SIAM J. CONTROL OPTIM.
Vol. 56, No. 6, pp. 4203–4227
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SECOND-ORDER ANALYSIS AND NUMERICAL APPROXIMATION FOR BANG-BANG BILINEAR CONTROL PROBLEMS*

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Abstract. We consider bilinear optimal control problems whose objective functionals do not depend on the controls. Hence, bang-bang solutions will appear. We investigate sufficient second-order conditions for bang-bang controls, which guarantee local quadratic growth of the objective functional in L^1. In addition, we prove that for controls that are not bang-bang, no such growth can be expected. Finally, we study the finite-element discretization and prove error estimates of bang-bang controls in L^1-norms.

Key words. bang-bang control, bilinear controls, second-order conditions, sufficient optimality conditions, error analysis

AMS subject classifications. 49K20, 49K30, 35J61

DOI. 10.1137/17M1139953

1. Introduction.

In this article, we consider optimal control problems of the following type: Minimize the cost functional

$$j(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to the elliptic equation

$$Ly + b(y) + \chi_\omega uy = f \tag{1.2}$$

and control constraints

$$\alpha \le u \le \beta. \tag{1.3}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, L is a second-order elliptic operator, and b is a monotone nonlinearity. The presence of the nonlinear coupling $\chi_\omega uy$ motivates us to call this problem "bilinear"; sometimes the term "control affine problem" is used. Many important processes in engineering, biology, socio-economics, and ecology may be modeled by bilinear systems; see [5] and [21]. This bilinear coupling complicates the analysis considerably.

Since j does not depend explicitly on the control, a typical situation is one in which locally optimal controls $\bar{u}$ are of bang-bang type, that is, $\bar{u}(x) \in \{\alpha, \beta\}$ for a.a. $x \in \Omega$. However, no simple conditions are known that ensure this property for optimal control of elliptic equations, and singular parts may appear. In this paper,

*Received by the editors July 21, 2017; accepted for publication (in revised form) September 4, 2018; published electronically November 20, 2018.
http://www.siam.org/journals/sicon/56-6/M113995.html
Funding: The first author was partially supported by the Spanish Ministerio de Economía, Industria y Competitividad under research projects MTM2014-57531-P and MTM2017-83185-P. The second author was partially supported by DFG under grant Wa 3626/1-1.
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we will *assume* that the considered reference control $\bar{u}$ is bang-bang. If the objective functional satisfies a growth condition with respect to the L^1-norm, then the optimal control has to be bang-bang, as we show in subsection 2.1.

We are interested in sufficient second-order optimality conditions and discretization error estimates for problem (1.1)–(1.3). To this end, we develop an abstract framework in section 2. The analysis relies on a structural assumption on the behavior of the reduced gradient on the active set. This allows us to prove a second-order condition; see Theorem 2.4. The abstract results are then applied in section 3 to bilinear distributed and bilinear boundary control problems for elliptic equations.

In addition, we investigate the discretization of the original problems using finite elements. Here, we show that under the sufficient second-order condition, we obtain an error estimate of the type

$$\|\bar{u} - \bar{u}_h\|_{L^1} \leq c\,h; \tag{1.4}$$

see Theorem 4.4. This extends earlier results for linear-quadratic bang-bang problems [13, 31] and regularized nonlinear control problems [2, 7].

To motivate the abstract theory, let us consider an optimal control problem with the elliptic differential equation

$$-\Delta y + \chi_\omega y u = f$$

on Ω with $f \in L^2(\Omega)$ and supplied with homogeneous Dirichlet boundary conditions. Let us suppose that the lower control bound α is nonnegative. Let us define $\mathcal{U}_{\mathrm{ad}} := \{u \in L^\infty(\omega) : \alpha \leq u \leq \beta \text{ a.e. in } \omega\}$. Then for every feasible control $u \in \mathcal{U}_{\mathrm{ad}}$, the elliptic equation has a unique solution $y_u \in H^1(\Omega)$ by the Lax–Milgram theorem. In addition, $y_u \in L^\infty(\Omega)$ holds by regularity results for elliptic equations. Hence, we can introduce the reduced objective function $J : \mathcal{U}_{\mathrm{ad}} \to \mathbb{R}$ via

$$J(u) := j(y_u, u).$$

In order to derive error estimates of the type (1.4), growth conditions on the functional near the locally optimal control $\bar{u}$ of the type

$$J(\bar{u}) + \nu\|u - \bar{u}\|_{L^1(\Omega)} \leq J(u) \quad \forall u \in U_{ad} : \ \|u - \bar{u}\|_{L^1(\Omega)}$$

are indispensable. In subsection 2.1 we show that such a growth condition can be satisfied only if the control is bang-bang. This is due to the fact that the reduced cost functional J is weak* sequentially continuous from $\mathcal{U}_{\mathrm{ad}} \subset L^\infty(\omega)$ to $\mathbb{R}$. Note that the presence of an additional regularization term $\|u\|^2_{L^2(\omega)}$ in the objective j would yield only weak* lower semicontinuity of the reduced objective J.

In our analysis we rely on an assumption on the behavior of the adjoint state on the active set; see (2.5) and (3.12). This enables us to obtain the lower bound

$$J'(\bar{u})(u - \bar{u}) \geq \kappa\|u - \bar{u}\|^2_{L^1(\omega)} \quad \forall u \in \mathcal{U}_{\mathrm{ad}};$$

cf. Theorem 2.3. Here, it is necessary to choose ω with positive distance to the Dirichlet boundary $\partial\Omega$. We comment on this in Remark 3.4.

In order to perform a second-order analysis of the optimal control problem, we will use the following differentiability properties. The control-to-state-map $u \mapsto y_u$ and, consequently, the reduced objective J are twice continuously differentiable with respect to u in $L^\infty(\Omega)$. The second derivative of J is given by

$$J''(u)(v_1, v_2) = \int_\Omega z_{v_1}\, z_{v_2}\, \mathrm{d}x - \int_\omega \varphi_u \left(v_1\, z_{v_2} + v_2\, z_{v_1}\right) \mathrm{d}x,$$

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where z_{v_i} are the solutions of the linearized state equation

$$-\Delta z_v + \chi_\omega u\, z_v + \chi_\omega v\, y_u = 0$$

with $v = v_i$, and the adjoint state φ_u solves

$$-\Delta \varphi_u + \chi_\omega u\, \varphi_u = y_u - y_d;$$

see Theorem 3.2. Using these formulas and standard regularity results for elliptic partial differential equations (PDEs), one can check that $J''(u)$ can be extended from $L^\infty(\Omega)^2$ to $L^q(\Omega)^2$ for some $q \in (1, \infty)$ close to 1. Due to the presence of the bilinear terms $v_1\, z_{v_2}$ and $v_2\, z_{v_1}$ in J'', it is not possible to work with $q = 1$. If the control appears linearly in the PDE, then an analysis of the optimal control problem in $L^1(\omega)$ is possible. This strategy was applied in our previous paper [10]. In addition, one obtains a continuity property of J'' in weaker norms. For the precise calculations, we refer the reader to subsection 3.1, where we consider a PDE in the general setting (1.2).

These properties of the reduced objective J are our starting point for the abstract setting in section 2. Indeed, it will be sufficient that the reduced objective J satisfies these differentiability properties in order to derive second-order optimality conditions and approximation error estimates. Therefore, we are not only able to produce results for the control problem (1.1)–(1.3), but our abstract setting is also applicable to many other problems. We outline the applicability to a bilinear boundary control problem in subsection 3.2, and we expect that the abstract setting also applies to other bilinear control problems; see the examples at the end of [10, section 3].

Let us comment on the existing literature for bang-bang control problems. The present paper continues our research on bang-bang problems. It extends earlier works [8, 10], which focused on problems with the control appearing linearly, to the bilinear case. In the literature on control problems governed by ordinary differential equations (ODEs) there are many contributions dealing with second-order conditions in the bang-bang case, e.g., [15, 17, 18, 20, 22, 23, 24]. In these contributions one typically assumes that the (differentiable) switching function $\sigma : [0, T] \to \mathbb{R}$ has finitely many zeros. Our structural assumption (2.5) can be considered as an extension to the distributed parameter case.

Bilinear control problems for time-dependent equations were studied, e.g., in [4, 3]; see also the references in these papers. By means of the Goh transform, the bilinear control problem is transferred into a problem where the control appears linearly. It is an open problem whether the idea of Goh transform can be applied to control of elliptic (and thus time-independent) equations.

2. Abstract framework. Throughout this section we assume that $(X, \mathcal{B}, \eta)$ is a finite and complete measure space. We consider the abstract optimization problem

$$\text{(P)} \qquad \begin{aligned} &\text{minimize} \quad J(u) \\ &\text{subject to} \quad u \in \mathcal{U}_{\text{ad}}, \end{aligned}$$

where

$$\mathcal{U}_{\text{ad}} = \{u \in L^\infty(X) : \alpha \le u(x) \le \beta \quad \text{a.e. in } X\} \tag{2.1}$$

with $-\infty < \alpha < \beta < +\infty$, and $J : \mathcal{U}_{\text{ad}} \to \mathbb{R}$ is a given function.

In what follows, we will denote the open ball with respect to the $L^p(X)$-norm of radius $r > 0$ around $v \in L^p(X)$ by $B^p_r(v)$.

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2.1. A negative result in the non–bang-bang case. In this subsection, we prove that we cannot expect *any* growth of the objective if the optimal control is not of bang-bang type.

THEOREM 2.1. *Let us assume that the measure space $(X, \mathcal{B}, \eta)$ is additionally separable and nonatomic. Suppose that $\bar{u}$ is a local minimizer of* (P) *in the sense of $L^1(X)$, which is not bang-bang. Further, we assume that J is weak* sequentially continuous from $L^\infty(X)$ to $\mathbb{R}$. Then, there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ and for any $\varepsilon > 0$, there exists $u \in \mathcal{U}_{\mathrm{ad}}$ with*

$$\|u - \bar{u}\|_{L^1(X)} = \delta \quad \text{and} \quad J(u) \le J(\bar{u}) + \varepsilon. \tag{2.2}$$

Before proving the theorem, we give some remarks and an auxiliary lemma. First, the theorem implies that a growth of type

$$J(u) \ge J(\bar{u}) + \nu \|u - \bar{u}\|^\gamma_{L^p(X)} \quad \forall u \in \mathcal{U}_{\mathrm{ad}} \cap B^1_\delta(\bar{u})$$

for some $\nu, \delta, \gamma > 0$ and $p \in [1, \infty]$ is *impossible*. Indeed, let us argue by contradiction. Without loss of generality, we can assume that the above growth holds for some $\delta < \delta_0$. Then, according to the theorem, for every $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{U}_{\mathrm{ad}}$ such that (2.2) holds. This implies with the assumed growth condition and Hölder's inequality that

$$\begin{aligned} \delta = \|u_\varepsilon - \bar{u}\|_{L^1(X)} &\le \eta(X)^{1-\frac{1}{p}} \|u_\varepsilon - \bar{u}\|_{L^p(X)} \\ &\le \eta(X)^{1-\frac{1}{p}} \Big(\frac{J(u_\varepsilon) - J(\bar{u})}{\nu} \Big)^{1/\gamma} \le \frac{\eta(X)^{1-\frac{1}{p}}}{\nu^{1/\gamma}} \varepsilon^{1/\gamma}. \end{aligned}$$

Finally, making $\varepsilon \to 0$ we get a contradiction.

Furthermore, even a growth of type $f(\|u - \bar{u}\|_{L^p(X)})$ cannot be satisfied, as long as f is a nondecreasing function and $f(t) > 0$ for $t > 0$.

Recall that the measure space is nonatomic if, for all $A \in \mathcal{B}$ with $\eta(A) > 0$, there is $B \in \mathcal{B}$ with $B \subset A$ and $0 < \eta(B) < \eta(A)$. The measure space is called separable if there is a countable subset $\{A_n\} \subset \mathcal{B}$ such that

$$\forall A \in \mathcal{B} \quad \text{and} \quad \forall \varepsilon > 0 \ \exists A_n : \eta\big((A \setminus A_n) \cup (A_n \setminus A)\big) < \varepsilon$$

holds. It is easy to check that this is equivalent to the separability of $L^p(X)$ for all $p \in [1, \infty)$. In particular, all regular Borel measures are separable measures.

Before proving the theorem, we need to state a lemma.

LEMMA 2.2. *Let the measure space $(X, \mathcal{B}, \eta)$ be as in Theorem 2.1. Let a measurable set $B \subset X$ be given. Then, there exists a sequence $\{v_k\} \subset L^\infty(X)$ such that $v_k(x) = 0$ for a.a. $x \in X \setminus B$, $v_k(x) \in \{-1, 1\}$ for a.a. $x \in B$, and $v_k \overset{*}{\rightharpoonup} 0$ in $L^\infty(X)$.*

Proof. We define the set

$$\mathbb{F} = \{v \in L^2(B) : v(x) \in \{-1, 1\} \text{ for a.a. } x \in B\}.$$

Then, according to [25, Proposition 6.4.19], we have

$$\overline{\mathbb{F}}^w = \{v \in L^2(B) : v(x) \in [-1, 1] \text{ for a.a. } x \in B\},$$

where $\overline{\mathbb{F}}^w$ is the closure of $\mathbb{F}$ with respect to the weak topology of $L^2(B)$. The space $L^2(B)$ is reflexive and separable since $(X, \mathcal{B}, \eta)$ is assumed to be separable. Hence, the weak topology is metrizable on the bounded set $\overline{\mathbb{F}}^w$. Thus, there is a sequence $\{v_k\} \subset L^2(B)$ with $v_k \in \mathbb{F}$ and $v_k \rightharpoonup 0$ in $L^2(B)$. Since $\{v_k\}$ is bounded in $L^\infty(B)$, the density of $L^2(B)$ in $L^1(B)$ implies $v_k \overset{*}{\rightharpoonup} 0$ in $L^\infty(B)$. Finally, the result follows if v_k is extended by 0 to X. ∎

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Now we are in position to prove Theorem 2.1.

Proof of Theorem 2.1. Since $\bar{u}$ is not bang-bang, the set $B = \{x \in X : \alpha + \rho \leq \bar{u} \leq \beta - \rho\}$ has positive measure for some $\rho > 0$. We apply Lemma 2.2 and obtain a sequence $\{v_k\} \subset L^\infty(X)$ with the properties stated in Lemma 2.2. Set $\delta_0 = \rho\eta(B)$. Then, given $\delta \leq \delta_0$, we consider the controls $u_k = \bar{u} + \frac{\delta}{\eta(B)} v_k$ and obtain $u_k \in \mathcal{U}_{\text{ad}}$. Moreover, we have $\|u_k - \bar{u}\|_{L^1(X)} = \delta$ for all k. The weak* sequential continuity of J implies $J(u_k) \to J(\bar{u})$. Thus, for any $\varepsilon > 0$ there exists $k_\varepsilon \geq 1$ such that $J(u_k) - J(\bar{u}) < \varepsilon$ for all $k \geq k_\varepsilon$, which implies (2.2). ∎

2.2. Second-order analysis. In this section, we consider the second-order analysis of problem (P). To this end, let $\bar{u} \in \mathcal{U}_{\text{ad}}$ be a fixed control. We make the following assumptions on J and $\bar{u}$.

(H1) The functional J can be extended to an $L^\infty(X)$-neighborhood $\mathcal{A}$ of $\mathcal{U}_{\text{ad}}$. It is twice continuously Fréchet differentiable with respect to $L^\infty(X)$ in this neighborhood. Moreover, we assume that $\bar{u}$ satisfies the first-order condition $J'(\bar{u})(u - \bar{u}) \geq 0$ for all $u \in \mathcal{U}_{\text{ad}}$.

(H2) The second derivative $J''(\bar{u}) : L^\infty(X)^2 \to \mathbb{R}$ can be extended continuously to $L^q(X)^2$ for some $q \in [1, 3/2)$. In particular, there is a constant $C > 0$ such that

$$|J''(\bar{u})(v_1, v_2)| \leq C \, \|v_1\|_{L^q(X)} \, \|v_2\|_{L^q(X)} \tag{2.3}$$

holds for all $v_1, v_2 \in L^\infty(X)$.

(H3) For each $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$\left|[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2\right| \leq \varepsilon \, \|u - \bar{u}\|^2_{L^1(X)} \tag{2.4}$$

holds for all $u \in \mathcal{U}_{\text{ad}} \cap B^1_{\delta_\varepsilon}(\bar{u})$, $u_\theta = \bar{u} + \theta(u - \bar{u})$, and any $0 \leq \theta \leq 1$.

(H4) There exists a function $\bar{\psi} \in L^1(X)$ such that $J'(\bar{u})\, v = \int_X \bar{\psi}\, v \, \mathrm{d}\eta$ for all $v \in L^\infty(X)$.

(H5) There exists a constant $K > 0$ such that

$$\eta(\{x \in X : |\bar{\psi}(x)| \leq \varepsilon\}) \leq K \, \varepsilon \tag{2.5}$$

is satisfied for all $\varepsilon > 0$.

We will see below that (H3) is satisfied for bilinear elliptic control problems in the case of distributed controls (for dimensions $n \leq 3$) and for boundary controls (for dimension $n = 2$).

The conditions (H1), (H4), and (H5) imply that $\bar{u}$ is a bang-bang control. In fact, we have

$$\bar{\psi}(x) > 0 \quad \Rightarrow \quad \bar{u}(x) = \alpha \quad \text{and} \quad \bar{\psi}(x) < 0 \quad \Rightarrow \quad \bar{u}(x) = \beta \tag{2.6}$$

for a.a. $x \in X$. Together with (H5), this yields $\bar{u}(x) \in \{\alpha, \beta\}$ for a.a. $x \in X$.

Under the previous assumptions, we can prove some sufficient second-order optimality conditions for the local optimality of a given bang-bang control $\bar{u}$. To this end, we introduce the following cone of critical directions: For every $\tau > 0$, we define

$$C^\tau_{\bar{u}} := \left\{v \in L^2(X) : v(x) = 0 \text{ if } |\bar{\psi}(x)| > \tau \text{ and } v \text{ satisfies (2.8)}\right\} \tag{2.7}$$

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with

$$v(x)\begin{cases}\geq 0 & \text{if } \bar{u}(x)=\alpha\\ \leq 0 & \text{if } \bar{u}(x)=\beta\end{cases}\quad \text{for a.a. } x\in X. \tag{2.8}$$

Before establishing the second-order conditions, we state the following result, whose proof can be found in [10, Proposition 2.7].

THEOREM 2.3. *Let us assume that* (H1), (H4), *and* (H5) *hold; then*

$$J'(\bar{u})(u-\bar{u})\geq \kappa\|u-\bar{u}\|^2_{L^1(X)}\quad \forall u\in\mathcal{U}_{\text{ad}}, \tag{2.9}$$

where $\kappa=(4(\beta-\alpha)K)^{-1}$.

The next theorem provides a second-order condition which allows us to prove a quadratic growth of the objective J in the neighborhood of $\bar{u}$. In particular, $\bar{u}$ is a strict local solution under this assumption. Note that condition (2.10) is slightly weaker than the corresponding results [10, Theorems 2.8 and 3.3], which required $\kappa'<\kappa$ in (2.10). This improvement has been possible by some slightly more refined estimates in the proof.

THEOREM 2.4. *Suppose that the above assumptions* (H1)–(H5) *are satisfied. Let* κ *be as in Theorem* 2.3. *Further, assume that*

$$\exists\tau>0,\ \exists\kappa'<2\kappa:\quad J''(\bar{u})v^2\geq -\kappa'\|v\|^2_{L^1(X)}\quad \forall v\in C^\tau_{\bar{u}}. \tag{2.10}$$

Then, there exist $\nu>0$ *and* $\delta>0$ *such that*

$$J(\bar{u})+\nu\|u-\bar{u}\|^2_{L^1(X)}\leq J(u)\quad \forall u\in\mathcal{U}_{\text{ad}}\cap B^1_\delta(\bar{u}). \tag{2.11}$$

The following lemma will be used to prove this theorem.

LEMMA 2.5. *Suppose that the above assumptions* (H1)–(H5) *are satisfied. Let* κ *be as in Theorem* 2.3. *Further, we assume that there exist* $\tau>0$ *and* $\kappa'\geq 0$ *such that*

$$J''(\bar{u})v^2\geq -\kappa'\|v\|^2_{L^1(X)}\quad \forall v\in C^\tau_{\bar{u}}. \tag{2.12}$$

Then, for every $\gamma\in(0,3\kappa)$, *there is a* $\delta>0$ *such that*

$$J'(\bar{u})(u-\bar{u})+J''(u_\theta)(u-\bar{u})^2\geq(\kappa-\kappa'-\gamma)\|u-\bar{u}\|^2_{L^1(X)}\quad \forall u\in\mathcal{U}_{\text{ad}}\cap B^1_\delta(\bar{u}), \tag{2.13}$$

where $u_\theta=\bar{u}+\theta(u-\bar{u})$ *and* $0\leq\theta\leq 1$ *is arbitrary.*

Proof. We follow the idea of the proofs of [10, Theorems 2.8 and 3.3]. First, we note that (2.3) implies that

$$|J''(\bar{u})(v_1,v_2)|\leq C\,\|v_1\|^{1/q}_{L^1(X)}\,\|v_2\|^{1/q}_{L^1(X)}\,\|v_1\|^{(q-1)/q}_{L^\infty(X)}\,\|v_2\|^{(q-1)/q}_{L^\infty(X)} \tag{2.14}$$

holds for all $v_1,v_2\in L^\infty(X)$. Now, let $u\in\mathcal{U}_{\text{ad}}$ with $\|u-\bar{u}\|_{L^1(X)}\leq\delta$ be given, where $\delta>0$ will be specified later. We define

$$u_1(x):=\begin{cases}\bar{u}(x) & \text{if } x\in X_\tau,\\ u(x) & \text{otherwise}\end{cases}\quad\text{and}\quad u_2(x):=\begin{cases}u(x)-\bar{u}(x) & \text{if } x\in X_\tau,\\ 0 & \text{otherwise,}\end{cases}$$

where $X_\tau=\{x\in X:|\bar{\psi}(x)|>\tau\}$. Then we have that $u=u_1+u_2$, $(u_1-\bar{u})\in C^\tau_{\bar{u}}$, and $|u_1-\bar{u}|\leq|u-\bar{u}|$ a.e. in X. Let $\gamma\in(0,3\kappa)$ be given. Now, we can use (2.12), (2.14),

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and Young's inequality (with exponents $s = 2\,q$ and $s' = 2q/(2q-2)$) to obtain for generic positive constants C

$$\begin{aligned} J''(\bar u)(u-\bar u)^2 &= J''(\bar u)(u_1-\bar u)^2 + 2\,J''(\bar u)(u_1-\bar u, u_2) + J''(\bar u)u_2^2 \\ &\geq -\kappa'\,\|u_1-\bar u\|^2_{L^1(X)} - C\,\|u_1-\bar u\|^{1/q}_{L^1(X)}\,\|u_2\|^{1/q}_{L^1(X)} - C\,\|u_2\|^{2/q}_{L^1(X)} \\ &\geq -\Big(\kappa' + \frac{\gamma}{3}\Big)\,\|u_1-\bar u\|^2_{L^1(X)} - C\,\|u_2\|^{2/(2q-1)}_{L^1(X)} - C\,\|u_2\|^{2/q}_{L^1(X)}. \end{aligned}$$

Owing to the construction of u_1 and u_2, we have for δ small enough

$$J''(\bar u)(u-\bar u)^2 \geq -\Big(\kappa' + \frac{\gamma}{3}\Big)\|u-\bar u\|^2_{L^1(X)} - C\|u-\bar u\|^{\bar q}_{L^1(X_\tau)} \tag{2.15}$$

with $\bar q = \min(2/(2q-1), 2/q) = 2/(2q-1) > 1$ since $1 \leq q < 3/2$. Next, we use Theorem 2.3 and (2.6) to infer

$$\begin{aligned} J'(\bar u)(u-\bar u) &= \Big(1 - \frac{\gamma}{3\,\kappa}\Big)\,J'(\bar u)(u-\bar u) + \frac{\gamma}{3\,\kappa}\,J'(\bar u)(u-\bar u) \\ &\geq \Big(\kappa - \frac{\gamma}{3}\Big)\,\|u-\bar u\|^2_{L^1(X)} + \frac{\gamma}{3\,\kappa}\int_{X_\tau} |\bar\psi|\,|u-\bar u|\;\mathrm{d}\eta \\ &\geq \Big(\kappa - \frac{\gamma}{3}\Big)\,\|u-\bar u\|^2_{L^1(X)} + \frac{\gamma\,\tau}{3\,\kappa}\|u-\bar u\|_{L^1(X_\tau)}. \end{aligned} \tag{2.16}$$

Furthermore, assumption (H3) implies

$$\big|[J''(u_\theta) - J''(\bar u)]\,(u-\bar u)^2\big| \leq \frac{\gamma}{3}\,\|u-\bar u\|^2_{L^1(X)} \tag{2.17}$$

if δ is chosen small enough. Now, by adding the inequalities (2.15), (2.16), and (2.17), we have

$$\begin{aligned} J'(\bar u)(u-\bar u) + J''(u_\theta)(u-\bar u)^2 \geq{}& (\kappa - \kappa' - \gamma)\,\|u-\bar u\|^2_{L^1(X)} \\ &+ \frac{\gamma\,\tau}{3\,\kappa}\,\|u-\bar u\|_{L^1(X_\tau)} - C\|u-\bar u\|^{\bar q}_{L^1(X_\tau)}. \end{aligned}$$

Note that the sum of the terms on the second line is nonnegative if δ is small enough since $\bar q > 1$. ∎

Now we are in position to prove Theorem 2.4.

Proof of Theorem 2.4. Let $\tau > 0$ and $\kappa' < 2\,\kappa$ be given such that (2.10) is satisfied. Without loss of generality, we assume that $\kappa' \geq 0$. We choose $\gamma \in (0, 2\,\kappa - \kappa')$. We apply Lemma 2.5 and get $\delta > 0$ such that (2.13) holds. Now, we choose an arbitrary $u \in \mathcal{U}_{\mathrm{ad}} \cap B^1_\delta(\bar u)$. Using a Taylor expansion, we get

$$J(u) - J(\bar u) = J'(\bar u)(u-\bar u) + \frac{1}{2}J''(u_\theta)(u-\bar u)^2$$

for $u_\theta = \bar u + \theta(u-\bar u)$ and $0 \leq \theta \leq 1$. Now, we apply (2.9) from Theorem 2.3 and (2.13) to conclude that

$$\begin{aligned} J(u) - J(\bar u) &= \frac{1}{2}J'(\bar u)(u-\bar u) + \frac{1}{2}J'(\bar u)(u-\bar u) + \frac{1}{2}J''(u_\theta)(u-\bar u)^2 \\ &\geq \frac{\kappa}{2}\,\|u-\bar u\|^2_{L^1(X)} + \frac{1}{2}(\kappa - \kappa' - \gamma)\,\|u-\bar u\|^2_{L^1(X)} \\ &\geq \frac{1}{2}(2\,\kappa - \kappa' - \gamma)\,\|u-\bar u\|^2_{L^1(X)}. \end{aligned}$$

Since $\nu := (2\,\kappa - \kappa' - \gamma)/2 > 0$, the assertion follows. ∎

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2.3. Approximation results. The rest of this section is dedicated to the numerical approximation of the optimization problem (P). To this end, we make the following assumptions. First, we fix an approximation of the underlying set X.

(D1) There is a sequence of measurable subsets $X_h \subset X$ such that $\eta(X \setminus X_h) \to 0$ as $h \to 0$.

We define the following two notions of convergence associated with the approximation X_h of X. For a sequence $u_h \in L^1(X_h)$ and $u \in L^1(X)$, we say that $u_h \to u$ in $L^1(X)$ if and only if $\|u_h - u\|_{L^1(X_h)} \to 0$ as $h \to 0$. Similarly, for a sequence $u_h \in L^\infty(X_h)$ and $u \in L^\infty(X)$, we say that $u_h \overset{*}{\rightharpoonup} u$ in $L^\infty(X)$ if and only if $\int_{X_h} v\, u_h \,\mathrm{d}\eta \to \int_X v\, u \,\mathrm{d}\eta$ as $h \to 0$ for all $v \in L^1(X)$. Due to $\eta(X_h \setminus X) \to 0$, both notions of convergence are equivalent to $(u_h + f\,\chi_{X\setminus X_h}) \to u$ in $L^1(X)$ and $(u_h + f\,\chi_{X\setminus X_h}) \overset{*}{\rightharpoonup} u$ in $L^\infty(X)$, respectively, where $f \in L^\infty(X)$ is an arbitrary but fixed extension of u_h.

Next, we state assumptions to define the approximation of our problem (P).

(D2) The sets $\mathcal{U}_{\mathrm{ad},h} \subset L^\infty(X_h)$ are closed, convex, and contained in the set $\{u_h \in L^\infty(X_h) : \alpha \le u_h \le \beta \text{ a.e. in } X_h\}$. Moreover, for every $u \in \mathcal{U}_{\mathrm{ad}}$ there exists a sequence $u_h \in \mathcal{U}_{\mathrm{ad},h}$ such that $u_h \to u$ in $L^1(X)$ as $h \to 0$.

(D3) $\{J_h\}_h$ is a sequence of functions $J_h : \mathcal{U}_{\mathrm{ad},h} \longrightarrow \mathbb{R}$ that are weakly lower semicontinuous with respect to the $L^2(X)$ topology.

(D4) The following properties hold for sequences $u_h \in \mathcal{U}_{\mathrm{ad},h}$ and $u \in \mathcal{U}_{\mathrm{ad}}$:

$$\text{If } u_h \overset{*}{\rightharpoonup} u \ \text{ in } \ L^\infty(X), \ \text{ then } \ J(u) \le \liminf_{h\to 0} J_h(u_h); \tag{2.18}$$

$$\text{if } u_h \to u \ \text{ in } \ L^1(X), \ \text{ then } \ J(u) = \lim_{h\to 0} J_h(u_h). \tag{2.19}$$

(D5) The functions J_h have C^1 extensions $J_h : \mathcal{A}_h \longrightarrow \mathbb{R}$, where $\mathcal{A}_h \subset L^\infty(X_h)$ is a neighborhood of $\mathcal{U}_{\mathrm{ad},h}$. Moreover, for all $u_h \in \mathcal{U}_{\mathrm{ad},h}$ and for all $u \in \mathcal{U}_{\mathrm{ad}}$, $J_h'(u_h)$ and $J'(u)$ are linear and continuous forms on $L^1(X_h)$ and $L^1(X)$, respectively. Hence, there exist elements $\psi_h \in L^\infty(X_h)$, $\psi \in L^\infty(X)$ such that the following identifications hold: $J_h'(u_h) = \psi_h$ and $J'(u) = \psi$.

Now, we define the approximating problems

$$(\mathrm{P}_h) \qquad \begin{aligned} &\text{minimize} \ \ J_h(u_h) \\ &\text{subject to} \ \ u_h \in \mathcal{U}_{\mathrm{ad},h}. \end{aligned}$$

First, we state a lemma which provides a partial converse to (D2).

LEMMA 2.6. *Let us assume that* (D1) *and* (D2) *hold. Let $u_h \subset \mathcal{U}_{\mathrm{ad},h}$ be a sequence with $u_h \overset{*}{\rightharpoonup} u$ in $L^\infty(X)$ for some $u \in L^\infty(X)$. Then, $u \in \mathcal{U}_{\mathrm{ad}}$ holds. If, additionally, $\|u_h - \bar{u}\|_{L^1(X_h)} \le \delta$ for some $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$, for some $\delta > 0$, and for all $h > 0$, then we get $\|u - \bar{u}\|_{L^1(X)} \le \delta$.*

Proof. We argue by contradiction. Assume that $u \le \beta$ is not satisfied a.e. on X. Then, there is a measurable set $B \subset X$ with $\eta(B) > 0$ and $\varepsilon > 0$ such that $u \ge \beta + \varepsilon$ a.e. in B. If h is small enough, we have $\eta(X \setminus X_h) < \eta(B)/2$, and hence $\eta(B \cap X_h) > \eta(B)/2$. Together with $u_h \le \beta$, this implies

$$\int_{X_h} \chi_B\,(u_h - u)\,\mathrm{d}\eta = \int_{B\cap X_h} (u_h - u)\,\mathrm{d}\eta \le \int_{B\cap X_h} [\beta - (\beta+\varepsilon)]\,\mathrm{d}\eta \le -\frac{1}{2}\,\eta(B)\,\varepsilon,$$

which contradicts $u_h \overset{*}{\rightharpoonup} u$ in $L^\infty(X)$. Similar arguments can be used if $u \ge \alpha$ is violated.

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It remains to check the second assertion. By extending u_h with $\bar{u}$ on $X \setminus X_h$, we get $u_h \overset{*}{\rightharpoonup} u$ in $L^\infty(X)$, in particular, $u_h \rightharpoonup u$ in $L^1(X)$. Now, the assertion follows from the weak lower semicontinuity of the norm of $L^1(X)$. ∎

The following theorem proves that (P_h) realizes a convergent approximation of (P).

THEOREM 2.7. *Let us assume that* (D1)–(D4) *hold. Then for every h, the problem* (P_h) *has at least a global solution $\bar{u}_h$. Furthermore, if $\{\bar{u}_h\}_h$ is a sequence of global solutions of* (P_h)*, and $\bar{u}_h \overset{*}{\rightharpoonup} \tilde{u}$ in $L^\infty(X)$, then $\tilde{u}$ is a global solution of* (P)*. Conversely, if $\bar{u}$ is a bang-bang strict local minimum of* (P) *in the $L^1(X)$ sense, then there exists a sequence $\{\bar{u}_h\}_h$ of local minimizers of problems* (P_h) *in the sense of $L^1(X_h)$ such that $\bar{u}_h \to \bar{u}$ in $L^1(X)$.*

Proof. The existence of a global solution $\bar{u}_h$ of (P_h) follows from the boundedness, convexity, and closedness of $\mathcal{U}_{\mathrm{ad},h}$ and from the weak lower semicontinuity of J_h; see assumptions (D2) and (D3). Now, consider a subsequence, denoted in the same way, such that $\bar{u}_h \overset{*}{\rightharpoonup} \tilde{u}$ in $L^\infty(X)$. Since $\bar{u}_h \in \mathcal{U}_{\mathrm{ad},h}$ for every h, the inclusion $\tilde{u} \in \mathcal{U}_{\mathrm{ad}}$ holds by Lemma 2.6. Furthermore, given an element $u \in \mathcal{U}_{\mathrm{ad}}$, according to assumption (D2) we can take a sequence $\{u_h\}_h$ with $u_h \in \mathcal{U}_{\mathrm{ad},h}$ such that $u_h \to u$ in $L^1(X)$. Then, using (D4) and the global optimality of every $\bar{u}_h$, we infer

$$J(\tilde{u}) \le \liminf_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(\bar{u}_h) \le \limsup_{h\to 0} J_h(u_h) = J(u).$$

Hence, $\tilde{u}$ is a solution of (P).

Conversely, we assume that $\bar{u}$ is a bang-bang strict local minimum of (P). Then, there exists $\delta > 0$ such that

$$J(\bar{u}) < J(u) \quad \forall u \in \mathcal{U}_{\mathrm{ad}} \cap B^1_\delta(\bar{u}) \ \text{ with } \ \bar{u} \ne u.$$

Then, we consider the problems

$$(\mathrm{P}_{\delta,h}) \qquad \begin{array}{l} \text{minimize} \ \ J_h(u_h) \\ \text{subject to} \ \ u_h \in \mathcal{U}_{\mathrm{ad},h} \text{ and } \|u_h - \bar{u}\|_{L^1(X_h)} \le \delta. \end{array}$$

From (D2) we deduce the existence of a sequence $\{u_h\}_h$ with $u_h \in \mathcal{U}_{\mathrm{ad},h}$ such that $u_h \to \bar{u}$ strongly in $L^1(X)$. Hence, for every h small enough we have that $u_h \in \mathcal{U}_{\mathrm{ad},h} \cap B^1_\delta(\bar{u})$. Therefore, the feasible set of $(\mathrm{P}_{\delta,h})$ is not empty for every h small enough, and arguing as before we have that $(\mathrm{P}_{\delta,h})$ has a solution $\bar{u}_h$ for every h small enough. Moreover, the sequence $\{\bar{u}_h\}$ is bounded in $L^\infty(X)$. Thus, there exists a weak* converging subsequence. Additionally, for any subsequence converging to $\tilde{u}$ in $L^\infty(X)$ weak*, we get that $\tilde{u} \in \mathcal{U}_{\mathrm{ad}} \cap B^1_\delta(\bar{u})$ by Lemma 2.6, and as above, $J(\tilde{u}) \le J(\bar{u})$. The strict local optimality of $\bar{u}$ in $\mathcal{U}_{\mathrm{ad}} \cap B^1_\delta(\bar{u})$ implies that $\tilde{u} = \bar{u}$. Moreover, we conclude that the whole sequence $\{\bar{u}_h\}_h$ converges to $\bar{u}$ in $L^\infty(X)$ weak*. In addition, by using the bang-bang property of $\bar{u}$, we get

$$\|\bar{u}_h - \bar{u}\|_{L^1(X_h)} = \int_{\{x\in X_h : \bar{u}(x)=\alpha\}} (\bar{u}_h - \bar{u})\ \mathrm{d}\eta + \int_{\{x\in X_h : \bar{u}(x)=\beta\}} (\bar{u} - \bar{u}_h)\ \mathrm{d}\eta \to 0 \text{ as } h \to 0.$$

From here we get that $\|\bar{u}_h - \bar{u}\|_{L^1(X_h)} < \delta$ for all h small enough. Hence, $\bar{u}_h$ is a local minimum of (P_h) for every small h. ∎

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We finish this section by proving an estimate of $\bar{u}_h - \bar{u}$ in terms of the order of the approximations of $\bar{u}$ by elements of $\mathcal{U}_{\mathrm{ad},h}$ and J' by J_h'. To perform this estimate, it will be beneficial to use a specific extension $\hat{u}_h$ of a discrete control $u_h \in \mathcal{U}_{\mathrm{ad},h}$ to X. In fact, we set $\hat{u}_h(x) = u_h(x)$ for $x \in X_h$ and $\hat{u}_h(x) = \bar{u}(x)$ for $x \in X \setminus X_h$. Then, for every $u_h \in \mathcal{U}_{\mathrm{ad},h}$, this extension $\hat{u}_h$ belongs to $\mathcal{U}_{\mathrm{ad}}$, hence $\hat{u}_h \in \mathcal{A}$ as well. Moreover, this specific extension of the elements u_h is quite convenient for the derivation of the error estimate. We will also see in section 4 below, that this will not impede the applicability of our abstract framework to derive discretization error estimates for optimal control problems.

THEOREM 2.8. *Let us assume that* (H1)–(H5) *and* (D1)–(D5) *hold. Additionally, we suppose that $\bar{u}$ satisfies the second-order condition* (2.10) *with $\kappa' \in (0,\kappa)$. Let $\{\bar{u}_h\}_h$ be a sequence of local solutions of problems* (P_h) *converging to $\bar{u}$ in $L^1(X)$. Then, for $\gamma = (\kappa - \kappa')/2$ we obtain that the estimate*

$$\begin{aligned}\|\bar{u}_h - \bar{u}\|^2_{L^1(X_h)} \le{}& \frac{\gamma+1}{\gamma^2}\|J_h'(\bar{u}_h) - J'(\hat{\bar{u}}_h)\|^2_{L^\infty(X_h)} \\ &+ \frac{1}{\gamma}\inf_{u_h\in\mathcal{U}_{\mathrm{ad},h}}\Big(\|u_h - \bar{u}\|^2_{L^1(X_h)} + 2J'(\hat{\bar{u}}_h)(\hat{u}_h - \bar{u})\Big)\end{aligned} \tag{2.20}$$

holds for all h small enough, where $\hat{\bar{u}}_h$ and $\hat{u}_h$ denote the extensions of $\bar{u}_h$ and u_h by $\bar{u}$ to X, respectively.

Proof. Let $u_h \in \mathcal{U}_{\mathrm{ad},h}$, and denote by $\hat{u}_h$ its extension to X by $\bar{u}$. Since $\bar{u}_h$ is a local minimum of (P_h), $J_h'(\bar{u}_h)(u_h - \bar{u}_h) \ge 0$. Due to (D5) this inequality can be written in the form

$$J'(\hat{\bar{u}}_h)(\hat{\bar{u}}_h - \bar{u}) \le [J_h'(\bar{u}_h) - J'(\hat{\bar{u}}_h)](\chi_{X_h}(\hat{u}_h - \hat{\bar{u}}_h)) + J'(\hat{\bar{u}}_h)(\hat{u}_h - \bar{u}). \tag{2.21}$$

Note that our choice of extension is crucial for the above rearrangement. Next, we rewrite the left-hand side, and by the mean value theorem and by denoting $u_\theta = \bar{u} + \theta_h(\hat{\bar{u}}_h - \bar{u})$ with $0 \le \theta_h \le 1$, we infer

$$\begin{aligned}J'(\hat{\bar{u}}_h)(\hat{\bar{u}}_h - \bar{u}) &= J'(\bar{u})(\hat{\bar{u}}_h - \bar{u}) + [J'(\hat{\bar{u}}_h) - J'(\bar{u})](\hat{\bar{u}}_h - \bar{u}) \\ &= J'(\bar{u})(\hat{\bar{u}}_h - \bar{u}) + J''(u_\theta)(\hat{\bar{u}}_h - \bar{u})^2.\end{aligned}$$

Taking $\gamma = (\kappa - \kappa')/2$ in Lemma 2.5, we get for h small enough

$$\gamma\|\bar{u}_h - \bar{u}\|^2_{L^1(X_h)} = \gamma\|\hat{\bar{u}}_h - \bar{u}\|^2_{L^1(X)} \le J'(\hat{\bar{u}}_h)(\hat{\bar{u}}_h - \bar{u}).$$

This estimate is now used in (2.21). After applying Young's inequality we obtain

$$\begin{aligned}\gamma\|\bar{u}_h - \bar{u}\|^2_{L^1(X_h)} &\le \|J_h'(\bar{u}_h) - J'(\hat{\bar{u}}_h)\|_{L^\infty(X_h)}\|u_h - \bar{u}_h\|_{L^1(X_h)} + J'(\hat{\bar{u}}_h)(\hat{u}_h - \bar{u}) \\ &\le \|J_h'(\bar{u}_h) - J'(\hat{\bar{u}}_h)\|_{L^\infty(X_h)}\big(\|u_h - \bar{u}\|_{L^1(X_h)} + \|\bar{u} - \bar{u}_h\|_{L^1(X_h)}\big) \\ &\quad + J'(\hat{\bar{u}}_h)(\hat{u}_h - \bar{u}) \\ &\le \left(\frac{1}{2} + \frac{1}{2\gamma}\right)\|J_h'(\bar{u}_h) - J'(\hat{\bar{u}}_h)\|^2_{L^\infty(X_h)} + \frac{1}{2}\|u_h - \bar{u}\|^2_{L^1(X_h)} + \frac{\gamma}{2}\|\bar{u}_h - \bar{u}\|^2_{L^1(X_h)} \\ &\quad + J'(\hat{\bar{u}}_h)(\hat{u}_h - \bar{u}).\end{aligned}$$

From this inequality we deduce

$$\begin{aligned}\|\bar{u}_h - \bar{u}\|^2_{L^1(X_h)} \le{}& \frac{\gamma+1}{\gamma^2}\|J_h'(\bar{u}_h) - J'(\hat{\bar{u}}_h)\|^2_{L^\infty(X_h)} \\ &+ \frac{1}{\gamma}\|u_h - \bar{u}\|^2_{L^1(X_h)} + \frac{2}{\gamma}J'(\hat{\bar{u}}_h)(\hat{u}_h - \bar{u}).\end{aligned}$$

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Since u_h is an arbitrary element of $\mathcal{U}_{\mathrm{ad},h}$, this inequality implies (2.20). ∎

In section 4 we will provide precise estimates for the right-hand side of (2.20) for some distributed optimal control problems, including bilinear controls.

3. Second-order analysis for bilinear control problems. In this section, we apply the second-order analysis results proved in the abstract framework in section 2 to the study of some optimal control problems. The first part of this section will be devoted to the analysis of a bilinear distributed control problem associated with a semilinear elliptic equation. In the second part, we will consider a bilinear Neumann control problem.

In what follows, Ω denotes a bounded open subset of $\mathbb{R}^n$, $1 \le n \le 3$, with a Lipschitz boundary Γ. In Ω we consider the elliptic partial differential operator

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}[a_{ij}\partial_{x_i} y] + a_0 y, \tag{3.1}$$

where $a_{ij}, a_0 \in L^\infty(\Omega)$ and $a_0 \ge 0$ in Ω. Associated with this operator, the usual bilinear form $a : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ is defined as

$$a(y,z) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i} y(x)\partial_{x_j} z(x) + a_0(x)y(x)z(x) \right) \mathrm{d}x. \tag{3.2}$$

Let Γ_D be a closed subset of Γ, possibly empty, and set $\Gamma_N = \Gamma \setminus \Gamma_D$. We define the space

$$V = \{y \in H^1(\Omega) : y = 0 \text{ on } \Gamma_D\},$$

equipped with the usual norm of $H^1(\Omega)$ and the operator $L : V \longrightarrow V^*$ via

$$\langle Ly, z\rangle = a(y,z) \quad \forall y, z \in V,$$

and we assume its coercivity.

(A1) We assume the existence of $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \ge \alpha\,|\xi|^2 \quad \forall \xi \in \mathbb{R}^n$$

holds for a.a. $x \in \Omega$. Further, we assume $\int_\Omega a_0 \,\mathrm{d}x \ne 0$ in the case when the surface measure of Γ_D is zero.

Note that (A1) implies

$$\exists \Lambda > 0 \text{ such that } \Lambda \|y\|_V^2 \le a(y,y) \quad \forall y \in V. \tag{3.3}$$

Moreover, we consider a Carathéodory function $b : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ of class C^2 with respect to the second variable, such that the following assumptions are satisfied.

(A2) We assume that $b(\cdot, 0) = 0$,

$$\frac{\partial b}{\partial y}(x,y) \ge 0 \quad \text{for a.a. } x \in \Omega \text{ and } \forall y \in \mathbb{R},$$

and that for all $M > 0$ there exists a constant $C_{b,M} > 0$ such that the boundedness estimate

$$\left|\frac{\partial b}{\partial y}(x,y)\right| + \left|\frac{\partial^2 b}{\partial y^2}(x,y)\right| \le C_{b,M} \text{ for a.e. } x \in \Omega \text{ and } \forall\ |y| \le M,$$

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and that for all $\varepsilon > 0$ and $M > 0$ there exists $\rho_{\varepsilon,M} > 0$ such that for a.e. $x \in \Omega$,

$$\left|\frac{\partial^2 b}{\partial y^2}(x,y_2) - \frac{\partial^2 b}{\partial y^2}(x,y_1)\right| < \varepsilon \text{ and } \forall\ |y_1|,|y_2| \le M \text{ with } |y_2 - y_1| < \rho_{\varepsilon,M}$$

are satisfied. In what follows we use the notation

$$b' = \frac{\partial b}{\partial y} \quad \text{and} \quad b'' = \frac{\partial^2 b}{\partial y^2}.$$

3.1. A bilinear distributed control problem. In this section, we consider the state equation

$$Ly + b(\cdot, y) + \chi_\omega u y = f \quad \text{in } V^*, \tag{3.4}$$

where ω is an open subset of Ω, and u and f satisfy the following assumptions.

(A3) We fix $\bar{p} > n$, and $\bar{p}' = \bar{p}/(\bar{p}-1)$ is its conjugate. We assume that $f \in W^{1,\bar{p}'}(\Omega)^*$.

(A4) We assume that $u \in \mathcal{A}$, where the open set $\mathcal{A} \subset L^\infty(\omega)$ is given by

$$\mathcal{A} = \left\{ v \in L^\infty(\omega) : \exists \varepsilon_v > 0 \text{ such that } v(x) > -\frac{\Lambda}{2} + \varepsilon_v \ \text{ for a.a. } x \in \omega \right\},$$

where Λ was introduced in (A1).

Assumption (A3) is used to obtain the boundedness of the states; see Theorem 3.1 below. The set $\mathcal{A}$ from assumption (A4) is an open neighborhood of $\mathcal{U}_{\rm ad}$ in $L^\infty(\Omega)$. This is required to state the differentiability properties of the control-to-state map and of the reduced objective.

In the next theorem, we analyze the equation (3.4).

THEOREM 3.1. *The following statements hold.*

(1) *For any $u \in \mathcal{A}$ there exists a unique solution $y_u \in Y := V \cap L^\infty(\Omega)$ of the state equation* (3.4). *Moreover, there exists a constant C such that*

$$\|y_u\|_Y = \|y_u\|_{L^\infty(\Omega)} + \|y_u\|_V \le C \quad \forall u \in \mathcal{A}. \tag{3.5}$$

(2) *The control-to-state mapping $G : \mathcal{A} \longrightarrow Y$ defined by $G(u) = y_u$ is of class C^2. Here, $\mathcal{A}$ is equipped with the $L^\infty(\Omega)$-norm. Moreover, for $v \in L^\infty(\Omega)$, $z_v = G'(u)\,v$ is the unique solution of*

$$L\,z_v + b'(\cdot, y_u)\,z_v + \chi_\omega u\,z_v + y_u\,\chi_\omega v = 0, \tag{3.6}$$

and given $v_1, v_2 \in L^2(\Omega)$, $w_{v_1,v_2} = G''(u)(v_1,v_2)$ is the unique solution of

$$\begin{aligned} L\,w_{v_1,v_2} + b'(\cdot, y_u)\,w_{v_1,v_2} + \chi_\omega u\,w_{v_1,v_2} \\ + b''(\cdot, y_u)\,z_{v_1}\,z_{v_2} + \chi_\omega v_1\,z_{v_2} + \chi_\omega v_2\,z_{v_1} = 0, \end{aligned} \tag{3.7}$$

where $z_{v_i} = G'(u)\,v_i$, $i = 1, 2$.

Proof. For the proof of existence and uniqueness of a solution of (3.4) in Y, first we observe that the linear operator $L + \chi_\omega u$ is coercive in V for all $u \in \mathcal{A}$ due to the fact that $u \ge -\frac{\Lambda}{2}$ and to assumption (A1). Then, the arguments are standard;

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see, for instance, [30, section 4.1]. We recall that the boundedness of y needed in this proof is a consequence of Stampacchia's result [29, Theorem 4.2]. To prove the differentiability of the mapping G, we use the implicit function theorem as follows. We define

$$Y_{\bar{p}} = \{y \in Y : Ly \in W^{1,\bar{p}'}(\Omega)^*\},$$

which is a Banach space when it is endowed with the graph norm. Now, we consider the mapping $\mathcal{L} : Y_{\bar{p}} \times \mathcal{A} \longrightarrow W^{1,\bar{p}'}(\Omega)^*$ given by

$$\mathcal{L}(y,u) = Ly + b(\cdot, y) + \chi_\omega uy - f.$$

From assumption (A2) we get that $\mathcal{L}$ is of class C^2 and that

$$\frac{\partial \mathcal{L}}{\partial y}(y_u, u)z = Lz + b'(\cdot, y_u)z + \chi_\omega uz$$

defines an isomorphism between $Y_{\bar{p}}$ and $W^{1,\bar{p}'}(\Omega)^*$ for all $u \in \mathcal{A}$. Indeed, it is obvious that $\frac{\partial \mathcal{L}}{\partial y}(y_u, u) : Y_{\bar{p}} \longrightarrow W^{1,\bar{p}'}(\Omega)^*$ is a continuous linear mapping. The bijectivity is a consequence of the Lax–Milgram theorem and, once again, [29, Theorem 4.2]. Hence, a straightforward application of the implicit function theorem implies that G is of class C^2 and that (3.6) and (3.7) hold. ∎

Associated with the state equation (3.4) is the bilinear distributed control problem

$$\text{(BDP)} \qquad \begin{aligned} &\text{minimize} \quad J(u) = \frac{1}{2}\|y_u - y_d\|^2_{L^2(\Omega)} \\ &\text{subject to} \quad u \in \mathcal{U}_{\text{ad}}, \end{aligned}$$

where

$$\mathcal{U}_{\text{ad}} = \{u \in L^\infty(\omega) : \alpha \le u(x) \le \beta \text{ for a.a. } x \in \omega\}$$

with $0 \le \alpha < \beta < \infty$. For y_d we assume the following.

(A5) $y_d \in L^2(\Omega)$ holds.

This problem is included in the abstract framework considered in section 2 by taking $X = \omega$ and η equal to the Lebesgue measure.

The next theorem is an immediate consequence of Theorem 3.1 and the chain rule.

THEOREM 3.2. *The reduced objective $J : \mathcal{A} \to \mathbb{R}$ is twice Fréchet differentiable, and the first and second derivatives are given by*

$$J'(u)\,v = \int_\Omega (y_u - y_d)\, z_v \,\mathrm{d}x = -\int_\omega \varphi_u\, y_u\, v \,\mathrm{d}x, \tag{3.8}$$

$$J''(u)(v_1, v_2) = \int_\Omega \left[z_{v_1}\, z_{v_2} + (y_u - y_d)\, w_{v_1,v_2}\right] \mathrm{d}x \tag{3.9}$$

$$= \int_\Omega \left[(1 - \varphi_u\, b''(\cdot, y_u))\, z_{v_1}\, z_{v_2}\right] \mathrm{d}x - \int_\omega \varphi_u \left(v_1\, z_{v_2} + v_2\, z_{v_1}\right) \mathrm{d}x, \tag{3.10}$$

where $\varphi_u \in Y$ is the unique solution of

$$L^*\,\varphi_u + b'(\cdot, y_u)\,\varphi_u + \chi_\omega u\,\varphi_u = y_u - y_d \quad \text{in } V^*, \tag{3.11}$$

and $y_u, z_{v_1}, z_{v_2}, w_{v_1,v_2}$ are defined as in Theorem 3.1.

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Using Theorems 3.1 and 3.2 we infer the next result by standard arguments.

THEOREM 3.3. (BDP) *has at least one global solution. Moreover, any local solution $\bar{u}$ in the sense of $L^p(\omega)$, for some $p \in [1,\infty]$, satisfies*

$$\int_\omega \bar{\varphi}\bar{y}(u-\bar{u})\,\mathrm{d}x \le 0 \quad \forall u \in \mathcal{U}_{\mathrm{ad}}, \tag{3.12}$$

where $\bar{y}$ and $\bar{\varphi}$ are the state and adjoint state, respectively, corresponding to $\bar{u}$.

In the rest of this subsection, $\bar{u}$ will denote a fixed element of $\mathcal{U}_{\mathrm{ad}}$ satisfying (3.12). We are going to apply the results obtained in the abstract framework in section 2. To this end, we observe that (H1) obviously holds with $X = \omega$, and (H4) is fulfilled with $\bar{\psi} = -(\bar{\varphi}\bar{y})|_\omega$. Assumption (H5) is formulated in our setting as follows: There exists a constant K such that

$$|\{x \in \omega : |\bar{\varphi}(x)\bar{y}(x)| \le \varepsilon\}| \le K\varepsilon \quad \forall \varepsilon > 0, \tag{3.13}$$

where $|\cdot|$ denotes the Lebesgue measure in ω. Then, (2.9) holds.

Remark 3.4. We remark that (3.13) does not hold in a neighborhood of the boundary Γ_D due to the fact that $\bar{y} = \bar{\varphi} = 0$ on Γ_D. Hence, the assumption (3.13) implies that the distance between ω and Γ_D must be strictly positive. However, in the pure Neumann case, we can take $\omega = \Omega$.

For the second-order analysis, we introduce the cone $C_{\bar{u}}^\tau$ as in (2.7). The rest of this section is devoted to proving that the quadratic growth condition (2.11) holds under the second-order condition (2.10). To do this, we apply Theorem 2.4. Therefore, we need only verify that assumptions (H2) and (H3) hold. The following lemma will be used for this verification.

LEMMA 3.5. *Given $c \in L^\infty(\Omega)$ with $c \ge 0$, we consider the equation*

$$Ly + cy = f \quad \text{in } V^*. \tag{3.14}$$

Then, the following statements hold:

$$\|y\|_{L^6(\Omega)} \le C_L \|f\|_{L^{6/5}(\Omega)} \quad \forall f \in L^{6/5}(\Omega), \tag{3.15}$$

$$\forall p > \frac{d}{2},\ p \ge 1,\ \exists C_p > 0 : \|y\|_{L^\infty(\Omega)} \le C_p \|f\|_{L^p(\Omega)} \qquad \forall f \in L^p(\Omega), \tag{3.16}$$

$$\forall p \in [1,3)\ \exists C_p > 0 :\ \|y\|_{L^p(\Omega)} \le C_p \|f\|_{L^1(\Omega)} \qquad \forall f \in V^* \cap L^1(\Omega), \tag{3.17}$$

where $y \in V$ denotes the unique solution of (3.14).

Proof. Inequality (3.15) is an immediate consequence of the continuous embeddings $V \subset L^6(\Omega)$ and $L^{6/5}(\Omega) \subset V^*$ for $n \le 3$. Inequality (3.16) is proved in [29, Theorem 4.2]. We argue by transposition to prove (3.17). For an arbitrary $g \in L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, we denote by $z \in V$ the solution of the adjoint equation

$$L^* z + c\, z = g \ \text{ in } V^*.$$

Since $p' > \frac{3}{2}$, we can apply again (3.16) to the adjoint equation and obtain

$$\|z\|_{L^\infty(\Omega)} \le C_{p'}\, \|g\|_{L^{p'}(\Omega)}.$$

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Now, we have

$$\begin{aligned}\int_\Omega y\, g\ \mathrm{d}x &= \langle y, L^*\, z + c\, z\rangle_{V,V^*}\\ &= \langle z, L\, y + c\, y\rangle_{V,V^*} = \int_\Omega z\, f\ \mathrm{d}x\\ &\le \|z\|_{L^\infty(\Omega)}\, \|f\|_{L^1(\Omega)} \le C_{p'}\, \|g\|_{L^{p'}(\Omega)}\, \|f\|_{L^1(\Omega)}.\end{aligned}$$

This implies $\|y\|_{L^p(\Omega)} \le C_{p'}\, \|f\|_{L^1(\Omega)}$. ∎

Of course, better estimates can be obtained in the previous lemma for dimensions $n < 3$, but we do not need them here.

Remark 3.6. Let us show that the solution z_v of (3.6) satisfies the estimates (3.15)–(3.17) for $f = -\chi_\omega v y_u$. It is enough to take $c(x) = b'(x, y_u(x)) + \chi_\omega(x)u(x)$. Note that $c(x) \ge 0$ due to $\bar{u} \ge \alpha \ge 0$. Moreover, using (3.5), we get that $\{y_u\}_{u\in\mathcal{U}_{\mathrm{ad}}}$ is uniformly bounded in $L^\infty(\Omega)$. Hence, the mentioned estimates for z_v can be written in terms of the norm of v in ω.

Additionally, if $u_1, u_2 \in \mathcal{U}_{\mathrm{ad}}$, then the estimates (3.15)–(3.17) are valid for $e = y_{u_2} - y_{u_1}$ in terms of $u_2 - u_1$. Indeed, subtracting the equations for y_{u_2} and y_{u_1}, and using the mean value theorem, we get that

$$Le + b'(\cdot, y_\theta)e + \chi_\omega u_1 e = \chi_\omega(u_1 - u_2)y_{u_2} \quad \text{in } V^*,$$

where $y_\theta = y_{u_1} + \theta(y_{u_2} - y_{u_1})$ for some measurable function $0 \le \theta(x) \le 1$. Now, we apply Lemma 3.5 with $c(x) = b'(x, y_\theta(x)) + \chi_\omega(x)u_1(x)$ and $f = \chi_\omega(u_1 - u_2)y_{u_2}$, and we observe that y_{u_2} is bounded in $L^\infty(\Omega)$.

The same comments apply to the difference of the adjoint states $\phi = \varphi_{u_2} - \varphi_{u_1}$. Indeed, ϕ satisfies the equation

$$\begin{aligned}L^*\phi + b'(\cdot, y_{u_1})\phi + \chi_\omega u_1\phi = {}& [b'(\cdot, y_{u_1}) - b'(\cdot, y_{u_2})]\varphi_{u_2}\\ &+ \chi_\omega(u_1 - u_2)\varphi_{u_2} + (y_{u_2} - y_{u_1}) \quad \text{in } V^*.\end{aligned}$$

Besides the fact that $\varphi_{u_2} \in L^\infty(\Omega)$, we have with assumption (A2) that

$$\|b'(\cdot, y_{u_2}) - b'(\cdot, y_{u_1})\|_{L^r(\Omega)} \le C\|y_{u_2} - y_{u_1}\|_{L^r(\Omega)} \quad \forall r \ge 1.$$

Then, we apply the convenient inequality of Lemma 3.5 to estimate $\|y_{u_2} - y_{u_1}\|_{L^r(\Omega)}$ in terms of $\|u_2 - u_1\|_{L^p(\omega)}$.

Verification of (H2). We prove that (H2) holds with $q = \frac{6}{5}$. Since $\bar{\varphi}$ and $b(\cdot, \bar{y})$ are bounded functions, according to the expression for J'' in (3.10) we need only the estimates

$$\begin{aligned}\int_\Omega |z_{v_1} z_{v_2}|\ \mathrm{d}x &\le \|z_{v_1}\|_{L^2(\Omega)}\|z_{v_2}\|_{L^2(\Omega)}\\ &\overset{(3.17)}{\le} C\|v_1\|_{L^1(\omega)}\|v_2\|_{L^1(\omega)} \le C|\omega|^{1/3}\|v_1\|_{L^{6/5}(\omega)}\|v_2\|_{L^{6/5}(\omega)}\end{aligned}$$

and

$$\int_\omega |v_1 z_{v_2}|\ \mathrm{d}x \le \|v_1\|_{L^{6/5}(\omega)}\|z_{v_2}\|_{L^6(\Omega)} \overset{(3.15)}{\le} C\|v_1\|_{L^{6/5}(\omega)}\|v_2\|_{L^{6/5}(\omega)}.$$

Hence, (H2) holds with $q = 6/5$.

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Verification of (H3). Let us fix $\varepsilon > 0$. For some δ that we will specify later, we take $u \in \mathcal{U}_{\rm ad} \cap B^1_\delta(\bar u)$, and set $u_\theta = \bar u + \theta(u - \bar u)$ for some $\theta \in [0,1]$. Let us denote $v = u - \bar u$, $y_\theta = G(u_\theta)$, $z_\theta = G'(u_\theta)v$, and φ_θ as the adjoint state corresponding to u_θ. Analogously, we denote by $(\bar y, \bar z, \bar\varphi)$ the associated functions to $\bar u$. With this notation, from (3.10) we obtain

$$\begin{aligned}
[J''(u_\theta) - J''(\bar u)]v^2 &= \int_\Omega \left[(1-\varphi_\theta b''(\cdot,y_\theta))z_\theta^2 - (1-\bar\varphi b''(\cdot,\bar y))\bar z^2\right] \,\mathrm{d}x \\
&\quad - 2\int_\omega (\varphi_\theta v z_\theta - \bar\varphi v \bar z)\,\mathrm{d}x \\
&= \int_\Omega [(1-\bar\varphi b''(\cdot,\bar y))](z_\theta^2 - \bar z^2)\,\mathrm{d}x + \int_\Omega (\bar\varphi - \varphi_\theta) b''(\cdot,y_\theta) z_\theta^2\,\mathrm{d}x \\
&\quad + \int_\Omega \bar\varphi[b''(\cdot,\bar y) - b''(\cdot,y_\theta)]z_\theta^2\,\mathrm{d}x - 2\int_\omega (\varphi_\theta - \bar\varphi) v z_\theta\,\mathrm{d}x - 2\int_\omega \bar\varphi v(z_\theta - \bar z)\,\mathrm{d}x.
\end{aligned}$$

We have to estimate these five integrals, which we denote by I_1–I_5. From our assumption (A2) and (3.5) we deduce that y_θ, $\bar y$, $b''(\cdot,y_\theta)$, and $b''(\cdot,\bar y)$ are bounded by a constant independent of $\theta \in [0,1]$ and $u \in \mathcal{U}_{\rm ad}$. Moreover, from [29, Theorem 4.2] or, alternatively, (3.16) and (A5), we infer the uniform boundedness of the adjoint states φ_θ and $\bar\varphi$.

As further preparation, we provide an estimate for the difference $e = z_\theta - \bar z$. By taking the difference of the equations (3.6) corresponding to z_θ and $\bar z$, we find that e solves the equation

$$Le + b'(\cdot,\bar y)e + \chi_\omega \bar u e = (b'(\cdot,\bar y) - b'(\cdot,y_\theta))z_\theta + \chi_\omega(\bar u - u_\theta)z_\theta + \chi_\omega(\bar y - y_\theta)v.$$

Owing to Lemma 3.5, we can estimate $\|e\|_{L^6(\Omega)}$ by the $L^{6/5}(\Omega)$-norm of the right-hand side. Together with Hölder's inequality, this gives the estimate

$$\begin{aligned}
\|z_\theta - \bar z\|_{L^6(\Omega)} &\le C_L \|b'(\cdot,\bar y) - b'(\cdot,y_\theta)\|_{L^{12/5}(\Omega)} \|z_\theta\|_{L^{12/5}(\Omega)} \\
&\quad + C_L\|\bar u - u_\theta\|_{L^{3/2}(\omega)}\|z_\theta\|_{L^6(\Omega)} + C_L\|\bar y - y_\theta\|_{L^6(\Omega)}\|v\|_{L^{3/2}(\omega)}.
\end{aligned}$$

Now, we can use (A2) and Remark 3.6, and we arrive at

$$\begin{aligned}
\|z_\theta - \bar z\|_{L^6(\Omega)} &\le C\|\bar u - u_\theta\|_{L^1(\omega)}\|v\|_{L^1(\omega)} + C\|\bar u - u_\theta\|_{L^{3/2}(\omega)}\|v\|_{L^{6/5}(\omega)} \\
&\quad + C\|\bar u - u_\theta\|_{L^{6/5}(\omega)}\|v\|_{L^{3/2}(\omega)}.
\end{aligned}$$

Using $u_\theta - \bar u = \theta v$, and taking into account that $\|u_\theta - \bar u\|_{L^1(\omega)} \le \|v\|_{L^1(\omega)} \le \delta$ and that

$$\|v\|_{L^q(\omega)} \le \|v\|^{1/q}_{L^1(\omega)}\|v\|^{1-1/q}_{L^\infty(\omega)} \le C\|v\|^{1/q}_{L^1(\omega)}, \tag{3.18}$$

for $\delta \le 1$ the above estimate becomes

$$\|z_\theta - \bar z\|_{L^6(\Omega)} \le C\|v\|^{3/2}_{L^1(\omega)}. \tag{3.19}$$

Now, we are in position to estimate the above integrals. For the first integral, we have

$$\begin{aligned}
|I_1| &= \left|\int_\Omega [(1-\bar\varphi b''(\cdot,\bar y))](z_\theta^2 - \bar z^2)\,\mathrm{d}x\right| \\
&\le \|1-\bar\varphi b''(\cdot,\bar y)\|_{L^\infty(\Omega)}\|z_\theta + \bar z\|_{L^2(\Omega)}\|z_\theta - \bar z\|_{L^2(\Omega)} \le C\,\|v\|_{L^1(\omega)}\|v\|^{3/2}_{L^1(\omega)},
\end{aligned}$$

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where we used Remark 3.6 and (3.19). Next,

$$|I_2| = \Big|\int_\Omega (\bar\varphi - \varphi_\theta) b''(\cdot, y_\theta) z_\theta^2 \,\mathrm{d}x\Big| \le C\,\|\bar\varphi - \varphi_\theta\|_{L^6(\Omega)} \|b''(\cdot, y_\theta)\|_{L^\infty(\Omega)} \|z_\theta\|^2_{L^6(\Omega)}$$
$$\le C\,\|v\|^3_{L^{6/5}(\omega)} \le C\,\|v\|^{5/2}_{L^1(\omega)},$$

where again Remark 3.6 and (3.18) have been utilized. For the next integral, we remark that $\|b''(\cdot,\bar y) - b''(\cdot, y_\theta)\|_{L^\infty(\Omega)}$ can be estimated by any small positive number if $\|\bar y - y_\theta\|_{L^\infty(\Omega)}$ is small enough; cf. (A2). For this, it is sufficient that δ is small enough since $u_\theta \in B^1_\delta(\bar u) \cap \mathcal{U}_{\mathrm{ad}}$; see again Remark 3.6. This, along with (3.17), leads to the estimate

$$|I_3| = \Big|\int_\Omega \bar\varphi [b''(\cdot,\bar y) - b''(\cdot, y_\theta)] z_\theta^2 \,\mathrm{d}x\Big| \le \|\bar\varphi\|_{L^\infty(\Omega)} \|b''(\cdot,\bar y) - b''(\cdot, y_\theta)\|_{L^\infty(\Omega)} \|z_\theta\|^2_{L^2(\Omega)}$$
$$\le \frac{\varepsilon}{5}\,\|v\|^2_{L^1(\omega)}.$$

Finally, by using similar arguments we obtain the estimates

$$|I_4| = \Big|\int_\omega (\varphi_\theta - \bar\varphi) v z_\theta \,\mathrm{d}x\Big| \le \|\varphi_\theta - \bar\varphi\|_{L^6(\Omega)} \|v\|_{L^{3/2}(\omega)} \|z_\theta\|_{L^6(\Omega)}$$
$$\le C\|v\|_{L^{6/5}(\omega)} \|v\|_{L^{3/2}(\omega)} \|v\|_{L^{6/5}(\omega)} \le C\|v\|^{7/3}_{L^1(\omega)}$$

and

$$|I_5| = \Big|\int_\omega \bar\varphi v (z_\theta - \bar z) \,\mathrm{d}x\Big| \le \|\bar\varphi\|_{L^\infty(\Omega)} \|v\|_{L^{6/5}(\omega)} \|z_\theta - \bar z\|_{L^6(\Omega)}$$
$$\le C\|v\|_{L^{6/5}(\omega)} \|v\|^{3/2}_{L^1(\omega)} \le C\|v\|^{7/3}_{L^1(\omega)},$$

where we used additionally (3.19). Putting these inequalities together, we obtain the desired estimate

$$\big|[J''(u_\theta) - J''(\bar u)]v^2\big| \le |I_1| + |I_2| + |I_3| + |I_4| + |I_5| \le \varepsilon\,\|u - \bar u\|^2_{L^1(\omega)}$$

if $\delta > 0$ is chosen small enough. Hence, we verified (H3) in our current setting.

Application of Theorem 2.4. We have verified that the assumptions (H1)–(H4) are satisfied in the setting of the bilinear distributed control problem (BDP). Thus, we can apply Theorem 2.4, and we obtain the following sufficient second-order condition.

THEOREM 3.7. *Let us assume that* (A1)–(A5) *are satisfied. Moreover, we suppose that there is a constant $K > 0$ such that* (3.13) *holds and that there exist $\tau > 0$ and $\kappa' < 2\,\kappa$ such that*

$$J''(\bar u)v^2 \ge -\kappa' \|v\|^2_{L^1(\omega)} \quad \forall v \in C^\tau_{\bar u}, \tag{3.20}$$

where $\kappa = (4(\beta - \alpha)K)^{-1}$. Then, there exist $\nu > 0$ and $\delta > 0$ such that

$$J(\bar u) + \nu\|u - \bar u\|^2_{L^1(\omega)} \le J(u) \quad \forall u \in \mathcal{U}_{\mathrm{ad}} \cap B^1_\delta(\bar u).$$

3.2. A bilinear boundary control problem. In this subsection we assume that $n = 2$. We outline the main steps necessary to transfer the analysis of subsection 3.1 to a bilinear boundary control problem. We follow the notation introduced in

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the beginning of section 3 and assume that (A1)–(A3) hold. Further, we take $\omega = \Gamma_N$ equipped with the surface measure. We define the operator $S_\omega : L^2(\omega) \longrightarrow V^*$ by

$$\langle S_\omega(g), z\rangle = \int_\omega g(x)z(x)\,\mathrm{d}x \quad \forall z \in V,$$

where we are denoting the trace of z on ω by z as well. It is well known that there exists a constant C_ω depending on Ω such that

$$\|z\|_{L^2(\omega)} \le C_\omega \|z\|_V \quad \forall z \in V. \tag{3.21}$$

Now, we consider the state equation

$$Ly + b(\cdot, y) + S_\omega(uy) = f \quad \text{in } V^*, \tag{3.22}$$

with $u \in \mathcal{A}$. Here, $\mathcal{A}$ is defined as

$$\mathcal{A} = \left\{ v \in L^\infty(\omega) : \exists \varepsilon_v > 0 \text{ such that } v(x) > -\frac{\Lambda}{2C_\omega^2} + \varepsilon_v \ \text{ for a.a. } x \in \omega \right\},$$

where Λ is as introduced in (A1). From the assumptions (A1) and (A3) along with (3.21) we get

$$\langle Ly, y\rangle + \langle S_\omega(uy), y\rangle \ge \Lambda \|y\|_V^2 - \frac{\Lambda}{2C_\omega^2}\|y\|_{L^2(\omega)}^2 \ge \frac{\Lambda}{2}\|y\|_V^2 \quad \forall y \in V.$$

Then, Theorem 3.1 holds with the obvious modifications. In particular, the equations (3.6) and (3.7) are modified as follows:

$$Lz_v + b'(\cdot, y_u)z_v + S_\omega(uz_v) + S_\omega(vy_u) = 0 \tag{3.23}$$

and

$$\begin{aligned} L\, w_{v_1,v_2} + b'(\cdot, y_u)\, w_{v_1,v_2} + S_\omega(u\, w_{v_1,v_2}) \\ + b''(\cdot, y_u)\, z_{v_1}\, z_{v_2} + S_\omega(v_1\, z_{v_2}) + S_\omega(v_2\, z_{v_1}) = 0. \end{aligned} \tag{3.24}$$

Associated with the state equation (3.4) is the bilinear boundary control problem

$$\begin{aligned} &\text{minimize} \quad J(u) = \frac{1}{2}\|y_u - y_d\|_{L^2(\Omega)}^2 \\ &\text{subject to} \quad u \in \mathcal{U}_{\mathrm{ad}}, \end{aligned} \tag{BBP}$$

where

$$\mathcal{U}_{\mathrm{ad}} = \{u \in L^\infty(\omega) : \alpha \le u(x) \le \beta \text{ for a.a. } x \in \omega\}$$

with $0 \le \alpha < \beta < \infty$. We suppose that y_d satisfies the assumption (A5). Then, Theorem 3.2 holds, and we need only change the adjoint state equation (3.11) by

$$L^*\,\varphi_u + b'(\cdot, y_u)\,\varphi_u + S_\omega(u\,\varphi_u) = y_u - y_d \quad \text{in } V^*. \tag{3.25}$$

We also have that Theorem 3.3 holds. To get the sufficient second-order conditions, we assume that (3.13) is fulfilled. Then, to check that Theorems 2.3 and 2.4 hold we need to check that assumptions (H1)–(H5) are satisfied. As in subsection 3.1, it is enough to verify (H2) and (H3). To this end, we will use the following lemma.

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LEMMA 3.8. *Let $c \in L^\infty(\Omega)$ be nonnegative, and let $u \in \mathcal{A}$. For $(f,g) \in L^2(\Omega) \times L^2(\omega)$ let $y \in V$ be the solution of the equation*

$$Ly + cy + S_\omega(uy) = f + S_\omega(g) \quad \text{in } V^*. \tag{3.26}$$

Then, for every $p \in [1,\infty)$ and $q > 1$ there exist constants C_p and M_q independent of (f,g), c, and u such that

$$\|y\|_{L^p(\Omega)} \le C_p\big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\omega)}\big), \tag{3.27}$$

$$\|y\|_{L^\infty(\Omega)} \le M_q\big(\|f\|_{L^q(\Omega)} + \|g\|_{L^q(\omega)}\big). \tag{3.28}$$

Proof. Since $L^1(\Omega)$ and $L^1(\omega)$ are subspaces of the space of real and regular Borel measures in Ω and ω, respectively, we can apply the well-known results for measures to deduce that the solution y of (3.26) satisfies

$$\|y\|_{W^{1,s}(\Omega)} \le C_s\big(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\omega)}\big)$$

for every $s \in [1, \frac{n}{n-1})$ and some constant C_s independent of (f,g), c, and u; see, for instance, [1, 6] or [19].

Since we have assumed $n = 2$, for every $p \in [1,\infty)$ there exists $s < \frac{n}{n-1}$ such that $W^{1,s}(\Omega) \subset L^p(\Omega)$, and hence (3.27) follows from the above estimate. The estimate (3.28) is proved in [1, Theorem 2]. ∎

Hence, though simpler estimates can be used, the estimates used in subsection 3.1 are valid to verify (H2) and (H3). As a consequence, we obtain a second-order sufficient condition analogously to Theorem 3.7 in the distributed case.

Finally, we mention that the same technique cannot be used to address the case $n > 2$. The verification of (H2) and (H3) for bilinear boundary control problems in more than two spatial dimensions remains an open problem.

4. Numerical approximation of distributed control problems. In this section, we consider the boundary value problem

$$\begin{cases} Ay + b(\cdot,y) + \chi_\omega uy = f & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where A is given by (3.1) with coefficients $a_{ij} \in C^{0,1}(\bar\Omega)$ satisfying the ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \Lambda|\xi|^2 \quad \forall x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^n.$$

We also assume that $a_0 \in L^\infty(\Omega)$, $a_0 \ge 0$, b satisfies the assumption (A2), and $f \in L^{\bar p}(\Omega)$ with $\bar p > n$. We follow the notation introduced in section 3. Hence, by Theorem 3.1 we know that (4.1) has a unique solution $y_u \in Y = H_0^1(\Omega) \cap L^\infty(\Omega)$ for all $u \in \mathcal{A}$.

We also introduce the adjoint state equation associated to the control u:

$$\begin{cases} A^*\varphi + b'(\cdot,y_u)\varphi + \chi_\omega u\varphi = y_u - y_d & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{4.2}$$

Now, we consider the control problem (BDP) associated to (4.1). Here we suppose that $y_d \in L^{\bar p}(\Omega) \cap L^2(\Omega)$. We also assume that $\bar\omega \subset \Omega$. If this condition does not hold,

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then the assumption (3.13) can be fulfilled only in some extreme cases. This is due to the fact that $\bar{y}$ and $\bar{\varphi}$ vanish on Γ, and hence the $\{x \in \Omega : |\bar{y}(x)\bar{\varphi}(x)| \le \varepsilon\}$ contains a strip along the boundary with a measure of order $\sqrt{\varepsilon}$. The situation is different for Neumann boundary problems.

Since assumptions (A1)–(A5) are satisfied, Theorems 3.2 and 3.3 are valid for the control problem (BDP) associated to the state equation (4.1). In what follows, $\bar{u}$ will denote a local solution of (BDP) satisfying the regularity condition (3.13). Therefore, Theorem 3.7 holds as well.

The goal of this section is to prove error estimates for the numerical approximation of (BDP) based on a finite-element discretization. To this end, we assume that Ω is convex and Γ is of class $C^{1,1}$. Therefore, we have additional regularity for the states y_u and adjoint states φ_u for every $u \in \mathcal{A}$, namely $y_u, \varphi_u \in W^{2,\bar{p}}(\Omega) \cap W_0^{1,\bar{p}}(\Omega)$; see [16, Chapter 2]. Since $\bar{p} > n$, we have that $W^{2,\bar{p}}(\Omega) \subset C^1(\bar{\Omega})$. If $n = 2$, this regularity holds for a convex and polygonal domain Ω assuming that the coefficients a_{ij} are of class C^1 in $\bar{\Omega}$. In dimension $n = 3$, the regularity result is valid for rectangular parallelepipeds under the same C^1 regularity of the coefficients; see [16, Chapter 4] and [12, Corollary 3.14].

Let $\{\mathcal{T}_h\}_{h>0}$ be a quasi-uniform family of triangulations of $\bar{\Omega}$; see [11]. We set $\overline{\Omega}_h = \cup_{T\in\mathcal{T}_h} T$, with Ω_h and Γ_h its interior and boundary, respectively. We assume that the vertices of $\mathcal{T}_h$ placed on the boundary Γ_h are also points of Γ, and there exists a constant $C_\Gamma > 0$ such that $\operatorname{dist}(x, \Gamma) \le C_\Gamma h^2$ for every $x \in \Gamma_h$. This always holds if Γ is a C^2 boundary and $n = 2$. From this assumption we know [27, section 5.2] that

$$|\Omega \setminus \Omega_h| \le C_\Omega h^2, \tag{4.3}$$

where $|\cdot|$ denotes the Lebesgue measure. Let us denote by $\mathcal{T}_{\omega,h}$ the family of all elements $T \in \mathcal{T}_h$ such that $T \subset \bar{\omega}$. We set $\bar{\omega}_h = \bigcup_{T\in\mathcal{T}_{\omega,h}} T$, and ω_h is its interior. We also assume that $|\omega \setminus \omega_h| \le C_\omega h^{p_\omega}$ with $p_\omega > n/2$.

We define the following spaces associated with this triangulation:

$$\begin{aligned} \mathcal{U}_h &= \{u_h \in L^\infty(\omega_h) : u_{h|_T} \in \mathcal{P}_0(T) \ \ \forall T \in \mathcal{T}_{\omega,h}\}, \\ Y_h &= \{y_h \in C(\bar{\Omega}) : y_{h|_T} \in \mathcal{P}_1(T) \ \ \forall T \in \mathcal{T}_h \text{ and } y_h = 0 \text{ in } \bar{\Omega} \setminus \Omega_h\}, \end{aligned}$$

where $\mathcal{P}_k(T)$ denotes the polynomial of degree k in T with $k = 0, 1$. Now, for every $u \in \mathcal{A}$ we consider the discrete system of nonlinear equations

$$\begin{aligned} &\text{find } y_h \in Y_h \text{ such that} \quad \forall z_h \in Y_h, \\ &a(y_h, z_h) + \int_\Omega [b(\cdot, y_h) + \chi_{\omega_h} u y_h] z_h \,\mathrm{d}x = \int_\Omega f z_h \,\mathrm{d}x, \end{aligned} \tag{4.4}$$

where the bilinear form a is as defined in (3.2). Using our assumptions on b and the ellipticity of the operator $y \to Ay + \chi_\omega uy$, we see that the existence and uniqueness of a solution of (4.4) follows by standard arguments. This solution will be denoted by $y_h(u)$. We also consider the discrete adjoint state equation

$$\begin{aligned} &\text{find } \varphi_h \in Y_h \text{ such that} \quad \forall z_h \in Y_h, \\ &a(z_h, \varphi_h) + \int_\Omega [b'(\cdot, y_h(u)) + \chi_{\omega_h} u] \varphi_h z_h \,\mathrm{d}x = \int_\Omega (y_h(u) - y_d) z_h \,\mathrm{d}x. \end{aligned} \tag{4.5}$$

The solution of this adjoint equation is denoted by $\varphi_h(u)$.

The following approximation results are needed for the numerical analysis of the discrete control problem.

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LEMMA 4.1. *Let $u \in \mathcal{A}$ fulfill $\|u\|_{L^\infty(\omega)} \le M$, and let y, y_h, φ, and φ_h be the solutions of* (4.1), (4.4), (4.2), *and* (4.5), *respectively. Then, for some constant C depending on M we have*

$$\|y - y_h\|_{L^\infty(\Omega)} + \|\varphi - \varphi_h\|_{L^\infty(\Omega)} \le Ch. \tag{4.6}$$

Proof. Let us denote $u_h = \chi|_{\omega_h} u$ and by y_{u_h} its continuous associated state. From Lemma 3.5 and Remark 3.6, and using the classical L^∞-estimates for finite-element approximations, see [2, 9, 26, 28], we get

$$\begin{aligned}\|y - y_h\|_{L^\infty(\Omega)} &\le \|y - y_{u_h}\|_{L^\infty(\Omega)} + \|y_{u_h} - y_h\|_{L^\infty(\Omega)}\\ &\le C_1(\|u - u_h\|_{L^{p_\omega}(\omega)} + h^{2-n/\bar{p}}|\log h|) \le Ch,\end{aligned}$$

where we have used that $|\omega \setminus \omega_h| \le C_\omega h^{p_\omega}$. From this estimate we deduce the corresponding estimate for $\varphi - \varphi_h$ by using similar arguments. ∎

Finally, we define the discrete control problem

$$\text{(BDP}_h\text{)} \qquad \begin{aligned}&\text{minimize} \quad J_h(u_h) = \frac{1}{2}\|y_h(u_h) - y_d\|^2_{L^2(\Omega_h)} + \frac{\alpha_h}{2}\|u_h\|^2_{L^2(\omega_h)}\\ &\text{subject to} \quad u_h \in \mathcal{U}_{\mathrm{ad},h},\end{aligned}$$

where

$$\mathcal{U}_{\mathrm{ad},h} = \{u_h \in \mathcal{U}_h : \alpha \le u_h(x) \le \beta \text{ for a.a. } x \in \omega_h\}.$$

Moreover, we included a Tikhonov parameter $\alpha_h \ge 0$ and require $\alpha_h \to 0$ as $h \to 0$. This regularization term is beneficial for the numerical solution of (BDP$_h$), and we will prove that the choice $\alpha_h = c\,h$ yields the same order of convergence as $\alpha_h = 0$; see (4.9) below.

Let us check that these approximations of (BDP) fit into the framework described in subsection 2.3. To this end, we have to check the assumptions (D1)–(D5). Taking $X = \omega$, $X_h = \omega_h$, and $\eta =$ Lebesgue measure in ω, (D1) follows from our assumption $|\omega \setminus \omega_h| \to 0$ as $h \to 0$.

Assumption (D2) is immediate. Indeed, it is enough to observe that given $u \in \mathcal{U}_{\mathrm{ad}}$ we can take u_h as the projection of u on $\mathcal{U}_h$,

$$\Pi_h u = u_h = \sum_{T \in \mathcal{T}_{\omega,h}} u_T \chi_T \quad \text{with} \quad u_T = \frac{1}{T}\int_T u \,\mathrm{d}x, \tag{4.7}$$

where χ_T denotes the characteristic function of T. It is well known that $u_h \to u$ strongly in $L^p(\omega)$ under the assumption $u \in L^p(\omega)$; see [14].

Now, (D3) is obvious. (D4) is a straightforward consequence of the following lemma.

LEMMA 4.2. *If $u_h \rightharpoonup u$ weakly in $L^1(\omega)$ with $u_h \in \mathcal{A} \cap \mathcal{U}_h$ and $u \in \mathcal{A}$, and there exists a constant $M > 0$ such that $\|u_h\|_{L^\infty(\omega_h)} \le M$ for all $h > 0$, then $y_h(u_h) \to y_u$ and $\varphi_h(u_h) \to \varphi_u$ in $L^\infty(\Omega)$ as $h \to 0$ strongly, and $J(u) = \lim_{h\to 0} J_h(u_h)$.*

Proof. Let us extend every u_h to ω by setting $u_h(x) = 0$ for all $x \in \omega \setminus \omega_h$. From (4.6) we get

$$\|y_u - y_h(u_h)\|_{L^\infty(\Omega)} \le \|y_u - y_{u_h}\|_{L^\infty(\Omega)} + \|y_{u_h} - y_h(u_h)\|_{L^\infty(\Omega)} \le \|y_u - y_{u_h}\|_{L^\infty(\Omega)} + Ch.$$

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Now, we prove that $\|y_u - y_{u_h}\|_{L^\infty(\Omega)} \to 0$ as $h \to 0$. Since $\|u_h\|_{L^\infty(\omega)} \le M$ for all $h > 0$, then $\{y_{u_h}\}_h$ is bounded in $W^{2,\bar{p}}(\Omega)$. Using the compactness of the embedding $W^{2,\bar{p}}(\Omega) \subset L^\infty(\Omega)$, we deduce the convergence $y_{u_h} \to \tilde{y}$ in $L^\infty(\Omega)$ along a subsequence. Let us show that the limit $\tilde{y}$ equals y_u. Recall that y_{u_h} solves

$$Ly_{u_h} + b(\cdot, y_{u_h}) + \chi_\omega u_h y_{u_h} = f \quad \text{in } V^*.$$

The convergence of y_{u_h} implies $Ly_{u_h} \rightharpoonup L\tilde{y}$ in V^*, $b(\cdot, y_{u_h}) \to b(\cdot, \tilde{y})$ in $L^\infty(\Omega)$, and $\chi_\omega u_h y_{u_h} \rightharpoonup \chi_\omega u\tilde{y}$ in $L^1(\Omega)$. Thus, for $z \in V \cap L^\infty(\Omega)$, we infer

$$\langle L\tilde{y} + b(\cdot, \tilde{y}) + \chi_\omega u\tilde{y}, z\rangle = \langle f, z\rangle.$$

By density, this holds for all $z \in V$, and thus $\tilde{y} = y_u$. In particular, the limit is unique, and thus the entire sequence y_{u_h} converges to y_u in $L^\infty(\Omega)$ as $h \to 0$. The convergence $J_h(u_h) \to J(u)$ follows easily by using $\alpha_h \to 0$. ∎

To check (D5) we take

$$\mathcal{A}_h = \left\{ v \in L^\infty(\omega_h) : \exists \varepsilon_v > 0 \text{ such that } v(x) > -\frac{\Lambda}{2} + \varepsilon_v \ \text{ for a.a. } x \in \omega_h \right\}.$$

It is easy to prove that $J_h : \mathcal{A}_h \longrightarrow \mathbb{R}$ is of class C^2 and its first derivative is given by

$$J_h'(u)v = -\int_{\omega_h} \varphi_h(u) y_h(u)\, v \, \mathrm{d}x + \alpha_h \int_{\omega_h} u\, v \, \mathrm{d}x \quad \forall u \in \mathcal{A}_h \text{ and } \forall v \in L^\infty(\omega_h), \tag{4.8}$$

where $y_h(u)$ and $\varphi_h(u)$ are the solutions of (4.4) and (4.5), respectively. Hence, it is enough to take $\psi_h = -(\varphi_h(u) y_h(u))|_{\omega_h} + \alpha_h\, u$. Concerning the function $J : \mathcal{A} \to \mathbb{R}$, we already know that it is of class C^2 (Theorem 3.2), and according to (3.8) we can take $\psi = -(\varphi_u y_u)|_\omega$.

Therefore, Theorems 2.7 and 2.8 hold. Observe that Theorem 2.7 is formulated as follows.

THEOREM 4.3. *Assume that* (A1)–(A5) *hold. For every h, the problem* (BDP$_h$) *has at least a global solution $\bar{u}_h$. If $\{\bar{u}_h\}_h$ is a sequence of global solutions of* (BDP$_h$) *and $\bar{u}_h \overset{*}{\rightharpoonup} \tilde{u}$ in $L^\infty(\omega)$, then $\tilde{u}$ is a global solution of* (BDP). *Conversely, if $\bar{u}$ is a bang-bang strict local minimum of* (BDP) *in the $L^1(\omega)$ sense, then there exists a sequence $\{\bar{u}_h\}_h$ of local minimizers of problems* (BDP$_h$) *with respect to the same topology such that $\bar{u}_h \to \bar{u}$ in $L^1(\omega)$.*

Now, we apply Theorem 2.8 to get the following result.

THEOREM 4.4. *Assume that* (A1)–(A5) *hold. Additionally, we suppose that* (3.13) *is fulfilled and $\bar{u}$ satisfies the second-order condition* (3.20) *with $\kappa' \in (0, \kappa)$. Let $\{\bar{u}_h\}_h$ be a sequence of local solutions of problems* (BDP$_h$) *converging to $\bar{u}$ in $L^1(\omega)$. Then, there exists a constant C independent of h such that*

$$\|\bar{u} - \bar{u}_h\|_{L^1(\omega_h)} \le C\,(h + \alpha_h). \tag{4.9}$$

Proof. To prove this theorem we will estimate the three terms in the right-hand side of (2.20). First, we observe that

$$\begin{aligned} \|J_h'(\bar{u}_h) - J'(\hat{\bar{u}}_h)\|_{L^\infty(X_h)} &= \|\bar{\varphi}_h \bar{y}_h - \varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h} + \alpha_h\, u_h\|_{L^\infty(\omega_h)} \\ &\le \|\bar{\varphi}_h \bar{y}_h - \varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h}\|_{L^\infty(\omega_h)} + C_0\, \alpha_h, \end{aligned} \tag{4.10}$$

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where $\bar{y}_h$ and $\bar{\varphi}_h$ are the discrete state and adjoint state associated with $\bar{u}_h$, and $y_{\hat{\bar{u}}_h}$ and $\varphi_{\hat{\bar{u}}_h}$ are the continuous state and adjoint state corresponding to $\hat{\bar{u}}_h$, which is the extension of $\bar{u}_h$ to ω by $\bar{u}$. Now, using Lemma 4.1 we obtain

$$\begin{aligned}\|\bar{\varphi}_h\bar{y}_h - \varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h}\|_{L^\infty(\omega_h)} &\le \|\bar{\varphi}_h\|_{L^\infty(\omega_h)}\|\bar{y}_h - y_{\hat{\bar{u}}_h}\|_{L^\infty(\omega_h)} \\ &\quad + \|y_{\hat{\bar{u}}_h}\|_{L^\infty(\omega_h)}\|\bar{\varphi}_h - \varphi_{\hat{\bar{u}}_h}\|_{L^\infty(\omega_h)} \le C_1 h.\end{aligned} \tag{4.11}$$

Now, we estimate the second term of (2.20). To this end, we take u_h as the projection of $\bar{u}$ on $\mathcal{U}_h$; see (4.7). Since $\bar{u}$ is bang-bang by assumption, it holds that $\bar{u} = u_h$ on all elements, where $\bar{u}$ is constant. It remains to estimate $|u_h - \bar{u}|$ on elements T, where $\bar{u}$ takes the values α and β on some points of T. Let us denote the family of such elements by $\mathcal{T}_{h,\bar{u}}$. Let us take $T \in \mathcal{T}_{h,\bar{u}}$. This means that $\bar{\varphi}\bar{y}$ changes the sign in T. Since $\bar{\varphi}\bar{y}$ is continuous in $\bar{\Omega}$, there exists a point $\xi_T \in T$ such that $\bar{\varphi}(\xi_T)\bar{y}(\xi_T) = 0$. Since $\bar{\varphi}\bar{y} \in W^{2,\bar{p}}(\Omega) \subset C^1(\bar{\Omega})$, we get the existence of constant $\bar{L}$ such that

$$|\bar{\varphi}(x)\bar{y}(x)| = |\bar{\varphi}(x)\bar{y}(x) - \bar{\varphi}(\xi_T)\bar{y}(\xi_T)| \le \bar{L}|x - \xi_T| \le \bar{L}h \quad \forall x \in T.$$

This inequality implies that

$$\bigcup_{T\in\mathcal{T}_{h,\bar{u}}} T \subset \{x \in \omega_h : |\bar{\varphi}(x)\bar{y}(x)| \le \bar{L}h\}.$$

This, along with (3.13), leads to

$$\sum_{T\in\mathcal{T}_{h,\bar{u}}} |T| \le K\bar{L}h.$$

Hence, we infer

$$\|u_h - \bar{u}\|_{L^1(\omega_h)} = \sum_{T\in\mathcal{T}_{h,\bar{u}}} \|u_h - \bar{u}\|_{L^1(T)} \le (\beta - \alpha)K\bar{L}h = C_2 h. \tag{4.12}$$

We finish the proof with the estimate of the third term of (2.20). Note that by construction it holds that $\hat{u}_h = \bar{u}$ on $\omega \setminus \omega_h$. Using that u_h is the projection of $\bar{u}$, we get with (3.8) and (4.12) that

$$\begin{aligned}|J'(\hat{\bar{u}}_h)(\hat{u}_h - \bar{u})| &= \Big|\int_\omega \varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h}(\hat{u}_h - \bar{u})\,\mathrm{d}x\Big| = \Big|\int_{\omega_h} \varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h}(u_h - \bar{u})\,\mathrm{d}x\Big| \\ &= \Big|\int_{\omega_h} (\varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h} - \Pi_h(\varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h}))(u_h - \bar{u})\,\mathrm{d}x\Big| \\ &\le \|\varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h} - \Pi_h(\varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h})\|_{L^\infty(\omega_h)}\|u_h - \bar{u}\|_{L^1(\omega_h)} \\ &\le Ch\|\varphi_{\hat{\bar{u}}_h} y_{\hat{\bar{u}}_h}\|_{C^1(\bar{\Omega})} C_2 h \le C_3 h^2.\end{aligned} \tag{4.13}$$

Here, we used that $\{\varphi_{\hat{\bar{u}}_h}\}$ and $\{y_{\hat{\bar{u}}_h}\}$ are uniformly bounded in $W^{2,p}(\Omega)$. Finally, (4.9) follows from (2.20), (4.10)–(4.13), and Young's inequality. ∎

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