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Convexity in semi-metric spaces, decision theory and consumer theory

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Abstract

Convexity is a concept usually developed in vector spaces. However since the seventies of the 20th century, convexity is also study in metric spaces in general. In this paper we develop a theory of convexity - including convex real functions - in semi-metric spaces and we suggest some possible applications in Economics and Decision Theory.

João Ferreira do Amaral[1]

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Introduction

Mathematics in several of its fields of research formalizes and quantifies certain functions of the mind. One of those functions is the differentiation function between two objects that is, between two elements of a given set.

If it is possible to distinguish two elements x and y of a given set E and quantify for each pair of distinct elements the degree of the differentiation this means that we have defined a *semi-metric* which is a real function define on the set E x E. In the case where the function differentiates the elements of the set according to their position in space we have a *distance* between the elements of the set and a semi-metric that is a

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metric. As it's well known many other operations of differentiation and not only the differentiation in space have characteristics of a distance and can be quantified using a metric.

The objective of the present paper is the study the property of convexity on sets where a semi-metric is defined that is on a semi-metric space.

Convexity of sets or functions was traditionally studied in vector spaces, therefore as an algebraic property. However since the seventies of the last century there is growing interest in the topological features of convexity, generalizing the concept of convexity to metric spaces even if they aren't vector spaces.

In that vein Takahashi (Takahashi, 1970) defined a *convex metric space* as a metric space (E,d) where a *convex structure* is defined that is, a function W from [0, 1] to E defined for all the pairs x, y of E and every real number λ (0 ≤λ ≤ 1)) verifying

$d(u, W(x, y, \lambda)) \leq \lambda d(u, x)+(l-\lambda)d(u, y)$ for all the u of E.

This definition has been useful for obtaining some generalizations of fixed point theorems (see Agarwal et al 2009 and Abdelhakim, 2016).

We define in this paper convexity for more general spaces, the semi-metric spaces.

Our analysis is more general that the one of Takahashi, not only because the space itself is a semi-metric one and not a metric space but also because the condition equivalent to the convex structure demands quasi-convexity and not convexity.

On the other hand we define the concept of convex function for semi-metric spaces.

The benefit of considering semi-metric spaces instead of metric ones is easily seen when we feel the necessity of quantify the differentiation between two elements of a given set but we don't see any reason to suppose that the properties of the differentiation function are the properties of a metric.

As is well known the specific property of a metric is the so-called triangular inequality, that is, being d(x,y) the value of the differentiation function between x and y of a given set E we have the following property.

(Triangular inequality). For every x, y, and z of the set E we have

$d(x,y) \leq d(x,z) + d(z,y)$

This type of differentiation functions was introduced in mathematics for solving problems of Physics associated with space locations and the possibility of access to one point from another. As distance between two points was considered to be the shortest trajectory that gives access from one of these points to the other, the differentiation function necessarily verifies the triangular inequality

But this is not the case in other type o problems, v.g. in what concerns economic and general decision problems that we exemplify in the conclusions of the present paper where the appropriate differentiation function to use is not a metric.

In section 1 we define semi-metric spaces and the property of convexity both for spaces and for functions in these spaces. In section 2 we apply the concept of convexity to set functions, generalising definitions and results first obtained in 1971 (Albuquerque et al, 1969)[2] and also in 1982 (Amaral, 1982). We end the paper with a section of conclusions where a possible application to Economics is mentioned.

1. Semi-metric spaces and convexity

1.1 Semi-metric

The concepts of semi-metric and semi-metric spaces were first developed both by Fréchet and by Karl Menger in the first decades of the XX th century (Wilson 1931, Sims 1962).

[2] The official date of the review was 1969 but the real date of publication was 1971.

Let E be a set and f a real function defined in the set $E^2 \equiv E \times E$ that verifies the following properties:

1- To each pair (x y) de E^2 corresponds one and only one real number $f(x,y) \geq 0$

2- $f(x,y) = f(y,x)$ for every x and y of E

3- $f(x,y) = 0$ if and only if $x = y$

Every function $f: E^2 \rightarrow R$ that verifies these properties is a *semi-metric* and the pair (E, f) is a semi-metric space, (*sms*). Note that contrary to the usual definition of space we don't impose ex-ante any topological conditions.

Remarks

1. Some authors, instead of semi-metric space when no topological conditions are imposed *ex-ante* call E a semi-metric set. We don't see the need for such a terminological distinction. Later on we go back to topology in a *sms*.

2. Obviously a semi-metric that verifies the triangular inequality is a metric. To define convexity we don't need a metric. A semi-metric will do.

3. The term *semi-metric* is not universally used (see for example Deza et al, 2009, p.3) where the term used is *symmetric*.

Definition (*segment*). *Let (E,f) be a sms. Let x e y be elements of E. Segment of extremities x and y is the set S(x,y) of all the z of E such that*

$z \in S(x,y)$ if and only if $f(x,y) = f(x,z) + f(z,y)$

Remark

Obviously, given property 2 above we have, S(x,y) = S(y,x)

Definition *(length). The length of S(x,y) is the number f(x,y)*

Defintion *(non-hollow). A segment S(x,y) is non-hollow if and only if* $S(x,y) - \{x,y\} \neq \emptyset$

1.2 Basic correspondence of a sms

For each *sms* we can define a correspondence – that we call the *basic correspondence* – which can be useful for characterizing the *sms*. The correspondence is defined as follows:

φ: $E^2 \rightarrow 2^E$ such that $\varphi(x,y)$ is the set of all the z of E such that $f(x,y) > f(x,z) + f(z,y)$.

It is easy to see that:

- for any *sms* (E,f) end every x,y of E , $\varphi(x,x) = \emptyset$, $x, y \notin \varphi(x,y)$ and $\varphi(x,y) = \varphi(y,x)$

- if f is a metric, for all the (x,y), $\varphi(x,y) = \emptyset$

- For all the segments S(x,y), $S(x,y) \cap \varphi(x,y) = \emptyset$

- If $z \in \varphi(x,y)$, then $x \notin \varphi(z,y)$ and $y \notin \varphi(x,z)$,

- If we define the real function H(x,y,z) such that H (x,y.z) = 0 if $z \notin \varphi(x,y)$ and

  H (x,y,z) = 1 if $z \in \varphi(x,y)$ we have $H(x,y,z) + H(x,z,y) + H(z,y,x) \leq 1$.

Definition *(sms bounded (strictly bounded) from above). A sms (E,f) is bounded (strictly bounded) from above if sup {f(x,y)} is finite (there is a pair (x*,y*) of E such that f(x*,y*) = max {f(x,y)}) for all the pairs (x,y) of* E^2.

We have the following result for basic correspondences.

Theorem 1. *If (E,f) is strictly bounded from above, that is* $f(x^*,y^*) = \max\{f(x,y)\}$, *if* $2f(x^*,z) < f(x^*,y^*)$ *and* $2f(y^*,z) < f(x^*,y^*)$ *for all the* $z \neq x^*,y^*$ *then the maximum (in the sense of the relation of inclusion between sets) of the sets* $\varphi(x,y)$ *is* *φ_(x*,y*) = E - {x*, y*}.*

Proof:

Immediate

Example

Consider E ={x, y z} with d(x,z) = 1, d(y,z) =2, d(x,y) = 5. Obviously the maximum set is φ (x,y) ={z} = E- {x, y}. On the other hand, φ (x,z) = φ (z,y) = Ø.

Definition *(sms bounded (strictly bounded) from below). A sms (E,f) is bounded (strictly bounded) from below if inf {f(x, y)}) > 0 (there is a pair (x**,y**) of E^2, x**≠ y** such that f(x**,y**) = min {f(x, y)}) for all the x≠y of* E.

Theorem 2 *If (E,f) is a sms strictly bounded from below, then there is at least one*

φ (x,y), x≠y such that φ (x,y) = Ø

Proof:

Immediate

We can find also some results about the sub-set of E where f(x,y) is a metric. The following is one of those results.

Theorem 3. *Let (E, f) be a sms. If it exists a non-empty set E*⊂E such that there is a positive number a such that 2a ≥ f(x,y) ≥ a for every pair (x,y) of E^{*2}, x≠y, then f is a metric on E*.*

Proof

Suppose that for a given pair (x,y) of E* there is a z of E* z ≠ y , x such that

f(x, y) > f(x, z) + f(z, y).

By the assumption of the theorem f(x,z) ≥ a and f(z,y) ≥ a so that

f(x, y) > a + a which is impossible since by assumption 2a ≥ f(x,y).

Corollary. *In such a set E* every non-hollow segment has length 2a*.

Proof.

Consider a non-hollow segment S(x,y) and z any element of the segment, $z \neq x,y$.

We have $f(x,y) = f(x,z) + f(z,y) \geq 2a$ so that $f(x,y) = 2a$.

Corollary. *Suppose that E* is such that for all the pairs of* E^{*2},

$\inf\{f(x,y)\} > 0$, $\sup\{f(x,y)\} \leq 2\inf\{f(x,y)\}$. Then f is a metric on E.*

Proof.

Immediate

Corollary . *In such a set every non-hollow segment has the length 2inf{f(x,y)}.*

Proof.

Immediate.

Remark (an indicator for the approximation to being a metric):

If for a given finite set E with n elements, we consider the set T of all the pairs (x,y)

$x \neq y$ of E such that $\varphi(x,y) = \emptyset$ the ratio $R = \#T/(n^2 - n)$, where # stands for "cardinal" gives an indication of how close the semi-metric f is of a metric. If R = 1, f is a metric. The lower the value of R (it cannot reach the value 0 since a finite set f is strictly bounded from below, see theorem 2) the farther is f of being a metric.

If E is infinite but with measure m(E) finite we can take R = m(T)/m(E). If R =1 , f is a metric or m(E-T) = 0. If T is not measurable we can take an approximation, that is we can take two measurable sets S, U such that $S \subset T \subset U$ and put m(T) = [m(S)+m(U)]/2.

Example

Some cases of relation of the value of R to different semi-metrics f, for the same set E, are easily found.

For instance if p > 1 the semi- metric f^p has a value of R not larger that the R of the semi-metric f (for the same set E).

To see this consider any z ,element of φ_f (x,y).

We have f(x,y) > f(x,z)+f(z,y) so that $f(x,y)^p > f(x,z)^p + f(z,y)^p$ and φ_{f^p} (x,y), obtained with the semi-metric f^p, is such that φ_f (x,y) $\subset$ φ_{f^p} (x,y) where φ_f (x,y) is the set obtained with the semi-metric f. Therefore it's not possible to have φ_{f^p} (x,y) = Ø

when φ_f (x,y)≠ Ø.

Example of a semi-metric defined on the circle

Consider a circle C $\subset$ R^2 and its circumference S.

Consider the *sms* space (C,f) with f defined as follows:

If x ∈ C- S f(x,y) for all the y of C is the familiar Euclidean distance

If x , y ∈ S f(x,y) is the length of the minor arc with extremities x and y

It is easy to verify that f is not a distance and that:

If x , y ∈ S φ (x,y) ≠ Ø , φ (x,y) ⊂ C- S

If x or y belong to C-S, φ (x,y) = Ø.

In this case R = 1 since (C-S)-D ⊆ T ⊂ C (D≡{(x,x)}) and m [(C-S)-D] = m(C) where m is the standard measure.

1.3 Almost full and full spaces

One interesting property that a *sms* may have is that all its segments are non-hollow, that is for every x and y, S(x,y) –{x,y}≠ Ø.

A space that verifies this condition is sometimes called a convex space. In order to clear distinguish the concept from the concept of convexity in Takahashi instead of convex *sms* we call such a space an *almost-full sms (afsms*).

Example

The R line endowed with the semi-metric $f(x,y) = (x-y)^2$ is not almost full. It corresponds even to a more extreme situation: there is no segment $S(x,y)$ such that $S(x,y) - \{x,y\} \neq \emptyset$. However this is not the case for the semi-metric $f(x,y) = (x_1 - y_1)^2 + (x_2 - y_2)^2$ for R^2.

It is easy to prove the following theorem.

Theorem 4. A *afsms does not verify the conditions of theorem 3 and its corollaries*.

Proof

In spaces that verify the conditions of the theorem all the non-hollow segments have the same length. However this is not possible for a *afsms.*

This result is in fact a consequence of the following theorem.

Theorem 5. *If (E,f) is s afsms , then inf {f(x,y)} = 0 for (x,y) of* E^2

Proof.

Suppose that $\inf \{f(x,y)\} = h > 0$ *and* choose a segment $S(x,y)$ with length a. Since $S(x,y)$ is non-hollow there is at least a z_1 belonging to the segment. If $f(x, z_1) < h$ the proof is complete. If not we have $f(z_1,y) \leq a-h$.

There is then a non-hollow segment $S(z_1,y)$ with length $\leq$ a-h. Choose z_2 belonging to $S(z_1,y)$.

If $f(z_1,z_2) < h$ the proof stops. If not we have $f(z_2,y) \leq (a-h) - h = a-2h$. Repeating the process N times where N = intg [(a-h)/h] +1[3] we have $f(z_N,y) < h$ or $f(z_N,y) \leq 0$, both a contradiction and the theorem is proved.

The fact that a given space is a *afsms* may depend not only on the characteristics of the set E but also on the properties of the semi-metric f. For instance, a finite set cannot have all the segments almost full. On the other hand a metric such that

$f(x,y) = 1$ for every x and y, such that $x \neq y$, has no almost full segment in every set where it is defined.

[3] The function *intg(X)* for any real number X means the highest integer $\leq$ X

Note also that an infinite countable set may be almost full. Such is the case of the rational numbers set Q with the metric f(p,q) = |p - q|. For all the infinite countable sets since they are equivalent to Q is possible to find a metric that makes the space (E, f) a *afsms.*

In what follows we seldom require that a *sms* is almost full. Later on we will need the following stronger concept of full *sms*.

Definition *(full semi-metric space). A sms (E,f) is a full sms if and only if, for every x and y of E, and for every real number α ∈ (0, f(x,y)] there is a z ∈ S(x,y) such that f(x,z) = α*

As there are possibly several z,w. … belonging to S(x,y) such that f(x,z) = f(x, w) =... = α we define the concept of full sub-segment of S(x,y).

Definition (sub-segment). *A sub-segment S*(x,y) of a segment S(x,y) in a full sms is a set S*(x,y) of elements z such that f(x,y) = f(x,z) + f(z,y), S*(x,y) ⊂ S(x, y) and for every real number α ∈ (0, f(x,y)] there is one and only one z ∈ S*(x,y) such that f(x,z) = α*

A different but related concept to full *sms* is the concept of generalized linear space (*gls*).

Definition *(generalised linear space). A sms (E,f) is a gls if and only if E* = E where E* is the set of all the elements z of E such that there are x, y ≠ z such that z ∈ S(x, y).*

The following condition the property-H that will be used later on guarantees that the space is a *gls*.

Property H. (*E,f) verifies property H if and only if for every x and y of the space and for every real number a > f(x,y) there exists z of E and a segment S(x,z) such that y ∈ S(x, z) and f(x,z) = a*

Remark

Of course a full *gls* that verifies property H is not bounded from above and not bounded from bellow.

When a *afsms* is a metric space we may prove the following theorem

Theorem 6. *Let S(x,y) be a segment and z an element of S(x,y).*

Then S(x,z) ∪ S(z,y) ⊂ S(x,y).

Proof.

Let u be an element of S(x,z) (a similar proof for S(z,y))

Then

d(x,z) = d(x,u) + d(u,z)

We have also d(x,y) = d(x,z) + d(z,y)

So that

d(x,y) = d(x,u) + d(u,z) +d(z,y)

But by the triangular inequality

d(u,y) ≤ d(u,z) + d(z,y) So that

d(x,y) ≥ d(x,u) + d(u,y)

Which is possible only when u belongs to S(x,y), as we wanted to prove.

Remarks

1. It could be expected that in a metric space if u and v are elements of a segment S(x.y) we always have S(u,v) ⊂ S(x,y).

However that is not the case as the following counter-example shows.

Let x, y, u and v elements of S(x,y) and z an element of S(u,v), with the following values for the respective distances

d(x,y) = 10; d(x,u) = 2; d(u,y) = 8

d(u,z) = 3; d(v,z) = 4 ; d(u,v) = 7

d(x,v) = 7; d(v,y) = 3 ; d(x,z) = 4

d(y,z) = 7

Obviously z belongs to S(u,v) but not to S(x,y).

2. Although Theorem 6 is true for metric spaces is not necessarily so for other *sms*.

We can now define now the concept of a convex subset of a *sms*.

Definition *(convex set). Let K ⊂ E. K is a convex set of the sms (E,f) if and only if for every x and y of K we have S(x,y) ⊂ K.*

Remarks

1. By the precedent counter-example we see that not all the segments are convex sets.

2. *Convention*. By convention, the empty set Ø and singletons are considered convex.

3. The whole set E is convex.

Theorem 7. *The intersection of convex sets is a convex set*

Proof: immediate

*Remark***.**

Theorem 7 allows us to define the *convex hull* of any set K ⊂ E as being the set C(K) that is the intersection of all the convex sets that contain K.

Theorem 8. *Consider an afsms (E,f). If E* ⊂ E is convex then inf {f(x,y)} = 0 for*

(x,y) ∈ E^{*2}, x≠y.

Proof.

Suppose the contrary, that is inf{f(x,y)} = ε > 0 for (x,y) ∈ E^{*2}, x≠y.

For $\varepsilon > \lambda > 0$ there exits (x^*, y^*) of E^{*2} such that $f(x^*, y^*) \leq \varepsilon + \lambda$. Since E is an *afsms* there is z such that $f(x^*,z) + f(z, y^*) \leq \varepsilon + \lambda$. That means that

$f(x^*,z)$ or $f(z, y^*) \leq (\varepsilon + \lambda)/2 < \varepsilon$ so that $z \notin E^*$ and E^* is not convex.

Corollary. *A non-empty finite set of an afsms is never convex unless it is a singleton.*

Proof

Immediate

Remark. This corollary shows that in order to study convexity on a finite set we must extend it to its convex hull.

Definition *(open ball and closed ball). Let x be an element of E and r a non-negative real number. Open (closed) ball with centre at x and radius r is the set N(x,r) (N*(x,r)) of all the z of E such that f(x,z) < r (f(x,y) ≤ r).*

Remark. Of course, if r=0, $N(x,0) = \emptyset$ and $N^*(x,0)= \{x\}$

An important fact concerning open balls is established by the following theorem

Theorem 9 . *For each pair (x,y) of E^2 where (E,f) is a semi-metric space we have*

$\varphi (x,y) \subset [N(x, f(x,y)) \cap N(y, f(x,y))] - S(x,y)$

Proof.

The proof is straightforward.

We have, for z of φ (x,y)

$f(x,y) > f(x,z) + f(z,y)$ so that $f(x,z)$, $f(z,y) < f(x,y)$ so that z belongs to $N(x, f(x,y))$

and $N(y, f(x,y))$.

Since z $\notin$ S(x,y) we have

φ (x,y) $\subset$ [N(x, f(x,y))$\cap$ N(y, f(x,y))] – S(x,y) as we had to prove.

Remark

As x and y $\notin$ N(x, f(x,y))$\cap$ N(y, f(x,y))], S(x.y) - {x,y} $\subset$ [N(x, f(x,y))$\cap$ N(y, f(x,y))] and

φ (x,y) $\cap$ S(x,y) = $\emptyset$ we can put this result in the form

φ (x,y) $\cup$ S(x,y) $\subset$ N(x, f(x,y))$\cap$ N(y, f(x,y))] $\cup$ {x,y}

Definition *(quasi-convex semi-metric). A semi-metric f is quasi-convex if and only if for every x, y and z of E we have*

f(w,z) ≤ max {f(x,z), f(y,z)} for every w $\in$ S(x,y)

It is important to ensure that the concept of quasi-convex semi-metric is compatible with other characteristics of the *sms*. We have the following condition that is a necessary condition for the existence of a quasi-convex semi-metric:

Theorem 10. *It is a necessary condition for the a given semi-metric to be quasi-convex that there are no distinct x,y,z, w of E such that w$\in$S(x,y) and x, y$\in$ φ (z,w).*

Proof

Suppose that there are distinct x,y,z,w such that *w$\in$S(x,y) and x, y $\in$ φ (z,w).*

We have f(w,z) > f(w,x) + f(x,z) and f(w,z) > f(w,y) + f(y,z) so that f(w,z) > max {f(x,z), f(y,z)} which contradicts the definition of quasi-convexity.

Theorem 11. *A closed ball N*(x,r) of a afsms (E,f) is a convex set if and only if the semi-metrics f is quasi-convex*

Proof:

The condition is sufficient

Let f be quasi convex and let r be a positive real number, x any element of E and y and z any elements of the ball N*(x,r). Consider S(y,z) and any one w of S(y,z). Then we have f(w,x) ≤ max {f(x,y), f(x,z)} ≤ r so that w belongs to N*(x, r).

The condition is necessary

We assume that for every z of E and for every non-negative r the ball N*(z, r) is a convex set so that for any x, y and z of E, the ball N(z, r*), where r* = max {f(x,z), f(y,z)} is convex. Therefore S(x,y) ⊂ N(z, r*) so that for every w of S(x,y),

f(w,z) ≤ r* = max {f(x,z), f(y,z)} and the semi-metric is quasi-convex.

With a similar reasoning we can prove

Theorem 12. *It is a sufficient condition for every open ball N(x,r) of the sms (E,f) to be convex that f is a quasi-convex semi-metric.*

Remark. It is easy to see that for an open ball the condition is not necessary.

Using theorems 8, 11 and 12 we obtain the following result.

Theorem 13. *If (E,f) is an afsms and f is quasi-metric then every N*(x,r) and every N(x,r) that aren't singletons, are such that inf {f(w,y)} =0 for w ≠ y (w,y)∈ N*(x,r) or N(x,r) .*

1.4 Analysis on sms-K spaces

Although topology is not the main focus of this paper it is useful to develop some topological considerations in order to get a better understanding of the usefulness of the concept of *sms* for the study of convexity.

Similarly to the case of metric spaces we can associate for every point of *afsms* a family of neighbourhoods that is the family of all the open balls with centre at that point. However contrary to metric spaces where an open ball is an open set in a *sms* this is not necessarily the case .

With this attribution of neighbourhoods we can define the concept of limit of a sequence in a *sms*.

Definition *(limit of a sequence). Consider a sequence of elements x_n of a semi-metric space (E,f). The sequence converges to x, if for any $\delta > 0$ there is a n^* such that for all the $n > n^*$ we have $f(x_n ,x) < \delta$.*

A sequence with a limit is called a *convergent* sequence. However, in order to have an operative concept of limit it is necessary to guarantee its uniqueness. Such guarantee exists for every metric space but nor for every semi-metric space. A specific case of semi-metric space such that the uniqueness of limit is verified is the following *K-sms*.

Definition *(K-sms). A semi-metric space is a K-semi-metric space if and only it verifies the following condition:*

$f(x,y) \leq K[f(x,z) + f(z,y)]$

for a given $K \geq 1$, where $K = \inf \{M: f(x,y) \leq M[f(x,z) + f(z,y)]\}$

and for every x, y and z of the space.

Remarks

1. K-spaces are also called *quasi-metric* or *near-metric* spaces (Deza et al 2009, p. 6). It is also said that they corresponds to the *K-relaxation of the triangular inequality*. Of course the case K =1 corresponds to semi-metric spaces that are metric spaces.

2. A simple example of a *2-sms* is the space (E,f) where $f(x,y) \equiv d^2(x,y)$ and d is a distance.

Some simple and already known results for *K-sms* follow.

Theorem 14. *Consider two points x and y, with $x \neq y$ in a K-sms. There exist two neighbourhoods, that is two open balls and N_x e N_y with centres respectively at x and y such that N_x and N_y are disjoint.*

Proof:

f(x,y) with $x \neq y$, is positive and consider the real number $r = f(x,y)/2K$.

Let z be an element of N(x,r).

We have

$2Kr = f(x,y) \leq K[f(x,z) + f(z,y)] = Kf(x,z)+ Kf(z,y) < Kr + Kf(z,y)$

So that

$r < f(z,y)$.

That is if z belongs to the ball with centre at x and radius r it does not belong to the ball with centre at y and radius r. Therefore we have two disjoint balls as we had to prove.

With this theorem we can prove the uniqueness of the limit.

Theorem 15. *In a K-sms for every convergent sequence its limit is unique.*

Proof

Let $\{x_n\}$ be a sequence with two limits, x and y, $x \neq y$.

Consider two disjoint N_x and N_y that by the previous theorem exist.

Then by definition of limit for a certain n* and for all the n > n* we have $x_n \in N_x$. And for a certain n**, and for all the n > n** we have $x_n \in N_y$.

That is for all the n > max (n*, n**) we have all the terms of the sequence belonging to both balls, which is not possible. Therefore the limit is unique.

When a *sms* is such that for each convergent sequence the limit is unique, we can define continuity for the respective semi-metric.

Definition *(accumulation point). Consider a sms (E,f) such that any convergent sequence has a unique limit. Given a set* $A \subset E$ *, z is an accumulation point of A if and only if z is the limit of a sequence* $\{z_n\}$ *of elements of A.*

Definition *(closed set)*. *Let A' be the set of all the accumulation points of A*. *Then A is a closed set if and only if* $A' \subset A$.

Remark

Obviously by the definition a set with no accumulation points is closed.

Definition *(continuity). The semi-metric f of the space (E,f) is continuous if for each sequence* $\{y_n\}$ *such that* $\{y_n\} \to y$ *we have for all the x of E,* $f(x,y) = \lim_{n\to\infty} f(x,y_n)$.

Remarks

The semi-metric of the example of 1.2 p. 7 is not continuous.

A simple example of a continuous semi-metric is the semi-metric $f \equiv d^p$ where d is a metric (therefore is continuous) and p is any non-negative number.

Theorem 16. *If the semi-metric f of the space (E,f) such that any convergent sequence has a unique limit is continuous then for each pair (x,y) of* E^2 *the set* $\varphi(x,y) \cup S(x,y)$ *is closed.*

Proof.

Suppose that there is a sequence $\{z_n\}$ such that $\{z_n\} \to z$ with $z_n \in \varphi(x,y) \cup S(x,y)$ converging to z (if not $\varphi(x,y) \cup S(x,y)$ has no accumulation points and therefore is closed).

We have for each n $f(x, y) \geq f(x, z_n) + f(z_n, y)$, so that, f being continuous

$f(x, y) \geq \lim_{n\to\infty} f(x, z_n) + \lim_{n\to\infty} f(z_n, y) = f(x, z) + f(z, y)$, which means that

$z \in \varphi(x,y) \cup S(x,y)$ and the set is closed.

The condition K also provides the necessary framework for the definition of a limit of a function for *sms*.

Definition *(limit of a function). Let $F : A \to B$ be a function that to each element of $A \subset E$ associates an element of $B \subset G$ where (E, f) and (G, f^*) are two sms. We say that b is the limit of F at the point a of A if and only if for each $\delta > 0$ there exists an $\varepsilon > 0$ such that for every x of A such that $0 < f(x,a) < \varepsilon$ we have $f^*(F(x), b) < \delta$.*

The limit is represented by $\lim_{x=a} F(x) = b$

Based on this definition we can prove the uniqueness of limits of functions defined on *sms*.

Theorem 17. *(uniqueness of the limit of a function).Under the conditions of the previous definition if the sms (G, f^*) is a K-sms then the limit is unique*.

Proof:

Suppose we had two distinct limits b and b*. Then $f^*(b, b^*) > 0$ and for very real number $\delta/2K$, ε and ε^* would exist such that for x and x* verifying $0 < f(x,a) < \varepsilon$ and $0 < f(x^*,a) < \varepsilon^*$, we would have respectively

$f^*(F(x), b) < \delta/2K$ and $f^*(F(x^*), b^*) < \delta/2K$.

For the smallest of the ε, we can choose $x = x^*$ verifying both inequalities

$f^*(F(x), b) < \delta/2K$ and $f^*(F(x), b^*) < \delta/2K$

But as (G, f^*) is a *K-sms* we would have

$f^*(b, b^*) \leq K [f^*(F(x), b) + f^*(F(x), b^*)] < \delta$, for every positive δ which implies

$f^*(b, b^*) = 0$ and we have a contradiction.

As for metric spaces we can define the concept of a complete *sms*. For that purpose we have to define the concept of a Cauchy sequence in a similar way as for metric spaces (the study of Cauchy sequences in a special kind of *sms* different from *K-sms* was developed by Burke (1972)).

Definition *(Cauchy sequence).A sequence $\{x_n\}$ of elements of the sms (E,f) is a Cauchy sequence if and only if for every $\delta > 0$ there is a natural number p such that for all n, n* not smaller then p we have $f(x_n, x_{n^*}) < \delta$.*

As it is easy to see in the case of a *K-sms* every convergent sequence is a Cauchy sequence because being x the limit of the sequence for every $\delta > 0$, we have a p such that for $n \geq p$ we have $f(x_n, x) < \delta/2K$. Being the space a *K-sms*, for n and n* not smaller then p we have

$f(x_n, x_{n^*}) \leq K [f(x_n, x) + f(x, x_{n^*})] < K [\delta/2K + \delta/2K] = \delta$.

However not all the Cauchy sequences are convergent.

Definition *(complete K-sms). A K-sms is complete if and only if every Cauchy sequence is convergent.*

For obtaining a complete *K-sms* the following theorem may be useful.

Theorem 18. *Let (E, f) be a K-sms such that f(x,y) ≡ h(d(x,y)) where h is a real function, strictly increasing and such that h(0) = 0 and d is a metric. It is necessary and sufficient for (E, f) to be complete that the metric space (E, d) is complete*.

Proof:

Necessity.

Let (E,f) be a complete *K-sms* and $\{x_n\}$ any Cauchy sequence of (E,d). Then for every δ > 0, for the number h^{-1} (δ), there exists p such that for n, n*≥ p we have

$d(x_n, x_{n^*}) < h^{-1}(\delta)$ so that

$f(x_n, x_{n^*}) = h(d(x_n, x_{n^*})) < \delta$ and the sequence is a Cauchy sequence in the space (E,f). Being this space a complete space the sequence has a limit. Let x be that limit. Then for every δ > 0 and h(δ) there exists N such that , for n > N, we have $f(x_n, x) < h(\delta)$. Therefore

$d(x_n, x) = h^{-1}[f(x_n, x)] < \delta$ and x is the limit of the sequence in (E,d).

Sufficiency.

Consider a Cauchy sequence of elements of (E,f).

For every δ > 0 and the corresponding number h (δ), exists p such that for n,n*≥ p we have

$f(x_n, x_{n^*}) < h(\delta)$, so that

$d(x_n, x_{n^*}) = h^{-1}(x_n, x_{n^*}) < \delta$ and the sequence is a Cauchy sequence in the space (E,d). Being this space complete the sequence has a limit. Let x be that limit. Then for every δ > 0 and h^{-1} (δ) there exists N such that for n > N, we have $d(x_n, x) < h^{-1}(\delta)$. Therefore

$f(x_n,x) = h[d(x_n, x)] < \delta$ and x is é limit of the sequence in the space (E,f)

The following corollary is also immediate.

Corollary. *Under the conditions of the previous theorem the sequence $\{x_n\}$ converges to x in the space (E,f) if and only if converges to x in the space (E,d)*

We can find *sms* that are not *K-sms* where we can prove the uniqueness limit of a convergent sequence. Consider the following theorem.

Theorem 19. *Under the conditions of theorem 18 concerning the real function h and even if the space (E,f) is not a K-sms the space (E,f) is such that every convergent sequence has one limit only.*

Proof:

Suppose that there was a sequence that had two limits in space (E,f). Then as those limits are limits of the sequence in the space (E,d) there would be a sequence with two limits in a metric space which is impossible.

Having defined the concept of limit of a function we can now move to continuity.

Definition (*continuous function). A function F defined as in theorem 17 (and the respective definition) - where (G,f*) is a K-sms - is continuous at point a if F(a) exists and* $\lim_{x=a} F(x) = F(a)$.

Theorem 20. *If F(x) is a continuous function at a of A then for every sequence $\{x_n\}$ of elements of A converging to a, the sequence $\{F(x_n)\}$converges to F(a)*

Proof

Let $\delta > 0$. Then as the function is continuous at a there exists ε such that for all the x of A such

$0 < f(x, a) < \varepsilon$ we have $f^*(F(x), F(a)) < \delta$.

As $\{x_n\}$ converges to a there is a n* such that for n > n* we have $f(x_n,a) < \varepsilon$ so that

$f^*(F(x_n), F(a)) < \delta$, and $\{F(x_n)\}$ converges to $F(a)$.

We can define a Lipschitz condition.

Definition *(Lipschitz condition). Let F : A → B be a function that to each element of*

$A \subset E$ associates an element of $B \subset G$ where (E ,f) and (G, f) are two sms. We say that the function F verifies the Lipschitz condition if and only if or every x and y of A there exists a real number*

$L > 0$ (independent of x and y) such that

$f^*(F(x), F(y)) \leq L\, f(x, y)$

Remark

Of course a function defined in two (E,f) and (G,f*) *K-sms* that verifies the Lipschitz condition is continuous in all the points of its domain. To see is enough to choose for each δ in the definition of limit a number $\varepsilon = \delta/L$.

As for metric spaces we can prove for *K-sms* a Banach fixed point theorem relative to mappings that are contractions. However we need a more strong condition. Instead of the contraction constant L verifying $0 < L < 1$ we need the double inequality $0 < KL <1$. When $K = 1$, that is when the *K-sms* is a metric space we obtain the original condition of the Banach fixed point theorem. Of course, as should be expected, $0 < KL <1$ is a much stronger condition than $0 < L <1$.

1.5 Convex functions in full K-sms

Let (E, f) be a full *sms* and $S^*(x,y)$ a sub-segment of the segment $S(x,y)$.

Definition *(convex function). A real function F defined on a convex set $A \subset E$ is a convex function if and only if for every x, y of A , for every sub-segment S*(x,y) and for every z of S*(x,y) we have*

$F(z) \leq [f(y,z)/f(x,y)]F(x) + [f(x,z)/f(x,y)]F(y)$

Or under another form

$[F(z)- F(x)]/f(x,z) \leq [F(y)- F(x)]/f(x,y)$

Remark.

A convex function is a *strictly convex* one when we have

$F(z) < [f(y,z)/f(x,y)]F(x) + [f(x,z)/f(x,y)]F(y)$ ($z \neq x, y$).

The importance of convex functions relates frequently with the concept of local minimum.

Definition *(local minimum). Let F be a function defined on a set A of a full sms. F(x*), with x* of A, is a local minimum of F if and only if there is a N(x*, r*) such that*

$F(x^*) = \min\{F(x)\}$ for all the $x \in A \cap N(x^*, r^*)$

Theorem 21. *A convex function F defined on A, convex subset of a full sms (E,f), can't have two different local minima in A.*

Proof

Suppose that there were two (local) minimizing x* e x** in A such that $F(x^{**}) < F(x^*)$. Consider the open ball N(x*, r*) where F(x*) is a minimum. Let us form the sub-segment S*(x*, x**). The sub-segment is contained in A because A is convex. Since E is full there is

$y \in S^*(x^*, x^{**})$, $y \neq x^{**}$ such that $0 < f(x^*, y) < r^*$. Then, as F is convex and

$F(x^{**}) < F(x^*)$ we have $F(y) < F(x^*)$, where $y \in A \cap N(x^*, r^*)$, which contradicts the definition of local minimum.

If the *sms* is full and F is convex in the strict sense we have the following theorem.

Theorem 22. *If (E,f) is a full sms and F a strictly convex function defined on the convex set $A \subset E$, then there are no two different points x^* and x^{**} of A such that*

$F(x^*) = F(x^{**}) = \min\{F(x)\}$ for all the x of A.

Proof.

If there is no minimum the theorem is proved.

Suppose now that there were two points x^* and x^{**} of A such that $F(x^*) = F(x^{**}) = \min\{F(x)\}$

Since the *sms* is full and A convex there is a $y \in A$ different from x^* and x^{**} such that

$y \in S^*(x^*, x^{**})$. As F is strictly convex, $F(y) < F(x^*) = F(x^{**})$, which is a contradiction.

A useful concept that will be needed is the next remark is the concept of compatibility between two semi-metrics.

Definition (*compatibility of semi-metrics*). Two semi-metrics, f_1 and f_2 , defined on the same set E are compatible if and only if, for each x of E and $\varepsilon > 0$, there are $\varepsilon^*, \varepsilon^{**} > 0$ such that

$N_2(x, \varepsilon^*) \subset N_1(x, \varepsilon)$ and $N_1(x, \varepsilon^{**}) \subset N_2(x, \varepsilon)$

Where N_i stands for open ball in the space (E, f_i) .

Theorem 23. *If x* is a local minimum of the function F defined on A, subset of the sms (E,f) then x* is a local minimum of F in A for every sms (E,g) that is full and such that g is compatible with f.*

Proof:

Immediate from the definition of local minimum.

Important remark

Convexity is a relative property of a function defined on a convex set A *vis a vis* the metric or semi-metric that is chosen. But global maxima and minima of a real function in A are not relative to semi-metrics, that is they don´t change with the semi-metric. That means that we can study global maxima and minima of a real function using different alternative semi-metrics provided in most cases that the corresponding spaces are full and the set A is convex with all the semi-metrics used. An interesting case is one where the function is strictly convex or concave according to the semi-metric chosen.

For instance, a real function defined on the whole set E that is strictly convex when we chose a specific semi-metric and strictly concave when we chose another one has at most one global maximizing element x* and one global minimizing element x** (of course the proofs of the theorems for concavity follow the same lines of convexity).

The exigence of compatibility between alternative semi-metrics may be needed for other studies like local maxima or minima.

Another important characteristic of convex functions has to do with the concept of radial derivative. We consider *a K-sms* because we need to ensure the uniqueness of a limit.

Let F be a real function continuous defined on a convex set A of a *full K-sms*.

Definition *(radial derivative). F has a radial derivative along the sub-segment S*(x,y) ⊂ A at the point x if and only if there exists the limit lim [F(z_n)-F(x)]/f(z_n,x), for a sequence {z_n} n= 1,… of elements z_n converging to x , such that the z_n belong to the sub-segment S(*x,z_{n-1}), where z_0 = y.*

Theorem 24. *Any convex function defined on a convex set A of a full K-sms has a radial derivative (finite or infinite) along some sub-segment of A at each point of A.*

Proof:

Let S*(x,y) be a sub-segment and F convex with x and y belonging to A. Since A is convex S*(x,y) ⊂ A

Let z_1 be an element of the segment different from x and y (which exists since the *sms* is full).

F is convex so that

[F(z_1) - F(x)]/f(x,z_1) ≤ [F(y)- F(x)]/f(x,y).

Consider now z_2 belonging to S*(x,z_1) and not coincident with x and z_1. As x and z_1 belong to A and A is convex, S*(x,z_1) ⊂ A, so that we have

[F(z_2) - F(x)]/f(x,z_2) ≤ [F(z_1)- F(x)]/f(x,z_1).

As the *sms* is full it is possible to form a sequence {z_n} of elements of S*(x,z_{n-1}) where F is convex converging to x. As the sequence [F(z_{n+1}) - F(x)]/f(x,z_{n+1}) ≤ [F(z_n)- F(x)]/f(x,z_n) is non-increasing monotonous it is convergent to a finite limit or to -∞ .

It is time now to look at an important specific case: semi-metric and families of sets.

2. An important application: spaces of sets

2.1 semi-metrics on families of sets

Consider a set E and the family 2^E of subsets of E and let Π be a family of sets such that Π ⊂ 2^E .

Let Ꞁ(Π) be the smallest ring that contains Π.

Let μ be a real function defined on Ꞁ(Π) satisfying the following properties :

1 μ (X) ≥ 0; μ (X)=0 if and only if X= Ø

2 μ is additive, that is if X and Y are disjoint, μ (X ∪ Y) = μ(X) + μ(Y)

With these two properties is easy to show that if X ⊂ Y, we have μ (Y) ≥ μ (X) and if X ⊂ Y, X ≠Y, we have μ (Y) > μ (X).

For each pair of sets X, Y of Ꞁ(Π) define f such that

f (X,Y) ≡ μ (X -Y) + μ (Y-X)

It is easy to see that (Ꞁ(Π) ,f) is a *sms*.

Theorem 25. *If μ is additive then*

S(E,F) ={G ∈Ꞁ(Π) : E∩F ⊂G ⊂E∪F}

Proof:

G ∈ Ꞁ(Π) belongs to S(E,F) if and only if

μ (E -F) + μ (F-E) = μ (E -G) + μ (G-E) + μ (F -G) + μ (G-E)

As μ is additive we can write

$\mu(E) + \mu(F) - 2\mu(E\cap F) = \mu(E) + \mu(G) - 2\mu(E\cap G) + \mu(F) + \mu(G) - 2\mu(F\cap G)$

and also

$\mu(E\cap G) + \mu(F\cap G) = \mu(G\cap (E \cup F)) + \mu(E\cap F\cap G)$

Combining the two equalities we get

$\mu(G) + \mu(E\cap F) = \mu(G\cap (E \cup F)) + \mu(E\cap F\cap G)$

But as

$\mu(G) \geq \mu(G\cap (E \cup F))$ and

$\mu(E\cap F) \geq \mu(G\cap E\cap F)$

We obtain necessarily

$E\cap F \subset G \subset E \cup F$

As we had to prove.

Remark.

If $E\subset F$ (which is the case we consider in the next theorem), the segment is the family of all the sets G of Ⅎ(Π) such that $E\subset G \subset F$. In previous papers (Albuquerque et al 1968 and Amaral 2000) we defined a convex family of sets as a family **P** such that for every two sets E and F of **P** with $E\subset F$, we have for all the G such that $E\subset G \subset F$, $G\in$**P**. The present definition of convex family that has to do with segments, although not very far from the previous one, seems to be more fruitful.

2.2 Convex supremum Weierstrass functions

We can now proceed to study the convexity of set functions. One important case is the case of the Weierstrass functions that is the real function WF(X) defined for a bounded function F

$WF(X) \equiv \sup \{F(x)\}$, x of X.

In the following we relate the convexity of functions of point to convexity of set functions.

Note that the family of sets that we consider is simpler than Ǝ(Π) . That is why we don't need conditions 1 and 2 to be verified by the function μ that we use in the following specific question. We a full *K-sms* of sets but we need a further condition, condition-H a kind of "Archimedean" property-H already mentioned at the beginning of the present paper.

Before that consider a *sms* (E , f) and let x and y be two points of E. Let $\mathbf{N_{xa}}$ be the family of all closed balls N*(x,a) such that $0 \leq a \leq f(x,y)$.

If we define for each N*(x,a), the real set function μ such that

μ (N*(x,a)) ≡ a, then ($\mathbf{N_{xa}}$, f_1) is a full *sms* endowed with the semi-metric

f_1 (N*(x,a), N*(x,b)) = |a-b|, a , b ≤ f(x,y)

The segments of $\mathbf{N_{xa}}$ with extremities in N_{xa} are S(N*(x,a), N*(x,b)) ={G_c: G_c = N* (x,c) with a ≤ c ≤ b }, if b ≥ a and therefore $\mathbf{N_{xa}}$ is a convex family. This is a simpler case that the previous one, since there is no need to consider the smallest ring that contains $\mathbf{N_{xa}}$.

Let us now restate the above mentioned property H.

Property H. *(E,f) verifies property H if and only if for every x and y of the space and for every real number a > f(x,y) there exists z of E and a segment S(x,z) such that y ∈ S(x, z) and f(x,z) = a*

With this property we can prove the following theorem which is a generalization of a theorem in Amaral (1982) that was proved for the Euclidean distance in R^n.

Theorem 26. *Let (E,f) be a full sms and F a bounded, real, strictly convex function defined on a convex set $A \subset E$. We suppose that there is at least a pair (x,y) of A^2 such that N*(x, f(x,y)) $\subset$ A. For every (x,y) of A^2 that verifies this relation let N*(x a) be a closed ball with radius a with $0 \leq a \leq f(x,y)$ and* $\mathbf{N_{xa}}$ *the family of all of sets N*(x a),*

$0 \leq a \leq f(x,y)$ endowed with the semi-metric f_1 (N(x a) , N*(x b)) ≡ |a-b|. Then the set function WF defined on* $\mathbf{N_{xa}}$ *is convex.*

Proof

First we show that for every N*(x,a) and for every w belonging to N(x,a) we have

WF(N*(x,a)) > F(w).

Suppose that WF(N*(x,a)) = F(u) with f(x, u) = a* < a. Since property H is verified we can form the sub-segment S*(x,z) - where F is convex - with u belonging to the sub-segment and f(x,z) = a. As F is strictly convex we can write (for u ≠ x, z)

F(u) < [f(u,z)/f(x,z)]F(x) + [f(x,u)/f(x,z)]F(z),

Which is not possible since F(u) = WF(N*(x,a)) .

Then for every w belonging to N(x,a) we have WF(N*(x,a)) > F(w) so that either WF(N*(x,a)) = sup {F(w)} for all the w of N(x,a) or WF(N*(x,a)) = sup {F(u)} for all the u of N*(x,a) such that f(x,u) = a (and since property H is verified there is at least one u verifying this equality). This conclusion that was drawn for number a is valid also for any a*< a, that is either WF(N*(x,a*)) = sup{F(w)} for all the w of N(x,a*) or WF(N*(x,a*))= sup{F(u)} for all the u of N*(x,a) such that f(x,u) = a*.

On the other hand, as was seen above

F(u) < [f(u,z)/f(x,z)]F(x) + [f(x,u)/f(x,z)]F(z)

for every u of the sub-segment S*(x,z) with 0 < f(u,x) < a and z such that f(x,z) = a. Then as z belongs to N*(x,a) and F(x) ≡ WF(N*(x,0)) we have

1) F(u) < [f(u,z)/f(x,z)] WF(N*(x,0)) + [f(x,u)/f(x,z)] WF(N*(x,a))

Since property-H is verified, for each u such that 0 < f(u,x) < a we can find a z such that f(x,z) = a and such that u belongs to the sub-segment S*(x,z) .Therefore this inequality 1) is valid for all the u such that 0 < f(u,x) < a , so that it is valid for any u such that f(x,u) = a* for every 0 < a*< a.

Let us return to the two possibilities: either $WF(N^*(x,a^*)) = \sup\{F(w)\}$ for all the w of $N(x,a^*)$ or $WF(N^*(x,a^*)) = \sup\{F(u)\}$ for all the u such that $f(x,u) = a^*$.

Consider the first possibility. Let $\varepsilon > 0$ be as small as you wish. Then there exists w of $N(x,a^*)$ such that $f(x,w) < a^*$ and $F(w) > WF(N^*(x,a^*)) - \varepsilon$.

Using property H let us form the sub-segment $S^*(x,w^*)$ where w belongs to the sub-segment and $f(x,w^*) = a^*$. As F is strictly convex we have $F(w) < F(w^*)$ or $F(w) < F(x)$.

If $F(w) < F(w^*)$, based on 1) above and noting that $a^* = f_1(N^*(x,a^*),N^*(x,0))$,

$a = f_1(N^*(x,a),N^*(x,0))$ and $a-a^* = f_1(N^*(x,a),N^*(x,a^*))$, we have

$$WF(N^*(x,a^*)) - \varepsilon < F(w) < F(w^*) < [(a-a^*)/a]WF(N^*(x,0)) + (a^*/a)WF(N^*(x,a))$$

Making ε converge to 0 we get

2) $WF(N^*(x,a^*)) \leq [(a-a^*)/a]WF(N^*(x,0)) + (a^*/a)WF(N^*(x,a))$

If $F(w) < F(x)$, making ε converge to 0 we get $WF(N^*(x,a^*)) = F(x)$ and the inequality 2) is still valid, since by the definition of function WF, $F(x) \equiv WF\{x\} \equiv WF(N^*(x,0)) \leq WF(N^*(x,a))$.

Consider now the second possibility that is, $WF(N^*(x,a^*)) = \sup\{F(u)\}$ for all the u such that $f(x,u) = a^*$. We get immediately from 1) that

$$WF(N^*(x,a^*)) \leq [(a-a^*)/a]WF(N^*(x,0)) + (a^*/a)WF(N^*(x,a))$$

and the theorem is proved.

Corollary *The limit* $\lim_{a^*\to 0} [WF(A\cap N^*(x,a^*)) - WF(A\cap N^*(x,0))]/a^* \equiv WF'(A\cap N^*(x,0)) \equiv WF'(\{x\}) \geq 0$ *exists*

Proof.

Immediate, since we have from 2 and the definition of WF the double inequality

$$0 \leq [WF(N^*(x,a^*)) - WF(N^*(x,0))] / a^* \leq [WF(N^*(x,a)) - WF(N^*(x,0))] / a$$

for $a^* < a$.

Since [WF(N*(x,a*)) - WF(N*(x,0))]/a* is a non-decreasing real function of a* the limit

$\text{Lim}_{a^* \to 0}$ [WF(N*(x,a*)) - WF(N*(x,0))] /a* ≡ WF'(N*(x,0)) ≡ WF'({x}) ≥ 0 exists.

Remark on minimax decision criterion

This corollary may be useful in certain problems of decision theory, for instance when minimax criteria of decision are involved.

Suppose for instance that F(x,m) is a loss function for a given decision-maker. If F is convex and for each x the respective value depends on the value of a parameter m that the decision-maker can control we can write for small values of a the expression of the variation of the value of the loss relative to the value of the present situation, F(x,m*), as

ΔWf ((N*(x,0),m) ≡ [WF((N*(x,a),m) – F(x,m)] + [F(x,m)-F(x,m*)] ≈ a.WF'(N*(x,0),m) + [F(x,m)-F(x,m*)].

If the decision-maker has the criterion of minimizing the highest potential increase of loss starting from the present level of loss F(x) he chooses the value m, m** such that

WF'(N*(x,0),m**) = $\min_{(m)}$ {WF'(N*(x,0),m))}

3. Conclusions. Decision theory, economic theory, convexity and *sms*

The results collected above show that important theorems about convexity can be proved for semi-metric spaces where the triangular inequality is eliminated (or K-relaxed).

Defining a semi-metric is frequently an adequate procedure to help a decision process.

Suppose for instance that in a macroeconomic policy framework the government intends to reach a situation S from the present situation T. As is frequently the case the political cost of a policy option is higher when the difference between the present situation and the intended one is larger. Most people don't like change. So a prudent policy would prefer to reach S from an intermediate situation U. That is T→U→S will minimize the political costs provided that electors quantify the difference between T and S, f(T,S) as larger than the sum f(T,U) + f(U,S), which is quite possible in subjective evaluations.

Another example, similar to the previous one, is the differentiation of possible alternative bundles of goods by a consumer. If X, Y and Z are three vectors with *n* components measured in the same unities, each component *i* being a quantity of good *i* the distance between the bundles can be calculated using the Euclidean metric in R^n. However the metric may not correspond to the way the consumer quantifies the difference between alternatives of consumption that, on the contrary, may well violate the triangular inequality.

For instance, it is usually assumed in consumer theory that there is for each consumer a utility function that is concave (or quasi-concave) relatively to the Euclidean metric. Actually the reasons for assuming this are convincing. Perhaps concavity relative to another semi-metric (the one that really differentiates alternatives) could be a better choice.

It is possible for instance to consider the utility u(A) that the consumer gates from obtaining the alternative A and suppose that u is a concave function that is

$[u(A) - u(B)]/f(B,A) \geq [u(B^*)- u(B))]/f(B,B^*)$ where $f(B,B^*) = f(B, A) + f(A,B^*)$ and A, B, B* are the characteristics that differentiate the alternatives, those characteristics not necessarily being the vectors of the quantities of each good consumed as in traditional consumer theory.

Convexity (and concavity) are crucial concepts for decision theory and economic theory. That is why is so important to study them in spaces *sms*. We think this may reveal itself as a promising new field of research.

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