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Life cycles with Endogenous Time Allocation and Age-Dependent Mortality

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Abstract

The negative effect of population aging on the economy can be mitigated by a behavioral effect of people as a reaction to a higher life expectancy. We analyze the optimal life-cycle of individuals that allocate time at the intensive margin between leisure, human capital accumulation, and labor supply while facing an age-dependent mortality. This allows to enhance effects of changes in life expectancy on labor supply and human capital accumulation and to uncover trade-offs between time allocations at different stages of the life-cycle. Our life-cycles are characterized by on the job training throughout all the working life with a possibility of a temporary exit from the labor market. We simulate the model numerically and find that with a higher life expectancy, labor supply increases at the intensive margin and the individual invests more in human capital. We also find a willingness to increase labor supply at the extensive margin.

Keywords: Life-Cycle; Age-Dependent Mortality; Aging; Time Allocation.

JEL Classification: D15; J22; J24; H55.

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1 Introduction

We may roughly divide the impact of demographic changes in the economy on an accounting effect and a behavioral effect. The accounting effect concerns the change of the age structure of the population. In the case of population aging it is negative. It can be measured by the deterioration of the ratio of the dependable population on active population. Everything else constant, a decrease in the proportion of the working-age population puts a strain on the economy as pension systems, health care, and other social programs have to be funded by a smaller fraction of the population.

The behavioral effect concerns the reaction of the various agents in the economy to the demographic transition. As many economies are experiencing population aging, studying this effect is important because if people invest more in human capital when they expect to live longer, this may mitigate substantially the negative accounting effect.

In this paper we obtain endogenously the planned life-cycle profiles of consumption, leisure, work effort, accumulation of human capital and financial assets for an individual facing a mortality law that is age dependent. We set up a model in continuous time in which the life of an individual starts and ends with zero labor supply, but allows an intermediate stage in which the three time activities may coexist. Consumers differ only in their age and make their decisions taking into consideration the existence of an unfunded pension system.

This study can be regarded as the consumer block of a general equilibrium model.[1] Since we want to study how the consumer reacts to changes in life expectancy, we chose to make the consumer optimize expected lifetime utility, facing in each moment in time an age-dependent survival law. Besides being realistic, Heijdra and Romp (2008) show that for a life-cycle small open economy, impulse response functions are very different from those of an economy where a constant survival law like in Blanchard (1985) is used.

We include some features in the model in order to boost the effects of the consumer reaction to changes in life expectancy in human capital and labor supply. One of this features is a human capital externality as in Lucas (1988) and Azariadis and Drazen (1990), which is also used in studies more close to our what we are doing, as in Boucekkine et al. (2002) and in Heijdra and Romp (2009a). Another feature is a subsidy for time allocated to schooling, an idea borrowed from Heijdra and Romp (2009a). This will be useful to simulate economic policies that can act directly on the creation of human capital. The government has at its disposal several taxes, the gross replacement rate, the retirement age and the years of mandatory schooling in order to achieve its policy goals.

This is a model of time allocation at the intensive margin. The consumer takes as exogenously defined by the government, the years of mandatory schooling and the retirement age. Consistent with our goal to highlight behavioral effects related to a higher life expectancy, either in labor supply or in human capital investment, we defined time allocated to learning in the intermediate phase as on-the-job training (OJT). Compared to a model of learning by

[1] In fact it is part of a general equilibrium model developed in Pereira (2018), too large for publication in one single article.

doing, using OJT allows the investment in human capital in that phase to be separated from labor supply.[2]

Early studies in modeling the consumer's life-cycle time allocation at the intensive margin between study, work and leisure are Heckman (1976), Blinder and Weiss (1976), Driffill (1980) and Ryder et al. (1976). Blinder and Weiss (1976) obtain a large range of possibilities for the combination of activities and also obtain the possibility of cycling between phases. Driffill (1980) proves that, in the Blinder and Weiss's model, if we impose retirement as the last stage of life, then the possibility of cycling is always ruled out. Ryder et al. (1976) show that a model with leisure provides a better empirically supported profile of labor supply than the Ben-Porath (1967) model that maximizes income instead of utility.

More recently, there are several models in which time allocation in learning and work is considered, but what we are particularly interested are in studies where a survival law is taken into consideration. For example, Boucekkine et al. (2002) use an age-dependent survival law in a model with time allocation at the extensive margin. The individual chooses the length of schooling and the retirement age. Heijdra and Reijnders (2016) work with a fully rectangular law in a model of learning by doing. Besides choosing the level of schooling and the retirement age, they introduce the decision to supply labor at the intensive margin. In Heijdra and Reijnders (2018), life uncertainty is introduced with an age-dependent survival law. However, now time allocation is only at the extensive margin.

Our modeling of time allocation is closer to Ludwig et al. (2012) and Vogel et al. (2017). In their models, there is an age-dependent survival law and consumers make decisions at an age where mandatory schooling already happened, deciding on time allocation at the intensive margin. Human capital accumulation relies upon an OJT process. We extend these papers by introducing a schooling decision at the intensive margin which makes investment in human capital endogenous in the entire life-cycle and allows us to uncover some trade-offs between schooling and OJT. Moreover, we also perform a sensitivity analysis to changes in some key parameters.

Our analytical results show that we never obtain a phase of only work, as in the "normal case" of Blinder and Weiss (1976). There is, however, the possibility to withdraw temporarily from the labor market in order to increase the investment in human capital. After schooling, the consumer will always pursue OJT till withdrawing completely from the labor market.

As it is possible to reduce labor supply to zero before the retirement age and once this happens, it will be permanent, this decision at the intensive margin translates to a decision at the extensive margin and we may say that the retirement decision is partially endogenous. The model is flexible enough to allow for a schooling decision at the extensive margin. The consumer may postpone the entry in the labor market, while having positive time allocated to studying. Different dynamics and age profiles of time allocation can appear but, with the parameters we use in the numerical analysis, labor supply, labor income, and human capital

[2] Although Kuruscu (2006) finds that OJT investments have small effects in increasing lifetime income we still saw as advantageous for our purposes, the possibility of separating time allocated to training from time allocated to labor supply. Some examples of models with learning by doing are Imai and Keane (2004) and Peterman (2016).

will be concave on age and leisure will be convex.

The numerical simulations suggest that the consumer will react to a longer life expectancy by increasing labor supply at the intensive margin and be willing to increase it at the extensive margin and also will increase time investment in human capital. The numerical exercise test also influences in the individual life-cycle of several changes in parameters and variables, with an emphasis on government instruments.

The article is organized as follows. Section 2 lays down the model. The optimization problem is solved in Section 3. Analytical results are shown in Section 4. In Section, 5 we do a numerical simulation of the model. Section 6 concludes.

2 The model

The economy is populated by overlapping generations of consumers, that are only heterogeneous in their age. The only source of uncertainty in this economy is the individual's time of death. We assume the individuals know the age-dependent survival law. What they do not know is if this survival law will change in the future.

As this is a partial equilibrium model, there are variables that are determined in aggregate markets which will be considered as given.[3] When making her plans we assume that the individual has myopic expectations on these variables and on the mortality law, i.e. she expects them to remain the same throughout her planning horizon.

Regarding notation, we will denote time, generically, as τ with t being the current time, the time when the planning is made. A subscript $_{*t}$ indicates a quantity being forecast or planned at time t for some time into the future. E.g., $X(\tau)_{*t}$ denotes the value that variable X is expected or planned to attain at time $\tau \geq t$, according to forecasts or plans made at time t. For the variables in which we apply directly the myopic expectations, we have $X(\tau)_{*t} = X(t)$.

We also assume the existence of a perfectly competitive institutional sector that combines a banking sector and an insurance sector which lends and borrows from consumers. The cost of borrowing is the same as the return on deposits and is equal to the interest rate plus a premium risk encompassing the mortality rate. There are no bequests as consumers contract with the institutional sector and give their net assets position at the time of death as collateral.

2.1 Mortality

The time of death of an individual born at time ν is a continuous random variable T_ν with density ϕ_ν:

$$\Pr\{T_\nu \leq t\} = \Phi_\nu(t) = \int_\nu^t \phi_\nu(\tau)d\tau.$$

[3]In Pereira (2018) we show that most results hold in general equilibrium.

Consider the functions $M_\nu : \mathbb{R} \mapsto [0, +\infty]$ defined as

$$M_\nu(t) = -\ln\left(1 - \Phi_\nu(t)\right), \qquad \text{i.e.,} \qquad 1 - \Phi_\nu(t) = e^{-M_\nu(t)},$$
$$m_\nu(t) = \dot{M}_\nu(t) = \begin{cases} +\infty & \text{for } \Phi_\nu(t) = 1 \\ \frac{\phi_\nu(t)}{1-\Phi_\nu(t)} & \text{for } \Phi_\nu(t) < 1 \end{cases}$$

Assumption 1 *The forecasts $\Phi_\nu(\tau)_{*t}$, $\phi_\nu(\tau)_{*t}$, $M_\nu(\tau)_{*t}$, $m(\tau)_{*t}$, are consistent in the sense that*

$$\Phi_\nu(\tau)_{*t} = \Phi_\nu(\tau), \quad \phi_\nu(\tau)_{*t} = \phi_\nu(\tau), \quad M_\nu(\tau)_{*t} = M_\nu(\tau), \quad m_\nu(\tau)_{*t} = m_\nu(\tau)$$

hold for every $\tau \leq t$, while

$$M_\nu(\tau)_{*t} = \int_{-\infty}^{\tau} m_\nu(s)_{*t} ds,$$
$$\Phi_\nu(\tau)_{*t} = 1 - e^{-M_\nu(\tau)_{*t}},$$
$$\phi_\nu(\tau)_{*t} = \frac{d}{d\tau}\left(\Phi_\nu(\tau)_{*t}\right)$$

hold for every $\tau > t$.

2.2 Time allocation

Total time endowment is normalized to 1. At time τ, for an individual born at time ν we have $s^l_\nu(\tau)$, $s^h_\nu(\tau)$ and $s^w_\nu(\tau)$ as, respectively, the fraction of time allocated to leisure, learning, and work. The following applies:

$$s^l_\nu(\tau) + s^w_\nu(\tau) + s^h_\nu(\tau) \equiv 1, \qquad\qquad s^i_\nu(\tau) \geq 0 \ \text{ for } i = h, l, w.$$

We define S_ν as the age at which mandatory schooling ends (assuming no grade repetition) and R_ν as the imposed retirement age. The consumer's life-cycle is divided into three phases, with different properties. In phase 1, for $\tau \in [\nu, S_\nu + \nu]$, she does not work and specializes in formal schooling. In phase 2, for $\tau \in [S_\nu + \nu, R_\nu + \nu]$, she may work and pursue on-the-job training.[4] Phase 3, for $\tau \in [R_\nu + \nu, +\infty[$, is the retirement period; the consumer does not work but we do not restrict her from pursuing education[5]. Therefore,

$$s^w_\nu(\tau) = 0 \ \text{ for } \ \tau \in [\nu, S_\nu + \nu] \cup [R_\nu + \nu, +\infty[$$

2.3 Preferences

There is no bequest motive. At each time $t \geq \nu$, the consumer plans her future seeking to maximize her forecast of the expected utility discounted by her time preference and weighted

[4] $S_\nu + \nu$ is the date at which working activity is allowed to start, but the consumer may choose to pursue more training and effectively start working later.

[5] Nevertheless, since in this model the sole purpose of accumulating human capital is to enhance labor income, there will be no reason for the consumer to accumulate human capital once retired, since it will have no effect on her earnings.

by her probability of survival into the next period.

$$\Lambda_{\nu*t} = \int_t^{+\infty} U\left(c_\nu(\tau)_{*t}, s_\nu^l(\tau)_{*t}\right) e^{-\int_t^\tau \rho_\nu(\theta)_{*t} d\theta} \Pr\{T_\nu > \tau | T_\nu > t\}_{*t} d\tau =$$
$$= \int_t^{+\infty} U\left(c_\nu(\tau)_{*t}, s_\nu^l(\tau)_{*t}\right) e^{-\int_t^\tau \rho_\nu(\theta)_{*t} + m_\nu(\theta)_{*t} d\theta} d\tau \tag{1}$$

Where $c_\nu(\tau)$ is consumption at time τ for a consumer born at time ν, and $\rho_\nu(.)$ is the pure time preference of a consumer born at time ν.

We assume that preferences are described by the following additive instantaneous utility function:

$$U\left[c_\nu(\tau)_{*t}, s_\nu^l(\tau)_{*t}\right] = \frac{c_\nu(\tau)_{*t}^{1-\sigma}}{1-\sigma} + \varphi \frac{s_\nu^l(\tau)_{*t}^{1-\epsilon}}{1-\epsilon} \qquad \text{with } \varphi, \sigma, \epsilon > 0$$

Lifetime maximization of consumption and leisure is constrained by the dynamics of the balance sheet, human capital, and a solvency condition.

2.4 Balance sheet

Consider $r(\tau)$ the market interest rate at time τ, that consumers have access to. We assume the existence of a risk-neutral institutional lender. The expected present value at time $t \geq \nu$ of a cash flow $s : [t, +\infty[\mapsto \mathbb{R}$ paid (or received) by a mortal born at time ν is

$$V_s(t) = \int_t^{+\infty} e^{-\int_t^\tau r(\theta) d\theta} s(\tau) \Pr\{T_\nu > \tau | T_\nu > t\} d\tau =$$
$$= \int_t^{+\infty} e^{-\int_t^\tau r(\theta) d\theta} s(\tau) \frac{1 - \Phi_\nu(\tau)}{1 - \Phi_\nu(t)} d\tau =$$
$$= \int_t^{+\infty} e^{-\int_t^\tau r(\theta) d\theta} s(\tau) e^{M_\nu(t) - M_\nu(\tau)} d\tau =$$
$$= \int_t^{+\infty} e^{-\int_t^\tau r(\theta) + m_\nu(\theta) d\theta} s(\tau) d\tau$$

Therefore, the risk-neutral lender is willing to lend/borrow a given amount for a mortal agent born at time ν with a premium over the pure financial market rate that is equal to the instantaneous death rate, m_ν.

Until the event of death, the balance sheet evolves according to the dynamics

$$\begin{aligned} \dot{a}_\nu(\tau)_{*t} = & [r(\tau)_{*t} + m_\nu(\tau)_{*t}]\, a_\nu(\tau)_{*t} - c_\nu(\tau)_{*t} - z_0(\tau)_{*t} + d(\tau)_{*t} + \\ & + e_\nu(\tau)_{*t} \chi_{[\nu, S_{\nu*t}+\nu]}(\tau) + \\ & + (1 - z_l(\tau)_{*t}) \omega_\nu(\tau)_{*t} \chi_{[S_{\nu*t}+\nu, R_{\nu*t}+\nu]}(\tau) + \\ & + (1 - z_p(\tau)_{*t}) p_{\nu*t} \chi_{[R_{\nu*t}, +\infty[}(\tau), \qquad \text{for } \tau \leq T_\nu. \end{aligned} \tag{2}$$

During the period $[\nu, S_\nu + \nu]$ the consumer receives an educational grant ($e_\nu(\tau)$), in the period $[S_\nu + \nu, R_\nu + \nu]$ the consumer receives labor income ($\omega_\nu(\tau)$) and, when retired, $[R(\nu) + \nu), +\infty[$, receives a pension benefit (p_ν). In case the consumer chooses to withdraw from the labor market before the retirement age, she will have no source of income between that period and the retirement age.[6] The variable $d(\tau)$ represents flows from aggregate markets, like dividends or other financial returns from the capital market, and $z_0(\tau)$, $z_l(\tau)$, $z_p(\tau)$ are, respectively, a lump sum tax, a labor income tax and a tax on pension benefits.

The consumer has the following myopic expectations:

$$r(\tau)_{*t} = r(t), \quad d(\tau)_{*t} = d(t), \quad S_{v*t} = S_v, \quad R_{v*t} = R_v, \quad z_0(\tau)_{*t} = z_0(t),$$
$$z_l(\tau)_{*t} = z_l(t), \quad z_p(\tau)_{*t} = z_p(t)$$

The educational grant is proportional to the fraction of the time dedicated to study and the wage rate per unit of human capital,[7]

$$e_v(\tau)_{*t} = \gamma_e(\tau)_{*t} w(\tau)_{*t} s_v^h(\tau)_{*t}$$

With $\gamma_e(\tau)$ being an educational subsidy and $w(\tau)$ the wage rate. For both the consumer has myopic expectations:

$$\gamma_e(\tau)_{*t} = \gamma_e(t), \quad w(\tau)_{*t} = w(t)$$

Labor income is proportional to the fraction of time dedicated to work and to the accumulated human capital:

$$\omega_v(\tau)_{*t} = h_v(\tau)_{*t} s_v^w(\tau)_{*t} w(\tau)_{*t} \tag{3}$$

The pension function depends on accumulated labor earnings up to the exogenous retirement date, being averaged by the period between the end of the mandatory schooling and the retirement,

$$p_{v*t} = \frac{\theta(\tau)_{*t} \pi_{R*t}}{R_{v*t} - S_{v*t}}$$

With π_{R*t} representing expected accumulated gross labor income. We have:

$$\pi_{R*t} = \int_{S_{v*t}+v}^{R_{v*t}+v} \omega_v(\tau)_{*t} d\tau = \int_{S_{v*t}+v}^{R_{v*t}+v} h_v(\tau)_{*t} s_v^w(\tau)_{*t} w(\tau)_{*t} d\tau$$
$$\dot{\pi}(\tau)_{*t} = h_v(\tau)_{*t} s_v^w(\tau)_{*t} w(\tau)_{*t}$$

Where $\theta(\tau) \geq 0$ is a policy parameter akin to the gross replacement rate, and R_v, the legal retirement age, is the minimum age of entitlement to a pension benefit. The consumer has myopic expectations on the parameter:

$$\theta(\tau)_{*t} = \theta(t)$$

[6] We follow, here, Heijdra and Romp (2009b).

[7] The purpose of introducing the educational grant $e_v(\tau)$, following Heijdra and Romp Heijdra and Romp (2009a), in the model is to mimic educational expenses of the government, which can be used as a tool to provide a further incentive to accumulate human capital.

2.5 Human capital

We assume that the production function of human capital is the same during the entire life-cycle. This means that, qualitatively, human capital acquired during the period of specialization in schooling is the same as the human capital acquired during the phase of work. This implies that OJT in phase 2 must have a generic nature and not be firm-specific (the same as schooling). Together with the assumptions of a perfectly competitive labor market and perfectly competitive production sectors in which labor is used, there is no incentive for firms to pay for employees training, as these may leave the firm and move on to other firms. Hence, all costs of accumulating human capital would be supported by the worker, in the form of foregone earnings and direct outlays. For simplification purposes we assume that there are no direct outlays and the only cost incurred by the worker has the form of foregone earnings or less leisure.

The accumulation of human capital for an agent born in υ follows the differential equation:

$$\dot{h}_\nu(\tau)_{*t} = \xi_h.\eta(\tau)_{*t}.s_\nu^h(\tau)_{*t}^{\phi_h} - \delta_h.h_\nu(\tau)_{*t} \qquad h_\nu(\nu) = 0, \text{ or}$$
$$h_\nu(t)_{*t} = h_\nu(t) \text{ for } t > \nu. \tag{4}$$

Where $\xi_h > 0$ is a productivity parameter, $\delta_h > 0$ measures the human capital erosion, $\phi_h \in (0, 1)$ measures the extent of marginal returns to the time of study, and all these parameters are constants. $\eta(\tau)$ introduces an intergenerational externality linked to the average per capita human capital ($\bar{h}$) in the economy, with strength ϕ_η. We allow average human capital to contribute to the accumulation of human capital with a different impact than time dedicated to studying. There are myopic expectations in the externality,

$$\eta(\tau)_{*t} = \eta(t)$$

The externality is computed as

$$\eta(t) = \bar{h}(t)^{\phi_\eta} = \left[\int_{-\infty}^{t} l_\nu(t)h_\nu(t)d\nu\right]^{\phi_\eta}$$

with $l_\nu(t) = L_\nu(t)/L(t)$ being the weight of the cohort born in ν that survived to time t, ($L_\nu(t)$), on total population at time t, ($L(t)$).

2.6 Solvency condition

The consumer is subject to a solvency condition. The expected present value of the balance sheet at the time of her death, given that she survived till time $t \geq \nu$ is:

$$\begin{aligned}
E\left[a_\nu(T_\nu)_{*t}e^{-\int_t^{T_\nu} r(\theta)_{*t}d\theta}\middle| T_\nu > t\right] &= \int_t^{+\infty} a_\nu(\tau)_{*t}e^{-\int_t^{\tau} r(\theta)_{*t}d\theta}\frac{\phi_\nu(\tau)_{*t}}{1-\Phi_\nu(t)}d\tau = \\
&= \int_t^{+\infty} a_\nu(\tau)_{*t}e^{-\int_t^{\tau} r(\theta)_{*t}d\theta} \frac{\phi_\nu(\tau)_{*t}}{1-\Phi_\nu(\tau)_{*t}}\frac{1-\Phi_\nu(\tau)_{*t}}{1-\Phi_\nu(t)}d\tau = \\
&= \int_t^{+\infty} a_\nu(\tau)_{*t}e^{-\int_t^{\tau} r(\theta)_{*t}d\theta} m_\nu(\tau)_{*t}e^{-\int_t^{\tau} m_\nu(\theta)_{*t}d\theta}d\tau
\end{aligned}$$

Thus, a risk-neutral lender will agree to finance the life-cycle strategy of a consumer only if:

$$\tau \longmapsto \min\left(0, m_\nu(\tau)_{*t} e^{-\int_t^\tau r(\theta)_{*t} + m_\nu(\theta)_{*t} d\theta} a_\nu(\tau)_{*t}\right) \text{ is Lebesgue-integrable} \tag{5}$$

$$\int_t^{+\infty} a_\nu(\tau)_{*t} m_\nu(\tau)_{*t} e^{-\int_t^\tau r(\theta)_{*t} + m_\nu(\theta)_{*t} d\theta} d\tau \geq 0 \tag{6}$$

Definition 2 *The generation* ν *is said to be solvent at time* $t > \nu$ *if there is some remaining lifetime strategy* $\left(c_\nu(\tau)_{*t}, s_\nu^l(\tau)_{*t}, s_\nu^h(\tau)_{*t}, s_\nu^w(\tau)_{*t}\right)$*,* $\tau > t$*, satisfying (5)-(6) with* $c_\nu(\tau)_{*t} > 0$*, a.e.* $\tau > t$.

3 Optimization problem

3.1 Statement of the problem

The consumer's strategy at time t is to solve the optimization problem:

$$\Lambda_{\nu*t} = \int_t^{+\infty} \left(\frac{c_\nu(\tau)_{*t}^{1-\sigma}}{1-\sigma} + \varphi \frac{s_\nu^l(\tau)_{*t}^{1-\epsilon}}{1-\epsilon}\right) e^{-\int_t^\tau \rho_\nu(\theta)_{*t} + m_\nu(\theta)_{*t} d\theta} d\tau \to \max, \tag{7}$$

with dynamics:

$$\dot{h}_\nu(\tau)_{*t} = \xi_h.\eta(\tau)_{*t}.s_\nu^h(\tau)_{*t}^{\phi_h} - \delta_h.h_\nu(\tau)_{*t}; \tag{8}$$

$$\dot{\pi}_\nu(\tau)_{*t} = h_\nu(\tau)_{*t} s_\nu^w(\tau)_{*t} w(\tau)_{*t} \chi_{[S_{\nu*t}+\nu, R_{\nu*t}+\nu]}(\tau); \tag{9}$$

$$\begin{aligned} \dot{a}_\nu(\tau)_{*t} = {} & [r(\tau)_{*t} + m_\nu(\tau)_{*t}]\, a_\nu(\tau)_{*t} - c_\nu(\tau)_{*t} - z_0(\tau)_{*t} + d(\tau)_{*t} \\ & + \gamma_e(\tau)_{*t} s_\nu^h(\tau)_{*t} w(\tau)_{*t} \chi_{[\nu, S_{\nu*t}+\nu]}(\tau) + \\ & + (1 - z_l(\tau)_{*t}) h_\nu(\tau)_{*t} s_\nu^w(\tau)_{*t} w(\tau)_{*t} \chi_{[S_{\nu*t}+\nu, R_{\nu*t}+\nu]}(\tau) + \\ & + (1 - z_p(\tau)_{*t}) \frac{\theta(\tau)_{*t} \pi_{R*t}}{R_{\nu*t} - S_{\nu*t}} \chi_{[R_{\nu*t}+\nu, +\infty[}(\tau); \end{aligned} \tag{10}$$

$$\dot{b}_\nu(\tau)_{*t} = m_\nu(\tau)_{*t} a_\nu(\tau)_{*t} e^{-\int_t^\tau r(\theta)_{*t} + m_\nu(\theta)_{*t} d\theta}; \tag{11}$$

with initial (consistency) conditions:

$$h_\nu(t)_{*t} = h_\nu(t), \quad \pi_\nu(t)_{*t} = \pi_\nu(t), \quad a_\nu(t)_{*t} = a_\nu(t), \quad b_\nu(t)_{*t} = 0, \tag{12}$$

and terminal (solvency) condition:

$$\lim_{\tau \to +\infty} b_\nu(\tau)_{*t} \geq 0 \tag{13}$$

The controls satisfy:

$$c_\nu(\tau)_{*t} \in [0, +\infty[, \quad \text{a.e. } \tau \in [t, +\infty[; \tag{14}$$

$$s_\nu^l(\tau)_{*t}, s_\nu^w(\tau)_{*t}, s_\nu^h(\tau)_{*t} \in [0, +\infty[, \quad \text{a.e. } \tau \in [t, +\infty[; \tag{15}$$

$$s_\nu^l(\tau)_{*t} + s_\nu^h(\tau)_{*t} + s_\nu^w(\tau)_{*t} \chi_{[S(\nu)_{*t}+\nu, R_\nu+\nu]}(\tau) = 1, \quad \text{a.e. } \tau \in [t, +\infty[; \tag{16}$$

We now proceed to solve the consumer's optimization problem via the Pontryagin's Maximum Principle. Due to the fairly simple dynamics in the retirement period, we opted to separate the maximization problem in two periods. In the next section we compute the value of maximized utility for the period $\tau \in [R_\nu, +\infty[$. Then we use this result, which can be interpreted as the value of retirement, to solve an optimization problem with free terminal value and fixed terminal time for $\tau \in [t, R_\nu[$.

3.2 Consumption allocation after retirement

In the following, we omit all subscripts ν, $_{*t}$, being understood that we are dealing with a given (fixed) demographic cohort ν and all quantities relative to a moment $\tau \geq t$ are forecasts (and hence the subscript $_{*t}$ is implicit). Variables to which the static expectation applies directly (are a function of t and do not change during the planning period), will be written as constants.

After retirement, all income is provided by pensions and interest. Since human capital is worth only for its use in the labor market, maximal utility after retirement is obtained with $s^w \equiv s^h \equiv 0$ or, equivalently $s^l \equiv 1$. Then the optimal problem for the period $\tau \in [R, +\infty[$ can be reduced to:

$$\int_R^{+\infty} \frac{c(\tau)^{1-\sigma}}{1-\sigma} e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} d\tau \to \max, \tag{17}$$

$$\dot{a}(\tau) = [r + m(\tau)]\, a(\tau) - c(\tau) - z_0 + d + \frac{(1-z_p)\theta\pi_R}{R-S}\chi_{[R,+\infty[}; \tag{18}$$

$$\dot{b}(\tau) = a(\tau)m(\tau)e^{-\int_t^\tau r(\theta)+m(\theta)d\theta}; \tag{19}$$

with π_R given and initial conditions

$$a(R) = a_R,\ b(R) = b_R \tag{20}$$

Since we are no longer concerned with maximizing leisure, we may introduce a running utility

$$\Lambda(\tau) = \int_R^\tau \frac{c(\tau)^{1-\sigma}}{1-\sigma} e^{-\int_t^{\theta_1} \rho(\theta_2)+m(\theta_2)d\theta_2} d\theta_1$$

i.e., we add to the dynamics (18), (19) an additional equation

$$\dot{\Lambda}(\tau) = \frac{c(\tau)^{1-\sigma}}{1-\sigma} e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} \tag{21}$$

Now, for any locally essentially bounded $c : [R, +\infty[\mapsto [0, +\infty[$ let (Λ_c, a_c, b_c) denote the trajectories of (18)-(19), (21) with $\Lambda_c(R) = 0$. Suppose that $\lim_{\tau \to +\infty} b_0(\tau) > 0$ (i.e., the consumer has some margin for consumption after retirement) and let $\hat{c}$ be an optimal control for problem (17)-(19), (20), (13), over the set of measurable functions (14). For any $T_1 > R$, $\hat{c}|_{[R,T_1]}$ maximizes the functional:

$$\Lambda_{(R,T_1)} = \int_R^{T_1} \frac{c(\tau)^{1-\sigma}}{1-\sigma} e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} d\tau \tag{22}$$

subject to (18)-(19), (13), (14), (20), and boundary conditions

$$a(T_1) = a_{\hat{c}}(T_1), \quad b(T_1) = b_{\hat{c}}(T_1) \tag{23}$$

Hence $\hat{c}|_{[R,T_1]}$ satisfies the Pontryagin's Maximum Principle with boundary conditions (20), (23). The Hamiltonian function is

$$\begin{aligned} &H(\tau, a(\tau), b(\tau), \lambda_0(\tau), \lambda_3(\tau), \lambda_4(\tau), c(\tau)) = \\ &\qquad = \lambda_0(\tau)\frac{c(\tau)^{1-\sigma}}{1-\sigma}e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} \\ &\qquad + \lambda_3(\tau)\Big([r+m(\tau)]\, a(\tau) - c(\tau) - z_0 + d + \frac{(1-z_p)\theta\pi_R}{R-S}\chi_{R,+\infty]}\Big) \\ &\qquad + \lambda_4(\tau)a(\tau)m(\tau)e^{-\int_t^\tau r+m(\theta)d\theta} \end{aligned} \tag{24}$$

By the PMP, there is an absolutely continuous curve $(\lambda_0(\tau), \lambda_3(\tau), \lambda_4(\tau))$ with $\tau \in [R, T_1]$ solving the Hamiltonian System

$$\begin{aligned} &\dot{\Lambda}_{\hat{c}}(\tau) = \frac{\hat{c}(\tau)^{1-\sigma}}{1-\sigma}e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}; \\ &\dot{a}_{\hat{c}}(\tau) = [r+m(\tau)]\, a_{\hat{c}}(\tau) - \hat{c}(\tau) - z_0 + d + \frac{(1-z_p)\theta\pi_R}{R-S}\chi_{[R,+\infty]}; \\ &\dot{b}_{\hat{c}}(\tau) = a_{\hat{c}}(\tau)m(\tau)e^{-\int_t^\tau r+m(\theta)d\theta}; \\ &\dot{\lambda}_0(\tau) = -\frac{\partial H}{\partial \Lambda} = 0; \\ &\dot{\lambda}_3(\tau) = -\frac{\partial H}{\partial a} = -[r(\tau)+m(\tau)]\,\lambda_3(\tau) - m(\tau)e^{-\int_t^\tau r+m(\theta)d\theta}\lambda_4(\tau); \\ &\dot{\lambda}_4(\tau) = -\frac{\partial H}{\partial b} = 0; \end{aligned} \tag{25}$$

and

$$\begin{aligned} &H(\tau, a_{\hat{c}}(\tau), b_{\hat{c}}(\tau), \lambda_0(\tau), \lambda_3(\tau), \lambda_4(\tau), \hat{c}(\tau)) = \\ &\qquad = \max_{c\geq 0} H(\tau, a_{\hat{c}}(\tau), b_{\hat{c}}(\tau), \lambda_0(\tau), \lambda_3(\tau), \lambda_4(\tau), c) \quad a.e. \quad \tau \in [R, T_1]. \end{aligned} \tag{26}$$

$(\lambda_0(T_1), \lambda_3(T_1), \lambda_4(T_1))$ can be set equal to any vector supporting the Pontryagin cone at time T_1. Since $\dot{\lambda}_0 \equiv \dot{\lambda}_4 \equiv 0$, we have $\lambda_0 \equiv \lambda_0(T_1)$, $\lambda_4 \equiv \lambda_4(T_1)$.

Let $(\tilde{\Lambda}, \tilde{a}, \tilde{b})$ denote a solution of (18)-(19), (21) in the interval $[T_1, +\infty[$ with $c(\tau) = \hat{c}(\tau)$ a.e. $\tau \geq T_1$.

Then, for any $T_2 > T_1$, we have

$$\begin{aligned} &\tilde{\Lambda}(T_2) - \Lambda_{\hat{c}}(T_2) = \tilde{\Lambda}(T_1) - \Lambda_{\hat{c}}(T_1) \\ &\tilde{a}(T_2) - a_{\hat{c}}(T_2) = \big(\tilde{a}(T_1) - a_{\hat{c}}(T_1)\big)e^{\int_{T_1}^{T_2} r(\theta)+m(\theta)d\theta} \\ &\tilde{b}(T_2) - b_{\hat{c}}(T_2) = \tilde{b}(T_1) - b_{\hat{c}}(T_1) + \big(\tilde{a}(T_1) - a_{\hat{c}}(T_1)\big)e^{-\int_t^{T_1} r(\theta)+m(\theta)d\theta}\int_{T_1}^{T_2} m(\tau)d\tau \end{aligned} \tag{27}$$

Suppose there is a control $\tilde{c} : [R, T_1] \mapsto [0, +\infty[$ such that

$$\Lambda_{\tilde{c}}(T_1) > \Lambda_{\hat{c}}(T_1) \quad \text{and} \quad a_{\tilde{c}}(T_1) > a_{\hat{c}}(T_1)$$

Then (27) shows that the control

$$c(\tau) = \begin{cases} \tilde{c}, & \tau \in [R, T_1] \\ \hat{c}, & \tau > T_1 \end{cases}$$

satisfies

$$\lim_{\tau \to +\infty} \Lambda_c(\tau) = \lim_{\tau \to +\infty} \Lambda_{\hat{c}}(\tau) + \Lambda_{\tilde{c}}(T_1) - \Lambda_{\hat{c}}(T_1)$$
$$\lim_{\tau \to +\infty} b_c(\tau) = +\infty$$

i.e., c satisfies the solvency condition and provides larger lifetime utility, a contradiction to the optimality of $\hat{c}$. Hence, any point $(\tilde{\Lambda}, \tilde{a}, \tilde{b})$ with $\tilde{\Lambda} > \Lambda_{\hat{c}}(T_1)$ and $\tilde{a} > a_{\hat{c}}(T_1)$ cannot be reached from $(0, a_R, b_R)$ by a trajectory of the system (18), (19), (21). It follows that the Pontryagin cone and the cone $[0, +\infty[^2 \times \mathbb{R}$ have non-overlapping interiors. Therefore we can pick $(\lambda_0, \lambda_3(T_1), \lambda_4(T_1))$ such that

$$\lambda_0 \nu_0 + \lambda_3(T_1)\nu_3 + \lambda_4 \nu_4 \geq 0 \qquad \forall\ \nu_0 \geq 0, \nu_3 \geq 0, \nu_4 \in \mathbb{R}, \tag{28}$$

which is equivalent to $\lambda_0 \geq 0, \quad \lambda_3(T_1) \geq 0, \quad \lambda_4 = 0$

If $\lambda_0 = 0$, then (26) implies $\hat{c} \equiv 0$, and therefore $\Lambda_{\hat{c}}(T_1) = 0$. This is non-optimal because we are assuming the consumer has some margin for consumption after retirement. The existence of this margin implies that $\Lambda_{\hat{c}}(T_1) > 0$, for some T_1 which is contradictory with $\lambda_0 = 0$. Thus, we conclude that we must have $\lambda_0 > 0$, but since the Hamiltonian System is homogeneous with respect to $(\lambda_0, \lambda_3, \lambda_4)$, we can normalize and set $\lambda_0 = 1$.

Since $\lambda_4 \equiv 0$ we have

$$\dot{\lambda}_3(\tau) = -(r + m(\tau))\lambda_3(\tau) \quad a.e. \quad \tau \in [R, T_1] \tag{29}$$

Therefore, $\lambda_3(T_1) = 0$ implies $\lambda_3 \equiv 0$, but in this case, (26) has no solution. This shows that $\lambda_3(T_1) > 0$. Now, (26) and (29) give

$$\begin{aligned}\hat{c}(\tau) &= \left(\lambda_3(\tau) e^{\int_t^\tau \rho(\theta)+m(\theta)d\theta}\right)^{-1/\sigma} \\ &= \left(\lambda_3(R) e^{-\int_R^\tau r+m(\theta)d\theta} e^{\int_t^\tau \rho(\theta)+m(\theta)d\theta}\right)^{-1/\sigma} \\ &= \lambda_3(R)^{-1/\sigma} e^{-\int_t^R \frac{\rho(\theta)+m(\theta)}{\sigma} d\theta} e^{\int_R^\tau \frac{r-\rho(\theta)}{\sigma} d\theta}\end{aligned} \tag{30}$$

Since T_1 is arbitrary, we see that $\hat{c}$ satisfies (25) and (30) with $\lambda_0 = 0, \lambda_4 = 0$ in all the interval

$[R, +\infty[$, i.e., the optimal consumption is known up to the parameter $\lambda_3(R)$. Substituting $\hat{c}$ in the Hamiltonian System, one obtains

$$a_{\hat{c}}(\tau) = e^{\int_R^\tau r+m(\theta)d\theta} \left[a_R + \Psi_1(\tau)\pi_R - \Psi_2(\tau) - \Psi_3(\tau)\lambda_3(R)^{-1/\sigma}\right]$$

with

$$\Psi_1(\tau) = \int_R^\tau e^{-\int_R^{\theta_1} r+m(\theta_2)d\theta_2} \frac{(1-z_p)\theta}{R-S} d\theta_1$$
$$\Psi_2(\tau) = \int_R^\tau e^{-\int_R^{\theta_1} r+m(\theta_2)d\theta_2} (z_0 - d)d\theta_1$$
$$\Psi_3(\tau) = \int_R^\tau e^{\frac{1-\sigma}{\sigma}\int_R^{\theta_1} r+m(\theta_2)d\theta_2} e^{-\int_t^{\theta_1} \frac{\rho(\theta_2)+m(\theta_2)}{\sigma}d\theta_2} d\theta_1$$

Also,

$$b_{\hat{c}}(\tau) = b_R + e^{-\int_t^R r+m(\theta)d\theta} \int_R^\tau m(\theta) \left[a_R + \Psi_1(\theta)\pi_R - \Psi_2(\theta) - \Psi_3(\theta)\lambda_3(R)^{-1/\sigma}\right] d\theta$$

Let $\Psi_i = \lim\limits_{\tau\to+\infty} \Psi_i(\tau),\ i = 1, 2, 3$. These limits are finite and hence

$$\left[a_R + \Psi_1\pi_R - \Psi_2 - \Psi_3\lambda_3(R)^{-1/\sigma}\right] \in \mathbb{R}$$

Now, if $\left[a_R + \Psi_1\pi_R - \Psi_2 - \Psi_3\lambda_3(R)^{-1/\sigma}\right] < 0$ then $\lim\limits_{\tau\to+\infty} b_{\hat{c}}(\tau) = -\infty$ i.e. $\hat{c}$ does not satisfy the solvency constraint.

If $\left[a_R + \Psi_1\pi_R - \Psi_2 - \Psi_3\lambda_3(R)^{-1/\sigma}\right] > 0$, then $\lim\limits_{\tau\to+\infty} b_{\hat{c}}(\tau) = +\infty$ i.e. $\hat{c}$ satisfies the solvency constraint and still leaves some margin for further consumption.

Finally, in the case $\left[a_R + \Psi_1\pi_R - \Psi_2 - \Psi_3\lambda_3(R)^{-1/\sigma}\right] = 0$, this is not incompatible with any value of $\lim\limits_{\tau\to+\infty} b_{\hat{c}}(\tau)$ (including $\pm\infty$), thus our problem (17)-(19), (13) may fail to have a solution. In other words, it is not guaranteed that there is not a solution that returns either $\lim\limits_{\tau\to+\infty} b(\tau) = +\infty$ (not optimal) or $\lim\limits_{\tau\to+\infty} b(\tau) = -\infty$ (not solvent).

This means that we must trade the solvency condition (13) by a weaker version. Hence, we shall assume that starting with

$$\left[a_R + \Psi_1\pi_R - \Psi_2 - \Psi_3\lambda_3(R)^{-1/\sigma}\right] > 0$$

in order to exclude the possibility of $\lim\limits_{\tau\to+\infty} b(\tau) = -\infty$ the individual will pursue a strategy that, by a satiation argument, will increase consumption until all the slackness has been used. This weaker solvency condition is:

$$\left[a_R + \Psi_1\pi_R - \Psi_2 - \Psi_3\lambda_3(R)^{-1/\sigma}\right] = 0 \tag{31}$$

From this expression we arrive at $\lambda_3(R)$:

$$\lambda_3(R) = \left[\frac{a_R + \Psi_1\pi_R - \Psi_2}{\Psi_3}\right]^{-\sigma} \tag{32}$$

Since consumption is restricted to be non-negative, (30) and (32), imply that

$$a_R + \Psi_1 \pi_R - \Psi_2 \geq 0 \tag{33}$$

Plugging (32) in the utility function we obtain the optimal utility accumulated after retirement:

$$\begin{aligned} G(\pi_R, a_R) &= \int_R^{+\infty} \left(\frac{\left(\lambda_3(\tau) e^{\int_t^\tau \rho(\theta)+m(\theta)d\theta}\right)^{-(1-\sigma)/\sigma}}{1-\sigma} + \frac{\varphi}{1-\epsilon} \right) e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} d\tau \\ &= \frac{\left(a_R + \Psi_1\pi_R - \Psi_2\right)^{1-\sigma}\Psi_3^\sigma}{1-\sigma} + \frac{\varphi}{1-\epsilon}\Psi_0 \end{aligned} \tag{34}$$

with:

$$\Psi_0 = \int_R^{+\infty} e^{-\int_t^{\theta_1} \rho(\theta_2)+m(\theta_2)d\theta_2} d\theta_1$$

3.3 The optimization problem reformulated

Using the result from the previous section the optimization problem stated in Section 3.1 is rewritten as:

$$\begin{aligned} \Lambda_{\nu*t} = &\int_t^{R_{\nu*t}} \left(\frac{c_\nu(\tau)_{*t}^{1-\sigma}}{1-\sigma} + \varphi \frac{s_\nu^l(\tau)_{*t}^{1-\epsilon}}{1-\epsilon} \right) e^{-\int_t^\tau \rho_\nu(\theta)_{*t}+m_\nu(\theta)_{*t}d\theta} d\tau + \\ &+ G(\pi_{R*t}, a_{R*t}) \to \max, \end{aligned} \tag{35}$$

with dynamics:

$$\dot{h}_\nu(\tau)_{*t} = \xi_h.\eta(\tau)_{*t}.s_\nu^h(\tau)_{*t}^{\phi_h} - \delta_h.h_\nu(\tau)_{*t}; \tag{36}$$

$$\dot{\pi}_\nu(\tau)_{*t} = h_\nu(\tau)_{*t} s_\nu^w(\tau)_{*t} w(\tau)_{*t} \chi_{[S_{\nu*t}+\nu, R_{\nu*t}+\nu]}(\tau); \tag{37}$$

$$\begin{aligned} \dot{a}_\nu(\tau)_{*t} = &\left[r(\tau)_{*t} + m(\tau)_{*t}\right] a_\nu(\tau)_{*t} - c_\nu(\tau)_{*t} - z_0(\tau)_{*t} + d(\tau)_{*t} \\ &+ \gamma_e(\tau)_{*t} s_\nu^h(\tau)_{*t} w(\tau)_{*t} \chi_{[\nu, S_{\nu*t}+\nu]}(\tau) + \\ &+ (1 - z_l(\tau)_{*t}) h_\nu(\tau)_{*t} s_\nu^w(\tau)_{*t} w(\tau)_{*t} \chi_{[S_{\nu*t}+\nu, R_{\nu*t}+\nu]}(\tau); \end{aligned} \tag{38}$$

With the initial conditions in (12) and the controls satisfying:

$$c_\nu(\tau)_{*t} \in [0, +\infty[, \quad \text{a.e. } \tau \in [t, R_{\nu*t}]; \tag{39}$$

$$s_\nu^l(\tau)_{*t}, s_\nu^w(\tau)_{*t}, s_\nu^h(\tau)_{*t} \in [0, +\infty[, \quad \text{a.e. } \tau \in [t, R_{\nu*t}]; \tag{40}$$

$$s_\nu^l(\tau)_{*t} + s_\nu^h(\tau)_{*t} + s_\nu^w(\tau)_{*t} \chi_{[\nu+S(\nu)_{*t}, R_\nu+\nu]}(\tau) = 1, \quad \text{a.e. } \tau \in [t, R_{\nu*t}]; \tag{41}$$

For this new time frame we have to consider a different solvency condition. Let T_m denote the continuous random variable time of death. We can write the solvency condition as:

$$\begin{aligned} &\Pr\left(T_m \leq R_{\nu*t} | T_m > t\right) E\left[a_\nu(T_m)_{*t} e^{-\int_t^{T_m} r(\theta)_{*t}d\theta} \middle| t < T_m \leq R_{\nu*t}\right] + \\ &\qquad + \Pr\left(T_m > R_{\nu*t} | T_m > t\right) K > 0 \end{aligned} \tag{42}$$

with

$$K = \sup_u E\left[a_\nu(T_m)_{*t} e^{-\int_t^{T_m} r(\theta)_{*t} d\theta} \middle| T_m > R_{\nu*t}\right] =$$
$$= E\left[a_\nu(T_m)_{*t} e^{-\int_t^{T_m} r(\theta)_{*t} d\theta} \middle| T_m > R_{\nu*t}, c.\chi_{[R_{\nu*t},+\infty[} \equiv 0\right]$$

The inequality (42) cannot be removed by any saturation argument because its slackness results from the trade-off between utility before and after retirement, i.e. we need to exclude the strategy that consumes everything up to reform with zero consumption afterward. Therefore, the additional constraint,

$$E\left[a_\nu(T_m)_{*t} e^{-\int_t^{T_m} r(\theta)_{*t} d\theta} \middle| T_m > R_{\nu*t}, c.\chi_{[R_{\nu*t},+\infty[} \equiv 0\right] > 0 \tag{43}$$

seems necessary to guarantee that the plan can be extended into post retirement. This last constraint translates into:

$$\int_{R_{\nu*t}}^{+\infty} a_\nu(\tau)_{*t} m_\nu(\tau)_{*t} e^{-\int_t^\tau r(\theta)_{*t} + m_\nu(\theta)_{*t} d\theta}, c.\chi_{[R_{\nu*t},+\infty[} \equiv 0 d\tau > 0$$
$$\Leftrightarrow \int_{R_{\nu*t}}^{+\infty} \Bigg(a_\nu(R_{\nu*t})_{*t} + \int_{R_{\nu*t}}^{\tau} e^{-\int_{R_{\nu*t}}^{\theta_1} r(\theta_2)_{*t} + m_\nu(\theta_2)_{*t} d\theta_2}.$$
$$.\Big(\frac{(1 - z_p(\theta_1)_{*t})\theta(\theta_1)_{*t}\pi_{R*t}}{R_{\nu*t} - S_{\nu*t}} \chi_{[R_{\nu*t},+\infty[} - z_0(\theta_1)_{*t} + d(\theta_1)_{*t}\Big) d\theta_1 \Bigg).$$
$$.m_\nu(\tau)_{*t} e^{-\int_t^{R_{\nu*t}} r(\theta)_{*t} + m_\nu(\theta)_{*t} d\theta} d\tau > 0$$

In which consumption is absent. It implies:

$$a_\nu(R_{\nu*t}) + \int_{R_{\nu*t}}^{+\infty} e^{-\int_{R_{\nu*t}}^{+\theta_1} r(\theta_2)_{*t} + m_\nu(\theta_2)_{*t} d\theta_2} \Big(\frac{(1 - z_p(\theta_1)_{*t})\theta(\theta_1)_{*t}\pi_{R*t}}{R_{\nu*t} - S_{\nu*t}} \chi_{[R_{\nu*t},+\infty[} -$$
$$- z_0(\theta_1)_{*t} + d(\theta_1)_{*t}) \, d\theta_1 > 0$$

or

$$a_\nu(R_{\nu*t}) + \Psi_{1*t}\pi_{R*t} - \Psi_{2*t} > 0 \tag{44}$$

The following proposition formally expresses the solution for the present problem.

Proposition 3 *If the cohort ν is solvent then for the Pontryagin's Maximum Principle applying to (35), (36)-(38), (39)-(41) we have that if $(\hat{h}, \hat{\pi}, \hat{a}, \hat{c}, \hat{s}^l, \hat{s}^h, \hat{s}^w)$ is an optimal solution to (7)-(16), there is an absolutely continuous curve $(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, not identically null, solving the Hamiltonian system:*[8]

[8] For notational convenience we drop the subscript ν.

$$
\begin{aligned}
\dot{\hat{h}} &= \frac{\partial H}{\partial \lambda_1} = \xi_h.\eta.\hat{s}^{h\ \phi_h} - \delta_h.\hat{h};\\
\dot{\hat{\pi}} &= \frac{\partial H}{\partial \lambda_2} = \hat{h}.\hat{s^{w}}.w.\chi_{[S,R]};\\
\dot{\hat{a}} &= \frac{\partial H}{\partial \lambda_3} = [r + m(\tau)]\,\hat{a} - \hat{c} - z_0 + d + \gamma_e.\hat{s}^h.w.\chi_{[v,S]} + \\
&\quad + (1 - z_l).\hat{h}.\hat{s}^w.w.\chi_{[S,R]};
\end{aligned}
$$

$$
\begin{aligned}
\lambda_0 &= 1;\\
\dot{\lambda}_1 &= \delta_h \lambda_1 - [\lambda_2 + \lambda_3\,(1 - z_l)]\, w s^w \chi_{[S,R]};\\
\dot{\lambda}_2 &= 0;\\
\dot{\lambda}_3 &= -\,[r(\tau) + m(\tau)]\,\lambda_3;
\end{aligned}
$$

with transversality conditions for the adjoints

$$
\begin{aligned}
\lambda_0\,(R) &= 1\\
\lambda_1\,(R) &= 0\\
\lambda_2\,(R) &= \lambda_3\,(R) \int_R^{+\infty} e^{-\int_R^{\theta_1} r + m(\theta_2) d\theta_2} \frac{(1 - z_p)\,\theta}{R - S} d\theta_1\\
\lambda_3\,(R)^{-1/\sigma} & \int_R^{+\infty} e^{\frac{1-\sigma}{\sigma}\int_R^{\theta_1} r + m(\theta_2) d\theta_2} e^{-\int_t^{\theta_1} \frac{\rho(\theta_2) + m(\theta_2)}{\sigma} d\theta_2} d\theta_1 = a\,(R) + \\
&+ \int_R^{+\infty} e^{-\int_R^{\theta_1} r + m(\theta_2) d\theta_2} \frac{(1 - z_p)\theta\pi_R}{R - S} d\theta_1\\
&- \int_R^{+\infty} e^{-\int_R^{\theta_1} r + m(\theta_2) d\theta_2} (z_0 - d) d\theta_1
\end{aligned}
\tag{45}
$$

The optimal controls are:

$$
\hat{c} = \left(\lambda_3 e^{\int_t^{\tau} \rho(\theta) + m(\theta) d\theta}\right)^{-1/\sigma}, \quad \text{for} \quad \forall \tau \in [t, +\infty[
$$

For $\tau \in [t, S]$*:*

$$
\hat{s}^l = \left(\frac{\varphi e^{-\int_t^{\tau} \rho(\theta) + m(\theta) d\theta}}{\lambda_3 \gamma_e w}\right)^{1/\epsilon} \wedge\ 1 \qquad \text{and} \qquad \hat{s}^h = 1 - \hat{s}^l \ \text{ if } \ \lambda_1 = 0
$$

$$
\frac{\varphi e^{-\int_t^{\tau} \rho(\theta) + m(\theta) d\theta}}{(s^l)^{\epsilon}} = \frac{\lambda_1 \xi_h \eta \phi_h}{(1 - s^l)^{1 - \phi_h}} + \lambda_3 \gamma_e w \qquad \text{and} \qquad \hat{s}^h = 1 - \hat{s}^l \ \text{ if } \ \lambda_1 > 0
$$

For $\tau \in [S, R]$*:*

$$
\begin{aligned}
&\lambda_1(\tau) = 0 \Rightarrow \tau = R, \\
&\text{for } \lambda_1(\tau) > 0 \textit{ we have } \quad \textit{either,} \\
&\hat{s}^l = \left(\frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{(\lambda_2 + (1-z_l)\lambda_3)\, wh} \right)^{1/\epsilon}, \quad \hat{s}^h = \left(\frac{\lambda_1 \xi_h \eta \phi_h}{(\lambda_2 + (1-z_l)\lambda_3)\, wh} \right)^{1/1-\phi}, \\
&\hat{s}^w = 1 - \hat{s}^l - \hat{s}^h \\
&\text{or}: \\
&\frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{(s^l)^\epsilon} = \frac{\lambda_1 \xi_h \eta \phi_h}{(1-s^l)^{1-\phi_h}}, \quad \hat{s}^h = 1 - \hat{s}^l, \quad \hat{s}^w = 0
\end{aligned}
$$

satisfying the maximal condition

$$
\begin{aligned}
&H(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \hat{h}, \hat{\pi}, \hat{a}, \hat{c}, \hat{s}^l, \hat{s}^h, \hat{s}^w) = \\
&\qquad = \max_{c, (s_v^l), (s_v^h), (s_v^w)} H(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \hat{h}, \hat{\pi}, \hat{a}, c, (s_v^l), (s_v^h), (s_v^w)).
\end{aligned} \tag{46}
$$

Plus, the Erdmann condition:

$$
\frac{dH(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \hat{h}, \hat{\pi}, \hat{a}, \hat{c}, \hat{s}^l, \hat{s}^h, \hat{s}^w)}{d\tau} = \frac{\partial H(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \hat{h}, \hat{\pi}, \hat{a}, \hat{c}, \hat{s}^l, \hat{s}^h, \hat{s}^w)}{\partial \tau}
$$

The proof is in Appendix A.

4 Life-cycle properties

Our model is complex enough to not allow many analytical deductions. Most results will be derived from numerical simulation. There are, however, some results that can be deduced analytically. This section deals with them.

4.1 There is no phase of pure work and OJT lasts till retirement

In this model there will never be a phase of pure work ($s_l > 0, s_h = 0, s_w > 0$). This is the case $\lambda_1 = 0$ for $\tau \in [S, R]$ in the proof to Proposition 3 but we explain it here in more detail. We can see this by starting from the transversality condition $\lambda_1(R) = 0$. By inspection of the differential equation $\dot{\lambda}_1 = \delta_h \lambda_1 - [\lambda_2 + \lambda_3(1-z_l)]\, w.s^w \chi_{[S,R]}$ we see that we must have $\lambda_1(\tau) \geq 0$ for $\tau < R$. If λ_1, became negative it could never recover to zero at the retirement age, because it would require a negative s^w which is impossible. From (52) we can see that the expression for time s^h, when the three time activities are positive, depends on λ_1 in a way that it will have the same sign of λ_1. So, starting in a position where $s^h > 0$, if the consumer wanted to leave OJT, $s^h = 0$ requires $\lambda_1 = 0$. But since λ_1 cannot be negative, if it reaches zero, it will require $s^w = 0$, meaning the consumer can only stop OJT if she stops working simultaneously. But, when this happens, $s^w > 0$ can never happen again, otherwise $\lambda_1(\tau) \geq 0$ would be violated. This means that the consumer will always pursue OJT till a point where she retires, either at R or before, if she chooses to withdraw the labor market. When she withdraws from the labor market, it will be permanent.

4.2 Temporary exits from the labor market

The consumer may choose to leave temporarily the labor market to pursue a specialization in training ($s_l > 0, s_h > 0, s_w = 0$). Following our discussion in the previous section, it remains to see the condition in which she will choose to leave temporarily the labor market. Due to high non-linearities and the great number of free parameters it is not possible to provide unambiguously analytical answers to this question.

We may state, however, the general conditions for temporary entries and exits from the labor market. For $S < \tau < R$, let:

$$Z(\tau) = \left(\frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{(\lambda_2(\tau) + (1-z_l)\lambda_3(\tau))\, wh(\tau)} \right)^{\frac{1}{\epsilon}} + \left(\frac{\lambda_1(\tau)\xi_h \eta \phi_h}{(\lambda_2(\tau) + (1-z_l)\lambda_3(\tau))\, wh(\tau)} \right)^{\frac{1}{1-\phi_h}}$$

Where $Z(\tau)$ is the sum of leisure and OJT ($s^l + s^h$). Suppose we arrive at a given point in time from a regime in which $s_w > 0$, then a regime with $s_w = 0$ is entered if:

$$\begin{aligned} &Z(\tau) = 1, \quad \dot{Z}(\tau) > 0, \qquad \text{or} \\ &Z(\tau) = 1, \quad \dot{Z}(\tau) = 0 \quad \wedge \quad \ddot{Z}(\tau) \geq 0 \end{aligned}$$

On the other hand, suppose we arrive at a given point in time from a regime in which $s_w = 0$, then a regime with $s_w > 0$ is entered if:

$$\begin{aligned} &Z(\tau) = 1, \quad \dot{Z}(\tau) < 0, \qquad \text{or} \\ &Z(\tau) = 1, \quad \dot{Z}(\tau) = 0 \quad \wedge \quad \ddot{Z}(\tau) \leq 0 \end{aligned}$$

If we derive $Z(\tau)$, using the fact that at this particular point, $Z(\tau) = 1$, time allocation is ($s^w = 0, s^l = 1 - s^h$), using the differential equations ((48)-(50)) and the transversality condition for λ_2 in (45), we arrive, after some algebra at:

$$\begin{aligned} \dot{Z}(\tau) = {} & \delta_h \left[2s^h(\tau)\epsilon + (1-s^h(\tau))(1-\phi_h) \right] - \left[s^h(\tau)\epsilon + (1-s^h(\tau))(1-\phi_h) \right] . \\ & . \frac{\xi_h \eta s^h(\tau)^{\phi_h}}{h(\tau)} + \frac{1}{\Psi_1 e^{-\int_\tau^R r(\theta)+m(\theta)d\theta} + (1-z_l)} . \\ & . \left[\left((1-z_l)(r(\tau)-\rho(\tau)) - (\rho(\tau)+m_v(\tau))\Psi_1 e^{-\int_\tau^R r(\theta)+m(\theta)d\theta} \right) . \right. \\ & \left. .(1-s^h(\tau))(1-\phi_h) + s^h(\tau)\epsilon(r(\tau)+m(\tau))(1-z_l) \right] \end{aligned}$$

For a temporary exit from the labor market to occur the expression must have a positive sign, but we cannot determine its sign and is difficult to make conclusions on the parameters influence, because of the feedback effects they will have on variables s^h and h. However, it seems to suggest that if it happens, it is likely to be early in life, because the exponential will have a lower value. A lower η, a higher δ and $r > \rho$ can make it more likely if the feedback effects are not too strong.

4.3 Time allocation profiles

Here, we provide some insights into the time allocation age profiles. Throughout this section, we assume that we are in a life-cycle without temporary exits from the labor market.

We start by considering leisure:

$$s^l(\tau) = \left(\frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{(\lambda_2(\tau) + (1 - z_l)\lambda_3(\tau))\, wh(\tau)} \right)^{1/\epsilon}$$

The minimum of leisure occurs when $\dot{s}^l = 0$, or:

$$\frac{\dot{h}(\tau)}{h(\tau)} = \frac{(r(\tau) + m(\tau))(1 - z_l)}{\Psi_1 e^{-\int_\tau^R r(\theta)+m(\theta)d\theta} + (1 - z_l)} - (\rho(\tau) + m(\tau))$$

For the above result we used the differential equations ((49)-(50)) and the transversality condition for λ_2 in (45).

When the growth rate of human capital is high enough, leisure will be decreasing. When the growth rate of human capital decreases sufficiently, leisure will increase. It is useful to simplify the above expression further.

$$\frac{\dot{h}(\tau)}{h(\tau)} = \frac{(r(\tau) - \rho(\tau))(1 - z_l) - (\rho(\tau) + m(\tau))\Psi_1 e^{-\int_\tau^R r(\theta)+m(\theta)d\theta}}{\Psi_1 e^{-\int_\tau^R r(\theta)+m(\theta)d\theta} + (1 - z_l)}$$

Our analysis will be restricted to the case where ρ and r are constant. For $r \leq \rho$ the turning point of leisure will always involve a decreasing growth rate of human capital, while if $r > \rho$ it is possible to have leisure growth changing sign while human capital has an increasing growth rate.

Regarding the evolution of labor supply, our numerical simulations result in a single peak. In this case, since $s^w = 1 - s^l - s^h$, in the peak we have $\dot{s}^w = -\dot{s}^l - \dot{s}^h = 0$ or $\dot{s}^l = -\dot{s}^h$. If, at that point, $\frac{\dot{h}(\tau)}{h(\tau)}$ is sufficiently small, then leisure will be increasing and time devoted to study must be decreasing. In this case, as leisure is increasing it means that the peak in labor supply will occur after the minimum in leisure.

With a single peak in labor supply, labor income will tend to have also a single peak. As labor income is $h.w.s^w$ the peak will occur when $\frac{\dot{h}(\tau)}{h(\tau)} + \frac{\dot{s}^w(\tau)}{s^w(\tau)} = 0$. In this case, if the growth rate of human capital is negative, which occurs when human capital production is lower than depreciation, the peak in labor income will be consistent with an increase in labor supply, which means the peak in labor income will occur before the peak in labor supply.

Summarizing we have the following results. If $\frac{\dot{h}(\tau)}{h(\tau)}$ is small enough when labor supply peaks, this peak happens after the minimum in leisure. Furthermore, if this growth rate of human capital is negative when the peak in labor income occurs, this peak occurs before the peak in labor supply. A pattern we will see often, as in Table 2 is the minimum in leisure being followed by the peak in labor income and afterward by the peak in labor supply.

5 Numerical simulation

We now present numerical simulations for the consumer's life-cycle model. In Subsection 5.2 we simulate various life-cycles with different parameter sets. Our aim is just to illustrate how these parameters influence the life-cycle planning of the consumer. In Subsection 5.3 we analyze how planned life-cycles change in response to a different mortality law, specifically one associated with a higher life expectancy.

As the model has many free parameters, the type of analysis performed in this section does not aim to provide a definitive answer on the typical life-cycle of a consumer and its variations. A more robust analysis would require a much more intensive numerical exercise covering a wider range in the domain of the parameters set.

5.1 Calibration

As Portugal is a small open economy undergoing a process of population aging, the choice of some parameters will rely on the Portuguese economy.

In Table 1 we present various sets of parameters related to our baseline and alternative scenarios. The choice of parameters for this scenario was based on a mix of values obtained from data, from existing literature and also with a view for the aggregate economy to display the desired levels for some key ratios.

The interest rate r was set to 3%. This value is in line with the average long-term real interest for Portugal since 1993 (10-year government bonds) extracted from the AMECO database, which is the annual macroeconomic database of the European Commission's Directorate General for Economic and Financial Affairs. We use a constant value of 0.02 for the subjective time preference which is a value commonly used in the economic literature.

The parameter σ is the inverse of the intertemporal elasticity of consumption. We set $\sigma = 2$ which is also commonly used in the literature. The parameter ϵ determines the labor supply elasticity, so its value will depend on the level of labor supply elasticity intended. The literature, however, provides estimates of the Frisch elasticity that differ a lot, depending on whether they are obtained from micro or macro data. Micro estimates tend to be much lower than macro estimates. Chetty et al. (2011) and Chetty (2012) tackle this issue and make the case for Frisch elasticities to be low, more in accordance to the values usually provided by micro estimates.[9] The former study recommends calibrating the models in order to obtain Frisch elasticities at the intensive margin of 0.5. We set $\epsilon = 4$, resulting on an average Frisch elasticity for employed cohorts of 0.437.

Regarding the human capital elasticity ϕ_h, Browning et al. (1999) (Table 2.3) report values ranging from 0.5 to 1. Heckman et al. (1998), for a human capital model including OJT, estimate values for this elasticity of 0.945 and 0.939, depending on the level of education. Hansen and Imrohoroğlu (2008), in order to replicate some of the estimates of Heckman et al. (1998) regarding lifetime time allocation between OJT and labor supply, set the elasticity at

[9] Imai and Keane (2004) find that labor elasticity is higher once human capital accumulation is endogenized, but they use a model with learning by doing.

0.001. In our model ϕ_h represents elasticity not only for schooling but also to OJT. Taking into consideration that schooling represents a major part of the human capital accumulated and the low estimate in Hansen and Imrohoroğlu (2008) we decided to set it at 0.5, the lower bound of the estimates surveyed by Browning et al. (1999).

We introduced a human capital externality to capture the idea that more developed economies will probably have better institutions, namely teaching institutions. The literature on human capital externalities is reviewed in Moretti (2004b) which reports some mixed evidence. We follow here his exposition for models based on wage and land prices. Some studies, as Acemoglu and Angrist (2001) and Ciccone and Peri (2002), do not find statistically significant externalities. Rauch (1993) finds externalities ranging from 3% to 5% but does not take into consideration endogeneity and complementarity issues. Moretti (2004a) tackles the issue of the presence of unobservable characteristics of individuals and cities and finds externalities ranging from 0.4% to 1.9% depending on the education level. We decided to set ϕ_η at 1%, within the range of these last estimates.

Regarding δ_h, Arrazola and de Hevia (2004) estimate the depreciation rate of human capital taking into consideration OJT for a sample of the Spanish population and find depreciation rates around 1% to 1.5%. Weber (2014) applies the same model as in Arrazola and de Hevia for Switzerland, with a further disaggregation by education level, and finds depreciation rates of 0.6%-0.7% for general education and slightly higher for specific education (0.9%-1%). Taking these studies in consideration we chose a value of 1% for human capital depreciation.

We use data from Portugal for the fiscal parameters. Our parameter z_l represents the wedge between the costs of employment to a firm and the net labor income received by the worker. It includes labor tax, contributions to social security by the worker and also by the firm. We set $z_l = 0.1223 + 0.11 + 0.2375 = 0.4698$, with 0.1223 being the approximate effective income tax, 0.11 the contributions of employees to social security and 0.2375 the contribution of employers to social security.[10] We set $z_p = 0.1223$, the same effective income tax used in z_l.

Our measure of the gross replacement rate needs an adjustment to be comparable to what is obtained in the data, as gross labor income received by the worker in a real economy would be related to the one in this model by the term $(1 - 0.2375)$. We take the gross replacement rate value of 0.74 for Portugal from the OECD (2017) and set $\theta = 0.74 * (1 - 0.2375) = 0.56425$.

Mandatory schooling in Portugal has been raised to 12 years of schooling which students will conclude with 18 years old in case of no grade retention ($S = 18$). Regarding the minimum eligible retirement age for Portugal, a benchmark value in the private sector was, for many years, 65 years. The retirement age has been rising due to the introduction of a sustainable factor that has the effect of linking it to the evolution of life expectancy. It sits currently at 66 years and a few months. We decided to use the value $R = 66$.

We are still left with several parameters to determine $(\varphi, \xi_h, \gamma_e, z_0)$ as well as the wage rate and the human capital externality. We made the calibration exercise for a general equilibrium

[10]The effective income tax rate was computed from the statistics of Autoridade Tributária e Aduaneira for the year 2015. The rates for social security contributions are the statutory ones.

model in which we tried to obtain the economy's ratios consistent to those of a real economy and a specific goal for the government's deficit (below 3%). We also wanted to achieve a concave profile for average labor supply and a population's plausible average labor supply around $1/3$ of total time. The full calibration exercise is in Pereira (2018) and we arrived at the values $\varphi = 0.8$, $\xi_h = 0.0381$, $\gamma_e = 0.288$, $z_0 = 0.0485$, $\eta \approx 0.9926$[11] and $w = 4.5$. With these values we achieve the economy's ratios over GDP of 62.3% for consumption, 19.1% for investment, and -2.1% for the government's deficit as well as an average labor supply of 34.7% of total time, slightly higher than our initial goal.

Table 1 presents the result of the calibration for the base scenario and some alternative parameter values that will be tested.

Table 1: Parameter sets

	Base	Alternative tested values	
σ	2		
φ	0.8		
ϵ	4		
r	0.03	0.025	
ρ	0.02		
ξ_h	0.0381	0.0419	
η	0.9926		
ϕ_h	0.5		
ϕ_η	0.01		
δ_h	0.01	0.0	
w	4.5	6	
S	18	21	
R	66	68	
γ_e	0.288	0.317	
z_0	0.0485		
z_l	0.4698	0.5198	0.3598
z_p	0.1223		
θ	0.56425	0.488	0

[11] We use the symbol $\approx$ because the numerical program runs with many more decimal points.

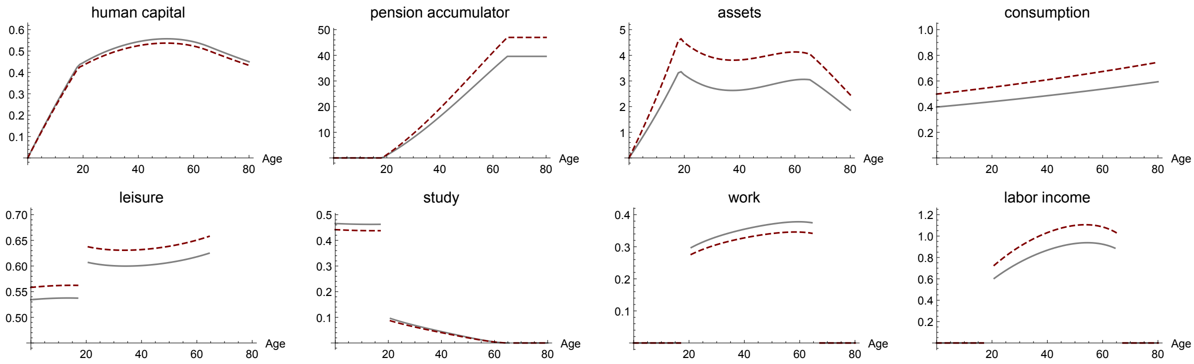


Figure 1: Base scenario and scenario with a higher wage rate ($w = 6.0$)

5.2 Analysis of life-cycles under different parameterizations

In Table 2 we present a set of indicators for selected scenarios which will guide us through the following discussion. The base scenario is characterized by no temporary exits from the labor market, with the consumer starting to work immediately after the mandatory age of compulsory schooling, (S) and continuing till the legal retirement age. This scenario is shown in Figure 1 (grey plot). The life-cycle displays hump-shaped patterns in labor supply and labor income and convexly shaped profile in leisure which are in accordance with empirical findings. These patterns are pervasive to all scenarios we tested. In phase 2 of the life-cycle, the peaks in human capital, labor income, and labor supply occur by this order and the minimum of leisure occurs before the maximum in labor income. This sequence also occurs in all scenarios we tested and it matches our discussion in Subsection 4.3. In the base scenario, minimum leisure occurs at age 35, the peak in human capital at age 51, the peak in labor income at age 55 and the peak in labor supply at age 60.

Comparing our results with the "normal" case in Blinder and Weiss (1976) we have a similar age profile of leisure and a single peak in labor supply. However the sequence of the peaks is different. In their paper, the peak in labor supply occurs before the one in human capital. For the case of no depreciation, they find that human capital peaks after labor income and we obtain the same result when we tested $\delta_h = 0$.

Consumption is monotonous, not replicating empirical findings that show that lifetime consumption is hump-shaped. This is a consequence of our choice to work with an additively separable utility function in consumption and leisure.

We start by considering a scenario that we would see materializing as the economy develops, which is a higher wage rate. This scenario is presented in Figure 1 (dashed red), in which a wage rate of 6 was considered, an increase of 33.3% when compared to the base scenario. As expected, labor income is higher, lifetime consumption is higher and so is lifetime utility. Assets also increase. More interesting is to notice that when the wage rate increases, people will substitute labor supply and time dedicated to study for leisure which will result in a lower human capital. This partial equilibrium result goes in the same direction as the general equi-

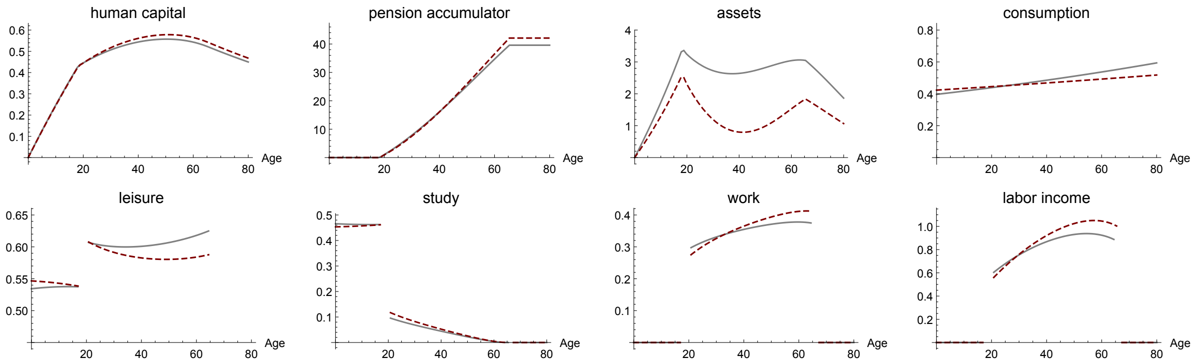


Figure 2: Base scenario and scenario with a lower interest rate ($r = 0.025$)

librium result of Vogel et al. (2017).[12] This substitution is mild compared to the increase in the wage rate, not preventing the increase in lifetime consumption. Peaks in human capital, labor supply, labor income and the trough in leisure all occur earlier than before.

In Figure 2 we show the effect of a reduction in the interest rate, from 0.03 to 0.025. Since it will still remain above the subjective time preference, consumption still rises during the lifetime, although at a slower rate. However, since the consumer had a positive net assets balance, their return will be lower. That is why total lifetime consumption will be lower. Regarding time allocation, the consumer substitutes leisure for time allocated to study and labor. A reduction in this interest rate can have a positive effect on GDP growth as labor supply and human capital increase. There is a higher investment in OJT at the start of the work-life, resulting in an initial lower labor supply that is recovered later in the life-cycle. This means that labor supply is postponed and study and leisure are anticipated, in comparison to the base scenario. This can be seen in Table 2. Labor supply and labor income have their peak 3 years later and the trough in leisure occurs at age 49.

Another area with interest is related to the human capital parameters. The higher the human capital externality η and the productivity parameter ξ_h, the more productive is the time dedicated to studying. Also, the lower the depreciation rate of human capital the longer knowledge will stick. So we tested scenarios in which ξ_h is higher (Figure 3) and δ_h is lower (Figure 4).[13]

In both cases, human capital increases throughout the life-cycle, resulting in higher labor income and higher lifetime consumption. However, an increase in the productivity parameter ξ_h by 10% has little effect on the time allocation profiles (there is a slight trade-off of schooling for OJT) while the decrease in δ_h has some similar effects to a decrease in the interest rate in the sense that it anticipates leisure and postpones labor supply. A reduction in the depreciation rate decreases time dedicated to schooling with little effect in OJT. The consumer allocates less time to accumulating human capital because now it does not depreciate and she

[12] Ludwig et al. (2012) report an increase in the average labor supply as the economy progresses but that result is affected by changes in the age composition of the population.

[13] Since, in partial equilibrium, the effects of an increase in ξ_h are the same as for an increase in η, we will not run a scenario for the latter case.

Table 2: Selected scenarios indicators

	Base	$w = 6$	$\xi = 0.0419$	$\delta_h = 0$	$S = 21$	$r = 0.025$
Lifetime Utility	-139.7	-115.2	-135.6	-129.2	-135.2	-141.4
Age entry lab mkt.	19	19	19	19	22	19
Retirement age	66	66	66	66	66	66
Yrs in lab mkt.	47	47	47	47	44	47
Age at Peak h	51	50	51	66	49	51
Age at Peak s^w	60	59	60	65	60	63
Age at Peak ω	55	54	55	65	54	58
Age at Min s^l	35	34	35	53	31	49
Av. s^l phase2	0.606	0.637	0.604	0.600	0.611	0.587
Av. s^h phase 2	0.047	0.043	0.048	0.047	0.040	0.056
Av. s^w phase 2	0.347	0.319	0.348	0.353	0.349	0.357
Av. h phase 2	0.524	0.506	0.575	0.678	0.549	0.539
π at R	39.59	46.97	43.65	52.94	39.06	42.10
a at R	3.05	4.04	2.78	2.00	3.06	1.84
	Base	**$z_l = 0.5198$**	**$\theta = 0.488$**	**$R = 68$**	**$\theta = 0$**	**$\theta = 0, z_l = 0.3598$**
Lifetime Utility	-139.7	-143.4	-140.3	-140.0	-144.3	-136.36
Age entry lab mkt.	19	19	19	19	19	19
Retirement age	66	66	66	68	66	66
Yrs in lab mkt.	47	47	47	49	47	47
Age at Peak h	51	51	51	52	50	50
Age at Peak s^w	60	60	59	61	53	53
Age at Peak ω	55	55	55	56	51	52
Age at Min s^l	35	35	34	34	28	29
Av. s^l phase2	0.606	0.608	0.607	0.609	0.614	0.608
Av. s^h phase 2	0.047	0.047	0.047	0.047	0.043	0.044
Av. s^w phase 2	0.347	0.345	0.346	0.344	0.343	0.347
Av. h phase 2	0.524	0.526	0.523	0.527	0.518	0.516
π at R	39.59	39.47	39.49	41.17	38.67	39.00
a at R	3.05	2.79	3.78	2.90	8.37	8.97

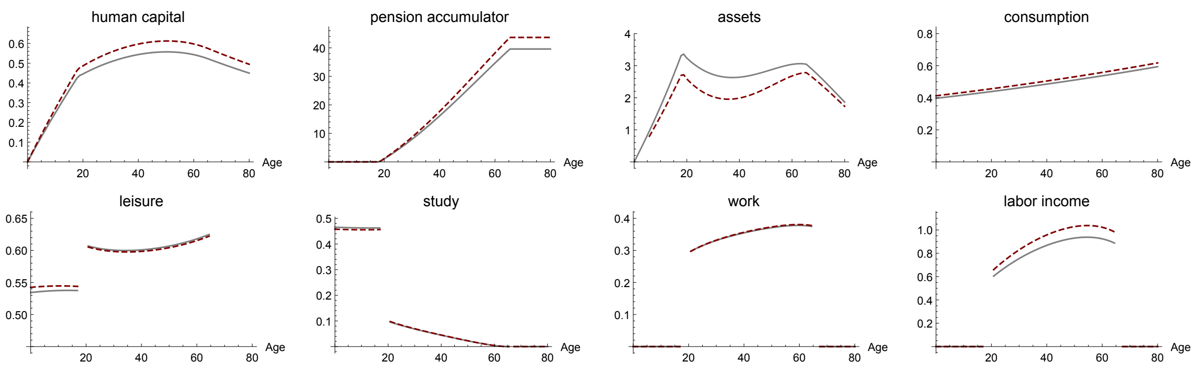


Figure 3: Base scenario and scenario with higher productivity in human capital ($\xi_h = 0.0419$)

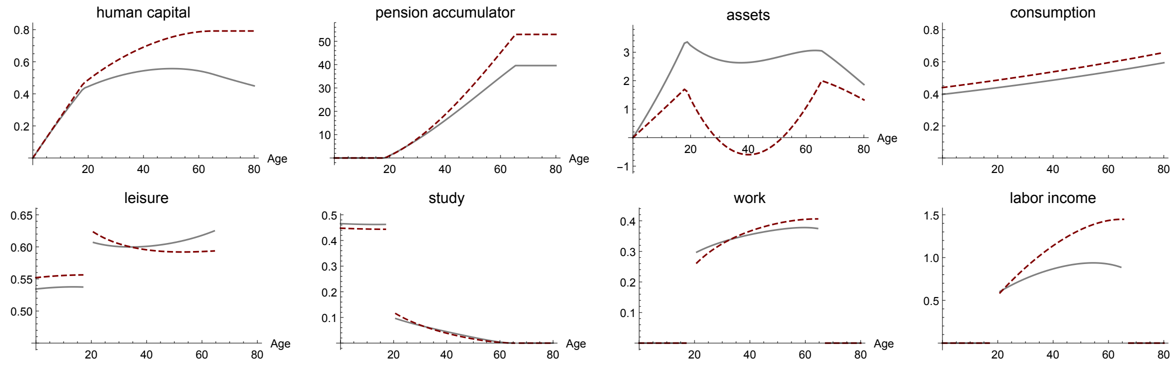


Figure 4: Base scenario and scenario without human capital depreciation ($\delta_h = 0$)

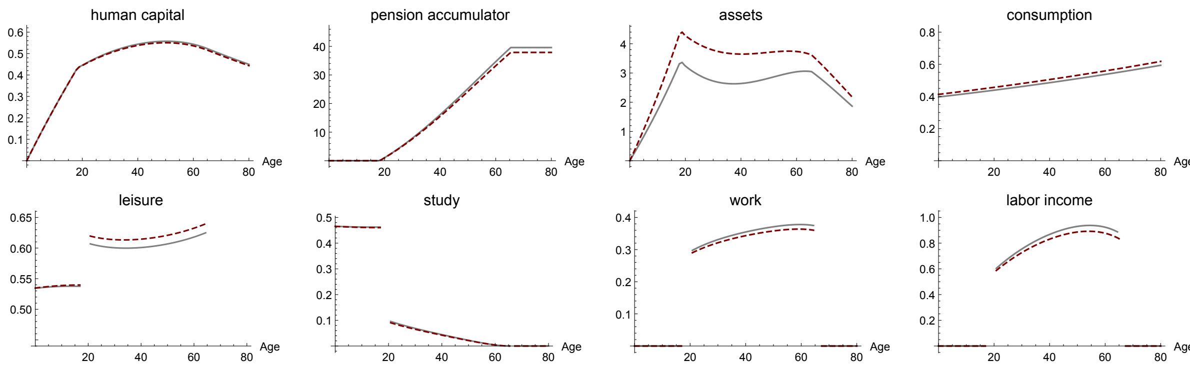


Figure 5: Base scenario and scenario with a higher educational subsidy ($\gamma_e = 0.317$)

does not need to compensate for the depreciation.

There are two more situations that we can consider to impact human capital accumulation, such as an increase in the years of mandatory schooling which translates to an increase in S or a more generous educational grant. Unlike the previous analysis, these changes do not affect the productivity of learning, they are exogenous incentives targeted to raise human capital.

We tested an increase in the educational subsidy of 10%, setting $\gamma_e = 0.317$, which is shown in Figure 5 and an increase in the mandatory schooling of 3 years, which students will complete ate age 21 as we assume no grade retention, shown in Figure 6.

With this 10% increase in the educational subsidy, the consumer accumulates slightly less human capital, as time allocated to learning, either for schooling or OJT decreases slightly. This is a result of the income effect being stronger than the substitution effect. Human capital is used to produce labor income in order to fund consumption. The increase in the educational subsidy would make the consumer willing to substitute leisure for studying in phase 1 of the life-cycle, but at the same time, it has a strong impact on income which makes the increase of lifetime consumption possible with slightly less human capital. The consumer takes advantage of this by reducing the time dedicated to study and labor in favor of leisure. In Figure 5, we see that the consumer is better off as she can increase simultaneously consumption and leisure. Hence, this policy measure, although having strong welfare effects fails as an incentive

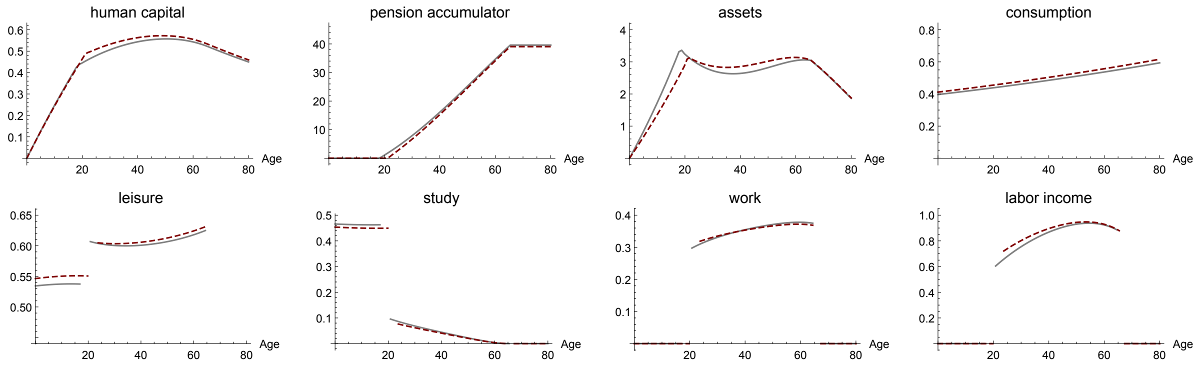


Figure 6: Base scenario and scenario with more mandatory schooling ($S = 21$)

to human capital accumulation as human capital supplied to market decreases by around 1%.

An exogenous increase in the years of mandatory schooling has a similar effect of reducing time allocated to study at any age, except the ages at which schooling was increased. Since there is a big difference between time allocated to learning in schooling and in OJT, these 3 extra years give a boost to human capital at the initial of the work life, that will tend to fade as the consumer approaches retirement. Still, with this parametrization, an increase in mandatory schooling increases average human capital supplied to the market by 4.7%. Also, contrary to the case of an increase in the educational grant, labor supply increases.

Summing up the analysis of the human capital parameters, a higher ξ_h (or η), a lower δ_h and an increase in S have all the same effect of raising human capital and labor supply, and if this holds in general equilibrium, are growth enhancing, while an increase in γ_e has the opposite effect. Since, in this model, only S and γ_e are policy instruments, the government can promote human capital accumulation by increasing S and decreasing γ_e.

We now turn to scenarios related to the case where the government has a problem of sustainability of the pension system due to an adverse demography and simulate some policy measures that can be used. The goal, at this point, is just to observe the effects on people's life-cycles. We consider four base policies that the government can use to keep public debt under control: an increase in the contribution rate, a decrease in the gross replacement rate, an increase in the retirement age and a decrease in educational subsidy. This last case has the opposite effect of the increase in γ_e, which was already analyzed above, so we focus on the remaining three.

We do an exercise for $z_l = 0.5198$, five percentage points higher than the base scenario and present this case in Figure 7. Although labor supply is reduced slightly, it does not have much impact on the time allocation profiles. When we tested a decrease of the gross replacement rate to 0.488, the more significant change in time allocation is a substitution of OJT for schooling. In this scenario, which is presented in Figure 8, the consumer saves more during the pre-retirement period in order to finance consumption in retirement, because now the pension is less generous. This effect can raise substantially domestic savings with a strong impact on foreign debt moving the economy towards a net creditor position.

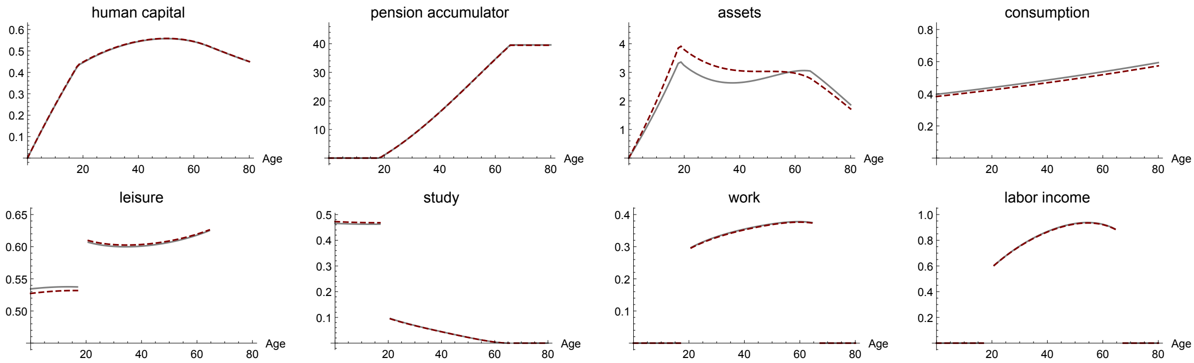


Figure 7: Base scenario and scenario with higher labor income taxes ($z_l = 0.5198$)

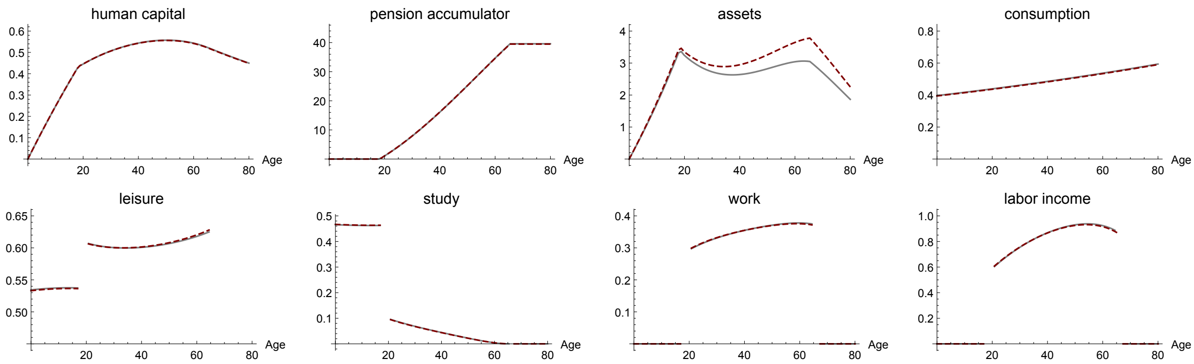


Figure 8: Base scenario and scenario with a cut in pensions ($\theta = 0.488$)

An increase in the retirement age to 68 years (Figure 9), slightly decreases labor supply at the intensive margin but this is more than compensated by the increase at the extensive margin. It also promotes human capital accumulation by an increase in OJT, although this is not clear in Table 2 as the average is computed for a longer period, in which OJT is slowing down. The combination of a slightly higher human capital and a slightly lower labor supply at the intensive margin makes taxable gross labor income not very different in both scenarios till age 66, but the government may collect two extra years of taxes and will save two years in pensions. Also, because the pension is a function of average labor income, it will be slightly lower as labor income in the extra two years is on a descending trajectory. These three effects combined suggest that this measure could be particularly effective in tackling social security sustainability problems.

Finally, we want to consider the case where a pure capitalization pension system is in place, which is shown in Figure 10. We assume that this is a voluntary system where workers would contribute the same amount as what they would save in case no pension system existed. We set $\theta = 0.0$ and we reduce labor income taxes ($z_l = 0.3598$) reducing z_l by the fraction that we considered to be the contribution of the worker (11%). We do not consider a further reduction of 23.75%, the contribution of the employer, because this will not accrue to the net income of the worker.

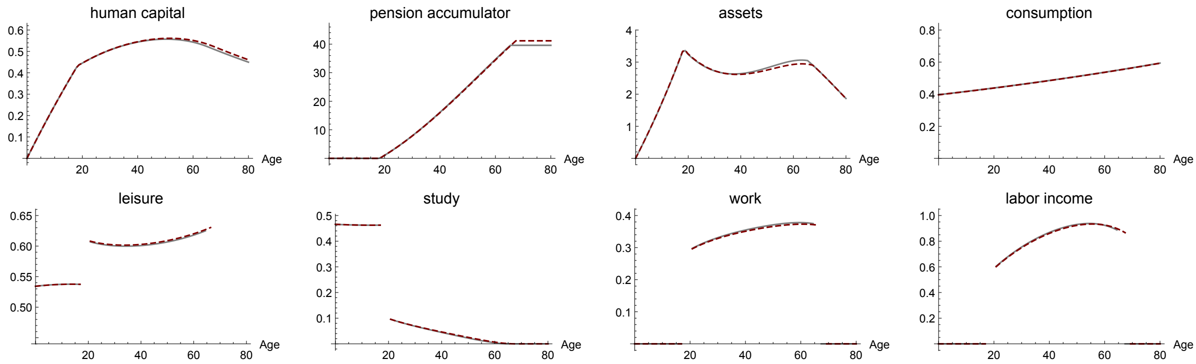


Figure 9: Base scenario and scenario with an increase in the retirement age ($R = 68$)

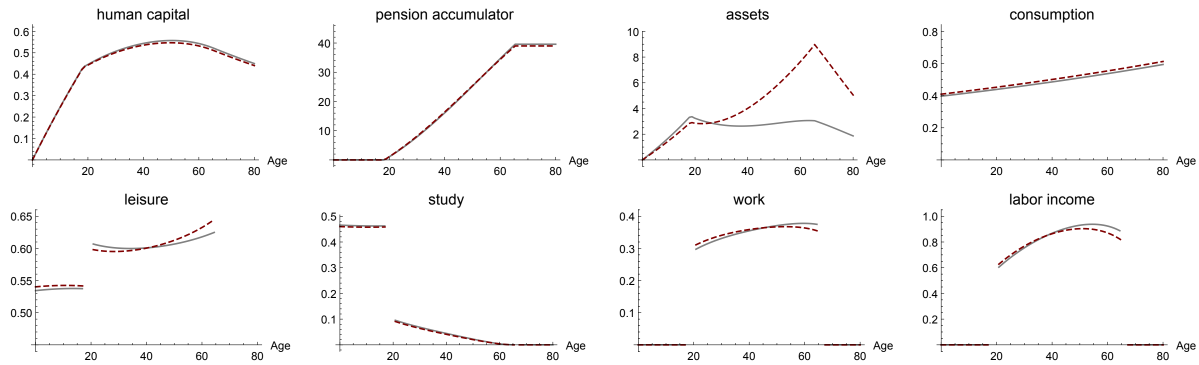


Figure 10: Base scenario and scenario with a pure capitalization pension system ($\theta = 0, z_l = 0.3598$)

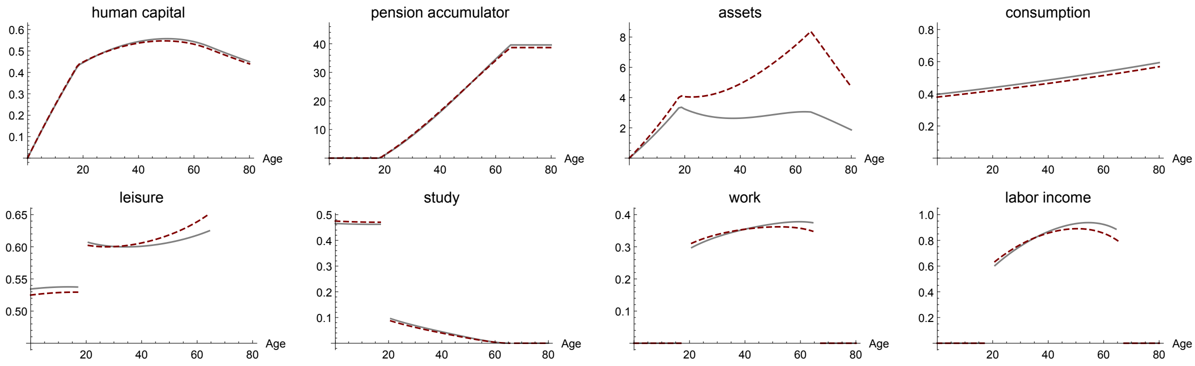


Figure 11: Base scenario and scenario with the transition from a PAYG pension system to a capitalization pension system ($\theta = 0$)

With this pension system, naturally, savings will be higher. Gross labor income will be lower because, faced with fewer taxes, this income effect will translate into a lower time allocated to study (with a higher impact on schooling). Average human capital supplied to the market will be lower (0.516 versus 0.524). Nevertheless, since net labor income is higher, the consumer accumulates enough assets for retirement and will enjoy higher lifetime consumption. Average labor supply is virtually the same although there is a transference of work effort from older to younger ages.

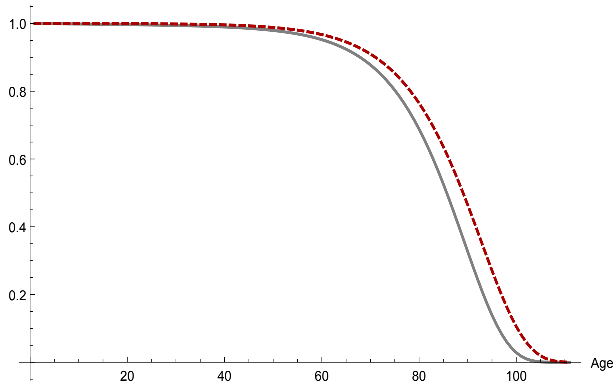
We also analyzed the case of a fully funded pension but without the reduction in z_l. This scenario, presented in Figure 11, could be a proxy for the transition phase from a PAYG pension system to a fully funded one. Comparing with the previous scenario, the decrease in human capital is now due solely to OJT as there is no decrease in schooling. If we compare with the base scenario, there is a trade-off between OJT to schooling. Another difference is that occurs a decrease in labor supply. This transition phase reduces the consumer lifetime utility because of a strong reduction in consumption that is not sufficiently compensated by the increase in leisure.

5.3 Analysis of life-cycles under mortality changes

We now study how a mortality law that generates a higher life expectancy affects the consumer plans. We will not test all scenarios of the previous section. Instead, we apply it to the base scenario and check the robustness of the results with a couple of other scenarios. These are Gompertz-Makeham mortality functions based on data and projections for Portugal. Figure 12 compares the survival laws produced by both mortality functions. The mortality function used in the previous subsection is labeled as the mortality function for the year 2020 and now we use the mortality function estimated for Portugal for 2080.[14]. With these survival laws, life expectancy at birth increases from 82.1 in 2020 to 85.7 in 2080, while life expectancy at 65 increases from 18.1 to 21.3. There is an increase of 4.4% on life expectancy at birth and an

[14]We computed a mortality function with data from the Human Mortality Database for the year 2015 and a mortality function for the year 2080 based on Eurostat's population projections. The mortality function for 2020 was computed by a linear interpolation from the mortality functions for 2015 and 2080.

increase of 17.7% of life expectancy at 65.



Year 2020 - gray line. Year 2080 - dashed red line

Figure 12: Survival laws

In Figure 13 we present the results of the consumer optimization problem for the base scenario under both mortality laws.[15] We observe that average labor supply and average time spent in OJT increase slightly. This is accompanied by an increase in time dedicated to schooling. As a result, average human capital and labor supply will be higher resulting in higher labor income. We can identify, in this partial equilibrium analysis, a positive behavioural effect associated with an increase in life expectancy. The consumer will invest more in human capital accumulation and increase her labor supply at the intensive margin.

Although our model features time allocation at the intensive margin, we can make an exercise to test if the consumer would be willing to increase labor supply at the extensive margin, taking advantage of the fact that retirement is partially endogenous. We tested this for scenarios in which the consumer chose to withdraw from the labor market before the retirement age R. We created two such scenarios. One of them, (Figure 14), is a case of a consumer with

[15] Throughout this section the gray line will always refer to the scenario with the 2020 mortality law and the dashed red line to the scenario with the 2080 mortality law

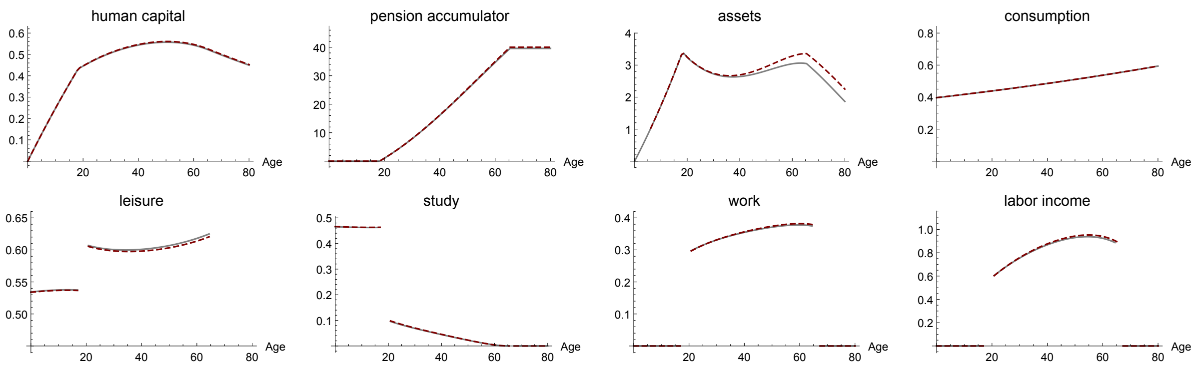


Figure 13: Base scenario under different mortality laws

a very high preference for leisure. We tested for $\varphi = 30$. In this case, with the 2020 mortality law, the consumer would want to retire at age 62. Another case, shown in Figure 15 is a scenario in which the educational subsidy is very generous (we set it at $\gamma_e = 1.01$), allowing the consumer to plan to retire at age 64.

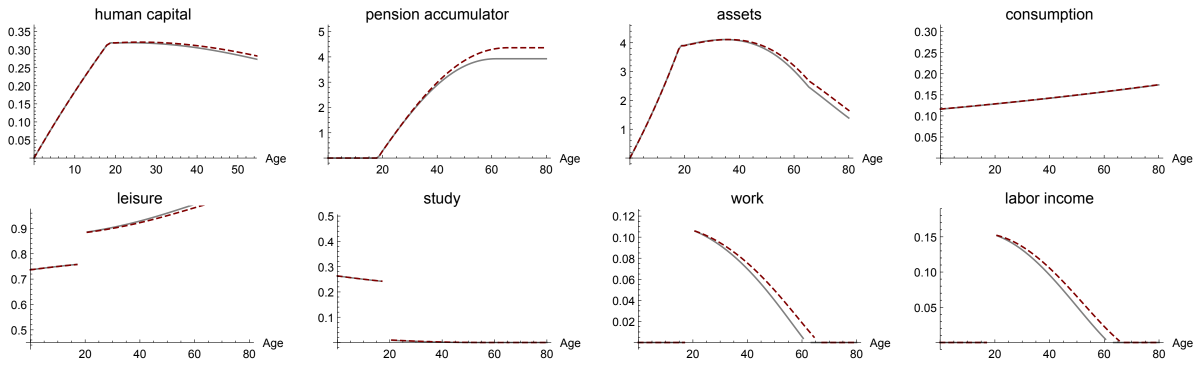


Figure 14: Higher preference for leisure under different mortality laws

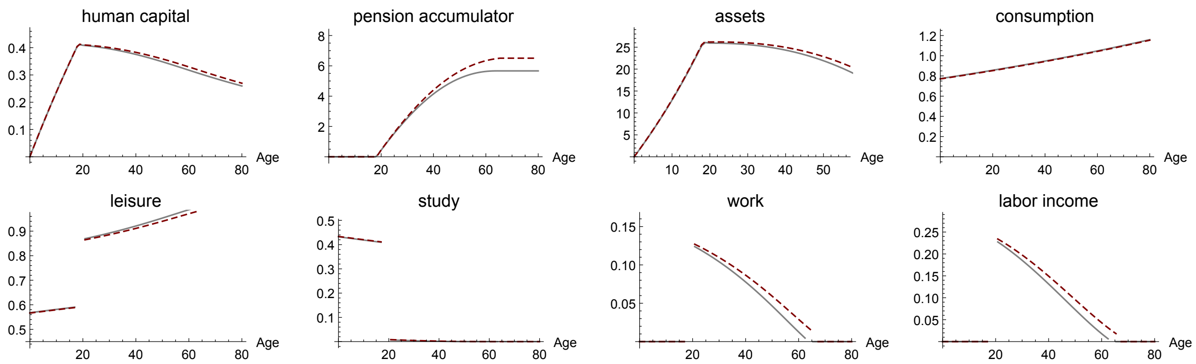


Figure 15: Generous educational grant under different mortality laws

In both cases, the consumer decides to retire later, retiring at the legal retirement age of 66. By inspection of Table 3 we see the same pattern observed in the base scenario; an increase in human capital investment and labor supply. This confirms the positive behavioural effect on a longer life expectancy.

Table 3 also presents indicators for the case of a capitalization pension system with lower taxes. We find qualitatively similar results as in the base scenario with a PAYG system, although the positive behavioural effect is milder, this may be just a result of the particular parameters used.

This section presented some results on the effects that a lower mortality rate may induce in the consumer's planing. Our emphasis was on the effects of a lower mortality on labor supply and on human capital accumulation. The effect on human capital is particularly important since in many growth models it has a pivotal role in ensuring long-run growth. If it is shown that when people live longer they will invest more in human capital, this higher investment

Table 3: Scenarios under different mortality laws

	Base		$\theta = 0.0, z_l = 0.3598$		$\varphi = 30$		$\gamma_e = 1.01$	
	2020	2080	2020	2080	2020	2080	2020	2080
Lifetime Utility	-139.7	-141.8	-136.4	-139.0	-966.1	-982.7	-74.0	-75.44
Age entry lab mkt.	19	19	19	19	19	19	19	19
Retirement age	66	66	66	66	62	66	64	66
Yrs in lab mkt.	47	47	47	47	43	47	45	47
Age at Peak h	51	51	50	50	24	26	19	19
Age at Peak s^w	60	61	53	54	19	19	19	19
Age at Peak ω	55	55	52	52	19	19	19	19
Age at Min s^l	35	36	29	29	19	19	19	19
Av. s^l phase2	0.606	0.603	0.608	0.606	0.931	0.930	0.923	0.916
Av. s^h phase 2	0.047	0.048	0.044	0.045	0.005	0.006	0.007	0.007
Av. s^w phase 2	0.347	0.348	0.347	0.349	0.064	0.065	0.071	0.077
Av. h phase 2	0.524	0.527	0.516	0.518	0.299	0.301	0.369	0.373
π at R	39.59	39.99	39.00	39.32	3.93	4.36	5.67	6.50
a at R	3.05	3.36	8.97	9.93	2.89	2.68	16.55	17.11

may mitigate the potential adverse effects of high dependency ratios. In our model, this increase in human capital would be even more beneficial since it increases the average human capital externality and thereby increases future human capital accumulation. In all the cases we tested human capital increased.

The model also suggests that, if the economy has a PAYG pension system, people will want to work more, either at the intensive margin or at the extensive margin.

6 Conclusions

We built a life-cycle model for a consumer that has to choose endogenously the trajectory of consumption and the time allocation among three margins while facing an age-dependent mortality. In particular, time investing in home capital is decided throughout the entire life-cycle.

Our main analytical findings are that there is on the job training throughout all of phase two of the life-cycle. There is no phase characterized by pure work, but it is possible to obtain a life-cycle with temporary exits from the labor market and pure learning. For a sufficiently decreasing human capital accumulation and a life-cycle without temporary exits from the labor market, the peak in labor income occurs after the minimum in leisure and if the growth rate of human capital is negative when the peak in labor income happens then it will happen before the peak in labor supply.

On the numerical exercise, we found for the parameterizations that were tested, that the model displays a hump-shaped profile in labor supply and labor income and a convexly shaped profile in leisure, while consumptions is monotonous. We found that when the interest rate decreases, the consumer substitutes leisure for human capital investment and labor supply. As the economy develops one should see an increase in the wage rate. A higher wage rate leads to less labor supply and lower human capital as people will reap the benefits of a higher wage rate by enjoying more leisure time.

An increase in mandatory schooling is effective in raising human capital and labor supply, while an increase in the educational subsidy has the opposite effect as there is a strong income substitution effect associated with this instrument. We tested three policies targeted to the sustainability of the pension system: an increase in the contribution rate a decrease in the gross replacement rate and an increase in the retirement age. All of them have negative effects on labor supply at the intensive margin. Decreasing the gross replacement rate has also a negative effect on human capital accumulation. The model suggests that a particularly effective policy is to increase the retirement age. It increases labor supply at the intensive margin and average human capital, generating higher taxable labor income, it postpones the payment of pensions and the resulting pension is slightly lower.

A capitalization pension system was also considered. We have seen that this pension system if accompanied by a reduction in contributions, raises accumulated assets and consumption, but human capital will be lower. However, in the transition phase between both systems, where one generation pays a contribution rate and does not receive a pension, that generation would substitute consumption for leisure, supplying less labor supply and investing less in human capital which creates additional problems on the ability to raise revenues to fund the transition phase.

Regarding the consumer's reaction to a higher longevity, we have seen that the labor supply increases at the intensive margin and if the consumer is not retiring at the rigidity R she will plan to retire later. The finding that people would be willing to work more at the extensive margin is an argument in favor of increases in the retirement age towards the sustainability of the pension system.

The consumer also invests more in human capital when faced with a longer life expectancy. Hence, our partial equilibrium analysis reveals a positive behavioral effect associated with aging that could counteract, at least partially, the negative accounting effect arising from an unfavorable demography.

Appendix

A Proof of Proposition 3

Proof.
The proof is divided in subsections.

I) Adjoint differential equations

Consider the Hamiltonian function:

$$\begin{aligned}
H =&\lambda_0\left(\frac{c^{1-\sigma}}{1-\sigma}+\varphi\frac{s^{l\ 1-\epsilon}}{1-\epsilon}\right)e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}+ \\
&+\lambda_1\left(\xi_h.\eta.(s^h)^{\phi_h}-\delta_h.h\right) \\
&+\lambda_2.h.s^w.w\left(\tau\right).\chi_{[S,R]}(\tau)+ \\
&+\lambda_3\Big([r+m(\tau)]\,a-c-z_0+d \\
&\qquad+\gamma_e.w.s^h\chi_{[\nu,S]}(\tau)+ \\
&\qquad+(1-z_l)w.h.s^w\chi_{[S,R]}(\tau)\Big)
\end{aligned} \tag{47}$$

The adjoint differential equations are:

$$\dot{\lambda}_1=-\frac{\partial H}{\partial h}=\delta_h\lambda_1-[\lambda_2+\lambda_3\left(1-z_l\right)]\,ws^w\chi_{[S,R]}; \tag{48}$$

$$\dot{\lambda}_2=-\frac{\partial H}{\partial \pi}=0 \tag{49}$$

$$\dot{\lambda}_3=-\frac{\partial H}{\partial a}=-\left[r+m(\tau)\right]\lambda_3; \tag{50}$$

II) $\lambda_0 = 1$

For $\lambda_0=0$, a $\lambda_3(R)>0$ would imply that there $\exists\ t_1<R:c(\tau)=0\quad\forall\tau\in[t_1,R]$ which is a non-optimal solution, a $\lambda_3(T)<0$ would imply that there$\exists\ t_1<R:c(\tau)=+\infty\quad\forall\tau\in[t_1,R]$ which is a not a maximum. Then: $\lambda_0=0$ implies $\lambda_3(R)=0$ and consequently $\Rightarrow\lambda_2(R)=0$. Since $\lambda_1(R)=0$ this would violate Pontryagin's Maximum Principle. Then, by contradiction, we must have $\lambda_0=1$.

III) Feedback controls

The expression for consumption is the same regardless of the phase:

$$\frac{\partial H}{\partial c}=0\Leftrightarrow c^{-\sigma}e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}-\lambda_3=0\Leftrightarrow c=\left(\lambda_3e^{\int_t^\tau \rho(\theta)+m(\theta)d\theta}\right)^{-1/\sigma} \tag{51}$$

The time allocation controls depend on the phase of the agent's life-cycle and on the sign of λ_1.

III a) $\tau \in [S, R]$

Case $\lambda_1 = 0$

$\lambda_1(\tau) = 0 \Rightarrow \dot{\lambda}_1(\theta) \leq 0$ a.e. $\theta \in [\tau, R] \Rightarrow s^w(\theta) \leq 0$ a.e $\theta \in [\tau, R] \Rightarrow$ behaviour in $\tau = R$.

Case $\lambda_1 > 0$

The problem for this period is to maximize the Hamiltonian,

$$H = \left(\frac{c^{1-\sigma}}{1-\sigma} + \varphi \frac{s^{l1-\epsilon}}{1-\epsilon}\right) e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} + \lambda_1 \Big(\xi_h.\eta.(s^h)^{\phi_h} - \delta_h.h\Big) +$$
$$+ \lambda_2.h.(1-s^l-s^h).w + \lambda_3\Big(\,[r+m(\tau)]\,a - c - z_0 + d + (1-z_l)w.h.(1-s^l-s^h)\Big)$$

subject to $s^l + s^h \leq 1$ and $s^l > 0$, $s^h > 0$.

Let μ be the multiplier attached to s^w and ζ be the multiplier attached to the constraint $s^l + s^h \leq 1$, then our first order conditions return:

$$\begin{aligned}
&\varphi(s^l)^{-\epsilon} e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} - (\lambda_2 + (1-z_l)\lambda_3)\, wh + \zeta = 0\\
&\lambda_1 \xi_h \eta (s^h)^{\phi_h - 1}\phi_h - (\lambda_2 + (1-z_l)\lambda_3)\, wh + \zeta = 0\\
&\zeta\mu = 0; \quad \zeta \geq 0; \quad \mu \geq 0\\
&s^l + s^h + \mu^2 \leq 1
\end{aligned}$$

We have then:

$$\begin{aligned}
&\text{For } \zeta = 0 && \text{For } \mu = 0\\
&s^l = \left(\frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{(\lambda_2 + (1-z_l)\lambda_3)\, wh}\right)^{1/\epsilon} && s^l : \frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{(s^l)^\epsilon} = \frac{\lambda_1 \xi_h \eta \phi_h}{(1-s^l)^{1-\phi_h}}\\
&s^h = \left(\frac{\lambda_1 \xi_h \eta \phi_h}{(\lambda_2 + (1-z_l)\lambda_3)\, wh}\right)^{1/1-\phi_h} && s^h = 1 - s^l\\
&s^w = 1 - s^l - s^h && s^w = 0
\end{aligned} \tag{52}$$

There are, then, two possible regimes at the second phase of the life-cycle, $\tau \in [S, R]$ when $\lambda_1 > 0$ and that are distinguishable by wether $s^w = 0$ or $s^w \geq 0$.

III b) $\tau \in [t, S]$

For this period the Hamiltonian is,

$$H = \left(\frac{c^{1-\sigma}}{1-\sigma} + \varphi \frac{s^{l1-\epsilon}}{1-\epsilon} \right) e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta} + \lambda_1 \Big(\xi_h.\eta.(s^h)^{\phi_h} - \delta_h.h \Big) +$$
$$+ \lambda_3 \Big([r + m(\tau)]\, a - c - z_0 + d + \gamma_e.w.(1 - s^l) \Big)$$

For $\lambda_1 = 0$

$$s^l = \left(\frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{\lambda_3 \gamma_e w} \right)^{1/\epsilon} \wedge 1$$
$$s^h = 1 - s^l$$

For $\lambda_1 > 0$

$$s^l : \frac{\varphi e^{-\int_t^\tau \rho(\theta)+m(\theta)d\theta}}{(s^l)^\epsilon} = \frac{\lambda_1 \xi_h \eta \phi_h}{(1-s^l)^{1-\phi_h}} + \lambda_3 \gamma_e w$$
$$s^h = 1 - s^l$$

IV) Transversality conditions

We write the problem as:

$$\int_t^R L(\tau, x, u)d\tau + G(x_R) \to max$$
$$s.t.$$
$$\dot{x} = f(\tau, x, u) \quad x_t = \bar{x}$$

Let $\tilde{y}$ denote a perturbation on a variable y. At the optimum, a perturbed solution must not yield a superior result, hence we must have:

$$\int_t^R L(\tau, \tilde{x}, \tilde{u})d\tau + G(R, \tilde{x}_R) - \int_t^R L(\tau, x, u)d\tau - G(R, x_R) \leq 0$$
$$\approx \Delta I + \frac{\partial G(R, x_R)}{\partial x} \Delta x_R \leq 0$$

Then the interior of the Pontryagin cone (K) and the accessible set must not overlap

$$int(K) \cap \left\{ (\Delta I, \Delta x) : \Delta I + \frac{\partial G(R, x_R)}{\partial x} \Delta x_R > 0 \right\} = \varnothing$$

From here we obtain

$$(\lambda_0, \lambda(R)) = \left(1, \frac{\partial G(R, x_R)}{\partial x} \right)$$

which proves (45).

■

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