THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Uniform cycle decompositions of complete multigraphs

Duncan Berry
BSc (Hons)

A thesis submitted for the degree of Master of Philosophy at The University of Queensland in 2018

School of Mathematics and Physics

Abstract

A decomposition of a graph G is a set $\{D_1, D_2, ..., D_r\}$ of subgraphs of G whose edge sets partition the edge set of G. A graph decomposition is uniform if the union of any two distinct subgraphs of the decomposition is isomorphic to the union of any other two distinct subgraphs, that is, $D_i \cup D_j \cong D_k \cup D_l$ whenever $1 \leq i < j \leq r, 1 \leq k < l \leq r$. This is a natural extension of the notion of uniformity of 1-factorisations (graph decompositions in which each subgraph is a 1-factor). In this project, we initiate the study of uniform decompositions of complete multigraphs into cycles, stars and paths.

A complete multigraph μK_n is a graph with n vertices and precisely μ edges between each pair of vertices. An m-cycle is a connected graph with m vertices in which each vertex has degree two, that is, a graph with the vertex set $\{v_1, v_2, ..., v_m\}$ and the edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{m-1}, v_m\}, \{v_m, v_1\}\}$. We show that if there exists a uniform decomposition of μK_n into m-cycles then (A) $n = m$ and $n \leq 7$, or (B) $\mu = 2$ and $m = n - 1$, or (C) $\mu = 1$, $m = (n-1)/2$ and $n \equiv 3 \pmod 4$ or (D) $\mu = 1$ and $2m(m+1) = n(n-1)$. We fully characterise the complete multigraphs which admit uniform cycle decompositions in case (A). In cases (B) and (C), we construct uniform decompositions for infinitely many values of n and categorise those decompositions into isomorphism classes. In case (D) we have no examples of uniform decompositions, but we prove that the existence of such a decomposition would imply the existence of a large quasi-residual design which is not residual.

We discuss the computational methods and algorithms used to examine small cases of these problems, including the proof that there is no uniform decomposition in case (A) when $n = 9$ or $n = 15$, or in case (C) when $n = 15$. We also present some tables of results from those algorithms, giving the uniform decompositions in case (B) when $n \leq 11$.

A k-star is a connected graph in which one vertex has degree k and k vertices have degree one. An m-path is a connected graph with $m + 1$ vertices and m edges in which two vertices have degree one and the remaining vertices have degree two, that is, a graph with vertex set $\{v_1, v_2, ..., v_m, v_{m+1}\}$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_m, v_{m+1}\}\}$. We show that there exists a uniform star decomposition of μK_n with $n > 2$ precisely when $\mu = 2$, or $\mu = 1$ and there exists a skew Hadamard design of order n or order $n-1$. Finally, we prove that uniform m-path decompositions of μK_n exist only when $n \leq 6$, and construct all such decompositions.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

I have clearly stated the contribution of others to my thesis as a whole, including statistical assistance, survey design, data analysis, significant technical procedures, professional editorial advice, financial support and any other original research work used or reported in my thesis. The content of my thesis is the result of work I have carried out since the commencement of my higher degree by research candidature and does not include a substantial part of work that has been submitted to qualify for the award of any other degree or diploma in any university or other tertiary institution. I have clearly stated which parts of my thesis, if any, have been submitted to qualify for another award.

I acknowledge that an electronic copy of my thesis must be lodged with the University Library and, subject to the policy and procedures of The University of Queensland, the thesis be made available for research and study in accordance with the Copyright Act 1968 unless a period of embargo has been approved by the Dean of the Graduate School.

I acknowledge that copyright of all material contained in my thesis resides with the copyright holder(s) of that material. Where appropriate I have obtained copyright permission from the copyright holder to reproduce material in this thesis and have sought permission from co-authors for any jointly authored works included in the thesis.

Publications during candidature

D. Berry, D. Bryant, M. Dean, B. Maenhaut, Uniform decompositions of complete multigraphs into cycles, submitted to J. Combin. Des. on 22/01/18, revision sent to J. Combin. Des. on 20/06/18.

Publications included in this thesis

The paper listed above is incorporated into this thesis as Chapter 2.

Contributor	Statement of contribution
Author Duncan Berry (Candidate)	Carried out computational work (100%) Identified useful existing results (40%) Proved new results (55%) Wrote and edited paper (40%)
Author Darryn Bryant	Identified useful existing results (20%) Proved new results (15%) Wrote and edited paper (40%)
Author Matthew Dean	Identified useful existing results (20%) Proved new results (15%) Wrote and edited paper (10%)
Author Barbara Maenhaut	Identified useful existing results (20%) Proved new results (15%) Wrote and edited paper (10%)

Contributions by others to the thesis

Darryn Bryant, Matthew Dean and Barbara Maenhaut contributed to the conception of the project and to proving of results. These authors also contributed to the paper incorporated in Chapter 2 as acknowledged on the previous page. Barbara Maenhaut contributed to the editing of this thesis for clarity.

Statement of parts of the thesis submitted to qualify for the award of another degree

None.

Acknowledgements

I would like to extend special thanks to Barbara Maenhaut for her good advice, helpful direction, ideas, and general academic support as my principal advisor. I would also like to thank Darryn Bryant, my associate advisor, for his advice and ideas. In addition, I would like to thank Matt Dean for letting me bounce attempted proofs off him.

I would like to acknowledge Murray Kane for his administrative support, and to thank my cousin John Barlow for helping me to establish myself in Brisbane for the duration of my candidature.

Financial support

This research was supported by an Australian Government Research Training Program Scholarship.

Keywords

graph decomposition, uniform decomposition, hamilton decomposition, cycle decomposition

Australian and New Zealand Standard Research Classifications (ANZSRC)

ANZSRC code: 010104 Combinatorics, 100%

Fields of Research Classification

FOR code: 0101, Pure Mathematics, 100%

Contents

List of Figures

List of Tables

List of Abbreviations used in the thesis

BIBD: Balanced incomplete block design.

(v, b, r, k, λ)**-BIBD:** Balanced incomplete block design with v vertices, b blocks, replication number r, block size k and index λ.

(v, k, λ)**-design:** Balanced incomplete block design with v vertices, block size k, index λ.

Chapter 1

Introduction

This thesis is structured as a sequence of four chapters. Chapter 1 introduces the problem of uniform graph decompositions, and provides a review of existing literature on cycle decompositions and uniform factorisations. Chapter 2 is a paper which was submitted to the Journal of Combinatorial Designs on 22 January 2018, revised and resubmitted to the same journal on 20 June 2018 (revised version included). The bibliography has been merged with the bibliography of this thesis. Chapter 3 discusses the use of computer programs to solve problems in uniform graph decomposition. Chapter 4 consists of a conclusion, a discussion of minor results obtained during the project, and a discussion of potential future directions for this research.

1.1 Definitions and notation

1.1.1 Graphs

In order to talk about graph decompositions, we must begin by defining graphs. A *graph* is a pair (V, E) such that the following are satisfied:

(1) V is a non-empty set of elements called *vertices*, and

(2) E is a multiset of elements called *edges* in which each edge is a multiset consisting of two (possibly identical) vertices.

The vertex set of a graph G is denoted by $V(G)$, and the edge multiset of G is denoted by $E(G)$.

The *union* of two graphs $G_1 \cup G_2$ is the graph $\{V(G_1) \cup V(G_2), E(G_1) \uplus E(G_2)\}$ (where $\uplus$ denotes multiset union, keeping multiple edges consisting of the same vertices). When using single-character vertex labels, we may refer to an edge $\{x, y\}$ as xy.

An edge consisting of two identical vertices is called a *loop*. A *simple graph* has no loops, and its edge multiset contains no duplicates; i.e. in a simple graph G, there cannot be $e_i = xy$ and $e_j = xy$ where $e_i, e_j \in E(G), i \neq j$ and $x, y \in V(G)$. Note that we use the term "graph" to include both simple graphs and multigraphs (graphs that may have loops and duplicate edges).

Consider an arbitrary graph G. Then a graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We then write $H \subseteq G$. If H is a subgraph of G and $V(H) = V(G)$, we say that H is a *spanning subgraph* of G.

Two graphs G, H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $\phi : V(G) \to V(H)$ which maps $E(G)$ to $E(H)$ under the operation $e\phi = \{v_i, v_j\}\phi = \{v_i\phi, v_j\phi\}$ for $e = \{v_i, v_j\} \in E(G)$ and $v_i, v_j \in V(G)$. Then ϕ is called an *isomorphism from G to H*. An isomorphism from G to G is called an *automorphism of G*, and the set of all automorphisms of a graph G with the operation composition of functions forms the *automorphism group of G*.

The *complete graph on n vertices*, denoted K_n, is a graph with n vertices where there is exactly one edge between each pair of distinct vertices. The *complete multigraph on n vertices with multiplicity μ*, denoted μK_n, is a graph with n vertices where there are exactly μ edges between each pair of distinct vertices. A complete multigraph does not contain loops.

A graph G is *bipartite* if there exists some partition of $V(G)$ into two sets V_1, V_2 such that every edge of G contains one vertex of V_1 and one vertex of V_2. A bipartite graph G is *complete* if $E(G)$ contains every edge v_1v_2 where $v_1 \in V_1, v_2 \in V_2$; a complete bipartite graph in which $|V_1| = n, |V_2| = m$ is denoted $K_{n,m}$.

A *path* $[v_1, v_2, ..., v_r]$ is a graph with vertex set $\{v_1, v_2, ..., v_r\}$ ($v_i \neq v_j$ whenever $i \neq j, 1 \leq i, j \leq r$) and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{r-1}, v_r\}\}$. We say that v_1 and v_r are the *endpoints* of the path, and that the path is *between* its endpoints. The *length* of a path is equal to its number of edges. A path with n vertices is denoted P_n, and is also referred to as an $(n-1)$-path (as it has length $n-1$). A graph G is *connected* if, for each pair of vertices $v_i, v_j \in V(G), i \neq j$, there is some path between v_i and v_j.

The *degree* of a vertex v in a loopless graph is the number of edges containing v. If every vertex in a graph G has degree k, we say G is *k-regular*. A *cycle* is a connected 2-regular graph. A cycle with vertex set $\{v_1, v_2, ..., v_r\}$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{r-1}, v_r\}, \{v_r, v_1\}\}$ is denoted $(v_1, v_2, v_3, ..., v_r)$. If G is a graph, H is a cycle, and H is a spanning subgraph of G, then H is a *Hamilton cycle of G*. We say that a cycle has *length* equal to its number of edges.

1.1.2 Factorisations and decompositions

Consider an arbitrary graph G. A *k-factor* of G is a k-regular spanning subgraph of G. If G has $2m$ vertices, a 1-factor of G consists of m nonadjacent edges. If G has $2m+1$ vertices, a subgraph consisting of m disjoint edges is a *near 1-factor*; note that there can be no 1-factor in a graph with an odd number of vertices. In the context of graph decompositions, we use the symbol I to denote an arbitrary 1-factor.

A *decomposition* of a graph G is a set $\{D_1, D_2, ..., D_r\}$ of subgraphs of G whose edge sets partition $E(G)$. If H is a graph, then a decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of a graph is an H-decomposition if each $D_i \in \mathcal{D}$ is isomorphic to H. A k-factorisation $\mathcal{F} = \{F_1, F_2, ..., F_r\}$ of a graph G is a decomposition of G where each $F_i \in \mathcal{F}$ is a k-factor of G. A decomposition of a graph G into cycles is a *cycle decomposition of G*, and a decomposition of G into Hamilton cycles is a *Hamilton decomposition of G*. It is clear that a Hamilton decomposition is a 2-factorisation.

Let $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ and $\mathcal{F} = \{F_1, F_2, ..., F_s\}$ be decompositions of graphs G and H respectively. Then $\mathcal{D}$ and $\mathcal{F}$ are *isomorphic*, denoted $\mathcal{D} \cong \mathcal{F}$, if there exists an isomorphism ϕ from G to H which maps $\{D_1, D_2, ..., D_r\}$ to $\{F_1, F_2, ..., F_s\}$ under the operation $D_i\phi = \{V(D_i)\phi, E(D_i)\phi\}$ for each $D_i \in \mathcal{D}$. Then ϕ is called an *isomorphism from $\mathcal{D}$ to $\mathcal{F}$*. An isomorphism from $\mathcal{D}$ to $\mathcal{D}$ is called an *automorphism of $\mathcal{D}$*, and the set of all automorphisms of a graph decomposition $\mathcal{D}$ with the operation composition of functions forms the *automorphism* *group of $\mathcal{D}$*. It is clear that the automorphism group of a decomposition of a graph G is a subgroup of the automorphism group of G.

1.1.3 Perfect and uniform 1-factorisations

A 1-factorisation $\{F_1, F_2, ..., F_r\}$ of a graph G is *perfect* if the union of any two distinct 1-factors $F_i \cup F_j$ is a Hamilton cycle in G, and *uniform* if the union of any two distinct 1-factors $F_i \cup F_j$ is isomorphic to the union of any other two, i.e. $F_i \cup F_j \cong F_k \cup F_l$ whenever $1 \leq i < j \leq r, 1 \leq k < l \leq r$. Clearly, every perfect 1-factorisation is uniform.

Similarly, a decomposition $\{D_1, D_2, ..., D_r\}$ of a graph is *uniform* if the union of any two distinct subgraphs $D_i \cup D_j$ is isomorphic to the union of any other two, i.e. $D_i \cup D_j \cong D_k \cup D_l$ whenever $1 \leq i < j \leq r, 1 \leq k < l \leq r$. There is no obvious way to extend the definition of perfect 1-factorisations to general graph decompositions.

Kotzig's conjecture [40], which remains unresolved, states that every complete graph of even order has a perfect 1-factorisation. Two infinite families of complete graphs are known to have perfect 1-factorisations: Kotzig [39] constructed a perfect 1-factorisation of K_{p+1}, and Anderson [3] and Nakamura [49] independently constructed perfect 1-factorisations of K_{2p} where p is an odd prime. Kobayashi [38] proved that Anderson's and Nakamura's constructions are isomorphic.

More recently, Bryant, Maenhaut and Wanless [16] constructed a new perfect 1-factorisation of each K_{p+1} where p is prime and $p \geq 11$, and proved that these perfect 1-factorisations are not isomorphic to those produced by Kotzig's construction.

Perfect 1-factorisations of K_n have also been constructed for all other even $n \leq 52$. Anderson constructed a perfect 1-factorisation of K_{16} [4] and K_{28} [5], Seah and Stinson constructed perfect 1-factorisations of K_{36} and K_{40} [55] [56], Ihrig, Seah and Stinson constructed a perfect 1-factorisation of K_{50} [33], and Wolfe constructed a perfect 1-factorisation of K_{52} [66]. Perfect 1-factorisations of some other large complete graphs have also been constructed. Dinitz and Stinson [26] constructed perfect 1-factorisations of K_{126}, K_{170}, K_{730}, K_{1370}, K_{1850}, K_{2198} and K_{3126}, and Kobayashi and Kiyasu-Zen'iti [37] constructed perfect 1-factorisations of K_{1332} and K_{6860}. Wanless [63] identified several complete graphs with perfect 1-factorisations and maintains an updated list at [64].

Wagner [61] took a different approach to Kotzig's conjecture. Consider all 1-factorisations $\mathcal{F}$ of K_n where n is even. For each $\mathcal{F}$, let $t_n(\mathcal{F})$ be the number of Hamilton cycles formed by unions of two 1-factors in $\mathcal{F}$. Then Wagner defined $c(n)$ to be the maximum value of $t_n(\mathcal{F})$ over all $\mathcal{F}$. Wagner proved that $c(n) \geq \frac{(n-1)\phi(n-1)}{2}$ where ϕ is the Euler totient, while Kotzig's conjecture is equivalent to saying $c(n) = \binom{n-1}{2}$ for each even n.

In 1994 Dinitz, Garnick and Mckay [25] counted all the nonisomorphic 1-factorisations of K_{12}, finding that there are 526 915 620 of them. Of these, they found that exactly 6 were uniform, 5 of which were perfect; hence, there is exactly one non-perfect uniform 1-factorisation of K_{12}, up to isomorphism. In this uniform 1-factorisation the union of two factors forms two disjoint 6-cycles. More recently, Meszka and Rosa [48] extended this result by enumerating all the uniform 1-factorisations of K_n where n is even and $n \leq 16$, except for possibly some perfect 1-factorisations of K_{16}.

Dinitz and Dukes [24] proved that if $k, m \in \mathbb{Z}$ such that k is even, $k \geq 6$ and m is positive, then there exists a uniform 1-factorisation of some large complete graph where the union of each pair of 1-factors has at least m cycles of length k.

Complete bipartite graphs are closely related to complete graphs. A complete bipartite graph $K_{n,m}$ is only regular if $n = m$, and as such can only have a 1-factorisation if $n = m$. Furthermore, Laufer [41] proved that if $K_{n,n}$ has a perfect 1-factorisation, then $n = 2$ or n is odd. It has been proved by several authors that if K_{n+1} has a perfect 1-factorisation, so does $K_{n,n}$ (see, for example, [62]); as such, if Kotzig's conjecture is true then $K_{n,n}$ has a perfect 1-factorisation for every odd integer n. For each odd prime p and $n = p^2$, Bryant, Maenhaut and Wanless [15] constructed a family of $\frac{p-1}{2}$ nonisomorphic perfect 1-factorisations of $K_{n,n}$. Bryant, Maenhaut and Wanless [16] later constructed new families of perfect 1-factorisations of $K_{p,p}$ where p is prime, in part using their new family of perfect 1-factorisations of K_{p+1}.

In 2013, perfect and uniform 1-factorisations of Cayley graphs began to be considered. Herke and Maenhaut [30] characterised the connected 3-regular circulant graphs with perfect 1-factorisations, and proved that if G is a connected 3-regular circulant graph on n vertices where $n > 6$, then there exists a perfect 1-factorisation of G if and only if $n = 2 \pmod 4$ and G is bipartite. They also proved that $\mathrm{Circ}(n, \{1, 2\})$ does not have a uniform 1-factorisation when $n > 6$, a slight extension of their result on perfect 1-factorisations.

Herke [31] constructed perfect 1-factorisations of several infinite classes of 4-regular circulant graphs, and proved that for all even $n > 6$ the circulant graph $\mathrm{Circ}(n, \{1, 4\})$ does not have a perfect 1-factorisation. Herke and Maenhaut [32] defined a class of graphs, provided necessary and sufficient conditions for an element of that class to form a Cayley graph, and used that class to construct an infinite family of connected bipartite 4-regular circulant graphs of order congruent to 2 (mod 4) which do not have perfect 1-factorisations. These results do not completely characterise the set of connected 4-regular circulant graphs with perfect 1-factorisations.

1.1.4 Cycle decompositions of complete multigraphs

Uniform cycle decompositions of complete multigraphs form a major component of this project. Here we discuss the existing literature on cycle decompositions of complete multigraphs.

We begin by restating a theorem of Bryant et al [12], which shows that the obvious necessary conditions for the existence of an m-cycle decomposition of μK_n are sufficient.

Theorem 1.1. [12]. *Let n, m, μ be integers such that $n \geq 2$ and $m, \mu \geq 1$. Then there exists an m-cycle decomposition of μK_n if and only if the following are true:*

(A) $\mu(n-1)$ *is even*

(B) $2 \leq m \leq n$

(C) $m | \frac{\mu n(n-1)}{2}$, *and*

(D) if $m = 2$, *then* μ *is even.*

In addition, if $n \geq 3$, there exists an m-cycle decomposition of $\mu K_n \setminus I$ if and only if the following are true:

(A) $\mu(n-1)$ *is odd*

(B) $3 \leq m \leq n$, *and*

(C) $m | \frac{\mu n(n-1)-n}{2}$.

While this result is relatively recent (2011), some partial results on cycle decompositions are substantially older. In 1892 Walecki [43] proved Theorem 1.1 in the specific case where $m = n$ and $\mu = 1$, i.e. for Hamilton decompositions of the complete graph, and in 1847 Kirkman [36] considered the case where $m = 3$.

We now consider two special properties that cycle decompositions can have. A cycle decomposition of a graph G is *cyclic* if the decomposition has an automorphism that permutes $V(G)$ in a single cycle. The existence of a cyclic Hamilton decomposition of K_n or $K_n - I$ where n is odd or even respectively, was resolved by Buratti and Del Fra [17] and Jordon and Morris [34] respectively. Building on these results, Buratti, Capparelli and Del Fra [18] proved that λK_n has a cyclic Hamilton decomposition if and only if $n \geq 3$, $\lambda(n-1)$ is even, $(n, \lambda) \neq (15, 1)$ and $(n, \lambda) \neq (p^\alpha, \lambda)$ where p is an odd prime and $\lambda \leq 2 \leq \alpha$. A cycle decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of a graph G is 2*-transitive on the cycles* if for each $D_i, D_j, D_k, D_l \in \mathcal{D}$ such that $D_i \neq D_j, D_k \neq D_l$, there exists some automorphism ϕ of G such that $\phi D_i = D_k$ and $\phi D_j = D_l$. Mazzuoccolo [46] proved that if $n > 3$ and K_n has a Hamilton decomposition which is 2-transitive on the cycles, then $n = 5$. Note that 2-transitivity on the cycles in a cycle decomposition implies uniformity, but is a stronger property than uniformity.

1.1.5 Balanced incomplete block designs

Graph decompositions have some connections with balanced incomplete block designs (henceforth abbreviated as BIBDs). In the course of this project we used these connections and existing results for BIBDs to construct some uniform graph decompositions and to prove that others do not exist.

Let v, b, r, k, λ be positive integers where $v > k \geq 2$. A *balanced incomplete block design* or *BIBD* with the parameters (v, b, r, k, λ) is a pair $(V, \mathcal{B})$ where the following are true:

(A) V is a set of points with $|V| = v$.

(B) $\mathcal{B}$ is a collection of sets with $|\mathcal{B}| = b$. We say that the members of $\mathcal{B}$ are the *blocks* of the design.

(C) Each $B_i \in \mathcal{B}$ is a k-subset of V.

(D) Each point in V occurs in precisely r blocks. We say that r is the *replication number* of the design.

(E) Each pair of distinct points in V occurs in precisely λ blocks. We say that λ is the *index* of the design.

We will see that the values of b and r can be determined from v, k, λ. Thus we say a (v, k, λ)-*design* is a BIBD with v vertices, block size k, and index λ.

Theorems 1.2 and 1.3 and Lemmas 1.5, 1.6 and 1.8 are well known and can be found in many combinatorics textbooks, for example [58].

Theorem 1.2. *Let $(V, \mathcal{B})$ be a (v, b, r, k, λ)-BIBD. Then:*

(A) $r = \frac{\lambda(v-1)}{k-1}$,

(B) $b = \frac{\lambda v(v-1)}{k(k-1)}$*, and*

(C) $b \geq v$ *(Fisher's inequality).*

Proof. Consider a point $x \in V$. We will use two different methods to count the number of occurrences of x in pairs of points where each pair is a subset of a block, in order to prove (A).

For each $y \in V \setminus \{x\}$, $\{x, y\}$ occurs in λ blocks. Since $|V \setminus \{x\}| = v - 1$, it follows that there are $\lambda(v-1)$ occurrences of pairs $\{x, y\}$ in blocks.

The point x occurs in r blocks. Each block containing x contains $k - 1$ other vertices. Thus there are $r(k-1)$ occurrences of pairs $\{x, y\}$ in blocks.

It follows that $\lambda(v-1) = r(k-1)$ and so $r = \frac{\lambda(v-1)}{k-1}$ as required.

There are v points in the design, each occurring in r blocks. There are b blocks each containing k points. Thus $vr = bk$ and so by (A), $b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k(k-1)}$ as required.

We have now proved (A) and (B). In order to prove (C) we need to define the incidence matrix of a design. The *incidence matrix* of a (v, b, r, k, λ)-BIBD is a $v \times b$ matrix $\mathbf{M} = (m_{i,j})$ such that $m_{i,j} = 1$ if the vertex i is in the block j, or 0 otherwise.

Then $\mathbf{M}\mathbf{M}^T$ is a $v \times v$ matrix in which every value on the diagonal is r and every value not on the diagonal is λ; since $\lambda \neq r$, it follows that $\mathbf{M}\mathbf{M}^T$ has non-zero determinant and so the rank of $\mathbf{M}\mathbf{M}^T$ is v. However, the rank of $\mathbf{M}$ must be at least the rank of $\mathbf{M}\mathbf{M}^T$ and at most b (since $\mathbf{M}$ is a $v \times b$ matrix); thus $v \leq b$. $\square$

A (v, b, r, k, λ)-BIBD is *symmetric* if $r = k$ (and equivalently $v = b$, the extreme case of Fisher's inequality). It follows from Theorem 1.2 that in a symmetric design, $v = \frac{k(k-1)}{\lambda} + 1$.

Some balanced incomplete block designs can be constructed from others. We introduce three constructions of BIBDs from other BIBDs: dual designs, complements of designs, and residual designs. We use dual designs solely to prove Theorem 1.3. The complement and residual designs are used later in this thesis.

Let $(V, \mathcal{B})$ be a (v, b, r, k, λ)-BIBD and let $V = \{v_1, v_2, ..., v_v\}$ and $\mathcal{B} = \{B_1, B_2, ..., B_b\}$. The *dual* of $(V, \mathcal{B})$ is the pair $(\mathcal{B}, V^*)$ where V^* is the set $\{V_i : V_i = \{B_j : v_i \in B_j, B_j \in \mathcal{B}\}$ for each $v_i \in V\}$. We will show in the proof of Theorem 1.3 that the dual of a symmetric BIBD is a symmetric BIBD.

The *complement* of a BIBD $(V, \mathcal{B})$ is the pair $(V, \mathcal{B}^*)$ where $\mathcal{B}^* = \{V \setminus B_i : B_i \in \mathcal{B}\}$. We will prove in Lemma 1.5 that the complement of a (v, k, λ)-BIBD is a BIBD whenever $k \leq v - 2$.

Let $(V, \mathcal{B})$ be a symmetric BIBD and let B be a member of $\mathcal{B}$. In addition, let $\mathcal{B}^*$ denote the set $\{B_i \cap B^c : B_i \in \mathcal{B} \setminus \{B\}\}$. Then the pair $(V \setminus B, \mathcal{B}^*)$ is a *residual design* of $(V, \mathcal{B})$. We will prove in Lemma 1.6 that a residual design of a symmetric BIBD is a BIBD.

Theorem 1.3. *A* (v, k, λ)*-BIBD is symmetric if and only if each pair of blocks intersects in* λ *points.*

Proof. Let $(V, \mathcal{B})$ be a (v, b, r, k, λ)-BIBD in which $V = \{v_1, v_2, ..., v_v\}, \mathcal{B} = \{B_1, B_2, ..., B_b\}$, and each pair of blocks intersects in μ points. Then in the dual $(\mathcal{B}, V^*)$, each pair of points $B_i, B_j \in \mathcal{B}$ occurs precisely in the μ blocks $V_l \in V^*$ such that $v_l \in B_i \cap B_j$. In addition, each $V_i \in V^*$ is an r-subset of $\mathcal{B}$, as there are r occurrences of the vertex v_i in blocks of $\mathcal{B}$, and each

$B_i \in \mathcal{B}$ occurs in precisely the k blocks $V_j \in V^*$ where $v_j \in B_i$. It follows that $(\mathcal{B}, V^*)$ is a (b, v, k, r, μ)-BIBD, and so $b \leq v$ by Fisher's inequality. Thus $b = v$ and $r = k$, and so $(V, \mathcal{B})$ is symmetric.

Let $(V, \mathcal{B})$ be a symmetric (v, b, r, k, λ)-BIBD and let $\mathbf{M}$ be the incidence matrix of $(V, \mathcal{B})$. In addition, let $\mathbf{J}$ be the $v \times v$ matrix in which every element is 1, and $\mathbf{I}$ be the $v \times v$ identity matrix. Since each vertex is in exactly r blocks, each row of $\mathbf{M}$ contains exactly r elements with the value 1, and $v - r$ elements with the value 0; thus $r\mathbf{J} = \mathbf{MJ}$. By the proof of Theorem 1.2, $\mathbf{M}$ is non-singular and so $\mathbf{MJ} = \mathbf{JM} = k\mathbf{J}$.

We proved in Theorem 1.2 that $\mathbf{MM}^T = \lambda\mathbf{J} + (r - \lambda)\mathbf{I}$. Thus $\mathbf{M}^T\mathbf{M} = \mathbf{M}^{-1}\mathbf{MM}^T\mathbf{M} = \mathbf{M}^{-1}(\lambda\mathbf{J} + (r - \lambda)\mathbf{I})\mathbf{M} = \mathbf{M}^{-1}\mathbf{M}(\lambda\mathbf{J} + (r - \lambda)\mathbf{I}) = (\lambda\mathbf{J} + (r - \lambda)\mathbf{I})$. It follows that the dual of $(V, \mathcal{B})$ is a symmetric (v, k, λ)-BIBD, and so each pair of blocks in $(V, \mathcal{B})$ intersects in λ points as required. □

Corollary 1.4. *In a non-symmetric (v, k, λ)-BIBD, there exist two pairs of distinct blocks $\{B_i, B_j\}, \{B_k, B_l\}$ of the design such that $|B_i \cap B_j| \neq |B_k \cap B_l|$.*

Lemma 1.5. *Let $(V, \mathcal{B})$ be a (v, b, r, k, λ)-BIBD with $k \leq v - 2$. Then the complement of $(V, \mathcal{B})$ is a $(v, b, b - r, v - k, b - 2r + \lambda)$-BIBD.*

Proof. Let $(V, \mathcal{B}^*)$ be the complement of $(V, \mathcal{B})$. Then each block $B_i^* = V \setminus B_i$ in $\mathcal{B}^*$ consists of $v - k$ points, each point is in $b - r$ blocks of $\mathcal{B}^*$, and each pair of points $\{x, y\}$ occurs in $b - 2r + \lambda$ blocks of $\mathcal{B}^*$ (as there are r blocks containing x and r containing y, but λ of these blocks contain both x and y).

In addition, $v - k \geq 2$ (as $k \leq v - 2$), $b - r \geq 2$ (as $b - r = \frac{vr}{k} - r = \frac{(v-k)r}{k} \geq \frac{2r}{k}$ and $r \geq k$), and $b - 2r + \lambda \geq 1$ (as every pair of points occurs in $b - 2r + \lambda$ blocks of $(V, \mathcal{B}^*)$ and at least one pair occurs in each block B_i^*). It follows that $(V, \mathcal{B}^*)$ is a BIBD with the parameters $(v, b, b - r, v - k, b - 2r + \lambda)$. □

Lemma 1.6. *A residual of a BIBD $(V, \mathcal{B})$ is a BIBD if and only if $(V, \mathcal{B})$ is symmetric.*

Proof. Let $(V, \mathcal{B})$ be a (v, k, λ)-BIBD. Suppose $(V, \mathcal{B})$ is symmetric and let B be a member of $\mathcal{B}$. For each block $B_i \in \mathcal{B} \setminus \{B\}$, let $B_i^* = B_i \setminus B$, and let $\mathcal{B}^*$ be the set of all such B_i^* (and so $|\mathcal{B}^*| = b - 1 = v - 1$). Then $|B_i^*| = k - \lambda$ for each i, since $|B_i| = k$ and $|B \cap B_i| = \lambda$ by Theorem 1.3. In addition, it is clear that each point $a \notin B$ or each pair of points $a, b \notin B$ occurs in precisely those (r or λ, respectively) blocks B_i^* where $a \in B_i$ or $\{a, b\} \subseteq B_i$, respectively. It follows that any residual of $(V, \mathcal{B})$ is a $(v - k, v - 1, k, k - \lambda, \lambda)$-BIBD.

Suppose $(V, \mathcal{B})$ is non-symmetric and let B be a member of $\mathcal{B}$. For each block $B_i \in \mathcal{B} \setminus \{B\}$, let $B_i^* = B_i \setminus B$, and let $\mathcal{B}^*$ be the set of all B_i^*. Then there exist two blocks B_i, B_j such that $|B_i \cap B| \neq |B_j \cap B|$, since $(V, \mathcal{B})$ is not symmetric. It follows that $|B_i^*| \neq |B_j^*|$ and so the pair $(V \setminus B, \mathcal{B}^*)$ is not a BIBD. Thus any residual of $(V, \mathcal{B})$ is not a BIBD. □

From this proof we obtain the following:

Corollary 1.7. *A residual (v, b, r, k, λ)-BIBD must have $r = k + \lambda$.*

Since the condition $r = k + \lambda$ considerably restricts the possible values of v, k, λ, there is a term for designs meeting this condition: a (v, b, r, k, λ)-design is *quasi-residual* if $r = k + \lambda$.

Several authors have considered quasi-residual designs which are not residual. For example, Mackenzie-Fleming constructed an infinite family of quasi-residual Hadamard designs which are not residual [45], and an infinite family of quasi-residual $(2(3^{d+1}), 2(3^d), 3^d)$-BIBDs (where $d > 1$) which are not residual [44].

Hadamard designs are a particular type of symmetric BIBD. We encountered Hadamard designs repeatedly in the course of this project. A *Hadamard design* is a BIBD with parameters $(v, b, r, k, \lambda) = (n, n, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-3}{4})$ for some integer n.

Since λ is a positive integer for any (v, k, λ)-BIBD, it follows that any Hadamard design must have $n \equiv 3 \pmod 4$ and $n \geq 7$. It has been proved that whenever n is a prime, $n \equiv 3 \pmod 4$ and $n \geq 7$, there exists a Hadamard design with parameters $(n, \frac{n-1}{2}, \frac{n-3}{4})$; see, for example, [58]. The standard proof relies on difference sets. Let G be an additive group of order v and let k, λ be integers less that v. Then a (v, k, λ)-*difference set* in G is a k-subset D of G where the multiset $\{a - b : a, b \in D, a \neq b\}$ contains each non-identity element of G exactly λ times.

A difference set can be used to construct a symmetric BIBD. Let D be a (v, k, λ)-difference set of G, let V be the underlying set of G and let $\mathcal{B} = \{D + g : g \in G\}$. Then it can be proved (see for example [58]) that the pair $(V, \mathcal{B})$ is a symmetric (v, k, λ)-BIBD. When n is a prime power and $n \equiv 3 \pmod 4$, the set $\{i^2 : i \in \mathbb{Z}_n \setminus 0\}$ is an $(n, \frac{n-1}{2}, \frac{n-3}{4})$-difference set and so the set $\{\{i^2 : i \in \mathbb{Z}_n \setminus 0\} + j : j \in \mathbb{Z}_n\}$ gives the blocks of a Hadamard design. The construction of $\mathcal{Y}_{q,\omega}$ given in Definition 2.9 of this thesis is based on this difference set.

Hadamard designs are associated with Hadamard matrices. A *Hadamard matrix of order n* is an $n \times n$ matrix $\mathbf{H}$ such that each entry of $\mathbf{H}$ is ± 1 and $\mathbf{H}\mathbf{H}^T = n\mathbf{I}$ where $\mathbf{I}$ is the $n \times n$ identity matrix. It is clear that multiplying any row or column of a Hadamard matrix by -1 produces another Hadamard matrix, and so any Hadamard matrix can be transformed into one where every element of the first row and first column is 1. We say such a Hadamard matrix is *standardised*. We will see that a standardised Hadamard matrix of order $n+1 > 4$ is equivalent to the incidence matrix of a Hadamard design with parameters $(n, \frac{n-1}{2}, \frac{n-3}{4})$.

Lemma 1.8. *There exists a Hadamard matrix of order $n + 1$ where $n > 3$ if and only if there exists a Hadamard design with parameters $(n, \frac{n-1}{2}, \frac{n-3}{4})$.*

Proof. Suppose $\mathbf{H} = (h_{i,j})$ is a standardised Hadamard matrix of order $n + 1$ where $n > 3$, and let r_i denote the i'th row of $\mathbf{H}$ for $1 \leq i \leq n + 1$. Then $r_1 \cdot r_i = 0$ for any $1 < i \leq n + 1$, and so r_i contains $\frac{n+1}{2}$ elements with the value 1 and $\frac{n+1}{2}$ with the value -1.

Let $\mathbf{M} = (m_{i,j})$ be the matrix formed by deleting the first row and column of $\mathbf{H}$ and replacing each -1 entry with 0. Then each row and column of $\mathbf{M}$ contains $\frac{n-1}{2}$ elements with the value 1 and $\frac{n+1}{2}$ elements with the value 0. In addition, since $\mathbf{H}\mathbf{H}^T$ has entries $h_{i,j} = 0$ whenever $i \neq j$, it is clear that each $r_i \cdot r_j = 0$ for $i \neq j$. It follows that $\frac{n+1}{2}$ of the pairs $(h_{i,k}, h_{j,k})$ (where $1 \leq k \leq n + 1$) are either $(1, -1)$ or $(-1, 1)$, while the other half are $(1, 1)$ or $(-1, -1)$. Since r_i and r_j each contain $\frac{n+1}{2}$ ones, it is clear that precisely one-quarter of the pairs $(h_{i,k}, h_{j,k})$ are $(1, 1)$. Furthermore, since $h_{i,1} = h_{j,1} = 1$, it follows that each pair of distinct rows of $\mathbf{M}$ have ones in exactly $\frac{n-3}{4}$ of the same columns. By the same argument, each pair of distinct columns of $\mathbf{M}$ have ones in exactly $\frac{n-3}{4}$ of the same rows, and each column of $\mathbf{M}$ has $\frac{n-1}{2}$ ones.

Let $V = \{1, 2, ..., n\}$ and let $B_i = \{j : m_{i,j} = 1\}$ for each $i \in V$. Then $|B_i| = \frac{n-1}{2}$ and $|B_i \cap B_j| = \frac{n-3}{4}$ for each $i, j \in V$ where $i \neq j$. In addition, each point i is in $\frac{n-1}{2}$ blocks and each pair of points occurs in λ blocks in the set $\mathcal{B} = \{B_i : 1 \leq i \leq n\}$. It follows that $(V, \mathcal{B})$ is an $(n, \frac{n-1}{2}, \frac{n-3}{4})$-BIBD, that is, a Hadamard design.

Conversely, suppose $(V, \mathcal{B})$ is an $(n, \frac{n-1}{2}, \frac{n-3}{4})$-BIBD. Let $\mathbf{M} = (m_{i,j})$ be the incidence matrix of $(V, \mathcal{B})$, and let $\mathbf{H}$ be the $(n+1) \times (n+1)$ matrix $(h_{i,j})$ in which

$$h_{i,j} = \begin{cases} 1, & \text{if } i = 1 \text{ or } j = 1 \\ -1, & \text{if } m_{i-1,j-1} = 0 \\ 1, & \text{if } m_{i-1,j-1} = 1 \end{cases}$$

Let r_i denote the i'th row of $\mathbf{H}$. Then $r_i \cdot r_i = n + 1$, as each row of $\mathbf{H}$ contains $\frac{n+1}{2}$ ones and $\frac{n+1}{2}$ minus ones (or $n+1$ ones and no minus ones if $i = 1$). When $i \neq 1$, we have $r_i \cdot r_1 = 0$, as r_1 consists entirely of ones and r_i contains $\frac{n+1}{2}$ ones and $\frac{n+1}{2}$ minus ones. When $i \neq j$ and $i, j > 1$, we have $r_i \cdot r_j = 0$, as r_i and r_j contain $\frac{n+1}{2}$ ones each and $\frac{n+1}{2}$ minus ones each, and precisely half of the ones in r_i are multiplied by ones in r_j by the dot product (since $|B_{i-1} \cap B_{j-1}| = \frac{n-3}{4}$). It follows that $\mathbf{H}\mathbf{H}^T = n\mathbf{I}$ and so $\mathbf{H}$ is a Hadamard matrix. □

Corollary 1.9. *If there exists a Hadamard matrix of order $n > 4$ then $n \equiv 0 \pmod 4$.*

It can then be shown (see for example [58]) that if there exist Hadamard matrices of order n_1 and n_2, then there exists a Hadamard matrix of order $n_1 n_2$ and thus a Hadamard $(n_1 n_2 - 1, \frac{n_1 n_2}{2} - 1, \frac{n_1 n_2}{4} - 1)$-design.

Chapter 2

Uniform decompositions of complete multigraphs into cycles

This chapter is the publication D. Berry, D. Bryant, M. Dean, B. Maenhaut, Uniform decompositions of complete multigraphs into cycles (originally submitted to the Journal of Combinatorial Designs on 22/01/18, revised version, which is included here, resubmitted to the Journal of Combinatorial Designs on 20/06/18).

2.1 Introduction

A decomposition of a graph into edge-disjoint subgraphs is *uniform* if the union of any two distinct subgraphs is isomorphic to the union of any other two. The notion of uniformity has been considered in the context of 1-factorisations, with perfect 1-factorisations being a special case where the union is required to be a Hamilton cycle. Here, we investigate uniform decompositions of complete multigraphs into cycles. These have interesting connections with Hadamard designs, quasiresidual designs that are not residual, and orthogonal double covers of graphs. We make use of the Bruck-Ryser-Chowla Theorem and Fisher's Inequality in our investigations, and we encounter Pell's equation in connection with simultaneously triangular and centred triangular numbers.

The m-cycle with vertices $x_1, x_2, \ldots, x_m$ and edges $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}$ and $\{x_m, x_1\}$ will be denoted $(x_1, x_2, \ldots, x_m)$, and the complete graph with vertex set N and multiplicity μ, which has μ edges joining each pair of distinct vertices in N, will be denoted μK_N. Throughout the paper we assume $\mu \geq 1$ is an integer, and when $\mu = 1$ we may write just K_N instead of $1K_N$. If $n = |N|$, then the notation μK_n is used to denote any graph isomorphic to μK_N. A *decomposition* of a graph K is a set $\{X_1, X_2, \ldots, X_t\}$ of subgraphs of K such that $E(X_i) \cap E(X_j) = \emptyset$ for $1 \leq i < j \leq t$ and $E(X_1) \cup E(X_2) \cup \cdots \cup E(X_t) = E(K)$. A decomposition $\{X_1, X_2, \ldots, X_t\}$ is said to be *uniform* if $X_i \cup X_j \cong X_k \cup X_l$ for $1 \leq i < j \leq t$ and $1 \leq k < l \leq t$.

The topic of this paper is uniform decompositions of complete multigraphs into m-cycles. In Section 2.2, we establish several necessary conditions for existence, thereby showing that for $n > 3$, any uniform decomposition of μK_n into m-cycles falls into one of the following four cases.

(A) uniform decompositions of μK_n into n-cycles where $(n, \mu) \in \{(5, 1), (7, 1), (4, 2), (5, 3)\}$.

(B) uniform decompositions of $2K_n$ into m-cycles where $m = n - 1$.

(C) uniform decompositions of K_n into m-cycles where $n \equiv 3 \pmod 4$ and $m = (n-1)/2$.

(D) uniform decompositions of K_n into m-cycles where $2m(m+1) = n(n-1)$.

For $n \leq 3$, uniform decompositions of μK_n into m-cycles are easily classified. The case $n = 1$ is trivial. There is a decomposition of μK_2 into 2-cycles if and only if μ is even, and these decompositions are uniform. There is a decomposition of μK_3 into 2-cycles if and only if μ is even, and the decomposition is uniform if and only if $\mu = 2$. Finally, there is a decomposition of μK_3 into 3-cycles for all $\mu \geq 1$, and these decompositions are uniform. In the remainder of the paper we focus on decompositions of μK_n where $n \geq 4$.

In Section 2.3, we construct uniform decompositions of μK_n into n-cycles for each of the four values of (n, μ) in Case (A). In each of Cases (B) and (C), we show that there is a uniform decomposition whenever n is a power of a prime, and also that there is a uniform decomposition of $2K_6$ into 5-cycles. We do not have any examples of uniform decompositions in Case (D). In Section 2.4 we determine the isomorphism classes of the uniform decompositions presented in Section 2.3. Somewhat surprisingly, we find that in constructions based on the finite field $\mathbb{F}_q$, distinct (non-isomorphic) uniform decompositions of $2K_q$ into $(q-1)$-cycles, and of K_q into $((q-1)/2)$-cycles, can be obtained by choosing distinct primitive elements of $\mathbb{F}_q$.

Uniform decompositions into m-cycles have interesting connections with balanced incomplete block designs. In Lemma 2.3, we show that any uniform decomposition of μK_n into m-cycles gives rise to a balanced incomplete block design, or (v, b, r, k, λ)-BIBD, which we call the *associated design* of the decomposition. See [22, 58] for background on design theory. In Section 2.2, we use this result together with Fisher's Inequality, to rule out the existence of uniform decompositions in many cases, see Corollary 2.4.

The associated designs of the uniform decompositions in Cases (A) and (B) are uninteresting, with each block containing all of the points for decompositions in Case (A), and with the blocks being every $(n-1)$-subset of an n-set for decompositions in Case (B). The associated designs of the uniform decompositions in Case (C) are Hadamard designs (a Hadamard design is a $(v, v, (v-1)/2, (v-1)/2, (v-1)/4)$-BIBD), see Theorem 2.6. The associated designs of decompositions in Case (D) are quasiresidual designs which are not residual, see Lemma 2.5. A BIBD is *residual* if it can be obtained from a symmetric BIBD by deleting a block and all of the points belonging to that block. The residual of a (v, v, k, k, λ)-BIBD is a $(v-k, v-1, k, k-\lambda, \lambda)$-design. A (v, b, r, k, λ)-BIBD with $r = k + \lambda$ is called *quasiresidual*.

As mentioned above, the topic of uniform decompositions of graphs has been considered previously in the context of 1-factorisations (decompositions into 1-factors), and interest has focused on the special case of *perfect* 1-factorisations, where the union of any two distinct 1-factors is a Hamilton cycle. Kotzig's 1964 conjecture [40] that there is a perfect 1-factorisation of K_n for all even n remains unresolved, and it is also unknown whether there is a uniform 1-factorisation of K_n for all even n. Several infinite families of perfect or uniform 1-factorisations of complete graphs, and several sporadic examples, are known. See [16, 24, 48, 66] for some of the more recent results on uniform and perfect 1-factorisations of complete graphs, and see [6] for a survey.

If the requirement of uniformity is removed, then for $n, m \geq 3$ and $\mu \geq 1$ there exists a decomposition of μK_n into m-cycles if and only if $m \leq n$, $\mu(n-1)$ is even, and m divides $\mu\binom{n}{2}$ [12]. The case $\mu = 1$ was settled in [1, 53]. The more general problem of decomposing μK_n into cycles of arbitrary specified lengths has also been settled [13, 14].

2.2 Necessary conditions for existence

We begin this section with the following lemma which shows that in any uniform decomposition of μK_n into m-cycles, the number of triangles in the union of any two of the m-cycles from the decomposition is given by the values of μ, n and m.

Lemma 2.1. *For $m \geq 4$, if $\mathcal{X}$ is a uniform decomposition of μK_n into m-cycles and X and X' are distinct cycles in $\mathcal{X}$, then the number of triangles in $X \cup X'$ is*

$$\frac{4\mu m^2}{\mu n(n-1) - 2m}.$$

In particular, if there exists a uniform decomposition of μK_n into m-cycles, then the above quantity is an integer.

Proof. Let $\mathcal{X}$ be a uniform decomposition of μK_n into m-cycles and consider those triangles of μK_n whose edge sets intersect exactly two cycles of $\mathcal{X}$. For any cycle $X \in \mathcal{X}$, there are μm such triangles that have exactly two edges in X. Since $|\mathcal{X}| = \mu n(n-1)/2m$, it follows that there are exactly $\mu^2 n(n-1)/2$ triangles of K_n whose edges lie in exactly two cycles of $\mathcal{X}$.

Now, since $\mathcal{X}$ is uniform, for any distinct $X, X' \in \mathcal{X}$, the number of triangles in $X \cup X'$ is a constant C, independent of X and X'. Since there are $\binom{|\mathcal{X}|}{2} = \binom{\mu n(n-1)/2m}{2}$ pairs of distinct cycles in $\mathcal{X}$, and since these pairs contain all the triangles considered in the previous paragraph, it follows that

$$C = \frac{\mu^2 n(n-1)}{2} / \binom{\frac{\mu n(n-1)}{2m}}{2} = \frac{4\mu m^2}{\mu n(n-1) - 2m}$$

is an integer. □

We now consider the case $m = n$, that is, uniform decompositions of μK_n into Hamilton cycles. Lemma 2.1 will be used to show that uniform decompositions of μK_n into Hamilton cycles may exist for only for a few small values of μ and n, and existence of two of these is ruled out by exhaustive computer search.

Corollary 2.2. *For $n \geq 4$, if there exists a uniform decomposition of μK_n into Hamilton cycles, then*

$$(n, \mu) \in \{(5,1), (7,1), (4,2), (5,3)\}.$$

Proof. If there exists a uniform decomposition of μK_n into Hamilton cycles with $n \geq 4$, then by Lemma 2.1 we have that $(4\mu n^2)/(\mu n(n-1) - 2n) = (4\mu n)/(\mu(n-1) - 2)$ is an integer. It is routine to check that the only values of n and μ (with $n \geq 4$) for which $(4\mu n)/(\mu(n-1) - 2)$ is an integer are

$$(n, \mu) \in \{(5,1), (7,1), (4,2), (5,3)\}$$

and

$$(n, \mu) \in \{(4,1), (6,1), (9,1), (15,1), (6,2), (10,2), (7,5), (4,6), (6,10)\}.$$

Thus, to complete the proof we need only rule out those values of n and μ in the second of these sets.

Firstly, there are no decompositions of K_4 nor K_6 into Hamilton cycles (and so certainly none that are uniform). Secondly, the number of 2-cycles in μK_n is $\binom{\mu}{2}\binom{n}{2}$. Thus, if there exists a uniform decomposition of μK_n into Hamilton cycles, then these 2-cycles are distributed equally amongst the $\binom{\mu(n-1)/2}{2}$ pairs of Hamilton cycles in the decomposition. Thus, $\binom{\mu(n-1)/2}{2}$ divides $\binom{\mu}{2}\binom{n}{2}$, but this is not the case when $(n,\mu) \in \{(6,2),(10,2),(4,6),\ (6,10)\}$.

Now suppose $\mathcal{X}$ is a uniform decomposition of $5K_7$ into Hamilton cycles. For each pair $\{x,y\}$ of distinct vertices in $5K_7$, let

$$B_{x,y} = \{X \in \mathcal{X} : x \text{ and } y \text{ are adjacent in } X\},$$

and let $\mathcal{B} = \{B_{x,y} : x,y \in V(5K_7), x \neq y\}$. Using the same counting as in the preceding paragraph, for any two distinct cycles X and X' in $\mathcal{X}$, there are exactly two 2-cycles in $X \cup X'$. Since $|\mathcal{X}| = 15$, it follows that $(\mathcal{X},\mathcal{B})$ is a $(15,21,7,5,2)$-BIBD. It is well known that no such design exists (see [22]), and so there is no uniform decomposition of $5K_7$ into Hamilton cycles.

This leaves only the cases $(n,\mu) = (9,1)$ and $(n,\mu) = (15,1)$, and we have shown by exhaustive computer search that there is no uniform decomposition in either of these cases. $\square$

We remark that up to isomorphism, there are 122 distinct decompositions of K_9 into Hamilton cycles [21, 27]. Our computer search shows that none of these is uniform. In Section 2.3 we present uniform decompositions of μK_n into Hamilton cycles for $(n,\mu) = (5,1),(7,1),(4,2)$ and $(5,3)$.

Our next result shows that a uniform decomposition of μK_n into m-cycles implies the existence of a (v,b,r,k,λ)-BIBD, where the values of v, b, r, k and λ are functions of μ, n and m.

Lemma 2.3. *If $n \geq 4$ and there exists a uniform decomposition of μK_n into m-cycles, then there exists a (v,b,r,k,λ)-BIBD with $v = \mu n(n-1)/2m$, $b = n$, $r = m$, $k = \mu(n-1)/2$, and $\lambda = m^2(\mu(n-1)-2)/(\mu n(n-1)-2m)$.*

Proof. Let N be a set such that $|N| = n$ and let $\mathcal{X} = \{X_1, X_2, \ldots, X_t\}$ be a uniform decomposition of μK_N into m-cycles. We construct a design $(V,\mathcal{B})$ with point set $V = \mathcal{X}$, and where for each vertex $x \in N$ we have a block $B_x \in \mathcal{B}$ such that the points in B_x are the cycles of $\mathcal{X}$ containing x. That is, for each vertex $x \in N$ we let $B_x = \{X \in \mathcal{X} : x \in V(X)\}$, and we let $\mathcal{B} = \{B_x : x \in N\}$.

It is clear that the design $(V,\mathcal{B})$ has $\mu n(n-1)/2m$ points, n blocks, constant block size $\mu(n-1)/2$, and replication number m. Moreover, since $\{X_1, X_2, \ldots, X_t\}$ is uniform, $|V(X_i) \cap V(X_j)|$ is a constant λ for $1 \leq i < j \leq t$, independent of i and j, and it follows that the design $(V,\mathcal{B})$ is balanced.

It remains only to show that $\lambda = m^2(\mu(n-1)-2)/(\mu n(n-1)-2m)$. Since there are n vertices and $\binom{\mu(n-1)/2}{2}$ pairs of distinct cycles on each vertex, and since there are $\binom{\mu n(n-1))/2m}{2}$ pairs of distinct cycles, each pair of distinct cycles has exactly

$$n\binom{\frac{\mu(n-1)}{2}}{2}\Big/\binom{\frac{\mu n(n-1)}{2m}}{2}$$

common vertices. Simplifying this expression we obtain $\lambda = m^2(\mu(n-1)-2)/(\mu n(n-1)-2m)$. $\square$

Lemma 2.3 allows us to use results from design theory to rule out the existence of uniform decompositions of μK_n into m-cycles for many values of μ, n and m. In particular, we can use Fisher's Inequality to prove the following result.

Corollary 2.4. *If $n \geq 4$ and there exists a uniform decomposition of μK_n into m-cycles, then*

(a) $m = n$ *and* $(n, \mu) \in \{(5,1), (7,1), (4,2), (5,3)\}$*; or*

(b) $\mu = 1$ *and* $(n-1)/2 \leq m \leq n-1$*; or*

(c) $\mu = 2$ *and* $m = n-1$.

Proof. If $m = n$, then $(n, \mu) \in \{(5,1), (7,1), (4,2), (5,3)\}$ by Corollary 2.2. Thus, we can assume $m < n$. By Lemma 2.3, if there exists a uniform decomposition of μK_n into m-cycles, then there exists a BIBD with $\mu n(n-1)/2m$ points and n blocks. Thus, by Fisher's Inequality we have $\mu n(n-1)/2m \leq n$. That is, $\mu(n-1) \leq 2m$. It is easy to see that if $n \geq 4$, $m < n$, and $\mu(n-1) \leq 2m$, then either $\mu = 1$ and $(n-1)/2 \leq m \leq n-1$, or $\mu = 2$ and $m = n-1$. □

It turns out that the values of μ, n, and m satisfying (b) of Corollary 2.4 fall into two cases, namely Cases (C) and (D) mentioned in the introduction. Case (C) is where $m = (n-1)/2$, and Lemma 2.5 shows that any other values of μ, n, and m satisfying (b) of Corollary 2.4 fall into Case (D).

Lemma 2.5. *If there exists a uniform decomposition of K_n into m-cycles with $(n-1)/2 < m < n$, then $n \geq 697$, $2m(m+1) = n(n-1)$, and there exists a quasiresidual $(m+1, n, n-m, (2m-n+3)/2, (3n-4m-3)/2)$-BIBD that is not residual.*

Proof. If $m = 3$, then by the hypothesis $(n-1)/2 < m < n$ we have $3 < n < 7$. But there is no decomposition of K_n into 3-cycles for $3 < n < 7$, and so we can assume $m \geq 4$. Let $\mathcal{X}$ be a uniform decomposition of K_n into m-cycles with $(n-1)/2 < m < n$, and let T be the number of triangles in the union of two distinct cycles of $\mathcal{X}$. By Lemma 2.1, $T = 4m^2/(n(n-1) - 2m)$.

If $m = n-1$, then we have $T = 4(n-1)/(n-2)$ and $n \geq 5$ (because $m \geq 4$), from which it follows that $n = 6$. However, n must be odd for a decomposition of K_n into cycles to exist, and we conclude that $m \neq n-1$. Thus, we have $(n-1)/2 < m \leq n-2$.

Now, it follows from $(n-1)/2 < m$ that

$$T = \frac{4m^2}{n(n-1) - 2m} > \frac{(n-1)^2}{n(n-1) - (n-1)} = 1,$$

and it follows from $m \leq n-2$ that

$$T = \frac{4m^2}{n(n-1) - 2m} \leq \frac{4(n-2)^2}{n(n-1) - 2(n-2)} < \frac{4(n-2)^2}{n(n-2) - 2(n-2)} = 4.$$

Thus, we have $T \in \{2, 3\}$.

For a contradiction, assume $T = 3$. Then using $T = 4m^2/(n(n-1) - 2m)$ it is straightforward to obtain

$$\binom{2m+2}{2} = 3\binom{n}{2} + 1. \tag{2.1}$$

We note as an aside that solutions to (2.1) correspond to numbers that are simultaneously triangular and centred triangular numbers (the right-hand side of (2.1) is the $(n-1)$-th centred triangular number), see [54]. By Lemma 2.3, $\lambda = m^2(n-3)/(n(n-1) - 2m)$ is an integer, and combining this with $4m^2/(n(n-1)-2m) = 3$, we obtain $4\lambda = 3(n-3)$ and hence $n \equiv 3 \pmod{4}$. Thus, $\binom{n}{2}$ is odd, and so by (2.1) we have $\binom{2m+2}{2}$ is even and m is odd.

Now, it follows from (2.1) that $(4m+3)^2 = 3(2n-1)^2+6$. Putting $x = 4m+3$ and $y = 2n-1$ we obtain

$$x^2 - 3y^2 = 6. \tag{2.2}$$

Equation (2.2) is a generalised Pell equation with positive integral solution (x_i, y_i) where $x_i + y_i\sqrt{3} = (3+\sqrt{3})(2+\sqrt{3})^i$ for each $i \in \mathbb{Z}$, or alternatively

$$x_i = 4x_{i-1} - x_{i-2}$$

with $x_0 = 3$ and $x_1 = 9$ (see Section 3 of [23], and see [20] or [57] for background on Pell's equation). However, each term x_i is congruent to 1 or 3 (mod 8), and we have $x = 4m+3$ with m odd, which implies $x \equiv 7 \pmod 8$. This is a contradiction and we conclude that $T \neq 3$.

Thus, we are left with $T = 2$, which implies

$$2m(m+1) = n(n-1). \tag{2.3}$$

Equation (2.3) can be rewritten as $(2n-1)^2 - 2(2m+1)^2 = -1$. With substitutions $x = 2n-1$ and $y = 2m+1$, this is again a generalised Pell equation

$$x^2 - 2y^2 = -1$$

with positive integral solution (x_i, y_i) where $x_i + y_i\sqrt{2} = (1+\sqrt{2})(3+2\sqrt{2})^i$ for each $i \in \mathbb{Z}$, or alternatively

$$x_i = 6x_{i-1} - x_{i-2} \tag{2.4}$$

with $x_0 = 1$ and $x_1 = 7$ (again, see Section 3 of [23], and see [20] or [57] for background on Pell's equation). Equation (2.4), together with the requirement that n is odd, imply that n belongs to $\{1, 21, 697, 23661, \ldots\} = \{n_0, n_1, \ldots\}$ where $n_0 = 1$, $n_1 = 21$ and $n_i = 34n_{i-1} - n_{i-2} - 16$ for $i \geq 2$.

It remains to show that $n \neq 21$ and that there exists a quasiresidual $(m+1, n, n-m, (2m-n+3)/2, (3n-4m-3)/2)$-BIBD that is not residual. By Lemma 2.3, since $\mathcal{X}$ is a uniform decomposition of K_n into m-cycles with $2m(m+1) = n(n-1)$, there exists an $(m+1, n, m, (n-1)/2, (n-3)/2)$-BIBD. Here, the expressions for the number of points and the index are simplified using $2m(m+1) = n(n-1)$. The complementary design of this BIBD is an $(m+1, n, n-m, (2m-n+3)/2, (3n-4m-3)/2)$-BIBD, and is quasiresidual because the sum of the block size and the index equals the replication number. If $n = 21$, then $m = 14$ and this complementary design is a $(15, 21, 7, 5, 2)$-BIBD. Since it is well known that there does not exist a $(15, 21, 7, 5, 2)$-BIBD (see [22]), we can conclude that $n \neq 21$.

Finally, we show that our quasiresidual $(m+1, n, n-m, (2m-n+3)/2, (3n-4m-3)/2)$-BIBD is not residual. If it were residual, then it would be the residual of an $(n+1, n+1, n-m, n-m, (3n-4m-3)/2)$-BIBD, but we can use the Bruck-Ryser-Chowla Theorem (see [58]) to show that no such design exists. Since the number of points is even, it suffices to show that the block size minus the index is not a perfect square. That is, we only need to show that $(2m-n+3)/2$ is not a perfect square.

Now, it is routine to check that

$$(n-m-1)^2 = m(m+1) - n + n(n-1) + m - 2nm + 1,$$

from which we obtain

$$(n-m-1)^2 = \tfrac{1}{2}n(n-1) - n + 2m(m+1) + m - 2nm + 1$$

by using $2m(m+1) = n(n-1)$. Thus, we have

$$(n-m-1)^2 = \tfrac{1}{2}(n^2-3n) + 2m^2 + 3m - 2nm + 1$$

and from here it follows easily that

$$(n-m-1)^2 = \tfrac{1}{2}(2m-n+1)(2m-n+2).$$

Since $(2m-n+1)/2$ and $2m-n+2$ are relatively prime integers, and since $(n-m-1)^2$ is their product, $(2m-n+1)/2$ is a perfect square. Thus, $(2m-n+1)/2+1 = (2m-n+3)/2$ is not a perfect square, and this completes the proof. $\square$

We summarise the necessary conditions we have proved for the existence of a uniform decomposition of μK_n into m-cycles in the following theorem.

Theorem 2.6. *For $n \geq 4$, if there exists a uniform decomposition of μK_n into m-cycles, then*

(A) *$m = n$ and $(n, \mu) \in \{(5,1),(7,1),(4,2),(5,3)\}$; or*

(B) *$\mu = 2$ and $m = n-1$; or*

(C) *$\mu = 1$, $n \equiv 3 \pmod 4$ and $m = (n-1)/2$; or*

(D) *$\mu = 1$, $n \geq 697$ and $2m(m+1) = n(n-1)$.*

Moreover, in case (B) $n \neq 10$, in case (C) $n \neq 15$ and there exists a Hadamard design with n points, and in case (D) there exists a quasiresidual $(m+1, n, n-m, (2m-n+3)/2, (3n-4m-3)/2)$-BIBD that is not residual.

Proof. The non-existence of uniform decompositions of $2K_{10}$ into 9-cycles and K_{15} into 7-cycles has been proved by exhaustive computer search. By Corollary 2.4 and Lemma 2.5, it remains only to show that if $\mu = 1$ and $m = (n-1)/2$, then $n \equiv 3 \pmod 4$ and there exists a Hadamard design with n points. This follows immediately from Lemma 2.3. $\square$

2.3 Some existence results

For $n \geq 4$, any uniform decomposition of μK_n into m-cycles falls into one of the four cases, (A), (B), (C) or (D), which are listed in Theorem 2.6. In this section we give our existence results for uniform decompositions of μK_n into m-cycles in each of cases (A)–(C) (we have no such results for Case (D)). The following lemma gives decompositions for each of the four values of (n, μ) in Case (A).

Lemma 2.7. *There exist uniform decompositions of K_5, K_7, $2K_4$ and $3K_5$ into Hamilton cycles.*

Proof. Up to isomorphism, there is a unique decomposition of K_5 into Hamilton cycles and it is trivially uniform. Any decomposition of K_7 into Hamilton cycles is uniform because the union of any two of the three cycles is necessarily the complement of a 7-cycle. Thus,

$$\{(1,2,3,4,5,6,7),(1,3,5,7,2,4,6),(1,4,7,3,6,2,5)\}$$

(for example) is a uniform decomposition of K_7 into Hamilton cycles. The set

$$\{(1,2,3,4),(1,2,4,3),(1,3,2,4)\}$$

of all Hamilton cycles of K_4 is a uniform decomposition of $2K_4$ into Hamilton cycles. A uniform decomposition of $3K_5$ into Hamilton cycles is given by

$$\mathcal{X}=\{(1,2,3,4,5),(1,3,2,5,4),(2,4,3,1,5),(3,5,4,2,1),(4,1,5,3,2),(5,2,1,4,3)\}.$$

To see that $\mathcal{X}$ is indeed a uniform decomposition of $3K_5$, observe that $\mathcal{X}$ is the orbit of the cycle $(1,2,3,4,5)$ under the action of A_5, the alternating group acting on $\{1,2,3,4,5\}$, and that A_5 acts 2-transitively on the cycles of $\mathcal{X}$.

Of course, if there is a 2-transitive action on the cycles of any decomposition into cycles, then the decomposition is uniform. To see this, observe that if X_i and X_j are distinct cycles, and X_k and X_ℓ are distinct cycles, then 2-transitivity tells us that there is an automorphism f such that $f(X_i)=X_k$ and $f(X_j)=X_\ell$. Thus, f is a graph isomorphism between $X_i\cup X_j$ and $X_k\cup X_\ell$. □

In the next lemma, we present an isolated specific example of a uniform decomposition of $2K_6$ into 5-cycles.

Lemma 2.8. *There exists a uniform decomposition of* $2K_6$ *into* 5*-cycles.*

Proof. A uniform decomposition of $2K_6$ into 5-cycles is given by

$$\mathcal{X}=\{(1,2,3,4,5),(4,2,3,1,6),(5,3,4,2,6),(1,4,5,3,6),(2,5,1,4,6),(3,1,2,5,6)\}.$$

To see that $\mathcal{X}$ is indeed a uniform decomposition of $2K_6$, observe that $\mathcal{X}$ is the orbit of the cycle $(1,2,3,4,5)$ under the action of the group $G=\langle(1\ 2\ 3\ 4\ 5),(1\ 4)(5\ 6)\rangle$, and that $G\cong A_5$ acts 2-transitively on the cycles of $\mathcal{X}$. □

In Definition 2.9 we define various sets of $(q-1)$-cycles and $((q-1)/2)$-cycles in $K_{\mathbb{F}_q}$, where $\mathbb{F}_q$ denotes the field of order q. We show in Lemmas 2.10 and 2.11 that many of these form uniform decompositions of $2K_q$ into $(q-1)$-cycles or of K_q into $((q-1)/2)$-cycles. Isomorphisms of these decompositions are discussed in Section 2.4. The decomposition $\mathcal{X}_{q,\omega,0}$ was introduced in [2], where it was used in the context of decompositions of complete symmetric digraphs on n vertices into directed cycles of length $n-1$ such that any two distinct cycles have exactly one oppositely directed edge in common. The notation introduced in Definition 2.9 will be used throughout the rest of the paper.

Definition 2.9. *Let $q\geq 4$ be a prime power and let ω be primitive in $\mathbb{F}_q$. For each $s\in\mathbb{F}_q$, we define $X^s_{q,\omega,0}$ to be the cycle*

$$X^s_{q,\omega,0}\ =(1+s,\omega+s,\omega^2+s,\dots,\omega^{q-2}+s),$$

and we define $\mathcal{X}_{q,\omega,0}=\{X^s_{q,\omega,0}:s\in\mathbb{F}_q\}$. Further, if q is odd then for each non-zero integer r and each $s\in\mathbb{F}_q$ we define $X^s_{q,\omega,r}$ and $Y^s_{q,\omega}$ to be the cycles

$$\begin{aligned}X^s_{q,\omega,r}&=(1+s,\omega^{2r+1}+s,\omega^2+s,\omega^{2r+3}+s,\omega^4+s,\dots,\omega^{-2}+s,\omega^{2r-1}+s),\\ Y^s_{q,\omega}&=(1+s,\omega^2+s,\omega^4+s,\dots,\omega^{-2}+s),\end{aligned}$$

and we define $\mathcal{X}_{q,\omega,r}=\{X^s_{q,\omega,r}:s\in\mathbb{F}_q\}$ and $\mathcal{Y}_{q,\omega}=\{Y^s_{q,\omega}:s\in\mathbb{F}_q\}$.

Lemma 2.10. *If $q \geq 4$ is a prime power and ω is primitive in $\mathbb{F}_q$, then*

- *$\mathcal{X}_{q,\omega,0}$ is a uniform decomposition of $2K_q$ into $(q-1)$-cycles;*
- *if $q \equiv 3 \pmod 4$ and $r \in \mathbb{Z}$, then $\mathcal{X}_{q,\omega,r}$ is a uniform decomposition of $2K_q$ into $(q-1)$-cycles; and*
- *if $q \equiv 1 \pmod 4$ and $r \equiv 0 \pmod{(q-1)/4}$, then $\mathcal{X}_{q,\omega,r}$ is a uniform decomposition of $2K_q$ into $(q-1)$-cycles.*

Proof. For each $a, b \in \mathbb{F}_q$ with $a \neq 0$, let $g_{a,b} : \mathbb{F}_q \to \mathbb{F}_q$ be given by $g_{a,b}(x) = ax + b$ for all $x \in \mathbb{F}_q$, let G_q be the group $G_q = \{g_{a,b} : a, b \in \mathbb{F}_q, a \neq 0\}$, and if q is odd let H_q be the subgroup of G_q given by $H_q = \{g_{a,b} : a \in Q, b \in \mathbb{F}_q\}$ where Q is the set of non-zero quadratic residues in $\mathbb{F}_q$. It is well known, and easily verified, that G_q is sharply 2-transitive on $\mathbb{F}_q$, and that if $q \equiv 3 \pmod 4$, then H_q is sharply 2-homogeneous on $\mathbb{F}_q$. In particular, G_q is transitive on the edges of $K_{\mathbb{F}_q}$ and the stabilizer in G_q of an edge has order 2, and if $q \equiv 3 \pmod 4$, then H_q is transitive on the edges of $K_{\mathbb{F}_q}$ and the stabilizer in H_q of an edge is trivial.

Let $r = 0$ if q is even, let $r \in \mathbb{Z}$ if q is odd, and let $f = g_{\omega,0}$. It can be seen that f^2 is an automorphism of the cycle $X^0_{q,\omega,r}$, and that f is an automorphism of the cycle $X^0_{q,\omega,0}$. It is also clear that if $q = 1 \pmod 4$ and $r \equiv (q-1)/4 \pmod{(q-1)/2}$, then f is an automorphism of the cycle

$$X^0_{q,\omega,r} = (1, \omega^{\frac{q+1}{2}}, \omega^2, \ldots, \omega^{\frac{q-1}{2}}, \omega, \omega^{\frac{q+3}{2}}, \ldots, \omega^{q-3}, \omega^{\frac{q-3}{2}}) = (1, -\omega, \omega^2, -\omega^3, \ldots, \omega^{-2}, -\omega^{-1}).$$

It follows from these observations that $H_q \leq \mathrm{Aut}(\mathcal{X}_{q,\omega,r})$ when q is odd, and that $G_q \leq \mathrm{Aut}(\mathcal{X}_{q,\omega,r})$ when $r = 0$ and when $q \equiv 1 \pmod 4$ and $r \equiv 0 \pmod{(q-1)/4}$ (when $r \equiv 0 \pmod{(q-1)/2}$ we have $X^0_{q,\omega,r} = X^0_{q,\omega,0}$).

Now, note that in general if V is any set of vertices, $\mathcal{X}$ is any set of m-cycles in K_V and $G \leq \mathrm{Aut}(\mathcal{X})$ such that G is transitive on the edges of K_V and transitive on the cycles of $\mathcal{X}$, then each edge of K_V occurs λ times in the cycles of $\mathcal{X}$ where $\lambda = m|\mathrm{Stab}_G(\{x,y\})|/|\mathrm{Stab}_G(X)|$, $\{x, y\}$ is any edge of K_V, and X is any cycle of $\mathcal{X}$. Thus, if $q \equiv 3 \pmod 4$, then $\mathcal{X}_{q,\omega,r}$ is a decomposition of $2K_{\mathbb{F}_q}$ because $H_q \leq \mathrm{Aut}(\mathcal{X}_{q,\omega,r})$ is transitive on the edges of $K_{\mathbb{F}_q}$ and on the cycles of $\mathcal{X}_{q,\omega,r}$, and because $|\mathrm{Stab}_{H_q}(\{x,y\})| = 1$ and $|\mathrm{Stab}_{H_q}(X^0_{q,\omega,r})| = (q-1)/2$. Also, if $r = 0$, or if $q \equiv 1 \pmod 4$ and $r \equiv 0 \pmod{(q-1)/4}$, then $\mathcal{X}_{q,\omega,r}$ is a decomposition of $2K_{\mathbb{F}_q}$ because $G_q \leq \mathrm{Aut}(\mathcal{X}_{q,\omega,r})$ is transitive on the edges of $K_{\mathbb{F}_q}$ and on the cycles of $\mathcal{X}_{q,\omega,r}$, and because $|\mathrm{Stab}_{G_q}(\{x,y\})| = 2$ and $|\mathrm{Stab}_{G_q}(X^0_{q,\omega,r})| = q-1$.

It remains to show that $\mathcal{X}_{q,\omega,r}$ is uniform. Since s is the only element of $\mathbb{F}_q$ that is not a vertex of the cycle $X^s_{q,\omega,r}$, if g is any automorphism of $\mathcal{X}_{q,\omega,r}$, then g maps $X^s_{q,\omega,r}$ to $X^{g(s)}_{q,\omega,r}$. Thus, if $q \equiv 3 \pmod 4$, then H_q acts transitively on unordered pairs of cycles of $\mathcal{X}_{q,\omega,r}$, and if $r = 0$, or if $q = 1 \pmod 4$ and $r \equiv 0 \pmod{(q-1)/4}$, then G_q acts transitively on unordered pairs of cycles of $\mathcal{X}_{q,\omega,r}$ (in fact, G_q acts transitively on ordered pairs of cycles of $\mathcal{X}_{q,\omega,r}$). Thus, $\mathcal{X}_{q,\omega,r}$ is uniform. □

Lemma 2.11. *If $q \geq 7$ is a prime power such that $q \equiv 3 \pmod 4$ and ω is primitive in $\mathbb{F}_q$, then $\mathcal{Y}_{q,\omega}$ is a uniform decomposition of K_q into $((q-1)/2)$-cycles.*

Proof. As in Lemma 2.10, let $g_{a,b} : \mathbb{F}_q \to \mathbb{F}_q$ be given by $g_{a,b}(x) = ax + b$ for all $x \in \mathbb{F}_q$, let G_q be the group $G_q = \{g_{a,b} : a, b \in \mathbb{F}_q, a \neq 0\}$, and let H_q be the subgroup of G_q given by $H_q = \{g_{a,b} : a \in Q, b \in \mathbb{F}_q\}$ where Q is the set of non-zero quadratic residues in $\mathbb{F}_q$. Let $m = (q-1)/2$. We claim that $\mathcal{Y}_{q,\omega}$ is a uniform decomposition of $K_{\mathbb{F}_q}$ into m-cycles.

Noting that $g_{a,0}$ is an automorphism of $Y^0_{q,\omega}$ when $a \in Q$, it is routine to verify that if $g_{a,b} \in H_q$, then $g_{a,b}$ maps $Y^s_{q,\omega}$ to $Y^{as+b}_{q,\omega}$. It follows that the stabiliser in H_q of $Y^0_{q,\omega}$ is $\{g_{a,0} : a \in Q\}$, and that $|\mathrm{orb}_{H_q}(Y^0_{q,\omega})| = q$. Now, G_q has a regular action on the set of ordered pairs of distinct elements of $\mathbb{F}_q$, the unique element of G_q that maps (x, y) to (y, x) is $g_{-1,x+y}$, and $g_{-1,x+y} \notin H_q$ because $q \equiv 3 \pmod 4$ implies $-1 \notin Q$. It follows that H_q has a regular action on the set of unordered pairs of distinct elements of $\mathbb{F}_q$. This, together with the fact that $|\mathrm{orb}_{H_q}(Y^0_{q,\omega})| = q$ guarantees that $\mathrm{orb}_{H_q}(Y^0_{q,\omega}) = \mathcal{Y}_{q,\omega}$ is a decomposition of $K_{\mathbb{F}_q}$ into m-cycles. Moreover, together with the fact that $g_{a,b} \in H_q$ implies $g_{a,b}$ maps $Y^s_{q,\omega}$ to $Y^{as+b}_{q,\omega}$, it guarantees that H_q has a regular action on the set of unordered pairs of distinct cycles of $\mathcal{Y}_{q,\omega}$. Thus, $\mathcal{Y}_{q,\omega}$ is uniform. □

The combined results of this section are stated in the following theorem.

Theorem 2.12. *Let $n \geq 4$. There exist uniform decompositions of μK_n into m-cycles in each of the following cases.*

(A) *$m = n$ and $(n, \mu) \in \{(5, 1), (7, 1), (4, 2), (5, 3)\}$;*

(B) *$m = n - 1$, $\mu = 2$ and $n = 6$ or n is a prime power;*

(C) *$m = (n - 1)/2$, $\mu = 1$ and $n \equiv 3 \pmod 4$ is a prime power.*

2.4 Isomorphism classes

In this section we determine the isomorphism classes of the uniform decompositions given by Lemma 2.10 and Lemma 2.11, which belong to Cases (B) and (C) respectively from Theorem 2.6. However, first we briefly discuss isomorphism classes of uniform decompositions of complete multigraphs into Hamilton cycles (Case (A) of Theorem 2.6), and uniform decompositions of $2K_6$ into 5-cycles (see Lemma 2.8).

Up to isomorphism, there exists only one decomposition of K_5 into Hamilton cycles, and it is trivially uniform. The number of pairwise non-isomorphic decompositions of K_7 into Hamilton cycles is 2 [21, 27], and, as noted in the proof of Lemma 2.7, both of these are uniform. It is easily verified that the uniform decompositions of $2K_4$ and $3K_5$ into Hamilton cycles given in the proof of Lemma 2.7 are unique up to isomorphism. For $3K_5$, simple counting shows that the union of any pair of cycles must contain exactly two 2-cycles, and then it is easily seen that these 2-cycles must be vertex disjoint, and that the decomposition is unique. Also, we have shown by a computer search that up to isomorphism the decomposition given in Lemma 2.8 is the only uniform decomposition of $2K_6$ into 5-cycles.

We now determine the isomorphism classes of the uniform decompositions $\mathcal{Y}_{q,\omega}$ given by Lemma 2.11. We will use these results later in this section to determine the isomorphism classes of the uniform decompositions $\mathcal{X}_{q,\omega,r}$ given by Lemma 2.10.

Lemma 2.13. *Let p be prime, let $q = p^\alpha \geq 4$ such that $q \equiv 3 \pmod 4$, and let ω_1 and ω_2 be primitive in $\mathbb{F}_q$. Then the decompositions $\mathcal{Y}_{q,\omega_1}$ and $\mathcal{Y}_{q,\omega_2}$ are isomorphic if and only if $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$.*

Proof. We begin by noting that we have $\mathcal{Y}_{q,\omega_1} = \mathcal{Y}_{q,\omega_2}$ if and only if $\omega_1 \in \{\omega_2, \omega_2^{-1}\}$. Now, if $f : \mathbb{F}_q \to \mathbb{F}_q$ is the well-known Frobenious automorphism given by $f(x) = x^p$ (recall that

$q = p^\alpha$) for each $x \in \mathbb{F}_q$, then it is easily verified that for any primitive element ω, we have $f(Y^s_{q,\omega}) = Y^{s^p}_{q,\omega^p}$. Thus, we have $\mathcal{Y}_{q,\omega} \cong \mathcal{Y}_{q,\omega^p}$ and hence $\mathcal{Y}_{q,\omega} \cong \mathcal{Y}_{q,\omega^{\pm p^k}}$ (since $\mathcal{Y}_{q,\omega} = \mathcal{Y}_{q,\omega^{-1}}$).

Conversely, suppose there exists an isomorphism f from $\mathcal{Y}_{q,\omega_1}$ to $\mathcal{Y}_{q,\omega_2}$. When $q = 7$ the result is trivial, and the result also holds when $q = 11$ because the decompositions $\mathcal{Y}_{11,2}$ and $\mathcal{Y}_{11,7}$ are easily seen to be non-isomorphic. Thus, we can assume $q > 11$. For any primitive element ω, if we consider the elements of $\mathbb{F}_q$ as points and the vertex sets of the cycles of $\mathcal{Y}_{q,\omega}$ as blocks, then we obtain the Paley design on q points (the Hadamard design whose blocks are the translates of the set of quadratic residues in $\mathbb{F}_q$). Thus, f is a Paley design automorphism and so by Theorem 8.1 of [35], we know that our isomorphism f is given by $f(x) = a^2\sigma(x) + b$ for all $x \in \mathbb{F}_q$ where $a, b \in \mathbb{F}_q$, $a \neq 0$, and $\sigma(x) = x^{p^k}$ for some k. Noting that any permutation of the form $x \mapsto a^2x + b$ is an automorphism of $\mathcal{Y}_{q,\omega_2}$, it follows that $\sigma(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$. But $\sigma(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\sigma(\omega_1)}$, and it follows that $\omega_2 = \omega_1^{\pm p^k}$. □

The preceding result gives us a lower bound on the number of non-isomorphic uniform decompositions of K_q into $((q-1)/2)$-cycles when $q \equiv 3 \pmod 4$ is a prime power. In what follows, ϕ denotes Euler's totient function. Thus, $\phi(q-1)$ is the number of primitive elements in $\mathbb{F}_q$.

Theorem 2.14. *If p is prime and $q = p^\alpha \geq 7$ such that $q \equiv 3 \pmod 4$, then there are at least $\phi(q-1)/2\alpha$ non-isomorphic uniform decompositions of K_q into $((q-1)/2)$-cycles. In particular, the number of isomorphism classes of uniform decompositions of K_q into $((q-1)/2)$-cycles in $\{\mathcal{Y}_{q,\omega} : \omega$ is primitive in $\mathbb{F}_q\}$ is exactly $\phi(q-1)/2\alpha$.*

Proof. By Lemma 2.13, for primitive elements ω_1 and ω_2 in $\mathbb{F}_q$, the decompositions $\mathcal{Y}_{q,\omega_1}$ and $\mathcal{Y}_{q,\omega_2}$ are isomorphic if and only if $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$. There are $\phi(q-1)$ primitive elements in $\mathbb{F}_q$, and for each primitive element ω_1, there are 2α primitive elements ω_2 such that $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$ ($\omega_1^{\pm p^k}$ for each $k \in \{0, 1, \ldots, \alpha - 1\}$). Thus, the number of isomorphism classes of uniform decompositions of K_q into $((q-1)/2)$-cycles in $\{\mathcal{Y}_{q,\omega} : \omega$ is primitive in $\mathbb{F}_q\}$ is exactly $\phi(q-1)/2\alpha$. □

We now investigate isomorphism classes of our decompositions $\mathcal{X}_{q,\omega,r}$, which belong to Case (B) (see Definition 2.9 and Lemma 2.10). In Lemma 2.10 we showed that $\mathcal{X}_{q,\omega,r}$ (q a prime power, ω primitive in $\mathbb{F}_q$) is a uniform decomposition of $2K_q$ into $(q-1)$-cycles where $r = 0$ when q is even, $r \equiv 0 \pmod{(q-1)/4}$ when $q \equiv 1 \pmod 4$, and $r \in \mathbb{Z}$ when $q \equiv 3 \pmod 4$. However, many of these decompositions are identical. It can be verified easily that

- $\mathcal{X}_{q,\omega,r} = \mathcal{X}_{q,\omega,r+\frac{q-1}{2}}$ when q is odd;
- $\mathcal{X}_{q,\omega,r} = \mathcal{X}_{q,\omega^{-1},-r}$; and
- $\mathcal{X}_{q,\omega,r} = \mathcal{X}_{q,-\omega,r+\frac{q-1}{4}}$ when $q \equiv 1 \pmod 4$.

It follows that when $q \equiv 1 \pmod 4$ (and when q is even), any one of the uniform decompositions of $2K_q$ into $(q-1)$-cycles given in Lemma 2.10 is equal to a decomposition $\mathcal{X}_{q,\omega,r}$ with $r = 0$, and that when $q \equiv 3 \pmod 4$, any one of the uniform decompositions of $2K_q$ into $(q-1)$-cycles given in Lemma 2.10 is equal to a decomposition $\mathcal{X}_{q,\omega,r}$ with $r \in \{0, 1, \ldots, (q-3)/4\}$. Thus, from here on we consider only these values of r.

Lemma 2.15. *Let p be prime, let $q = p^\alpha \geq 4$, let ω be primitive in $\mathbb{F}_q$, and let $r \in \{0, 1, \ldots, (q-3)/4\}$ such that $r = 0$ if q is even or $q \equiv 1 \pmod 4$. Then*

(I) $\mathcal{X}_{q,\omega,r} \cong \mathcal{X}_{q,\omega^p,r}$; *and*

(II) $\mathcal{X}_{q,\omega,r} \cong \mathcal{X}_{q,\omega^{-1},r}$.

Proof. (I) We have $\mathcal{X}_{q,\omega,r} = \{(1+s, \omega^{2r+1}+s, \omega^2+s, \omega^{2r+3}+s, \ldots) : s \in \mathbb{F}_q\}$. Thus, since the mapping $x \mapsto x^p$ for all $x \in \mathbb{F}_q$ is an automorphism of $\mathbb{F}_q$, we know that

$$\begin{aligned}\mathcal{X}_{q,\omega,r} &\cong \{(1^p+s^p, (\omega^{2r+1})^p+s^p, \omega^{2p}+s^p, (\omega^{2r+3})^p+s^p, \ldots) : s^p \in \mathbb{F}_q\}\\ &= \{(1+s, (\omega^p)^{2r+1}+s, (\omega^p)^2+s, (\omega^p)^{2r+3}+s, \ldots) : s \in \mathbb{F}_q\}\\ &= \mathcal{X}_{q,\omega^p,r}.\end{aligned}$$

(II) If $r=0$, then $X^0_{q,\omega,r} = X^0_{q,\omega^{-1},r}$ and so $\mathcal{X}_{q,\omega,r}$ is actually equal to $\mathcal{X}_{q,\omega^{-1},r}$. Thus, we can assume that $q \equiv 3 \pmod 4$, and in this case it is routine to check that the mapping $x \mapsto -x$ for all $x \in \mathbb{F}_q$ is an isomorphism from $\mathcal{X}_{q,\omega,r}$ to $\mathcal{X}_{q,\omega^{-1},r}$, because the image of $X^s_{q,\omega,r}$ under this mapping is $X^{-s}_{q,\omega^{-1},r}$. □

The next lemma states that there are no isomorphisms between $\mathcal{X}_{q,\omega_1,0}$ and $\mathcal{X}_{q,\omega_2,0}$ other than those generated by the isomorphisms of Lemma 2.15.

Lemma 2.16. *Let p be prime, let $q = p^\alpha \geq 4$, and let ω_1 and ω_2 be primitive in $\mathbb{F}_q$. Then $\mathcal{X}_{q,\omega_1,0} \cong \mathcal{X}_{q,\omega_2,0}$ if and only if $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$.*

Proof. By Lemma 2.15, if $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$, then $\mathcal{X}_{q,\omega_1,0} \cong \mathcal{X}_{q,\omega_2,0}$. Thus, we only need to prove the converse; namely that $\mathcal{X}_{q,\omega_1,0} \cong \mathcal{X}_{q,\omega_2,0}$ implies $\omega_2 = \omega_1^{\pm p^k}$.

Assume $\mathcal{X}_{q,\omega_1,0} \cong \mathcal{X}_{q,\omega_2,0}$. We noted in the proof of Lemma 2.10 that the automorphism group of $\mathcal{X}_{q,\omega_1,0}$ is 2-transitive (on the vertex set). Thus there is an isomorphism f from $\mathcal{X}_{q,\omega_1,0}$ to $\mathcal{X}_{q,\omega_2,0}$ such that $f(0)=0$ and $f(1)=1$. For any primitive element ω and any $s \in \mathbb{F}_q$, the cycle $X^s_{q,\omega,0}$ contains every vertex of $K_{\mathbb{F}_q}$ except s, and it follows that for all $s \in \mathbb{F}_q$ we have $f(X^s_{q,\omega_1,0}) = X^{f(s)}_{q,\omega_2,0}$. In particular, we have $f(X^0_{q,\omega_1,0}) = X^0_{q,\omega_2,0}$. That is, $f((1, w_1, w_1^2, \ldots, w_1^{-1})) = (1, w_2, w_2^2, \ldots, w_2^{-1})$. Since $f(1)=1$, this implies that either $f(\omega_1^i) = \omega_2^i$ for $i = 0,1,\ldots,q-2$ or $f(\omega_1^i) = (\omega_2^{-1})^i$ for $i = 0,1,\ldots,q-2$. We first deal with the case $f(\omega_1^i) = \omega_2^i$ for $i = 0,1,\ldots,q-2$.

It is easy to see that $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{F}_q$. We now proceed to show that $f(x+y) = f(x)+f(y)$ for all $x,y \in \mathbb{F}_q$, thereby showing that f is a field automorphism. Since $f(0)=0$, we have $f(x+y) = f(x)+f(y)$ when either x or y is 0.

Now, let $i \in \{0,1,\ldots,q-2\}$ and consider the cycle

$$X^{\omega_1^i}_{q,\omega_1,0} = (1+\omega_1^i, \omega_1+\omega_1^i, \omega_1^2+\omega_1^i, \ldots, \omega_1^{i-1}+\omega_1^i, \omega_1^i+\omega_1^i, \omega_1^{i+1}+\omega_1^i, \ldots).$$

We have

$$f(X^{\omega_1^i}_{q,\omega_1,0}) = X^{\omega_2^i}_{q,\omega_2,0} = (1+\omega_2^i, \omega_2+\omega_2^i, \omega_2^2+\omega_2^i, \ldots, \omega_2^{i-1}+\omega_2^i, \omega_2^i+\omega_2^i, \omega_2^{i+1}+\omega_2^i, \ldots).$$

If q is even, then $f(\omega_1^i+\omega_1^i) = f(0) = 0 = \omega_2^i+\omega_2^i$. If q is odd, then the vertex opposite $\omega_1^i+\omega_1^i$ in $X^{\omega_1^i}_{q,\omega_1,0}$ is 0 (this can be seen by noting that ω_1^i is opposite $-\omega_1^i$ in $X^0_{q,\omega_1,0}$ because $\omega^{(q-1)/2} \equiv -1 \pmod q$), and similarly the vertex opposite $\omega_2^i+\omega_2^i$ in $X^{\omega_2^i}_{q,\omega_2,0}$ is 0. Thus, we have also $f(\omega_1^i+\omega_1^i) = \omega_2^i+\omega_2^i$ when q is odd.

Looking at the neighbours of $\omega_1^i+\omega_1^i$ in $X^{\omega_1^i}_{q,\omega_1,0}$ and the neighbours of $\omega_2^i+\omega_2^i$ in $f(X^{\omega_1^i}_{q,\omega_1,0})$ (see the above displayed cycles), it follows from $f(\omega_1^i+\omega_1^i) = \omega_2^i+\omega_2^i$ that either

(1) $f(\omega_1^{i-1}+\omega_1^i)=\omega_2^{i-1}+\omega_2^i$ and $f(\omega_1^{i+1}+\omega_1^i)=\omega_2^{i+1}+\omega_2^i$; or

(2) $f(\omega_1^{i-1}+\omega_1^i)=\omega_2^{i+1}+\omega_2^i$ and $f(\omega_1^{i+1}+\omega_1^i)=\omega_2^{i-1}+\omega_2^i$.

Assume, for a contradiction, that (2) holds and now consider the cycle

$$X_{q,\omega_1,0}^{\omega_1^{i+1}}=(1+\omega_1^{i+1},\omega_1+\omega_1^{i+1},\omega_1^2+\omega_1^{i+1},\ldots,\omega_1^i+\omega_1^{i+1},\omega_1^{i+1}+\omega_1^{i+1},\omega_1^{i+2}+\omega_1^{i+1},\ldots)$$

and its image under f, namely

$$X_{q,\omega_2,0}^{w_2^{i+1}}=(1+\omega_2^{i+1},\omega_2+\omega_2^{i+1},\omega_2^2+\omega_2^{i+1},\ldots,\omega_2^i+\omega_2^{i+1},\omega_2^{i+1}+\omega_2^{i+1},\omega_2^{i+2}+\omega_2^{i+1},\ldots).$$

Similarly to previously, we have $f(\omega_1^{i+1}+\omega_1^{i+1})=\omega_2^{i+1}+\omega_2^{i+1}$ and it follows that either

(3) $f(\omega_1^i+\omega_1^{i+1})=\omega_2^i+\omega_2^{i+1}$ and $f(\omega_1^{i+2}+\omega_1^{i+1})=\omega_2^{i+2}+\omega_2^{i+1}$; or

(4) $f(\omega_1^i+\omega_1^{i+1})=\omega_2^{i+2}+\omega_2^{i+1}$ and $f(\omega_1^{i+2}+\omega_1^{i+1})=\omega_2^i+\omega_2^{i+1}$.

However, the first equation of (2) and the first equation of (3) together imply that $\omega_1\in\{1,-1\}$, which is a contradiction (recall that $q\geq 4$). Similarly, the first equation of (2) and the second equation of (4) together imply that $\omega_1^{i-1}+\omega_1^i=\omega_1^{i+2}+\omega_1^{i+1}$, and again we have the contradiction that $\omega_1\in\{1,-1\}$.

We conclude that (1) holds. Looking again at the cycles $X_{q,\omega_1,0}^{\omega_1^i}$ and $f(X_{q,\omega_1,0}^{\omega_1^i})$, this implies that we have $f(\omega_1^i+\omega_1^j)=\omega_2^i+\omega_2^j$ for all $i,j\in\{0,1,\ldots,q-2\}$. Since we are in the case $f(\omega_1^i)=\omega_2^i$ for $i=0,1,\ldots,q-2$ (and since we already noted that $f(x+y)=f(x)+f(y)$ when either x or y is 0) this completes the proof that $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{F}_q$, thereby showing that f is a field automorphism. Thus, $\omega_2=\omega_1^{p^k}$ for some $k\in\mathbb{Z}$. In the other case, where $f(\omega_1^i)=(\omega_2^{-1})^i$ for $i=0,1,\ldots,q-2$, the same argument yields $\omega_2=\omega_1^{-p^k}$ for some $k\in\mathbb{Z}$. This completes the proof □

Lemma 2.17. *Let p be prime, let $q=p^\alpha\geq 7$ such that $q\equiv 3\pmod 4$, let ω_1 and ω_2 be primitive in $\mathbb{F}_q$, and let $r_1,r_2\in\{0,1,\ldots,(q-3)/4\}$. Then $\mathcal{X}_{q,\omega_1,r_1}\cong\mathcal{X}_{q,\omega_2,r_2}$ if and only if $r_1=r_2$ and $\omega_2=\omega_1^{\pm p^k}$ for some $k\in\mathbb{Z}$.*

Proof. By Lemma 2.15, if $\omega_2=\omega_1^{\pm p^k}$ for some $k\in\mathbb{Z}$ and $r_1=r_2$, then $\mathcal{X}_{q,\omega_1,r_1}\cong\mathcal{X}_{q,\omega_2,r_2}$. We now prove the converse. For $q=7$ it is easily verified that the decompositions $\mathcal{X}_{7,3,0}$ and $\mathcal{X}_{7,3,1}$ are non-isomorphic, and for $q=11$ it is easily verified that the decompositions $\mathcal{X}_{11,2,0},\mathcal{X}_{11,2,1},\mathcal{X}_{11,2,2},\mathcal{X}_{11,7,0},\mathcal{X}_{11,7,1}$ and $\mathcal{X}_{11,7,2}$ are pairwise non-isomorphic (for distinct $(\omega_1,r_1),(\omega_2,r_2)\in\{(2,0),(2,1),(2,2),(7,0),(7,1),(7,2)\}$, the graphs $X^0_{11,\omega_1,r_1}\cup X^1_{11,\omega_1,r_1}$ and $X^0_{11,\omega_2,r_2}\cup X^1_{11,\omega_2,r_2}$ are not isomorphic). This proves the result for $q\in\{7,11\}$ and thus we can assume $q>11$.

Suppose there exists an isomorphism f from $\mathcal{X}_{q,\omega_1,r_1}$ to $\mathcal{X}_{q,\omega_2,r_2}$. For each $\omega\in\{\omega_1,\omega_2\}$ and each $s\in\mathbb{F}_q$, let $Z^s_{q,\omega}$ be the cycle $(\omega+s,\omega^3+s,\ldots,\omega^{-1}+s)$, and let $\mathcal{Z}_{q,\omega}=\{Z^s_{q,\omega}:s\in\mathbb{F}_q\}$. As in the proof of Lemma 2.16, we have $f(X^s_{q,\omega_1,r_1})=X^{f(s)}_{q,\omega_2,r_2}$ for all $s\in\mathbb{F}_q$. It follows that for each $s\in\mathbb{F}_q$ we have $f(Y^s_{q,\omega_1})=Y^{f(s)}_{q,\omega_2}$ or $f(Y^s_{q,\omega_1})=Z^{f(s)}_{q,\omega_2}$.

We will show that either $f(Y^s_{q,\omega_1})=Y^{f(s)}_{q,\omega_2}$ for all $s\in\mathbb{F}_q$ or $f(Y^s_{q,\omega_1})=Z^{f(s)}_{q,\omega_2}$ for all $s\in\mathbb{F}_q$. That is, we will show that either $f(\mathcal{Y}_{q,\omega_1})=\mathcal{Y}_{q,\omega_2}$ or $f(\mathcal{Y}_{q,\omega_1})=\mathcal{Z}_{q,\omega_2}$. To do this we observe the following properties of Paley designs.

Recall that the Paley design on q points, which we denote by $\mathcal{P}_q$, has the elements of $\mathbb{F}_q$ as its points and the translates of the set Q of quadratic residues in $\mathbb{F}_q$ as its blocks. It is a symmetric $(q, \frac{q-1}{2}, \frac{q-3}{4})$-design, and as such any two distinct blocks intersect in exactly $(q-3)/4$ points. The translates of the set Q' of quadratic non-residues in $\mathbb{F}_q$ also form a symmetric $(q, \frac{q-1}{2}, \frac{q-3}{4})$-design, which we will denote by $\mathcal{P}'_q$, and the mapping $x \mapsto -x$ for all $x \in \mathbb{F}_q$ is an isomorphism from $\mathcal{P}_q$ to $\mathcal{P}'_q$.

It is easy to verify that if $B = Q + s$ is a block of $\mathcal{P}_q$ and $B' = Q' + s'$ is a block of $\mathcal{P}'_q$, then $|B \cap B'| = (q-3)/4$ if and only if $s' - s \in Q$ (for this, observe that any two blocks of $\mathcal{P}'_q$ intersect in exactly $(q-3)/4$ points, and that the additive inverse of a quadratic residue is a quadratic non-residue). Thus, for each block B in $\mathcal{P}_q$ there are exactly $(q-1)/2$ blocks of $\mathcal{P}'_q$ that intersect B in exactly $(q-3)/4$ points, and for each block B' in $\mathcal{P}'_q$ there are exactly $(q-1)/2$ blocks of $\mathcal{P}_q$ that intersect B' in exactly $(q-3)/4$ points. It follows that the only $(q, \frac{q-1}{2}, \frac{q-3}{4})$-designs with each block being either a block of $\mathcal{P}_q$ or a block of $\mathcal{P}'_q$ are $\mathcal{P}_q$ or $\mathcal{P}'_q$ themselves.

Now, as we saw in the proof of Lemma 2.13, for any primitive element ω in $\mathbb{F}_q$, if we consider the elements of $\mathbb{F}_q$ as points and the vertex sets of the cycles of $\mathcal{Y}_{q,\omega}$ as blocks, then we obtain $\mathcal{P}_q$. Similarly, the vertex sets of the cycles of $\mathcal{Z}_{q,\omega}$ form the design $\mathcal{P}'_q$. Thus, since the vertex sets of the cycles of $\mathcal{Y}_{q,\omega_1}$ form a $(q, \frac{q-1}{2}, \frac{q-3}{4})$-design, so do the vertex sets of the cycles of $f(\mathcal{Y}_{q,\omega_1})$. From our above observations concerning Paley designs, this means that either $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$ or $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Z}_{q,\omega_2}$.

If $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Z}_{q,\omega_2}$, then consider the mapping $f' : \mathbb{F}_q \to \mathbb{F}_q$ given by $f'(x) = -f(x)$ for all $x \in \mathbb{F}_q$. As noted in the proof of Lemma 2.15, the mapping $x \mapsto -x$ is an isomorphism from $\mathcal{X}_{q,\omega_2,r_2}$ to $\mathcal{X}_{q,\omega_2^{-1},r_2}$, and it is easy to see that $x \mapsto -x$ is an isomorphism from $\mathcal{Z}_{q,\omega_2}$ to $\mathcal{Y}_{q,\omega_2}$. Thus, when $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Z}_{q,\omega_2}$, we have $f'(\mathcal{X}_{q,\omega_1,r_1}) = \mathcal{X}_{q,\omega_2^{-1},r_2}$ and $f'(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$. Thus, in either case (the case $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$ or the case $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Z}_{q,\omega_2}$), there exists a mapping $h : \mathbb{F}_q \to \mathbb{F}_q$ ($h = f$ if $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$, and $h = f'$ if $f(\mathcal{Y}_{q,\omega_1}) = \mathcal{Z}_{q,\omega_2}$) such that $h(\mathcal{X}_{q,\omega_1,r_1}) \in \{\mathcal{X}_{q,\omega_2,r_2}, \mathcal{X}_{q,\omega_2^{-1},r_2}\}$ and $h(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$.

As in the proof Lemma 2.13, since $h(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$, the mapping h is a Paley design automorphism, and so by Theorem 8.1 of [35], we know that h is given by $h(x) = a^2\sigma(x) + b$ for all $x \in \mathbb{F}_q$ where $a, b \in \mathbb{F}_q$, $a \neq 0$, and σ is a Frobenius automorphism. Noting that any permutation of the form $x \mapsto a^2x + b$ is an automorphism of each of the decompositions $\mathcal{Y}_{q,\omega_2}$, $\mathcal{X}_{q,\omega_2,r_2}$ and $\mathcal{X}_{q,\omega_2^{-1},r_2}$, it follows that $\sigma(\mathcal{X}_{q,\omega_1,r_1}) \in \{\mathcal{X}_{q,\omega_2,r_2}, \mathcal{X}_{q,\omega_2^{-1},r_2}\}$ and $\sigma(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\omega_2}$.

It is easily verified that $\sigma(\mathcal{Y}_{q,\omega_1}) = \mathcal{Y}_{q,\sigma(\omega_1)}$, and so we have $\mathcal{Y}_{q,\sigma(\omega_1)} = \mathcal{Y}_{q,\omega_2}$. This last equality tells us that $\sigma(\omega_1) \in \{\omega_2, \omega_2^{-1}\}$ (as noted at the beginning of the proof of Lemma 2.13). Thus, we have $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$, and it remains only to prove that $r_1 = r_2$. For this, we note that $\sigma(\mathcal{X}_{q,\omega_1,r_1}) = \mathcal{X}_{q,\sigma(\omega_1),r_1}$. Since $\sigma(\omega_1) \in \{\omega_2, \omega_2^{-1}\}$, and recalling that $\sigma(\mathcal{X}_{q,\omega_1,r_1}) \in \{\mathcal{X}_{q,\omega_2,r_2}, \mathcal{X}_{q,\omega_2^{-1},r_2}\}$, this implies that

$$\mathcal{X}_{q,\omega_2,r_1} \in \{\mathcal{X}_{q,\omega_2,r_2}, \mathcal{X}_{q,\omega_2^{-1},r_2}\} \quad \text{or} \quad \mathcal{X}_{q,\omega_2^{-1},r_1} \in \{\mathcal{X}_{q,\omega_2,r_2}, \mathcal{X}_{q,\omega_2^{-1},r_2}\},$$

and in particular that

$$X^0_{q,\omega_2,r_1} \in \{X^0_{q,\omega_2,r_2}, X^0_{q,\omega_2^{-1},r_2}\} \quad \text{or} \quad X^0_{q,\omega_2^{-1},r_1} \in \{X^0_{q,\omega_2,r_2}, X^0_{q,\omega_2^{-1},r_2}\}.$$

However, it is easily verified (for example by considering the vertex adjacent to both vertex 1 and vertex ω^2 in the cycles $X^0_{q,\omega,r}$ and $X^0_{q,\omega^{-1},r}$ for $r = 0, 1, \ldots, (q-3)/4$) that for any primitive ω, the cycles

$$X^0_{q,\omega,1}, X^0_{q,\omega^{-1},1}, X^0_{q,\omega,2}, X^0_{q,\omega^{-1},2}, \ldots, X^0_{q,\omega,\frac{q-3}{4}}, X^0_{q,\omega^{-1},\frac{q-3}{4}}$$

are pairwise distinct, and also distinct from $X^0_{q,\omega,0} = X^0_{q,\omega^{-1},0}$. It follows immediately that $r_1 = r_2$. □

As a consequence of Lemmas 2.16 and 2.17, we obtain the following theorem.

Theorem 2.18. *If p is prime and $q = p^\alpha \geq 4$, then the number of non-isomorphic uniform decompositions of $2K_q$ into $(q-1)$-cycles is*

- *at least $(q+1)\phi(q-1)/8\alpha$ when $q \equiv 3 \pmod 4$ and*
- *at least $\phi(q-1)/2\alpha$ otherwise.*

In particular, the number of isomorphism classes of uniform decompositions of $2K_q$ into $(q-1)$-cycles in $\{\mathcal{X}_{q,\omega,0} : \omega$ is primitive in $\mathbb{F}_q\}$ is exactly $\phi(q-1)/2\alpha$, and when $q \equiv 3 \pmod 4$ the number of isomorphism classes of uniform decompositions of $2K_q$ into $(q-1)$-cycles in $\{\mathcal{X}_{q,\omega,r} : \omega$ is primitive in $\mathbb{F}_q, r \in \{0,1,2,\ldots,(q-3)/4\}\}$ is exactly $(q+1)\phi(q-1)/8\alpha$.

Proof. By Lemma 2.16, for primitive elements ω_1 and ω_2 in $\mathbb{F}_q$, we have $\mathcal{X}_{q,\omega_1,0} \cong \mathcal{X}_{q,\omega_2,0}$ if and only if $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$. There are $\phi(q-1)$ primitive elements in $\mathbb{F}_q$, and for each primitive element ω_1, there are 2α primitive elements ω_2 such that $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$. Thus, the number of isomorphism classes of uniform decompositions of $2K_q$ into $(q-1)$-cycles in $\{\mathcal{X}_{q,\omega,0} : \omega$ is primitive in $\mathbb{F}_q\}$ is exactly $\phi(q-1)/2\alpha$.

By Lemma 2.17, when $q \equiv 3 \pmod 4$, ω_1 and ω_2 are primitive elements in $\mathbb{F}_q$, and $r \in \{0,1,2,\ldots,(q-3)/4\}$, we have $\mathcal{X}_{q,\omega_1,r_1} \cong \mathcal{X}_{q,\omega_2,r_2}$ if and only if $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$ and $r_1 = r_2$. There are $\phi(q-1)$ primitive elements in $\mathbb{F}_q$ and $(q+1)/4$ values of r_1 in $\{0,1,\ldots,(q-3)/4\}$, and for each primitive element ω_1, there are 2α primitive elements ω_2 such that $\omega_2 = \omega_1^{\pm p^k}$ for some $k \in \mathbb{Z}$. Thus, the number of isomorphism classes of uniform decompositions of $2K_q$ into $(q-1)$-cycles in $\{\mathcal{X}_{q,\omega,r} : \omega$ is primitive in $\mathbb{F}_q, r \in \{0,1,2,\ldots,(q-3)/4\}\}$ is exactly $(q+1)\phi(q-1)/8\alpha$. □

2.5 Concluding Remarks

In this section we summarise our results, make some observations, and mention some open questions. Theorem 2.6 shows that for $n \geq 4$, any uniform decomposition of μK_n into m-cycles falls into one of the following four cases.

(A) $m = n$ and $(n,\mu) \in \{(5,1),(7,1),(4,2),(5,3)\}$.

(B) $\mu = 2$ and $m = n-1$.

(C) $\mu = 1$, $n \equiv 3 \pmod 4$ and $m = (n-1)/2$.

(D) $\mu = 1$ and $2m(m+1) = n(n-1)$.

We discuss each of these in turn.

Case (A) covers uniform decompositions of μK_n into Hamilton cycles, and such decompositions exists for K_5, K_7, $2K_4$ and $3K_5$ only (and for μK_2 for all even μ, and for μK_3 for all μ). Up to isomorphism, the uniform decompositions of K_5, $2K_4$ and $3K_5$ into Hamilton cycles are unique. There are exactly two decompositions of K_7 into Hamilton cycles, and they are both uniform.

In Case (B), we know from Theorems 2.6 and 2.12 that uniform decompositions of $2K_n$ into $(n-1)$-cycles exist for $3 \leq n \leq 9$ but not for $n = 10$, and that such decompositions exist whenever n is a prime power. Indeed, if n is a prime power, say $n = p^\alpha$ where p is prime, then we have constructed $\phi(n-1)/2\alpha$ non-isomorphic uniform decompositions of $2K_n$ into $(n-1)$-cycles when $n \equiv 0$ or $1 \pmod 4$, and $(n+1)\phi(n-1)/8\alpha$ non-isomorphic uniform decompositions of $2K_n$ into $(n-1)$-cycles when $n \equiv 3 \pmod 4$, see Theorem 2.18. The existence of uniform decompositions of $2K_n$ into $(n-1)$-cycles remains an open question for all $n \geq 12$ such that n is not a prime power. Exhaustive computer searches have shown that up to isomorphism the uniform decomposition of $2K_6$ into 5-cycles given in Lemma 2.8 is unique, and that for $n \in \{3, 4, 5, 7, 8, 9, 11\}$ every uniform decomposition of $2K_n$ into $(n-1)$-cycles is isomorphic to $\mathcal{X}_{n,\omega,r}$ for some primitive $\omega \in \mathbb{F}_n$ and some $r \in \{0, 1, \ldots, (q-3)/4\}$.

If $\mathcal{X}$ is any uniform decomposition of $2K_n$ into $(n-1)$-cycles, then $|\mathcal{X}| = n$ and the union of any two distinct cycles of $\mathcal{X}$ contains exactly one 2-cycle (because there are $\binom{n}{2}$ pairs of cycles in $\mathcal{X}$, and also $\binom{n}{2}$ 2-cycles in $2K_n$) and exactly four triangles (by Lemma 2.1). Thus, any uniform decomposition of $2K_n$ into $(n-1)$-cycles is an orthogonal double cover of K_n with $(n-1)$-cycles. An *orthogonal double cover* of K_n with a graph G is a set of subgraphs of K_n, each isomorphic to G, such that every edge of K_n occurs in exactly two of the subgraphs, and any two of the subgraphs have exactly one edge in common. See [28] for a survey on orthogonal double covers of graphs. Not all orthogonal double covers of K_n with $(n-1)$-cycles are uniform. For example, there is an orthogonal double cover of K_{10} with 9-cycles, but no uniform decomposition of $2K_{10}$ into 9-cycles. It is conjectured that there exists an orthogonal double cover of K_n with $(n-1)$-cycles for all $n \geq 4$ (see [29]), and it is known that such an orthogonal double cover exists for $4 \leq n \leq 102$ (see [42]), but the conjecture is unresolved in general.

We now discuss Case (C), uniform decompositions of K_n into $((n-1)/2)$-cycles, the existence of which requires $n \equiv 3 \pmod 4$. We know that such decompositions exist when n is a prime power, and (by exhaustive computer search) that there is no uniform decomposition of K_{15} into 7-cycles, but the existence of uniform decompositions of K_n into $((n-1)/2)$-cycles is unresolved for all other $n \equiv 3 \pmod 4$, with the smallest unresolved case being the existence of a uniform decomposition of K_{35} into 17-cycles. Up to isomorphism, the uniform decomposition of K_7 into 3-cycles is unique, and a computer search has shown that there are no uniform decompositions of K_{11} into 5-cycles other than $\mathcal{Y}_{11,2}$ and $\mathcal{Y}_{11,7}$. If $n \equiv 3 \pmod 4$ is a prime power, say $n = p^\alpha$ where p is prime, then we have constructed $\phi(n-1)/2\alpha$ non-isomorphic uniform decompositions of K_n into $((n-1)/2)$-cycles, see Theorem 2.14.

As noted in the introduction (also see Lemma 2.3 and Theorem 2.6), the associated design of a uniform decomposition of K_n into $((n-1)/2)$-cycles is a Hadamard design on n points, and the well-known Hadamard Conjecture asserts the existence of these for all $n \equiv 3 \pmod 4$ (see [22] page 274). However, the existence of a Hadamard design on n points does not guarantee the existence of a uniform decomposition of K_n into $((n-1)/2)$-cycles. There is no uniform decomposition of K_{15} into 7-cycles, despite the existence of Hadamard designs on 15 points.

The final case to discuss is uniform decompositions of K_n into m-cycles where $2m(m+1) = n(n-1)$, Case (D). The first four integer values of n and m that satisfy $2m(m+1) = n(n-1)$, with $n > 1$ odd, are $(n, m) = (21, 14), (697, 492), (23\,661, 16\,730), (803\,761, 568\,344)$. We saw in Lemma 2.5 that the non-existence of a $(15, 21, 7, 5, 2)$-BIBD rules out a uniform decomposition of K_{21} into 14-cycles. Whether or not there exists a uniform decomposition of K_n into m-cycles for the larger values of n and m in this case remains open, with the existence of the associated quasiresidual designs that are not residual also unresolved. We think that constructing uniform decompositions in Case (D) will be a very difficult problem, especially if they don't exist!

We saw in the proof of Lemma 2.5 that the odd values of n for which there is an integer solution to $2m(m+1) = n(n-1)$ are given by $n \in \{n_1, n_2, \dots\}$ where $n_1, n_2, \dots$ is the sequence given by $n_1 = 1$, $n_2 = 21$ and the recurrence relation $n_i = 34n_{i-1} - n_{i-2} - 16$ for $i \geq 3$. We remark that the integer values of n and m satisfying $2m(m+1) = n(n-1)$ with n odd can alternatively be parameterized by $n = 2s + 2k - 1$ and $m = s + 2k - 2$ where

$$k \in \{x \geq 1 : x - 1 \text{ and } 2x - 1 \text{ are both perfect squares}\} = \{1, 5, 145, 4901, 166465, \dots\}$$

and $s = (2k-1)^{\frac{1}{2}}(k-1)^{\frac{1}{2}}$; with the associated quasiresidual designs being $(s+2k-1, 2s+2k-1, s+1, k, s-k+1)$-BIBDs.

We observe below that all of the examples of uniform decompositions of complete graphs into cycles presented in this paper have an automorphism group that acts 2-homogeneously on the cycles. This raises the question of whether or not there exist any uniform decompositions of complete graphs into cycles for which the automorphism group does not have a 2-homogeneous action on the cycles. We know of no such decompositions of complete graphs, but there does exist a uniform decomposition into Hamilton cycles of $K_{12} - I$, the graph obtained from K_{12} by removing the edges of a 1-factor, such that the automorphism group does not act 2-homogeneously on the cycles.

It is easily verified that the orbit of the cycle $X = (0, 2, 1, 6, 3, 9, 7, 4, 8, \infty_1, 5, \infty_2)$ under the permutation $\rho = (0\ 2\ 4\ 6\ 8)(1\ 3\ 5\ 7\ 9)(\infty_1)(\infty_2)$ is a uniform decomposition of $K_{12} - I$ into Hamilton cycles (one only needs to check that $X \cup \rho(X)$ is isomorphic to $X \cup \rho^2(X)$). To see that the automorphism group of this decomposition does not act 2-homogeneously on the cycles, observe that $X \cup \rho(X)$ and $X \cup \rho^2(X)$ each contain exactly one copy of $K_4 - e$, the graph obtained from K_4 by removing an edge. Denote the copy of $K_4 - e$ in $X \cup \rho(X)$ by Z_1 and the copy of $K_4 - e$ in $X \cup \rho^2(X)$ by Z_2. Now, X contains three edges of Z_1 and $\rho^2(X)$ contains three edges of Z_2, but $\rho(X)$ contains two adjacent edges of Z_1 and X contains two non-adjacent edges of Z_2. Thus, there can be no automorphism of the decomposition that maps $\{X, \rho(X)\}$ to $\{X, \rho^2(X)\}$.

For our decompositions $\mathcal{X}_{q,\omega,r}$ of $2K_n$ into $(n-1)$-cycles and our decompositions $\mathcal{Y}_{q,\omega}$ of K_n into $((n-1)/2)$-cycles, the existence of this 2-homogeneous action on the cycles is shown in the proofs of Lemmas 2.10 and 2.11. Indeed, in the proof of Lemma 2.10 it is noted that the automorphism group of $\mathcal{X}_{q,\omega,r}$ acts 2-transitively on the cycles when $r = 0$. The automorphism groups of the uniform decompositions of $3K_5$ into Hamilton cycles (see Lemma 2.7) and $2K_6$ into 5-cycles (see Lemma 2.8) are each isomorphic to the alternating group A_5, and in each case the group acts 2-transitively on the six cycles of the decomposition. The automorphism groups of the uniform decompositions of $2K_4$ and K_5 into Hamilton cycles also have a 2-transitive action on the cycles.

Up to isomorphism, there are exactly two non-isomorphic decompositions of K_7 into Hamilton cycles [21, 27],

$$\mathcal{W} = \{(1,2,3,4,5,6,7), (1,3,5,7,2,4,6), (1,4,7,3,6,2,5)\}$$

and

$$\mathcal{W}' = \{(1,2,3,4,5,6,7), (1,3,5,7,4,2,6), (1,4,6,3,7,2,5)\}.$$

It can be seen that $(1\ 2\ 4)(3\ 6\ 5)(7)$ is an automorphism of $\mathcal{W}$ and that $(1\ 7\ 4)(2\ 5\ 6)(3)$ is an automorphism of $\mathcal{W}'$, and that in each case the automorphism permutes the three cycles of the decomposition in a single cycle. It follows that the automorphism groups act 2-homogeneously on the cycles of the decompositions. It is also easy to verify that for each of the two decompositions of K_7 into Hamilton cycles, the automorphism group does not act 2-transitively on the cycles.

As a final comment, we observe that in both cases (B) and (C) the number of cycles in a uniform decomposition is equal to the number of vertices in the graph being decomposed, and that in both cases (B) and (D) the number of cycles is one more than the cycle length.

Acknowledgement: The authors acknowledge the support of the Australian Research Council via grants DP150100530, DP150100506, DP120100790 and DP130102987, and thank Victor Scharaschkin for helpful discussions. The first author is supported by an Australian Government Research Training Program Scholarship. The program *nauty* [47] was used in some of our computations.

Chapter 3

Computational analysis and programs

Computation was a significant part of this project. In this section, we discuss the algorithms we used to search for uniform graph decompositions. These algorithms allowed us to construct small uniform graph decompositions, and to prove that certain graph decompositions do not exist.

We define a *partial decomposition* of a graph G to be a decomposition of a subgraph of G. Thus any subset of a decomposition of G is a partial decomposition of G. Since a partial decomposition of G is still a graph decomposition, the definitions of a uniform partial decomposition and a partial cycle decomposition are clear.

3.1 Graph isomorphism

In order to determine whether a graph decomposition is uniform, we need to be able to check whether two graphs are isomorphic. This is the well-known graph isomorphism problem. There is no known worst-case polynomial-time graph isomorphism algorithm, although the problem is also not known to be NP-complete [47]. In particular, the lowest worst-case time complexity yet proved is $e^{O(\sqrt{n \log n})}$ [8]. However, Babai [9] recently published a graph-isomorphism algorithm claimed to have time complexity $e^{(\log n)^{O(1)}}$. Babai then retracted the claim, but has revised the algorithm and now claims that the revision restores that time complexity [10].

Since there is no deterministic polynomial-time algorithm for graph isomorphism, a number of non-deterministic algorithms have been developed. One of these is *nauty*, written by McKay and Piperno [47]. We refer to *nauty* in several of the algorithms in this chapter. *nauty* is a *canonical labelling algorithm*. A canonical labelling algorithm is one which relabels graph vertices in such a way that any two isomorphic graphs will be identical after relabelling; we call the relabelled graph the *canonical labelling* or *canonical form* of the original graph. McKay and Piperno's paper also provides a summary of some other competitive graph isomorphism algorithms, not all of which involve canonical labelling. As McKay and Piperno write, "Although none of the programs tested have the best performance on all graph classes, it is clear that Traces is currently the leader on the majority of difficult graph classes tested, while *nauty* is still preferred for mass testing of small graphs." [47]

Since we could only search for uniform decompositions of small graphs, and each search involved a potentially large number of graph isomorphism tests, this made *nauty* a suitable graph isomorphism algorithm for our problem.

3.2 Algorithms for constructing decompositions of small graphs

In this project, we often needed to construct decompositions of small graphs computationally. Finding uniform decompositions of small graphs provided a starting point for identifying more general constructions. By making the construction algorithms exhaustive, we were also able to prove the nonexistence of certain decompositions (such as uniform Hamilton decompositions of K_9 and K_{15}). The algorithm we used most often is given below as Algorithm 4. Algorithm 4 is a nested depth-first search for uniform cycle decompositions (i.e. a depth-first search for cycles in such a decomposition, where each cycle is generated by a depth-first search). However, in order to understand this algorithm, it is simpler to start with a nested depth-first search for cycle decompositions in general, which we describe in Algorithm 2.

Let G be a graph with vertices $\{0, 1, 2, ..., n-1\}$, and let H be a subgraph of G. We wish to construct all H-decompositions of G. We start with an empty partial decomposition of G. For any given partial H-decomposition $\{D_0, D_1, ..., D_x\}$ of G, we first check if the partial decomposition is a decomposition of G. If so, we print that decomposition and then backtrack to the partial decomposition $\{D_0, D_1, ..., D_{x-2}\}$ (as there exists no other H-decomposition of G containing $\{D_0, D_1, ..., D_{x-1}\}$). If not, we use a depth-first search to find the lexicographically least subgraph $E \cong H$ of $G \setminus (D_0 \cup D_1 \cup ... \cup D_x)$, and recurse to the partial decomposition $\{D_0, D_1, ..., D_x, E\}$, continuing on that branch until all possible H-decompositions of G containing the partial decomposition $\{D_0, D_1, ..., D_x, E\}$ have been found. We then backtrack and try the lexicographically next least subgraph $E \cong H$ of $G \setminus (D_0 \cup D_1 \cup ... \cup D_x)$. Once all such subgraphs have been considered, we backtrack to the previous partial decomposition, $\{D_0, D_1, ..., D_{x-1}\}$. Once all branches of the search have concluded, every H-decomposition of G will have been printed.

This search will construct every H-decomposition of G. However, we only need at least one representative from each isomorphism class of such decompositions. As such, we can optimize the search in two ways; firstly, in searching for each candidate subgraph E we can assume that E contains the lexicographically least edge in $G \setminus (D_0 \cup D_1 \cup ... \cup D_x)$, as any decomposition of G contains some subgraph with that edge and the ordering of subgraphs is arbitrary. Thus we need not backtrack to before the first edge of E. Secondly, if $G = K_n$ then the vertex labelling of G is arbitrary and so we need only consider the lexicographically least choice of D_0 (and so need not backtrack to other candidates for D_0).

Algorithm 2 is an example of this, and is a depth-first search on possible cycles for a decomposition of a graph G into m-cycles. Given a partial decomposition $\{D_0, D_1, ..., D_x\}$ of G, Algorithm 2 uses Algorithm 1 to identify the possible cycles E, and then calls itself to examine each partial decomposition $\{D_0, D_1, ..., D_x, E\}$. Algorithm 1 is a depth-first search through $G \setminus \{D_0, D_1, ..., D_x\}$ to identify each possible cycle E.

Algorithm 1: Build cycle (G, m, P)

Input: a graph G
Input: an integer m, the length of the cycle to build.
Input: a path $P = [p_1, p_2, ..., p_i]$ of length no greater than m.

if $i = m$
  if p_1 is adjacent to p_i in G
    Store the cycle $C = (p_1, p_2, ..., p_i)$.
else
  for each vertex v of G which is not in P and is adjacent to p_i in G
    Build cycle (G, m, $P \cup [p_i, v]$)

Algorithm 2: Build cycle decomposition (G, m, r, $\mathcal{D}$)

Input: a graph G
Input: an integer m, the length of the cycles in the decomposition of G
Input: an integer r, the number of cycles in the decomposition of G ($r = |E(G)|/m$)
Input: a partial decomposition $\mathcal{D}$ of G into cycles of length m.

if $|\mathcal{D}| = r$
  Store the decomposition $\mathcal{D}$.
else
  Set x_1, x_2 to be the lexicographically least edge in G which is not in any cycle of $\mathcal{D}$
  Set $G' = G$
  For each cycle D of $\mathcal{D}$:
    Remove the edges of D from G'
  For each cycle C produced by **Build cycle** (G', m, $[x_1, x_2]$)
    Build cycle decomposition (G, m, r, $\mathcal{D} \cup \{C\}$)

Given the graph G, cycle length m, number of cycles r and an empty partial decomposition $\mathcal{D} = \emptyset$, Algorithm 2 will produce at least one member of each isomorphism class of decompositions of G into cycles of length m. In order to find only uniform decompositions, we can modify Algorithm 2 by only accepting those cycles C such that $C \cup D$ is isomorphic to $D_0 \cup D_1$ for all $D \in \mathcal{D}$, where D_0 and D_1 are the first two cycles of $\mathcal{D}$. We apply those modifications in Algorithm 4.

If G is a complete multigraph μK_n, we can further restrict the search for uniform m-cycle decompositions of G by considering the number of triangles and 2-cycles in $C \cup D$. By Lemma 2.1, the number of triangles in $C \cup D$ is equal to $\frac{4\mu m^2}{\mu n(n-1)-2m}$. By the proof of Corollary 2.2, the number of 2-cycles in $C \cup D$ is $\frac{\binom{n}{2}\binom{\mu}{2}}{\binom{(\mu(n-1))/2}{2}} = \frac{n(\mu-1)}{\mu(n-1)-2}$. Algorithm 3 is used to filter out partial decompositions which do not meet this restriction; it returns failure if G is a complete multigraph and C and D do not form the appropriate number of triangles and double edges, or success otherwise.

Algorithm 3: Check union graph (G, C, D)

Input: A graph G
Input: An m-cycle C in G
Input: An m-cycle D in G

if G is not a complete multigraph μK_n
    Return success.
else
    if the number of triangles in $C \cup D$ is $\frac{4\mu m^2}{\mu n(n-1)-2m}$ **and** the number of 2-cycles in $C \cup D$ is $\frac{n(\mu-1)}{\mu(n-1)-2}$
        Return success.
    else
        Return failure.

Algorithm 4: Build uniform cycle decomposition (G, m, r, $\mathcal{D}$)

Input: a graph G
Input: an integer m, the length of the cycles in the decomposition of G
Input: an integer r, the number of cycles in the decomposition of G ($r = |E(G)|/m$)
Input: a partial decomposition $\mathcal{D}$ of G into cycles of length m.

if $|\mathcal{D}| = r$
    Store the decomposition $\mathcal{D}$.
else
    Set x_1, x_2 to be the lexicographically least edge in G which is not in any cycle of $\mathcal{D}$
    Set $G' = G$
    For each cycle D in $\mathcal{D}$:
        Remove the edges of D from G'
    For each cycle C produced by **Build cycle** (G', m, $[x_1, x_2]$)
        if $\mathcal{D}$ is not empty
            Set D_0 to be the first cycle of $\mathcal{D}$
            For each cycle D in $\mathcal{D} \setminus \{D_0\}$
                if $D \cup D_0 \ncong C \cup D$ **or Check union graph** (G, C, D) returns failure
                    Discard C.
        if C was not discarded
            Build uniform cycle decomposition (G, m, r, $\mathcal{D} \cup \{C\}$)

3.2.1 Nonexistence of uniform Hamilton decompositions of K_9 and K_{15}

We showed in Chapter 2, Corollary 2.2 that there exists a uniform Hamilton decomposition of K_n only if $n = 3, 5, 7, 9, 15$. In addition, we claimed that there does not exist a uniform Hamilton decomposition of K_9 or K_{15}, based on the use of computer searches. In this section we describe those exhaustive computer searches.

We used Algorithm 4 to search for uniform Hamilton decompositions of K_9. No such decompositions were found. Since Algorithm 4 is exhaustive, this is sufficient to show that K_9 does not have any uniform Hamilton decompositions. To independently verify this result, we wrote a program based on a different algorithm which we now describe.

Up to isomorphism, there are 122 distinct Hamilton decompositions of K_9 [21, 27]. Thus in order to disprove the existence of a uniform Hamilton decomposition of K_9, it suffices to prove that none of these 122 decompositions are uniform. By Lemma 2.1, any uniform Hamilton decomposition of K_9 has 6 triangles in the union of any two cycles. Thus we can use Algorithm 5, below, to examine the Hamilton decompositions of K_9 for this restriction.

Algorithm 5: Check triangle count ($\mathcal{D}$)
Input: a Hamilton decomposition $\mathcal{D} = \{D_0, D_1, D_2, D_3\}$ of K_9

```
For each cycle D_i in D
        For each cycle D_j ≠ D_i in D
                if D_i ∪ D_j does not contain exactly 6 triangles
                        Return failure.
Return success.
```

Applying Algorithm 5 to each of the Hamilton decompositions of K_9 shows that only four such decompositions contain exactly 6 triangles in the union of each pair of cycles. It is easily verified by hand that those four decompositions are not uniform.

The existence or nonexistence of uniform Hamilton decompositions of K_{15} cannot be determined in either of these ways; no list of the Hamilton decompositions of K_{15} exists, and Algorithm 4 is too slow. We had to apply other optimisations in order to perform an exhaustive search in a reasonable amount of time. In particular, we constructed categories of 15-cycles such that if there exists a uniform Hamilton decomposition of K_{15}, there exists one which contains cycles from only one of those categories. Then we searched through each individual category instead of searching through all possible cycles.

Assume there is a uniform Hamilton decomposition of K_{15}. Then (as relabelling the vertices is an automorphism of K_{15}) there is a uniform Hamilton decomposition $\mathcal{D}$ of K_{15} with the cycle $D_0 = (0, 1, 2, 3, ..., 14)$. In addition, the union of any two distinct cycles D_i, D_j in $\mathcal{D}$ must be isomorphic to $D_0 \cup D_k$ for all $D_k \in \mathcal{D}, k \neq 0$, and must contain exactly five triangles by Lemma 2.1. Consequently, we can construct all possible cycles D_i in $K_n \setminus D_0$ and categorise them by the graph $D_0 \cup D_i$. Any uniform decomposition will consist only of D_0 and cycles within one such category. All the cycles can be constructed using Algorithm 1: **Build cycle** $(K_{15} \setminus D_0, 15, [0])$.

There are 805491 connected 4-regular graphs on 15 vertices [51][52], and of these, 162645 contain exactly five triangles [52]. For each of these graphs, we wish to construct the category of all cycles D_i such that $D_0 \cup D_i$ is isomorphic to that graph. Thus for each graph $D_0 \cup D_i$ with five triangles, we must determine to which of the 162645 graphs it is isomorphic. A linear search would require an average of 81323 comparisons. Fortunately, as we test isomorphisms using canonical labelling, the canonical forms of the listed graphs can be sorted into a lexicographical ordering. Thus the canonical form of $D_0 \cup D_i$ can be found in that list using a binary search in only $log_2(162645) \approx 18$ graph-equality comparisons. D_i is then placed in the category corresponding to the 4-regular graph isomorphic to $D_0 \cup D_i$. Algorithm 6 describes the categorisation process.

Algorithm 6: Categorise cycles for decomposition ($\mathcal{G}$, $\mathcal{S}$)
Input: The set $\mathcal{G}$ of 4-regular graphs on 15 vertices with five triangles.
Input: A set $\mathcal{S}$ of Hamilton cycles in K_{15} which are disjoint from D_0 and form exactly five triangles in union with D_0.
1:**For** each 4-regular graph G_i in $\mathcal{G}$
2: Use *nauty* to get the canonical form of G_i
3: Store the canonical form of G_i in a list $\mathcal{L}$
4:Sort $\mathcal{L}$ in ascending order.
5:**For** each canonical graph in $\mathcal{L}$
6: Create a file with number equal to the position of the canonical graph in $\mathcal{L}$.
5:**For** each cycle S_i in $\mathcal{S}$
6: Use *nauty* to get the canonical form T_i of $S_i \cup D_0$.
7: Use a binary search to find T_i in $\mathcal{L}$.
8: Print S_i to the file corresponding to the position of T_i in $\mathcal{L}$.

With this optimisation in place, we were able to categorise all possible 15-cycles in $K_{15} \setminus D_0$ whose unions with D_0 contain exactly five triangles. The categorisation took approximately 1 week of running time, without parallelisation. The largest two categories held 9360 cycles each. Five categories held no cycles, from which we can conclude those 4-regular graphs have no Hamilton decomposition. We estimate the total number of cycles categorised to be approximately 10^9.

We then searched the cycles of each category for a Hamilton decomposition of K_{15} such that the union of any two cycles contains exactly five triangles (a necessary condition for the decomposition to be uniform). We used a depth-first search as described by Algorithm 7, below.

Algorithm 7: Find decompositions in category ($\mathcal{D}$, $\mathcal{S}$)
Input: A partial Hamilton decomposition $\mathcal{D}$ of K_{15}
Input: A set of cycles $\mathcal{S}$ such that $S_i \cup D_0 \cong S_j \cup D_0$ for all $S_i, S_j \in \mathcal{S}$
1:**if** $|\mathcal{D}| = 7$
2: Print $\mathcal{D}$
3: Terminate
4:Set $\{v_1, v_2\}$ to be the lexicographically least edge not in any cycle of $\mathcal{S}$
5:**For** each cycle S in $\mathcal{S}$ containing $\{v_1, v_2\}$ and edge-disjoint from $\mathcal{D}$
6: **For** each cycle D_i in $\mathcal{D}$
7: **if** $S \cup D_i$ does not contain exactly five triangles
8: Continue to the next iteration of the loop at line 5
9: Create $\mathcal{D}' = \mathcal{D} \cup \{S\}$
10: **Find decompositions in category** $(\mathcal{D}', \mathcal{S})$

The use of Algorithm 7 is easily parallelised, since the result from each category depends only on the cycles in that category. Running on a single computer with a 4-core processor, we were able to search all categories in approximately 1 week.

Only one category contained cycles which formed a Hamilton decomposition in which every union of two distinct cycles contained exactly five triangles. Ten such decompositions were found, but none of them are uniform.

3.2.2 Uniform $(\frac{n-1}{2})$-cycle decompositions of K_n

Suppose there exists a uniform decomposition of K_n into $(\frac{n-1}{2})$-cycles. Then we proved in Theorem 2.6 that there exists a Hadamard design of order n. Thus $n \equiv 3 \pmod 4$. In addition, we proved that the union of any two cycles in the decomposition contains precisely one triangle when $n > 7$.

We proved in Lemma 2.11 that there exists a uniform $(\frac{n-1}{2})$-cycle decomposition of K_n whenever $n \equiv 3 \,(\mathrm{mod}\ 4)$ and n is a prime power, using the construction $\mathcal{Y}_{q,\omega}$. We claimed every uniform 5-cycle decomposition of K_{11} is isomorphic to $\mathcal{Y}_{11,2}$ or $\mathcal{Y}_{11,7}$ and there is no uniform 7-cycle decomposition of K_{15}. Both of these claims relied on computer searches which we now describe.

We used Algorithm 4 to search for uniform 5-cycle decompositions of K_{11}. We found 2880 distinct decompositions, of which 1440 were isomorphic to $\mathcal{Y}_{11,2}$ and 1440 were isomorphic to $\mathcal{Y}_{11,7}$. Thus there exists no other uniform 5-cycle decomposition of K_{11}.

We found that, in practice, Algorithm 4 was too slow to search for a uniform 7-cycle decomposition of K_{15}. Consequently, we applied a modified version of Algorithm 6 and Algorithm 7 to this problem. Unlike the problem of searching for uniform Hamilton decompositions of K_{15}, there was no pre-existing list of possible union graphs. Thus we needed to construct such a list.

Assume there is a uniform decomposition of K_{15} into 7-cycles. Then there is a uniform 7-cycle decomposition $\mathcal{D}$ of K_{15} with the cycle $D_0 = (0, 1, 2, 3, 4, 5, 6)$. In addition, we proved in Lemma 2.1 that $D_0 \cup D_1$ contains precisely one triangle and in Lemma 2.3 that D_0 intersects D_1 in precisely 3 vertices, where D_1 is in $\mathcal{D} \setminus D_0$. In addition, we can assume without loss of generality that D_1 contains the lexicographically least edge in $\mathcal{D} \setminus D_0$. It follows that Algorithm 1 can be used to construct all possible candidates for D_1. Since we are interested in the graph $D_0 \cup D_1$, we removed any candidate cycles D_1^* where $D_0 \cup D_1^* \cong D_0 \cup D_1'$ and D_1' is a candidate cycle lexicographically less than D_1^*. The resultant set of candidate cycles for D_1 is $\{(0, 2, 5, 7, 8, 9, 10), (0, 2, 7, 4, 8, 9, 10), (0, 2, 7, 5, 8, 9, 10), (0, 2, 7, 6, 8, 9, 10), (0, 2, 7, 8, 1, 9, 10),$ $(0, 2, 7, 8, 3, 9, 10), (0, 2, 7, 8, 4, 9, 10)\}$. Let this set be T, and let $\mathcal{G} = \{D_0 \cup D_1 : D_1 \in T\}$. Then if there exists a uniform 7-cycle decomposition of K_{15}, then the union of any two cycles in that decomposition must be isomorphic to some graph G in $\mathcal{G}$.

Since $|\mathcal{G}| = 7$, it was not necessary to sort the set of canonical forms of graphs in $\mathcal{G}$ or to use a binary search to find them. Thus we used a simplified version of Algorithm 6 to categorise all possible cycles in a uniform decomposition (as generated by Algorithm 1) by their union with D_1 in $\mathcal{G}$. Then we used a modified version of Algorithm 7 to search for uniform decompositions in each category. No uniform decompositions were found. Since this search is exhaustive, it follows that there exist no uniform 7-cycle decompositions of K_{15}.

When n is not a prime power, $n > 15$ and $n \equiv 3 \pmod 4$, it is not known whether there exists a uniform $(\frac{n-1}{2})$-cycle decomposition of K_n. When n is a prime power, $n > 11$ and $n \equiv 3$ (mod 4), it is not known whether every uniform $(\frac{n-1}{2})$-cycle decomposition of K_n is isomorphic to some $\mathcal{Y}_{n,\omega}$.

3.3 Specific graph decompositions

In Chapter 2 we have already enumerated all the uniform Hamilton decompositions of μK_n, and proved that they are unique up to isomorphism, except for the case of K_7, which has two uniform Hamilton decompositions (both enumerated in Chapter 2). Here we will enumerate some other uniform decompositions found using computational methods.

3.3.1 Uniform $(n-1)$-cycle decompositions of $2K_n$

Suppose there exists a uniform decomposition $\mathcal{D} = \{D_0, D_1, ..., D_{n-1}\}$ of $2K_n$ into $(n-1)$-cycles, where $V(K_n) = \{0, 1, 2, ..., n-1\}$. We can assume without loss of generality that D_i does not contain the vertex i, since each cycle misses precisely one vertex and the ordering of the cycles is arbitrary. Furthermore, we can assume that $D_0 = (1, 2, 3, ..., n-1)$ since vertex labelling is arbitrary. Finally, for each cycle $D_i = (v_0, v_1, ..., v_{n-2})$, we can assume without loss of generality that v_0 is the lexicographically least vertex of D_i and v_1 is the lexicographically least neighbour of v_0 in D_i. Using these assumptions, we wrote a program implementing Algorithm 4 to construct uniform decompositions of $2K_n$ into $(n-1)$-cycles when $5 \leq n \leq 11$, which was exhaustive in that its output contained at least one representative from each isomorphism class of such decompositions. The program did not check whether any two decompositions produced were isomorphic; thus, the output included multiple isomorphic uniform $(n-1)$-cycle decompositions of $2K_n$ in some cases. The output was subsequently checked for isomorphism and the tables below include only one representative from each isomorphism class.

All of the decompositions found are isomorphic to those generated by the construction $\mathcal{X}_{q,\omega,r}$ given in Chapter 2, except for the decomposition of $2K_6$ which is also given in Chapter 2. Since Algorithm 4 is exhaustive, it follows that the construction $\mathcal{X}_{q,\omega,r}$ generates all uniform $(n-1)$-cycle decompositions of $2K_n$ when $n \leq 11$ and n is a prime power.

$2K_5$, $2K_6$, $2K_7$:

Uniform decomposition of $2K_5$ into 4-cycles	Uniform decomposition of $2K_6$ into 5-cycles	Uniform decompositions of $2K_7$ into 6-cycles	
(1 2 3 4)	(1 2 3 4 5)	(1 2 3 4 5 6)	(1 2 3 4 5 6)
(0 2 4 3)	(0 2 4 3 5)	(0 2 4 5 3 6)	(0 2 5 4 6 3)
(0 3 1 4)	(0 1 4 5 3)	(0 1 4 6 5 3)	(0 1 3 5 6 4)
(0 1 2 4)	(0 2 5 1 4)	(0 2 5 1 6 4)	(0 4 1 6 2 5)
(0 1 3 2)	(0 3 1 2 5)	(0 3 6 2 1 5)	(0 3 5 1 2 6)
	(0 1 3 2 4)	(0 4 1 3 2 6)	(0 1 4 2 3 6)
		(0 1 3 4 2 5)	(0 2 4 3 1 5)
Isomorphic to $\mathcal{X}_{5,2,0}$	Cannot be constructed as $\mathcal{X}_{6,\omega,0}$	Isomorphic to $\mathcal{X}_{7,3,0}$	Isomorphic to $\mathcal{X}_{7,3,1}$

$2K_8$ and $2K_9$:

Uniform decompositions of $2K_8$ into 7-cycles	Uniform decompositions of $2K_9$ into 8-cycles
(1 2 3 4 5 6 7)	(1 2 3 4 5 6 7 8)
(0 3 5 6 2 7 4)	(0 3 2 7 5 8 4 6)
(0 4 6 7 3 1 5)	(0 4 3 8 6 1 5 7)
(0 5 7 1 4 2 6)	(0 5 4 1 7 2 6 8)
(0 6 1 2 5 3 7)	(0 1 7 3 8 2 5 6)
(0 1 4 6 3 2 7)	(0 2 8 4 1 3 6 7)
(0 1 3 4 7 5 2)	(0 3 1 5 2 4 7 8)
(0 2 4 5 1 6 3)	(0 1 8 5 3 6 2 4)
	(0 2 1 6 4 7 3 5)
Isomorphic to $\mathcal{X}_{8,\omega,0}$ for primitive ω in $\mathbb{F}_8$	Isomorphic to $\mathcal{X}_{9,\omega,0}$ for primitive ω in $\mathbb{F}_9$

$2K_{10}$:

By exhaustive computer search, there exist no uniform decompositions of $2K_{10}$ into 9-cycles.

$2K_{11}$:

Uniform decompositions of $2K_{11}$ into 10-cycles	
(1 2 3 4 5 6 7 8 9 10)	(1 2 3 4 5 6 7 8 9 10)
(0 2 5 7 3 10 4 9 8 6)	(0 3 2 6 9 4 10 8 5 7)
(0 3 6 8 4 1 5 10 9 7)	(0 4 3 7 10 5 1 9 6 8)
(0 4 7 9 5 2 6 1 10 8)	(0 5 4 8 1 6 2 10 7 9)
(0 5 8 10 6 3 7 2 1 9)	(0 6 5 9 2 7 3 1 8 10)
(0 6 9 1 7 4 8 3 2 10)	(0 1 9 2 4 8 3 10 6 7)
(0 1 3 4 9 5 8 2 10 7)	(0 2 10 3 5 9 4 1 7 8)
(0 2 4 5 10 6 9 3 1 8)	(0 3 1 4 6 10 5 2 8 9)
(0 3 5 6 1 7 10 4 2 9)	(0 4 2 5 7 1 6 3 9 10)
(0 4 6 7 2 8 1 5 3 10)	(0 1 10 4 7 2 8 6 3 5)
(0 1 4 6 2 9 3 8 7 5)	(0 2 1 5 8 3 9 7 4 6)
Isomorphic to:	
$\mathcal{X}_{11,2,0}$	$\mathcal{X}_{11,7,0}$

Uniform decompositions of $2K_{11}$ into 10-cycles			
(1 2 3 4 5 6 7 8 9 10)	(1 2 3 4 5 6 7 8 9 10)	(1 2 3 4 5 6 7 8 9 10)	(1 2 3 4 5 6 7 8 9 10)
(0 3 6 4 10 5 9 2 7 8)	(0 3 7 10 9 4 2 5 8 6)	(0 3 5 7 2 10 9 6 4 8)	(0 2 7 9 8 5 3 10 6 4)
(0 6 8 1 10 4 7 5 3 9)	(0 6 1 7 5 3 10 4 8 9)	(0 7 6 3 9 5 10 4 1 8)	(0 3 8 1 7 10 4 6 5 9)
(0 5 8 6 2 7 1 4 9 10)	(0 5 9 2 1 6 4 7 10 8)	(0 5 7 9 4 2 1 8 6 10)	(0 4 9 1 10 7 5 2 8 6)
(0 1 5 7 9 6 2 3 10 8)	(0 1 10 6 2 5 7 9 3 8)	(0 9 8 5 1 7 2 6 3 10)	(0 1 7 8 6 2 9 3 10 5)
(0 2 1 6 3 9 4 8 10 7)	(0 7 1 4 3 8 6 9 2 10)	(0 2 8 10 3 4 6 1 9 7)	(0 6 1 3 2 9 7 4 10 8)
(0 3 7 9 1 8 4 5 2 10)	(0 3 2 8 4 7 9 1 5 10)	(0 1 10 7 3 9 4 8 5 2)	(0 3 9 10 8 4 1 5 2 7)
(0 4 3 8 5 1 6 10 2 9)	(0 2 4 1 8 10 5 6 3 9)	(0 4 10 2 5 6 8 3 1 9)	(0 8 3 5 4 1 9 6 2 10)
(0 2 4 7 6 10 3 1 9 5)	(0 2 7 3 1 9 6 10 4 5)	(0 3 2 9 5 1 6 10 7 4)	(0 5 1 2 10 6 3 7 4 9)
(0 1 4 2 8 3 7 10 5 6)	(0 1 5 8 7 2 10 3 6 4)	(0 1 3 5 10 8 7 4 2 6)	(0 2 4 8 1 3 6 7 5 10)
(0 4 6 9 8 2 5 3 1 7)	(0 4 9 5 3 1 8 2 6 7)	(0 5 4 1 7 3 8 2 9 6)	(0 1 6 9 5 8 2 4 3 7)
Isomorphic to:			
$\mathcal{X}_{11,2,1}$	$\mathcal{X}_{11,2,2}$	$\mathcal{X}_{11,7,1}$	$\mathcal{X}_{11,7,2}$

$2K_{12}$ is too large for our computational approach.

3.3.2 Uniform Hamilton decompositions of $K_n \setminus I$

When n is even, K_n has odd degree. It follows that there is no cycle decomposition of K_n. However, as noted in Subsection 1.1.4, Walecki [43] proved that there exists a Hamilton decomposition of $K_n \setminus I$ when n is even. Thus it is natural to ask: When n is even, does there exist a uniform Hamilton decomposition of $K_n \setminus I$?

We used Algorithm 4 to search for uniform Hamilton decompositions of $K_n \setminus I$. We chose to search for a partial uniform decomposition of K_n consisting of $\frac{n-2}{2}$ Hamilton cycles; such a partial decomposition is a uniform Hamilton decomposition of $K_n \setminus I$. This allowed us to choose the first cycle $D_0 = (0, 1, 2, ..., n-1)$, instead of choosing a particular 1-factor I. Since $K_n \setminus I$ is not a complete graph, Algorithm 3 does not apply.

There is only one Hamilton decomposition (up to isomorphism) of $K_6 \setminus I$, and it is uniform, as is the only Hamilton decomposition (up to isomorphism) of $K_4 \setminus I$. Thus we consider only $K_n \setminus I$ where $n = 8, 10, 12$. When $n \geq 14$, Algorithm 4 is too slow and so we do not know of any uniform Hamilton decompositions of $K_n \setminus I$. We found that there are exactly 6 uniform Hamilton decompositions of $K_8 \setminus I$ (up to isomorphism), there are no uniform Hamilton decompositions of $K_{10} \setminus I$ and there is exactly one uniform Hamilton decomposition of $K_{12} \setminus I$ (up to isomorphism). The uniform Hamilton decompositions of $K_8 \setminus I$ and $K_{12} \setminus I$ are given in the tables below.

Uniform Hamilton decompositions of $K_8 \setminus I$		
(0 1 2 3 4 5 6 7)	(0 1 2 3 4 5 6 7)	(0 1 2 3 4 5 6 7)
(0 2 4 6 1 3 7 5)	(0 2 4 6 3 7 1 5)	(0 2 4 7 3 6 1 5)
(0 3 6 2 5 1 7 4)	(0 3 1 4 7 5 2 6)	(0 3 5 2 7 1 4 6)
(0 1 2 3 4 5 6 7)	(0 1 2 3 4 5 6 7)	(0 1 2 3 4 5 6 7)
(0 2 4 7 5 1 6 3)	(0 2 5 7 3 1 6 4)	(0 2 6 4 7 5 1 3)
(0 4 1 7 3 5 2 6)	(0 3 6 2 7 4 1 5)	(0 4 2 5 3 7 1 6)

Uniform Hamilton decomposition of $K_{12} \setminus I$
(0 1 2 3 4 5 6 7 8 9 10 11)
(0 2 8 6 11 3 7 1 5 9 4 10)
(0 3 9 7 2 5 8 11 1 10 6 4)
(0 5 11 9 6 2 10 3 1 8 4 7)
(0 6 1 9 2 4 11 7 10 5 3 8)

Chapter 4

Conclusions and other directions

As discussed in Section 2.5, we have proved that when $n \geq 4$ any uniform decomposition of μK_n into m-cycles falls into one of the following four cases:

(A) $m = n$ and $(n, \mu) \in \{(5, 1), (7, 1), (4, 2), (5, 3)\}$.

(B) $\mu = 2$ and $m = n - 1$.

(C) $\mu = 1$, $n \equiv 3 \pmod 4$ and $m = (n - 1)/2$.

(D) $\mu = 1$ and $2m(m + 1) = n(n - 1)$.

However, there remain some open questions in the problem of uniform cycle decompositions of complete multigraphs. In Case (B) we have constructed uniform decompositions of $2K_n$ into $(n - 1)$-cycles when n is a prime power or $n = 6$. However, the existence of uniform decompositions of $2K_n$ into $(n - 1)$-cycles when n is not a prime power remains unsolved except when $n = 4, n = 6$ or $n = 10$. In addition, when n is a prime power it is not known whether every such uniform decomposition is isomorphic to those produced by our construction $\mathcal{X}_{q,\omega,r}$ as defined in Definition 2.9.

Similarly, in Case (C) the existence of uniform decompositions of K_n into $(\frac{n-1}{2})$-cycles has not been established or disproved when $n > 15$ and n is not a prime power. When n is a prime power it is not known whether every such uniform decomposition is isomorphic to those produced by our construction $\mathcal{Y}_{q,\omega}$ as defined in Definition 2.9.

In Case (D) the existence of uniform decompositions of K_n into m-cycles where $2m(m + 1) = n(n - 1)$ has not been established or disproven when $n \geq 697$ and n is odd. This problem is of interest mostly because the existence of any such uniform decomposition would imply the existence of a large quasi-residual BIBD which is not residual, as proved in Lemma 2.5.

Consider a complete multigraph μK_n of odd degree. We raise the question: When does $\mu K_n \setminus I$ have a uniform m-cycle decomposition (where I is a 1-factor in K_n)? In Subsection 3.3.2 we proved that $K_n \setminus I$ has a uniform Hamilton decomposition when $n = 4, 6, 8, 12$, but not when $n = 10$. The general problem remains unresolved.

In the course of this project we also investigated uniform decompositions of complete multigraphs into complete subgraphs, stars and paths. It is well-known that a K_m-decomposition of μK_n is equivalent to an (n, m, μ)-BIBD, in which each block is the vertex set of a complete subgraph. Such a decomposition is uniform if and only if the number of points in the intersection of two blocks is constant, independent of the choice of blocks; by Theorem 1.3 it follows that a uniform K_m-decomposition of μK_n is equivalent to a symmetric (n, m, μ)-BIBD. Uniform decompositions into stars and paths are discussed in the rest of this chapter.

4.1 Uniform decompositions of complete multigraphs into stars

A *k-star* is a graph consisting of $k+1$ vertices and k edges, such that k of the vertices have degree 1 and one vertex has degree k. We call the vertex of degree k the *centre* of the star, and the vertices of degree 1 its *leaves*.

It is possible to have a uniform star decomposition of a graph with two different-order stars only if there are exactly two stars in the decomposition (as the number of edges per union of two stars must be constant). Since at least one end of each edge must be the centre of a star, the only complete multigraph μK_n with a uniform star decomposition into different-order stars is K_3, with the two stars being a 1-star and a 2-star. Thus we only need to consider the cases where each star in the decomposition has the same number of edges, i.e. uniform decompositions of μK_n into k-stars where k is constant.

In 1974, Cain [19] proved that K_n can be decomposed into k-stars if and only if $n = km$ where k is odd or m is even, or $n = km+1$ where k is odd or m is even. In 1979, Tarsi [59] proved that if $\mu > 1$, then μK_n can be decomposed into k-stars if and only if $k|\mu\binom{n}{2}$ and either $n \geq k+1$ (if μ is even) or $n \geq k + 1 + \frac{k}{\mu}$ (if μ is odd). In this section we will prove that uniform k-star decompositions of K_n are equivalent to skew Hadamard designs. Hadamard designs have been discussed earlier in this thesis. A *skew* Hadamard design has an associated Hadamard matrix $\mathbf{H}$ such that $\mathbf{H} - \mathbf{I}$ is skew-symmetric (where $\mathbf{I}$ is the identity matrix). Ó Catháin [50] proved that a Hadamard design is skew if and only if it is possible to add one point to each block of the design and form another BIBD.

Skew Hadamard designs are known to exist for all orders $(2^n q_1 q_2 ... q_t) - 1$ where n is a non-negative integer and $q_i \cong 3 \pmod 4$ is a prime power for each i in $\{1, 2, ..., t\}$ [65]. In addition, for any odd $u > 1$ and any skew Hadamard design of order h, there exists a skew Hadamard design of order h^u [60].

It is trivial to show that μK_1 has a uniform k-star decomposition for any k, and μK_2 has a uniform decomposition into 1-stars but not into k-stars for any $k \neq 1$. In this section we will prove the following theorem:

Theorem 4.1. *Let $n \geq 3$. Then there exists a uniform decomposition of μK_n into k-stars if and only if one of the following is true:*
(A) $\mu = 2$ and $k = (n-1)$,
(B) $\mu = 1, k = \frac{n-1}{2}$ and there exists a skew Hadamard design of order n, or
(C) $\mu = 1, k = \frac{n}{2}$ and there exists a skew Hadamard design of order $n-1$.

When $n \geq 3$, any star decomposition of μK_n must contain two stars with distinct centres, since for each vertex there is at least one edge in μK_n disjoint from that vertex. It follows that any uniform k-star decomposition of μK_n where $k > 1$ may not contain any two stars with the same centre, as the union of two k-stars with the same centre has a vertex of degree $2k$ and the union of two k-stars with distinct centres does not. If $k = 1$ then a uniform k-star decomposition of μK_n is equivalent to a uniform edge decomposition, and it is trivial to show that such a decomposition exists only when $\mu K_n = K_3$ or $n < 3$. It is also easy to see that K_3 has a unique uniform decomposition into edges.

We first consider the case $\mu > 2$, and prove that there is no uniform star decomposition of μK_n in this case.

Lemma 4.2. *For $n > 2, \mu > 2$, there exists no uniform star decomposition of μK_n.*

Proof. Suppose there exists a uniform k-star decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of μK_n where $\mu > 2, n > 2$ and $k \leq n-1$. As we showed in the comments above, no two stars have the same centre; thus $r \leq n$. The total number of edges in stars of $\mathcal{D}$ is then $rk \leq n(n-1)$. Since μK_n has $\frac{\mu n(n-1)}{2}$ edges and $\mu > 2$, this is a contradiction. $\square$

Lemma 4.2 shows that when $n > 2$ and there exists a uniform star decomposition of μK_n, we have $\mu \leq 2$. We now consider the case where $\mu = 2$.

Lemma 4.3. *For $n > 2$, there exists a unique uniform star decomposition of $2K_n$. Furthermore, this is a decomposition into k-stars where $k = n-1$.*

Proof. Suppose there exists a uniform star decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of $2K_n$. As we showed in the previous lemma, the number of edges in stars of $\mathcal{D}$ is rk where $r \leq n$ and $k \leq (n-1)$. In addition, $2K_n$ has $n(n-1)$ edges. It follows that $r = n$ and $k = n-1$, and so each vertex is the centre of a star with vertex set $V(2K_n)$. This uniquely defines the decomposition $\mathcal{D}$, and it is easy to see that $\mathcal{D}$ is uniform. $\square$

We are now left only with the case where $\mu = 1$. In this case we find that uniform star decompositions of K_n are equivalent to skew Hadamard designs of order n or $n-1$.

Lemma 4.4. *If $n \geq 3$ and there exists a uniform k-star decomposition of K_n, then $n \equiv 3 \pmod 4$, $k = \frac{n-1}{2}$, and there exists a uniform $(k+1)$-star decomposition of K_{n+1}, or $n \equiv 0 \pmod 4$, $k = \frac{n}{2}$, and there exists a uniform $(k-1)$-star decomposition of K_{n-1}.*

Proof. Suppose $n \geq 3$ and there exists a uniform k-star decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of K_n. First consider the case where $r < n$. Then there exists a vertex v which is not the centre of any star in $\mathcal{D}$. Since v is adjacent to all other vertices of K_n, it follows that every other vertex is the centre of some star in $\mathcal{D}$, and v is a leaf of every star in $\mathcal{D}$. Thus $r = (n-1)$ and $k = \frac{n}{2}$ (as K_n has $\frac{n(n-1)}{2}$ edges). For each star D_i, we define a $(k-1)$-star D'_i which has the same centre as D_i, where $V(D'_i) = V(D_i) \setminus \{v\}$. It is clear that the set $\mathcal{D}' = \{D'_1, D'_2, ..., D'_r\}$ is a $(k-1)$-star decomposition of K_{n-1} where $V(K_{n-1}) = V(K_n) \setminus \{v\}$. We will show that $\mathcal{D}'$ is uniform.

For each $a, b, c, d \in \{1, 2, ..., r\}$ such that $a < b, c < d$, there exists a permutation f of $V(K_n)$ such that $f(D_a \cup D_b) = D_c \cup D_d$ (as $\mathcal{D}$ is uniform). Furthermore, all the degree 2 vertices in $D_c \cup D_d$ are adjacent only to the centres of D_c and D_d, and $f(v)$ has degree 2 (as v has degree 2 in both $D_a \cup D_b$ and $D_c \cup D_d$). Thus the transposition which swaps v and $f(v)$ is an automorphism of $D_c \cup D_d$. Let g be the transposition $(f(v), v)$. Then gf is an isomorphism from $D_a \cup D_b$ to $D_c \cup D_d$ which fixes v. Thus gf is an isomorphism from $D'_a \cup D'_b$ to $D'_c \cup D'_d$. It follows that $\mathcal{D}'$ is a uniform $(k-1)$-star decomposition of K_{n-1}.

Now consider the case where $r = n$. If $n = 3$ then we can arbitrarily assign the centre of each star to be a distinct vertex (as the centre of a 1-star is indistinguishable from its leaf). If $n > 3$

then we have already proved that no two stars have the same centre. Then every vertex is the centre of a star, and since $|E(K_n)| = \frac{n(n-1)}{2}$, we have $k = \frac{(n-1)}{2}$. Now consider a new vertex v where $v \notin V(K_n)$. For each star D_i we define a $(k+1)$-star D_i^* which has the same centre as D_i, where $V(D_i^*) = V(D_i) \cup \{v\}$. Furthermore, we define the set $\mathcal{D}^* = \{D_1^*, D_2^*, ..., D_r^*\}$. Since every vertex of K_n is the centre of some D_i, it is clear that $\mathcal{D}^*$ is a $(k+1)$-star decomposition of K_{n+1}. In addition, for any $D_a, D_b, D_c, D_d \in \mathcal{D}$ with $a < b$ and $c < d$, there exists a permutation f of $V(K_n)$ such that $f(D_a \cup D_b) = D_c \cup D_d$. Then let $f^* : V(K_n) \cup \{v\} \to V(K_n) \cup \{v\}$ be the function where $f^*(x) = f(x)$ when $x \in V(K_n)$ and $f^*(v) = v$. Since v is adjacent to only the centres of $D_a^*, D_b^*, D_c^*, D_d^*$ in those stars, it follows that f^* is an isomorphism from $D_a^* \cup D_b^*$ to $D_c^* \cup D_d^*$. Thus $\mathcal{D}^*$ is a uniform $(k+1)$-star decomposition of K_{n+1}.

We now need only show that $r = n$ implies that $n \equiv 3 \pmod 4$. We will do this by counting the number of degree 2 vertices (or equivalently, triangles) in the union of two stars.

Suppose $r = n, n > 3$ and there exists a uniform k-star decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of K_n. Then the union of each pair of distinct stars contains the same number of degree 2 vertices. Each vertex occurs as a degree 2 vertex in the union of two stars precisely when it is a leaf of both stars. Since each vertex is the centre of a k-star, each vertex is a leaf of $(n-1)-k$ stars. Since $k = \frac{n-1}{2}$, it follows that each vertex is a leaf of $\frac{n-1}{2}$ stars and so has degree 2 in $\binom{\frac{n-1}{2}}{2} = \frac{(n-1)(n-3)}{8}$ unions of pairs of stars. Thus there are $\frac{n(n-1)(n-3)}{8}$ degree 2 vertices in pairs of stars. However, there are $\frac{n(n-1)}{2}$ pairs of stars; thus the union of each pair has $\frac{n-3}{4}$ degree 2 vertices. Since this must be an integer, it follows that $n \equiv 3 \pmod 4$. In this paragraph we assumed that $n > 3$, but it is also clear that $n \equiv 3 \pmod 4$ when $n = 3$.

Now suppose there exists a uniform k-star decomposition of K_n with $r = n - 1$. Then there exists a uniform $(k-1)$-star decomposition of K_{n-1} with r stars, and so $(n-1) \equiv 3 \pmod 4$. This proves the result. $\square$

In Lemma 4.4 we observed that if there is a uniform k-star decomposition of K_n with the number of stars equal to the number of points, then each pair of stars have $\frac{n-3}{4}$ leaves in common. Since each star has $k = \frac{n-1}{2}$ leaves and each vertex is a leaf of $\frac{n-1}{2}$ stars, it makes sense to ask the question: Considering the stars as points and letting each vertex define a block, do we obtain an $(n, \frac{n-1}{2}, \frac{n-3}{4})$-BIBD, i.e. a Hadamard design? In answering this question, we will show that uniform star decompositions of this type are equivalent to skew Hadamard designs.

Lemma 4.5. *For $n > 3$ and $n \equiv 3 \pmod 4$, there exists a uniform star decomposition of K_n if and only if there exists a skew Hadamard design of order n.*

Proof. Let $\mathcal{D} = \{D_1, D_2, ..., D_n\}$ be a uniform star decomposition of K_n where $n \equiv 3 \pmod 4$, $n > 3$, and $V(K_n) = \{v_1, v_2, ..., v_n\}$. For each vertex v_i in K_n, we define a block $B_i = \{D_a : v_i$ is a leaf of $D_a, a \in \{1, 2, ..., n\}\}$. Each vertex is a leaf of $\frac{n-1}{2}$ stars, so $|B_i| = \frac{n-1}{2}$ for each $i \in \{1, 2, ..., n\}$. Similarly, every star has $\frac{n-1}{2}$ leaves, so each star D_a occurs in $\frac{n-1}{2}$ blocks. Each pair of distinct stars D_a, D_b occurs in precisely those blocks whose vertices are leaves of both stars; there are $\frac{n-3}{4}$ vertices which are leaves of both D_a and D_b, so each pair of distinct stars occurs in $\frac{n-3}{4}$ blocks. Since the number of stars is equal to the number of blocks, this is sufficient to prove that $\mathcal{B} = \{B_1, B_2, ..., B_n\}$ is a Hadamard design with parameters $(n, \frac{n-1}{2}, \frac{n-3}{4})$.

Now let $\mathcal{B}^* = \{B_1^*, B_2^*, ..., B_n^*\}$ where each $B_i^* = B_i \cup \{D_a :$ the centre of D_a is $v_i\}$. It is clear that $\mathcal{B}^*$ is an $(n, \frac{n+1}{2}, \frac{n+1}{4})$-BIBD, as each block of B_i^* contains the $\frac{n-1}{2}$ stars in B_i and the star centred at v_i, and the stars D_a and D_b occur together in the $\frac{n-3}{4}$ blocks in $\mathcal{B}^*$ corresponding to their blocks in $\mathcal{B}$, and the block B_i^* such that v_i is the centre of one of D_a, D_b and a leaf of the other (which must be unique since the edge between the centres of D_a and D_b occurs in precisely one of D_a, D_b). Ó Catháin [50] proved that this is sufficient to show that $\mathcal{B}$ is a skew design.

Conversely, suppose there exists a skew Hadamard design $(\mathcal{D}, \mathcal{B})$ with parameters $(n, \frac{n-1}{2}, \frac{n-3}{4})$, where $\mathcal{D} = \{D_1, D_2, ..., D_n\}$, and $\mathcal{B} = \{B_1, B_2, ..., B_n\}$. Then by Ó Catháin [50], there exists a design $(\mathcal{D}, \mathcal{B}^*)$ where $\mathcal{B}^* = \{B_1^*, B_2^*, ..., B_n^*\}$ and $B_i^* = B_i \cup D_a$ where D_a is a point not in B_i, and each B_i^* has a distinct new point (i.e. $\{B_1^* \setminus B_1, B_2^* \setminus B_2, ..., B_n^* \setminus B_n\} = \{\{D_1\}, \{D_2\}, ..., \{D_n\}\}$). Note that $(\mathcal{D}, \mathcal{B}^*)$ is a symmetric $(n, \frac{n+1}{2}, \frac{n+1}{4})$-BIBD. Then let each D_a be a star with vertices corresponding to the blocks D_a occurs in, i.e. $V(D_a) = \{v_i : D_a \in B_i^*\}$, and the centre of D_a is the vertex v_i such that $D_a \in B_i^* \setminus B_i$. It is clear (since a symmetric BIBD has block size equal to its replication number) that each D_a has $\frac{n-1}{2}$ leaf vertices and so is an $(\frac{n-1}{2})$-star. In addition, every pair of stars D_a, D_b intersects in $\frac{n-3}{4}$ leaf vertices, since D_a, D_b share a leaf vertex v_j if and only if B_j contains D_a and D_b (and B_j is a block in a Hadamard design of order n). Furthermore, D_a and D_b intersect in exactly $\frac{n+1}{4}$ vertices, since $(\mathcal{D}, \mathcal{B}^*)$ is a symmetric $(n, \frac{n+1}{2}, \frac{n+1}{4})$-design. This is sufficient to show that $D_a \cup D_b$ is unique up to isomorphism and so $\mathcal{D}$ is uniform. $\square$

4.2 Uniform decompositions of complete multigraphs into paths

We now discuss uniform decompositions of the complete multigraph into paths. It is trivial to show that μK_2 has a uniform 1-path decomposition for any $\mu \geq 1$, and that μK_1 has a uniform m-path decomposition for any m. Thus we consider only cases when $n \geq 3$.

Lemma 4.6. *Let $n \geq 3$. Then there exists a uniform decomposition of μK_n into m-paths if and only if one of the following is true:*
(A) $n = 3$, $\mu \in \{1, 2\}$ and $m = \mu$
(B) $n = 4$, $\mu = 1$, and $m = \frac{n}{2}$
(C) $n = 4$ or $n = 6$, $\mu = 1$, and $m = n - 1$.

Proof. There are only three distinct 1-paths and three distinct 2-paths in K_3. In addition, any uniform path decomposition of μK_3 must contain all three 1-paths or all three 2-paths (as otherwise such a decomposition would not contain the same number of edges between each pair of points). Since the set of 1-paths forms a uniform decomposition of K_3 and the set of 2-paths forms a uniform decomposition of $2K_3$, it follows that μK_3 has a uniform m-path decomposition if and only if $\mu \in \{1, 2\}$ and $m = \mu$.

We now break the remaining problem into three distinct cases:

(i) μK_n has even degree.

(ii) μK_n has odd degree and we look for a decomposition where two paths share an endpoint.

(iii) μK_n has odd degree and we look for a decomposition where no two paths share an endpoint.

We will prove the lemma for each case in turn.

(i)
Suppose $n \geq 4, \mu K_n$ has even degree and there exists a uniform m-path decomposition $\mathcal{D} =$

$\{D_1, D_2, ..., D_r\}$ of μK_n. Let a, b be the endpoints of D_1. Since the endpoints of a path have degree 1 and all other vertices of the path have degree 2, and since μK_n has even degree, it follows that some other path (without loss of generality D_2) has endpoints a, c. Then $D_1 \cup D_2$ has exactly two vertices of odd degree, or zero vertices of odd degree if $b = c$. It follows that $D_1 \cup D_x$ has at most two vertices of odd degree for all $D_x \in \mathcal{D}$. Thus every path D_x has one endpoint in common with D_1 (and similarly with D_2, D_3, etc.)

Now suppose $b \neq c$ and some path (without loss of generality D_3) does not have the endpoint a. Then D_3 must have endpoints b, c (since D_3 shares an endpoint with each of D_1 and D_2). It follows that any other path $D_x \notin \{D_1, D_2, D_3\}$ must have exactly one endpoint in $\{a, b\}$, exactly one endpoint in $\{b, c\}$, and exactly one endpoint in $\{a, c\}$. No pair of endpoints satisfies these three conditions. Thus, $\mathcal{D} = \{D_1, D_2, D_3\}$. Since μK_n has $\mu\frac{n(n-1)}{2}$ edges and $D_1 \cup D_2 \cup D_3$ has $3m \leq 3(n-1)$ edges, it follows that $\mu\frac{n(n-1)}{2} = 3m \leq 3(n-1)$ and so $\mu\frac{n}{2} \leq 3$. Since we have assumed that $n \geq 4$, it follows that $\mu = 1, n \in \{4, 6\}$. However, K_4 and K_6 have odd degree - a contradiction.

Conversely, assume every path has the endpoint a. Then the degree of a must be the number of paths in the decomposition. The degree of a is $\mu(n-1)$, while the number of paths is $r = \frac{\mu n(n-1)}{2m}$ (that is, the number of edges in μK_n divided by the number of edges per path). Thus $\frac{n}{2m} = 1$ and so $n = 2m$. This implies that μ is even, since μK_n has degree $\mu(n-1)$ and n is even. If the paths in $\mathcal{D}$ do not all have the same two endpoints, then no two paths in $\mathcal{D}$ share both endpoints; it follows that there are vertices which are the endpoint of only one path, and so have odd degree - a contradiction. Thus every path has the endpoints a, b. However, if $m > 1$ (and correspondingly $n > 2$), there exists no m-path with the edge $\{a, b\}$ and endpoints a, b, and so $\mathcal{D}$ is not a decomposition of μK_n.

(ii)
Suppose $n \geq 4, \mu K_n$ has odd degree and there exists a uniform m-path decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of μK_n in which two paths share an endpoint. It is immediately clear, since μK_n has odd degree, that every vertex is an endpoint of an odd number of paths. Let a, b be the endpoints of D_1. Without loss of generality we can say that a, c are the endpoints of D_2. Then $D_1 \cup D_2$ has exactly two vertices of odd degree, or zero vertices of odd degree if $b = c$. It follows that every path has endpoint a, or $r = 3$ and D_3 has endpoints b, c. However in the latter case every vertex has even degree - a contradiction. Thus every path has a as an endpoint, and every other vertex is an endpoint of exactly one path (as no two paths share both endpoints). Then $r = (n-1)$, since each path has exactly one endpoint other than a. Since no path may have more than $(n-1)$ edges, and μK_n has $\mu\frac{n(n-1)}{2}$ edges, it follows that $\mu = 1, r = n-1$ and $m = \frac{n}{2}$.

The union of two paths contains precisely two odd-degree vertices. We will show that these odd-degree vertices are adjacent in some, but not all, of the unions of two paths when $n > 4$. Suppose the path D_i has the endpoint $v_i \neq a$ and the edge $\{v_j, v_i\}$. Then there is precisely one other path D_j with the endpoint v_j. Thus there is precisely one union of two paths in which D_i has an edge between the two odd-degree vertices. Since D_i is arbitrary, it follows that there are precisely $(n-1)$ pairs of paths whose unions have two adjacent odd-degree vertices. There are $\binom{(n-1)}{2}$ pairs of paths in the decomposition; thus $(n-1) = \frac{(n-1)(n-2)}{2}$ and so $n = 4$. The set $\{[0, 1, 2], [0, 2, 3], [0, 3, 1]\}$ is a uniform 2-path decomposition of K_4, thus in case (ii) there exists a uniform $\frac{n}{2}$-path decomposition of K_n if and only if $n = 4$.

(iii)
Suppose $n \geq 4, \mu K_n$ has odd degree and there exists a uniform m-path decomposition $\mathcal{D} = \{D_1, D_2, ..., D_r\}$ of μK_n in which no two paths share an endpoint. It is immediately clear that

every vertex is an endpoint of precisely one path, and so $r = \frac{n}{2}$. Since $|E(\mu K_n)| = \frac{\mu n(n-1)}{2}$ and $m \leq n-1$, it follows that $\mu = 1$ and $m = n-1$.

We will show that the existence of such a decomposition implies the existence of a uniform Hamilton decomposition of K_{n+1}. In particular, let $V(K_n) \cap \{v_{n+1}\} = \emptyset$. Then for each path D_i with endpoints d, e, let D_i^* be the cycle with $V(D_i^*) = V(D_i) \cup \{v_{n+1}\}$ and $E(D_i^*) = E(D_i) \cup \{\{d, v_{n+1}\}, \{e, v_{n+1}\}\}$. Since each vertex of K_n is an endpoint of precisely one path in $\mathcal{D}$, it follows that $\mathcal{D}^* = \{D_i^* : D_i \in \mathcal{D}\}$ is an $(m+2)$-cycle decomposition of K_{n+1} - a Hamilton decomposition. In addition, $\mathcal{D}^*$ is uniform; in any union of two distinct cycles D_i^*, D_j^*, the graph $D_i^* \cup D_j^*$ consists of $D_i \cup D_j$, the vertex v_{n+1}, and the edges from the odd degree vertices of $D_i \cup D_j$ to v_{n+1}. Since there exists no uniform Hamilton decomposition of K_{n+1} with $n > 6$ by Corollary 2.2, it follows that $n = 4$ or $n = 6$. It is trivial to construct a uniform 3-path decomposition of K_4, and the set $\{[0,2,5,4,3,1], [2,4,1,0,5,3], [4,0,3,2,1,5]\}$ is a uniform 5-path decomposition of K_6 (with vertex set $\mathbb{Z}_6$). It follows that there exists a uniform decomposition of K_n into $(n-1)$-paths where $n > 3$ and K_n has odd degree if and only if $n = 4$ or $n = 6$. $\square$

4.3 Conclusion

We have proved that there exist uniform decompositions of μK_n into m-cycles, m-stars or m-paths only under restricted conditions. These results have interesting connections with design theory, as the high level of symmetry required for uniformity is often sufficient to produce well-known designs. A natural extension of our work would be to investigate the uniform graph decompositions of $\mu K_n \setminus I$ and of complete bipartite multigraphs $\mu K_{n,m}$. It would also be worthwhile to examine uniform decompositions of $\mu K_n, \mu K_n \setminus I$ and $\mu K_{n,m}$ into Cayley graphs. Finally, in cases where there exists an H-decomposition of G but not necessarily a uniform one, the number of isomorphic unions of pairs of subgraphs may be counted - similar to Wagner's [61] approach to perfect 1-factorisations.

Bibliography

[1] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B*, **81** (2001), 77–99.

[2] B. Alspach, K. Heinrich and M. Rosenfeld, Edge partitions of the complete symmetric directed graph and related designs, *Israel J. Math.*, **40** (1981), 118–128.

[3] B. A. Anderson, Finite topologies and Hamiltonian paths. *J. Combin. Theory Ser. B* **14** (1973), 87–93.

[4] B. A. Anderson, A perfectly arranged Room square. *Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1973)* 141–150. *Utilitas Math., Winnipeg, Man.* 1974.

[5] B. A. Anderson, A class of starter-induced 1-factorizations. *Graphs and Combinatorics (Proc. Capital Conf., George Washington Univ., Washington, D.C., 1973)* 180–185. Lecture notes in Math., Vol. 406, *Springer, Berlin*, 1974.

[6] L. D. Andersen, Factorizations of Graphs, in: C. J. Colbourn, J. H. Dinitz (Eds.), The Handbook of Combinatorial Designs, second ed., CRC Press, Boca Raton, 2007, pp. 740–755.

[7] B. Auerbach, R. Laskar, On decomposition of r-partite graphs into edge-disjoint hamilton circuits, *Discrete Math.* **14** (1976), 265–268.

[8] L. Babai, W. Kantor, E. Luks, Computational complexity and the classification of finite simple groups, *Proceedings of the 24th Annual Symposium on the Foundations of Computer Science* (1983), 162–171.

[9] L. Babai, Graph isomorphism in quasipolynomial time, *Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing* (2016), 684–697.

[10] L. Babai, László Babai's Home Page, *Departments of Computer Science and Mathematics, University of Chicago* `http://people.cs.uchicago.edu/~laci/update.html` (2017).

[11] D. Bryant, P. Danziger, W. Pettersson, Bipartite 2-factorizations of complete multipartite graphs, *J. Graph Theory* **78** no. 4 (2015), 287–294.

[12] D. Bryant, D. Horsley, B. Maenhaut and B. R. Smith, Cycle decompositions of complete multigraphs, *J. Combin. Des.*, **19** (2011), 42–69.

[13] D. Bryant, D. Horsley, B. Maenhaut and B. R. Smith, Decompositions of complete multigraphs into cycles of varying lengths, *J. Combin. Theory Ser. B*, in press, available online 29th September 2017, ⟨`https://doi.org/10.1016/j.jctb.2017.09.005`⟩.

[14] D. Bryant, D. Horsley and W. Pettersson, Cycle decompositions V: Complete graphs into cycles of arbitrary lengths, *Proc. London Math. Soc.*, **108** (2014), 1153–1192.

[15] D. Bryant, B. Maenhaut, I. M. Wanless, A family of perfect 1-factorisations of complete bipartite graphs. *J. Combin. Theory* Series A, **98** no. 2 (2002), 328–342.

[16] D. Bryant, B. M. Maenhaut and I. M. Wanless, New families of atomic Latin squares and perfect one-factorisations, *J. Combin. Theory A*, **113** (2006), 608–624.

[17] M. Buratti, A. Del Fra, Cyclic Hamiltonian cycle systems of the complete graph. In honour of Zhu Lie, *Discrete Math.* **279** no. 1–3 (2004), 107–119.

[18] M. Buratti, S. Capparelli, A. Del Fra, Cyclic Hamiltonian cycle systems of the λ-fold complete and cocktail party graphs, *European Journal of Combinatorics* **31** (2010), 1484–1496.

[19] P. Cain, Decomposition of complete graphs into stars, *Bull. Austral. Math. Soc.* **10** (1974), 23–30.

[20] H. Cohn, A second course in number theory. Wiley, New York, 1962.

[21] C. J. Colbourn, Hamiltonian decompositions of complete graphs, *Ars Combin.*, **14** (1982), 261–269.

[22] C. J. Colbourn, J. H. Dinitz (Eds.) The Handbook of Combinatorial Designs, second ed., CRC Press, Boca Raton, 2007.

[23] K. Conrad, Pell's Equation, II Online lecture notes available at `http://www.math.uconn.edu/~kconrad/blurbs/ugradnumthy/pelleqn2.pdf`

[24] J. H. Dinitz and P. Dukes, On the structure of uniform one-factorizations from starters in finite fields, *Finite Fields App.*, **12** (2006), 283–300.

[25] J. H. Dinitz, D. K. Garnick, B. D. McKay, There are 526, 915, 620 nonisomorphic one-factorizations of K_{12}, *J. Combin. Des.* **2** no. 4 (1994), 273–285.

[26] J. H. Dinitz, D. R. Stinson, Some new perfect one-factorizations from starters in finite fields, *J. Graph Theory* **13** no. 4 (1989), 405–415.

[27] F. Franek and A. Rosa, Two-Factorizations of Small Complete Graphs, *J. Statist. Plann. Inference*, **86** (2000), 435–442.

[28] H-D. O. F. Gronau, M. Grüttmüller, S. Hartmann, U. Leck, V. Leck, On orthogonal double covers of graphs, *Des. Codes Cryptogr.*, **27** (2002), 49–91.

[29] S. Hartmann, U. Leck, V. Leck, More orthogonal double covers of complete graphs by Hamiltonian paths, *Discrete Math.*, **308** (2008), 2502–2508.

[30] S. Herke, B. Maenhaut, Perfect 1-factorisations of circulants with Small Degree, *Electron. J. Combin.* **20** no. 1, Paper 58 (2013).

[31] S. Herke, On the perfect 1-factorisation problem for circulant graphs of degree 4, *Australas. J. Combin.* **60** (2014), 79–108.

[32] S. Herke, B. Maenhaut, Perfect 1-factorisations of a family of Cayley graphs, *J. Combin. Des.* **23** no. 9 (2015), 369–399.

[33] E. C. Ihrig, E. S. Seah, D. R. Stinson, A perfect one-factorization of K_{50}, *J. Combin. Math. Combin. Comput.* **1** (1987), 217–219.

[34] H. Jordon, J. Morris, Cyclic Hamiltonian cycle systems of the complete graph minus a 1-factor, *Discrete Math.* **308** no. 12 (2008), 2440–2449.

[35] W. M. Kantor, 2-transitive symmetric designs, *Trans. Amer. Math. Soc.*, **146** (1969), 1–28.

[36] T. Kirkman, On a problem in Combinatorics, *Cambridge and Dublin Math. J.* **2** (1847), 191–204.

[37] M. Kobayashi, Kiyasu-Zen'iti, Perfect one-factorizations of K_{1332} and K_{6860}, *J. Combin. Theory* Series A **51** no. 2 (1989), 314–315.

[38] M. Kobayashi, On perfect one-factorization of the complete graph K_{2p}, *Graphs Combin.* **5** no. 4 (1989), 351–353.

[39] A. Kotzig, Hamilton graphs and Hamilton circuits. *Theory of Graphs and its Applications (Proc. Sympos. Smolenice 1963)*, Nakl. CSAV, Praha: 63–82, 1964.

[40] A. Kotzig, Problem 20. *Theory of Graphs and its Applications (Proc. Sympos. Smolenice 1963)*, Nakl. CSAV, Praha: 162, 1964.

[41] P. J. Laufer, On strongly Hamiltonian complete bipartite graphs, *Ars Combinatoria* **9** (1980), 43–46.

[42] V. Leck, On orthogonal double covers by Hamilton paths, *Congr. Numer.*, **135** (1998), 153–157.

[43] E. Lucas, *Récréations Mathématiques*, Vol. 2 (1892).

[44] K. Mackenzie-Fleming, An infinite family of non-embeddable quasi-residual designs with $k < v/2$, *J. Combin. Theory Ser. A* **74** no. 2 (1996), 345-350.

[45] K. Mackenzie-Fleming, Infinite families of non-embeddable quasi-residual Hadamard designs, *J. Geom.* **67** no. 1–2 (2000), 173-179.

[46] G. Mazzuoccolo, On 2-factorizations whose automorphism group acts doubly transitively on the factors, *Discrete Mathematics* **308** no. 5–6 (2008), 931–939.

[47] B. D. McKay and A. Piperno, Practical Graph Isomorphism, II, *J. Symbolic Comput.*, **60** (2014), 94–112.

[48] M. Meszka and A. Rosa, Perfect 1-factorizations of K_{16} with nontrivial automorphism group, *J. Combin. Math. Combin. Comput.*, **47** (2003), 97–111.

[49] G. Nakamura, Dudney's round table problem and the edge-coloring of the complete graph (in Japanese), *Sūgaku Seminar* **159** (1975), 24–29.

[50] P. Ó Catháin, Nesting symmetric designs, *Irish Math. Soc. Bull.* **72** (2013), 71–74.

[51] R. C. Read, R. J. Wilson, Atlas of Graphs, Oxford University Press, New York, 2005.

[52] G. Royle, personal communication (Nov 2015)

[53] M. Šajna, Cycle decompositions III: complete graphs and fixed length cycles, *J. Combin. Des.*, **10** (2002), 27–78.

[54] S. J. Schlicker, Numbers Simultaneously Polygonal and Centered Polygonal, *Math. Mag.*, **84** (2011), 339–350.

[55] E. S. Seah, D. R. Stinson, A perfect one-factorization for K_{36}, *Discrete Math.* **70** no. 2 (1988), 199–202.

[56] E. S. Seah, D. R. Stinson, A perfect one-factorization for K_{40}, Eighteenth Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, MB, 1988). *Cong. Numer.* **68** (1989), 211–213.

[57] J. E. Shockley, Introduction to Number Theory, Holt, Rinehart and Winston, Inc., New York-Toronto, Ont.-London, 1967.

[58] D. R. Stinson, Combinatorial designs : constructions and analysis, Springer, 2004.

[59] M. Tarsi, Decomposition of complete multigraphs into stars, *Discrete Math.* **26** no. 3 (1979), 273-278.

[60] R. J. Turyn, On C-matrices of arbitrary powers, *Canad. J. Math.* **23** (1971), 531–535.

[61] D. G. Wagner, On the perfect one-factorization conjecture, *Discrete Math.* **104** no. 2 (1992), 211–215.

[62] I. M. Wanless, Perfect factorisations of bipartite graphs and Latin squares without proper subrectangles, *Electron. J. Combin.* **6**, Research Paper 9 (1999).

[63] I. M. Wanless, Atomic Latin Squares based on Cyclotomic Orthomorphisms, *Electron. J. Combin.* **12**, Research Paper 22 (2005).

[64] I. M. Wanless, Perfect 1-factorisations,
`http://users.monash.edu.au/~iwanless/data/P1F/newP1F.html`.

[65] J. Williamson, Hadamard's determinant theorem and the sum of four squares, *Duke Math. J.* **11** no. 1 (1944), 65–81.

[66] A. Wolfe, A perfect one-factorization of K_{52}, *J. Combin. Des.*, **17** (2009), 190–196.