Vortex pseudomomentum and dissipation in a superfluid vortex lattice

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Abstract. We propose an alternative approach to the dissipative vortex dynamics occurring in a superfluid vortex lattice at finite temperatures. Focusing upon the pseudomomentum of a vortex and its surrounding quasiparticles, we derive an equation of motion which, in spite of yielding the same evolution as the usual one for massless vortices, does not involve the vortex mass. This picture could provide further insights into the controversy about the nature of the vortex mass.

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1. Introduction

Below the lambda transition, at temperatures below $T_\lambda = 2.172$ K, liquid helium is known as *helium II* and can be regarded as a mixture of a normal fluid with mass density ρ_n and a superfluid with mass density ρ_s. Such densities are temperature dependent, so that $\rho_s(T_\lambda) = \rho_n(0) = 0$, while the total density $\rho = \rho_s + \rho_n$ remains nearly constant. The most striking property of such a superfluid component is, perhaps, that it can only rotate through vortices having microscopic cores and quantized circulations. In fact, the circulation of the superfluid velocity field around each of such vortices is quantized in units of h/m_4, the so-called quantum of circulation κ, given by the ratio of Planck's constant and the mass of one ^{4}He atom. In practice, however, we only find configurations of one quantum per vortex, since they are favored by energetic considerations [1]. When a rotating sample of liquid helium is cooled below the lambda temperature, all the rotation of the superfluid becomes concentrated in such vortices, which eventually form a lattice consisting in a uniform array of vortex filaments parallel to the axis of rotation [1, 2, 3]. However, the *macroscopic* superfluid velocity field, which corresponds to spatial averages over regions large compared with the spacing between vortices, yields the usual configuration of solid body flow, $\mathbf{v}_s(\mathbf{r}) = \Omega_{\rm rot}\hat{\mathbf{z}} \times \mathbf{r}$ for a rotation frequency $\Omega_{\rm rot}$ around the z axis. On the other hand, the *microscopic* superfluid velocity field, ie without averaging, is irrotational except where a vortex is located. Since the circulation of such a field around a vortex yields the quantum of circulation κ, according to the Stokes' theorem we may write

$$\frac{1}{A}\int\int_A dx\, dy\, \hat{\mathbf{z}} \cdot \mathbf{w} = \kappa N_A/A, \tag{1}$$

where the area A in the x-y plane contains N_A vortices and $\mathbf{w}$ denotes the microscopic vorticity, ie the curl of the microscopic superfluid velocity. On the left-hand side of (1) we have an averaged vorticity which should be identified with the macroscopic value $\nabla \times \mathbf{v}_s = 2\Omega_{\rm rot}\,\hat{\mathbf{z}}$. Thus, the above equation yields a link between the macroscopic and the microscopic views of the superfluid component, namely the number of vortices per unit area corresponds to the ratio of the macroscopic vorticity to the quantum of circulation, $N_A/A = 2\Omega_{\rm rot}/\kappa$ (Feynman's rule [1, 4]).

Just as the superfluid flow is microscopically formed by vortices, the normal fluid consists of superfluid quasiparticle excitations, phonons and rotons, the average flow of which is characterized by a normal fluid velocity field $\mathbf{v}_n$ given by $\Omega_{\rm rot}\hat{\mathbf{z}} \times \mathbf{r}$. That is, both fluids are expected to move with the same velocity at equilibrium. It is important to recall, however, that there are remarkable exceptions to this result, eg the equilibrium configuration of the Hess-Fairbank experiment [5], or the *metastable* superflow states having effectively infinite lifetimes [6]. Now, keeping the focus on the simpler situation of having identical normal and superfluid velocity fields at an equilibrium state to be reached within a finite relaxation time, one may interpret that such a behaviour arises, from a microscopic viewpoint, from the vortex motion with the normal fluid velocity in order to avoid dissipation. That is, any relative motion of vortices with respect to the normal fluid in their vicinity, should be subjected to a friction force that causes such a motion to eventually cease. Such a *mutual friction force* [7] between the two fluids appears then as playing a central role in the mechanism which maintains the stability of the above equilibrium state. A phenomenological model for this macroscopic dynamics was proposed long ago through the so-called Hall-Vinen-Bekharevich-Khalatnikov (HVBK) equations [7, 8]. These basically consist of a Navier-Stokes equation for the normal fluid and an Euler equation for the superfluid, which, in the absence of pressure and temperature gradients, are coupled together only by a mutual friction term [1, 2]. The original proposal of the HVBK model was later rederived from first principles within the framework of classical continuum mechanics [9], but a derivation from a full microscopic theory is still lacking. On the other hand, with the exception of a few works [10, 11] and because of their complexity, the HVBK equations have been mainly utilized so far to model helium II with a spatially uniform configuration of vortices.

The simplest departure from the rotating equilibrium configuration is given by a normal fluid field of the form $(\Omega_{\rm rot} + w_0)\,\hat{\mathbf{z}} \times \mathbf{r}$, but unfortunately a simple ansatz like $\mathbf{v}_n(\mathbf{r}, t) = \Omega_n(t)\,\hat{\mathbf{z}} \times \mathbf{r}$ does not constitute an acceptable solution of the HVBK equations. Simplicity, however may be preserved by changing to a different geometry consisting in rectilinear flows of uniform vorticity,

$$\mathbf{v}_s(\mathbf{r}, t) = -\,2\,\Omega_s(t)\,y\,\hat{\mathbf{x}} \qquad (2)$$

$$\mathbf{v}_n(\mathbf{r}, t) = -\,2\,\Omega_n(t)\,y\,\hat{\mathbf{x}} \qquad (3)$$

($y < 0$), where $\Omega_s(t)$ and $\Omega_n(t)$ should converge for $t \to \infty$ to a common steady state value. Then, to make contact with the standard rotational configuration, we may identify such a value with the former angular velocity $\Omega_{\rm rot}$. Note that this assignment leads to a uniform vorticity $\nabla \times \mathbf{v}_s = 2\,\Omega_{\rm rot}\,\hat{\mathbf{z}}$, which coincides with that of the rotational scheme.

The above macroscopic view of the interaction between superfluid and normal fluid, ruled by the HVBK equations, is complemented by the microscopic picture of a dissipative vortex dynamics arising from the scattering of quasiparticles by vortex lines‡. In the usual approach, the vortex equation of motion arises simply from assuming a vanishing total force, which is given by the sum of a hydrodynamic Magnus force and a dissipative force. Such a neglect of the vortex inertia corresponds to the assumption that its mass is given by the hydrodynamic mass of a core of atomic dimensions [1]. This result, however, has never been experimentally confirmed owing to the difficulties that embodies a direct measurement of the vortex mass [14]. Moreover, there are different theories [15, 16, 17] that yield several orders of magnitude higher values for the vortex mass, casting doubt on models based on massless vortices§. On the other hand, it has been recently suggested that an unambiguous vortex mass may not exist, and that

‡ Vortex bending in the form of thermal excitation of vortex oscillation modes, or collective excitations as Tkachenko waves are not expected to be relevant to this discussion [1, 12, 13].

§ There have also been conflicting results for the vortex mass in superconductors [18].

inertial effects in vortex dynamics may be scenario-dependent [19]. Given such an open debate, it seems to be quite advisable to follow an eclectic procedure, assuming in what follows a finite vortex mass per unit length, which we shall denote through the parameter m_v.

There is a close analogy between the dynamics of massive rectilinear vortices and the well-known electrodynamical problem of point charges subjected to a uniform magnetic field and a perpendicular electric field. More precisely, there exists a whole mapping by which a 2-D homogeneous superfluid can be mapped onto a (2+1)-D electrodynamic system, with vortices and phonons playing the role of charges and photons, respectively [15, 20, 21, 22, 23]. Here we shall only make a restricted use of this mapping, which corresponds to the formal analogy between the Magnus and Lorentz forces. Thus we may assume that the dissipative dynamics of the vortex lattice should be ruled by three characteristic frequencies, namely the imposed rotational frequency $\Omega_{\rm rot}$, an initial departure w_0 from this frequency and the cyclotron frequency stemming from the electromagnetic analogy [13]

$$\Omega = \rho_s \kappa / m_v. \qquad (4)$$

Taking as a lowest estimate of the vortex mass the value arising from the above hydrodynamical model, we have $\Omega \lesssim 3 \times 10^{12}$ s^{-1} [1]. On the other hand, typical experimental values are of order $\Omega_{\rm rot} \sim 1$ s^{-1}, so we shall restrict our study to cyclotron frequency values fulfilling $\Omega_{\rm rot} \ll \Omega$.

The above assumption of massive vortices and rectilinear flows leads us to a very useful magnitude, which to the best of our knowledge has not been utilized so far, that is the concept of *vortex pseudomomentum*. In fact, such a pseudomomentum corresponds to the vortex generator of translation, as can be straightforwardly shown from the electromagnetic analogy [24]. More generally, the translation generated by a pseudomomentum corresponds to a motion of the physical state (vortex) but keeping the medium, the uniform superfluid in this case, fixed [25]. In addition, contrary to the other two momenta (*canonical* and *dynamical*) that can be ascribed to a vortex, the vortex pseudomomentum turns out to be free from the ambiguities carried by the vortex mass. Here it is worthwhile noticing that a similar situation occurs in the case of the normal fluid. In fact, it can be shown that the momentum of a sound wave (or of a phonon in quantum mechanics), turns out to be a very complicated object, which may not even have a well-defined value at all, while its pseudomomentum is a simple quantity and far more useful [25, 26, 27]. This led us to investigate a pseudomomentum approach to the dissipative vortex dynamics occurring in a uniform vortex lattice, finding that this formalism leads to the same evolution as that predicted by the usual approach, whereas it involves far less restrictive assumptions about the value of the vortex mass.

This paper is organized as follows. In the next section we study a solution of the HVBK equations for the above rectilinear flows. In section 3 we analyze the dissipative vortex dynamics from the viewpoint of the usual phenomenological approach. A Hamiltonian approach is proposed in section 4, which consists of a vortex Hamiltonian based on the analogy to electrodynamics (section 4.1) and a Hamiltonian for the quasiparticle gas representing the normal fluid (section 4.2). Finally a pseudomomentum equation of motion is obtained and discussed in section 4.3.

2. Two-fluid equations

The HVBK equations for the rectilinear flows (2)-(3) are very simple and read

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} = \mathbf{F} \tag{5}$$

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} = -\mathbf{F}, \tag{6}$$

where the mutual friction force [1, 2, 7],

$$\mathbf{F} = \frac{\rho_n \rho_s}{\rho}\, \Omega_s(t)[-B(\mathbf{v}_n - \mathbf{v}_s) + B'\, \hat{\mathbf{z}} \times (\mathbf{v}_n - \mathbf{v}_s)], \tag{7}$$

becomes the single coupling term between both fluids and the dimensionless coefficients B and B' weight dissipative and nondissipative contributions to such a force. If we assume that the system is not far from equilibrium, so that the factor $\Omega_s(t)$ in (7) can be approximated by Ω_{rot}, replacing in (5) to (7) the velocity fields according to (2) and (3), an elementary calculation leads to the solution

$$\Omega_n(t) = \Omega_{\mathrm{rot}} + w_0\, e^{-\Omega_{\mathrm{rot}} B t} \tag{8}$$

$$\Omega_s(t) = \Omega_{\mathrm{rot}} - \frac{\rho_n}{\rho}\, w_0\, e^{-\Omega_{\mathrm{rot}} B t}, \tag{9}$$

where we should assume temperatures below 1 K, so that the normal fluid density turns out to be much smaller than the superfluid one ($\rho_n/\rho_s < 10^{-3}$) and the last term in (9) becomes negligible, ie

$$\frac{\rho_n w_0}{\rho_s \Omega_{\mathrm{rot}}} \ll 1. \tag{10}$$

Note that this model of strictly rectilinear flows amounts to ignoring nondissipative contributions to the mutual friction force, ie it leads to $B' \equiv 0$.

3. Dissipative vortex dynamics: phenomenological approach

A microscopic view to the above lattice leads us to focusing on a single vortex dynamics. Thus, we assume that the motion of each vortex is ruled by [1]:

$$m_v \ddot{\mathbf{r}} = \rho_s \kappa\, \hat{\mathbf{z}} \times (\dot{\mathbf{r}} - \mathbf{v}_s) - D(\dot{\mathbf{r}} - \mathbf{v}_n), \tag{11}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ denotes the position of the vortex core and D denotes a microscopic friction coefficient arising from vortex-quasiparticle scattering. The right-hand side of (11) represents the total force acting on the vortex, namely the Magnus force (first term) plus the dissipative force (second term). Here we have disregarded again any contribution from a nondissipative component of the mutual friction force‖. Now, replacing (2), (3), (8) and $\Omega_s(t) \simeq \Omega_{\rm rot}$ in (11) we obtain,

$$\ddot{x} = -\,\Omega \frac{D}{\rho_s \kappa}\dot{x} - \Omega\,\dot{y} - 2\,\Omega_{\rm rot}\,\Omega \frac{D}{\rho_s \kappa} y \left(1 + \frac{w_0}{\Omega_{\rm rot}} e^{-\Omega_{\rm rot} B t}\right) \tag{12}$$

$$\ddot{y} = -\,\Omega \frac{D}{\rho_s \kappa}\dot{y} + 2\,\Omega\,\Omega_{\rm rot} y + \Omega\,\dot{x}. \tag{13}$$

The equation (12) can be formally solved in $\dot{x}(t)$,

$$\dot{x}(t) = e^{-(\Omega D/\rho_s \kappa)t}\,\dot{x}(0) - \int_0^t d\tau e^{-(\Omega D/\rho_s \kappa)\tau} \times \left[\Omega\,\dot{y}(t-\tau) + 2\,\Omega_{\rm rot}\,\Omega \frac{D}{\rho_s \kappa} y(t-\tau)\left(1 + \frac{w_0}{\Omega_{\rm rot}} e^{-\Omega_{\rm rot} B(t-\tau)}\right)\right] \tag{14}$$

and replacing the above expression in (13) we are led to a second-order integro-differential equation in $y(t)$. This problem, however, may be greatly simplified by utilizing the Markov approximation. In fact, we first note that the coefficients in the arguments of the exponentials in (14) define two very different time scales, namely a microscopic one given by $(\Omega D/\rho_s \kappa)^{-1}$ and a macroscopic one given by $(\Omega_{\rm rot} B)^{-1}$, ie

$$\frac{\rho_s}{\rho_n}\frac{\Omega_{\rm rot}}{\Omega} \ll 1, \tag{15}$$

where we have taken into account that the macroscopic and microscopic friction coefficients are related for $T \lesssim 1$ K through [1],

$$B = \frac{2D}{\rho_n \kappa}. \tag{16}$$

Note that the inequality (15) is expected to break down at extremely low temperatures. Now, assuming that t in (14) belongs to the macroscopic time scale, we have that the exponential factor in the first term will be negligible, while the same exponential function, which depends on τ in front of the integrand, will make that only the lowest portion of the integration domain ($\tau \ll t$) gives a nonnegligible contribution to the integral. That is, we may safely approximate all the dependencies on $t-\tau$ in (14) by t, and the upper limit of the integral by $+\infty$. Finally, the Markov approximation to such an equation reads as,

$$\dot{x}(t) = -\int_0^\infty d\tau e^{-(\Omega D/\rho_s \kappa)\tau}\left[\Omega\,\dot{y}(t) + 2\,\Omega_{\rm rot}\,\Omega \frac{D}{\rho_s \kappa} y(t)\left(1 + \frac{w_0}{\Omega_{\rm rot}} e^{-\Omega_{\rm rot} B\,t}\right)\right] = -\frac{\rho_s \kappa}{D}\dot{y} - 2\,\Omega_{\rm rot}\,y\left(1 + \frac{w_0}{\Omega_{\rm rot}} e^{-\Omega_{\rm rot} B\,t}\right), \tag{17}$$

which replaced in (13) and using (16) yields,

$$\frac{\rho_n B}{2\rho_s \Omega}\,\ddot{y} + \dot{y} + \frac{\rho_n}{\rho_s} B\,w_0\,e^{-\Omega_{\rm rot} B\,t}\,y = 0. \tag{18}$$

‖ Actually, according to some theories [28] and related experimental evidence [29, 30], such a component may in fact be non-existent.

Here it is useful to change to the adimensional macroscopic time variable $\mathcal{T} = \Omega_{\rm rot} B\, t$,

$$\frac{1}{2}\left(\frac{\rho_n B}{\rho_s}\right)^2 \left(\frac{\rho_s}{\rho_n}\frac{\Omega_{\rm rot}}{\Omega}\right)\frac{d^2 y}{d\mathcal{T}^2} + \frac{dy}{d\mathcal{T}} + \left(\frac{\rho_n w_0}{\rho_s \Omega_{\rm rot}}\right) e^{-\mathcal{T}} y = 0 \tag{19}$$

and recall that $\rho_n/\rho_s \lesssim 10^{-3}$ and $B \lesssim 1$ for $T \lesssim 1$ K. Then, assuming $w_0 \lesssim \Omega_{\rm rot}$ and taking into account (10) and (15), we realize that we may safely drop the term containing the second derivative in (19), since we may estimate that the coefficient in front of such a derivative will be a quantity of third order in comparison to the first-order small parameter in front of the exponential. Note that according to (4), this approximation turns out to be equivalent to the usual one of neglecting the vortex mass. Thus, we are led to a first-order differential equation which is easily integrated yielding,

$$y(t) = y(0)\, \exp[(\rho_n w_0/\rho_s \Omega_{\rm rot})(e^{-\Omega_{\rm rot} B t} - 1)], \tag{20}$$

ie a small vortex displacement in the direction perpendicular to the velocity of the background superflow, while (17) becomes

$$\dot{x} = -2\Omega_{\rm rot} y, \tag{21}$$

which simply states that the x-component of the vortex velocity will coincide with the superfluid velocity.

4. Dissipative vortex dynamics: Hamiltonian approach

4.1. Vortex Hamiltonian

We start from the vortex equation of motion (11) in the absence of normal fluid

$$\ddot{\mathbf{r}} = \Omega\, \hat{\mathbf{z}} \times (\dot{\mathbf{r}} - \mathbf{v}_s), \tag{22}$$

which turns out to be analogous to that ruling the two-dimensional motion of a negative point charge in the presence of magnetic and electric fields in the z and y directions, respectively [24]. Such an equation derives from the Hamiltonian

$$H_v = \frac{m_v}{2}(v_x^2 + v_y^2) - \Omega_{\rm rot}\, \rho_s \kappa\, y^2, \tag{23}$$

being (Landau gauge)

$$\begin{aligned} v_x &= \frac{p_x}{m_v} - \Omega\, y \\ v_y &= \frac{p_y}{m_v}, \end{aligned} \tag{24}$$

where $\mathbf{p} = p_x \hat{\mathbf{x}} + p_y \hat{\mathbf{y}}$ corresponds to the vortex canonical momentum. Note that such a momentum as well as the Hamitonian (23) are given per unit length of the vortex line. Then, from Hamilton equations it is easy to check that the expressions (24) correspond to the velocity of the vortex core $\dot{\mathbf{r}}$, while the acceleration is indeed given by (22). From the above electromagnetic analogy it is also useful to represent the coordinate $\mathbf{r}$ as the sum of the center coordinate $\mathbf{r}_0 = x_0\hat{\mathbf{x}} + y_0\hat{\mathbf{y}}$ of the cyclotron circle plus the relative coordinate $\mathbf{r}'$ from such a center [24], being

$$\begin{aligned} x_0 &= -\frac{v_y}{\Omega} + x \\ y_0 &= \frac{v_x}{\Omega} + y. \end{aligned} \tag{25}$$

Then, it is easy to extract the time evolution of such coordinates working to zero-th order in the small parameter $\Omega_{\rm rot}/\Omega$:

$$\dot{\mathbf{r}}_0 = -2\Omega_{\rm rot} y_0\,\hat{\mathbf{x}}, \tag{26}$$

which means that the center coordinate $\mathbf{r}_0$ will move with the superfluid velocity, while $\mathbf{r}'(t)$ will perform a counterclockwise circular cyclotron motion with angular frequency Ω. Finally, we may see from (25) that the limit of a vanishing vortex mass ($\Omega \to \infty$), which is often found in the literature [31], amounts to ignoring the cyclotron motion ($\mathbf{r}' \to 0$, $\mathbf{r} \equiv \mathbf{r}_0$).

4.2. Quasiparticle Hamiltonian

Although we have described the vortex Hamiltonian in classical terms, it is immediate by means of the electromagnetic analogy to switch to a quantum mechanical picture [24]. This constitutes a necessary generalization, since the normal fluid will be treated as a low-temperature quantum gas. In fact, such a fluid will be represented by the following Hamiltonian:

$$H_n = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}}\; a^\dagger_{\mathbf{q}}\, a_{\mathbf{q}}, \tag{27}$$

where $a^\dagger_{\mathbf{q}}$ denotes a creation operator of quasiparticle excitations of (pseudo-) momentum $\hbar\mathbf{q}$ and frequency $\omega_{\mathbf{q}}$. Measuring such a frequency from the lab frame, it is written as a Doppler-shifted frequency from the superfluid frame, $\omega_{\mathbf{q}} = \omega_q + \mathbf{q}\cdot\mathbf{v}_s$, where ω_q is the familiar (isotropic) dispersion relationship of ^{4}He quasiparticle excitations. Note also that we disregard any interaction between the quasiparticles themselves, since we shall work at low enough temperature, so that they remain dilute allowing their treatment as a noninteracting gas.

4.3. Equation of motion in terms of pseudomomentum

Note that according to the electromagnetic analogy, there are three kinds of momentum to be ascribed to the vortex, viz the canonical one $\mathbf{p}$, the dynamical one $m_v\mathbf{v} = \rho_s\kappa\hat{\mathbf{z}}\times\mathbf{r}'$ and the so-called *pseudomomentum* [24], which is given by $\mathbf{K} = -\rho_s\kappa\hat{\mathbf{z}}\times\mathbf{r}_0$ and should be regarded as the generator of translation. Then, adding such a pseudomomentum to the quasiparticle pseudomomentum $\sum_{\mathbf{q}} \hbar\mathbf{q}\; a^\dagger_{\mathbf{q}}\, a_{\mathbf{q}}$, we have the pseudomomentum of the whole system. Note that only the x-component of the vortex pseudomomentum $\rho_s\kappa\, y_0$ will commute with the vortex Hamiltonian (23), unless $\Omega_{\rm rot} = 0$. This result may be easily interpreted, since a superflow of velocity $\mathbf{v}_s = -2\,\Omega_{\rm rot}\, y\,\hat{\mathbf{x}}$ produces a translation symmetry breaking in the y-direction.

Next let us analize the conservation of the x-component of the pseudomomentum for a given vortex of the lattice and its surrounding quasiparticles¶:

$$\rho_s \kappa L \langle y_0 \rangle + \sum_{\mathbf{q}} n_{\mathbf{q}} \, \hbar \mathbf{q} \cdot \hat{\mathbf{x}} = \text{const.}, \tag{28}$$

where $n_{\mathbf{q}} = \langle a^{\dagger}_{\mathbf{q}} \, a_{\mathbf{q}} \rangle$ denotes the average number of quasiparticles with pseudomomentum $\hbar\mathbf{q}$ and L denotes the vortex line length. We shall assume that such a population is well described by a local equilibrium form:

$$n_{\mathbf{q}} = [e^{\hbar\omega_{\mathbf{q}}/k_BT} - 1]^{-1} \simeq [e^{\hbar\omega_q/k_BT} - 1]^{-1} + \frac{\hbar \, \mathbf{q} \cdot [\mathbf{v}_n - \mathbf{v}_s]}{4k_BT \sinh^2(\hbar\omega_q/2k_BT)}, \tag{29}$$

where the quasiparticle frequency is measured from a reference frame where the local normal fluid is at rest, $\omega_{\mathbf{q}} = \omega_q - \mathbf{q} \cdot [\mathbf{v}_n - \mathbf{v}_s]$, with $\mathbf{v}_s = -2\,\Omega_{\rm rot}\, y \,\hat{\mathbf{x}}$ and $\mathbf{v}_n = -2\,\Omega_n(t)\, y\, \hat{\mathbf{x}}$, [$\Omega_n(t)$ given by (8)]. Then, replacing (29) in (28) we have,

$$\rho_s \kappa \langle y_0 \rangle + A_v \, \rho_n (\mathbf{v}_n - \mathbf{v}_s) \cdot \hat{\mathbf{x}} = \text{const.}, \tag{30}$$

where

$$\rho_n = \frac{\hbar^2}{12 A_v \, L \, k_BT} \sum_{\mathbf{q}} \frac{q^2}{\sinh^2(\hbar\omega_q/2k_BT)} \tag{31}$$

yields the normal fluid density and $A_v = (2\Omega_{\rm rot}/\kappa)^{-1}$ corresponds to the area per vortex of the lattice (Feynman's rule). Taking into account that the velocity fields in (30) must be evaluated at $y = \langle y_0 \rangle$, and using the solutions (8) and (9) we obtain,

$$\langle y_0(t) \rangle = \langle y_0(0) \rangle \, [1 + (\rho_n w_0/\rho_s \Omega_{\rm rot})(e^{-\Omega_{\rm rot} B t} - 1)], \tag{32}$$

which according to (10), turns out to be equivalent to (20). In other words, starting from the conservation of pseudomomentum for a vortex and its surrounding quasiparticles, without any further assumptions regarding the vortex mass, we have shown that the y-component of $\mathbf{r}_0$ presents the same dissipative evolution as that obtained in section 3 for a vortex of negligible mass, for which $\mathbf{r} \equiv \mathbf{r}_0$. Thus, we may see how the dynamics in terms of the pseudomomentum makes the coordinate $\mathbf{r}_0$ the focus of attention, while the less relevant cyclotron dynamics of $\mathbf{r}'$, which was forced to vanish in the phenomenological treatment through the neglect of the vortex mass, is now naturally ignored.

To study the dynamics along the x-direction we must focus on the y-component of pseudomomentum, which is given by the nonconserved vortex contribution $-\rho_s \kappa \langle x_0 \rangle$ (cf (26)). Finally, taking into account the whole vector pseudomomentum $\mathbf{K}_{\rm tot}$ of the vortex plus surrounding quasiparticles, we may write the following equation:

$$\frac{d\mathbf{K}_{\rm tot}}{dt} = \mathbf{F}_{\rm ext}, \tag{33}$$

¶ Here it is worth comparing with a recently derived conservation theorem for wave pseudomomentum plus a suitably defined classical vortex impulse [27].

where

$$\mathbf{K}_{\mathrm{tot}} = -\rho_s \kappa \hat{\mathbf{z}} \times \langle \mathbf{r}_0 \rangle + \frac{\kappa}{2\Omega_{\mathrm{rot}}}\, \rho_n (\mathbf{v}_n - \mathbf{v}_s) \tag{34}$$

and

$$\mathbf{F}_{\mathrm{ext}} = -\rho_s \kappa \hat{\mathbf{z}} \times \mathbf{v}_s \tag{35}$$

represents the force exerted on the vortex by the superfluid current [3], which is responsible for the nonconserved y-component of pseudomomentum.

The equation of motion (33) constitutes the central result of this paper, which may be regarded as arising from a straightforward combination of the electromagnetic analogy and the fruitful concept of pseudomomentum. Here it is important to remark that contrary to the phenomenological equation of motion (11), the pseudomomentum equation (33) does not involve the vortex mass, a fact to be regarded as most welcome, given its apparent elusive and ambiguous nature [19].

To conclude we may observe that it would be important to generalize this picture for the rotating superfluid, where the key magnitude would be represented by an *angular pseudomomentum*. To achieve this goal, given the drawbacks of the HVBK equations pointed out in section 1, it would be helpful previously to seek for a validation of the present results without utilizing such equations as a starting point.

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