q$ is realizable since the binary supremum operation on Σ is effectively realizable. The following proposition tells us that the binary supremum operation on $\mathcal{L}$ is not continuously realizable.

Proposition 4.7. *The function $\vee : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ where $P \vee Q$ is the least closed subspace of $\mathcal{H}$ containing P and Q as subsets is not continuous and thus not a morphism in* **AdmRep**.

Proof. Suppose $\vee : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is continuous. Then it preserves suprema of ascending ω-chains in each argument. Since $(-)^\perp$ is an anti-automorphism this is equivalent to $P \wedge \bigvee Q_n = \bigvee P \wedge Q_n$ for all sequence (Q_n) in $\mathcal{L}$ with $Q_n \subseteq Q_{n+1}$.

The following counterexample, however, shows that this is not generally the case. Let Q_n be the closed linear subspace of $\mathcal{H}$ spanned by $e_0, \ldots, e_n$ and P a 1-dimensional subspace of $\mathcal{H}$ with $P \cap Q_n = 0$ for all n. Since $\bigvee Q_n = \mathcal{H}$ we have $P = P \wedge \bigvee Q_n$ although $\bigvee P \wedge Q_n = 0$. □

Moroever, biorthogonalisation is not continuous as a map from $\mathcal{C}(\mathcal{H}') \to \mathcal{L}$.

Proposition 4.8. *The biorthogonalization map $(-)^{\perp\perp} : \mathcal{C}(\mathcal{H}') \to \mathcal{L}$ sending $C \in \mathcal{C}(\mathcal{H}')$ to $C^{\perp\perp}$ is not continuous and thus not a morphism in* **AdmRep**.

Proof. Since the topology on $\mathcal{L}'$ is the sequentialization of the topology induced by the inclusion of $\mathcal{L}$ into $\mathcal{C}(\mathcal{H}')$ the map $(-)^{\perp\perp} : \mathcal{C}(\mathcal{H}') \to \mathcal{L}$ is a continuous if and only if $(-)^{\perp\perp} : \mathcal{C}(\mathcal{H}') \to \mathcal{C}(\mathcal{H}')$ is Scott continuous which, however, is not the case as the following counterexample shows.

Let (C_n) be a sequence in $\mathcal{C}(\mathcal{H}')$ with $C_{n+1} \subseteq C_n$ such that $\bigcap C_n = \{0\}$ and all $C_n^{\perp\perp}$ are the same 1-dimensional subspace of $\mathcal{H}'$. For example one may take $C_n = \{mx \mid m \geq n\}$ where x is some unit vector in $\mathcal{H}$. Then $(\bigcap C_n)^{\perp\perp} = 0$ although $C_n^{\perp\perp} = \mathbb{C}x$ for all n and thus $\bigcap C_n^{\perp\perp} = \mathbb{C}x \neq 0 = (\bigcap C_n)^{\perp\perp}$ providing the desired counterexample to (Scott) continuity of biorthogonalization. □

[5]We can reconstruct L from $B(\mathcal{H}) \cap L$ since any code of an element x of $\mathcal{H}'$ different from 0 can be transformed effectively into a code of an element $x' \in B(\mathcal{H}')$ which is different from 0 and a multiple of x and thus $x \in L$ iff $x' \in B(\mathcal{H}) \cap L$.

4.4. **States within AdmRep.** We want to identify states as particular maps in **AdmRep** from the Hilbert lattice $\mathcal{L}$ to the unit interval. Since such maps have to be Scott continuous w.r.t. the specialization order we must endow the unit interval with the *upper* topology, i.e. consider states as maps from $\mathcal{L}$ to $\mathcal{I}$. In the subsequent Theorem 4.10 we will characterize states as those $s \in \mathcal{I}^{\mathcal{L}}$ which validate the conditions (S1) and (S3) of Prop. 2.2.

But for this purpose we need the following result about pure states.

Lemma 4.9. *For all unit vectors $x \in \mathcal{H}$ the pure state $s_x : \mathcal{L} \to \mathcal{I} : p \mapsto \langle x|P_p x\rangle$ is a morphism in* **AdmRep***, i.e. continuous w.r.t. the induced topologies.*

Proof. The claim is an immediate consequence of Th. 4.6. □

Theorem 4.10. *A function $s : \mathcal{L} \to \mathbb{I}$ is a state iff $s \in \mathcal{I}^{\mathcal{L}}$ and it satisfies the conditions*

(S1) $s(0) = 0$ *and* $s(\mathcal{H}) = 1$
(S3) $s(P \vee Q) = s(P) + s(Q)$ *whenever* $P \perp Q$

of Prop. 2.2.

Proof. By Prop. 2.2 a function $s : \mathcal{L} \to \mathbb{I}$ is a state iff it satisfies conditions (S1), (S2) and (S3). Since $s \in \mathcal{I}^{\mathcal{L}}$ preserves suprema of ω-chains it always validates (S2) of Prop. 2.2. Thus, we have shown the backward direction of our claim.

By Lemma 4.9 every pure state is an element of $\mathcal{I}^{\mathcal{L}}$ and thus validates condition (S2). Obviously, it always validates conditions (S1) and (S3). By Proposition 2.1 every state arises as a countable convex combination of pure states. Any such countable convex combination satisfies conditions (S1) and (S3) and is moreover continuous. Thus, all states are elements of $\mathcal{I}^{\mathcal{L}}$ and validate conditions (S1) and (S3). □

Notice that at first sight condition (S3) cannot be expressed in the internal language since by Prop. 4.7 the operation $\vee$ on $\mathcal{L}$ is not a morphism of **AdmRep**. But under the assumption that $P \perp Q$ a unit vector x is in $P \vee Q$ iff $s(P)(x) + s(Q)(x) = 1$ and thus $x \notin P \vee Q$ iff $s(P)(x) + s(Q)(x) < 1$ in $\mathcal{I}$ which is a proposition in Σ. Thus, for $P \perp Q$ we may define $P \vee Q \in \mathcal{L}$ as $\lambda x{:}\mathcal{H}.\, \|x\| > 0 \wedge s(P)\big(\frac{x}{\|x\|}\big) + s(Q)\big(\frac{x}{\|x\|}\big) < 1$ in the internal language. In the light of these considerations the following definition makes sense in the internal language of $\mathbf{RT}(\mathcal{K}_2)$ and $\mathcal{KV}$.

Definition 4.11. Let St be the $\neg\neg$-closed subobject of $\mathcal{I}^{\mathcal{L}}$ consisting of those $s \in \mathcal{I}^{\mathcal{L}}$ which validate the conditions

(S1) $s(0) = 0$ and $s(\mathcal{H}) = 1$
(S3) $s(P \vee Q) = s(P) + s(Q)$ whenever $P \perp Q$.

Notice that St is a topological predomain since there are no nontrivial ascending chains in St. For this reason it also lacks a least element and thus is not a topological domain.

4.5. **Observables within in AdmRep.** In Proposition 2.3 we have characterized (quantum) observables as (Scott) continuous maps $\nu : \mathcal{C}(\mathbb{R}) \to \mathcal{I}$ such that for every unit vector $x \in \mathcal{H}$ the map $\nu_x : \mathcal{C}(\mathbb{R}) \to \mathcal{I} : C \mapsto \langle x|\nu(C)(x)\rangle$ is a probability valuation on $\mathbb{R}$. That ν is a family of probability valuations is axiomatized by the conditions (O1) and (O3) of Proposition 2.3. As follows from the subsequent Lemma 4.12 condition (O2) is equivalent to $\nu \in \mathcal{L}^{\mathcal{C}(\mathbb{R})}$.

Lemma 4.12. *A map $o : \mathcal{C}(\mathbb{R}) \to \mathcal{L}$ is continuous iff it is Scott continuous as a map from $\mathcal{C}(\mathbb{R})$ to $\mathcal{C}(\mathcal{H}')$.*

Proof. Recall that $\mathcal{C}(\mathbb{R}) \cong \Sigma^{\mathbb{R}}$ in **AdmRep** carries the Scott topology which, moreover, is sequential. Further recall that the topology of $\mathcal{L}$ is the sequentialization of the topology induced by the inclusion of $\mathcal{L}$ into $\mathcal{C}(\mathcal{H}') \cong \Sigma^{\mathcal{H}'}$.

The claim follows from the fact that sequentialization is a coreflection. □

Thus, observables are the (global) elements of the object Obs in **AdmRep** as introduced in the next definition.

Definition 4.13. Let Obs be the $\neg\neg$-subobject of $\mathcal{L}^{\mathcal{C}(\mathbb{R})}$ consisting of those $\nu \in \mathcal{L}^{\mathcal{C}(\mathbb{R})}$ such that $\nu_x = \lambda C \in \mathcal{C}(\mathbb{R}).s(\nu(C))x$ is a probability valuation for all unit vectors $x \in \mathcal{H}$ where s is the continuous map of Theorem 4.6.

Notice that Obs is a topological predomain since there are no nontrivial ascending chains in Obs. For this reason it also lacks a least element and thus is not a topological domain.

5. A Spectral Theorem for Bounded Observables

Von Neumann's Spectral Theorem establishes a 1-1-correspondence between observables o and unbounded self adjoint operators A on $\mathcal{H}$ which are defined on a dense subspace of $\mathcal{H}$ on which they are continuous in the sense that their graph is a closed subspace of $\mathcal{H}\times\mathcal{H}$, see e.g. [20]. The correspondence is given explicitly by

$$\langle x|Ax\rangle = \int \lambda \, do_x(\lambda)$$

for $x \in \mathcal{H}$ where o_x is the measure on $\mathbb{R}$ given by $o_x(B) = \langle x|o(B)x\rangle$. Notice that Ax is defined iff the integral $\int \lambda \, do_x(\lambda)$ exists. This restricts to a 1-1-correspondence between bounded self adjoint operators A on $\mathcal{H}$ and observables o which are **bounded** in the sense that $o([-c, c]) = \mathcal{H}$ for some $c \geq 0$.

This correspondence extends to observables formulated in terms of valuations since by [8] integration w.r.t. valuations can be developed within a fairly weak constructive theory certainly validated by $\mathbf{RT}(\mathcal{K}_2)$ and $\mathcal{KV}$.

We will show now that at least for bounded self adjoint operators this process can be inverted within $\mathbf{RT}(\mathcal{K}_2)$ and $\mathcal{KV}$. Clearly, by rescaling, it suffices to show this for self adjoint operators bounded by 1.

Given a self adjoint operator A bounded by 1 we consider the commutative subalgebra $\mathfrak{A}(A)$ of $\mathfrak{B}(\mathcal{H})$ generated by A. By the Gel'fand-Naimark theorem for commutative $\mathbb{C}^*$-algebras, which can be proven constructively and thus holds in $\mathbf{RT}(\mathcal{K}_2)$ and $\mathcal{KV}$, the algebra $\mathfrak{A}(A)$ is isomorphic to $C(\mathrm{Sp}(\mathfrak{A}(A)))$. Thus, to every unit vector $x \in \mathcal{H}$ we may associate the map $I_A(x) : C([-1, 1]) \to \mathbb{R} : f \mapsto \langle x|f(A)x\rangle$ which is the Daniell-Stone integral corresponding to $\nu_A(x)$ where ν_A is the observable corresponding to A by the Spectral Theorem. By the Portmanteau Theorem (Theorem 2.1 of [5]) these Daniell-Stone integrals with the weak topology (of pointwise convergence) correspond to probability valuations on $[-1, 1]$ with the weak topology (of pointwise convergence), which as shown by M. Schröder in [22] coincides with the natural $\mathbf{QCB}_0$ topology on probability valuations on $[-1, 1]$. Externally, the observable ν_A corresponding to A is given by

$$\nu_A(x)(C) = \inf\{I_A(x)(f) \mid \chi_C \leq f \in \mathbb{R}^{[-1,1]}\}$$

for $x \in \mathcal{H}$ and $C \in \mathcal{C}([-1, 1])$. That this correspondence is a homeomorphism follows from the above mentioned Portmanteau Theorem and the fact that the natural topology on

Daniell-Stone integrals on $[-1,1]$ considered as $\neg\neg$-subobject of $\mathbb{R}^{\mathbb{R}^{[-1,1]}}$ is the topology of pointwise convergence as follows from Theorem 4.1 since the natural topology on $\mathbb{R}^{[-1,1]}$ coincides with the metric one induced by the supremum norm.

It remains to show that we can get $\nu_A(x)$ from $I_A(x)$ in an effectively continuous way. For this purpose we show that ν_A can be defined from I_A in the internal language of $\mathbf{RT}(\mathcal{K}_2)$ and $\mathcal{KV}$ using an argument provided by D. Lešnik from Univ. of Ljubljana. First notice that we can also define ν_A as

$$\nu_A(x)(C) = \inf\{I_A(x)(f) \mid f \in \mathbb{R}^{[-1,1]} \text{ and } \forall x \in C.\, 1 < f(x)\}$$

and, accordingly, also as

$$\nu_A(x)(C) = \inf\{q \in \mathbb{Q} \mid \exists f \in \mathbb{R}^{[-1,1]}.\, I_A(x)(f) < q \wedge \forall x \in C.\, 1 < f(x)\}$$

Since closed subsets of $[-1,1]$ are compact we have $\forall x \in C.\, 1 < f(x) \in \Sigma$ for all $f \in \mathbb{R}^{[-1,1]}$. The proposition $I_A(x)(f) < q$ is in Σ anyway. Let D be some countable dense subset of $\mathbb{R}^{[-1,1]}$, e.g. piecewise linear continuous functions with rational "breakpoints". A Σ-subset of $\mathbb{R}^{[-1,1]}$ is non-empty iff it has non-empty intersection with D. Thus, we have

$$\nu_A(x)(C) = \inf\{q \in \mathbb{Q} \mid \exists f \in D.\, I_A(x)(f) < q \wedge \forall x \in C.\, 1 < f(x)\}$$

which is an element of $\mathcal{I}$ since it arises as infimum of a Σ-subset of $\mathbb{Q}$ (because Σ is closed under existential quantification over the countable set D).

Thus, we have shown that

Theorem 5.1. *The Spectral Theorem for self adjoint operators bounded by* 1 *holds in* $\mathbf{RT}(\mathcal{K}_2)$ *and* $\mathcal{KV}$ *and so does - by rescaling - also the Spectral Theorem for arbitrary bounded self adjoint operators.*

By Theorem 4.1 the natural $\mathbf{QCB}_0$ topology on the operator side is the sequentialization of the strong operator topology. On the side of observables a sequence (ν_n) converges to ν_∞ iff for all unit vectors $x \in \mathcal{H}$ and $C \in \mathcal{C}([-1,1])$ the sequence $\nu_n(C)(x)$ converges to $\nu_\infty(C)(x)$ in $\mathcal{I}$, i.e. $\limsup \nu_n(C)(x) \leq \nu_\infty(C)(x)$.

6. Conclusion and Future Work

We have constructed an admissible representation for the Hilbert lattice $\mathcal{L}$ and based on this admissible representations St and Obs for the sets of quantum states and quantum observables, respectively. Thus, these data types come endowed with a notion of computability in the sense of Weihrauch's Type Two Effectivity where a map between admissible representations is computable iff it is realized by an element of $\mathcal{K}_{2,\mathit{eff}}$, i.e. a computable element of Baire space *aka* a total recursive function. The corresponding category $\mathbf{AdmRep}_{\mathit{eff}}$ of admissible representations and effective, i.e. computable, maps between them arises as a full reflective subcategory of the so-called *Kleene-Vesley* topos $\mathcal{KV}$ which is the effective part of $\mathbf{RT}(\mathcal{K}_2)$ as described in [19].

Since $\mathbf{RT}(\mathcal{K}_2)$ and $\mathcal{KV}$ are models of Brouwerian intuitionistic mathematics, see [14,19], it appears as natural to develop basic quantum mechanics *synthetically* by identifying appropriate axioms holding in the internal language of the respective toposes.

We have arrived at our admissible representations of $\mathcal{L}$, St and Obs by the fairly abstract methods of topological domain theory. We have used our abstract account already for proving some basic negative results, namely that $(-)^{\perp\perp} : \mathcal{C}(\mathcal{H}') \to \mathcal{L}$ and $\vee : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ are not continuous and thus not computable. On the positive side in Theorem 5.1 we have mangaged to show that the Spectral Theorem for bounded observables does hold in $\mathbf{RT}(\mathcal{K}_2)$ and $\mathcal{KV}$ and thus is continuously and also effectively realizable. We conjecture that it can be extended to the general case since the Spectral Theorem for unitary operators follows constructively from the one for bounded self adjoint operators and from this the general spectral theorem can be obtained using the so-called Cayley transform. We expect that these results also hold in $\mathcal{KV}$, i.e. have not only continuous but also computable realizers.

It might be interesting to make the implicit representations more explicit as a basis for actual computation. A particularly challenging question is to which extent the results of [25] can be lifted to our more abstract approach.

Acknowledgements

We are grateful to A. Bauer, V. Brattka, K. Keimel, D. Lešnik, M. Schröder, A. Simpson and B. Spitters for discussions and helpful hints. In particular, we want to thank M. Schröder for pointing us out that a preliminary version of Theorem 4.6 was wrong (see Appendix A) and suggesting that we better define $\mathcal{L}$ as a $\neg\neg$-subobject of $\Sigma^{\mathcal{H}'}$ rather than $\Sigma^{\mathcal{H}}$.

References

[1] I. Battenfeld. *Topological domain theory.* PhD thesis, School of Informatics, University of Edinburgh, 2008.

[2] I. Battenfeld, M. Schröder, and A. Simpson. A convenient category of domains. In L. Cardelli, M. Fiore, and G. Winskel, editors, *Computation, Meaning and Logic, Articles dedicated to Gordon Plotkin*, pp.69-99 Electronic Notes in Computer Science, 2007.

[3] A. Bauer. *The Realizability Approach to Computable Analysis and Topology.* PhD thesis, School of Computer Science, Carnegie Mellon University, Pittsburgh, 2000.

[4] A. Bauer. Realizability as connection between constructive and computable mathematics. In T. Grubba, P. Hertling, H. Tsuiki, and K. Weihrauch, editors, *CCA 2005 - Second International Conference on Computability and Complexity in Analysis, August 25-29,2005, Kyoto, Japan*, volume 326-7/2005 of *Informatik Berichte*, pages 378–379. FernUniversität Hagen, Germany, 2005.

[5] P. Billingsley. *Convergence of Probability Measures.* Wiley 1999.

[6] V. Brattka and R. Dillhage. Computability of the spectrum of self-adjoint operators. *j-jucs*, 11(12):1884–1900, dec 2005.

[7] V. Brattka and M. Schröder. Computing with Sequences, Weak Topologies and the Axiom of Choice. Proc. of CSL 2005, Springer LNCS 3364, pp.462-476, 2005.

[8] T. Coquand and B. Spitters. Integrals and Valuations. Journal of Logic & Analysis, 2009.

[9] L. S. Dediu. *The constructive theory of operator algebras.* PhD thesis, University of Canterbury, 2000.

[10] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains.* Cambridge University Press, Cambridge, 2003.

[11] A. Gleason. Measures on the closed subspaces of a Hilbert space. *Indiana University Mathematics Journal*, 6:885–893, 1957.

[12] P. Halmos What Does the Spectral Theorem Say? *The American Mathematical Monthly*, 70(3):241–247, March 1963.

[13] P. Halmos *A Hilbert Space Problem Book.* Springer, 1967.

[14] S. C. Kleene and R. Vesley. *The foundations of intuitionistic mathematics. Especially in relation to recursive functions.* Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1965.

[15] P. Lietz. *From Constructive Mathematics to Computable Analysis via the Realizability Interpretation.* PhD thesis, Fachbereich Mathematik, TU Darmstadt, Darmstadt, 2004.

[16] G. Mackey. *The Mathematical Foundations of Quantum Mechanics: A Lecture-note Volume.* The mathematical physics monograph series. Princeton university, 1963.

[17] J. von Neumann. *Mathematische Grundlagen der Quantenmechanik*. Springer Verlag, Berlin, Germany, 1932.
[18] E. Neumann. Computational problems in metric fixed point theory and their Weihrauch degrees. Logical Methods in Computer Science, 11(4), 2015.
[19] J. van Oosten. *Realizability: An Introduction to its Categorical Side*. Studies in Logic and the Foundations of Mathematics. Elsevier, 2008.
[20] E. Prugovečki. *Quantum mechanics in Hilbert space*. Academic Press, 1971.
[21] P. Pták and S. Pulmannová. *Orthomodular structures as quantum logics*. Fundamental theories of physics. Kluwer Academic Publishers, 1991.
[22] M. Schröder. Admissible Representations of Probability Measures. MLQ Math. Log. Q. 53 (2007), no. 4-5, 431-445.
[23] T. Streicher. *Domain-theoretic foundations of functional programming*. World Scientific, 2006.
[24] A. Troelstra. Comparing the theory of representations and constructive mathematics. Computer science logic (Bern, 1991), 382-395, Lecture Notes in Comput. Sci., 626, Springer, Berlin, 1992.
[25] Weihrauch and Zhong. Computing Schrödinger propagators on type-2 Turing machines. *Journal of Complexity*, 22, 2006.
[26] K. Weihrauch. *Computable Analysis*. Springer, Berlin, 2000.

Appendix A. Why $\mathcal{L}$ should not be considered as a subobject of $\Sigma^{\mathcal{H}}$

In previous versions of this paper we considered the Hilbert lattice $\mathcal{L}$ as the $\neg\neg$-subobject of $\Sigma^{\mathcal{H}}$ consisting of all p for which $p^{-1}(\bot)$ is a subspace of $\mathcal{H}$. However, as pointed out to us by Matthias Schröder this at first sight compelling definition of $\mathcal{L}$ does not validate the crucial Theorem 4.6. We now describe the counterexample of Schröder.

Let A_n be the closed linear subspace of $\mathcal{H}$ spanned by the vector $e_0 + e_n$. We first show that the sequence (A_n) converges to $\{0\}$.

Suppose (x_n) is a sequence in $\mathcal{H}$ converging to x_∞ in $\mathcal{H}$ such that $\forall n \exists k \geq n\, x_n \in A_n$. Then there exists a subsequence (x_{n_k}) of (x_n) with $x_{n_k} \in A_{n_k}$ for all k. Then for all $m \geq 1$ it holds that $\lim\limits_{k\to\infty} \langle e_m | x_{n_k} \rangle = 0$ and thus $\langle e_m | x_\infty \rangle = 0$. Suppose $\varepsilon > 0$. Then there exists n_0 such that for all $k \geq n_0$ it holds that $\|x_k - x_\infty\| < \varepsilon$ and thus $\langle e_m | x_k \rangle < \varepsilon$ for all $m \geq 1$. But since $x_k \in A_k$ we have $\langle e_0 | x_k \rangle = \langle e_k | x_k \rangle$ and thus $\langle e_0 | x_k \rangle < \varepsilon$ for $k > n_0$. Thus, we have shown that $0 = \lim\limits_{k\to\infty} \langle e_0 | x_k \rangle = \langle e_0 | x_\infty \rangle$. Accordingly, we have $x_\infty = 0$ as desired.

For all n we have $s(A_n)(e_0) \geq \frac{1}{2}$, Thus, since $s(A_\infty)(e_0) = 0$ it cannot hold that $\lim\limits_{n\to\infty} s(A_n)(e_0) = s(A_\infty)(e_0)$, i.e. we have shown that s is not continuous.

Appendix B. Some Facts about Closed Balls and Half Spaces

For $c \in \mathcal{H}$ and $0 \leq r < 1$ we may consider the closed ball $\overline{B(c,r)} = \{x \in \mathcal{H} \mid \|x - c\| \ \leq r\}$ and the closed half space $h_{c,r} = \{x \in \mathcal{H} \mid \Re(\langle c|x\rangle) \geq \sqrt{1-r^2}\}$. We write $H_{c,r}$ for the open complement of $h_{c,r}$.

Lemma B.1. *Let $c \in S(\mathcal{H})$, $0 \leq r < 1$ and $x \in S(\mathcal{H})$. Then $|\Re(\langle c|x\rangle)| \geq \sqrt{1-r^2}$ iff $\lambda x \in \overline{B(c,r)}$ for some $\lambda \in [-1,1]$.*

Proof. We have $\lambda x \in \overline{B(c,r)}$ iff $\|c - \lambda x\| \leq r$ iff $\langle c - \lambda x | c - \lambda x\rangle \leq r^2$ iff

$$0 \geq \langle c - \lambda x | c - \lambda x \rangle - r^2 = \lambda^2 - 2\lambda\Re(\langle c|x\rangle) + 1 - r^2 =: f(\lambda)$$

Since $r^2 < 1$ we have $f(0) > 0$. Since f is continuous by the intermediate value theorem we have $f(\lambda) \leq 0$ for some real λ with $|\lambda| \leq 1$ iff $f(\lambda) = 0$ for some real λ with $|\lambda| \leq 1$.

Since $f(\lambda) = (\lambda - \Re(\langle c|x\rangle)^2 - \Re(\langle c|x\rangle)^2 + 1 - r^2$ the function f has a real zero iff $\Re(\langle c|x\rangle)^2 \geq 1 - r^2$, namely

$$\lambda = \Re(\langle c|x\rangle) - \pm\sqrt{\Re(\langle c|x\rangle)^2 - (1 - r^2)}$$

which always can be chosen to lie in $[-1, 1]$ since $|\Re(\langle c|x\rangle| \leq |\langle c|x\rangle| \leq 1$ and $|\Re(\langle c|x\rangle| \geq \sqrt{\Re(\langle c|x\rangle)^2 - (1 - r^2)}$. $\square$

Lemma B.2. *Let $c \in S(\mathcal{H})$ and $0 \leq r < 1$. If L is a closed linear subspace of $\mathcal{H}$ with $B(\mathcal{H}) \cap L \cap \overline{B(c, r)} = \emptyset$ then $B(\mathcal{H}) \cap L \cap h_{c,r} = \emptyset$.*

Proof. We argue by contraposition. So suppose L is a closed linear subspace of $\mathcal{H}$ and $B(\mathcal{H}) \cap L \cap h_{c,r} \neq \emptyset$ for some $c \in S(\mathcal{H})$ and $0 \leq r < 1$.

Choose some $y \in B(\mathcal{H}) \cap L \cap h_{c,r}$. Since $y \in h_{c,r}$ and $\sqrt{1 - r^2} > 0$ it follows that $y \neq 0$. Thus there exists a unique $\mu \geq 1$ with $x = \mu y$ and $x \in S(\mathcal{H})$. Thus we have $x \in S(\mathcal{H}) \cap L \cap h_{c,r}$ from which it follows by Lemma B.1 that for some $\lambda \in [-1, 1]$ we have $\lambda x \in \overline{B(c, r)}$. But we have also $\lambda x \in B(\mathcal{H}) \cap L$ since $x \in S(\mathcal{H}) \cap L$.

Thus $B(\mathcal{H}) \cap L \cap \overline{B(c, r)} \neq \emptyset$ as desired. $\square$

Lemma B.3. *Let $c \in S(\mathcal{H})$ and $0 \leq r < 1$. If L be a closed linear subspace of $\mathcal{H}$ with $B(\mathcal{H}) \cap L \cap h_{c,r} = \emptyset$ then $B(\mathcal{H}) \cap L \cap \overline{B(c, r)} = \emptyset$.*

Proof. We proceed by contraposition. So suppose $B(\mathcal{H}) \cap L \cap \overline{B(c, r)} \neq \emptyset$.

Then there exists a $y \in B(\mathcal{H}) \cap L \cap \overline{B(c, r)}$. Since $y \in \overline{B(c, r)}$ we have $y \neq 0$. Thus there exists $x \in S(\mathcal{H})$ and $\lambda \in [-1, 1]$ with $y = \lambda x$. Notice that $\lambda \neq 0$ since $y \neq 0$. Thus $x = \frac{1}{\lambda} y \in S(\mathcal{H}) \cap L$ from which it follows by Lemma B.1 that $|\Re(\langle c|x\rangle)| \geq \sqrt{1 - r^2}$. If $\Re(\langle c|x\rangle \geq 0$ then $x \in h_{c,r}$. Otherwise we have $-x \in h_{c,r}$ and thus since $B(\mathcal{H}) \cap L$ is closed under multiplication with -1 we also have $-x \in B(\mathcal{H}) \cap L$. Thus in any case $h_{c,r}$ and $B(\mathcal{H}) \cap L$ are not disjoint as desired. $\square$

Lemma B.4. *Let L be a closed linear subspace of $\mathcal{H}$ and $c \in S(\mathcal{H})$. Then we have*

$$B(\mathcal{H}) \cap L \cap h_{c,r} = \emptyset \quad \textit{iff} \quad L \cap \overline{B(c, r)} = \emptyset \quad \textit{iff} \quad d(c, L) > r$$

for all $0 \leq r < 1$.

Proof. Let $c \in S(\mathcal{H})$, $0 \leq r < 1$ and L a closed linear subspace of $\mathcal{H}$. The second equivalence is almost tautological. Thus we concentrate on the first equivalence.

From Lemma B.2 and lemma B.3 it follows that

$$B(\mathcal{H}) \cap L \cap h_{c,r} = \emptyset \quad \text{iff} \quad B(\mathcal{H}) \cap L \cap \overline{B(c, r)} = \emptyset$$

and thus for establishing our claim it remains to show that

$$L \cap \overline{B(c, r)} = \emptyset \quad \text{iff} \quad B(\mathcal{H}) \cap L \cap \overline{B(c, r)} = \emptyset$$

which, however, follows from the following consideration.

Since the forward direction is trivial it suffices to prove the contraposition of the backwards direction. For this purpose suppose $L \cap \overline{B(c, r)} \neq \emptyset$, i.e. $d(c, L) \leq r$. Thus $d(c, P_L(c)) = d(c, L) \leq r$, i.e. $P_L(c) \in \overline{B(c, r)}$. Since $P_L(c) \in B(\mathcal{H})$ it follows that $B(\mathcal{H}) \cap L \cap \overline{B(c, r)} \neq \emptyset$ as desired. $\square$

Theorem B.5. *Let L be a closed linear subspace of $\mathcal{H}$ and $c \in S(\mathcal{H})$. Then we have*

$$B(\mathcal{H}) \cap L \subseteq H_{c,r} \quad \textit{iff} \quad d(c, L) > r$$

for all $0 \leq r < 1$.

Proof. Immediate from Lemma B.4 by observing that for $0 \leq r < 1$ we have

$$B(\mathcal{H}) \cap L \subseteq H_{c,r} \quad \text{iff} \quad B(\mathcal{H}) \cap L \cap h_{c,r} = \emptyset$$

since by definition $H_{c,r}$ is the complement of $h_{c,r}$. □