Modelling the Joint Behaviour of Interest Rates and Foreign Exchange Rates

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Abstract

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Exposures to interest rate term structures in different currencies and their respective exchange rates are a challenge for risk management. In this paper we address this problem by extending the arbitrage-free Nelson Siegel model, an affine term structure model, to a multi-currency setting integrating exchange rate dynamics to allow forecasting of interest and exchange rates. We review the current state of research in term structure modelling and establish reasoning for using a three-factor model on interbank interest rates. Consequently, we provide the theoretical background for the dynamics of the state variables and the dependence of the exchange rate on the market risk premium. Moreover, to test the model empirically we establish an estimation framework using a Kalman filter. We show empirical results for different extensions of the arbitrage-free Nelson Siegel model. It is apparent that the forecasting performance is highly sensitive to the robustness of the estimation process.

A exposição a diversas estruturas de taxas de juros em moedas diferentes e a sua respectiva taxa de câmbio é um desafio para a gestão de risco. Nesta tese, este problema é analisado usando uma extensão do modelo de Nelson-Siegel sem arbitragem. Este modelo assume linearidade das taxas de juros e é extendido de forma a integrar as taxas de juros e a respectiva taxa de câmbio permitindo não só a previsão de taxas de juro como também da taxa de câmbio. Primeiro, a partir de uma revisão de literatura alargada, estabelecemos que se deve usar um modelo com três factores nas taxas de juro interbancárias. Consequentemente, nós definimos o contexto teórico para as dinâmicas das variáveis de estado e a dependência da taxa de câmbio no prémio de risco de mercado. Adicionalmente, um filtro de Kalman é utilizado para testar o modelo. Os resultados são apresentados para diferentes extensões do modelo Nelson-Siegel. Concluímos que a previsão depende da robustez do processo de estimação.

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List of Abbreviations

AFNS	**A**rbitrage-**F**ree **N**elson **S**iegel (model)
ATSM	**A**ffine **T**erm **S**tructure **M**odel
CNY	**C**hinese Renminbi
DNS	**D**ynamic **N**elson **Siegel** (model)
EFF	**E**xtended **F**und **F**acility
EURIBOR	**Eur**o **I**nter**b**ank **O**ffered **R**ate
FX	**F**oreign E**x**change
GBP	Pound Sterling
HIBOR	**H**ong Kong **I**nter**b**ank **O**ffered **R**ate
IBOR	**I**nter**b**ank **O**ffered **R**ate
IMF	**I**nternational **M**onetary **F**und
IRS	**I**nterest **R**ate **S**wap
JPY	**J**a**p**anese **Y**en
LIBOR	**L**ondon **I**nter**b**ank **O**ffered **R**ate
OLS	**O**rdinary **L**east **S**quares
PCA	**P**rincipal **C**omponent **A**nalysis
QTSM	**Q**uadratic **T**erm **S**tructure **M**odel
RMSE	**R**oot-**M**ean-**S**quare **E**rror
RMSFE	**R**oot-**M**ean-**S**quare **F**orecasting **E**rror
SDR	**S**pecial **D**rawing **R**ights
USD	**U**nited **S**tates **D**ollar

Chapter 1

Introduction

In this thesis we analyse the behaviour of yields across different maturities, condensed in the yield curve, over time for multiple currencies and the implied relationship to their respective exchange rates. The objective is to identify a model that captures said joint dynamics sufficiently in order to allow forecasting of both interest rates and foreign exchange rates. This can provide an instrument to gauge risks in fixed income instruments with a multi-currency exposure.

The motivation for this thesis stems from the exposure of financial institutions, sovereign governments and supranational organisations to interest rate and currency risk. An example of significant exposure for sovereign governments are the loans given by the International Monetary Fund (IMF) under the Extended Fund Facility (EFF) agreement. Within the Euro zone, currently Greece, Portugal and Cyprus are borrowers under such an agreement (International Monetary Fund, 2018).

The particular nature of IMF loans makes their risk management challenging. Obligations to the IMF are in its international reserve asset called Special Drawing Rights (SDR) which is essentially an artificial currency. An SDR represents a claim on a basket of five currencies, currently consisting of US-Dollar, Euro, Pound Sterling, Japanese Yen and Chinese Renminbi. The interest rate for these obligations is tied to reference rates in said currencies as well, resulting in a diverse exposure for the borrower. As of 30 April 2018 the outstanding amount under the EFF agreement in SDR for Greece, Portugal and Cyprus were SDR 9.04 bn., SDR 3.86 bn. and SDR 0.57 bn. respectively. The approximate equivalent value in US-Dollars were $13.00 bn., $5.6 and $0.82 bn.[1]

Obligations issued in foreign currencies are another possible application of the model as often their repayment is dependant on cash flows in a different currency. According to the European Central Bank (2018) approximately 14.53% of all debt securities outstanding within the Euro area as of March 2018 are in foreign currencies, which amounts to 2,433 bn. Euro. An example here is the $4.5 billion fixed coupon bond issued by Portugal in 2014 which marked the

[1] 1 SDR = US$1.43806 as of 30 April 2018 (International Monetary Fund, 2018)

country's return to the capital markets. IGCP, the Portuguese Treasury and Debt Management Agency, is in charge of hedging this risk using a range of derivatives.

In order to facilitate risk management, we apply an existing term structure model to this specific case and extend and adapt it to enable us to forecast short-term interest rate and foreign exchange rate movements. Primarily for institutions and sovereigns with significant expose to SDR this can be a useful tool to gauge the hedging requirements. First, we provide some theoretical background of term structure models and the definitions for the market price of risk, from which changes in foreign exchange rates can be deducted in chapter 2. In chapter 3 we provide reasoning for the model selection and adapt it to our specific case. Through principal component analysis we are able to see why three-factor models are useful to describe the term structure. We then extend the model to a multi-currency environment, taking into account foreign exchange rates and discuss the estimation framework using a Kalman filter.

Our results are presented in chapter 4 for several implementations of the model. First we estimate a two-currency model without incorporating foreign exchange rates. For robustness we estimate all models for different currency pairings, always considering the Euro as the domestic currency. We then move on to incorporating foreign exchange rates, which requires some changes in our estimation procedure. Finally, the model is extended to include four major currencies and their respective exchange rates.

We show that existing term structure models can be extended to capture multiple term structure and their respective exchange rates. While it is straightforward to achieve good in-sample fit, forecasting interest and exchange rates proves to be more difficult. It becomes apparent that the estimation of the model is the major challenge in term structure modelling with a large number of parameters.

Chapter 2

Literature Review

Interest rate modelling is a well-established topic in financial research. Piazzesi (2010) identifies four key reasons to study the dynamics of bond yields through term structure models: forecasting, monetary policy, debt policy as well as pricing and hedging of interest rate derivatives. The most widely adapted classes of term structure models are the affine term structure model (ATSM), the Nelson Siegel model and the quadratic term structure model (QTSM), with the ATSM class being the most popular and considered state-of-the-art (Ahn, Dittmar, & Gallant, 2002).

2.1 Evolution of Short Rate Models

First approaches to term structure models were taken with short-rate models, e.g. simple factor models as described by Vasicek (1977). The Vasicek model describes the instantaneous short rate as an Ornstein-Uhlenbeck process that is Gaussian and mean reverting with fixed parameters. As it allowed for negative interest rates, which had not been observed in the market at this time, Cox, Ingersoll, and Ross (1985) introduced another one factor model which avoided the issue, but has since fallen out of favour with practitioners as negative interest rates became reality after the financial crisis. According to Brigo and Mercurio (2006) these time-homogeneous models were the most successful due to their analytical tractability. They can be seen as the first affine term structure models (Piazzesi, 2010), which we discuss in more detail later on. Hull and White (1990) extended the Vasicek model by making the Vasicek parameter θ a deterministic, time-varying variable in order to allow for a better fit to the market-implied term structure. We refer to Brigo and Mercurio (2006) and Nawalkha, Beliaeva, and Soto (2007) who cover popular short rate models in great detail.

2.2 Affine Term Structure Models

Affine models are a class of term structure models, in which yields are simply an affine function of the latent state variables, which makes them particularly popular in research. They were introduced by Duffie and Kan (1996) and classified by Dai and Singleton (2000). Piazzesi (2010), who we refer to for a detailed introduction, defines the τ-period yield in an affine term structure model as an affine function on some state vector X_t

$$y(\tau) = A(\tau) + B(\tau)' X_t$$

where A and B depend on the period τ. In recent literature the terms "completely affine" and "essentially affine" have sprung up to differentiate between the definitions of the data-generating processes, specifically the market price of risk (Duffee, 2002). The famous Vasicek and Cox-Ingersoll-Ross models are examples of one-factor affine models, as they only depend on the short rate. The close relations of bond yields in the cross-section become apparent through cross-equation restrictions, which are reflected in the construction of A and B. As non-linear functions they can therefore not simply be estimated using ordinary least squares (OLS). The downside of the implementation of no-arbitrage is, as Piazzesi (2010) points out, the significantly more coding work and higher computational complexity. Furthermore, an internally consistent model does not necessarily perform better empirically (Diebold & Rudebusch, 2013) potentially caused by the high number of parameters. Diebold and Rudebusch (2013) point out that as a result statistically insignificant parameters are often set to zero without an underlying theoretical reasoning.

2.3 The Nelson-Siegel Class of Term Structure Models

The parametric model proposed by Nelson and Siegel (1987) is a well-known model for fitting the cross-section of yields to market data to obtain a smooth curve. They introduce a "simple, parsimonious model that is flexible enough to represent the range of shapes generally associated with yield curves: monotonic, humped, and S shaped" (p. 473). In the Nelson-Siegel model, the continuously compounded yield with maturity τ is given by

$$y(\tau) = \beta_0 + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) \beta_1 + \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \beta_2$$

The three factors β_0, β_1 and β_2, are commonly referred to as level, slope and curvature of the yield curve. The λ parameter scales how quickly the slope and curvature components approach zero in the limit, i.e. how rapidly the yield curve asymptotically approaches β_0 for longer maturities.

Svensson (1994) later extended the three-factor Nelson-Siegel model by a fourth term to allow for the representation of an even wider range of yield curves. A survey by the Bank for International Settlements (2005) suggests that this extension, known as extended Nelson-Siegel or (Nelson-Siegel-)Svensson model, is established among practitioners, especially central banks.

While the aforementioned Nelson Siegel model and the extension by Svensson only describe the cross-section of interest rates at a given point in time, Diebold and Li (2006) aim to make the factors time-dependent in order to establish accurate out-of-sample forecasts in a dynamic Nelson Siegel model. Their model does not enforce no-arbitrage principles. They add that parsimonious models often exhibit superior forecasting performance.

Christensen, Diebold, and Rudebusch (2011) essentially close the gap between the parsimonious dynamic Nelson-Siegel model and the more theoretically rigorous affine term structure models. They place the dynamic Nelson-Siegel model in the affine context as defined in the Duffie and Kan (1996) framework and find that a specific model of the Duffie-Kan class can be derived that has identical factor loadings as seen in the dynamic Nelson Siegel model. They show that under the affine framework, given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ under the customary conditions (cf. Øksendal, 2003; Williams, 1991), zero-coupon bond yields can be written as

$$y_t(\tau) = X_t^1 + \frac{1-e^{-\lambda\tau}}{\lambda\tau} X_t^2 + \left[\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right] X_t^3 - \frac{A(\tau)}{\tau}, \quad \tau = T - t \tag{2.1}$$

The term $\frac{A(\tau)}{\tau}$ stems from placing the model in the affine setting and is called "yield-adjustment term". It is not time-dependent, but only affected by the maturity and the choice of parameters for the state variable dynamics.

Christensen et al. (2011) assume an essentially affine specification of the market risk premium following Duffee (2002), which allows for flexible specification of the dynamics under the physical measure while keeping risk-neutral dynamics identical. They present both an independent- and correlated-factor Arbitrage-Free Nelson Siegel (AFNS) model, which differ in the construction of the parameter matrices under the physical measure. In their empirical evaluation they find that while the correlated-factor model exhibits superior in-sample fit, the independent-factor model has better forecasting abilities. It also outperforms the Dynamic Nelson Siegel (DNS) model as presented by Diebold and Li (2006).

Chapter 3

Data and Methodology

In this chapter we present the data used for empirical valuation of the model. Some data pre-processing is necessary to obtain the zero-coupon bond rates we use in the model. We provide justification for our model selection using principal component analysis and empirical proxies for the factors of the Nelson Siegel model. Lastly we explain the theoretical foundation for our model and its implementation in detail. All computations are conducted using Python. Python provides a wide arrange of libraries that facilitate a high speed of development. The most important libraries for our research are QuantLib for Python, Numpy and Pandas.

3.1 Data

For the purpose of this thesis we use Interbank Offered Rates (IBOR) to gauge the current state of yields for each currency, although SDR interest rates are linked to sovereign bonds. Due to the use of interbank rates as a reference for all kinds of fixed income instruments from floating rate notes to derivatives, their importance, especially in hedging interest rate risks, cannot be overstated. Furthermore, the EURIBOR and its respective swap curve provide us with a single yield curve for the entire Euro. Using treasury/government bond yields would leave us with either having to choose a single country's bonds or artificially constructing one covering all Euro zone members.

As mentioned in the introduction, the Portuguese government has significant exposure to foreign currencies, which is managed by the Portuguese Treasury and Debt Management Agency. In this dissertation we focus on the major currencies, i.e. Euro (EUR), US-Dollar (USD), Pound Sterling (GBP) and Japanese Yen (JPY). The currencies were the most traded as of April 2016 according to a survey by the Bank for International Settlements (2016). For these currencies, the following rates have been obtained:

Euro EURIBOR and Euro swap curve

US-Dollar	ICE LIBOR USD and US-Dollar swap curve
Japanese Yen	ICE LIBOR JPY and Yen swap curve
Pound Sterling	ICE LIBOR GBP and Pound Sterling swap curve

Since the admission of the Chinese Renminbi (CNY) to the Internation Monetary Funds Special Drawing Rights currency basket an exposure to this currency also exists. Due to limited data availability (the CNY HIBOR has only been established in 2013) this thesis does not consider this exposure.

IBORs are zero-coupon bond rates with maturities up to one year. The remainder of the yield curve must be inferred from the swap curves. Here we use a bootstrapping approach to obtain the missing zero-coupon bond yields and also adjust for different conventions in the swap contracts (e.g. day count conventions) to produce homogeneous data for the following analysis.[1] The tenors used for bootstrapping the zero-coupon bond rates are

IBOR	1, 2, 3, 6 and 12 months
IRS rates	2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 25 and 30 years

We have obtained monthly data for interbank rates, swap rates and foreign exchange rates for a time frame between August 2011 and January 2018 from Bloomberg. All data has been cross-checked with Thomson Reuters Eikon.

Figure 3.1 shows the evolution of the bootstrapped interest rates for Euro, US-Dollar, Pound Sterling and Japanese Yen for our data window. The Western currencies exhibit some common patterns. Leading up to the financial crisis the spreads between short- and long-term are significantly compressed for the Euro and US-Dollar. The GBP LIBOR yield curve starts to invert from 2005, the US LIBOR curve temporarily inverts leading up to the financial crisis and the EURIBOR curve shortly before the Lehman collapse. All three curves show a significant decline of short-term rates after the Lehman collapse due to the intervention of sovereign governments and central banks. The JPY LIBOR does not feature such a significant drop in absolute terms as yields in Yen have been at a low level for an extended time. Post-crisis all currencies show low volatility of short-term rates, the only significant changes being increase of the EURIBOR during the Euro zone crisis and the temporary inversion of the Yen curve in 2017.

[1] In the present multicurve interest rate environment bootstrapping zero-coupon rates swaps is technically outdated. For our purposes we assume it to be a sufficient approximation. More details on this topic can be found in Mercurio (2010).

FIGURE 3.1: Bootstrapped zero-coupon bond yields for the selected currencies; maturities are 6 months, 5, 10 and 30 years; financial crisis marked in grey

3.2 Principal Component Analysis

The widely-adopted Nelson Siegel describes the shape of the yield curve through three factors with clear financial and macroeconomic implications: level, slope and curvature. As we have presented, Christensen et al. (2011) present an arbitrage-free Nelson Siegel model as a specific case of the Duffie-Kan class of affine models with identical factor loadings.

To provide support for the choice of this model and its factors we conduct a Principal Component Analysis (PCA) on the gathered interest rate data in order to compare the results with standard empirical proxies for the three factors of Nelson-Siegel models. PCA is a method for dimension-reduction of correlated data by transforming them into a smaller number of uncorrelated principal components or factors that describe as much of the variation as possible. For a more detailed explanation of PCA we refer to Jolliffe (2002). The attribution of the first three factors as level, slope and curvature was established empirically by Litterman and Scheinkman (1991). The proxies for these factors are determined according to Diebold, Rudebusch, and Boragan Aruoba (2006).

$$
\begin{aligned}
L_t &= \frac{1}{3}(y_t(3) + y_t(24) + y_t(120)) \\
S_t &= y_t(3) - y_t(120) \\
C_t &= 2 \cdot y_t(24) - y_t(3) - y_t(120)
\end{aligned}
$$

where $y_t(m)$ represents the m months maturity yield.

In a principal component analysis conducted on the local interbank rates for each currency described in section 3.1 we find that three factors explain between 98.54% and 99.77% of the variation in each individual IBOR curve. Furthermore, the local empirical proxies are mostly correlated with the individual PCA factors. Hence it appears that the assumption of a three-factor model to describe the term structure is justified and that these three factors can be identified as level, slope and curvature.

In addition to the local PCA conducted, we analyse if we can establish a similar clear structure in global interbank rate data. Here we can observe that three factors already explain 94.46% of our data's variation (cf. table 3.4).

In order to provide some further insights into the behaviour of the proxies across currencies, we compute the correlation matrices. For the level proxy (cf. table 3.1), as defined before, we see that they generally show high correlation with the highest being between EUR and GBP with $\rho = 0.94$ and lowest between USD and JPY with $\rho = 0.68$. This establishes that the level

	EUR	USD	JPY	GBP
EUR	1.00	0.81	0.80	0.94
USD	0.81	1.00	0.68	0.89
JPY	0.80	0.68	1.00	0.72
GBP	0.94	0.89	0.72	1.00

TABLE 3.1: Correlation matrix for the level proxies

factors of the term structures across currencies are highly intertwined. From economic intuition this is reasonable as the level factor embeds expectations of economic activity and future short-term rates which are unlikely to diverge between major economies over a longer time horizon.

The correlation weakens for the slope proxies, foremost for pairings with the Japanese Yen (cf. table 3.2). Among the remaining currencies, correlations remain high. We therefore conclude that the behaviour of the slope factor is different between the major Western currencies and the Yen. This is possibly attributable to different central bank policies which affect the short end of the yield curve and therefore change the slope factor.

	EUR	USD	JPY	GBP
EUR	1.00	0.76	0.26	0.78
USD	0.76	1.00	0.14	0.65
JPY	0.26	0.14	1.00	-0.16
GBP	0.78	0.65	-0.16	1.00

TABLE 3.2: Correlation matrix for the slope proxies

In the correlation matrix for the curvature proxies (cf. table 3.3) we can observe a similar picture as for the slope proxy, although with overall lower correlations. The Yen again shows the lowest correlation with the remaining currencies.

	EUR	USD	JPY	GBP
EUR	1.00	0.64	0.38	0.65
USD	0.64	1.00	0.38	0.52
JPY	0.38	0.38	1.00	-0.09
GBP	0.65	0.52	-0.09	1.00

TABLE 3.3: Correlation matrix for the curvature proxies

Finally, we look at the correlations between the PCA factors and the proxies (cf. table 3.4). The first factor shows high correlation with all proxies, the least correlated being the Yen. We can conclude that changes in the (global) level factors therefore also result in changes in the each

	EUR	USD	JPY	GBP	Var.
PCA Factor 1	0.97	0.88	0.78	0.97	79.35%
PCA Factor 2	0.41	0.59	0.72	-0.02	8.88%
PCA Factor 3	-0.35	-0.43	0.36	-0.68	6.23%
Cum. Var. explained					94.46%

TABLE 3.4: Correlations between PCA factors and proxies, the last column represents the variance explained by the PCA factors

local level factor, which is also shown in figure 3.2. For the other two PCA factors, formerly identified as slope and curvature, the effects are much less pronounced. The slope and curvature proxies for Euro, US-Dollar and Pound Sterling seem be only correlated with either the second or third PCA factors. This leads to the conclusion that the second and third latent global PCA factors do not precisely capture the changes in the local proxies for slope and curvature. We come back to these results in section 3.3.

The arbitrage-free Nelson Siegel model is based on the three factors: level, slope and curvature. The analysis shows that three factors are sufficient to explain a high percentage of the variation within the interest rate data which provides support the use of a Nelson Siegel class model. Empirical proxies for the level factor are highly correlated among the major currencies. Furthermore, they show high correlation with the first PCA factor. The level can therefore be assumed to be a global factor in our model, which we discuss in more detail in the following chapter.

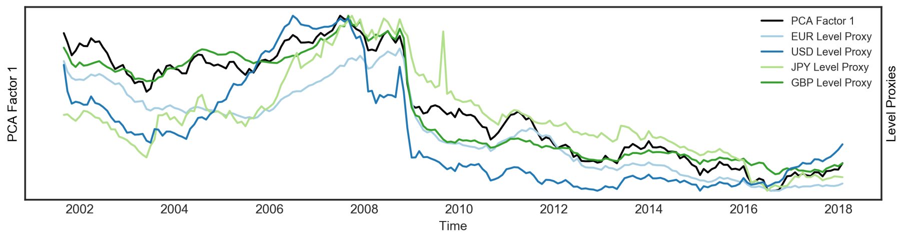

FIGURE 3.2: First PCA factor and level proxies for each currency over time

3.3 The Arbitrage-Free Nelson Siegel Model

After establishing the intuition about the three factor that drive the yield curve through PCA in section 3.2 we now move on to implementing an independent-factor arbitrage-free Nelson Siegel model for our specific case. Even though it is simpler in the structure of its parameters, Christensen et al. (2011) show that the independent-factor model has superior forecasting performance in comparison to the correlated-factor model. In addition it exhibits less complexity and we expect a simpler estimation process. Some changes need to be made to the model as presented by Christensen et al. (2011) as their model only describes a single term structure. We aim to adapt the model enabling it to caption the dynamics of both interest rates and foreign exchange rates in multiple currencies. We start out with a two-currency structure without taking into account foreign exchange rates, then implementing exchange rate dynamics and finally provide to means to model four currencies including their foreign exchange rates.

3.3.1 Model Implementation

In equation 2.1 we see the definition of zero-coupon bonds in the AFNS according to Christensen et al. (2011). To model several term structures at once we need to estimate this for all currencies' term structures. The state variables $X_t = (X_t^1, X_t^2, X_t^3)$ are described under the risk-neutral measure Q by the following SDEs (cf. Christensen et al., 2011)

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right] dt + \Sigma \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix} \qquad \lambda > 0 \tag{3.1}$$

The essentially affine definition of the market price of risk as in equation (3.2) allows for flexibility in the construction of the parameters under the physical measure while keeping affine dynamics. Duffee (2002) assumes that the market price of risk has the form

$$\Gamma_t = \gamma_0 + \gamma_1 X_t \tag{3.2}$$

An independent-factor AFNS assumes the following dynamics (with $\lambda > 0$) under the physical measure

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} \kappa_{11}^P & 0 & 0 \\ 0 & \kappa_{22}^P & 0 \\ 0 & 0 & \kappa_{33}^P \end{pmatrix} \left[\begin{pmatrix} \theta_1^P \\ \theta_2^P \\ \theta_3^P \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{pmatrix} \tag{3.3}$$

The SDEs shown in equation 3.3 are Gaussian Ornstein-Uhlenbeck processes. In these, the first parameter (K^P) can be interpreted as the mean reversion rate, the second (θ^P) as the mean reversion level. The Σ parameter controls the diffusion of the stochastic process.

As we established in section 3.2 we observe that in our data the first global PCA factor, identified as the level, is highly correlated with the proxies for all the currencies. Hence, we assume a global level factor X_t^1 in our joint model. Our reasoning for this assumption is twofold: first and foremost, this provides us with a mean to capture the mutual term structure dynamics across currencies and secondly it facilitates the estimation process by decreasing the dimensionality of the optimisation problem. As we show in table 3.4 the global PCA factors 2 and 3, identified as slope and curvature, do not capture the changes in the proxies sufficiently across all currencies, therefore we decide to define these aslocal factors, implying independent factor dynamics for each currency. In these equations the global parameters and state variables are distinguished from the local ones by the subscript g for global, d for domestic and f for foreign. In the multi-currency model, each foreign currency is driven by a set of SDEs in the form of equation 3.5.

Under this specification the SDEs for our case in an independent-factor AFNS under the P-measure become (for the domestic currency)

$$\begin{pmatrix} \mathrm{d}X_t^{1,g} \\ \mathrm{d}X_t^{2,d} \\ \mathrm{d}X_t^{3,d} \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{P,g} & 0 & 0 \\ 0 & \kappa_{22}^{P,d} & 0 \\ 0 & 0 & \kappa_{33}^{P,d} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{P,g} \\ \theta_2^{P,d} \\ \theta_3^{P,d} \end{pmatrix} - \begin{pmatrix} X_t^{1,g} \\ X_t^{2,d} \\ X_t^{3,d} \end{pmatrix} \right] \mathrm{d}t + \begin{pmatrix} \sigma_{11}^{g} & 0 & 0 \\ 0 & \sigma_{22}^{d} & 0 \\ 0 & 0 & \sigma_{33}^{d} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^{1,g} \\ \mathrm{d}W_t^{2,d} \\ \mathrm{d}W_t^{3,d} \end{pmatrix} \tag{3.4}$$

and for the foreign currency

$$\begin{pmatrix} \mathrm{d}X_t^{1,g} \\ \mathrm{d}X_t^{2,f} \\ \mathrm{d}X_t^{3,f} \end{pmatrix} = \begin{pmatrix} \kappa_{11}^{P,g} & 0 & 0 \\ 0 & \kappa_{22}^{P,f} & 0 \\ 0 & 0 & \kappa_{33}^{P,f} \end{pmatrix} \left[\begin{pmatrix} \theta_1^{P,g} \\ \theta_2^{P,f} \\ \theta_3^{P,f} \end{pmatrix} - \begin{pmatrix} X_t^{1,g} \\ X_t^{2,f} \\ X_t^{3,f} \end{pmatrix} \right] \mathrm{d}t + \begin{pmatrix} \sigma_{11}^{g} & 0 & 0 \\ 0 & \sigma_{22}^{f} & 0 \\ 0 & 0 & \sigma_{33}^{f} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^{1,g} \\ \mathrm{d}W_t^{2,f} \\ \mathrm{d}W_t^{3,f} \end{pmatrix} \tag{3.5}$$

Christensen et al. (2011) derive the yield-adjustment term, which we introduce in equation 2.1, for both the correlated and independent-factor AFNS in analytical form. For an independent-factor AFNS model the yield-adjustment term $\frac{A(t,T)}{T-t}$ has the form

$$\begin{aligned} \frac{A(t,T)}{T-t} = {} & \sigma_{11}^2 \frac{(T-t)^2}{6} + \sigma_{22}^2 \left[\frac{1}{2\lambda^2} - \frac{1}{\lambda^3} \frac{1-e^{-\lambda(T-t)}}{T-t} + \frac{1}{4\lambda^3} \frac{1-e^{-2\lambda(T-t)}}{T-t} \right] \\ & + \sigma_{33}^2 \left[\frac{1}{2\lambda^2} + \frac{1}{\lambda^2} e^{-\lambda(T-t)} - \frac{1}{4\lambda}(T-t) e^{-2\lambda(T-t)} - \frac{3}{4\lambda^2} e^{-2\lambda(T-t)} \right. \\ & \left. - \frac{2}{\lambda^3} \frac{1-e^{-\lambda(T-t)}}{T-t} + \frac{5}{8\lambda^3} \frac{1-e^{-2\lambda(T-t)}}{T-t} \right] \end{aligned}$$

3.3.2 Modelling Foreign Exchange Rates

In order to extend our model to foreign exchange rates we have to understand what drives exchange rate dynamics. The main relevant concepts to look at here are the pricing kernel (also known as stochastic discount factor) and the market price of risk. The market price of risk Γ_t, that we introduced in equation 3.2 reflects the change of drift between the risk-neutral and physical probability measure under the Girsanov theorem. A detailed review of the Girsanov theorem is beyond the scope of this thesis, for further reference we recommend Steele (2001) and Øksendal (2003).

According to Backus, Foresi, and Telmar (2001) pricing kernels are "stochastic processes governing the prices of state-contingent claims", that satisfy

$$1 = E_t(M_{t+1}R_{t+1})$$

where m is the pricing kernel and R_{t+1} represents the return on an asset between t and $t+1$. A pricing kernel must exist in a complete market that prohibits arbitrage opportunities, which is related to the existence of an equivalent martingale measure under the absence of arbitrage as established by Harrison and Kreps (1979). For the purpose of this thesis, we assume the following specification of the pricing kernel (cf. Egorov, Li, & Ng, 2011)

$$\frac{\mathrm{d}M_t}{M_t} = -r_t\,\mathrm{d}t - \Gamma_t'\,\mathrm{d}W_t \tag{3.6}$$

We define S_t as the domestic currency spot price of one unit of foreign currency and call S_{t+1}/S_t the depreciation rate of said domestic currency. Backus et al. (2001) define the relationship between exchange rates and pricing kernels as

$$\frac{M^f_{t+1}}{M^d_{t+1}} = \frac{S_{t+1}}{S_t} \tag{3.7}$$

Under the application of Itô's Lemma we can obtain the SDE governing the exchange rate dynamics from equations 3.6 and 3.7.

$$\frac{\mathrm{d}S_t}{S_t} = \left[(r^d_t - r^f_t) + \Gamma^{d\prime}_t(\Gamma^d_t - \Gamma^f_t)\right]\mathrm{d}t + (\Gamma^d_t - \Gamma^f_t)'\,\mathrm{d}W_t \tag{3.8}$$

Analysing equation 3.8 we observe that the exchange rate dynamics are governed by the interest rate differential between the domestic and foreign instantaneous interest rate and the quadratic differential of the market price of risk. Equation 2.1 implies that the instantaneous interest rate is the sum of the first two state variables, i.e. $r_t = X_t^1 + X_t^2$. Following Duffee (2002) we assume

$$\begin{aligned}\Gamma_t &= \gamma^0 + \gamma^1 X_t \\ \mathrm{d}X_t &= K^Q(\theta^Q - X_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t^Q\end{aligned}$$

Through change of measure under the Girsanov theorem

$$\begin{aligned}\mathrm{d}W^Q &= \mathrm{d}W^P + \Gamma_t \\ K^Q(\theta^Q - X_t)\,\mathrm{d}t + \Sigma\left(\mathrm{d}W_t^P + \Gamma_t\,\mathrm{d}t\right) &= K^P(\theta^P - X_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t^P \\ K^Q(\theta^Q - X_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t^P + \Sigma\Gamma_t\,\mathrm{d}t &= K^P(\theta^P - X_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t^P \\ K^Q(\theta^Q - X_t)\,\mathrm{d}t + \Sigma\Gamma_t\,\mathrm{d}t &= K^P(\theta^P - X_t)\,\mathrm{d}t \\ K^Q\theta^Q - K^Q X_t + \Sigma(\gamma^0 + \gamma^1 X_t) &= K^P(\theta^P - X_t) \\ K^Q\theta^Q - K^Q X_t + \Sigma\gamma^0 + \Sigma\gamma^1 X_t &= K^P(\theta^P - X_t) \\ -(K^Q - \Sigma\gamma^1)X_t + K^Q\theta^Q + \Sigma\gamma^0 &= K^P(\theta^P - X_t) \\ \left(K^Q - \Sigma\gamma^1\right)\left(\frac{K^Q\theta^Q + \Sigma\gamma^0}{K^Q - \Sigma\gamma^1} - X_t\right) &= K^P(\theta^P - X_t)\end{aligned}$$

This implies that parameters for the market price of risk (cf. equation 3.2) γ_0 and γ_1 are

$$\begin{aligned}K^P &= K^Q - \Sigma\gamma^1 \\ \gamma^1 &= \Sigma^{-1}(K^Q - K^P)\end{aligned}$$

and imposing $\theta^Q = 0$, which is equivalent to fixing the mean levels of the state variables (cf. Christensen et al., 2011)

$$\begin{aligned}\theta^P &= (K^P)^{-1}\,(K^Q\theta^Q + \Sigma\gamma^0) \\ \theta^P &= (K^P)^{-1}\,\Sigma\gamma^0 \\ \gamma^0 &= \Sigma^{-1}\,K^P\theta^P\end{aligned}$$

3.3.3 Estimation Framework

We continue with describing the estimation process for the two and four currency implementations of the model presented beforehand. We conduct a maximum likelihood estimation using a Kalman filter to determine the parameters following the approach of Christensen et al. (2011).

The exchange rate calibration requires the use of an extended Kalman filter due to the non-linear properties of the exchange rate dynamics. The number of parameters to be estimated is 20 for the two-currency model and 34 for the multi-currency model. The conditional mean and covariance matrix in the AFNS model have the following closed-form solution

$$
\begin{aligned}
E^P[X_T|\mathcal{F}_t] &= (\mathbb{I} - exp(-K^P\Delta t))\theta^P + exp(-K^P\Delta t)X_t \qquad \Delta t = T - t \\
V^P[X_T|\mathcal{F}_t] &= \int_0^{\Delta t} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)' s}\,\mathrm{d}s
\end{aligned}
$$

The state transition equation has the form

$$X_t = (\mathbb{I} - exp(-K^P\Delta t))\theta^P + exp(-K^P\Delta t)X_t + \eta_t \qquad \Delta t = t_i - t_{i+1}$$

Based on the equation 2.1 the AFNS measurement equation $y_t = A + BX_t + \varepsilon_t$ for each separate currency is

$$
\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} - \begin{pmatrix} \frac{A(\tau_1)}{\tau_1^2} \\ \frac{A(\tau_2)}{\tau_2^2} \\ \vdots \\ \frac{A(\tau_N)}{\tau_N^2} \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix}
$$

Here we denote the vector containing the yield-adjustment term as A and the factor loadings matrix as B. Both are dependent on the model parameters, which we denote as ψ. N indicates the number of observations. The transition and measurement error are diagonal matrices with the variance Q and H

$$
\begin{aligned}
\eta_t &\sim \mathcal{N}(0,\, Q) \\
\varepsilon_t &\sim \mathcal{N}(0,\, H) \\
Q &= \int_0^{\Delta t} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)' s}\,\mathrm{d}s \\
H &= \mathrm{diag}(\sigma_\varepsilon^2(\tau_1), \ldots, \sigma_\varepsilon^2(\tau_N))
\end{aligned}
$$

We initialise the state vector with

$$E[X_t|\mathcal{F}_0] = \theta^P \qquad var[X_t|F_0] = \int_0^{\inf} e^{-K^P s}\Sigma\Sigma' e^{-(K^P)' s}\,\mathrm{d}s$$

The prediction step is

$$
\begin{aligned}
E^P[X_t|\mathcal{F}_{t-1}] &= \Phi_t^{X,0}(\psi) + \Phi_t^{X,1}(\psi)X_{t-1} \\
\text{var}[X_t|\mathcal{F}_{t-1}] &= \Phi_t^{X,1}(\psi)\text{var}[X_{t-1}]\Phi_t^{X,1}(\psi)' + Q_t(\psi)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_t^{X,0}(\psi) &= (\mathbb{I} - exp(-K^P\Delta t))\theta^P \\
\Phi_t^{X,1}(\psi) &= exp(-K^P\Delta t)
\end{aligned}
$$

The state vector is then updated according to

$$
\begin{aligned}
X_t = E^P[X_t|\mathcal{F}_t] &= E^P[X_t|\mathcal{F}_{t-1}] + \text{var}[X_t|\mathcal{F}_{t-1}]B(\psi)'F_t^{-1}v_t \\
var[X_t] = \text{cov}(v_t) &= B(\psi)\text{var}[X_t|\mathcal{F}_{t-1}]B(\psi)' + H(\psi)
\end{aligned}
$$

where

$$
\begin{aligned}
v_t &= y_t - E[y_t|\mathcal{F}_{t-1}] = y_t - A(\psi) - B(\psi)E^P[X_t|\mathcal{F}_{t-1}] \\
F_t &= \text{cov}(v_t) = B(\psi)\text{var}[X_t|\mathcal{F}_{t-1}]B(\psi)' + H(\psi)
\end{aligned}
$$

The Gaussian log likelihood for a given set of parameters ψ is

$$
\log l(y;\psi) = -\frac{N}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\left[\log|F_t| - \frac{1}{2}v_t'F_t^{-1}v_t\right] \tag{3.9}
$$

The extended Kalman filter is required incorporate the exchange rates into the estimation. As we pointed out previously the exchange rate dynamics are a quadratic function of the market price of risk for each currency and therefore of the state variables X_t^d and X_t^f. The extended Kalman filter uses a first order Taylor series expansion around the prediction point to linearise the measurement equation. Assuming the same measurement error as for the interest rates ε_t, the measurement equation for the exchange rates is

$$
\frac{\Delta S_t}{S_{t-1}} = \Phi(X_t) + \varepsilon_t
$$

We consider a discretisation of the continuous time exchange rate dynamics to approximate the percentage change

$$E^P\left[\frac{\Delta S_t}{S_t}\middle|\mathcal{F}_t\right] = \Phi(E^P[X_t|\mathcal{F}_t]) = E^P\left[(r^d_{t+1}-r^f_{t+1})+\Gamma^{d\prime}_{t+1}(\Gamma^d_{t+1}-\Gamma^f_{t+1})\middle|\mathcal{F}_t\right]\Delta t$$

The measurement equation therefore becomes

$$\frac{\Delta S_t}{S_{t-1}} = \left[(r^d_t - r^f_t)+\Gamma^{d\prime}_t(\Gamma^d_t-\Gamma^f_t)\right]\Delta t + \varepsilon_t \qquad \Delta S_t = S_t - S_{t-1}$$

The first order Taylor series expansion takes the form

$$\Phi(X_t) = \Phi(E^P[X_t|\mathcal{F}_{t-1}]) + \left.\frac{\partial\Phi(X_t)}{\partial X_t}\right|_{E^P[X_t|\mathcal{F}_{t-1}]}(X_t - E^P[X_t|\mathcal{F}_{t-1}]) \tag{3.10}$$

$$\tag{3.11}$$

By rearranging our linear approximation of $\Phi(X_t)$ in equation 3.10 we find that

$$\begin{aligned}
A_{FX}(\psi) &= \Phi(E^P[X_t|\mathcal{F}_{t-1}]) + \left.\frac{\partial\Phi(X_t)}{\partial X_t}\right|_{E^P[X_t|\mathcal{F}_{t-1}]} E^P[X_t|\mathcal{F}_{t-1}]) \\
B_{FX}(\psi) &= \left.\frac{\partial\Phi(X_t)}{\partial X_t}\right|_{E^P[X_t|\mathcal{F}_{t-1}]}
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial\Phi(X_t)}{\partial X_t} &= \frac{\partial}{\partial X_t}\left[(r^d_t - r^f_t)+\Gamma^{d\prime}_t(\Gamma^d_t-\Gamma^f_t)\right]\Delta t \\
&= \frac{\partial}{\partial X_t}\Big[(X^{2,d}_t - X^{2,f}_t) + (\gamma^d_0)'(\gamma^d_0-\gamma^f_0) + (\gamma^d_0)'\gamma^d_1 X^d_t - (\gamma^d_0)'\gamma^f_1 X^f_t) \\
&\quad + (X^d_t)'(\gamma^d_1)'(\gamma^d_0-\gamma^f_0) + (X^d_t)'(\gamma^d_1)'\gamma^d_1 X^d_t - (X^d_t)'(\gamma^d_1)'\gamma^f_1 X^f_t\Big]\Delta t \\
&= \Big[(\gamma^d_1)'\gamma^d_0 - (\gamma^f_1)'\gamma^d_0 + (\gamma^d_1)'(\gamma^d_0-\gamma^f_0) + (\gamma_1 d)'\gamma^d_1 X^d + (\gamma^d_1)'\gamma_1 d X^d_t \\
&\quad - (\gamma^d_1)'\gamma^f_1 X^f_t - (\gamma^f_1)'\gamma^d_1 X^f_t\Big]\Delta t
\end{aligned}$$

With these results we can use the Kalman filter as described before by combing the terms factor loadings matrices B and B_{FX} and the (yield-)adjustment terms A and A_{FX}. Moreover the vector v_t has to be extended to incorporate $\Delta S_t/S_{t-1}$.

Unlike Christensen et al. (2011), who use a Nelder-Mead simplex algorithm, we numerically maximize our likelihood function using a particle swarm optimization algorithm as outlined by

Kennedy and Eberhart (1995). We find that in our extended model with a higher number of parameters this is computationally more efficient. An in-depth study of deployable algorithms for the given optimization problem is beyond the scope of this thesis.

Chapter 4

Results

In the following chapter we present our results given the data described in section 3.1 and our estimation framework given in section 3.3.3. We progressively extend the model starting with a plain two-currency AFNS model, then extending it to foreign exchange rates and lastly to a four-currency model. In order to evaluate forecasting performance of the models we use a backtesting approach. For the two-currency models we estimate the model quarterly and compare the forecasts to the real rates. Our forecasting horizon are 1, 3, 6 and 12 months. Due to the time-consuming estimation process, the multi-currency model is only estimated annually. For both cases the minimum training window for the model is limited to 5 years.

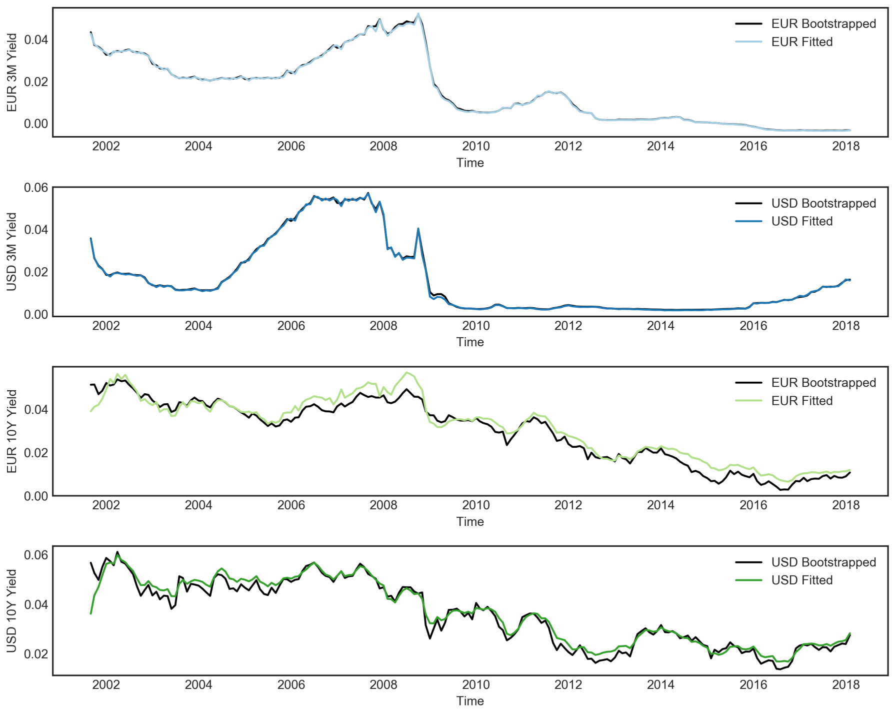

FIGURE 4.1: In-Sample fit for a two-currency (EUR/USD) AFNS model. Three maturities are shown: 3 months, 2 years and 10 years.

As we presented in section 3.1 we use bootstrapped zero-coupon bond interbank rates as the underlying data of our model. A crucial point from our perspective is the choice of maturities among the variety of rates available that we use to calibrate the model. For the purpose of this thesis the maturities we select are 1, 2, 3, 6 and 12 months as well as the 5, 10 and 25 year rates. Our intuition for overweighting short-term rates stems from the assumption that they are a stronger driver of foreign exchange spot rates. The trade-off is a potentially decreasing ability of the model to capture they dynamics of the long end of the yield curve. Limiting our choice of maturities reduces the computational burden in the estimation process. With higher computational capacities an extension of the fit of the model to a wider array of maturities is feasible.

4.1 Two-Currency AFNS

First, we take a look at the most simple extension of the original arbitrage-free Nelson Siegel model: an extension to two currencies. As introduced in chapter 3 we will assume a global level factor and local slope and curvature factors. We show in-sample fit of the model estimated as laid out in section 3.3.3 and then we evaluate forecasting performance.

Figure 4.1 shows in-sample fit for a two-currency model for three different maturities and compares them to the bootstrapped zero-coupon bond rates based on actual data. As we can observe, we obtain good in-sample performance of the model for the EUR/USD currency pairing. Table 4.1 shows and residual means and root-mean-square errors (RMSE) for a range on maturities. The RMSE is noticeably smaller for short-term maturities, which is in line with the choice of maturities to calibrate the model. As we chose to overweight short-term maturities for the estimation, we expect to see a better fit here. It is evident that the model is able to capture the abrupt changes of the short-term interest rates during the financial crisis of 2008 while at the same time replicating the following period of stable, low interest rates.

We conduct the same estimation process for the EUR/JPY and EUR/GBP for robustness and find similar results. The results are consistent with the in-sample fitting errors shown by Christensen et al. (2011). As expected our RMSEs are slightly higher, which is reasonable due to their estimation just capturing a single term structure. It has to be noted that a direct comparison is difficult due the different data used by them and can only serve as a mere guideline.

We investigate the out-of-sample forecasting performance of the two-currency AFNS empirically find that the root-mean-square forecasting errors (RMSFE) are significantly larger than the RMSE we observe in-sample. Table 4.2 shows the RMSFE for the EUR/USD pairing. As expected we see that the forecasting error increases for longer forecasting horizons. Curiously

	Euro		US-Dollar	
Maturity	Mean	RMSE	Mean	RMSE
1	**-0.78**	**2.76**	**0.01**	**2.54**
2	**-2.54**	**4.50**	**-1.98**	**4.15**
3	**1.50**	**4.02**	**1.82**	**4.51**
6	**1.97**	**4.47**	**0.73**	**3.67**
12	**0.48**	**2.78**	**-0.33**	**2.45**
24	-9.41	13.76	-13.17	18.68
36	-16.95	39.29	-16.13	45.82
48	-22.23	43.80	-19.57	45.35
60	**-25.61**	**45.39**	**-20.98**	**42.34**
72	-26.99	45.12	-20.52	38.34
84	-26.86	43.60	-18.84	34.42
96	-25.53	41.38	-16.42	31.19
108	-23.33	38.74	-13.23	28.32
120	**-20.41**	**36.03**	**-9.11**	**26.07**
144	-12.38	31.80	1.52	24.57
180	1.18	31.35	18.39	31.60
240	24.91	42.34	47.60	55.46
300	**51.85**	**62.83**	**80.29**	**86.82**

TABLE 4.1: Summary statistics for in-sample model fit. Maturities are in months; residual means and root-mean-square errors are in basis points. Maturities used to fit the model are printed in bold.

this is especially pronounced for the rates up to one year, which were overweighted in the estimation. Among the mid- to long-term maturities the errors increase significantly less for longer forecasting horizons. We obtain similar results for EUR/GBP and EUR/JPY, indicating that this is a general issue of the estimation process.

The one-month ahead forecast shown in figure 4.2 show that the forecasts are overall fairly close to the actual rates but are at single dates significantly off. For the Euro IBOR rates the model manages to forecast the sharp decline in short-term interest rates during the financial crisis. The forecasted direction of change is only correct in about half or less of the forecasts.

For the six-month ahead forecast in figure 4.3 we can observe the higher error as reflected in the RMSFE. The forecasts constantly overestimate the magnitude of changes, which becomes specifically apparent in the very stable interest rate environment of the short-term rates after the financial crisis.

The root-mean-square forecasting error results for our model are a magnitude higher than what Christensen et al. (2011) find for a single currency. For their data they show RMSFEs in

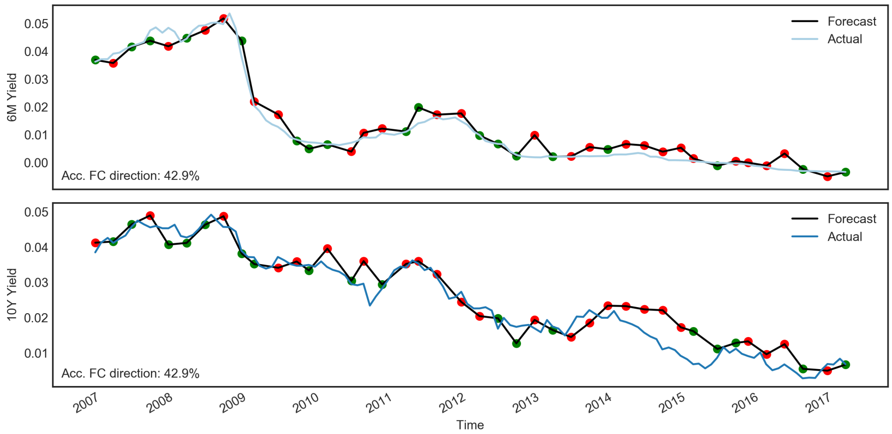

FIGURE 4.2: Interest rate forecasts with a forecasting horizon of $h = 1$ month conducted quarterly. The figure shows the 6 month and 10 year zero-coupon bond yields for Euro. The coloured dots indicate weather the direction of change is forecasted correctly.

a range between around 60 and 160 bps for a twelve-month ahead forecast. While they find their errors decreasing for longer maturities, for our data we find the RMSFE to be lowest in a maturity range of 5-15 years as displayed in table 4.2. The differing choice of maturities may contribute to the different distribution of RMSFEs across maturities between our and their results.

4.2 Two-Currency AFNS with Exchange Rates

As our focus is forecasting the joint behaviour of interest rates and foreign exchange rates, we now move on to an extended model including foreign exchange rates as described in section 3.3.2. We mainly limit the presentation of results to forecasting errors.

Table 4.4 shows the interest rate forecasting error for the model extended to capture the exchange rate dynamics. As previously in the interest-rate-only two-currency model we find increasing errors for longer forecasting horizons. Across maturities the errors are lowest for maturities between 5 and 15 years. Comparing the results to the interest-rate-only model we present in the previous section we find that the changes are mixed. For short-term maturities we find that results worsen slightly for Euro interest rate one- and three-month ahead forecasts, while for the longer horizons and especially the US-Dollar they are better. For maturities between one year and ten years the forecasts mostly improve and for the long end of the curve they worsen significantly.

FIGURE 4.3: Interest rate forecasts with a forecasting horizon of $h = 6$ month conducted quarterly. The figure shows the 6 month and 10 year zero-coupon bond yields for Euro.

	Euro				US-Dollar			
	h=1	h=3	h=6	h=12	h=1	h=3	h=6	h=12
1	**35.34**	**91.36**	**171.78**	**291.23**	**99.46**	**131.59**	**189.12**	**280.23**
2	**35.72**	**91.34**	**170.54**	**288.31**	**101.53**	**132.76**	**188.48**	**277.01**
3	**32.05**	**85.47**	**164.20**	**281.22**	**97.31**	**124.40**	**179.61**	**269.45**
6	**29.74**	**81.03**	**156.90**	**269.20**	**98.81**	**123.10**	**173.68**	**257.78**
12	**27.25**	**75.41**	**145.61**	**249.21**	**97.19**	**118.55**	**162.53**	**236.41**
24	32.00	71.71	133.95	226.69	100.73	116.89	151.93	212.19
36	57.62	85.31	135.03	213.11	104.23	116.69	148.47	200.36
48	60.20	83.78	129.10	200.96	92.80	103.54	133.56	182.78
60	**59.65**	**80.82**	**123.23**	**190.62**	**80.36**	**90.51**	**120.40**	**169.35**
72	57.16	76.98	117.39	181.80	68.90	78.97	109.84	159.62
84	53.62	72.83	112.18	174.33	59.58	70.05	102.21	153.42
96	49.81	69.13	107.66	168.35	53.14	64.34	97.75	150.08
108	46.40	66.18	104.16	163.72	49.44	61.50	95.63	148.66
120	**44.06**	**64.36**	**101.97**	**160.55**	**48.65**	**61.27**	**95.48**	**148.77**
144	43.95	64.86	100.62	157.32	53.98	67.26	99.78	152.22
180	54.12	73.77	105.83	159.06	70.56	84.09	112.82	162.26
240	79.48	96.88	123.52	171.16	105.47	118.34	142.04	186.10
300	**110.51**	**126.42**	**148.79**	**190.71**	**146.27**	**158.73**	**178.74**	**217.93**

TABLE 4.2: RMSFEs for out-of-sample interest rate forecasting. Maturities are in months; root-mean-square forecasting errors are in basis points. Maturities used to fit the model are printed in bold.

h	EUR/USD	EUR/JPY	EUR/GBP
1	0.0350	0.0398	0.0228
3	0.1018	0.1402	0.0539
6	0.2214	0.4373	0.1459
12	0.4550	1.0639	0.3773

TABLE 4.3: Exchange rate RMSFEs for out-of-sample forecasting. Forecasting horizons are in months; root-mean-square forecasting errors are in Euro.

The extended model allows to forecast foreign exchange rates as well. In a similar pattern to the interest rate forecasts the forecasting error rises for longer forecasting horizons. Table 4.3 displays forecasting error for exchange rates. Here each currency pairing is estimated separately. The forecasting errors are the highest for the Euro/Yen pairing, the best estimation is for the Euro and Pound pairing.

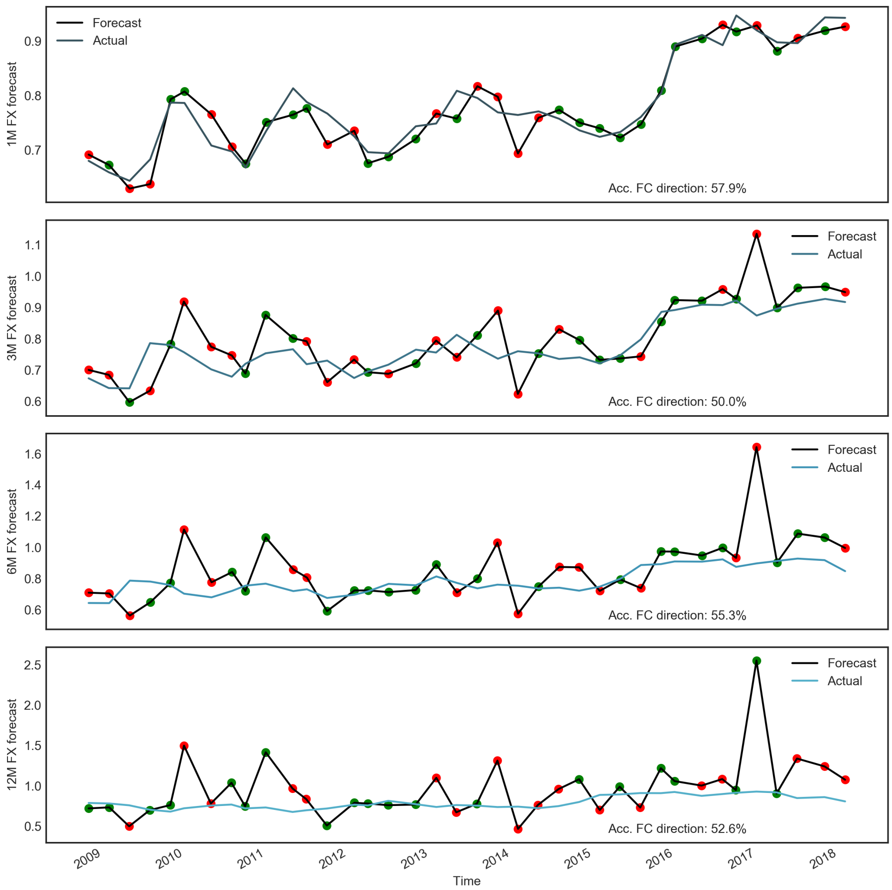

FIGURE 4.4: USD/EUR exchange rate forecasts with a forecasting horizons of 1, 3, 6 and 12 months conducted quarterly.

Figure 4.4 plots exchange rate forecasts and actual exchange rates for the US-Dollar and Euro currency pairing for different forecasting horizons. While for the one-month ahead forecast the forecasted rates still generally reflect the evolution of the market rates, the forecasts for longer horizons are much dispersed. One estimation especially stands out as an strong outlier, indicating that the estimation process for this forecasting date fails to find a good estimator for the parameters of the state variables. In the exchange rate dynamics this especially stands out due to the quadratic market price of risk term. This makes the exchange rate dynamics especially sensible to subpar estimations of the parameters, which enter the conditional expectation of the state variables. The forecasts for the EUR/GBP and EUR/JPY currency pairings display similar extreme outliers.

	Euro				US-Dollar			
	h=1	h=3	h=6	h=12	h=1	h=3	h=6	h=12
1	**98.40**	**134.15**	**180.23**	**261.01**	**77.35**	**111.70**	**158.85**	**243.56**
2	**92.76**	**129.93**	**176.87**	**257.67**	**71.78**	**108.52**	**156.89**	**241.38**
3	**81.76**	**120.27**	**169.07**	**250.45**	**62.13**	**97.59**	**148.27**	**235.78**
6	**65.60**	**106.52**	**157.58**	**237.81**	**46.85**	**86.69**	**140.41**	**228.60**
12	**45.02**	**90.06**	**142.82**	**216.87**	**29.44**	**74.25**	**128.29**	**214.82**
24	33.19	77.33	128.36	193.24	28.11	72.39	120.22	196.22
36	37.84	70.39	119.78	175.52	41.84	84.68	130.80	190.30
48	36.71	65.14	112.58	165.04	43.51	83.07	125.96	178.56
60	**36.14**	**61.25**	**107.30**	**157.03**	**43.34**	**80.04**	**120.89**	**168.27**
72	35.37	58.54	103.27	151.08	42.34	76.68	116.41	160.18
84	34.69	56.73	100.69	146.87	41.88	74.25	113.53	154.44
96	34.46	56.29	99.57	144.88	42.38	73.22	112.40	151.12
108	35.38	57.48	100.03	144.48	43.95	73.52	112.66	149.89
120	**38.05**	**60.39**	**102.01**	**145.70**	**46.63**	**75.34**	**114.36**	**150.56**
144	48.99	71.49	110.45	153.45	57.31	83.77	122.04	157.30
180	76.04	98.01	132.99	174.36	82.76	106.88	142.99	178.31
240	135.46	156.63	188.02	227.39	145.44	167.98	200.28	237.03
300	**213.27**	**234.01**	**263.59**	**301.92**	**232.54**	**254.55**	**284.26**	**322.54**

TABLE 4.4: RMSFEs for out-of-sample forecasting of interest rates in two-currency extended model with foreign exchange rates. Maturities are in months; root-mean-square forecasting errors are in basis points. Maturities used to fit the model are printed in bold.

The difficulty to find a good estimate is reflected in the results of the likelihood function as laid out in equation (3.9). Due to the extending window we expect a linear increase of the log likelihood given the same fit for the increasing number of observations. In practice we find the likelihood of the estimations highly volatile, which leads to the conclusion that the optimization of the estimated parameters is not sufficient.

4.3 Multi-Currency AFNS with Exchange Rates

Lastly, we apply the extended AFNS to a four-currency setting. As the estimation process for the larger amount of data is significantly more time-consuming we only evaluate the model annually. Again we only show forecasting results.

Table 4.6 provides root-mean-square forecasting errors for the joint model for all currencies. Here we limit our focus to one- and six-month ahead forecasting horizons for the interest rates. The errors for the joint multi-currency model do not exhibit a clear pattern, except being substantially high for the longest maturities (25 years). For almost all currencies and forecasting horizons the errors tend to be lowest for maturities around 10 years. Generally, the forecasting errors are about the same or lower for the six-month ahead forecasts compared to the one-month horizon. This is unlike the results we present previously for the two-currency models, where RMSFEs mostly increased for longer forecasting horizons.

A comparison of the results with table 4.4 suggests that an extension of the model to more currencies does not necessarily harm the quality of the forecasts as RMSFEs for the six-month ahead forecasts are lower and mostly in the same range for one-month ahead forecasts. We have to note that with respect to the smaller amount of forecasts conducted for the multi-currency model we cannot infer a general better forecasting performance in comparison to the two-currency model.

h	EUR/USD	EUR/JPY	EUR/GBP
1	0.0382	0.0487	0.0402
3	0.1370	0.1291	0.1218
6	0.2083	0.1808	0.2128
12	0.2574	0.2495	0.2818

TABLE 4.5: Exchange rate RMSFEs for out-of-sample forecasting. Forecasting horizons are in months; root-mean-square forecasting errors are in Euro.

Table 4.5 provides root-mean-square forecasting errors for the exchange rates between the currencies. It shows that forecasting errors are approximately in the same range for all currency pairings, suggesting there are no currency-specific features that would significantly lower forecasting performance. For all exchange rates the errors increase for longer forecasting horizons. Comparing the results to the two-currency model in table 4.3 we observe that for the shorter forecasting horizons the RMSFEs are in the same range, while actually being lower for longer horizons. This could be due to unobservable interrelations between the currency pairings captured by the model. We need to restate the limited amount of forecasts conducted for the multi-currency model, which does not provide enough data to validate this claim.

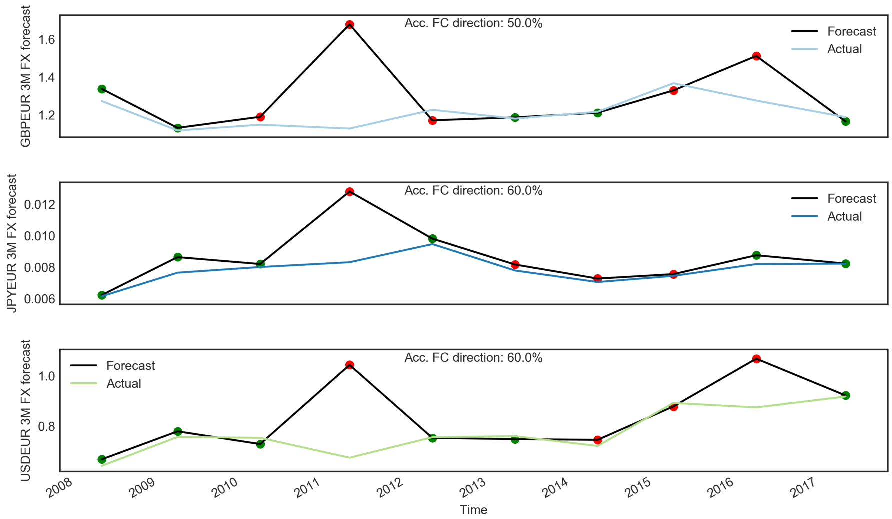

FIGURE 4.5: Exchange rate forecasts with a forecasting horizon of $h = 3$ months conducted annually.

Figure 4.5 plots three-month ahead FX forecasts for all currency pairings. The plot shows a similar picture to what we see for the two-currency model's exchange rate forecast in Figure 4.4. While the forecasted exchange rate is mostly in the range of the actual rate there are strong outliers in all currency pairings. As they happen at the same forecasting dates for all exchange rates, we deduct that this is likely due to an estimation issue.

	EUR		USD		JPY		GBP	
	h=1	h=6	h=1	h=6	h=1	h=6	h=1	h=6
1	**110.30**	**92.79**	**129.73**	**152.19**	**160.22**	**189.38**	**140.57**	**155.26**
2	**100.71**	**86.74**	**121.48**	**146.08**	**156.05**	**185.81**	**129.15**	**147.05**
3	**97.37**	**84.69**	**114.60**	**141.85**	**152.46**	**178.76**	**123.15**	**143.87**
6	**79.98**	**76.56**	**104.20**	**133.11**	**149.98**	**172.75**	**104.63**	**132.08**
12	**53.85**	**64.57**	**109.20**	**125.69**	**162.35**	**173.08**	**92.52**	**123.62**
24	35.54	57.88	140.54	125.75	202.12	191.77	102.01	123.14
36	55.79	52.96	161.72	131.65	224.61	198.86	121.98	126.58
48	62.99	54.57	154.44	125.56	232.05	200.19	124.24	128.45
60	**62.32**	**52.84**	**140.87**	**115.09**	**231.79**	**196.66**	**120.13**	**126.15**
72	55.95	51.81	124.60	104.77	226.66	189.53	113.53	123.52
84	47.11	54.68	108.28	97.38	218.44	180.54	107.39	121.90
96	39.93	63.68	93.51	93.91	208.31	170.75	103.15	122.59
108	39.88	78.11	81.26	94.65	197.36	161.27	102.32	126.38
120	**49.35**	**96.19**	**71.90**	**99.13**	**186.56**	**153.30**	**104.97**	**132.99**
144	86.09	139.91	67.14	117.30	168.24	143.46	120.18	154.03
180	155.66	212.65	92.49	155.90	154.00	149.07	157.43	195.82
240	282.11	339.89	168.17	235.19	179.45	205.00	235.77	278.43
300	**417.20**	**475.97**	**264.37**	**331.45**	**253.75**	**293.99**	**332.10**	**377.82**

TABLE 4.6: RMSFEs for out-of-sample interest rate forecasting. Maturities are in months; root-mean-square forecasting errors are in basis points. Maturities used to fit the model are printed in bold.

Chapter 5

Conclusion

The purpose of this thesis has been to establish a framework that allows forecasting of interest and exchange rates for multiple currencies in a joint model. The necessity for such a model arises due to the complex exposures of institutions to multiple sources of interest rate and foreign exchange risk, which are highly correlated across currencies and therefore must be modelled jointly. The Nelson Siegel class of term structure models, whose factors are level, slope and curvature, is popular among practitioners and researchers due to the clear financial and macroeconomic interpretation of its factors.

Our empirical analysis is based on interbank rates and spot exchange rates for four major currencies: Euro, US-Dollar, Japanese Yen and Pound Sterling. Interbank rates play a significant role in fixed income markets, especially for hedging through derivatives. We establish support for three-factor models, such as the Nelson Siegel class of term structure models through principal component analysis, which we conduct on a local basis for each separate currency and globally for all rates across currencies. On the whole, the results of the PCA and the correlations between the empirical proxies for level, slope and curvature lead us to assume a global level factor and local slope and curvature factors. This allows our model to exhibit similar correlations across term structures as observed empirically.

We therefore extend the independent-factor arbitrage-free Nelson Siegel model, which is a specific case of affine model in the Duffie-Kan framework exhibiting the Nelson Siegel factor loadings, to a multi-currency setting including exchange rate dynamics and provide an estimation framework. In the arbitrage-free Nelson Siegel model, the state variable dynamics are assumed to follow Ornstein–Uhlenbeck processes, which are Gaussian and mean reverting. In the independent-factor AFNS that we choose here, the factor dynamics under the physical measure are independent of each other. The market price of risk, which represents the change of drift between the risk-neutral and the physical measure under the Girsanov theorem, is assumed to be an essentially affine function on the state variables. This allows to put some constraints on the parameters of the market price of risk, which in turn is a parameter of the exchange rate dynamics. The estimation framework is based on a Kalman filter which provides us with a maximum likelihood estimate, which is maximised using particle swarm optimisation.

Overall the empirical results show that while good in-sample performance is readily achievable, the performance of out-of-sample forecasts varies substantially. We show results for three different extensions of the arbitrage-free Nelson Siegel model. An extension to two currencies is the simplest case, which is then extended to incorporate foreign exchange rates and finally placed in a four-currency setting. For the two-currency model we check robustness by separately estimating the models for all foreign currencies in combination with the Euro.

With the exception of the multi-currency model forecasting errors increase for longer forecasting horizons. In the multi-currency model we obtain similar results for both the one-month ahead and six-month ahead forecast. However, these findings cannot be taken as evidence for a similar forecasting ability for both forecasting horizons as the amount of forecasts conducted is to small.

In all models the root-mean-square forecasting errors are the highest on the long end of yield curve for a maturity of 25 years. Furthermore, we observe higher root-mean-square errors for the short-term maturities up to one year, although these are used to estimate the model parameters. This can be a possible effect of overfitting.

Plotting the forecasts over time shows that the forecasting errors are significantly influenced by extreme outliers, which are especially pronounced for exchange rate forecasts. In these cases, it is likely that the estimation process fails to find a good estimate of the model parameters, which in turn drive the conditional expectations of the state variables. While the arbitrage-free Nelson Siegel model is a parsimonious model for a single term structure, the complexity of a joint model increases considerably. The increased degrees of freedom lead to difficulty in finding a global maximum of the likelihood. The outliers are likely caused by the estimation process finding at a local maximum with economically insensitive model parameters.

The overall results indicate that a starting point to improve forecasting performance is the robustness of the estimation process. We see three approaches to improve this: choice of a different optimisation algorithm, running a more rigorous and time-consuming estimation process and improving computational efficiency. While the Kalman filter is a proven algorithm for parameter estimation, the methodology of the maximisation algorithm for the likelihood estimation has been beyond the scope of this thesis and is an extensive research topic itself. The parameter estimation process implemented in Python for the multi-currency model already has already required around 100 hours of runtime on a conventional computer. Therefore, we suggest on improving computational efficiency first. Ceteris paribus we expect significant runtime reduction under an implementation optimised for multi-core systems or in a compiled language.

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