PHYSICAL REVIEW A **87**, 043814 (2013)

# Experimental determination of the degree of polarization of quantum states

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(Received 27 July 2012; published 15 April 2013)

We demonstrate experimental excitation-manifold-resolved polarization characterization of quantum states of light ranging from the few-photon to the many-photon level. In contrast to the traditional characterization of polarization that is based on the Stokes parameters, we experimentally determine the Stokes vector of each excitation manifold separately. Only for states with a given photon number do the methods coincide. For states with an indeterminate photon number, for example Gaussian states, the employed method gives a richer and more accurate description. We apply the method both in theory and in experiment to some common states to demonstrate its advantages.

DOI: 10.1103/PhysRevA.87.043814 PACS number(s): 42.50.Dv, 42.25.Ja, 03.65.Ca

*Introduction.* Polarization is one of the key parameters of the electromagnetic field as demonstrated by the plethora of different applications. To mention a few, the classical polarization is used in thin-film ellipsometry [1], near-field microscopy [2], remote sensing [3], and light scattering [4]. In recent years, the concept of polarization has also found a footing in quantum optics and in quantum information science where the information is efficiently encoded in the polarization degree of freedom. This has led to demonstrations of polarization entanglement [5], teleportation of the quantum polarization [6], and quantum key distribution based on quantum polarization encoding [7,8]. Due to the importance of quantum polarization in these applications and others, it is important to be able to quantify the degree of quantum polarization.

Classically, the degree of polarization is a simple expression of the mean values of the Stokes parameters [9] which can be straightforwardly measured [10]. It has been suggested to use a similar expression in the quantum domain as a measure of the degree of quantum polarization [11–13]. However, it was soon realized that this polarization parameter is insufficient to characterize the degree of polarization for many quantum states since it classifies some states as being unpolarized although they are polarized, and vice versa (see, for example, [14], and references therein). This inconsistency calls for a new measure that more accurately characterizes the polarization of quantum states.

Several attempts have been made to quantify the degree of quantum polarization differently (see [15] for an overview), the most prominent ones being the distance-based [16,17] or $Q$-function-based measures [18]. While they all fully satisfy the requirements for a polarization measure their complexity makes them extremely hard to access in a time-efficient manner.

In this work we propose a simple measure of quantum polarization that is successful in quantifying the (first-order) degree of polarization for a large range of common quantum states. We use this polarization measure to experimentally determine the degree of quantum polarization of different quantum states with varying excitations that range from few to many photons. With these experiments we also close the existing gap in characterizing the polarization properties of Gaussian states in the intermediate-photon-number regime. Previous accounts have focused on the few-photon [14,19,20] or many-photon [10,21] regimes.

*Polarization measures.* The polarization of a classical electromagnetic field is uniquely described by the Stokes parameters which can be written as

$$\begin{aligned} S_0 &= |a_1|^2 + |a_2|^2, \quad S_1 = a_H^* a_V + a_H a_V^*, \\ S_2 &= -i(a_H^* a_V - a_H a_V^*), \quad S_3 = |a_H|^2 - |a_V|^2, \end{aligned} \tag{1}$$

where $a_H$ and $a_V$ denote the amplitudes of the field in two linearly polarized orthogonal modes $H$ and $V$. From these, the classical degree of polarization $\mathbb{P}^{\rm cl}$ is defined as

$$\mathbb{P}^{\rm cl} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}. \tag{2}$$

The degree of quantum polarization has previously been defined as a direct translation of the classical degree [11–13]:

$$\mathbb{P}_1^{\rm sc}(\hat{\rho}) = \frac{\sqrt{\langle \hat{S}_1\rangle^2 + \langle \hat{S}_2\rangle^2 + \langle \hat{S}_3\rangle^2}}{\langle \hat{S}_0\rangle}, \tag{3}$$

where $\hat{\rho}$ is the quantum state under scrutiny [22] and the Stokes operators $(\hat{S}_0, \hat{S}_1, \hat{S}_2, \hat{S}_3)$ are found by the canonical quantization of the field amplitudes in the expressions (1). As (3) is undefined for the two-mode vacuum state, the degree needs the supplementary definition $\mathbb{P}_1^{\rm sc}(|0\rangle_H|0\rangle_V) = 0$, as this state is invariant under any polarization transformation and is therefore unpolarized [25].

Despite the seemingly correct translation from classical to quantum polarization, the definition in (3) yields an inconsistent quantification of the degree of polarization [23]. This can be illustrated by some simple examples: According to the definition in (3), the state $|\Psi\rangle_H|0\rangle_V$ is fully polarized (that is, $\mathbb{P}_1^{\rm sc} = 1$) for any pure state $|\Psi\rangle_H \neq |0\rangle$. This implies the unpalatable consequence that states arbitrarily close to the two-mode vacuum are fully polarized and renders the measure discontinuous as a function of the state excitation. As another example of its failure, we consider the state $|\Xi\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|1,0\rangle_{H,V} + e^{i\varphi}|0,2\rangle_{H,V})$ where $\varphi \in [0;2\pi)$. Applying the semiclassical polarization (3) yields $\mathbb{P}_1^{\rm sc}(|\Xi\rangle) = 0$ and thus implies that $|\Xi\rangle$ is unpolarized. This means that the state should be polarization-transformation invariant [25]. However, $|\Xi\rangle$ is not polarization-transformation invariant: Under, a $\pi/2$

 

polarization rotation of the state, the state is transformed into $\frac{1}{\sqrt{3}}(\sqrt{2}|0,1\rangle_{H,V} - e^{i\varphi}|2,0\rangle_{H,V})$, which is orthogonal to $|\Xi\rangle$. Thus $|\Xi\rangle$ is not invariant under polarization transformation, and it is therefore clear that the definition in (3) falls short in quantifying the degree of quantum polarization. We note that a common property of the aforementioned examples is that the photon number $n$ is not a fixed quantity.

From the above discussion, it is clear that the semiclassical definition is unsuitable for determining the degree of polarization for many quantum states. As the main result of this paper, we propose a definition of the degree of polarization that circumvents the shortcomings of the previous definition:

$$\mathbb{P}_1(\hat{\rho}) = \sum_{N=1}^{\infty} p_N \frac{\sqrt{\langle \hat{S}_{1,N}\rangle^2 + \langle \hat{S}_{2,N}\rangle^2 + \langle \hat{S}_{3,N}\rangle^2}}{\langle \hat{S}_{0,N}\rangle}, \quad (4)$$

where $p_N = \mathrm{Tr}(\hat{\mathbb{1}}_N\hat{\rho})$, $\hat{\rho}_N = (\hat{\mathbb{1}}_N\hat{\rho}\hat{\mathbb{1}}_N)/p_N$, $\langle \hat{S}_{j,N}\rangle = \mathrm{Tr}(\hat{S}_j\hat{\rho}_N)$, and $\hat{\mathbb{1}}_N = \sum_{m=0}^{N}|m,N-m\rangle\langle m,N-m|$, so that $\hat{\rho}_N$ is the normalized $N$-photon projection of the state's density matrix. The polarization degree is quantified by a weighted sum of the semiclassical degree of polarization in each excitation manifold of the state (except for $N=0$). In other words, every excitation manifold is treated separately. As is clear from the definition, $\mathbb{P}_1$ coincides with $\mathbb{P}_1^{\mathrm{sc}}$ when the number of photons is a fixed quantity. The two definitions also become approximately equal for classical-like states such as coherent states with $\langle \hat{S}_0\rangle \gg 1$. For many other states the two definitions give different results, and in the following we argue that $\mathbb{P}_1$ gives a better assessment of the polarization properties of quantum states than $\mathbb{P}_1^{\mathrm{sc}}$. For example, we find the intuitively correct results $\mathbb{P}_1(|\Xi\rangle) = 1$ and $\mathbb{P}_1(|\Psi\rangle_H|0\rangle_V) \neq 1$.

We note that the idea of dividing polarization by its excitation manifolds was already suggested by some of us in [24]. However, at that time we were expecting it to be relevant only for few-photon, discrete-variable states (i.e., only in the first few excitation manifolds), and also thought that with present technology it was experimentally infeasible for photon numbers larger than a handful. Therefore, the theory was worked out explicitly only for the first four manifolds, and experimental investigations were demonstrated only for two-photon states in [24]. In the present paper we give theoretical considerations for the most common continuous-variable states. Furthermore, we have to change our previous assumption of the experimental infeasibility of excitation-manifold-resolved polarization characterization. Below, we show that this, on the contrary, is possible at least up to the fiftieth manifold. The experimental results show different features between the semiclassical and the excitation-manifold-resolved polarization and can thus, hopefully, be helpful in the task of finding quantitative measures indicating the usefulness for states to perform physical tasks where polarization encoding is used. We will examine, theoretically and experimentally, some properties of $\mathbb{P}_1$ for continuous-variable Gaussian states, and we will in particular focus on moderately excited coherent and squeezed states [26].

First we consider a two-mode state in which one of two orthogonal polarization modes is a vacuum state whereas the other one is a coherent state; $|\Psi(\alpha)\rangle = |\alpha\rangle_H \otimes |0\rangle_V$, where $\alpha = ae^{i\phi}$ is the complex amplitude of the coherent state

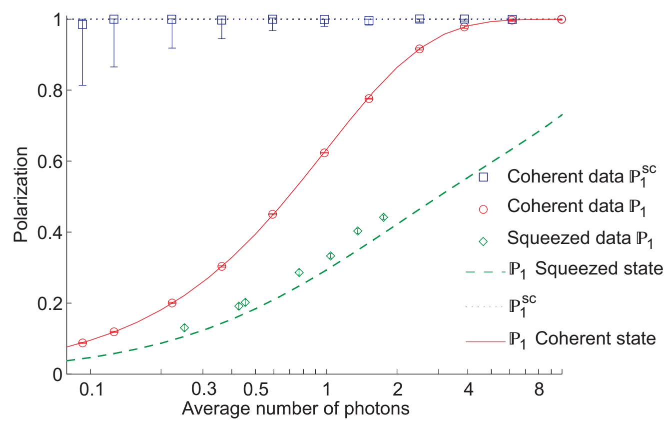


FIG. 1. (Color online) Comparison between $\mathbb{P}_1$ and $\mathbb{P}_1^{\mathrm{sc}}$ as functions of the average photon number for the states $|\Psi(\alpha)\rangle$ and $|\phi(\xi(r,\theta),0)\rangle$ as defined in the text, where the angle $\theta$ is arbitrary and does not influence the values of $\mathbb{P}_1$ and $\mathbb{P}_1^{\mathrm{sc}}$. The lines represent theoretical predictions while the circles indicate experimental values.

$\{a \in \mathbb{R}_0^+, \phi \in [0;2\pi)\}$. For this state, the semiclassical degree of polarization is unity, $\mathbb{P}_1^{\mathrm{sc}}[|\Psi(\alpha)\rangle] = 1$, for all values of $\alpha$ except $\alpha = 0$. On the other hand, using the new measure we find

$$\mathbb{P}_1[|\Psi(\alpha)\rangle] = 1 - e^{-|\alpha|^2} \quad (5)$$

which is continuous $\forall\, \alpha$; $\mathbb{P}_1[|\Psi(\alpha)\rangle] \to 0$ when $\alpha \to 0$, and for large amplitudes, $\mathbb{P}_1[|\Psi(\alpha)\rangle] \to 1$ when $|\alpha| = a \gg 1$. Therefore, the classical and quantum limits, respectively corresponding to large and small amplitudes, are smoothly connected. $\mathbb{P}_1[|\Psi(\alpha)\rangle]$ is illustrated in Fig. 1 by the solid line; $\mathbb{P}_1^{\mathrm{sc}}(\hat{\rho})$ is shown by the dotted line. Equation (5) can be easily generalized to any two-mode coherent state $|\alpha\rangle_H|\beta\rangle_V$, which after an appropriate transformation can be written as a one-mode coherent state $|\alpha'\rangle|0\rangle$ in some other polarization basis with $|\alpha'|^2 = |\alpha|^2 + |\beta|^2$. The degree of quantum polarization of any two-mode coherent state is therefore given by Eq. (5) with $|\alpha|^2 \to |\alpha|^2 + |\beta|^2$.

Next we consider the degree of quantum polarization for single-mode (displaced) squeezed states: $|\varphi(\xi,\alpha)\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle_H \otimes |0\rangle_V$ where $\hat{D}$ is the displacement operator and $\hat{S}(\xi) = \exp[(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})/2]$ is the squeezing operator with $\xi = re^{i\theta}$, $r \in \mathbb{R}_0^+$, $\theta \in [0;2\pi)$ being the squeezing parameter. For this state we find $\mathbb{P}_1^{\mathrm{sc}}[|\varphi(\xi,\alpha)\rangle] = 1$, whereas

$$\mathbb{P}_1[|\varphi(\xi,\alpha)\rangle] = 1 - \frac{1}{\cosh(r)} \times \exp\left[-|\alpha|^2 - \frac{1}{2}(\alpha^{*2}e^{i\theta} + \alpha^2 e^{-i\theta})\tanh(r)\right], \quad (6)$$

which is illustrated in Fig. 1 as a function of the average number of photons for a squeezed vacuum state [that is, $\alpha = 0$ and $\langle n_{\mathrm{sqz}}\rangle = \sinh(r)$]. We clearly see that $\mathbb{P}_1$ differs significantly from $\mathbb{P}_1^{\mathrm{sc}}$ for basically all practical squeezing values. We also note that the degree of first-order quantum polarization (according to the new measure) is not solely determined by the photons associated with the coherent excitation but also by the photons responsible for the squeezing. However, for a fixed number of photons the coherent state is significantly more polarized than a squeezed vacuum state. Finally, we note that for the generalized squeezed state in (6), the degree of polarization also depends on the squeezing angle $\theta$ relative to the phase of the displacement $\phi$: It is maximized for amplitude squeezing ($\theta - 2\phi = 0$) and minimized for phase squeezing ($\theta - 2\phi = \pi/2$).

Finally, we consider the generalized pure two-mode (displaced) squeezed vacuum state

$$\hat{D}(\alpha_H)\hat{S}(\xi_H)\,|0\rangle_H \otimes \hat{D}(\alpha_V)\hat{S}(\xi_V)\,|0\rangle_V\,, \tag{7}$$

and plot the degree of polarization [both $\mathbb{P}_1^{\text{sc}}$ (left column) and $\mathbb{P}_1$ (right column)] in Fig. 2 for three different states. In Fig. 2 (top row), a two-mode vacuum state ($\alpha_H = \alpha_V = 0$) is illustrated for different squeezing degrees. Both measures exhibit zero polarization degree for equal squeezing parameters whereas for different squeezing parameters, $\mathbb{P}_1$ gives lower values than $\mathbb{P}_1^{\text{sc}}$. If we now set $\xi_H = 0$ and $\alpha_V = 0$ (corresponding to a coherent state in the $H$ mode and a squeezed vacuum state in the $V$ mode), the behavior of the two polarization measures is very different as illustrated in Fig. 2 (middle row). Finally, we plot the two-mode displaced squeezed state (with $\xi_H = 0.2$ and $\xi_V = 0.6$) in Fig. 2 (bottom row). The plot for the semiclassical measure once again illustrates its inappropriateness to be a good measure of polarization:

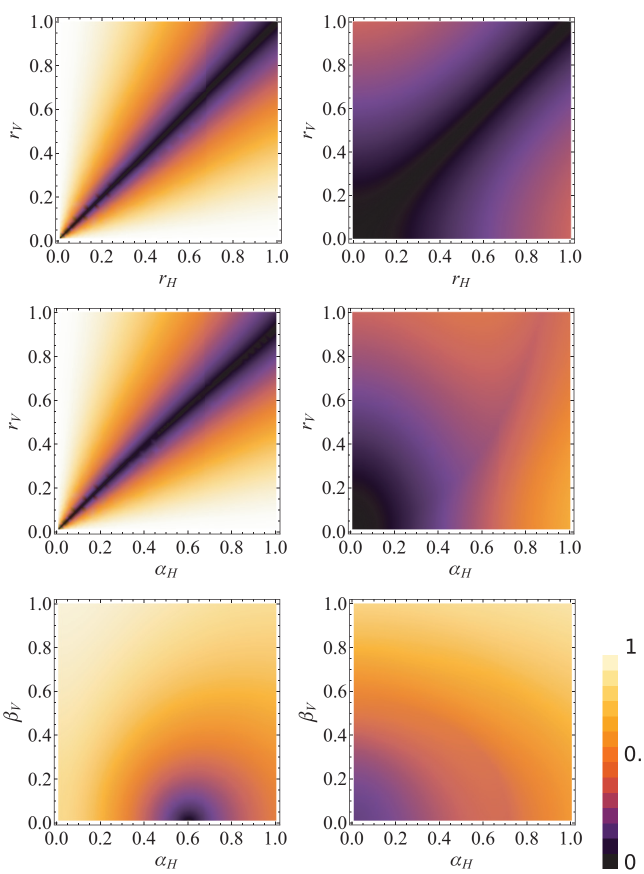


FIG. 2. (Color online) Theoretical plots of the degree of polarization. We plot the states $|\psi(\xi_H(r_H),\xi_V(r_V))\rangle$ (top row), $|\alpha_H\rangle_H \otimes |\psi(\xi_V(r_V))\rangle_V$ (middle row), and $\hat{D}(\alpha_H)\hat{S}(0.2)|0\rangle_H \otimes \hat{D}(\beta_V)\hat{S}(0.6)|0\rangle_V$ (bottom row). The left plots show $\mathbb{P}_1^{\text{sc}}$, whereas the right plots show $\mathbb{P}_1$.

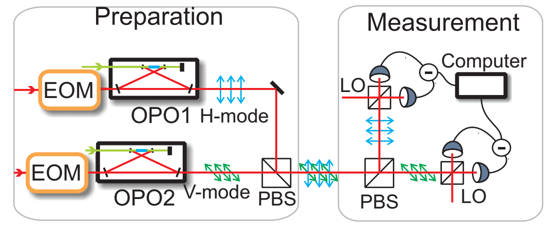


FIG. 3. (Color online) Setup for the production of the states in Eq. (7). A detailed description of the setup can be found in [29]. For the results of this paper we do not use the squeezing of the second OPO since the interesting features of $\mathbb{P}_1$ can be experimentally shown without it (see the text and Fig. 4).

According to $\mathbb{P}_1^{\text{sc}}$, by displacing a squeezed state further away from the vacuum, the state becomes more unpolarized. This incorrect behavior is not given by the new measure.

*Experimental realization.* Since $\hat{S}_0$ commutes with all other Stokes operators, the Stokes vectors per excitation manifold and thus $\mathbb{P}_1$ can be directly accessed by using a proper waveplate configuration, a polarizing beam splitter, and two photon-number-resolving detectors. Such detectors are currently capable of efficiently detecting more than six photons, and due to the rapid progress in developing such detectors more advanced versions with increased optical power range might soon become available [27]. For very high excitations, standard-intensity detectors can be used [10,21,28].

Since we wish to characterize the degree of polarization in different regimes from low to high photon numbers, we have chosen to use a homodyne detector. Using such a detection device, a full tomographic reconstruction of the state from low to relatively high photon numbers is possible, and from this reconstruction we deduce the degree of polarization.

We produce displaced two-mode squeezed states using the setup shown in Fig. 3. Two optical parametric oscillators (OPOs) based on nonlinear down-conversion in periodically poled $KTiOPO_4$ (KTP) crystals are used to generate vacuum squeezed states. The OPOs are injected with modulated coherent states to enable the production of displaced squeezed states [30]. To form the two-mode state, the outputs from the OPOs are combined on a polarizing beam splitter.

In contrast to previous realisations on continuous-variable polarization quantum states, we solely define our state to be residing at a sideband frequency of 4.9 MHz [33]. Such a definition of the polarization state enables us to investigate a large variety of different polarization states from a low excitation to a relatively high excitation. We measure each mode, $H$ and $V$, by splitting the polarization state on a polarizing beam splitter and using two homodyne detectors. The measured currents of the homodyne detectors are sampled at 500 kHz with a frequency bandwidth of 90 kHz, and subsequently sent to a computer for analysis. Since the generated states have Gaussian wave functions, it suffices to estimate the covariance matrix of the state for full characterization [34]. From this we calculate the first 40 excitation manifolds of the two-mode density matrix (which is sufficient to make the truncation error negligible for all states presented) and take the expectation values of the Stokes operators (per manifold), from which the degree of polarization is estimated.

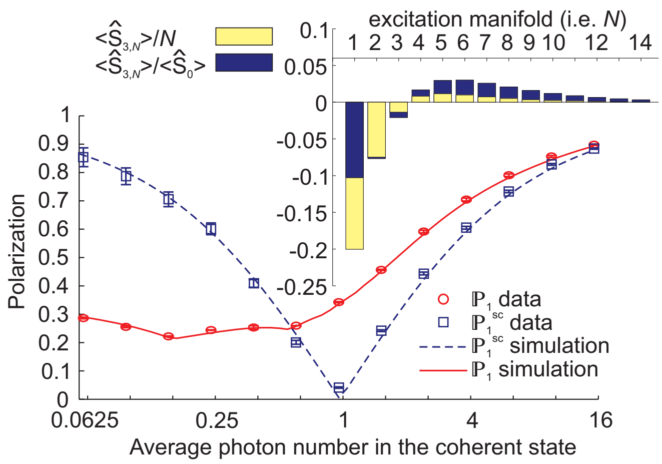


FIG. 4. (Color online) Degree of polarization for experimental data of a squeezed state in mode $H$ and a displaced coherent state in mode $V$. We vary the displacement from −6 dB ($\alpha = 0.25$, 0.0625 photons) to 6 dB ($\alpha = 4$, 16 photons). The squeezed state has a squeezing of 3.2 dB and an antisqueezing of 7.4 dB, corresponding to 1.0 photons. Error bars correspond to 1% uncertainty in the shot noise. Simulations starting from the initial squeezed state are shown with the solid lines. The inset shows the polarization contributions of the different manifolds for the 0 dB (one-photon) state. Only $\langle \hat{S}_3 \rangle$ contributes to the polarization of the states produced here (i.e., $\langle \hat{S}_1 \rangle = \langle \hat{S}_2 \rangle = 0$), and therefore one has $\mathbb{P}_1^{\rm sc} = |\sum \Gamma_b|$ and $\mathbb{P}_1 = \sum |\Gamma_y|$, where $\Gamma_b$ denotes the blue (dark) bars and $\Gamma_y$ denotes the yellow (light) bars.

We start our experimental analysis with one-mode squeezed or one-mode coherent states as defined in Eqs. (5) and (6) (when $\alpha = 0$), respectively. These states are produced by blocking OPO2 while operating either the electro-optic modulator (EOM) (for producing the coherent state) or the OPO1 (for producing the squeezed states). The excitation of the coherent state is controlled by the modulation depth of the EOM whereas the squeezing degree (or the average number of photons associated with the squeezing process) of the squeezed state is controlled by the pump power. Our results for $\mathbb{P}_1$ and $\mathbb{P}_1^{\rm sc}$ are plotted in Fig. 1, where the error bars indicate the 1% uncertainty in determining the shot-noise limit. The experimental values for the squeezed state deviate slightly from the theoretical prediction [Eq. (6) with $\alpha = 0$], which is a consequence of the small impurity of the generated state. As also predicted by theory, we see that both states become increasingly more polarized as the photon numbers from the coherent state or from the squeezed state increase.

Next, we investigate another particularly interesting state in which a coherent state is excited in the $H$ mode while the $V$ mode is a squeezed vacuum state corresponding to $\alpha_V = 0$ and $r_H = 0$. The squeezed state is squeezed by 3.2 dB below the shot-noise limit and the coherent excitation of the $H$ mode is varied.

We present the experimental results for this state in Fig. 4. For a coherent amplitude of 0.25, $\mathbb{P}_1^{\rm sc}$ yields a large degree of polarization of 0.88 although this state is very close to the vacuum state. Furthermore, when the coherent modulation is increased, $\mathbb{P}_1^{\rm sc}$ decreases to zero, which occurs when the number of photons in each polarization mode is unity. This result is erroneous as the state is not invariant to rotation (permutation of the $H$ and $V$ modes) for any value of the displacement. In contrast, the new measure is behaving as expected: The degree of polarization is reasonably small for low excitations and increases nearly monotonically for larger excitations. These different behaviors can be understood by looking at the contributions of the different manifolds in definition (4). $\hat{S}_1$ is the only operator contributing to the polarization and we plot the expectation value of this per manifold in the inset of Fig. 4. Here, we see that it points in opposite directions for the different manifolds which then sum up to zero for $\mathbb{P}_1^{\rm sc}$, and thus the polarization becomes hidden. However, for the $\mathbb{P}_1$ measure, the polarization is not hidden since in this case the absolute values of $\langle \hat{S}_1 \rangle$ from the different manifolds are added.

As a final experiment we operate both OPOs and modulators in order to produce the generalized state in Eq. (7). For these generalized states also we measure a degree of polarization which is monotonically increasing as a function of the coherent excitation.

*Conclusion.* We have proposed a measure of a state's first-order polarization (using the first moments of the Stokes operators) that overcomes the failings of the conventional measure. Specifically, in contrast to the conventional measure, it detects first-order hidden polarization and it is continuous. Due to its suitability in quantifying polarization and its extraordinary simplicity—it can be directly measured—we believe that this measure of polarization will have wide applicability in different sciences.

*Acknowledgments.* We acknowledge support from the Danish Research Council (Sapere Aude Project No. 10-081599). G.B. acknowledges financial support from the Swedish Research Council (VR) through its Linnæus Center of Excellence ADOPT and Contract No. 319-2010-7332. We acknowledge discussions with Jonas Söderholm.

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