Topics in Hyperbolic Groups

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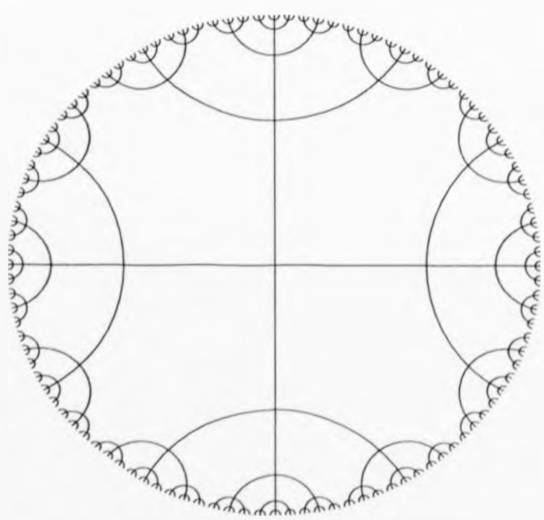

Acknowledgements

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I would also especially like to thank Andrew Clow for countless useful conversations. Also, my thanks go to Caroline Series, Swarup Gadde, Giles Robert, Alison Kay, Charles Walkden, Timothy Luffingham and Rob Foord for their help and interest.

Declaration

Chapter 1 is expository. To the best of my knowledge, the other chapters are original, except where stated. Some of the results in Chapter 3 are being submitted for publication as part of a joint paper with David Epstein and Derek Holt. The results which appear in Chapter 3 are my own work.

Summary

Hyperbolic groups are a class of groups introduced by Gromov in 1987, which form an important part of geometric group theory. In Chapter 1, we give an introduction to this subject.

In Chapter 2, we use the theory of complexes of groups to show that the integral homology and cohomology groups of a hyperbolic group are computable by a Turing machine.

In Chapter 3, we present the boundary of a hyperbolic group as an inverse limit of topological spaces and use this to give computable estimates for properties of the boundary.

In Chapter 4, we investigate symbolic dynamic properties concerning hyperbolic groups. In paricular, we give symbolic codings for the actions on the boundary of a hyperbolic and actions on the geodesic flow on a hyperbolic group.

In Chapter 5 we investigate the problem of determining when graphs are Cayley graphs. The graphs which we are concerned with are regular and semi-regular planar graphs.

Chapter 1

Background

Throughout this chapter, G is a finitely generated group with finite generating set X. An element g of G can be written as a product $x_1 \cdots x_n$, where each x_i is either in X or is the inverse of an element of X. Such a product is called a representative of g. The length of g (written $|g|$) is the minimum number of elements in a representative of g. We equip G with the *word metric*, $d(g,h) = |g-h| = |g^{-1}h|$.

Definition 1.0.1 (Cayley Graph): The *Cayley graph* $\Gamma(G,X)$, or Γ when there is no ambiguity, is the graph with, for each $g \in G$, a vertex labelled g and for each $g \in G$, $x \in X$ an edge from g to gx labelled by x. We give Γ a metric by giving each edge length 1 and then taking the path metric. This metric coincides with the word metric on the vertices of Γ. We use $d(x,y)$ or $|x-y|$ to denote the distance between two points $x, y \in \Gamma$.

Definition 1.0.2 (Geodesic): A *geodesic* in Γ is an isometric embedding of a closed interval into Γ. Suppose that $\gamma : [a,b] \to \Gamma$ is a geodesic with $\gamma(a) = x$ and $\gamma(b) = y$. We sometimes write $[x,y]$ for γ.

1.1 Hyperbolic Groups

Hyperbolic groups are a class of groups introduced by Gromov in [Gro87] whose Cayley graph satisfies a single axiom (see Definition 1.1.2). Hyperbolicity is independent of the choice of finite generating set. The definition applies to more general spaces than Cayley graphs and gives rise to the notion of a hyperbolic space. For simplicity, we stick to the case of a Cayley graph.

Good introductions to the theory of hyperbolic spaces and hyperbolic groups are [ABC+91], [CDP90] and [GdlH90]. All the information in this section has been taken from these books.

1.1.1 Definitions of Hyperbolicity

Definition 1.1.1 (Inner product [Gro87]): Fix a basepoint x_0 in Γ. For any two points $x, y \in \Gamma$, define the *inner product*

$$(x.y) = \frac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y)). \tag{1.1}$$

Definition 1.1.2 (Hyperbolic [Gro87]): The group G with generating set X is δ-hyperbolic if, for any three points $x, y, z \in \Gamma$,

$$(x.z) \geqslant \min((x.y), (y.z)) - \delta. \tag{1.2}$$

A group is called *hyperbolic* if there is a finite generating set and a constant δ such that G is δ-hyperbolic with respect to this generating set.

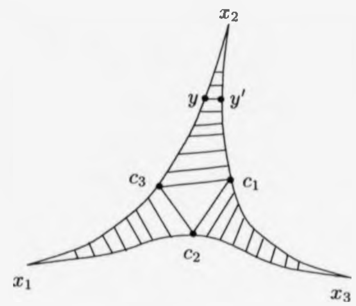


Figure 1.1. Thin triangles

Theorem 1.1.3 (Gromov): *If G is hyperbolic then G is hyperbolic with respect to any finite generating set.*

In the following definitions, Δ is a geodesic triangle in Γ with sides $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$.

Definition 1.1.4 (Thin triangles): Choose points $c_i \in [x_{i+1}, x_{i+2}]$ ($i = 1, 2, 3$, addition modulo 3) such that $d(c_i, x_{i+1}) = d(c_i, x_{i+2})$ (see Figure 1.1). Each point y on one of the edges of the triangle is on one of the 6 geodesics $[x_i, c_j]$ ($i \neq j$) and there is a corresponding point y' on $[x_i, c_k]$ ($k \neq j, i$) such that $d(x_i, y) = d(x_i, y')$. We say that Δ is *δ-thin* if for every $y \in \Delta$, $d(y, y') \leqslant \delta$. The Cayley graph Γ is *δ-thin* if all geodesic triangles in Γ are δ-thin. That is, there is a global bound on the thinness of the triangles. If there is such a δ, Γ is said to have *thin triangles*.

Note that, if we take $x_1 = x_0$, $x_2 = x$ and $x_3 = y$ in Figure 1.1, then $(x.y) = (x_2.x_3) = d(x_1, c_3) = d(x_1, c_2)$.

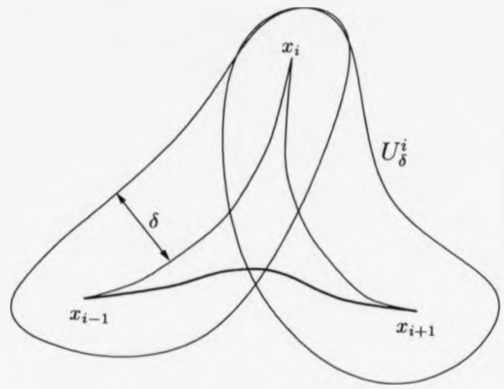


Figure 1.2. Slim triangles

Definition 1.1.5 (Insize): Let c_1, c_2, c_3 be as in Definition 1.1.4 (see Figure 1.1). Then the *insize* of Δ is diam $\{c_1, c_2, c_3\}$.

Definition 1.1.6 (Slim triangles): (Attributed to Rips.) Let U_δ^i be the closed δ neighbourhood of $[x_{i-1}, x_i] \cup [x_i, x_{i+1}]$ (addition modulo 3), that is,

$$U_\delta^i = \{z \mid \exists x \in [x_{i-1}, x_i] \cup [x_i, x_{i+1}] \text{ such that } d(x, z) \leqslant \delta\}.$$

Then Δ is *δ-slim* if $[x_{i-1}, x_{i+1}] \subset U_\delta^i$ (see Figure 1.2). Γ is said to be *δ-slim* if all geodesic triangles in Γ are δ-slim. If there is such a δ, Γ is said to have slim triangles.

Definition 1.1.7 (Minsize): The *minsize* of Δ is (see Figure 1.3)

$$\inf \{\text{diam}\, \{y_1, y_2, y_3\} \mid y_i \in [x_{i+1}, x_{i+2}]\}.$$

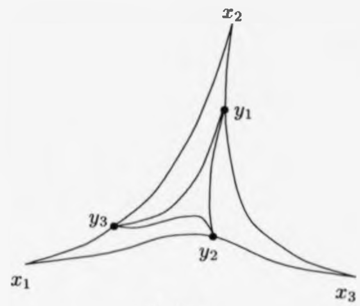


Figure 1.3. The minsize of a triangle

Theorem 1.1.8: *Let G be a group with finite generating set X. The following are equivalent:*

1. *G is hyperbolic.*
2. *G has thin triangles.*
3. *G has slim triangles.*
4. *There is a global bound on the insize.*
5. *There is a global bound on the minsize.*

For a proof, see [ABC$^+$91].

Some examples of hyperbolic groups are free groups, finite groups, surface groups, most small cancellation groups, groups which act properly discontinuously and cocompactly on hyperbolic space and free products of hyperbolic groups. The free Abelian group of rank 2 is not hyperbolic. Examples of Cayley graphs of hyperbolic groups are Graphs 1 and 2 (see also Graph 3 on page 54).

1.1.2 The Boundary of a Hyperbolic Group

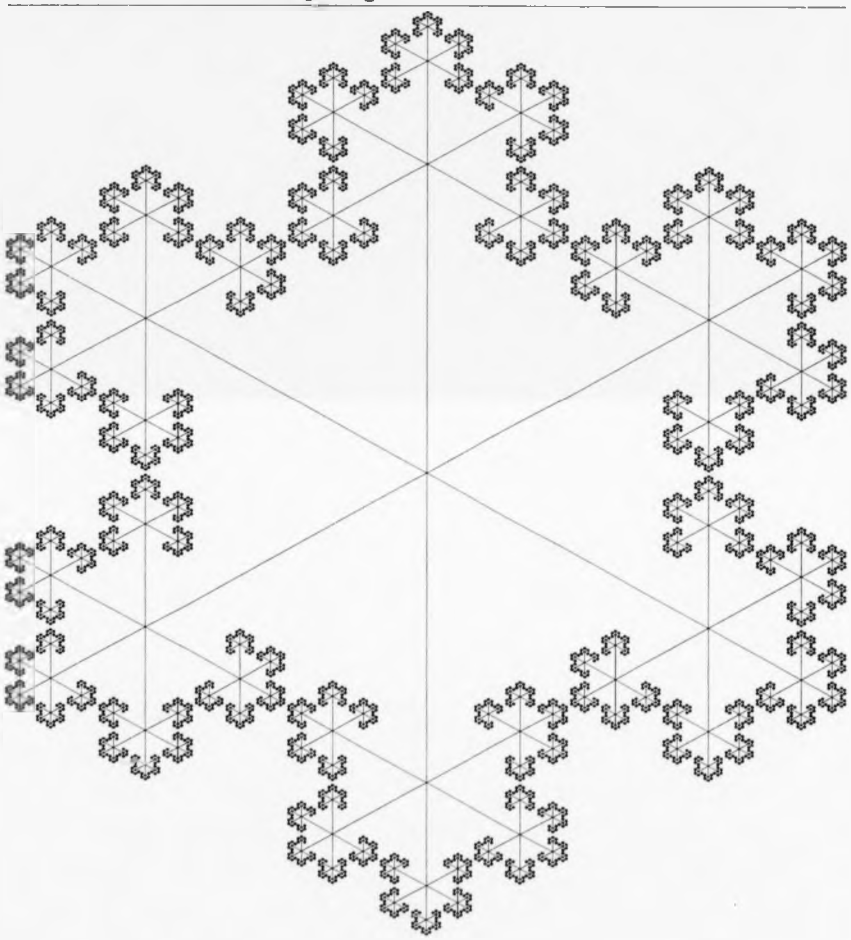
Hyperbolic groups have a boundary which is a quotient of the space of infinite geodesics. The boundary is usually described in terms of sequences and the Gromov inner product (1.1). We give here a more geometric approach using geodesics in the Cayley graph Γ. The resulting space defines a compactification of Γ.

Definition 1.1.9 (Geodesic ray): A *geodesic ray*, r, is an isometric embedding of the interval $[0, \infty)$ into Γ (compare with Definition 1.0.2). A *geodesic ray from* x is a geodesic ray r such that $r(0) = x$. A *biinfinite* geodesic ray is an isometric embedding of $(-\infty, \infty)$ in Γ.

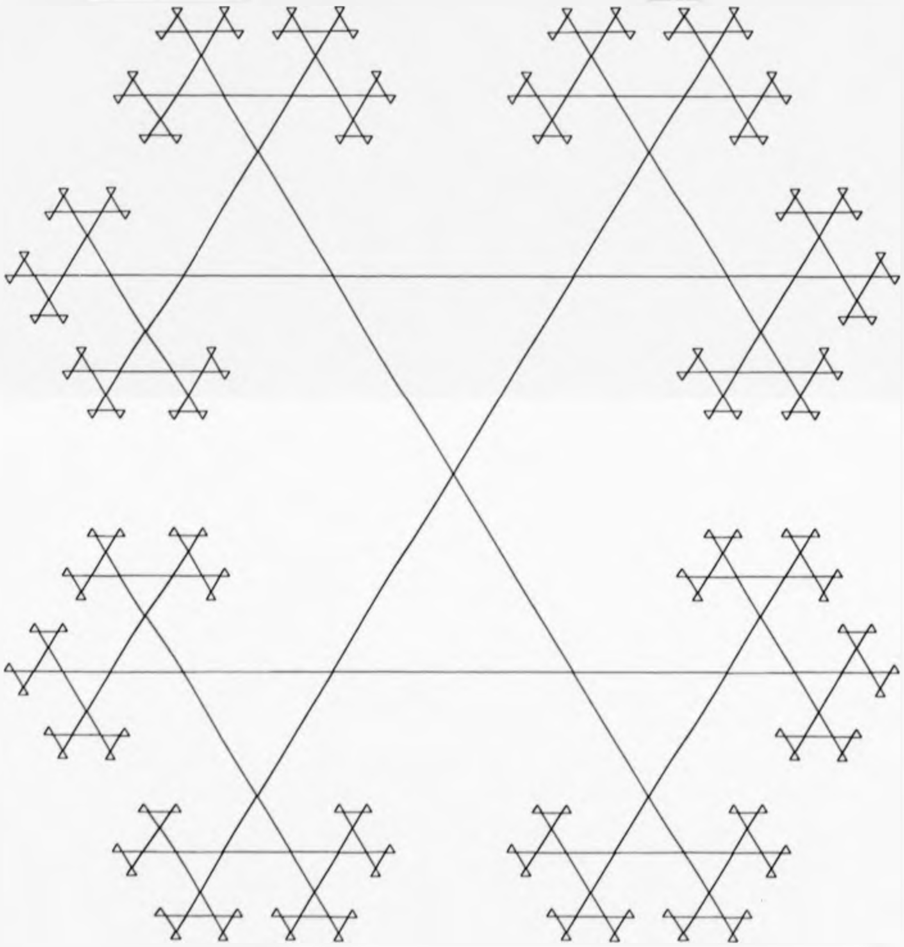
Let r, s be geodesic rays. We define the equivalence relation, $\sim$, by $r \sim s$ if there is a constant k such that, for all t, $d(r(t), s(t)) \leqslant k$. Suppose that r and s are geodesic rays from the identity. If Γ has δ-thin triangles (see Definition 1.1.4) and, for some t, $d(r(t), s(t)) > \delta$, then, for all $u \geqslant 0$, $d(r(t+u), s(t+u)) \geqslant 2u$ (see Figure 1.4). So, in this case, if such a constant k exists, we can take it to be δ.

Definition 1.1.10 (Boundary): We define the *boundary of* Γ, denoted by $\partial\Gamma$, to be the set of geodesics modulo $\sim$. A point on the boundary is an equivalence class of 'parallel' geodesic rays; geodesic rays which stay a bounded distance apart. If r is a geodesic ray in the equivalence class a, then

Cayley Graph 1. The free group on 3 generators, $\langle a, b, c \mid \rangle$. The a edges all travel up, the b edges all travel north-east and the c edges all travel south-east. This picture was drawn using `xfig`



Cayley Graph 2. $\mathbb{Z}_3 * \mathbb{Z}_3 = \langle a, b \mid a^3, b^3 \rangle$, the edges of a triangle are either all labelled by a or all labelled by b. Alternate triangles have different labels. This picture was drawn using `xfig`.



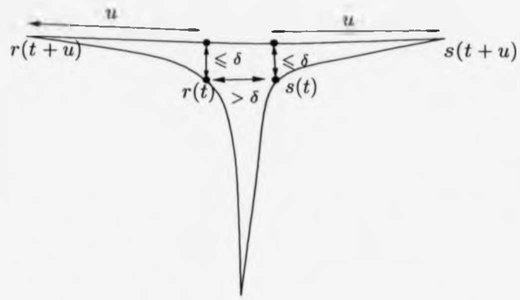


Figure 1.4. Geodesics which diverge

we say that r tends to a, $r \to a$, $r(t) \to a$ or $r(\infty) = a$. Similarly, if r is a biinfinite geodesic, then we write $r(-\infty) = b$ if the reverse of r tends to b.

Let $a, b \in \partial\Gamma$ with $a \neq b$. Then there is a biinfinite geodesic r such that $r(-\infty) = a$ and $r(\infty) = b$. We say that r *connects* a and b and write $r\colon (-\infty, \infty) \to (a, b)$.

Definition 1.1.11 (Visual metric): We give $\partial\Gamma$ the *visual metric*, $d_\varepsilon(\cdot,\cdot)$. Fix a base point x_0 in Γ and a constant $\varepsilon > 0$. The first approximation to the visual metric is

$$\rho_\varepsilon(a,b) = \begin{cases} \inf\limits_{r:(-\infty,\infty)\to(a,b)} \exp -(\varepsilon d(x_0, r)) & \text{if } a \neq b, \\ 0 & \text{if } a = b, \end{cases} \tag{1.3}$$

where $d(x_0, r)$ denotes the distance between x_0 and the image of r (see Figure 1.5). This is not a metric because it does not satisfy the triangle inequality. However, we can use it to define a topology by specifying open balls in the usual way. To make it a metric, we consider *chains* $a = a_0, a_1, \ldots, a_n = b$.

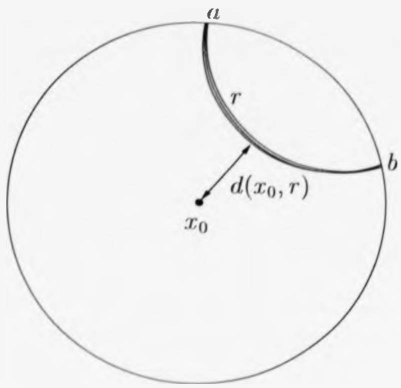


Figure 1.5. The visual metric on $\partial\Gamma$

Then

$$d_\varepsilon(a,b) = \inf_{\text{chains}} \sum_{i=1}^{n} \rho_\varepsilon(a_{i-1}, a_i). \tag{1.4}$$

This defines a metric if ε is sufficiently small ($\varepsilon < \frac{1}{4\delta}\log(\sqrt{2})$). The visual metric defines a topology which is independent of the choice of x_0 and ε. The topology from $\rho_\varepsilon(\cdot,\cdot)$ is the same as the topology from $d_\varepsilon(\cdot,\cdot)$ (See [CDP90, Chapter 11]).

In the sequential construction, points on the boundary are equivalence classes of sequences which tend to infinity. The visual metric is constructed by extending the Gromov product to the boundary by

$$(a.b) = \inf_{\substack{a_n \to a \\ b_n \to b}} \lim_{n \to \infty} (a_n.b_n).$$

Note that $(a.b) = \infty$ if and only if $a = b$. The construction is finished off in a similar way to above, with $(\cdot.\cdot)$ replacing $d(\cdot,\cdot)$ in (1.3). Given $a, b \in \partial\Gamma$

with $a \neq b$, let r be a biinfinite geodesic ray connecting a and b. Then $(a.b)$ satisfies $d(x_0, r) \geqslant (a.b) \geqslant d(x_0, r) - 4\delta$.

The topology is also independent of the choice of finite generating set. So the following definition is consistent:

Definition 1.1.12 (Boundary of G): The *boundary of G* is $\partial G = \partial \Gamma$.

Note that ∂G is compact and Hausdorff.

1.2 Automatic Groups

The information in this section is given in more detail in [ECH⁺92].

1.2.1 Finite State Automata

An *alphabet* is a finite set of symbols. A *word* is a string of symbols of the alphabet A. We use A^* to denote the set of all words over A (including the empty word). A *language* is a subset of A^*. If $w \in A^*$ and $n \leqslant |w|$, the length of w, then $w(n)$ denotes the word given by the the first n symbols of w.

A *finite state automaton* $\mathcal{A}$ is the following data:

1. A finite set of states, S.
2. An alphabet, A.
3. A set of edges, $E \subset S \times S \times A$. Let $e \in E$. The projection s onto

the first coordinate is called the source of e. The projection t onto the second coordinate is called the target e. The projection l onto the third coordinate is called the label of e.

4. Two subsets I and Y of S called the initial and accept states.

The automaton $\mathcal{A}$ can be drawn as a directed graph with each state being a vertex and each edge being a directed edge from its source to its target.

An edge path is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $t(e_i) = s(e_{i+1})$. The edge path $e_1, e_2, \ldots, e_n$ defines the word $w = l(e_1)l(e_2)\cdots l(e_n)$ in A^*. If $s(e_1) \in I$ and $t(e_n) \in Y$, then we say that the word w is *accepted* by $\mathcal{A}$. That is, there is a path in $\mathcal{A}$, labelled by w, starting at an initial state and ending at an accept state. The *language accepted by* $\mathcal{A}$ is the set of words accepted by $\mathcal{A}$.

1.2.2 Automatic Groups

Let G be a group with finite generating set X. We assume that X is closed under taking inverses. There is a natural projection from words over X onto G.

Definition 1.2.1 (automatic group): The group G is *automatic* if there are the following finite state automata:

1. The word acceptor with alphabet X which accepts at least one representative for each element of G.

2. For each generator x, the machine M_x over the padded alphabet $X \cup$

$\{\$\} \times X \cup \{\$\}$ which accepts all pairs of padded words (w_1, w_2) such that w_1 and w_2 are both accepted by the word acceptor and such that, in G, $w_1 x = w_2$.

1.2.3 Hyperbolic Groups

Hyperbolic groups are automatic. There are two special automatic structures which are used in this thesis.

Firstly, there is the geodesic automatic structure. The word acceptor accepts the set of geodesics. For each $g \in G$, any shortest representative of g is accepted. We call this word acceptor the *geodesic acceptor*. The property of having an automatic structure with a word acceptor which accepts the set of all geodesics is called *strongly geodesically automatic*. A group is strongly geodesically automatic if and only if it is hyperbolic.

Secondly, there is the ShortLex automatic structure. The generators are given an order. For each $g \in G$ we take the lexicographically least geodesic; that is, among shortest representatives for g, we take the one which comes first in the lexicographical order. There is an automatic structure whose word acceptor accepts the set of all ShortLex representatives.

Another important automaton is the *word difference machine*. This automaton accepts pairs of geodesics and keeps track of the word difference (in the group) at each stage. The word (w_1, w_2) is accepted if and only if, for every $n \leqslant |w_i|$, $d(w_1(n), w_2(n)) < N$. If $N > \delta$, then this machine forms part of an automatic structure for a hyperbolic group.

1.3 Computability Problems in Groups

We say that we can determine whether a group has a given property if there is an algorithm (which can be implemented by a Turing machine) which outputs 'yes' if the property is true and 'no' if the property is false. Partial algorithms, which output 'yes' if the answer is true but may not terminate if the answer is false (or vice-versa), are often much easier to find.

The question of whether a particular problem or property can be solved algorithmically is a difficult one. The most famous negative answer is the unsolvability of the word problem [Nov55, Boo57]. That is, there are groups such that there is no algorithm which takes as input a word in the generators and outputs whether the word equals the identity. A related question is; can one can determine whether a given presentation defines the trivial group? Again the answer is that there is no algorithm which takes as input a finite presentation and outputs whether the group defined by the presentation is trivial. However, given the automatic structure of a group, one *can* solve these problems.

If we know that a group is automatic we can set about trying to find the automatic structure using algorithms described in [ECH$^+$92] and, more practically, in [EHR91], in the knowledge that they will eventually terminate. So if we know that a group is automatic, we can compute its automatic structure (for example, by using Derek Holt's KBMAG computer package [Hol95]) and use this to compute other properties.

1.3.1 Computability in Hyperbolic groups

We have already seen that hyperbolic groups are a special kind of automatic group. We ask what properties of a hyperbolic group we can compute.

Suppose we are given a presentation $G = \langle X \mid R \rangle$ for a group which we know to be hyperbolic; then:

1: The word problem is solvable (see the discussion above).

2: The problem of deciding whether G is trivial is solvable (see the discussion above).

3: We can solve the conjugacy problem in G. That is, given $u, v \in G$, we can algorithmically determine whether u is conjugate to v.

4: We can compute the constant of hyperbolicity, δ (see [EH], also Theorem 2.1.4).

5: We can list the elements of finite order up to conjugation. That is, we can write a (finite) list such that for every finite order element $g \in G$, the list contains a conjugate of g. (We can refine the list so that it contains exactly one conjugate of each finite order element.)

Proof: Each element of finite order, g, is conjugate to an element h of length $\leqslant 4\delta + 1$ (by a corollary of Rips' theorem (Theorem 2.1.2)). Further, each power of h also has length $\leqslant 4\delta + 1$. So for each element of length $\leqslant 4\delta + 1$ we just need to compute powers and reduce until either we have the identity, or we have an element of length greater than $4\delta + 1$. ❑

6: We can list finite subgroups up to conjugation. That is, we can write a (finite) list such that, for every finite subgroup H of G, the list contains a conjugate of H.

Proof: Each finite subgroup is conjugate to a subgroup of diameter $\leqslant 10\delta+1$ ([BG95]). There are a finite number of such sets. We can systematically write down all such sets and check whether they form a group. ❑

7: We can compute the integral homology and cohomology of G. That is, given n, we can compute $H^n(G, \mathbb{Z})$ and $H_n(G, \mathbb{Z})$ (see Chapter 2, Theorem 2.2.12).

Recall that a group has 0, 1, 2 or infinitely many ends.

8: There is an algorithm to determine whether G has 0 ends and whether G has 2 ends. There is a partial algorithm to determine whether G has infinitely many ends.

Proof: Whether a group has 0 or 2 ends can be read from the structure of the ShortLex word acceptor. We can compute the ShortLex word acceptor (using [Hol95]). We can compute the strong components of the ShortLex word acceptor using an algorithm due to Tarjan (see, for example, [Bil96]). A group has 0 ends if and only if the ShortLex word acceptor has only trivial strong components. A group has 2 ends if and only if the group has 'linear' strong components with no path to another linear strong component. Both these conditions are detectable.

To detect if a group has infinitely many ends, make successive approximations to the boundary (see Chapter 3). If there is ever more than 1 component then there must be infinitely many ends (see Proposition 3.3.13). ❑

Remark: Andrew Clow has implemented an algorithm which detects if an automatic group has infinitely many ends (see [Clow]).

Question 1.3.1: Can we compute the number of ends of G?

Remark: From the discussion above, we are left with the problem of finding an algorithm which detects if a group has 1 end.

Question 1.3.2: Can we compute the dimension of the boundary of G? An upper bound can be computed (see Proposition 3.3.6).

Question 1.3.3: A theorem of Delzant's [Del91] says that a torsion free hyperbolic group has only finitely many conjugacy classes of non-free, two-generator subgroups. Can we algorithmically list these?

Question 1.3.4: Before we can answer Question 1.3.3 we must answer the following question. Given a hyperbolic group and a set of words $\{w_1, \ldots, w_n\}$, can we tell whether the subgroup generated by $\{w_1, \dots, w_n\}$ is free on these generators?

1.3.2 Unsolvable Problems in Hyperbolic Groups

Some problems in hyperbolic groups are known to be unsolvable. The results here are given in [BMS94] using a construction of Rips [Rip82]. Given an arbitrary finite presentation for a group H, the Rips construction gives a hyperbolic group $G(H)$ with a natural epimorphism onto H such that the kernel K is generated by 2 elements. That is, the following sequence is exact;

$$1 \to K \to G \to H \to 1,$$

where H has an arbitrary presentation, K can be generated by 2 elements and G is hyperbolic.

Unsolvable Problem 1.3.5 (Rips [Rip82]): The generalised word problem is unsolvable in hyperbolic groups. That is, there is no algorithm which takes as input a finite presentation of a group G which we know to be a hyperbolic, a subgroup H of G given by a finite set of elements of G which generate it and an arbitrary word w in G and outputs whether $w \in H$.

Unsolvable Problem 1.3.6 (Baumslag, Miller, Short [BMS94]): There is no algorithm which takes as input finite presentation of a hyperbolic group, a finitely generated normal subgroup N given by a finite generating set and an element $g \in G$ and outputs whether any power of g lies in N.

Unsolvable Problem 1.3.7 (Baumslag, Miller, Short [BMS94]): The rank (minimum number of generators) of a hyperbolic group is not computable. This follows from the unsolvability of the isomorphism problem for an arbitrary presentation. Let H be the group defined by an arbitrary presentation. Perform the Rips construction on $H * H * H$. The hyperbolic group $G(H * H * H)$ has rank 2 if and only if H is trivial, otherwise it has rank at least 3. If we can compute the rank of G, then we can compute whether H is trivial.

Unsolvable Problem 1.3.8 (Baumslag, Miller, Short [BMS94]): There is no algorithm which takes as input a finite presentation of a hyperbolic group G and a subgroup H of G given by a finite set of elements of G which generate it and outputs whether the subgroup H;

1. is G,
2. has finite index,
3. is finitely presented,
4. has finitely generated second integral homology group,
5. is normal,
6. is a maximal proper normal subgroup,
7. is root-closed,
8. has only finitely many conjugates.

Chapter 2

Computing the Homology of a Hyperbolic Group

Chapter Summary

We show that there is a Turing machine which has as input a finite presentation of a hyperbolic group G and as output the nth integral homology and cohomology groups $H_n(G, \mathbb{Z})$ and $H^n(G, \mathbb{Z})$, for any given n. In Section 2.1 we recall the Rips construction of a simplicial complex on which G acts rigidly, with finite stabilisers and with finite quotient. In Section 2.2 we amend the space using the theory of complexes of groups, introduced by Haefliger in [Hae91], so that the action is free and the quotient is finite in each dimension. The quotient space is computable by a Turing machine and so its homology groups are computable.

2.1 Constructing a Contractible Space with Rigid G-action, Finite Stabilisers and Finite Quotient

Let G be a hyperbolic group.

Definition 2.1.1 (Rips Complex): Fix a generating set for G and fix a constant d. The *Rips complex* P_d is the simplicial complex with simplices $(v_0, \ldots, v_n)$ for every set $\{v_0, \ldots, v_n \in G \mid \forall i, j, \ |v_i - v_j| \leqslant d\}$. The group G acts on P_d by $g(v_0, \ldots, v_n) = (gv_0, \ldots, gv_n)$. This action is simplicial and the action on the vertices is free and transitive.

Theorem 2.1.2 (Rips): *Let G be a δ-hyperbolic group. If $d \geqslant 4\delta + 1$ then the Rips complex P_d is contractible, finite dimensional and the stabiliser of each simplex is finite. If G is torsion free then the action is free. (See, for example, [GdlH90, Chapter 4].)*

Let P'_d denote the first barycentric subdivision of P_d. We say that a group action on a simplicial complex is *rigid* if whenever a group element stabilises a simplex, it acts as the identity on that simplex. This is a natural generalisation of acting without inversion on a graph.

Lemma 2.1.3: *The action of G on P'_d is rigid.*

Proof: Let σ be a simplex of P'_d. Any pair of vertices of σ come from simplices in P_d of different dimensions. Therefore an element of G cannot permute the vertices of σ non-trivially. ❑

Theorem 2.1.4 (Epstein, Holt [EH]): *If we are given a presentation for a group G and we know that the group is hyperbolic then we can compute δ.*

Corollary 2.1.5: *We can compute the quotient $G \backslash P'_d$.*

Proof: Theorem 2.1.4 tells us that we can compute δ. We can then fix $d = 4\delta + 1$. Every simplex in $G \backslash P_d$ is the quotient of a simplex in P_d containing the identity vertex (because the G-action on the vertices of P_d is transitive). The ball of radius d is finite; therefore we can list every simplex in $G \backslash P_d$ and the boundary maps—hence all the simplices in $G \backslash P'_d$. ❑

Note that, in the case when G is torsion free, the homology of G is the homology of $G \backslash P'_d$ and Corollary 2.1.5 tells us that we can compute this space. The fact that $G \backslash P'_d$ is finite enables us to compute the homology of G.

Lemma 2.1.6: *Given a simplex of P'_d, we can compute its stabiliser.*

Proof: By Lemma 2.1.3, if $g \in G$ stabilises a simplex, it stabilises each vertex. Each vertex of P'_d is a finite subset of G. Each element of the stabiliser of the vertex $\{g_1, \ldots, g_n\}$ has the form $g_i g_j^{-1}$, for some $1 \leqslant i, j \leqslant n$. We can check which of these elements stabilise the vertex and hence find the stabiliser of any vertex. The stabiliser of the simplex is just the intersection of the stabilisers of each of its vertices. ❑

Corollary 2.1.7: *We can write a list of stabilisers of simplices of P'_d which contains one representative from each G-orbit.*

Proof: There are finitely many G orbits in P'_d so we have finitely many conjugacy classes of stabilisers. We can compute a representative stabiliser for each conjugacy class. ❑

2.2 Constructing a Contractible Space with a Free G-action and Finite Quotient in Each Dimension

We now assume that we are given a contractible simplicial complex $\tilde{X}$ on which the hyperbolic group G acts rigidly, with finite stabilisers and finite quotient. In Section 2.1 we showed how to construct such a space. We amend this space so that the action becomes free while the space remains contractible. The amended space is no longer finite but it remains finite in each dimension. Our main tool will be complexes of groups which were introduced by Corson [Cor92] and Haefliger [Hae91].

Let $X = G\backslash\tilde{X}$. For every simplex $\sigma \subset X$ choose a simplex $\tilde{\sigma} \subset \tilde{X}$ lying above σ in the quotient map. Let $G_\sigma < G$ be the stabiliser of $\tilde{\sigma}$. We use $E(G_\sigma, 1)$ to denote a contractible space with free G_σ action. The space we construct is the barycentric subdivision $\tilde{X}'$ of $\tilde{X}$ except that we glue in an $E(G_\sigma, 1)$ for each vertex σ. The fact that each stabiliser group is finite enables us to easily construct $E(G_\sigma, 1)$'s which are finite in each dimension. We take the classifying space of the category $\bar{G}_\sigma$ defined by Segal in [Seg68, Section 3] and repeated below. The information for gluing in the $E(G_\sigma, 1)$'s is obtained from the simplicial cell complex of groups description for G given by its action on $\tilde{X}$.

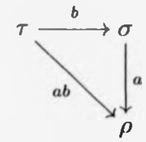
For a group G, the category $\bar{G}$ is defined as follows: The set of objects is G. There is a unique morphism between any pair of objects; for every $g, h \in G$, we have the morphism, $g^{-1}h$, from g to h. The morphisms correspond to multiplication in the group and are composed in the obvious way. This category is equivalent to the trivial category with one object and one morphism, therefore its classifying space is contractible (see Section 2.2.4).

2.2.1 Simplicial Cell Complexes of Groups

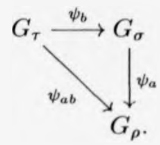
A simplicial cell complex of groups is a simplicial cell complex Y with a group associated to each simplex. We use G_σ to denote the group associated to the simplex σ. Let τ be a simplex of Y and let σ be a simplex in the closure of τ (so that σ is a simplex of lower dimension than τ which is incident to τ). Then there is an injection from G_τ to G_σ. We take the barycentric subdivision Y' of our simplicial cell complex Y so that we have a group at every vertex and an injection, ψ_a, for every directed edge, a. We require that the following conditions hold:

1. For every pair of composable edges a, b (read 'Y' for 'X' in Figure 2.1(a)

on page 33),

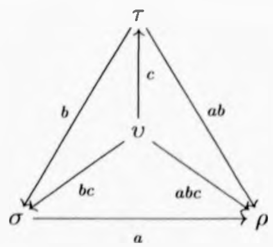


there is an element $g_{a,b} \in G_\rho$ such that the following diagram commutes up to conjugation by $g_{a,b}$;



That is, $\mathrm{Conj}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$ (where $\mathrm{Conj}(g) \colon x \mapsto gxg^{-1}$).

2. For every triple of connected edges a, b, c (a tetrahedron in Y'),



the 'conjugators' satisfy $\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$.

See [Hae91] for a more detailed exposition.

These conditions will be naturally satisfied by the way we construct our groups and injections from the G-action on $\bar{X}$.

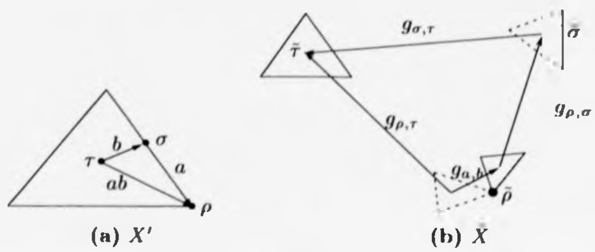


Figure 2.1. The conjugator $g_{a,b} = g_{\rho,\sigma}^{-1} g_{\sigma,\tau}^{-1} g_{\rho,\tau}$

2.2.2 Returning to our Situation

In the barycentric subdivision X' of X, with edges directed towards the vertices of X, we have a group G_σ at each vertex σ. For each directed edge b in X' (joining τ to σ), choose $g_{\sigma,\tau} \in G$ such that $g_{\sigma,\tau}(\bar{\sigma}) \subset \bar{\tau}$ in X (see Figure 2.1). Note that $g_{\sigma,\tau}$ is unique up to left multiplication by an element of G_τ. We define the injection $\psi_b : G_\tau \to G_\sigma$ by $\psi_b(h) = g_{\sigma,\tau}^{-1} h g_{\sigma,\tau}$. (Changing $g_{\sigma,\tau}$ is the same as composing ψ with an inner automorphism of G_τ.) For every pair of composable edges a, b in X' (a triangle b, a, ab in X' corresponding to a triple of simplices τ, σ, ρ in X with $\bar{\tau} \supset \bar{\sigma} \supset \bar{\rho}$, see Figure 2.1(a)), we define the *conjugator* $g_{a,b} = g_{\rho,\sigma}^{-1} g_{\sigma,\tau}^{-1} g_{\rho,\tau}$ (see Figure 2.1(b)).

Lemma 2.2.1 (Haefliger [Hae91]): *The above description defines a simplicial cell complex of groups presentation for G.* ❑

We now have a complex of groups presentation for G with simplicial cell complex X, simplex groups G_σ, injections ψ_b and conjugators $g_{a,b}$.

2.2.3 The Fundamental Group of a Complex of Groups

We have a group G_σ for each vertex σ of X' and an injection for each directed edge. We use $s(a)$ and $t(a)$ to denote the source and target vertices of the directed edge a. We associate to each edge, a, the element $g_a = g_{t(a),s(a)} \in G$, so that $g_a(\widetilde{t(a)})$ is incident to $\widetilde{s(a)}$. Occasionally we want to use a to denote not moving, in which case $s(a) = t(a)$ and $g_a = 1$. Note that, by construction, the elements g_a satisfy the relations $g_{ab} = g_b g_a g_{a,b}$ and that $\psi_a(g) = g_a^{-1} g g_a$.

A G-path in X' is a path in X' with additional information to define an element of G. It is defined as follows: Trace a path in X'. Every time we traverse an edge, a, we multiply on the right by g_a. When we reach a vertex (including the initial vertex), σ, we multiply on the right by an element of G_σ. This path can be written as a sequence $g_0 a_0 g_1 a_1 g_2 \cdots a_{n-1} g_n$, where a_i is an edge, $s(a_{j+1}) = t(a_j)$, $g_j \in G_{s(a_j)}$ and $g_n \in G_{t(a_n)}$. The set of closed G-paths can be multiplied in the obvious way to form a group. This group is the *fundamental group* of the complex of groups and it is isomorphic to G.

2.2.4 Categories and Classifying Spaces

We reconstruct the space $\tilde{X}'$ using the language of category theory. We give two naturally isomorphic categories whose classifying spaces are respectively $\tilde{X}'$ and a space with a free G-action. A result of Segal's (Proposition 2.2.2) tells us that they are homotopic; hence we have our desired space.

The *classifying space* of a category has vertices the set of objects, edges the set of morphisms, 2-simplices the set of commutative triangles, and so on (see [Seg68, Section 2]). A group action on a category extends to an action on the classifying space.

Two categories are *equivalent* if there are functors between them such that each composition, T, is a natural isomorphism from the category to itself. That is, there is a natural transformation (morphism of functors) between the identity functor and T such that each induced morphism is invertible. In other words; for every object, Ob, and for every morphism, f from Ob to Ob$'$, there are invertible morphisms, ν and ν', such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ob} & \xrightarrow{\nu} & T(\mathrm{Ob}) \\ \downarrow f & & \downarrow Tf \\ \mathrm{Ob}' & \xrightarrow{\nu'} & T(\mathrm{Ob}'). \end{array}$$

(See [Mac71, Page 16]).

Proposition 2.2.2 (Segal): *The classifying spaces of two equivalent categories are homotopic ([Seg68, Proposition 2.1]).*

Definition 2.2.3 (CC): Our first category has objects (gG_σ, σ) where σ is a simplex in X and gG_σ is a coset of the stabiliser group G_σ. The nontrivial morphisms from (gG_σ, σ) are labelled by edges a with $s(a) = \sigma$;

$$(gG_{s(a)}, s(a)) \xrightarrow{a} (gg_aG_{t(a)}, t(a)).$$

This is well defined because $G_{s(a)} \leqslant g_aG_{t(a)}g_a^{-1}$, so it is independent of the choice of coset representative, g. We call this category $CC(G(X))$ or CC ('coset category of $G(X)$').

Lemma 2.2.4: *The classifying space BCC of CC is* $\bar{X}'$.

Proof: Let $C(\bar{X})$ be the category associated to $\bar{X}$. Its objects are the simplices of $\bar{X}$ and its non-trivial morphisms are the directed edges of $\bar{X}'$. The classifying space of $C(\bar{X})$ is $\bar{X}'$.

Define the map $F : (gG_\sigma, \sigma) \mapsto g\tilde{\sigma}$ from the objects of CC to the objects of $C(\bar{X})$. Recall that g_a is such that $g_a(\widetilde{t(a)})$ is incident to $\widetilde{s(a)}$, so there is an edge $g(a)$ in $\bar{X}'$ connecting $\widetilde{s(a)}$ and $g_a(\widetilde{t(a)})$. Consequently, for each morphism a, we have the following commutative diagram:

$$\begin{array}{ccc} (gG_{s(a)}, s(a)) & \xrightarrow{a} & (gg_aG_{t(a)}, t(a)) \\ \downarrow F & & \downarrow F \\ g(\widetilde{s(a)}) & \xrightarrow{ga} & (gg_a(\widetilde{t(a)})). \end{array}$$

So the map F extends to a functor.

The inverse map $F^{-1} : g\tilde{\sigma} \mapsto (gG_\sigma, \sigma)$ is also a functor. Hence F and F^{-1} are natural isomorphisms of categories. Lemma 2.2.4 follows from Proposition 2.2.2 and from the fact that F is a bijection. ❑

Definition 2.2.5 (GC): The objects of our second category, which we call $GC(G(X))$ or GC ('group category of $G(X)$'), are pairs $(g, \sigma) \in G \times$ Simplices of X. The morphisms are

$$(g, s(a)) \xrightarrow{(a,k_a)} (gk_a, t(a)),$$

where $k_a(t(a)) \subset s(a)$ and a is either a directed edge in X', or $s(a) = t(a)$. Note that $k_a = g_a g_{t(a)}$ for some $g_{t(a)} \in G_{t(a)}$, so the morphisms can be thought of as (traverse an edge then) multiply by an element of the stabiliser group $G_{t(a)}$. The morphisms are composed in the obvious way; $(a, k_a) \circ (b, k_b) = (ab, k_a k_b)$ whenever it is defined (when $s(a) = t(b)$). Observe that, since $t(ab) = t(b)$, $s(ab) = s(a)$ and $t(a) = s(b)$, we have

$$k_a k_b(t(ab)) \subset k_a(t(a)) \subset s(ab).$$

If $s(a) = t(a)$ we sometimes write the morphism as k, where $k \in G_{s(a)}$. These are the only invertible morphisms in GC. There is a G-action on the objects of GC given by $g(g', \sigma) = (gg', \sigma)$. Clearly, this action is free.

Proposition 2.2.6: *The action of G on the classifying space BGC is free.*

Proof: Each simplex of BGC comes from an ordered set of objects of GC. So if g fixes a simplex, it must fix each of the objects (permuting them will change the order). But the action on the objects is free. ❑

We will now define the functors $P\colon GC \to CC$ (projection) and $R\colon CC \to GC$ (inclusion given by a choice of coset representatives, $\{h_{gG_\sigma}\}$, one representative for every coset of each stabiliser group).

Definition 2.2.7 (P): The projection functor, $P\colon GC \to CC$, is defined by $P\colon (g, \sigma) \mapsto (gG_\sigma, \sigma)$ on the objects and by $P\colon (a, k_a) \mapsto a$ on the morphisms. P is a functor because, for any morphism $(g, s(a)) \xrightarrow{(a,k_a)} (gk_a, t(a))$ in GC, the following diagram commutes;

$$\begin{array}{ccccc} GC & & (g, s(a)) & \xrightarrow{(a,k_a)} & (gk_a, t(a)) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} & \downarrow{\scriptstyle P} & \downarrow{\scriptstyle P} \\ CC & & (gG_{s(a)}, s(a)) & \xrightarrow{a} & (gk_aG_{t(a)}, t(a)). \end{array}$$

Note that, since $k_a = g_a g_{t(a)}$ for some $g_{t(a)} \in G_{t(a)}$, $gk_aG_{t(a)} = gg_aG_{t(a)}$.

Definition 2.2.8 (R): Choose a set of coset representatives $\{h_{gG_\sigma}\}$, one for each coset of each stabiliser group. If we write $h_1 = h_{gG_{s(a)}}$ and $h_2 = h_{gg_aG_{t(a)}}$, then $h_1 = gg_{s(a)}$, $h_2 = gk_a$ for some $g_{s(a)} \in G_{s(a)}$, and for some k_a such that $k_a(t(a)) \subset s(a)$. So $h_1^{-1}h_2 = g_{s(a)}^{-1}k_a$ and

$$h_1^{-1}h_2(t(a)) = g_{s(a)}^{-1}k_a(t(a)) \subset g_{s(a)}^{-1}s(a) = s(a).$$

We define $R : CC \to GC$ by $R : (gG_\sigma, \sigma) \mapsto (h_{gG_\sigma}, \sigma)$ on the objects and by $R : a \mapsto (a, h_1^{-1}h_2)$ on the morphisms. R is a functor because, for any morphism $(gG_{s(a)}, s(a)) \xrightarrow{a} (gg_aG_{t(a)}, t(a))$ in CC, the following diagram commutes;

$$\begin{array}{ccccc} CC & & (gG_{s(a)}, s(a)) & \xrightarrow{a} & (gg_aG_{t(a)}, t(a)) \\ \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} & \downarrow{\scriptstyle R} & \downarrow{\scriptstyle R} \\ GC & & (h_1, s(a)) & \xrightarrow{(a,h_1^{-1}h_2)} & (h_2, t(a)). \end{array}$$

Lemma 2.2.9: *The categories CC and GC are equivalent.*

Proof: The composition functor $PR = P \circ R : CC \to CC$ is equal to the identity and is therefore a natural isomorphism.

For the other composition functor $RP = R \circ P : GC \to GC$, $(g, \sigma) \mapsto (h_{gG_\sigma}, \sigma)$, for every morphism, we have the following commutative diagram;

$$\begin{array}{ccccc}
GC & (g, s(a)) & \xrightarrow{(a,k_a)} & (gk_a, t(a)) \\
\Big\downarrow RP & x \Big\downarrow RP & \Big\downarrow RP & z \Big\downarrow RP \\
CC & (h_{gG_{s(a)}}, s(a)) & \xrightarrow{(a,y)} & (h_{gk_aG_{t(a)}}, t(a)),
\end{array}$$

where $x = g^{-1}h_{gG_{s(a)}}$, which is in $G_{s(a)}$ because g and $h_{gG_{s(a)}}$ are in the same coset of $G_{s(a)}$, and similarly $z = (gk_a)^{-1}h_{gk_aG_{t(a)}} \in G_{t(a)}$. We need to check that $y = h^{-1}_{gG_s(a)}h_{gk_aG_{t(a)}}$ satisfies $y(t(a)) \subset s(a)$. Now, $h_{gG_{s(a)}} = gg_{s(a)}$ for some $g_{s(a)} \in G_{s(a)}$ and $h_{gk_aG_{t(a)}} = gk_a g_{t(a)}$ for some $g_{t(a)} \in G_{t(a)}$. So $y(t(a)) = g^{-1}_{s(a)}k_a g_{t(a)}(t(a)) \subset s(a)$.

Clearly, both x and z are invertible, so RP and Id are equivalent. ❑

Theorem 2.2.10: *The classifying space BGC of GC is contractible.*

Proof: By Lemma 2.2.9, the categories GC and CC are equivalent. By Proposition 2.2.2, the classifying spaces BGC and BCC are homotopic. Thus $BGC \simeq BCC = \bar{X}'$ (by Lemma 2.2.4) which is contractible by Rips' Theorem (Theorem 2.1.2). Therefore BGC is contractible. ❑

Proposition 2.2.11: *For each n, we can list the simplices of the n-skeleton of the quotient space $G\backslash BGC$.*

Proof: An n-simplex comes from a sequence of n composable edges. We can list all of these. ❑

Theorem 2.2.12:

Given an integer n and a finite presentation for a group G which we know to be hyperbolic, we can compute the nth (co)homology group of G.

Proof: The nth (co)homology group of G is the nth (co)homology group of $G\backslash BGC$, which is computable and finite in every dimension. Therefore we can explicitly write lists of the $n+1$, n and $n-1$ cells and the boundary maps $(G\backslash BGC)^{(n+1)} \xrightarrow{\partial} (G\backslash BGC)^{(n)} \xrightarrow{\partial} (G\backslash BGC)^{(n-1)}$. The problem reduces to the problem of finding the kernel and image of linear maps in free-Abelian groups, which is computable. ❑

The category $CG(X)$ in [Hae91] and [Hae92] is precisely $G\backslash GC$. Haefliger shows in [Hae92] that this category is naturally isomorphic to the category $G \ltimes C(X)$ (which we describe below) and that the classifying space of this category is homotopy equivalent to $BG \times_G \tilde{X} = G\backslash BG \times \tilde{X}$, where BG is an $E(G,1)$ and G has the diagonal action on $BG \times \tilde{X}$. $BG \times_G \tilde{X}$ is sometimes called the Borel construction or Borel homotopy quotient (see, for example, [Geo]). The category $G \ltimes C(\tilde{X})$ has objects the cells $\tilde{\sigma}$ of $\tilde{X}$ and, for every $g \in G$ and for every $\tilde{\sigma}$, it has a morphism $\tilde{\sigma} \xrightarrow{g} g\tilde{\sigma}$. When $\tilde{X}$ is contractible, $BG \times_G \tilde{X}$ is a $K(G,1)$. See Figure 2.2 for a map of the different proofs.

2.3 Examples

Example 2.3.1: An easy first example is $\mathbb{Z}_3 = \langle x \mid x^3 \rangle$. The Rips complex P_1 is a filled triangle. The action of the generator x is a rotation of $2\pi/3$

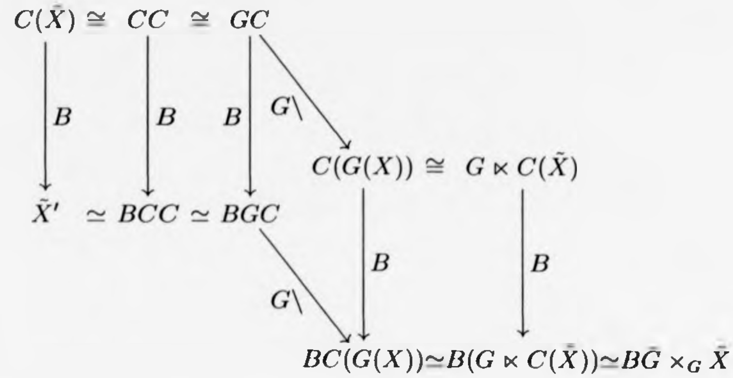


Figure 2.2. A diagram showing the relationships between the different categories in this chapter and in [Hae92]. The down arrows marked B are geometric realisations.

about the barycentre of the 2-simplex. The stabiliser of the 2-simplex is $\mathbb{Z}_3$, all other stabilisers are trivial. Taking the barycentric subdivision makes the action rigid with one vertex stabiliser group $\mathbb{Z}_3$ and all other stabilisers trivial (see Figure 2.3).

We choose representatives for each $\mathbb{Z}_3$-orbit in P_1' as shown in Figure 2.4. The stabiliser of v_3 is $\mathbb{Z}_3$, all other stabilisers are trivial.

The 1-skeleton of the space $BG(\mathbb{Z}_3)$ is shown in Figure 2.5. The black vertices lie above the chosen simplices of Figure 2.4 and so correspond to objects of the form (e, σ). The grey vertices come from objects of the form (x, σ) and the white vertices from objects of the form (x^2, σ). The 3 central vertices come from the stabiliser of the 2-simplex in the Rips complex.

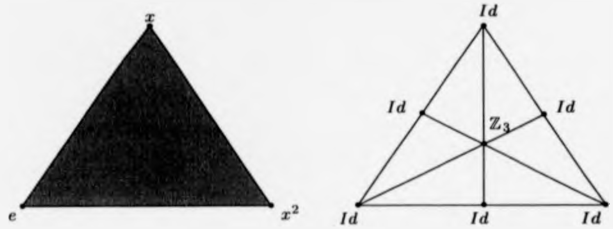


Figure 2.3. The Rips complex of $\mathbb{Z}_3 = \langle x \mid x^3 \rangle$ and the stabilisers of the simplices.

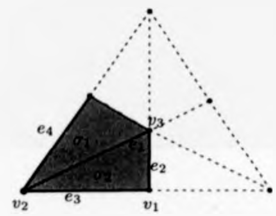


Figure 2.4. Choosing representatives in each orbit

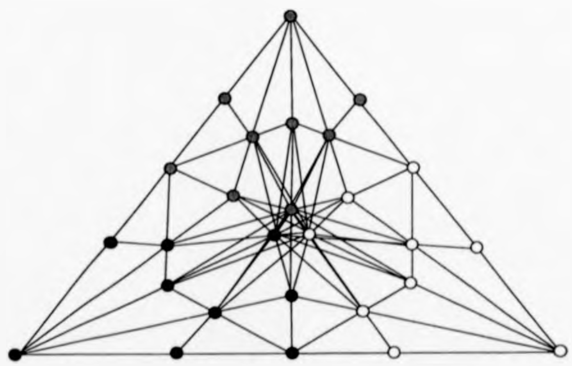


Figure 2.5. The 1-skeleton of $B(\mathbb{Z}_3)$

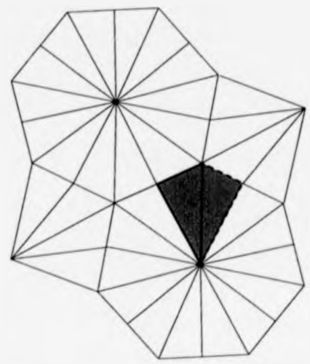


Figure 2.6. Part of the tiling by $(\pi/2, \pi/3, \pi/7)$-triangles. The shaded area is a fundamental domain for the action of (2,3,7)-triangle group on this tiling.

Example 2.3.2: A better example is the orientation preserving (2,3,7)-triangle group $\langle x, y, z \mid x^7, y^3, z^2, zyx \rangle$. We proceed from Section 2.2 using a tiling of the hyperbolic plane by $(\pi/2, \pi/3, \pi/7)$-triangles as our contractible space on which the group acts rigidly and with finite stabilisers. Call this simplicial complex $\tilde{X}$. The action is generated by clockwise rotations through $\pi/7$, $\pi/2$ and $\pi/3$ about $\tilde{v}_1$, $\tilde{v}_2$ and $\tilde{v}_3$ (the vertices in the shaded region of Figure 2.6) respectively.

A fundamental region for our action is a quadrilateral (for example, the shaded region in Figure 2.6). The quotient space is homeomorphic to a sphere (Figure 2.7).

For each simplex in the quotient $X = G\backslash\tilde{X}$, select a simplex lying above it in $\tilde{X}$. The simplices we choose lie in the shaded fundamental region shown in Figure 2.6. The stabilisers of the vertices are $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_7$. The other stabilisers are trivial.

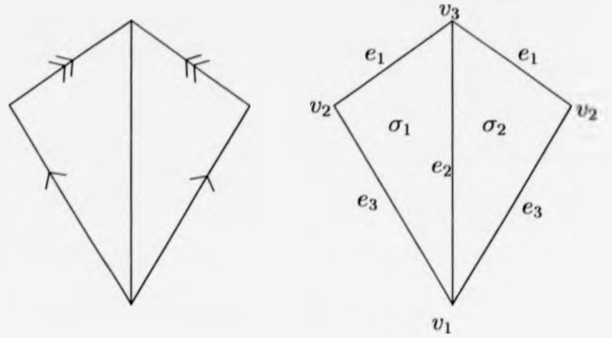
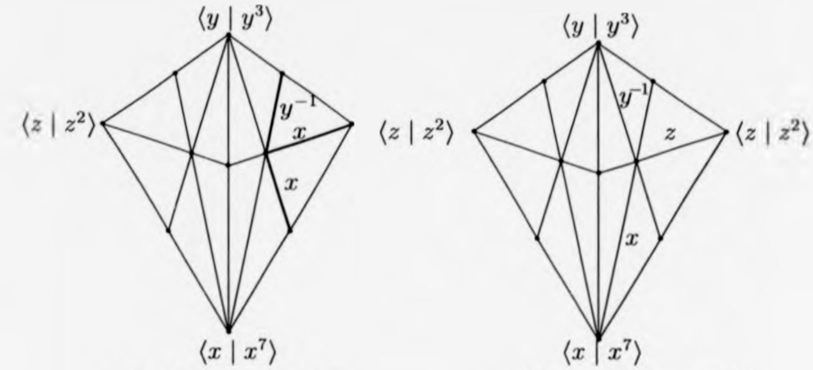


Figure 2.7. The quotient space for the $(\pi/2, \pi/3, \pi/7)$-triangle tiling under the action of the (2,3,7)-triangle group

For a pair of incident simplices σ, τ in X, we need to choose a $g_{\sigma,\tau} \in G$ such that $g_{\sigma,\tau}(\bar{\sigma})$ is incident to $\bar{\tau}$. In most cases we can choose the identity. The only exceptions being between the pairs (σ_2, e_1), (σ_2, v_2) and (σ_2, e_3) (see Figures 2.7 and 2.6). We take $g_{\sigma_2,v_2} = g_{\sigma_2,e_3} = x$ and $g_{\sigma_2,e_1} = y^{-1}$ and label edges in the barycentric subdivision of X accordingly (see Figure 2.8(a)). The triangle in X' with vertices σ_i, e_j, v_k has conjugator $g_{\sigma_i,e_j} g_{e_j,v_k} g_{\sigma_i,v_k}^{-1}$ and so most are trivial. The exceptions are shown in Figure 2.8(b).

We now construct the category GC. Recall that the objects are labelled by pairs (g, σ), where $g \in G$ and σ is a vertex in X'. The morphisms come from traversing an edge in X' (or staying put) followed by multiplying by an element of the stabiliser group of the destination vertex.

We can label each simplex lying above σ in the original complex by a coset of G_σ. The labelling by a group element g then becomes hg_σ where h is a coset



(a) Labelling edges in X' (b) Labelling faces in X'

Figure 2.8. Labelling edges and finding conjugators. Unmarked edges in the left picture and unmarked faces in the right picture are labelled by e.

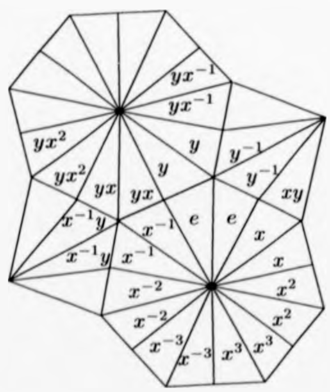


Figure 2.9. Labelling the faces

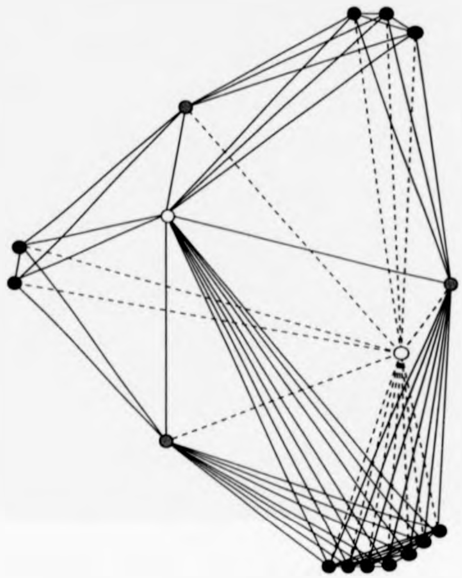


Figure 2.10. The quotient space $G\backslash BGC$ for the $(2, 3, 7)$-triangle group.

representative and $g_\sigma \in G_\sigma$. The faces of our tiling X have been labelled in Figure 2.9.

The 1-skeleton of the quotient space $G\backslash BGC$ is shown in Figure 2.10. The space $G\backslash BGC$ looks spherical, but the 2-cells don't 'match up' so there aren't any embedded spheres. The third homology group is $\mathbb{Z}$. The 2-cells have to be 'wrapped around' 42 times until they match up. This coincides with the second homology group being $\mathbb{Z}_{42}$. The first homology group, the abelianisation of G, is trivial.

Figure 2.11 shows the 1-skeleton of BGC. An inclusion of BCC in BGC for a particular choice of coset representatives is shown in Figure 2.12.

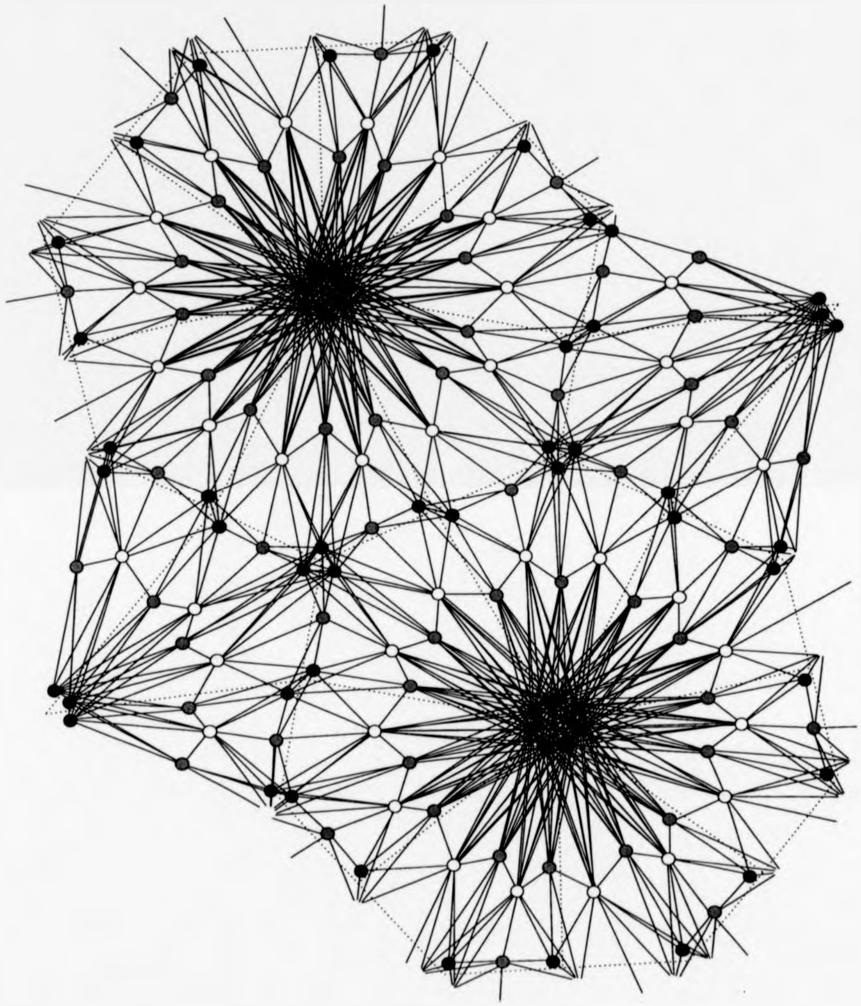


Figure 2.11. The 1-skeleton of BGC for the (2,3,7)-triangle group

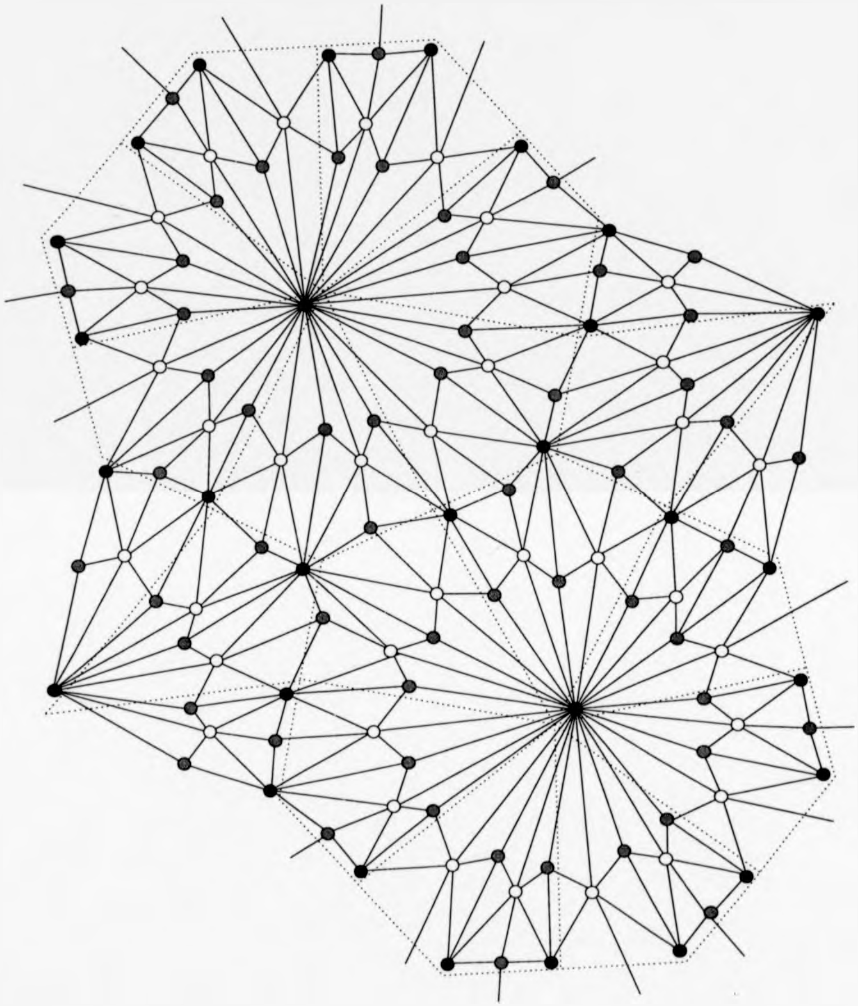


Figure 2.12. The 1-skeleton of BCC for the (2,3,7)-triangle group

Chapter 3

The Boundary of a Hyperbolic Group as an Inverse Limit of Finite Sets

Chapter Summary

Given a hyperbolic group G, we show how to construct an inverse system of finite topological spaces whose inverse limit is homeomorphic to ∂G, the boundary of G. Each of the finite spaces are computable and can be used to estimate topological properties of the boundary.

Let G be a hyperbolic group and fix a generating set. Let Γ be the corresponding Cayley graph. Recall (Section 1.2.3) that G has a ShortLex automatic structure and a geodesic automatic structure.

3.1 The Boundary as an Inverse Limit

3.1.1 Constructing Finite Topological Spaces

Let W_n denote the set of elements of length n in G. The set W_n can be thought of as the set of ShortLex geodesics of length n in Γ.

Definition 3.1.1 (cluster): A set of ShortLex geodesic rays (geodesic rays such that every prefix is a ShortLex geodesic) to the same boundary point is called a *cluster of geodesic rays* or a *cluster*.

Definition 3.1.2 (frond): The set of truncations of length n of the geodesics in a cluster is called the nth frond of the cluster. A *frond of length* n or, *frond* , is a subset of W_n such that each element of the subset can be extended ShortLex geodesically to the same boundary point. We call a frond with $k+1$ points a *k-frond*. Note that elements which cannot be extended ShortLex geodesically to the boundary are not fronds.

Definition 3.1.3 ($\mathbf{F_n}$): We use F_n to denote the set of fronds of length n. We give F_n a topology by defining the closure of each point. The closure of the point x is defined as $\overline{x} = \{y \in F_n \mid y \subset x\}$. Each cluster defines a frond

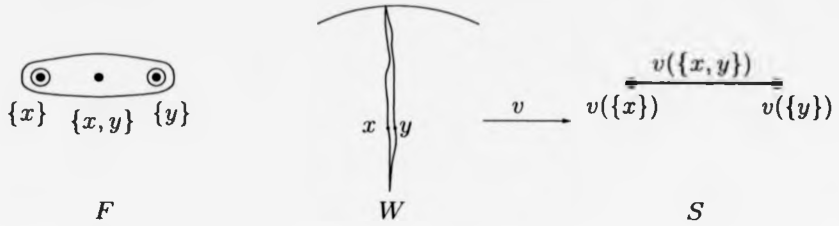


Figure 3.1. The relationship between F, W and S

of length n by truncating each element of the cluster at length n. Thus we have a map $p_n\colon \{\text{clusters}\} \to F_n$.

It is instructive to construct a simplicial complex S_n which graphically represents the topology of F_n. There is a k-simplex of S_n for every k-frond, so, for example, the set of vertices of S_n is the set of 0-fronds; that is, fronds of the form $\{w\}$. We use $v(x)$ to denote the simplex associated to the frond x. We give S_n the 'combinatorial topology', where a set is open if and only if it is a union of open simplices which is open in the CW-topology (the set of open stars of simplices forms a basis for this topology). The map $S_n \to F_n$, defined by $y \mapsto x$ if y is in the interior of $v(x)$, is a 'relational homeomorphism'; that is, it is continuous, surjective and the image of an open set is open. Figure 3.1 shows the relationship between F_n, W_n and S_n. Closures of points in F_n are indicated by ringing points.

3.1.2 The Restriction Map

We define the map $\rho_n \colon W_n \to W_{n-1}$ by

$$\rho_n \colon a_1 \cdots a_{n-1}a_n \mapsto a_1 \cdots a_{n-1},$$

where each a_i is in the generating set and $a_1 \cdots a_n$ traces a ShortLex geodesic in Γ. The map ρ restricts a geodesic of length n to itself truncated at length $n-1$.

Definition 3.1.4 (The restriction map): The map ρ_n extends to a map from F_n to F_{n-1} by

$$f_n \colon F_n \to F_{n-1},\ \{w_0, w_1, \ldots, w_k\} \mapsto \{\rho(w_0), \rho(w_1), \ldots, \rho(w_k)\}\,.$$

The map f_n is called the *restriction map.* The image of a frond is clearly a frond. A k-frond maps to a j-frond, for some $j \leqslant k$. Recall the map p_n from Definition 3.1.3. The restriction map satisfies $p_n \circ f_n = p_{n-1}$; that is, for each n, the following diagram commutes:

$$\begin{array}{ccc} \{\text{clusters}\} & \xrightarrow{p_n} & F_n \\ & \searrow^{p_{n-1}} & \downarrow f_n \\ & & F_{n-1}. \end{array} \tag{3.1}$$

Now, if $x, y \in F_n$ and $x \in \overline{y}$, then $f_n(x) \in \overline{f_n(y)}$. Since S_n exhibits the topology of F_n, we can think of f_n as a map from S_n to S_{n-1}.

Example 3.1.5: Let G be the fundamental group of a surface of genus 2 with the usual generating set ($G = \langle a, b, c, d \mid aca^{-1}c^{-1}bdb^{-1}d^{-1}\rangle$, see Graph 3).

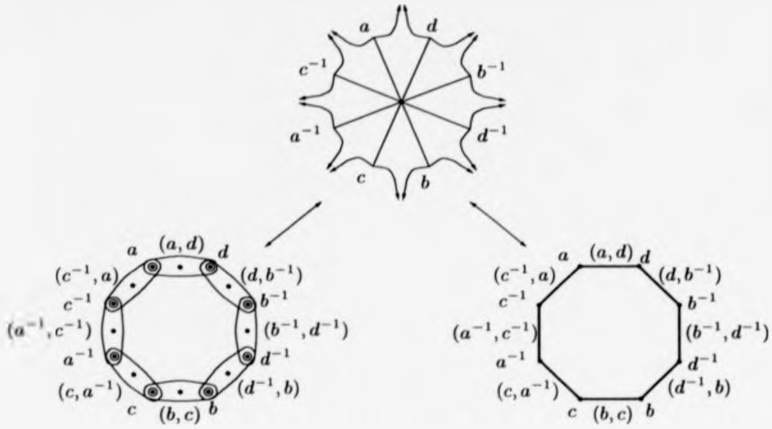


Figure 3.2. F_1, part of the Cayley graph and S_1 of the fundamental group of a surface of genus 2

The fronds of length 1 are the generators and pairs $\{x, y\}$ such that $y^{-1}x$ appears in the relator written cyclically (see Figure 3.2).

Part of the restriction map f_2 is shown, as a map $S_2 \to S_1$, in Figure 3.3.

3.1.3 The Inverse Limit

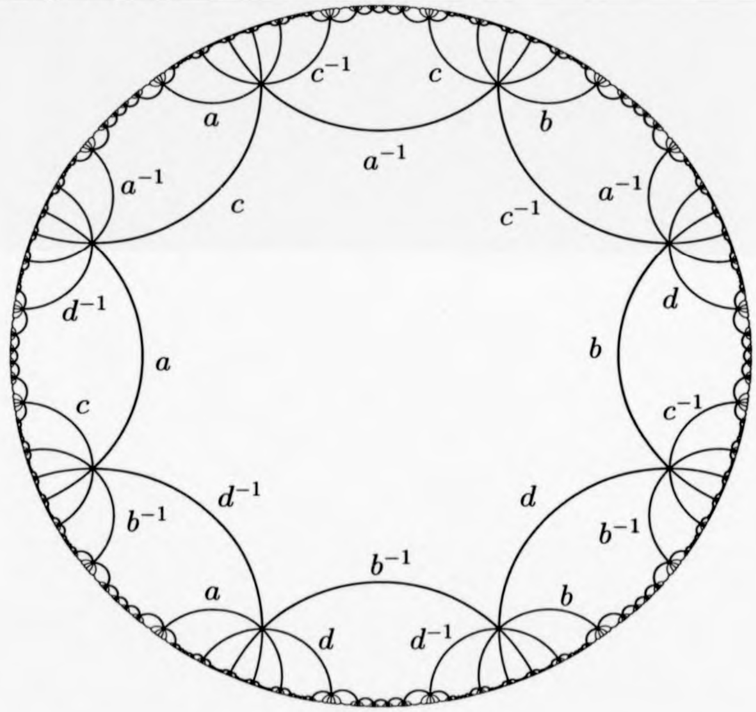
The finite sets F_n and the maps $f_n : F_n \to F_{n-1}$ form an inverse system, which we call F:

$$F_0 \xleftarrow{f_1} F_1 \xleftarrow{f_2} F_2 \xleftarrow{f_3} F_3 \xleftarrow{f_4} \cdots . \tag{3.2}$$

We now look at the inverse limit of F, $\varprojlim F$.

A point in the inverse limit is a sequence $x = (x_i)_{i=0}^{\infty}$, such that, for each $i \geqslant$

Cayley Graph 3. The fundamental group of the torus of genus 2. The edges in the central octagon should be read clockwise. The edges in the adjacent octagons should be read anticlockwise. This picture was drawn by Sascha Rogmann from the University of Düsseldorf.



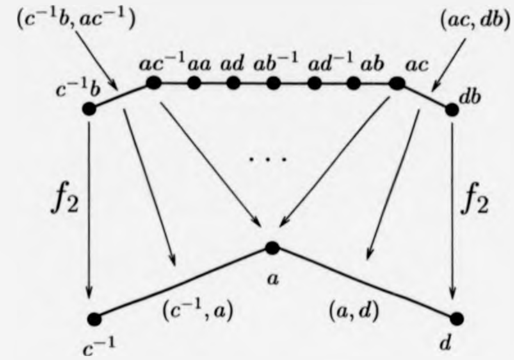


Figure 3.3. The restriction map f_2

1, $f_i(x_i) = x_{i-1}$. Since, for each i, $f_i \circ p_i = p_{i-1}$ (3.1), a cluster corresponding to x_{i-1} is a cluster corresponding to x_i. So we can extend the maps p_n to a map p: {clusters} $\to \varprojlim F$. The map p is a bijection, so a point in $\varprojlim F$ can be thought of as a cluster (that is, a set of geodesic rays to the same boundary point). The closure of a point $x = (x_i)_{i=1}^{\infty}$ in $\varprojlim F$ is given by

$$\overline{x} = \{y = (y_i)_{i=1}^{\infty} \mid \forall i,\ y_i \in \overline{x_i}\}.$$

Equivalently, $y \in \overline{x}$ if and only if $p^{-1}(y) \subset p^{-1}(x)$.

Definition 3.1.6 (The map k): The point x in $\varprojlim F$ naturally defines the cluster $p^{-1}(x)$, which is a set of geodesic rays to the same boundary point ξ. We define the map $k : \varprojlim F \to \partial G$ by $k(x) = \xi$. Every infinite ShortLex geodesic is a cluster, so the map k is a surjection.

Lemma 3.1.7: *For all $\xi \in \partial G$, $|k^{-1}(\xi)|$ is finite.*

Proof: There are only a finite number of ShortLex geodesic rays to each boundary point, so there are a finite number of clusters to each boundary point. ❑

In fact, there is a universal bound on the size of $|k^{-1}(\xi)|$ (see Lemma 3.3.2 on page 68).

3.1.4 Hausdorffifying the Inverse Limit

The space $\varprojlim F$ is not necessarily Hausdorff because the closure of a point is not necessarily itself.

To make $\varprojlim F$ Hausdorff, we amalgamate certain points. We amalgamate the points x and y if there is a $z \in \varprojlim F$ such that $x \in \bar{z}$ and $y \in \bar{z}$. If x and y define the same boundary point, then $p^{-1}(x)$ and $p^{-1}(y)$ are clusters to the same boundary point. Therefore $p^{-1}(x) \cup p^{-1}(y)$ is a cluster, so we can set $z = p[p^{-1}(x) \cup p^{-1}(y)]$. Then x and y are both in $\bar{z}$. So we amalgamate x and y if and only if they define the same boundary point. Equivalently, we amalgamate x and y if and only if $k(x) = k(y)$. Lemma 3.3.2 shows that this process is finite at each point.

Let H denote the space we obtain by performing the above operations and let $q\colon \varprojlim F \to H$ be the quotient map.

Lemma 3.1.8: *The pair H, q has the property that for all Hausdorff spaces, Y, and continuous maps, $c\colon \varprojlim F \to Y$, there is a unique continuous map,*

l, such that the following diagram commutes:

$$\begin{array}{ccc} \varprojlim F & \xrightarrow{q} & H \\ & \searrow^{c} & \downarrow l \\ & & Y. \end{array}$$

Proof: Let Y be a Hausdorff space and let c be a continuous map from $\varprojlim F$ to Y. Let $x, y \in \varprojlim F$ such that $q(x) = q(y)$. Then there is a z in $\varprojlim F$ such that $x, y \in \overline{z}$. Then $c(x) = c(z) = c(y)$ because c is continuous and $x \in \overline{z} \implies c(x) \in \overline{c(z)} = \{c(z)\}$. So the map l must be $l(a) = cq^{-1}(a)$. (Although q^{-1} is not a map, cq^{-1} is a map.) The map c is continuous and q is a quotient map, so l is continuous. ❑

In particular, in the case when $Y = \partial G$, we have the following commutative diagram;

$$\begin{array}{ccc} \varprojlim F & \xrightarrow{q} & H \\ & \searrow^{k} & \downarrow h \\ & & \partial G. \end{array} \tag{3.3}$$

We shall show that the map k is continuous and deduce that h is a homeomorphism (Theorem 3.1.15).

3.1.5 The Topology of H

It is useful to first examine the topology of $\varprojlim F$. Let $x_n \in F_n$. The set

$$C(x_n) = \left\{ (y_i)_{i=1}^{\infty} \in \varprojlim F \mid y_n = x_n \right\} \tag{3.4}$$

is called a cylinder (compare with Definition 4.1.2). Note that fixing the nth entry of a sequence in $\varprojlim F$ fixes all earlier entries. So a cylinder is fixed on a finite number of symbols but can then vary arbitrarily. We also refer to finite unions of cylinders as cylinders. The complement of a cylinder is itself a cylinder.

Recall that a basic open set in F_n corresponds to the star of a simplex in S_n; open sets are unions of simplices in S_n which are open in the CW-topology. That is, given a frond $x \in F_n$, there is a basic open set B_x in F_n given by $B_x = \bigcup_{y \supset x} y$. If U is an open subset of F_n, then $f_{n+1}^{-1}(U)$ is open in F_{n+1}, because, given a frond $x \in F_{n+1}$, $y \supset x$ if and only if $f_{n+1}(y) \supset f_{n+1}(x)$. So the inverse image of a star is a union of stars. So, if U is open in F_n, then its inverse image in F_{n+m} must always be open.

So a basic open set in $\varprojlim F$ is a set of the form

$$C(B_x) = \bigcup_{y \in B_x} C(y)$$

where $x \in F_n$, for some n; that is, a basic open set in $\varprojlim F$ is the cylinder of a basic open set in F_n. An open set in $\varprojlim F$ can be written as a union of cylinders of stars.

Lemma 3.1.9: *Any closed set of $\varprojlim F$ can be made by taking (finite unions and) arbitrary intersections of cylinder sets.*

Proof: Let E be a closed set. Then $E = U'$ for some open U, where U' denotes the complement of U. Now, $U = \bigcup_{\lambda \in \Lambda} C_\lambda$, where C_λ are cylinder sets.

So,

$$E = U' = \Big(\bigcup_{\lambda\in\Lambda} C_\lambda\Big)' = \bigcap_{\lambda\in\Lambda} C'_\lambda.$$

But C'_λ is a cylinder and the result follows. ❑

Note that the converse is not true; if you take an arbitrary intersection of (finite unions of) cylinders, then you do not necessarily get a closed set.

Observation 3.1.10: *An open set U is characterised by the property that if $C(x) \subset U$ (where $x \in F_n$) and $x \subset y$ then $C(y) \subset U$.*

Observation 3.1.11: *A closed set, E, is characterised by the property that if $C(x) \subset E$ and $x \supset y$ then $C(y) \subset E$.*

Now, a closed set in H is a set, E, such that $q^{-1}(E)$ is closed in $\varprojlim F$. Similarly, an open set, U, in H is a set such that $q^{-1}(U)$ is open in $\varprojlim F$.

We now examine the topology of ∂G.

Definition 3.1.12 (Shadow): Let $g \in G$. The *shadow of g*, which we denote by $S(g)$, is the subset of ∂G defined by the set of geodesic rays from the identity through g. Let $\omega = \{w_1, \ldots, w_k\} \in F_n$, then the *shadow of ω* is $S(\omega) = \bigcap_i S(w_i)$.

Lemma 3.1.13: *Shadows are closed. That is, for each $g \in G$, $S(g)$ is closed.*

Proof: Let $(\xi_i)_{i=1}^{\infty}$ be a convergent sequence in $S(g)$. For each ξ_i, pick a

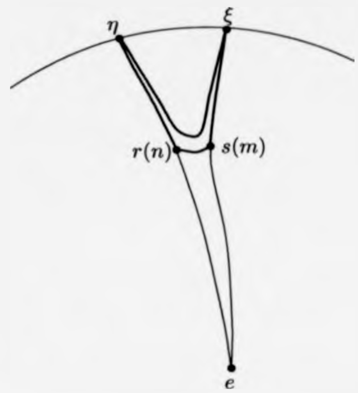


Figure 3.4. A geodesic quadrilateral

geodesic r_i from e to ξ_i through g. By passing to subsequences, for each $n > |g|$, we can ensure that the r_i agree on the first n symbols. This gives us a geodesic ray r from e to $\lim \xi_i$ through g. So $\lim \xi_i \in S(g)$. ❑

Lemma 3.1.14: *If $U \subset \partial G$ is open and r is a geodesic ray from the identity such that $r(\infty) \in U$, then there is an n such that $S(r(n)) \subset U$. Further, if $B = B(r(n), \delta)$ denotes the ball of radius δ about $r(n)$ then $\bigcup_{g \in B} S(g) \subset U$.*

Proof: Let $\eta = r(\infty)$ and let $\xi \in \bigcup_{g \in B(r(n))} S(g)$. Then there is a geodesic ray s from e to ξ which passes within δ of $r(n)$. So there is an $m \geqslant n - \delta$ such that $d(s(m), r(n)) \leqslant \delta$. Consider the quadrilateral with edges $[r(n), \eta) = r_{|[n,\infty)}$, $[s(m), \xi) = s_{|[m,\infty)}$, $[r(n), s(m)]$ and (η, ξ) (see Figure 3.4). The geodesic (η, ξ) lies in a 16δ-neighbourhood of the union of the other edges ([CP93, Chapter 1, Proposition 3.2]), so $d(e, (\eta, \xi)) \geqslant n - 17\delta$. Therefore, by (1.4)

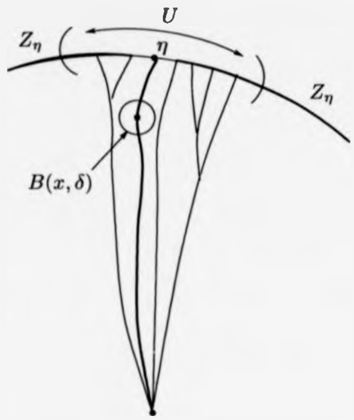


Figure 3.5. The set Z_η as in the proof of Theorem 3.1.15

and (1.3), $d_\varepsilon(\eta,\xi) \leqslant \exp -(\varepsilon(n-16\delta))$, which we can make as small as we like by increasing n. Therefore, for sufficiently large n, $\bigcup_{g\in B(r(n))} S(g) \subset U$. ❑

Theorem 3.1.15:

The map $h\colon H \to \partial G$ is a homeomorphism. That is, the Hausdorffification of the inverse system (3.2) is homeomorphic to ∂G.

Proof: We show that the map k (see (3.3) and Definition 3.1.6) is continuous. Let $U \subset \partial G$ be open and let $\eta \in U$. Let r be a ShortLex geodesic ray to η. By Lemma 3.1.14, there is an n such that $\bigcup_{g\in B(r(n),\delta)} S(g) \subset U$.

Now, $r(n) \in W_n$. Let $Z = \{z \in F_n \mid z \cap B(r(n),\delta) = \varnothing\}$ and let $Z_\eta = \bigcup_{z\in Z} S(z)$ (see Figure 3.5). Then Z_η corresponds to the geodesic rays from e which don't go within δ of $r(n)$, so $\eta \notin Z_\eta$.

Claim: *The set $k^{-1}(Z_\eta)$ is closed.*

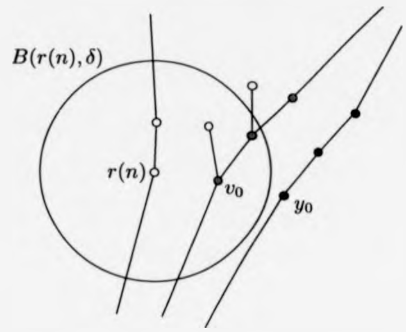


Figure 3.6. The difference between Y_m and V_m. Grey circles indicate elements of V_m which are not elements of Y_m. Black circles indicate elements of Y_m and V_m. White circles indicate elements which are in neither Y_m nor V_m.

Proof: Let $Y_n = \{y \in W_n \mid |y - r(n)| > \delta\}$ and, for each $m > n$, let $Y_m = \{w \in W_m \mid \rho^{m-n}(w) \in Y_n\}$—the set of geodesic extensions of Y_n of length m. Then $k(C(Y_m)) = Z_\eta$. However, $C(Y_m)$ is not the inverse image of Z_η. Let

$$V_m = \{v \in W_m \mid \exists y \in Y_m \text{ such that } \{v, y\} \in F_m\}.$$

The set V_m is the set of ShortLex geodesics of length m which can be extended to the same boundary point as an element of Y_m. There could be elements of V_n which are within δ of $r(n)$ (see Figure 3.6). Let T_m be the set of subsets of V_m which are also fronds. Then $C(T_m)$ is closed in $\varprojlim F$ because, if $w \in T_m$ and $v \subset w$, then $v \in T_m$ (Observation 3.1.11).

We show that $k^{-1}(Z_\eta) = \bigcap_m C(T_m)$. Suppose that $x = (x_i)_{i=0}^{\infty} \in k^{-1}(Z_\eta)$. Each x_i is a frond $\{w_0^i, w_1^i, \ldots, w_q^i\}$, where each $w_j^i \in W_i$ and q is a bounded non-decreasing function of i. Since $k(x) \in Z_\eta$, there is a geodesic ray s from the identity such that $s(\infty) = k(x)$ and $\mathrm{Im}(s) \cap B(r(n), \delta) = \varnothing$. Therefore

$y = (\{s(i)\})_{i=0}^{\infty} \in \varprojlim F$ and $s(m) \in Y_m$. So, for each $i \geqslant n$,

$$x_i \cup \{s(i)\} \in F_i \implies \{w_j^i, s(i)\} \in F_i \text{ for each } j$$
$$\implies w_j^i \in V_i \implies x_i \in T_i \implies x \in C(T_i) \implies x \in \bigcap_m C(T_m).$$

Therefore $k^{-1}(Z_\eta) \subset \bigcap_m C(T_m)$.

Conversely, suppose $x \in \bigcap_m C(T_m)$. Then, for each m, $x_m \in T_m$. Let s be a geodesic in the cluster $p^{-1}(x)$. Then $s(m) \in V_m$ so there is a $y_m \in Y_m$ such that y_m and $s(m)$ can be extended to a common boundary point. Note that the truncations $\rho^{m-i}(y_m)$ and $s(i)$ can also be extended to the same common boundary point. By passing to subsequences, for each m we can choose a y_m (in Y_m if $m > n$) such that $\rho(y_m) = y_{m-1}$. Therefore, $(y_m)_{m=0}^{\infty}$ traces a geodesic to a boundary point ξ. Now, each $y_m \in Y_m$, therefore $\xi \in Z_\eta$. Also, for each m, $\{y_m, s(m)\} \in F_m$, and thus $\xi = k(x)$. Therefore $\bigcap_m C(T_m) \subset k^{-1}(Z_\eta)$. Hence $\bigcap_m C(T_m) = k^{-1}(Z_\eta)$ and therefore $k^{-1}(Z_\eta)$ is closed. □

Now, $(Z_\eta)' \subset \bigcup_{g \in B(r(n),\delta)} S(g) \subset U$. So $\bigcup_{\eta \in U} (Z_\eta)' = U$ and

$$k^{-1}(U) = k^{-1}\Big(\bigcup_{\eta \in U} (Z_\eta)'\Big) = \bigcup_{\eta \in U} k^{-1}(Z_\eta') = \bigcup_{\eta \in U} (k^{-1}(Z_\eta))',$$

which is a union of open sets and is therefore open.

So k is continuous and hence so is h. Now, H is compact and ∂G is Hausdorff, so h is a continuous bijection from a compact space to a Hausdorff space and therefore is a homeomorphism. ❑

3.2 Similar Constructions

So far, we have talked about ShortLex geodesics which can be extended ShortLex geodesically to the same boundary point. We can produce a similar description for the set of ShortLex geodesics which can be extended geodesically to the same boundary point and for the set of geodesics which can be extended geodesically to the same boundary point.

Let F_n^S denote the finite set obtained by considering ShortLex geodesics of length n which can be extended geodesically to the same boundary point. Let F_n^G denote the finite set obtained by considering all geodesics of length n which can be extended geodesically to the same boundary point. As above, we can define inverse systems F^S and F^G with Hausdorffifications $\varprojlim F^S/\sim$ and $\varprojlim F^G/\sim$.

Lemma 3.2.1: *The spaces H and $\varprojlim F^S/\sim$ are equal.*

Proof: We have, $\varprojlim F = \varprojlim F^S$ because the points in both sets are in bijection with the set of infinite ShortLex geodesic rays and so the cylinder sets will be identical. The closures of a point are also the same, both consisting of clusters to the same boundary point. So the quotient maps will be the same and thus the two spaces are equal. ❑

Note that the finite sets F_n and F_n^S may be different but any differences disappear in the inverse limit (see Example 3.2.3).

Proposition 3.2.2: *The spaces H and $\varprojlim F^G/\sim$ are homeomorphic.*

Proof: We show that $\varprojlim F^G/\!\sim$ is homeomorphic to ∂G. Let $q_G : \varprojlim F^G \to \varprojlim F^G/\!\sim$ be the quotient map and let h_G be the induced map from $\varprojlim F^G/\!\sim$ to ∂G.

Let E be a closed subset of $\varprojlim F^G/\!\sim$. Then $q_G^{-1}(E)$ is closed in $\varprojlim F^G$ and so, by the corresponding result to Lemma 3.1.9, can be expressed as a finite union and arbitrary intersection of cylinder sets.

Let C be a cylinder in $\varprojlim F^G$. The cylinder, C, corresponds to all possible infinite extensions of a frond, so $k(C)$ is a shadow (of a frond) and therefore closed by Lemma 3.1.13.

Hence, $h_G(E) = kq_G^{-1}(E)$ can be made by finite unions and arbitrary intersections of closed sets, and therefore is closed. Thus h_G^{-1} is continuous.

Now, ∂G is compact and $\varprojlim F^G/\!\sim$ is Hausdorff, so h_G^{-1} is a continuous bijection from a compact space to a Hausdorff space and therefore is a homeomorphism. By Theorem 3.1.15, H is homeomorphic to ∂G and the result follows. ❑

The advantage of using F over F^S is two-fold. Firstly, it is computationally easier. Secondly, given any nth frond x, $f_{n+1}^{-1}(x)$ is non-empty. This is not the case for F^S (see Example 3.2.3).

F^G is computationally much harder because we have exponentially many excessive points (points in F_n^G corresponding to the same element of W_n). Further, the Hausdorffification process of $\varprojlim F^G$ may not be finite at each point as there may be infinitely many geodesic rays to the same boundary

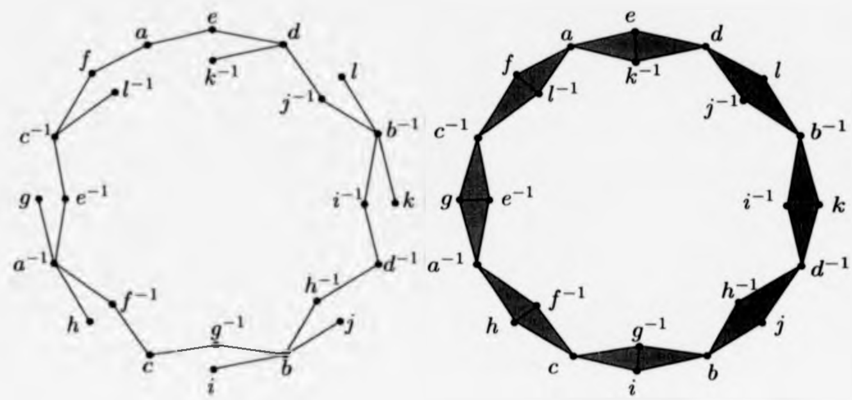


Figure 3.7. S_1 and S_1^S for Example 3.2.3

point.

Example 3.2.3: Consider the fundamental group of the surface of genus 2 (Example 3.1.5). Add the new generators

$$e = ac,\ f = ac^{-1},\ g = a^{-1}c^{-1},\ h = a^{-1}d,$$
$$i = bd,\ j = bd^{-1},\ k = b^{-1}d^{-1},\ l = b^{-1}c.$$

The new generators are those words of length 2 which appear in the relator or its inverse written cyclically. Graphically, they are two sides of an octagon.

The simplicial complexes, S_1 and S_1^S, corresponding to F_1 and F_1^S respectively, are shown in Figure 3.7. The vertices are the same, but the higher dimensional simplices differ. For example, there is an edge (e, k^{-1}) in S_1^S which does not appear in S_1. The reason for this is that there is a *geodesic* exten-

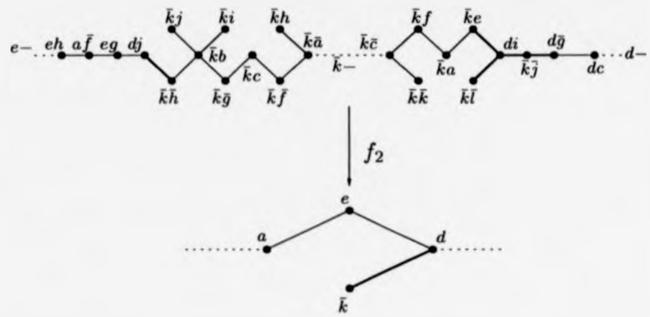


Figure 3.8. Straightening out a stranded edge. Bars are used to indicate inverses.

sion of both e and k^{-1} to the same boundary point, because $eg = k^{-1}i^{-1}$. Of these, eg is the ShortLex geodesic. The pair $\{e, k^{-1}\}$ does not have a preimage in F_2^S.

The complex S_2 has a similar appearance to S_1. The 'stranded' edges (such as (d, k^{-1})) get 'straightened out' in the preimage (although new stranded edges appear). Figure 3.8 shows a good example of this.

3.3 Applications

Definition 3.3.1 (Dimension): Let T be a topological space. The *dimension* of T, $\dim T \leqslant k$, if, given any finite open covering, there is a refinement whose nerve has dimension k.

Given a hyperbolic group, G, and a finite generating set, we use the Hausdorffification H of $\varprojlim F$, described above, to produce a computable upper bound for the dimension of ∂G.

Lemma 3.3.2: *Given a hyperbolic group, G, and a finite generating set, there is a universal bound on the dimension of the simplicial complexes S_n.*

Proof: A k-simplex $\sigma \subset S_n$ corresponds to a k-frond in F_n. Thus the dimension of S_n is bounded above by the size of the fronds. If x and y are elements of a k-frond then they must be within δ of each other. So, given a k-frond ω and an element $x \in \omega$, we know that $\omega \subset B(x, \delta)$ and thus $|\omega| \leqslant |B(x, \delta)|$ which is a constant independent of x. The size of a frond is (excessively) bounded above by the volume of the ball of radius δ. ❑

Lemma 3.3.3: *Let $k \in \mathbb{N}$. Then we can construct an automaton which accepts all k-fronds.*

Proof: We inductively construct the k-frond acceptor, which we call $\mathcal{A}_k$, from the $(k-1)$-frond acceptor and the word difference machine.

The 0-frond acceptor, $\mathcal{A}_0$, is the pruned ShortLex geodesic acceptor. The 1-frond acceptor, $\mathcal{A}_1$, is the pruned word difference machine.

The states of $\mathcal{A}_k$ have the form (s_{k-1}, w) where s_{k-1} is a state in $\mathcal{A}_{k-1}$ and w is a state in $\mathcal{A}_1$. The initial state i_k is (i_{k-1}, i_1), where i_j is the initial state of $\mathcal{A}_j$. The alphabet is the set of $(k+1)$-tuples of generators; $a = (a_0, a_1, \dots a_k)$, where each a_i is a generator. The k-tuple $(a_0, a_1, \dots, a_{k-1})$ is fed into the $\mathcal{A}_{k-1}$ part of the automaton, while the pair (a_0, a_k) is fed into the $\mathcal{A}_1$ part.

The automaton $\mathcal{A}_k$ is made by taking all edges from each state and then pruning.

The $(k+1)$-tuple $(w_0, w_1, \ldots, w_k)$ is accepted if $(w_0, w_1, \ldots, w_{k-1})$ is accepted by $\mathcal{A}_{k-1}$, (w_0, w_k) is accepted by the word difference machine and for each $i \neq j$, $w_i \neq w_j$. That is, if $\{w_0, w_1, \ldots, w_k\}$ is a set of $k+1$ elements which can be extended to a common boundary point. There has to be an extension common to both $(w_0, w_1, \ldots, w_{k-1})$ and (w_0, w_k) because $\mathcal{A}_k$ is pruned. The last condition is just a matter of removing some of the states from the list of accept states. ❑

Remark: The program of constructing a k-frond acceptor has been implemented (in a more direct way) in [BEH].

Lemma 3.3.4: *We can compute the size of the largest frond.*

Proof: Construct the k-frond acceptors, $\mathcal{A}_k$, as in Lemma 3.3.3. Since there is a bound on the size of fronds (Lemma 3.3.2), we will eventually obtain an $\mathcal{A}_n$ which accepts the empty language (it has no accept states). Let m be such that $\mathcal{A}_{m+2}$ is the first $\mathcal{A}_n$ which accepts the empty language. Then, for each $n \geqslant m+2$, $\mathcal{A}_n$ also accepts the empty language and thus the largest fronds will have $m+1$ elements. That is, the largest fronds will be m-fronds. ❑

Definition 3.3.5 (m(G)): Let $m(G)$ denote the largest m for which there is an m-frond. Lemma 3.3.4 shows that $m(G)$ can be explicitly computed. Note that $m(G)$ depends on the choice of generating set as well as the group.

We have that the simplicial complex S_n has dimension $\leqslant m(G)$. The bound is attained for large enough n because there are $m(G)$-fronds and all fronds survive under f_n^{-1}.

Proposition 3.3.6: *The dimension of the boundary of a hyperbolic group is bounded above by $m(G)$, which we can compute using the automatic structure.*

Proof: Let $\mathcal{T}$ be an open covering of ∂G. Then if we take n large enough, the closed covering $\mathcal{C} = \{S(x) \mid x \in W_n\}$ is a refinement of $\mathcal{T}$. Now, the nerve of $\mathcal{C}$ is precisely the simplicial complex S_n. The closed covering can be 'thickened' to make an open covering, with the same nerve as $\mathcal{C}$, such that it remains a refinement of $\mathcal{T}$. Thus $\dim \partial G \leqslant m(G)$. ❑

Unfortunately, the bound is not exact.

Example 3.3.7: Let G be a hyperbolic group with generating set A and consider $G \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by z. Take as generating set $\{z\} \cup A$ with lexicographical ordering such that, for each $a \in A$, $z < a$. Then G and $G \times \mathbb{Z}_2$ have the same boundary, but $m(G \times \mathbb{Z}_2) = 2m(G)$ because, given an $m(G)$-frond $\{x_0, \ldots, x_{m(G)}\}$ in G, there is an $m(G)$-frond $\{y_0, \ldots, y_{m(G)}\}$ in $f^{-1}(\{x_0, \ldots, x_{m(G)}\})$ and therefore there is a $2m(G)$-frond $\{(y_i, e), (x_i, z) \mid 0 \leqslant i \leqslant m(G)\}$ in $G \times \mathbb{Z}_2$.

Example 3.3.8: Let G be a hyperbolic group with generating set A. Let H be a finite group. Consider $G \times H$, with generating set $H \cup A$ which has lexicographical order such that, for each $h \in H$, $a \in A$, $h < a$. Then $m(G \times H) = |H| \cdot m(G)$.

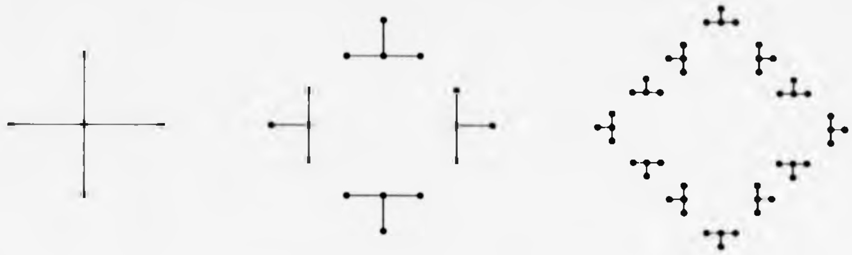


Figure 3.9. S_1, S_2 and S_3 of $F(a,b) \times \mathbb{Z}_2$

Example 3.3.9 (A case of Example 3.3.7): Consider $F(a,b) \times \mathbb{Z}_2 = \langle z, a, b \mid z^2, [a,z], [b,z] \rangle$. The complexes S_1, S_2 and S_3 are shown in Figure 3.9.

Example 3.3.10: Let G be a Fuchsian group and let a be a generator of infinite order. If we add the generators a^2 and a^3 with the ordering $a < a^2 < a^3$ then $\{a, a^2, a^3\}$ is a 2-frond and thus $\dim S_1 \geqslant 2$.

Example 3.3.11: Let G be a Fuchsian group with geometric generating set. Then there cannot be more than two geodesics to the same boundary point and so $\dim S_n \leqslant 1$. In this case, $m(G) = \dim \partial G$.

Note that $\dim \partial G$ is not uniform; there can be points at which the local dimension is less than the global dimension, even though the orbit of every boundary point is dense in ∂G.

Example 3.3.12: Let $G = \Pi_1(T_2) * \mathbb{Z}$ where T_2 is the torus of genus 2. So $G = \langle a, b, c, d, z \mid aca^{-1}c^{-1}bdb^{-1}d^{-1} \rangle$. Then there are 3 types of boundary

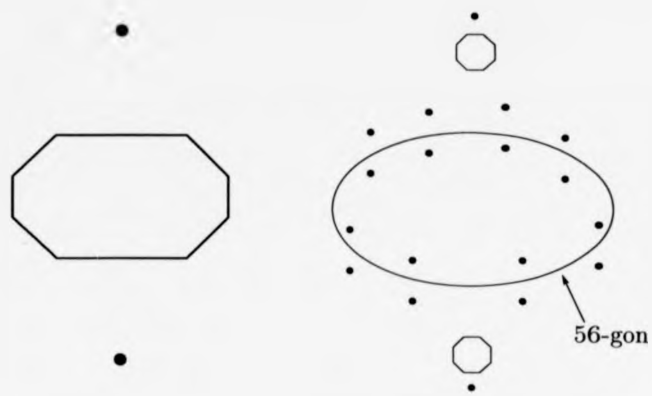


Figure 3.10. S_1 and S_2 of $\Pi_1(T_2) * \mathbb{Z}$

point:

1. A geodesic which eventually lies in $\mathbb{Z}$. In this case, the boundary point will have dimension 0.
2. A geodesic which eventually lies in $\Pi_1(T_2)$. In this case, the boundary point will have dimension 1.
3. A geodesic which switches infinitely often between $\mathbb{Z}$ and $\Pi_1(T_2)$. Here we have dimension 0 again.

The complexes S_1 and S_2 are shown in Figure 3.10. The inverse image of an isolated point is an octagon and a point, the inverse image of an octagon is a 56-gon and 16 points. Points correspond to elements currently lying in $\mathbb{Z}$, polygons correspond to elements currently lying in $\Pi_1(T_2)$. The boundary of Γ is a cantor set and a 'cantor set of circles'.

Proposition 3.3.13: *There is a partial algorithm which determines whether the boundary of a hyperbolic group has infinitely many connected components.*

Proof: The number of components in successive approximations is non-decreasing. If we ever get more than 2 components, we know that ∂G has more than 2 components, therefore infinitely many components. We can compute the number of components at each stage. ❑

Note that the converse can be detected in special cases. For example; if, for each n, the preimage under f_{n+1} of each vertex of S_n is connected, then the group has 1 end. This condition is detectable because the preimage of a vertex depends only on the state of the ShortLex geodesic acceptor to which it corresponds. This condition holds in Example 3.1.5 but doesn't hold in Example 3.2.3.

Chapter 4

Symbolic Dynamics in Hyperbolic Groups

Chapter Summary

We present here symbolic codings of the boundary of a hyperbolic group and the geodesic flow on a hyperbolic group, using its automatic structure. Firstly, we give a finite presentation of the boundary as a dynamical system, with 'time' being a hyperbolic group. Secondly, we give a uniformly finite-to-one presentation of the boundary as a semi-Markovian space. Thirdly, we give a uniformly finite-to-one presentation of the G-quotient of the geodesic flow as a subshift of finite type. Most of the results have been proved by Coornaert and Papadopoulos in [CP93], [CP98] and [CP99] using different coding systems. The main advantage of our method is that the symbol sets are much smaller. We are also able to give a semi-Markovian presentation of the boundary which works in the case of groups with torsion.

4.1 Symbolic Dynamics

The definitions that we give here are those given in [CP93] and are more general than in most texts. Traditionally, symbolic dynamics is the study of sequence spaces, and dynamical systems is the study of powers of a homeomorphism on a topological space. We generalise the definitions so that the sequences are indexed by an arbitrary (infinite) group (instead of $\mathbb{Z}$) and we have a family of homeomorphisms indexed by the same group. We think of this as a G-action by homeomorphisms. The definitions can also be applied to semigroups. Our main interest lies in the cases when the indexing set is a hyperbolic group or is $\mathbb{Z}$ or $\mathbb{N}$.

4.1.1 Subshifts of Finite Type Over a Group

Definition 4.1.1 (Bernoulli shift, shift map): Let S be a finite set of symbols and G be a group. The set of maps $G \to S$, denoted $\Sigma(G, S)$ or Σ, is called the *Bernoulli shift*.

Let $\sigma \in \Sigma$, $g, h \in G$. There is a natural action of G on Σ given by $(g*\sigma)(h) = \sigma(hg)$ (in particular, $g*\sigma(e) = \sigma(g)$ and $g*\sigma(g^{-1}) = \sigma(e)$). This is an action because $(g_1 * (g_2 * \sigma))(h) = (g_2 * \sigma)(hg_1) = \sigma(hg_1g_2) = ((g_1g_2) * \sigma)(h)$. This action is called the *shift action* (see also Definition 4.1.9). When $G = \mathbb{Z}$ or $\mathbb{N}$, and $g = 1$, this is called the right shift. A *subshift* is a subset of Σ which is invariant under this action.

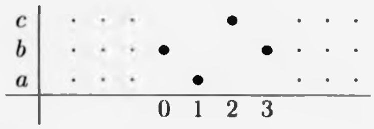


Figure 4.1. A cylinder in $\Sigma(\mathbb{Z}, \{a,b,c\})$; $F = \{0,1,2,3\}$ and A is the set of maps such that $(0,1,2,3) \mapsto (b,a,c,b)$.

Definition 4.1.2 (Cylinder, subshift of finite type): Let F be a finite subset of G and let A be a set of maps $F \to S$. Let $C = \{\sigma \in \Sigma \mid \sigma_{|_F} \in A\}$. A map σ is in C if, when restricted to our finite set F, the map is one of the specified possibilities A; elsewhere it can be anything (see Figure 4.1). Such a set C is called a *cylinder*. We say that $\Psi \subset \Sigma$ is a *subshift of finite type* if there is a cylinder, C, such that $\Psi = \bigcap_{g\in G} g * C = \bigcap_{g \in G} g^{-1} * C$. (Note that, if G is a semigroup, then Ψ is a subshift of finite type if $\Psi = \bigcap_{g\in G} g^{-1} * C$, where g^{-1} is the preimage of the shift map.) If G is finitely generated then, without loss of generality, we can take F to be a ball of finite radius n centred at the identity. Then Ψ is a subshift of finite type if it is one of the specified possibilities, A, on every ball of radius n. Now, Ψ is shift invariant, and, since cylinders are closed, $g^{-1} * C$ is also closed, and therefore so is Ψ.

4.1.2 Dynamical Systems

A *dynamical system* (Ω, G) is a topological space Ω with a G-action by homeomorphisms. In the case when $G = \mathbb{Z}$, the $\mathbb{Z}$-action is generated by a single homeomorphism f. In this case, we usually write the dynamical system

$(\Omega, \mathbb{Z})$ as (Ω, f).

We are interested in the case when there is a shift space $\Psi \subset \Sigma(G, S)$ and a continuous, surjective, G-equivariant map $h \colon \Psi \to \Omega$ such that, for every $g \in G$, the following diagram commutes:

$$\begin{array}{ccc} \Psi & \xrightarrow{g} & \Psi \\ {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow \\ \Omega & \xrightarrow{g} & \Omega, \end{array}$$

that is, for each $g \in G$, $\psi \in \Psi$, $h(g * \psi) = g \cdot h(\psi)$. Such a map, h, is called a *dynamical homomorphism*. We say that the map h *intertwines* the G actions. If the map h is a homeomorphism then it is called a *topological conjugacy*. Every subshift is itself a dynamical system under the shift action, with h the identity map.

4.1.3 Examples and non-examples of subshifts of finite type over $\mathbb{Z}$

If $G = \mathbb{Z}$, then $\Sigma(\mathbb{Z}, S)$ is the set of biinfinite sequences on the symbols S. The action of $\mathbb{Z}$ on $\Sigma(\mathbb{Z}, S)$ is generated by $(1 + \sigma)(n) = \sigma(1 + n)$; the (right) shift. A subset $\Psi \subset \Sigma(\mathbb{Z}, S)$ is a subshift of finite type if there is an n such that $\psi \in \Psi$ if and only if on every set of n consecutive symbols, it is one of a specified sequence of n symbols in S (Definition 4.1.2, see also Figure 4.1). Let A be the set of allowable sequences of n symbols and let $C = \left\{\sigma \mid \sigma|_{\{1,\ldots,n\}} \in A\right\}$ be the cylinder of A. Then $\Psi = \bigcap_{m \in \mathbb{Z}} m + C$.

Let $S = \{a, b\}$.

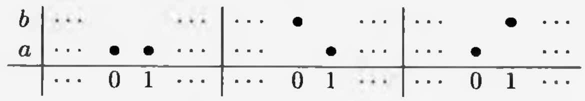


Figure 4.2. The cylinders for Example 4.1.3

Example 4.1.3: The set of sequences that never have two consecutive b's is a subshift of finite type. We can take $F = \{0, 1\}$ and A to be the set of maps $(0, 1) \mapsto (a, a)$, (a, b), (b, a) (every pair of consecutive integers doesn't map to the pair (b, b), see Figure 4.2).

Example 4.1.4: Let σ_n be the sequence $\cdots a^n b^n a^n b^n \cdots$ (that is, n consecutive a's followed by n consecutive b's repeated indefinitely). Let Ψ_n be the set of all shifts of σ_n (there are $2n$ of these). Let

$$\Psi = \bigcup_{n \in \mathbb{N}} \Psi_n.$$

Then Ψ is not a subshift of finite type.

Example 4.1.5: The set of sequences Ψ over S with at most one occurrence of b (sequences of the form $\cdots aaabaaa \cdots$ or $\cdots aaaa \cdots$) is not a subshift of finite type, because a b is allowed if and only if no b has appeared earlier in the sequence, which we cannot tell by looking at a finite set of symbols. However, we can keep track of whether or not we have had a b by adding a new symbol $\bar{a}$. Now consider the set of sequences Ψ' of the form $\cdots aaaa \cdots$, $\cdots \bar{a}\bar{a}\bar{a}\bar{a} \cdots$, $\cdots aaab\bar{a}\bar{a}\bar{a} \cdots$. As in Example 4.1.3, take $F = \{0, 1\}$. Let A be the set of maps $(0, 1) \mapsto (a, a)$, (a, b), $(b, \bar{a})$, $(\bar{a}, \bar{a})$. Then Ψ' is a subshift

Figure 4.3. Neighbouring states in a finite state automaton

of finite type. The map $h\colon \Psi' \to \Psi$, $\psi' \mapsto \psi$, given by

$$\psi(n) = \begin{cases} a & \text{if } \psi'(n) = a \text{ or } \psi'(n) = \bar{a}, \\ b & \text{if } \psi'(n) = b, \end{cases}$$

is surjective and shift invariant. Therefore there is a dynamical homomorphism from the subshift of finite type Ψ' to Ψ.

Example 4.1.6 (State sequences): Let $\mathcal{A}$ be a finite state automaton (see Section 1.2.1). Let S be the set of states of $\mathcal{A}$ and let $\Psi \subset \Sigma(\mathbb{Z}, S)$ be the set of biinfinite sequences of states that can be traced in $\mathcal{A}$. Then Ψ is a subshift of finite type. Again, we let $F = \{0, 1\}$. We can take A to be the set of maps $(0,1) \mapsto (s_0, s_1)$ such that s_0 and s_1 are adjacent states; that is, there is an edge in $\mathcal{A}$ from the state s_0 to the state s_1 (see Figure 4.3). Similarly, the set of infinite sequences of states of $\mathcal{A}$ is a subshift of finite type over $\mathbb{N}$.

Example 4.1.7 (Edge sequences): We can alter the automaton $\mathcal{A}$ to $\mathcal{A}'$ without changing the set of infinite sequences of edge labels that can be traced, so that a sequence of states in $\mathcal{A}'$ uniquely determines a sequence of edges in $\mathcal{A}$. The map from state sequences in $\mathcal{A}'$ to edge sequences in $\mathcal{A}$ is a dynamical homomorphism. Example 4.1.5 with the automaton in Figure 4.4 is an example of such a construction. A general method is as follows: The set of states is the set of pairs (s, e), where s is a state in the original automaton

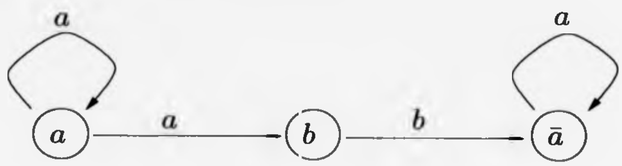


Figure 4.4. Automaton recognising the sequences in Example 4.1.5

and e is an edge label. For each edge f from s to t in the original automaton and for each label e, we have an edge labelled f from (s, e) to (t, f).

4.1.4 Finitely Presented Dynamical Systems

Given a subshift $\Psi \subset \Sigma(G, S)$, the Cartesian product $\Psi \times \Psi$ can be thought of as a subshift of $\Sigma(G, S \times S)$, where $(\psi_1, \psi_2) \colon g \mapsto (\psi_1(g), \psi_2(g))$. Suppose that (Ω, G) is a dynamical system and that there is a dynamical homomorphism h from the subshift Ψ to Ω. Consider the subshift $E \subset \Sigma(G, S \times S)$, given by $E = \{(\psi_1, \psi_2) \mid h(\psi_1) = h(\psi_2)\}$. If both Ψ and E are subshifts of finite type then we say that (Ω, G) is a *finitely presented dynamical system.*

An account of the case when $G = \mathbb{Z}$ is given in [Fri87]. In particular, finitely presented dynamical systems over $\mathbb{Z}$ have rational zeta-functions. This can be used to prove the rationality of a zeta-function related to hyperbolic groups [Aut].

Question 4.1.8: Can one sensibly generalise the notions of periodic point and zeta-function to dynamical systems over a hyperbolic group? Would such a zeta-function of a finitely presented dynamical system be rational?

Remark: Any such construction would have to depend on the choice of generating set. The situation when $\mathbb{Z}$ is generated by $\{2, 3\}$ is already much more complicated than $\mathbb{Z}$ generated by $\{1\}$.

4.1.5 Semi-Markovian Spaces

A subshift, Ψ, of the Bernoulli shift $\Sigma(\mathbb{N}, S)$ is called *semi-Markovian* if it can be written as $\Psi = \Phi \cap C$, where Φ is a subshift of finite type over $\mathbb{N}$ and C is a cylinder. A topological space Ω is called *semi-Markovian* if there is a homeomorphism f and a continuous surjective map π from a semi-Markovian subshift Ψ to Ω, such that the equivalence relation $R = \{(\psi_1, \psi_2) \mid \pi(\psi_1) = \pi(\psi_2)\} \subset \Psi \times \Psi$ is also a semi-Markovian subshift.

Any finitely presented dynamical system over $\mathbb{N}$ is semi-Markovian.

4.1.6 An Alternative G-Action on $\Sigma(G, S)$

Definition 4.1.9 (Inverse-shift, inverse-shift of finite type): There is a natural left action of G on Σ given by $(g \diamond \sigma)(h) = \sigma(g^{-1}h)$ (in particular, $g \diamond \sigma(e) = \sigma(g^{-1})$ and $g \diamond \sigma(g) = \sigma(e)$). This is an action because $(g_1 \diamond (g_2 \diamond \sigma))(h) = (g_2 \diamond \sigma)(g_1^{-1}h) = \sigma(g_2^{-1}g_1^{-1}h) = ((g_1 g_2) \diamond \sigma)(h)$. We call this action the *inverse shift* action.

We can generalise Definition 4.1.2 to the inverse shift action. With the same definition of cylinder, we say that the subshift Φ of Σ is an *inverse-shift of finite type*, if there is a cylinder, C, such that $\Phi = \bigcap_{g \in G} g \diamond C$.

The advantage that the inverse shift action has over the shift action is that left multiplication by inverses is an isometry of the Cayley graph, whereas right multiplication is not, in general, an isometry. The inverse shift action is more intuitive than the shift action.

Proposition 4.1.10: *Every inverse-shift of finite type is topologically conjugate to a subshift of finite type.*

Proof: Let $\Phi \subset \Sigma(G,S)$ be an inverse-shift of finite type. Then, there is a finite set $F \subset G$ and a set of maps $A \subset \Sigma(F,S)$ such that

$$\Phi = \bigcap_{g \in G} g \diamond C, \quad \text{where} \quad C = \left\{\sigma \mid \sigma_{|_F} \in A\right\}.$$

Without loss of generality, we can take F to be a ball about the identity so that F is closed under taking inverses.

Define the map $f \colon \Phi \to \Sigma(G,S)$ by $(f\phi)(g) = \phi(g^{-1})$. Let $\Phi' = \mathrm{Im} f$. Then $f \colon \Phi \to \Phi'$ is a homeomorphism. Further, for each $\phi \in \Phi$,

$$(f(g \diamond \phi))(h) = (g \diamond \phi)(h^{-1}) = \phi(g^{-1}h^{-1}) = (f\phi)(hg) = g * (f\phi)(h),$$

so f intertwines the shift and inverse-shift actions.

Let $\bar{f}$ be the restriction of f to A, so, since F is closed under taking inverses, $\bar{f} : A \to \Sigma(F,S)$. Let $A' = \bar{f}A = \left\{\bar{f}\sigma \colon F \to S \mid \sigma \in A\right\}$ and let $C' =$

$\{\sigma \mid \sigma_{|_F} \in A'\}$. Now,

$$\phi \in \Phi' \iff f^{-1}\phi \in \Phi \iff (f^{-1}\phi) \in \bigcap_{g \in g} g \diamond C,$$

which is true if and only if, for every $g \in G$,

$$g \diamond (f^{-1}\phi)_{|_F} \in A \iff \bar{f}(g \diamond (f^{-1}\phi))_{|_F} \in \bar{f}A = A' \iff (g * \phi)_{|_F} \in A', \tag{4.1}$$

because f is bijective and is intertwining. Since (4.1) holds for every $g \in G$,

$$\phi \in \Phi' \iff \phi \in \bigcap_{g \in G} g * C'.$$

Thus $\Phi' = \bigcap_{g \in G} g * C'$ (that is, Φ' is a subshift of finite type) and $f : \Phi \to \Phi'$ is a topological conjugacy. ❑

Definition 4.1.11 (Finitely presented inverse-system): Let $\Phi \subset \Sigma(G, S)$ be an inverse-shift of finite type, let Ω be a topological space with a G-action by homeomorphisms and let $a \colon \Phi \to \Omega$ be a continuous, surjective and G-equivariant map (that is, $g \cdot a(\phi) = a(g \diamond \phi)$, compare with Sections 4.1.2 and 4.1.4). Then Ω is called an *inverse-system of finite type*. If the equivalence relation

$$E = \{(\phi_1, \phi_2) \in \Phi \times \Phi \mid a(\phi_1) = a(\phi_2)\} \subset \Sigma(G, S \times S)$$

is also an inverse-shift of finite type, then Ω is called a *finitely presented inverse-system.*

Proposition 4.1.12: *Every finitely presented inverse-system is a finitely presented dynamical system.*

Proof: Let Ω be a finitely presented inverse system. Then there is an inverse-shift of finite type Φ and a continuous, surjective, G-equivariant map $a \colon \Phi \to \Omega$ such that the equivalence relation

$$E = \{(\phi_1, \phi_2) \in \Phi \times \Phi \mid a(\phi_1) = a(\phi_2)\} \subset \Sigma(G, S \times S)$$

is also an inverse-shift of finite type.

Proposition 4.1.10 implies that there is a subshift of finite type $\Psi \subset \Sigma(G, S)$ and a topological conjugacy $f : \Phi \to \Psi$. The map $b = a \circ f^{-1} : \Psi \to \Omega$ is continuous, surjective and G-equivariant. Let

$$E' = \{(\psi_1, \psi_2) \in \Psi \times \Psi \mid b(\psi_1) = b(\psi_2)\} \subset \Sigma(G, A \times A).$$

We show that E' is a subshift of finite type.

The map $f^\times : E \to \Sigma(G, S \times S)$, given by $f^\times(\phi_1, \phi_2) = (f\phi_1, f\phi_2)$, is a topological conjugacy. It remains to show that $E' = \mathrm{Im}(f^\times)$.

Suppose that $f^\times(\phi_1, \phi_2) = (f\phi_1, f\phi_2) \in \mathrm{Im}(f^\times)$. Then

$$b(f(\phi_1)) = af^{-1}f(\phi_1) = a(\phi_1) = a(\phi_2) = b(f(\phi_2)).$$

Therefore $(f\phi_1, f\phi_2) \in E'$, so $\mathrm{Im}(f^\times) \subset E'$.

Conversely; if $(\psi_1, \psi_2) \in E'$, then

$$\begin{array}{cc} & b(\psi_1) = b(\psi_2) \\ \Longrightarrow & a(f^{-1}(\psi_1)) = a(f^{-1}(\psi_2)) \\ \Longrightarrow & (f^{-1}(\psi_1), f^{-1}(\psi_2)) \in E \\ \Longrightarrow & f^\times(f^{-1}(\psi_1), f^{-1}(\psi_2)) = (\psi_1, \psi_2) \in \mathrm{Im}(f^\times). \end{array}$$

Therefore $E' \subset \mathrm{Im}(f^\times)$ and thus $E' = \mathrm{Im}(f^\times)$. ❑

Proposition 4.1.13: *Every subshift of finite type is topologically conjugate to an inverse-shift of finite type. Every finitely presented dynamical system is a finitely presented inverse system.*

Remark: A proof along similar lines to the one above works.

Propositions 4.1.10 and 4.1.12 show that inverse-shifts and subshifts are essentially the same, the only difference is that you need to be in a group to have an inverse-shift. In the following sections, we do not distinguish between inverse-shifts and subshifts, referring to them both as shifts. In Section 4.2, we use the inverse-shift.

4.2 The Boundary of a Hyperbolic Group as a Dynamical System

In this section we describe the action of an infinite hyperbolic group, G, on its boundary as a finitely presented dynamical system over G. We use the (inverse) shift map of Definition 4.1.9. A useful example to bear in mind throughout this section is the free group on 2 generators (see Section 4.2.4 on page 103); although, since there are no relators, it is by no means a typical example.

4.2.1 Geodesics

We use $[g, h]$ to denote a finite geodesic in Γ between the vertices g and h; $[g, \xi)$ to denote an infinite geodesic in Γ from g to the boundary point ξ; $[g, h]_{\mathrm{SL}}$ to denote a ShortLex geodesic in Γ and $[g, h] \cup [h, k]$ to denote the concatenation of two geodesics. Our notation will not distinguish geodesics from their images.

The motivation for the definitions in Section 4.2.2 and the key to proving that the subshift Φ (which we define in Definition 4.2.8) is non-empty is the following construction:

Theorem 4.2.1: *Let G be an infinite hyperbolic group and let ξ be a point on the boundary. Then there is a family of geodesic rays, $\{\alpha_g \mid g \in G\}$, where α_g starts at g, such that the following conditions hold:*

1. *Each geodesic ray, α_g, tends to ξ.*
2. *If $h \in \alpha_g$, then $\alpha_h \subset \alpha_g$ (that is, $\{\alpha_g\}$ is closed under removing prefixes).*

(See Figure 4.5.)

Proof: Let $\xi \in \partial G$ and fix a geodesic ray r from $r(0)$ to ξ. Let $[r(t), \xi)$ be the geodesic from $r(t)$ to ξ along r.

For every $g \in G$ consider the function $b_{r,g}(t) = d(g, r(t)) - t$, defined on $\mathbb{N}$ so that $r(t)$ is a vertex of r (compare with the Busemann function $h_r : \Gamma \to \mathbb{R}$, $\gamma \mapsto \lim_{t\to\infty}(|\gamma - r(t)| - t)$). The function $b_{r,g}$ is a non-increasing function

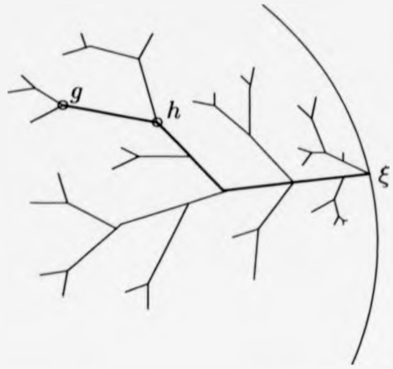
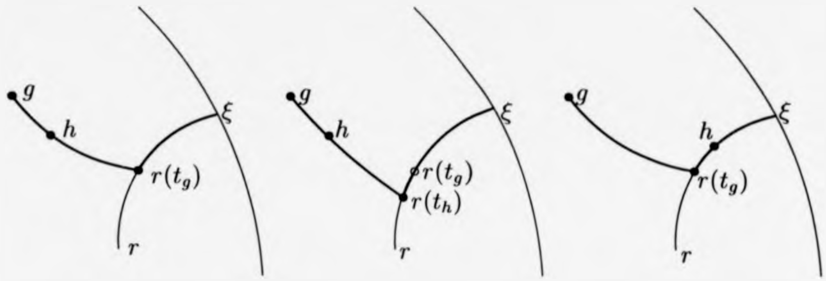


Figure 4.5. A family of geodesics which is closed under removing prefixes. All geodesics point towards ξ.

from $\mathbb{N}$ to $\mathbb{Z}$ which is bounded below by $-d(g, r(0))$, so it must eventually be constant. Let t_g be the first place at which $b_{r,g}$ attains its bound. So; if $t \geqslant t_g$, then $b_{r,g}(t) = b_{r,g}(t_g)$ and if $t < t_g$, then $b_{r,g}(t) > b_{r,g}(t_g)$. Note that, if $t \geqslant t_g$, then $[g, r(t_g)] \cup [r(t_g), r(t)]$ is a geodesic, because $d(g, r(t)) = d(g, r(t_g)) + t - t_g$; hence so is $[g, r(t_g)] \cup [r(t_g), \xi)$. Also, t_g is the smallest s such that $[g, r(s)] \cup [r(s), r(t)]$ is a geodesic for all $t > s$.

Now choose $\alpha_g = [g, r(t_g)]_{\text{SL}} \cup [r(t_g), \xi)$. We shall show that $\{\alpha_g \mid g \in G\}$ is closed under removing prefixes. That is, if $h \in \alpha_g$, then $\alpha_h \subset \alpha_g$.

Suppose $h \in [g, r(t_g)]_{\text{SL}}$ (see Figure 4.6(a)). Then $[h, r(t_g)]_{\text{SL}} \cup [r(t_g), \xi)$ is a geodesic so $t_h \leqslant t_g$. But $[g, r(t_h)] \cup [r(t_h), r(t_g)]$ (see Figure 4.6(b)) is also a



(a) The case when h is in $[g, r(t_g)]_{\mathrm{SL}}$ (b) The path from g to $r(t_g)$ via $r(t_h)$ (c) The case when h is in $[r(t_g), \xi)$

Figure 4.6. If h is in α_g, then α_h is a suffix of α_g

geodesic because its length is

$$\begin{aligned} d(g, r(t_h)) + t_g - t_h &\leqslant d(g,h) + d(h, r(t_h)) + t_g - t_h \\ &= d(g,h) + d(h, r(t_g)) \\ &= d(g, r(t_g)). \end{aligned}$$

So $t_g \leqslant t_h$ and thus $t_h = t_g$. Therefore $[g, r(t_g)]_{\mathrm{SL}} = [g,h]_{\mathrm{SL}} \cup [h, r(t_h)]_{\mathrm{SL}}$ Hence $\alpha_h \subset \alpha_g$.

Now suppose that $h \in [r(t_g), \xi)$ (see Figure 4.6(c)). Then $r(t_h) = h$ and hence $\alpha_h = [h, \xi) \subset [r(t_g), \xi) \subset \alpha_g$. So $\{\alpha_g \mid g \in G\}$ is a family of geodesics which is closed under removing prefixes. ❑

4.2.2 The Subshift Φ

Recall (see Section 1.2.3) that hyperbolic groups are strongly geodesically automatic. This means that there are the following automata;

the geodesic acceptor, which accepts all words in the generating set which are geodesic paths in the Cayley graph;

the ShortLex geodesic acceptor, which accepts all lexicographically minimal geodesics;

the geodesic pairs machine, which accepts all pairs of geodesics which start at the same point and stay within a uniformly bounded distance of each other;

the ShortLex geodesic pairs acceptor, which accepts all pairs of ShortLex geodesics which start at the same point and stay within a uniformly bounded distance of each other.

All of these automata can be made to be partially deterministic and with all states accept states. The first two are closed under removing prefixes or suffixes.

Let G be a hyperbolic group with generating set X, and let Γ be the corresponding Cayley graph. Let S be the set of states in the geodesic acceptor. We shall confuse elements of G with vertices of Γ and elements of X with labelled edges (of Γ and the geodesic acceptor). When there is an outgoing edge labelled x from the state s, we use s^x to denote its destination.

The states of the geodesic pairs machine can be characterised as a triple (s, t, w), where s and t are states in the geodesic word acceptor and w is a (short) word in the generators. An edge is a pair $(x, y) \in X \times X$. There is an edge from the state (s, t, w) to the state $(s^x, t^y, x^{-1}wy)$ whenever there are suitable edges in the geodesic acceptor and $x^{-1}wy$ is also a 'short' word.

The set of symbols which we use is $\mathcal{S} = \mathbb{P}(S) \setminus \{\varnothing\}$; the set of non-empty subsets of the set of states in the geodesic acceptor.

We shall construct a subshift of finite type of the Bernoulli shift $\Sigma(G, \mathcal{S})$ which maps surjectively and equivariantly onto ∂G. We show further that the equivalence relation given by mapping to the same boundary point is also a subshift of finite type.

Let

$$\phi \colon G \to \mathcal{S}, \ g \mapsto \phi(g).$$

Our goal is to put satisfiable and sufficient conditions on ϕ to define a surjective, equivariant map to ∂G. We will define a map by tracing a geodesic from any point $g \in G$ using rules determined by ϕ. The conditions will force any pair of geodesics defined in this way to go to the same boundary point. The conditions are local conditions so that the resulting construction is a subshift of finite type.

Definition 4.2.2 ($\mathbf{R}_\phi$): The relation R_ϕ on G is defined by $gR_\phi h$ if $h = gx$ for some $x \in X$ and, for each $s \in \phi(g)$, there is an edge labelled x in the geodesic acceptor from s to a state in $\phi(h)$ (that is, $s^x \in \phi(h)$).

$$\begin{array}{rrcl} \text{In } \Gamma; & g & \xrightarrow{x} & h = gx. \\ \text{In the geodesic acceptor;} & s \in \phi(g) & \xrightarrow{x} & s^x \in \phi(h). \end{array}$$

Definition 4.2.3 (ϕ-gradient): Let $\phi : G \to S$ and let $g_0 \in G$. A ϕ-gradient from the vertex g_0 in Γ is an edge path $x_1 x_2 x_3 \cdots$ such that for each i, $g_i R_\phi g_i x_{i+1} = g_{i+1}$.

We define a subshift of finite type of $\Sigma(G, S)$ such that there are always ϕ-gradients and, for a fixed ϕ, all ϕ-gradients converge to the same boundary point. We use two explicit geodesic pairs machine which are built from:

Definition 4.2.4 ($\mathcal{WD}'$): The automaton $\mathcal{WD}'$ is built as follows: Take a ball of sufficiently large radius N. ('Sufficiently large' means large enough for all word differences of pairs of geodesics that start distance $\leqslant 1$ apart and go to the same boundary point to be included, compare with *standard automata* in [ECH+92].) The set of states of $\mathcal{WD}'$ is the set of triples of the form (s, t, w) where $s, t \in S$ and w is in the ball of radius N about the identity. The edges from the state (s, t, w) are the pairs $(x, y) \in X \times X$ such that $|x^{-1}wy| \leqslant N$ and such that there are edges in the geodesic acceptor from s and t labelled x and y respectively. The initial and accept states are not important because this machine is only used as the starting point for building $\mathcal{WD}(X)$ and $\mathcal{WD}$.

Definition 4.2.5 ($\mathcal{WD}$(X)): The automaton $\mathcal{WD}(X)$ is obtained from $\mathcal{WD}'$ as follows: Take the states of the form (s, t, x), where $s, t \in S$ and $x \in X \cup \{e\}$,

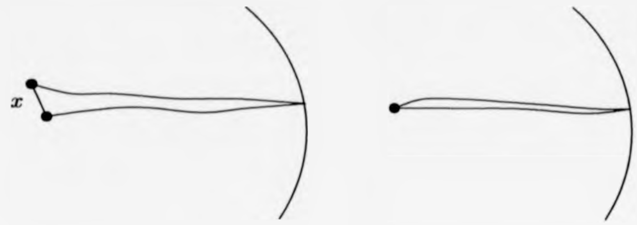


Figure 4.7. Words accepted by $\mathcal{WD}(X)$

as initial states. We take the maximal subautomaton of $\mathcal{WD}'$ that contains all states which can be reached from an initial state as part of an infinite word. All states are accept states. The automaton $\mathcal{WD}(X)$ accepts all prefixes of pairs of geodesic rays which start at a distance $\leqslant 1$ away and go to the same boundary point (see Figure 4.7). A geodesic which cannot be extended infinitely cannot be part of such a pair.

Definition 4.2.6 ($\mathcal{WD}$): Let $W \subset G$ be the set of third coordinates of the states in $\mathcal{WD}(X)$. The automaton $\mathcal{WD}$ is constructed from $\mathcal{WD}'$ as follows: Firstly, we restrict to states of the form (s, t, w) where $w \in W$. We then prune so that the remaining states are precisely those states which are part of an infinite path in the automaton. All states are initial and accept states. The automaton $\mathcal{WD}$ accepts all pairs of geodesics which differ by a word in W all the time. That is, geodesics r, s such that, for each n, $r(n)^{-1}s(n) \in W$. Every state in $\mathcal{WD}(X)$ is a state in $\mathcal{WD}$.

Remark: If r and s are geodesic rays to the same boundary point which start at distance $\leqslant 1$ away, then, for each n, $(r_{|[0,n]}, s_{|[0,n]})$ is accepted by $\mathcal{WD}(X)$, and, for each m, n, $(r_{|[m,n]}, s_{|[m,n]})$ is accepted by $\mathcal{WD}$. The language accepted by $\mathcal{WD}(X)$ is not closed under removing prefixes. However, if w is accepted by $\mathcal{WD}(X)$ and v is a suffix of w then v is accepted by $\mathcal{WD}$. The language accepted by $\mathcal{WD}$ is closed under removing prefixes.

The following definition is technical but important:

Definition 4.2.7 ($\mathbf{G}_\phi$): The subset G_ϕ of $G \times G$ is defined by $(g, h) \in G_\phi$ if and only if the following conditions hold:

1. For each $s \in \phi(g)$, $t \in \phi(h)$, the state $(s, t, g^{-1}h)$ is a state of $\mathcal{WD}$.
2. For some $s \in \phi(g)$, $t \in \phi(h)$, the state $(s, t, g^{-1}h)$ is a state in $\mathcal{WD}(X)$.

The second condition says that there are elements $g' \in G$ and $x \in X \cup \{e\}$ such that there are geodesics $[g', g]$ and $[g'x, h]$ of the same length which finish in states s and t respectively in the geodesic acceptor (see Figure 4.8) and that these geodesics can be extended infinitely to the same boundary point.

Definition 4.2.8 (Φ): Recall that W is the set of third coordinates of $\mathcal{WD}(X)$ (see Definition 4.2.5) and recall the relation R_ϕ (Definition 4.2.2). Let Φ denote the set of all maps $\phi \colon G \to \mathcal{S}$ for which:

1. For each $g \in G$, there is a $g_1 \in G$ such that $gR_\phi g_1$.
2. If $g \in G$, $x \in X$, then $(g, gx) \in G_\phi$.
3. If $(g, h) \in G_\phi$, $gR_\phi g_1$ and $hR_\phi h_1$, then $(g_1, h_1) \in G_\phi$ (Figure 4.8).

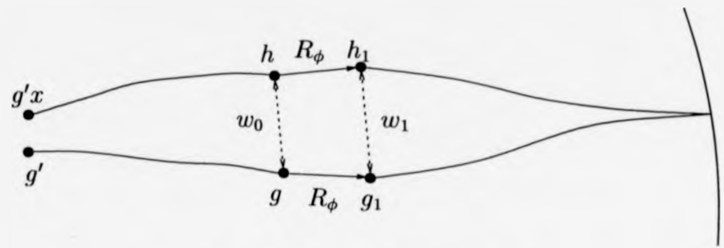


Figure 4.8. The condition for Φ

The second condition isn't trivial because some of the states (s, t, x) in $\mathcal{WD}'$ may not be part of an infinite word and would therefore have been pruned from $\mathcal{WD}(X)$.

Lemma 4.2.9: *For each $g \in G$ and for each $\phi \in \Phi$, there is a ϕ-gradient from g. Every ϕ-gradient is a geodesic.*

Proof: Fix $\phi \in \Phi$ and drop the subscript on the relation.

Since ϕ satisfies Definition 4.2.8 condition 1, we can define an infinite ϕ-gradient in Γ inductively from any point g_0 by, at each vertex g_i, choosing a neighbouring vertex $g_{i+1} = g_i x_{i+1}$, for some $x_{i+1} \in X$, such that $g_i R g_{i+1}$. This gives us the ϕ-gradient $g_0 \xrightarrow{x_1} g_1 \xrightarrow{x_2} g_2 \cdots$. The ϕ-gradient is a geodesic because, for each $s_i \in \phi(g_i)$, there is an edge in the geodesic acceptor, labelled x_{i+1}, such that $s_i^{x_{i+1}} \subset \phi(gx_{i+1})$. So, for each n, $x_1 x_2 \ldots x_n$ is accepted by the geodesic acceptor, so is a geodesic.

Any such path can be extended infinitely, so the sequence $(x_n)_{n=1}^{\infty}$ defines a geodesic ray from g_0. ❑

Lemma 4.2.10: *There is a well-defined map $a\colon \Phi \to \partial G$ given by tracing a ϕ-gradient from any vertex in Γ.*

Proof: Fix $\phi \in \Phi$. Any ϕ-gradient is a geodesic ray. Following a geodesic ray from a vertex defines a point on the boundary. To see that a is well-defined, we must check that all such geodesic rays define the same boundary point.

Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be ϕ-gradients from g_0 and h_0 respectively, where $h_0 = g_0 y_0$ for some $y_0 \in X$. Let $g_{i+1} = g_i x_{i+1}$ and let $h_{i+1} = h_i y_{i+1}$. Then, since ϕ satisfies conditions 3 and 2 of Definition 4.2.8 and by the definition of G_ϕ (Definition 4.2.7), we inductively see that for each n, $g_n^{-1} h_n \in W$. Thus the geodesics stay a bounded distance apart. So $(y_n)_{n=1}^{\infty}$ from h_0 and $(x_n)_{n=1}^{\infty}$ from g_0 define the same point on the boundary.

Now, let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be ϕ-gradients from g and h respectively. We can construct a finite sequence $g = z_0, z_1, z_2, \ldots, z_m = h$ such that $z_i^{-1} z_{i+1} \in X$. By our previous argument, any ϕ-gradients from z_i and z_{i+1} define the same point on the boundary; hence so do $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ from g and h respectively. ❑

Proposition 4.2.11: *The map $a : \Phi \to \partial G$ (defined in Lemma 4.2.10) is surjective.*

Proof: To show that a is surjective (and that Φ is non-empty), we fix a point $\xi \in \partial G$ and show that there is a $\phi \in \Phi$ such that $a(\phi) = \xi$.

By Theorem 4.2.1, there is a family of geodesic rays, $\{\alpha_g \mid g \in G\}$, all tending to ξ, which is closed under removing prefixes.

Fix $g \in G$. Then, α_g traces a path in the geodesic acceptor from the initial state. This defines a map f'_g from the vertices of the geodesic α_g to the set of states, S, of the geodesic acceptor. We can extend this to a map $f_g \colon G \to S' \cup \{\varnothing\}$, where $S' = \{\{s\} \mid s \in S\}$ is the set of singletons in $\mathbb{P}(S)$, by

$$f_g(h) = \begin{cases} \{f'_g(h)\} & \text{if } h \in \alpha_g, \\ \varnothing & \text{otherwise.} \end{cases}$$

Now we define

$$\phi \colon G \to \mathcal{S},\ h \mapsto \bigcup_{g \in G} f_g(h).$$

Note that $f_g(g)$ is the initial state in the geodesic acceptor. Therefore, for each g, $\phi(g) \neq \varnothing$. It remains to show that ϕ satisfies the conditions of Definition 4.2.8.

Claim 1: *For each $h \in G$, there is an $x \in X$ such that $hR_\phi hx$.*

Proof: Let $h \in G$. There is an $x \in X$ such that $hx = \alpha_h(1)$ (see Figure 4.9). Let $s \in \phi(h)$, then $s = f'_g(h)$ for some $g \in G$. So α_g traces a geodesic which passes through h in Γ at state s in the geodesic acceptor (see Figure 4.9);

$$\begin{array}{rccccccccccc} \text{in } \Gamma; & g & \to & \alpha_g(1) & \to & \cdots & \to & h & \to & hx & \to & \cdots \\ \text{in geodesic acceptor;} & s_0 & \to & s_1 & \to & \cdots & \to & s & \to & s^x & \to & \cdots, \end{array}$$

where s_0 is the initial state in the geodesic acceptor. Further, $s^x \in \phi(hx)$ and therefore $hR_\phi hx$. $\square$

Claim 2: *If $g \in G$ and $x \in X$ then $(g, gx) \in G_\phi$.*

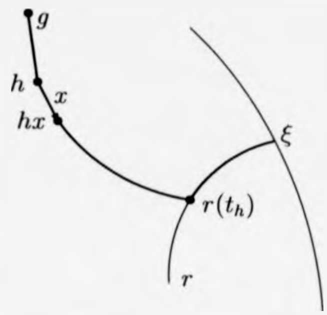


Figure 4.9. There is a ϕ- gradient from each vertex of Γ. Bold lines indicate α_g and α_h.

Proof: Given $s \in \phi(g)$, $t \in \phi(gx)$, the geodesics α_g and α_{gx} are a pair of geodesics which start at distance 1 from each other and go to the same boundary point. The geodesic α_g traces a path in the geodesic acceptor from the state s and the geodesic α_{gx} traces a path in the geodesic acceptor from t. Therefore the pair (α_g, α_{gx}) traces an infinite path in $\mathcal{WD}(X)$ from (s, t, x). Hence $(s, t, x) \in \mathcal{WD}(X)$ and thus $(g, gx) \in G_\phi$. □

Claim 3: *If $g, h \in G$, $x, y \in X$ are such that $(g, h) \in G_\phi$, $gR_\phi gx$ and $hR_\phi hy$, then $(gx, hy) \in G_\phi$.*

Proof: Since $(g, h) \in G_\phi$, there are $s \in \phi(g)$, $t \in \phi(h)$ such that $(s, t, g^{-1}h)$ is a state in $\mathcal{WD}(X)$. That is, there are $g_0 \in G$, $x_0 \in X \cup \{e\}$ and $s_0, t_0 \in S$ such that there is a path in $\mathcal{WD}(X)$ from (s_0, t_0, x_0) to $(s, t, g^{-1}h)$ in $\mathcal{WD}(X)$ which traces the geodesics $[g_0, g]$ and $[g_0x_0, h]$ in Γ (see Figure 4.10).

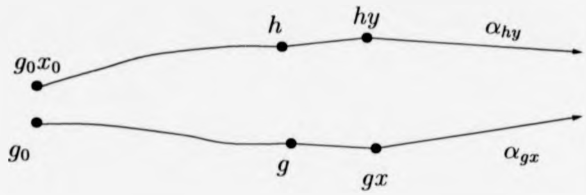


Figure 4.10. Geodesics in $\mathcal{WD}(X)$ through g and h

Since $gR_\phi gx$ and $hR_\phi hy$, $s^x \in \phi(gx)$ and $t^y \in \phi(hy)$. Now, α_{gx} and α_{hy} trace geodesics in the geodesic acceptor from s^x and t^y respectively. Therefore $[g_0, g] \cup [g, gx] \cup \alpha_{gx}$ and $[g_0x_0, h] \cup [h, hy] \cup \alpha_{hy}$ are a pair of geodesics which start distance $\leqslant 1$ apart and go to the same boundary point. Since $[g_0, g]$ and $[g_0x_0, h]$ have the same length, the state $(s^x, t^y, x^{-1}g^{-1}hy)$ is a state in $\mathcal{WD}(X)$.

Now suppose that s and t are any states in $\phi(gx)$ and $\phi(hy)$ respectively. The geodesics α_{gx} and α_{hy} trace paths in the geodesic acceptor from s and t respectively. The word differences are the same as for the geodesics $[g_0, g] \cup [g, gx] \cup \alpha_{gx}$ and $[g_0x_0, h] \cup [h, hy] \cup \alpha_{hy}$ above, and are therefore in W. Hence, there is an infinite path in $\mathcal{WD}$ from the state $(s, t, x^{-1}g^{-1}hy)$. Thus $(gx, hy) \in G_\phi$. □

The three claims above show that $\phi \in \Phi$. Hence, for each $\xi \in \partial G$, there is a $\phi \in \Phi$ such that $a(\phi) = \xi$. This concludes the proof of Proposition 4.2.11.

❑

Lemma 4.2.12: *The map $a\colon \Phi \to \partial G$ is G-equivariant.*

Proof: The shift action of G on Φ is given by $(g * \phi)(h) = \phi(g^{-1}h)$. Under this action, the map a of Lemma 4.2.10 is clearly G-equivariant. ❑

4.2.3 The Dynamical System $(\partial \mathrm{G}, \mathrm{G})$ is Finitely Presented

Let F be a finite subset of G and let $\sigma : F \to \mathcal{S}$. We make the following local definitions (compare with Definitions 4.2.2 and 4.2.7):

Definition 4.2.13 ($\mathbf{R}_\sigma$): The relation R_σ on F is defined by $gR_\sigma h$ if $h = gx$ for some $x \in X$ and, for each $s \in \sigma(g)$, $s^x \in \sigma(h)$.

Definition 4.2.14 ($\mathbf{F}_\sigma$): The subset F_σ of $F \times F$ is defined by $(g, h) \in F_\sigma$ if and only if the following conditions hold:

1. For each $s \in \sigma(g)$, $t \in \sigma(h)$, the state $(s, t, g^{-1}h)$ is a state of $\mathcal{WD}$.
2. For some $s \in \sigma(g)$, $t \in \sigma(h)$, the state $(s, t, g^{-1}h)$ is a state in $\mathcal{WD}(X)$.

Note that, if $(e, h) \in F_\sigma$, then $h \in W$.

Lemma 4.2.15: *The subshift Φ (see Definition 4.2.8) is a subshift of finite type.*

Proof: Let $F = \{e\} \cup W \cup WX$, where $WX = \{wx \mid w \in W,\ x \in X\}$. Let A be the set of maps $\sigma : F \to \mathcal{S}$ satisfying (compare with Definition 4.2.8);

1. $eR_\sigma x$, for some $x \in X$;
2. if $x \in X$, then $(e, x) \in F_\sigma$;

3. if $(e,h) \in F_\sigma$, $eR_\sigma x$ and $hR_\sigma hy$ then $(x,hy) \in F_\sigma$.

Then F is finite and A is a subset of the set of maps $F \to \mathcal{S}$, so $C = \{\sigma \in \Sigma(G,\mathcal{S}) \mid \sigma_{|_F} \in A\}$ is a cylinder. Now, Φ is precisely the set of maps from G to $\mathcal{S}$ satisfying this local condition everywhere (Definition 4.2.8) so $\Phi = \bigcap_{g \in G} g^{-1} * C$. Hence Φ is a subshift of finite type. ❑

We now consider the subshift

$$\Phi \times \Phi = \{(\phi_1,\phi_2) : G \to \mathcal{S} \times \mathcal{S}, g \mapsto (\phi_1(g),\phi_2(g)) \mid \phi_1,\phi_2 \in \Phi\}$$

of the Bernoulli shift $\Sigma(G,\mathcal{S} \times \mathcal{S})$. In particular, we are interested in the equivalence relation

$$E = \{(\phi_1,\phi_2) \in \Phi \times \Phi \mid a(\phi_1) = a(\phi_2)\}.$$

We shall show that E is a subshift of finite type.

Definition 4.2.16 ($\mathbf{G}_{\phi_1,\phi_2}$): (Compare with Definitions 4.2.7 and 4.2.14.) Let $\phi_1,\phi_2 \in \Phi$. The subset G_{ϕ_1,ϕ_2} of $G \times G$ is defined by $(g,h) \in G_{\phi_1,\phi_2}$ if and only if the following conditions hold:

1. For each $s \in \phi_1(g)$, $t \in \phi_2(h)$, the state $(s,t,g^{-1}h)$ is a state of $\mathcal{WD}$.
2. For some $s \in \phi_1(g)$, $t \in \phi_2(h)$, the state $(s,t,g^{-1}h)$ is a state in $\mathcal{WD}(X)$.

Lemma 4.2.17: *Let $\phi_1,\phi_2 \in \Phi$ and let a be the map defined in Lemma 4.2.10. Then $a(\phi_1) = a(\phi_2)$ if and only if the following conditions hold (compare with Definition 4.2.8):*

1. *If* $g \in G$, $x \in X$, *then* $(g, gx) \in G_{\phi_1,\phi_2}$.

2. *If* $(g, h) \in G_{\phi_1,\phi_2}$, $gR_{\phi_1}g_1$ *and* $hR_{\phi_2}h_1$ *then* $(g_1, h_1) \in G_{\phi_1,\phi_2}$.

Proof: Suppose that $a(\phi_1) = a(\phi_2)$.

First, we prove that condition 1 holds (compare with Claim 2 in Proposition 4.2.11): Let $g \in G$, $x \in X$, $s \in \phi_1(g)$ and $t \in \phi_2(gx)$. Let r_1 be any ϕ_1-gradient from g and let r_2 be any ϕ_2-gradient from gx. Then r_1 and r_2 trace infinite paths in the geodesic acceptor from states s and t respectively. Further, (r_1, r_2) is a pair of geodesic rays which start at distance $\leqslant 1$ away and go to the same boundary point. Therefore (s, t, x) is a state in $\mathcal{WD}(X)$ and a state in $\mathcal{WD}$. Hence $(g, gx) \in G_{\phi_1,\phi_2}$.

Next, we prove that condition 2 holds (compare with Claim 3 in Proposition 4.2.11): Let $g, h \in G$, $x, y \in X$, such that $(g, h) \in G_{\phi_1,\phi_2}$, $gR_{\phi_1}gx$ and $hR_{\phi_2}hy$. Then there are $g' \in G$ and $x' \in X$ such that the ϕ_1-gradient $[g', g] \cup [g, gx]$ and the ϕ_2-gradient $[g'x', h] \cup [h, hy]$ have the same length, start at distance $\leqslant 1$ apart and can be extended infinitely by a ϕ_1- and a ϕ_2-gradient respectively to the same boundary point (see Figure 4.11). By tracing these paths in the geodesic acceptor from any states $s_0 \in \phi_1(g')$ and $t_0 \in \phi_2(g'x')$, we find states $s \in \phi_1(gx)$ and $t \in \phi_2(hy)$ such that condition 2 of Definition 4.2.16 holds.

Let α_1 be any ϕ_1-gradient from gx and let α_2 be any ϕ_2-gradient from hy. We can express α_1 as a word, $x_1x_2\cdots$, in the generators. Now, $[g', g] \cup [g, gx] \cup \alpha_1$ is a ϕ_1-gradient, because $gxR_{\phi_1}gxx_1$ means that, for every state, $s \in \phi_1(gx)$, the state $s^{x_1} \in \phi_1(gxx_1)$. Inductively, we see that α_1 is a suffix of a ϕ_1-

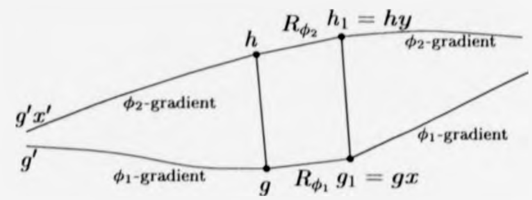


Figure 4.11. The condition on $\Phi \times \Phi$

gradient from g'. Similarly, α_2 is a suffix of a ϕ_2-gradient from $g'x$. Therefore, the pair of geodesic rays (α_1, α_2) differ by a word in W at each step. Given states $s \in \phi_1(gx)$, $t \in \phi_2(hy)$, the geodesic rays α_1 and α_2 can be traced from s and t respectively. Thus, $(s, t, x^{-1}g^{-1}hy)$ is a state in $\mathcal{WD}$. Therefore condition 1 of Definition 4.2.16 holds. Hence condition 2 in the statement of Lemma 4.2.17 holds.

To prove the converse; let $g \in G$, $x \in X$. Any ϕ_1-gradient from g and ϕ_2-gradient from gx go to the same boundary point. Therefore $a(\phi_1) = a(\phi_2)$. ❑

To prove that E is a subshift of finite type, we need the following definition (compare with Definition 4.2.14):

Definition 4.2.18 ($\mathbf{F}_{\sigma_1,\sigma_2}$): Let $\sigma_1, \sigma_2 \in \Sigma(G, S)$ and let $F = \{e\} \cup W \cup WX$ as in Lemma 4.2.15. The subset F_{σ_1,σ_2} of $F \times F$ is defined by $(g, h) \in F_{\sigma_1,\sigma_2}$ if and only if the following conditions hold:

1. For each $s \in \sigma_1(g)$, $t \in \sigma_2(h)$, the state $(s, t, g^{-1}h)$ is a state of $\mathcal{WD}$.
2. For some $s \in \sigma_1(g)$, $t \in \sigma_2(h)$, the state $(s, t, g^{-1}h)$ is a state in $\mathcal{WD}(X)$.

Proposition 4.2.19: *The equivalence relation* $E = \{(\phi_1, \phi_2) \mid a(\phi_1) = a(\phi_2)\}$ *is a subshift of finite type.*

Proof: Let F and A be as in the proof of Lemma 4.2.15. Let A' be the set of maps $\sigma_1 \times \sigma_2 \colon F \to \mathcal{S} \times \mathcal{S}$ such that $\sigma_1, \sigma_2 \in A$ and the following conditions hold:

1. If $x \in X$, then $(e, x) \in F_{\phi_1,\phi_2}$.
2. If $(e, h) \in F_{\phi_1,\phi_2}$, $eR_{\phi_1}x$ and $hR_{\phi_2}hy$ then $(x, hy) \in F_{\phi_1,\phi_2}$.

Then $C' = \{(\phi_1, \phi_2) \mid (\phi_1, \phi_2)_{|_F} \in A'\}$ is a cylinder. By Lemma 4.2.17, $E = \bigcap_{g \in G} g^{-1}C'$. Thus E is a subshift of finite type. ❑

The dynamical homomorphism $a \colon \Phi \to \partial G$, defined in Lemma 4.2.10, gives us the following:

Theorem 4.2.20:
The subshift of finite type Φ *in Definition 4.2.8 is a finite presentation of the dynamical system* $(\partial G, G)$ *with dynamical homomorphism* a. ❑

Remark: Coornaert and Papadopoulos give a different finite presentation for the dynamical system $(\partial G, G)$ in [CP93].

4.2.4 An Instructive Example

Consider F_2, the free group on two generators x and y. The geodesic accepting automaton is shown in Figure 4.12. The acceptable word differences

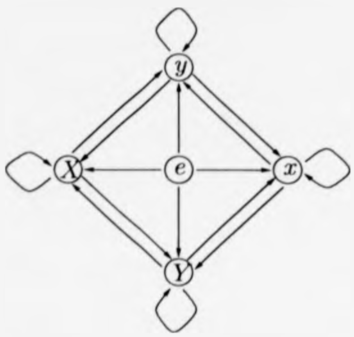


Figure 4.12. The geodesic acceptor for the free group with 2 generators. The label for each edge is the same as the label of its destination (where X is assumed to mean x^{-1} and Y means y^{-1}).

are $W = \{e,\ x,\ y,\ x^{-1},\ y^{-1}\}$. We show how to construct a map from F_2 to non-empty subsets of the set of states $\{e,\ x,\ X,\ y,\ Y\}$. In fact we will only use the sets $\hat{x} = \{e,\ X,\ y,\ Y\}$, $\hat{X} = \{e,\ x,\ y,\ Y\}$, $\hat{y} = \{e,\ x,\ X,\ Y\}$ and $\hat{Y} = \{e,\ x,\ X,\ y\}$.

We pick the the geodesic $xxx\cdots$ starting from the identity and going to the boundary point 'x^∞'. Following the proof (and notation) of Proposition 4.2.11, we get the following: Write $g \in F_2$ as $g = x^m w$, where w is a reduced word which starts with $y^{\pm 1}$. If $m \geqslant 0$, then $t_g = m$, otherwise $t_g = 0$. The geodesic $\alpha_g = w^{-1}x^\infty$ (the geodesics α can be seen by following the arrows in Figure 4.13). From this, we obtain a map as follows: Let v be a reduced word in the generators $\{x,\ y,\ x^{-1},\ y^{-1}\}$ and let v_{last} be the final letter of v. Then define $\phi\colon F_2 \to \Sigma$ by

$$\phi(v) = \begin{cases} \hat{X} & \text{if } v = x^n \ (n \geqslant 0), \\ \hat{v}_{\mathsf{last}} & \text{otherwise.} \end{cases}$$

We trace a geodesic in the Cayley graph by going in the direction z^{-1} whenever we reach a vertex labelled $\hat{z}$ (see Figure 4.13). For each state in $\hat{z}$, there is an outgoing edge labelled z^{-1}.

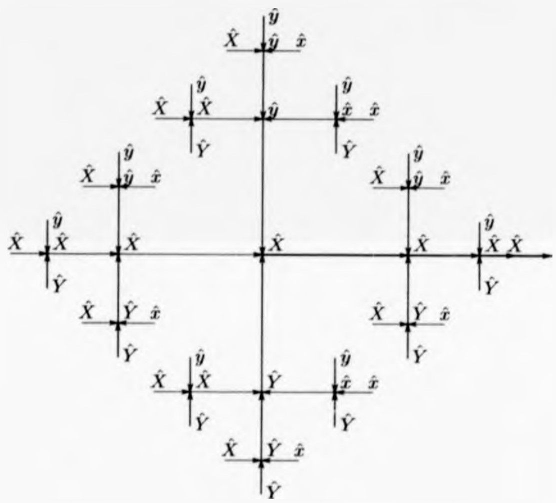


Figure 4.13. Tracing a geodesic in F_2

4.3 The Boundary as a Semi-Markovian Space

The boundary of a hyperbolic group can be given a uniformly finite-to-one semi-Markovian presentation using the ShortLex geodesic acceptor and the word difference machine.

Theorem 4.3.1: *The boundary of a hyperbolic group is a semi-Markovian space. Further, it has a uniformly finite-to-one semi-Markovian presentation.*

The case when the group has a torsion free subgroup of finite index was proved in [CP93, Chapter 7].

Proof: We take as our set of symbols the set S of states in the ShortLex geodesic acceptor and the set of infinite paths from the initial state in this automaton as our subshift. Without loss of generality, we assume that all incoming edges to a state have the same label (otherwise we can modify the automaton), so that a sequence of states uniquely determines a geodesic.

The set of infinite paths in the automaton is a subshift of finite type (see Example 4.1.6). The set of infinite ShortLex geodesics is the set of infinite paths which start at the initial state and is therefore a semi-Markovian subshift. The set of infinite ShortLex geodesics maps continuously to the boundary. Also, there are uniformly finitely many ShortLex geodesics which map to each boundary point. This follows from the fact that there is a unique ShortLex geodesic to each group element and from the uniform bound on the distance between geodesics to the same boundary point.

The equivalence relation given by pairs of sequences which define the same boundary point is also semi-Markovian—being the set of pairs of words accepted by the word difference machine. ❑

4.4 The Geodesic Flow

This section is essentially a summary of coding results concerning biinfinite geodesics on the Cayley graph of a hyperbolic group. All the results are stated in [CP99], the proof of Theorem 4.4.5 is original. The results in [CP99] are an attempt to understand Gromov's generalisation [Gro87, Section 8.3] of coding results of geodesics on a surface of constant negative curvature (for example [Mor21], [Koe29] and [Art65]). The symbols of such a coding set can be taken to be a generating set for the fundamental group of the surface. The symbolic structure of finite type is very similar to the automatic structure of such a group (see, for example, [Ser86] for a more detailed history). This naturally leads one to ask about geodesic flows on a Gromov hyperbolic space, in particular, the Cayley graph of a hyperbolic group.

Let G be an infinite hyperbolic group with (finite) generating set X. Let Γ be the corresponding Cayley graph.

Definition 4.4.1 (Geo(Γ)): Let Geo(Γ) be the set of all biinfinite geodesics in Γ; that is, Geo(Γ) is the set of all isometric embeddings of $\mathbb{R}$ into Γ. Note that two geodesics with the same image do not in general define the same element of Geo(Γ). The space of geodesics can be given a metric by

$$d(r,s) = \sum_{n=-\infty}^{\infty} d(r(n), s(n))2^{-|n|}. \tag{4.2}$$

The map $\mathrm{Geo}(\Gamma) \to \Gamma$, $r \mapsto r(0)$ is a quasi-isometry. Let $d = |r(0) - s(0)|$. Then $d \leqslant d(r,s) \leqslant \sum_{n=-\infty}^{\infty}(d+2|n|)2^{-|n|} = 3d + 2\sum_{n=1}^{\infty} 2n2^{-n} = 3d+8$. So $r \mapsto r(0)$ is a $(3,8)$-quasi-isometry.

Definition 4.4.2 (Geodesic flow): Let $r : (-\infty, \infty) \to \Gamma$ be in $\mathrm{Geo}(\Gamma)$. There are $\mathbb{R}$-, G- and $\mathbb{Z}_2$-actions by isometries on $\mathrm{Geo}(\Gamma)$. The $\mathbb{R}$-action, called the flow action, is denoted by $\mathcal{F}$ and is defined as follows: Let $t \in \mathbb{R}$. Then $\mathcal{F}_t r(u) = r(t+u)$. The *geodesic flow* is the space $\mathrm{Geo}(\Gamma)$ with the metric (4.2), together with the flow action $\mathcal{F}$ of $\mathbb{R}$. The G-action on Γ extends to a G-action on $\mathrm{Geo}(\Gamma)$. The $\mathbb{Z}_2 = \langle \iota \rangle$-action is given by reversing time; $(\iota r)(t) = r(-t)$. The G-action commutes with the flow, and $\mathbb{Z}_2$-actions. The flow and $\mathbb{Z}_2$ actions satisfy $\iota \mathcal{F}_t(r) = \mathcal{F}_{-t}\iota(r)$.

This construction is not ideal because between two boundary points there may be many geodesics. Gromov constructed, in [Gro87, Section 8.3], a space $\hat{G}$ which is homeomorphic to $\partial^2\Gamma \times \mathbb{R}$ (where $\partial^2\Gamma$ denotes the set of pairs $(a, b) \in \partial\Gamma \times \partial\Gamma$ such that $a \neq b$). (See [Cha94] for the proof.) The space $\hat{G}$ is unique up to $(G \times \mathbb{Z}_2)$-equivariant homeomorphism which takes $\mathbb{R}$-orbits to $\mathbb{R}$-orbits. The main drawback to this construction is the apparent non-existence of a canonical map $\mathrm{Geo}(\Gamma) \to \hat{G}$ (that is, a map such that each geodesic $r \in \mathrm{Geo}(\Gamma)$ is isometrically embedded into $\hat{G}$, with the endpoints of r going to the appropriate end points in $\partial^2\Gamma$). However, each geodesic can be quasi-isometrically embedded into $\hat{G}$.

Definition 4.4.3 (Integral geodesics $\mathrm{Geo}_{\mathbb{Z}}(\Gamma)$): Let $\mathrm{Geo}_{\mathbb{Z}}(\Gamma)$ denote the subset of geodesics in $\mathrm{Geo}(\Gamma)$ such that $\mathbb{Z}$ maps onto the vertices of Γ. We call $\mathrm{Geo}_{\mathbb{Z}}(\Gamma)$ the set of *integral geodesics*. The space $\mathrm{Geo}_{\mathbb{Z}}(\Gamma)$ has the metric (4.2) and natural G-, $\mathbb{Z}$- and $\mathbb{Z}_2$-actions given by restricting the actions of Definition 4.4.2. The $\mathbb{Z}$-action is generated by $\mathcal{F}_1$, the time 1 return map.

Conjecture 4.4.4: *It is not, in general, possible to code the dynamical system* $(\mathrm{Geo}_{\mathbb{Z}}(\Gamma), \mathcal{F}_1)$ *as a subshift of finite type. That is, there is an infinite hyperbolic group* G *such that there is no subshift of finite type,* Ψ, *with a dynamical homomorphism* $\psi \to \mathrm{Geo}_{\mathbb{Z}}(\Gamma)$ *(that is, a map,* $h\colon \Psi \to \mathrm{Geo}_{\mathbb{Z}}(\Gamma)$, *which intertwines the shift and flow actions).*

Remark: Using the geodesic automatic structure of G, one can easily obtain a symbolic coding of a geodesic in terms of the labelling of edges, but this does not tell us any points through which the geodesic passes.

However, since the G- and $\mathbb{Z}$-actions commute, $\mathcal{F}_1$ defines a $\mathbb{Z}$-action, $\mathcal{F}_1^*$, on the quotient $G\backslash \mathrm{Geo}_{\mathbb{Z}}(\Gamma)$. We can code this in the same way.

Theorem 4.4.5 (Coornaert, Papadopoulos [CP99]): *The quotient space* $G\backslash \mathrm{Geo}_{\mathbb{Z}}(\Gamma)$ *together with the flow action,* $\mathcal{F}_1^*$, *is a finitely presented dynamical system over* $\mathbb{Z}$.

Remark: Following [CP98] and [CP99], we can prove this result by reinterpreting the symbol space Φ of Section 4.2 firstly as a subshift over $\mathbb{N}$ and then as a subshift over $\mathbb{Z}$. Loosely speaking, the set of symbols used is the set of maps A in the proof of Lemma 4.2.15. Let Φ be as in Lemma 4.2.15 and let $\phi \in \Phi$. Let x be the lexicographically least generator x such that $\phi(e)^x \subset \phi(x)$. Then the shift map is defined by $1 + \phi = x * \phi$. This process can be applied forwards and backwards, so that each element of the subshift defines a ShortLex geodesic (which passes through the identity at time 0).

We present a different proof using the automatic structure directly.

Proof: Take our set of symbols, S, to be the set of states in the geodesic acceptor (we assume that the geodesic acceptor is partially deterministic with no failure states and such that all incoming edges to the same state have the same label). The set of biinfinite sequences of states read by the geodesic acceptor, $\Psi \subset \Sigma(\mathbb{Z}, S)$, is a subshift of finite type (see Example 4.1.6). Each sequence $\psi = (s_i)_{i=-\infty}^{\infty}$ naturally defines an integral geodesic $r = r(\psi)$ by $r(0) = e$, $r(n) = x_1 x_2 \cdots x_n$, $r(-n) = x_0^{-1} x_{-1}^{-1} \cdots x_{-n+1}^{-1}$ (where $n \in \mathbb{N}$ and x_i is the label of the edge connecting s_{i-1} and s_i in the geodesic acceptor). Let $b : \Psi \to G \backslash \mathrm{Geo}_{\mathbb{Z}}(\Gamma)$ denote the map $(s_i)_{i=-\infty}^{\infty} \mapsto G(r)$, where r is as above and $G(r)$ denotes the G-orbit of r. We show that b is a dynamical homomorphism.

Every G-orbit contains a unique geodesic which passes through the identity at time 0 (given by $r(0)^{-1}(r) \in G(r)$). Given an integral geodesic r such that $r(0) = e$, there is a sequence of states in the geodesic acceptor which traces r. Hence the map b is surjective.

We now show that, if $\psi \in \Psi$, then $b(1 + \psi) = \mathcal{F}_1^* b(\psi)$. In the following argument, the dot is used to keep track of time 0. We write geodesics as $\cdots (r(-1))(r(0)) \cdot (r(1))(r(2)) \cdots$ and sequences as $\cdots s_{-1} s_0 \cdot s_1 s_2 \cdots$. Let $\psi = \cdots s_{-1} s_0 \cdot s_1 s_2 \cdots$ be a sequence of states in the geodesic acceptor. Again we use x_i to denote the label of the edge between the states s_{i-1} and

s_i. Then

$$r(\psi) = \cdots (x_0^{-1}x_{-1}^{-1})(x_0^{-1})(e)\cdot(x_1)(x_1x_2)\cdots$$

$$\Longrightarrow \mathcal{F}_1 r(\psi) = \cdots (x_0^{-1})(e)(x_1)\cdot(x_1x_2)(x_1x_2x_3)\cdots .$$

Whereas

$$1+\psi = \cdots s_0 s_1 \cdot s_2 s_3 \cdots$$

$$\Longrightarrow r(1+\psi) = \cdots (x_1^{-1}x_0^{-1})(x_1^{-1})(e)\cdot(x_2)(x_2x_3)\cdots ,$$

$$\Longrightarrow x_1(r(1+\psi)) = \cdots (x_0^{-1})(e)(x_1)\cdot(x_1x_2)(x_1x_2x_3)\cdots .$$

So $r(1+\psi)$ is in the same G-orbit as $\mathcal{F}_1 r(\psi)$. So $b(1+\psi) = \mathcal{F}_1^* b(\psi)$; hence the map b is a dynamical homomorphism.

Two sequences $\psi_1, \psi_2 : \mathbb{Z} \to S$ define the same element of $G\backslash \mathrm{Geo}_{\mathbb{Z}}(\Gamma)$ if, for all n, the edge from $s_1(n)$ to $s_1(n+1)$ has the same label as the edge from $s_2(n)$ to $s_2(n+1)$. The set of pairs of sequences (ψ_1, ψ_2) which satisfy this condition is a subshift of finite type. Therefore the dynamical system $(G\backslash \mathrm{Geo}_{\mathbb{Z}}(\Gamma), \mathcal{F}_1^*)$ is finitely presented. ❑

Definition 4.4.6 (Mapping torus): The *mapping torus* of the dynamical system (E, f) over $\mathbb{Z}$ is the space $T(E,f) = (E \times I)/\{(x,1) = (f(x),0)\}$, where I denotes the unit interval. When the space E is a shift space and the homeomorphism f is the shift map, we use $T(E)$ to denote the mapping torus. Equivalently, $T(E) = (E \times \mathbb{R})/\mathbb{Z}$ where the $\mathbb{Z}$-action is generated by the homeomorphism $(e,t) \mapsto (f(e), t-1)$. We can define a flow on $E \times \mathbb{R}$ by $\mathcal{F}_t(e,u) = (e, u+t)$, which gives a flow on $T(E)$.

Theorem 4.4.7 (Coornaert, Papadopoulos [CP99]): *There is a dynamical homomorphism from the flow on the mapping torus of a subshift of finite type to the flow on $G\backslash\mathcal{G}$.*

The proof of this theorem follows exactly the same steps as the proof in [CP99]. The only difference is that we substitute the subshift of finite type Ψ from the proof of Theorem 4.4.5 for the subshift of finite type used in [CP99].

Proof: The dynamical homomorphism $b\colon \Psi \to G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma)$ from the proof of Theorem 4.4.5 induces the dynamical homomorphisms $b_{\mathbb{R}}: (\psi, t) \mapsto (b(\psi), t)$ and $b_T\colon T(\Psi) \to T(G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma), \mathcal{F}_1^*)$ (see (4.3)).

$$\begin{array}{ccccc} \Psi & \dashrightarrow & \Psi \times \mathbb{R} & \xrightarrow{/\mathbb{Z}} & T(\Psi) \\ \downarrow b & & \downarrow b_{\mathbb{R}} & & \downarrow b_T \\ G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma) & \dashrightarrow & G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma) \times \mathbb{R} & \xrightarrow{/\mathbb{Z}} & T(G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma), \mathcal{F}_1^*) \end{array} \tag{4.3}$$

The $\mathbb{Z}$-actions on $\Psi \times \mathbb{R}$ and $G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma) \times \mathbb{R}$ are generated by $(\psi, t) \mapsto (1 + \psi, t - 1)$ and $(r, t) \mapsto (\mathcal{F}_1^*(r), t - 1)$ respectively (see Definition 4.4.6).

Given $r \in \operatorname{Geo}_{\mathbb{Z}}(\Gamma)$, let $r_0 = r(0)^{-1}r$. Then r_0 depends only on the G-orbit of r, so there is a well-defined map $i\colon G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma) \times \mathbb{R} \to \operatorname{Geo}(\Gamma)$, $(r, t) \mapsto \mathcal{F}_t(r_0)$. Let $q_{\mathbb{R}} = p \circ i$, where p is the canonical projection $\operatorname{Geo}(\Gamma) \to G\backslash \operatorname{Geo}(\Gamma)$ (see

(4.4)). Then $q_{\mathbb{R}}$ is surjective and $\mathbb{Z}$-invariant.

$$\begin{array}{ccc} G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma) \times \mathbb{R} & \xrightarrow{\ i\ } & \operatorname{Geo}(\Gamma) \\ {\scriptstyle /}\big\downarrow{\scriptstyle \mathbb{Z}} & \searrow^{q_{\mathbb{R}}} & {\scriptstyle p}\big\downarrow{\scriptstyle G\backslash} \\ T(G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma), \mathcal{F}_1^*) & \xrightarrow{q_{\mathbb{R}}} & G\backslash \operatorname{Geo}(\Gamma). \end{array} \tag{4.4}$$

The map $q_{\mathbb{R}}$ induces the map $q_T : T(G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma), \mathcal{F}_1^*) \to G\backslash \operatorname{Geo}(\Gamma)$ (see (4.5)). This map is a topological conjugacy.

$$\begin{array}{ccc} G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma) \times \mathbb{R} & & \\ & \searrow^{q_{\mathbb{R}}} & \\ {\scriptstyle /\mathbb{Z}}\big\downarrow & & G\backslash \operatorname{Geo}(\Gamma). \\ & \nearrow_{q_T} & \\ G\backslash \operatorname{Geo}_{\mathbb{Z}}(\Gamma) \times \mathbb{R} & & \end{array} \tag{4.5}$$

Since G acts properly discontinuously and cocompactly on Γ, there is a continuous, surjective, G-equivariant map $Q : \operatorname{Geo}(\Gamma) \to \hat{G}$ ([Gro87, Theorem 8.3.C., Condition 5], see (4.6)). The map Q is surjective because the map $\operatorname{Geo}(\Gamma) \to \partial^2\Gamma$, $r \mapsto (r(-\infty), r(+\infty))$ is surjective and the flow orbit of a point is a quasi-isometric embedding of $\mathbb{R}$ in $\hat{G}$. This induces the continuous map $Q_G : G\backslash \operatorname{Geo}(\Gamma) \to G\backslash\hat{G}$.

$$\begin{array}{ccc} \operatorname{Geo}(\Gamma) & \xrightarrow{\ Q\ } & \hat{G} \\ {\scriptstyle p}\big\downarrow{\scriptstyle G\backslash} & & {\scriptstyle G\backslash}\big\downarrow \\ G\backslash \operatorname{Geo}(\Gamma) & \xrightarrow{Q_G} & G\backslash\hat{G}. \end{array} \tag{4.6}$$

Combining the above, we get that $Q_G \circ q_T \circ b_T$ is a dynamical homomorphism from the flow on $T(\Psi)$ to the flow on $G\backslash\hat{G}$. ❑

Chapter 5

Cayley Graphs which are Regular Tilings

Chapter Summary

We investigate the following problem: Given a regular tiling, is its 1-skeleton the Cayley graph of a group? In Section 5.2, we determine the complete picture for regular tilings of the sphere (platonic solids). In Section 5.3 we find an elegant condition for the tiling of s-gons meeting v to a vertex to be a Cayley graph. The answer is yes if and only if there is a non-trivial divisor of s which is smaller than v (Theorem 5.3.8). We then investigate some semi-regular tilings.

5.1 Introduction

By a regular tiling, we mean the tiling of 2-dimensional space (spherical, Euclidean or hyperbolic) by regular polygons, where each polygon has the same number of sides and the same angle at each vertex. We take the 1-skeleton of such a tiling and ask whether it can be the Cayley graph of a group. We use $\mathcal{G}(s; v)$ to denote the 1-skeleton of the tiling by s-gons meeting v at each vertex.

The problem as to which platonic solids (regular tilings of the sphere) could be Cayley graphs was posed by David Epstein in an M.Sc. course on Geometric Group Theory. We start by classifying which of these sphere-tilings can be Cayley graphs and give a comprehensive list of all groups and generating sets (up to automorphism and replacement of generators by their inverses) with platonic solids as Cayley graphs. We then classify precisely which regular tilings can be Cayley graphs.

Let G be a group with generating set X. Recall (Definition 1.0.1) that the Cayley graph $\Gamma(G, X)$, or Γ, is the graph with, for each $g \in G$, a vertex labelled g and, for each $g \in G$, $x \in X$, an edge from g to gx labelled by x, written $g \xrightarrow{x} gx$.

We adopt the convention that if a generator x has order 2, we have a single, undirected edge labelled x

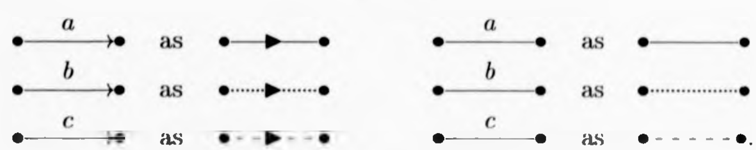
$$g \underset{x}{\overset{x}{\rightleftarrows}} gx \quad \text{becomes} \quad g \overset{x}{\text{---}} gx.$$

The group G acts on its Cayley graph on the left by left multiplication on the vertices. This action gives many strong symmetry results.

Observation 5.1.1: *The valency of each vertex of a Cayley graph is $m + 2n$, where m is the number of generators of order 2 and n is the number of generators which do not have order 2.*

Proof: At each vertex, for each generator x of order 2, there is a single edge labelled x. For every other generator y there is one incoming edge and one outgoing edge labelled y. ❑

Most of the Cayley graphs which we draw have less than or equal to 3 generators. To save excessive labelling of graphs, we suppress the labels on the vertices. We label our generators from the start of the alphabet and adopt the following conventions for labelling edges in our graphs;



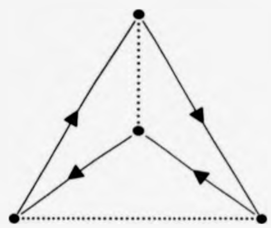
5.2 Platonic Solids

There are 5 regular tilings of the sphere. The 1-skeletons are isomorphic to the 1-skeletons of the platonic solids.

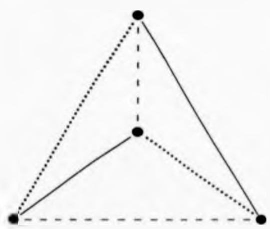
5.2.1 The Tetrahedron

We start with the simplest regular polyhedron—the tetrahedron. In this case it is easy to see that there are 2 possible pictures, one for each group of order 4 (Graphs 4, 5).

Cayley Graph 4. $\mathbb{Z}_4 = \langle a, b \mid a^4, b = a^2 \rangle$



Cayley Graph 5. $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b, c \mid a^2, b^2, ab = ba, c = ab \rangle$

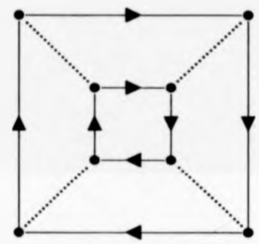


5.2.2 The Cube

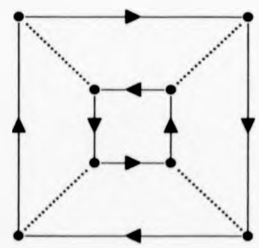
The cube has 8 vertices and valency 3. Therefore, there are either 3 generators of order 2 or there is 1 generator of order 2 and 1 generator not of order 2.

An easy first example is $\mathbb{Z}_4 \times \mathbb{Z}_2$ (Graph 6). Reversing the arrows on the inner square gives us the Cayley graph of the dihedral group D_8 generated by a rotation and reflection of a square (Graph 7). The dihedral group D_8 has another generating set which has the cube as Cayley graph. The second (Graph 8) is obtained from the presentation of D_8 as the Coxeter group $I_2(4) = \langle a, b \mid a^2, b^2, (ab)^4 \rangle$, with an extra generator $c = bab$.

Cayley Graph 6. $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a, b \mid a^4, b^2, ab = ba \rangle$

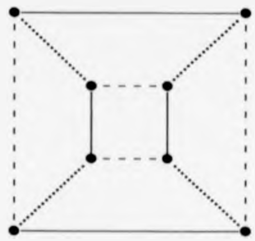


Cayley Graph 7. $D_8 = \langle a, b \mid a^4, b^2, (ab)^2 \rangle$

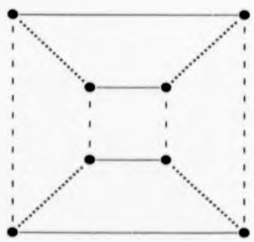


The group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with the standard generating set $a = (1, 0, 0)$, $b = (0, 1, 0)$, $c = (0, 0, 1)$ also has the cube as its Cayley graph (Graph 9).

Cayley Graph 8. $D_8 = \langle a, b, c \mid a^2, b^2, (ab)^4, c = bab \rangle$



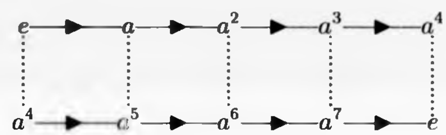
Cayley Graph 9. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b, c \mid a^2, b^2, c^2, ab = ba, bc = cb, ca = ac \rangle$



Lemma 5.2.1: *The cyclic group of order 8 ($\mathbb{Z}_8$) does not have a Cayley graph which is the 1-skeleton of the cube.*

Proof: If the Cayley graph is a cube, then it must have a generator of order 2 (because the valency at each vertex is odd). Three generators of order 2 is impossible, so $\mathbb{Z}_8$ is generated by elements a and b, where $b^2 = e$ and a is not of order 2. Clearly, a has order 8 and $a^4 = b$. Graph 10 shows the Cayley graph of $\mathbb{Z}_8$, generated by a and b, drawn on the Möbius band. ❑

Cayley Graph 10. $\mathbb{Z}_8 = \langle a, b \mid a^8, b = a^4 \rangle$



Lemma 5.2.2: *The quaternion group $Q_8 = \langle a, b \mid a^4, a^2 = b^2, ab = b^{-1}a \rangle$ does not have a Cayley graph which is the 1-skeleton of the cube.*

Proof: The group Q_8 has only 1 element of order 2 (and 6 of order 4). If the Cayley graph is the 1-skeleton of a cube, it must have one generator, x, of order 2 and one generator, y, of order 4. But $x = y^2$, so x and y do not generate Q_8 (they generate a cyclic subgroup of order 4). ❑

Lemma 5.2.3: *Graphs 6–9 are the only cuboid Cayley graphs (up to label preserving isometry).*

Proof: Lemma 5.2.1 tells us that we cannot have a generator of order 8.

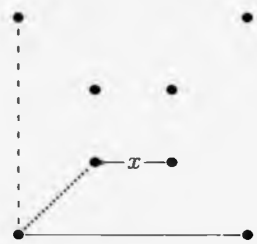


Figure 5.1. The choice of x determines the Cayley graph.

If we have a generator of order 4, then the vertices are partitioned into 2 cosets given by 2 faces of the cube. The other generator (of order 2) must connect these faces. The only choice is the relative orientation of these 4-cycles. These 2 choices correspond to Graphs 6 and 7.

If all the generators have order 2, then we must have 1 edge of each label from each vertex. Without loss of generality we can put in all the generators from one of the vertices (see Figure 5.1). Once we have chosen the edge labelled x in Figure 5.1, the rest of the edges are determined. There are 2 choices for x, corresponding to Graphs 8 and 9. ❑

5.2.3 The Octahedron

The octahedron has 6 vertices and both groups of order 6 can have the 1-skeleton of the octahedron as a Cayley graph. In each case, the generating set has a redundant generator of order 3. Both groups have generating sets which have the hexagon as a Cayley graph (namely $\mathbb{Z}_6 = \langle a \mid a^6 \rangle$ and $D_6 =$

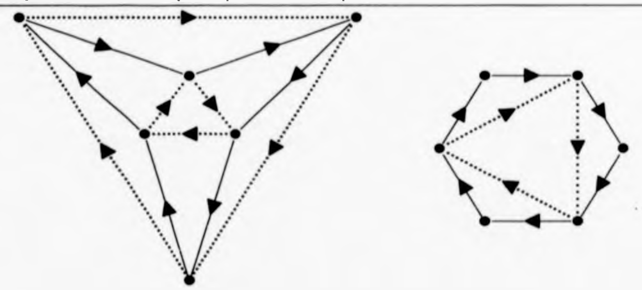
$\langle a, b \mid a^2, b^2, (ab)^3 \rangle)$. The generator of order 3 ($c = a^2$ and $c = ab$ respectively) turns the picture of a hexagon into the picture of an octahedron.

In the case of $\mathbb{Z}_6$, we cannot have a generator of order 2 because there is only one element of order 2 and we must have valency 4. The elements of order 3 are mutually inverse; the same goes for elements of order 6. So we must have a generator a of order 6 and a generator b of order 3. We can have either $b = a^2$ or $b = a^{-2}$. So we can reverse the arrows on the generator of order 3 to get a slightly different picture, but this is the only other picture we can have.

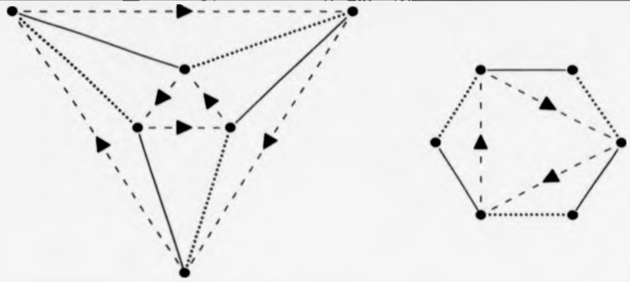
The group D_6 is not generated by elements of order 3, so must have one generator, c, of order 3 and 2 generators, a, b, of order 2. We have either $c = ab$ or $c = ba$, which give the same picture up to swapping the labels a and b.

Lemma 5.2.4: *The only octahedral Cayley graphs are Graphs 11 and 12 or Graphs 11 and 12 with the order 3 generator arrows reversed.* ❑

Cayley Graph 11. $\mathbb{Z}_6 = \langle a, b \mid a^6, b = a^2 \rangle$



Cayley Graph 12. $D_6 = \langle a, b, c \mid a^2, b^2, (ab)^3, c = ab \rangle$



5.2.4 Relator types

The two remaining platonic solids are harder. To cope with them we need to introduce some machinery. We assume that no generator is trivial. We work in the free group on the generators, but keep the involution conditions.

Definition 5.2.5 (Simply reduced): A word in the generators is *simply reduced* if no proper, non-empty subword equals the identity. Note that a simply reduced word is freely reduced.

Definition 5.2.6 (Relator): A *relator* is a cyclically simply reduced word in the generators which equals the identity in the group. Two relators are equal if they are identical as simply reduced words.

Definition 5.2.7 (Inverse): The *inverse* of a relator is the free inverse of the relator.

Definition 5.2.8 (Relator class): The *relator class* $R(r)$ of a relator r is the set of cycles and inverses of cycles of r. A relator class has *length* $|R|$ if each of its relators has length $|R|$. Given $r = x_1 \cdots x_n$, we use r_i to denote the cyclic conjugate $r_i = x_i \cdots x_n x_1 \cdots x_{i-1}$.

Definition 5.2.9 (Relator type): The *relator type* $k(R)$ (or $k(r)$) of the relator class $R(r)$ is half the cardinality of R. If a relator is in a relator class, then so is its inverse. A relator cannot equal its own inverse because that contradicts it being simply reduced ($x_1 \cdots x_n \equiv x_n^{-1} \cdots x_1^{-1} \implies x_1 = x_n^{-1}$). Therefore the cardinality of each relator class is even, so the relator type is an integer. The relator type is the number of distinct cycles of the relator r, unless one of the cycles is the inverse of r, in which case it's half this number.

Examples: Some examples of relators and their relator types:

1. The relator x^n has relator class $\{x^n, (x^{-1})^n\}$ and has type 1.
2. The relator xyz has relator class

$$\{xyz,\ yzx,\ zxy,\ z^{-1}y^{-1}x^{-1},\ x^{-1}z^{-1}y^{-1}, y^{-1}x^{-1}z^{-1}\}$$

   and has type 3.
3. The relator $xyzxyz$ has relator class

$$\{xyzxyz,\ yzxyzx,\ zxyzxy,\ z^{-1}y^{-1}x^{-1}z^{-1}y^{-1}x^{-1},\\ x^{-1}z^{-1}y^{-1}x^{-1}z^{-1}y^{-1},\ y^{-1}x^{-1}z^{-1}y^{-1}x^{-1}z^{-1}\}$$

   and has type 3.

4. If x and y are not both involutions then the relator $xyxy$ has relator class
$$\{xyxy,\ yxyx,\ x^{-1}y^{-1}x^{-1}y^{-1},\ y^{-1}x^{-1}y^{-1}x^{-1}\}$$
and has type 2.

5. If x and y are both involutions, then the relator $xyxy$ has relator class $\{xyxy,\ yxyx\}$ and has type 1.

6. The relator $xyx^{-1}y$, where y is an involution, has relator class
$$\{xyx^{-1}y,\ yx^{-1}yx, x^{-1}yxy,\ yxyx^{-1}\}$$
and has type 2.

Lemma 5.2.10: *The relator type $k(R)$ divides the length of R.*

Proof: The number of distinct cycles c of a relator r divides the length of r and the relator type is either c or $c/2$. In either case, $k(R) = k(r) \mid |r| = |R|$.
❑

Definition 5.2.11 (Simple loop): A *simple loop* in Γ is a closed path in Γ from a fixed base point such that no vertex is repeated (apart from the initial vertex, which is visited at the beginning and at the end, but is not visited in between). We count going backwards round the loop as being the same as going forwards round the loop.

Proposition 5.2.12: *Consider relator classes of the same length l. Then*
$$\sum_{|R|=l} k(R) = \text{The number of simple loops of length } l \text{ in } \Gamma.$$

Proof: A relator, r, is a closed loop in the Cayley graph from the identity. Taking its inverse reverses the direction of the loop so each (relator,inverse) pair coincides with a simple loop. Taking a cycle distinct from r and r^{-1} gives a different loop based at the identity. The relator type just counts the number of different simple loops obtained by taking cycles.

Conversely, a simple loop in the Cayley graph traces a word, w, in the generators which equals the identity. The 'visit no vertex twice' condition means that w is simply reduced, therefore, w is a relator. ❑

Lemma 5.2.13: *The relator classes of length l and type 1 are either x^l, where x is not an involution, or $(xy)^{l/2}$, where x and y are both involutions.*

Corollary 5.2.14: *The only relators of type 1 of odd length l are x^l where x is not an involution.*

Corollary 5.2.15: *If the generator x is in a relator of even length, l, and type 1, then x has order l or 2.*

5.2.5 The Dodecahedron

Lemma 5.2.16: *The 1-skeleton of the dodecahedron (Figure 5.2) cannot be the Cayley graph of a group.*

Proof: Consider relator classes R of length 5. By Lemma 5.2.10, R has type 1 or 5. There are 3 simple loops of length 5 in the dodecahedron, so by

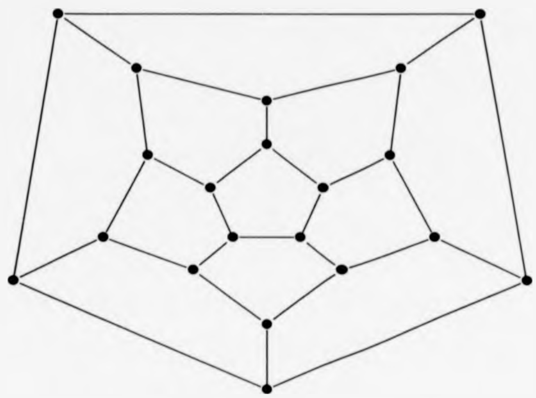


Figure 5.2. The 1-skeleton of the dodecahedron. This picture (along with Figure 5.3 and all the Cayley Graphs in this chapter) were drawn using Paul Taylor's commutative diagrams package, `diagrams`.

Proposition 5.2.12, $\sum_{|R|=5} k(R) = 3$. Therefore, each relator class of length 5 has type 1 (because $5 > 3$). So there must be 3 distinct relator classes of type 1 and hence at least 3 non-involution generators (by Corollary 5.2.14). Therefore, the Cayley graph has valency at least 6. But the dodecahedron has valency 3. ❑

5.2.6 The Icosahedron

We now apply the methods of the proof of Lemma 5.2.16 to the icosahedron (Figure 5.3).

Consider relator classes of length 3—they have type 1 or 3. There are 5 loops of length 3. There cannot be 5 relator classes of type 1, because then

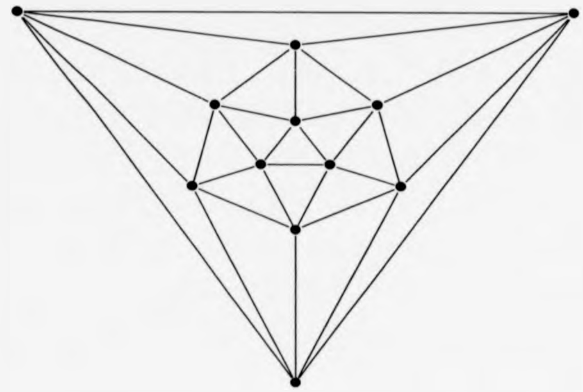


Figure 5.3. The 1-skeleton of the icosahedron.

we would have 5 non-involution generators giving a valency of 10. We cannot have 2 relator classes of type 3 because $2 \times 3 > 5$, contradicting Proposition 5.2.12. So we must have 1 relator class of type 3 and 2 relator classes of type 1.

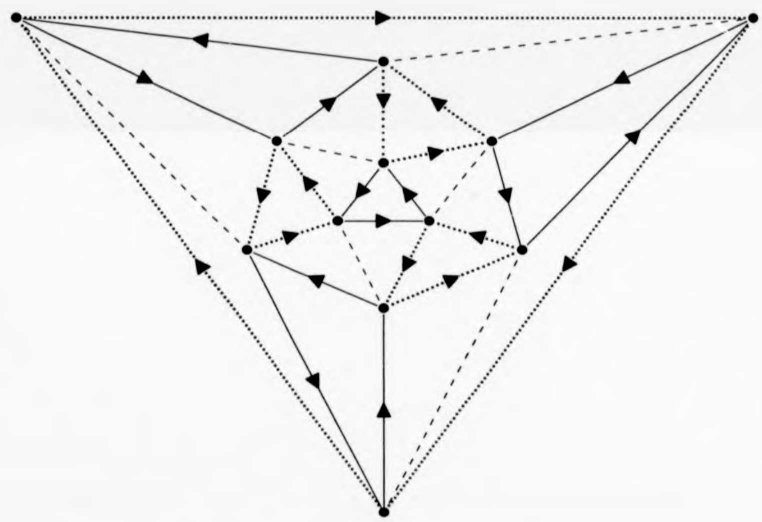
We have 2 non-involution generators, a, b, of order 3; 1 for each relator class of type 1. The valency of the icosahedron is 5, so we must have 1 generator, c, of order 2. The remaining relator must involve c. Each generator cannot appear more than once, otherwise one of the generators would be trivial. We choose the final relator to be abc. The relators of length 3 are a^3, b^3, abc. Trying the presentation $\langle a, b, c \mid a^3, b^3, c^2, abc \rangle$, we find that we get a group (the group of even permutations of 4 symbols) which has the icosahedron as a Cayley graph (see Graph 13).

The only choice we had was for the generator $c = ab$. We had to choose

$c = a^{\varepsilon_1} b^{\varepsilon_2}$ for $\varepsilon_1, \varepsilon_2 = \pm 1$ (we can put a before b because $c^{-1} = c$). The different choices correspond to replacing a generator with its inverse.

Lemma 5.2.17: *The alternating group A_4 is the only group with the icosahedron as a Cayley graph. There is a unique choice of generators (up to automorphism and replacement by inverses) which gives this Cayley graph.* ❑

Cayley Graph 13. $A_4 = \langle a, b, c \mid a^3, b^3, (ab)^2, c = ab \rangle$.



5.3 Regular Tilings

5.3.1 Regular Tilings Which are not Cayley Graphs

The method of proof for the dodecahedron can be generalised to 1-skeletons of some regular tilings of other surfaces. The tilings we consider are tilings of the Euclidean or hyperbolic planes by s-gons meeting v to a vertex ($v \geqslant 3$).

Observation 5.3.1: *Generators must be non-trivial and distinct, otherwise we would get subgraphs which are a single loop or are a loop of length 2;*

$$g \overset{x}{\underset{y}{\rightrightarrows}} gx.$$

Both would contradict the Cayley graph being a regular tiling.

Observation 5.3.2: *Every loop of length s must be the boundary of an s-gon. Therefore, each relator of length s must be the boundary of an s-gon.*

Proposition 5.3.3: *If all divisors of s are larger than v (that is, $d|s$, $d \neq 1 \implies d > v$), then the graph $\mathcal{G}(s;v)$ cannot be the Cayley graph of a group.*

Proof: First, note that if s is even, then $2|s$, so s has a divisor which is smaller than v. So we may assume that s is odd.

By Lemma 5.2.10, the relators of length s must have type d, where $d|s$. There are v simple loops of length s, so $d \leqslant v$ (by Proposition 5.2.12); hence $d = 1$. So there are v relators of length s each of which has type 1. By

Corollary 5.2.14, there are at least v generators of order greater than 2. But the valency of $\mathcal{G}(s;v)$ is v. ❑

In fact, these are the only regular tilings which cannot be the Cayley graph of a group. For the positive proofs we use:

Theorem 5.3.4 (Poincaré, [Poi82]): *Let Δ be a compact convex polygon with side pairings which satisfies the cycle conditions. Then the group generated by the side pairings is a discrete group with Δ as fundamental domain. A complete set of relations for the group consists of the reflection relations and the cycle relations. The Cayley graph for the group generated by the side transformations is the dual to the tiling by Δ. (See, for example, [EP94].)*

5.3.2 The Case When s is Even

Proposition 5.3.5: *If s is even, then $\mathcal{G}(s;v)$ is the Cayley graph of the group*

$$E(s,v) = \langle a_1, \ldots, a_v \mid \forall i,\ \ a_i^2, (a_i a_{i+1})^{(s/2)} \rangle,$$

where the index i in $a_i a_{i+1}$ is interpreted cyclically.

Proof: Suppose we have s-gons meeting v to a vertex. Consider the regular tiling of v-gons meeting s to a vertex (the angle at each vertex is $\frac{2\pi}{s} = \frac{\pi}{s/2}$). A group of symmetries of this tiling is generated by reflections in each side. The Cayley graph of this group is $\mathcal{G}(s;v)$ (see, for example, Figure 5.4). ❑

The group $E(4,3)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (Graph 9).

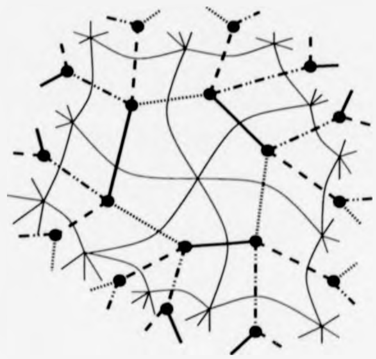


Figure 5.4. 4-gons meeting 6 to a vertex with side reflection generates $\langle a, b, c, d \mid a^2, b^2, c^2, d^2, (ab)^3, (bc)^3, (cd)^3, (da)^3 \rangle$, whose Cayley graph is $\mathcal{G}(6; 4)$.

5.3.3 The Block Construction

We now present a method of constructing Cayley graphs which are regular tilings.

Definition 5.3.6: Given a set of relator classes $\{R_j\}$, we define its *block* to be the set of loops from a vertex traced by relators in $\bigcup R_j$. A $(d_1, \ldots, d_n)$-*block* is a block such that, when the initial vertex is removed, there are n connected components, C_i, containing the remains of d_i of the loops in the block. A connected component C_i with the initial vertex restored is called a *slab*, or a d_i-*slab*. Note that $\sum d_i = \sum k(R_j)$ (see Definition 5.2.9). If we have a set of relators $\{r_j \mid j \in J\}$ which are in distinct relator classes, we talk of the block of $\{r_j \mid j \in J\}$ which is precisely the same as the block of $\{R(r_j) \mid j \in J\}$.

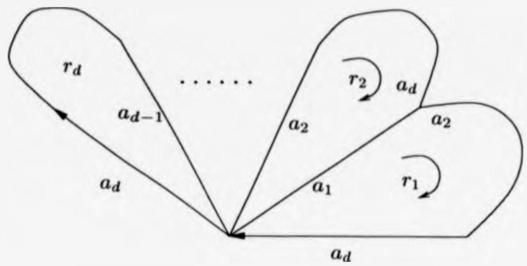


Figure 5.5. The d-block

Fix s and let $d \geqslant 3$ be a divisor of s.

The d-block Let $a_1, \ldots, a_d$ be generators with $a_1, \ldots, a_{d-1}$ being involutions. The standard d-block is the block of (the relator class of) the relator $r = (a_1 a_2 \cdots a_d)^{(s/d)}$. It consists of d s-gons joined together at a vertex with neighbouring pairs joined along an edge; the end s-gons have only 1 neighbour (see Figure 5.5). We say that this block is *flanked* by a_d.

The s-gons are glued together along the generators of order 2. We have an s-gon starting with each generator and an s-gon ending with each generator (so that the reverse of the s-gon starts with the inverse of a generator). Since generators of order 2 are their own inverse, the end of one s-gon is the start of the next one, so they stick along generators of order 2.

Splitting the d-block—the (d − 1, 1)-block Given generators $a_1, \ldots, a_d$ with $a_1, \ldots, a_{d-1}$ being involutions, and the relator $r = (a_1 \cdots a_d)^{s/d}$, we get the d-block as above. If we remove the condition that a_1 has order 2, so that

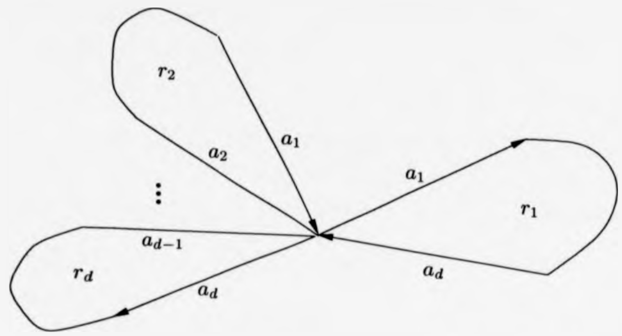


Figure 5.6. The $(d-1,1)$-block

it has (apparent) infinite order, then the relators no longer stick together along a_1 and the block splits into 2 parts, one containing a single relator, the other being a $(d-1)$-slab (Figure 5.6).

Note that, if we make a_j $(1 < j < d)$ not have order 2, instead of a_1, then the d-block splits into a $(d-j, j)$-block.

The (d + k)-block We can reconnect the $(d-1,1)$-block given above by adding in the relator a_1^s. The relator a_1^s joins the 2 components of the $(d-1,1)$-block giving a $(d+1)$-block (Figure 5.7).

We can repeat this procedure by changing the condition $a_2^2 = 1$ to $a_2^s = 1$. Inductively, we build a $(d+k)$-block with generators $a_1, \dots, a_d$ and relators $(a_1 \cdots a_d)^{s/d}$, $a_1^s, \dots, a_k^s$, $a_{k+1}^2, \dots, a_{d-1}^2$ (a_d has infinite order). The $(d+k)$-block is flanked by a_d (Figure 5.8).

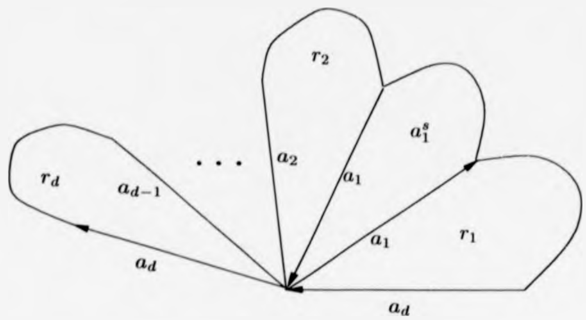


Figure 5.7. The $(d + 1)$-block

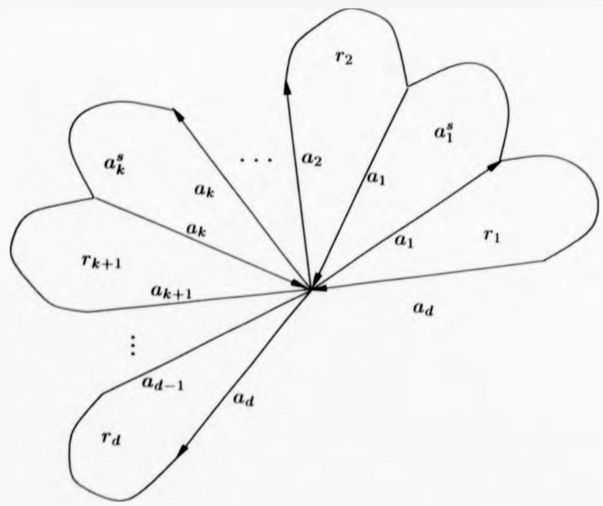


Figure 5.8. The $(d + k)$-block

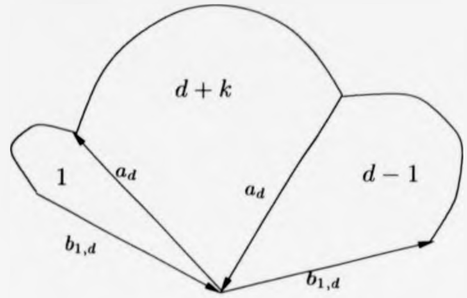


Figure 5.9. The $(2d+k)$-block

Gluing blocks—the (ld + k)-block ($l \geqslant 1$) Given $l \geqslant 1$ and $0 \leqslant k < d$, we can construct an $(ld+k)$-block using a $(d+k)$-block and $l-1$ $(d-1,1)$-blocks.

We start with a $(d+k)$-block given by the generators $a_1, \ldots, a_d$ and the relators $(a_1 \cdots a_d)^{s/d}$, $a_1^s, \ldots, a_k^s$, $a_{k+1}^2, \ldots, a_{d-1}^2$.

Take a $(d-1,1)$-block given by the generators $b_{1,1}, \ldots, b_{1,d}$ and the relators $(b_{1,1} \cdots b_{1,d})^{s/d}$, $b_{1,2}^2, \ldots, b_{1,d-1}^2$.

Glue them together by adding the relation $a_d = b_{1,1}$. This gives a $(2d+k)$-block (Figure 5.9).

We continue inductively: Add the $(d-1,1)$-blocks given by generators $b_{i,1}, \ldots, b_{i,d}$ and relators $(b_{i,1} \cdots b_{i,d})^{s/d}$, $b_{i,2}^2, \ldots, b_{i,d-1}^2$ by gluing with the relation $b_{i-1,d} = b_{i,1}$. We stop when we have an $(ld+k)$-block.

Having constructed an $(ld+k)$-block flanked by $b_{l-1,d}$, we add the relator

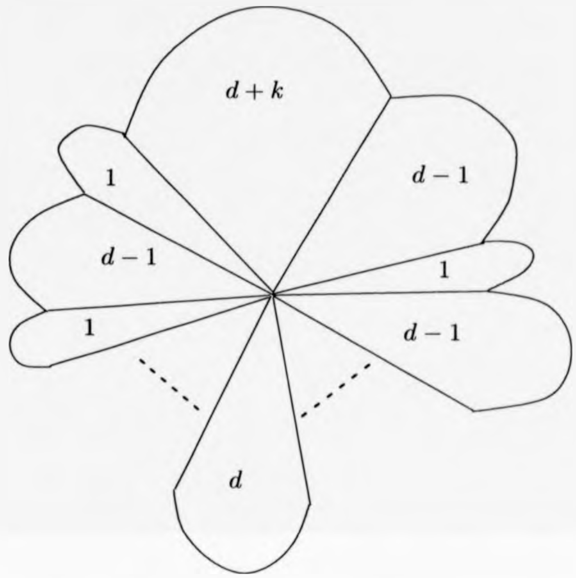


Figure 5.10. The block diagram (the 'unflanked' $(ld + k)$-block)

$b^2_{l-1,d}$. This 'zips' the $(ld+k)$-block to give an 'unflanked' $(ld+k)$-block; that is, $(ld+k)$ s-gons meeting at a vertex (Figure 5.10). We call Figure 5.10 the *block diagram.*

The dual polygon Given the local picture in Figure 5.10, we must show that this extends to the correct global picture.

Draw the polygon dual to this local picture. We have a vertex for each relator and an edge for each generator from the central vertex (Figure 5.11). Make each angle $2\pi/s$. Call this polygon P.

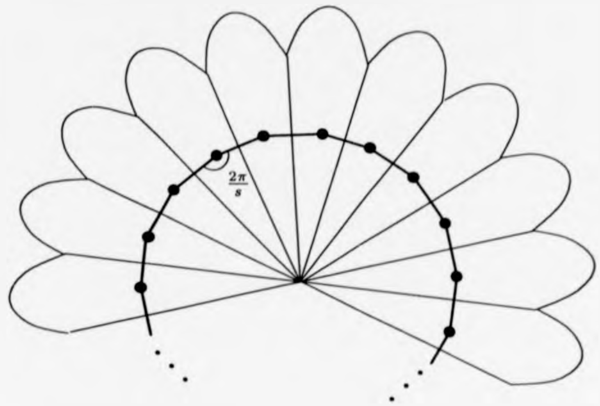


Figure 5.11. The dual polygon

Each generator (or inverse) x is naturally associated to the edge which crosses it. We give this edge the label $e(x)$. Each vertex, $v = v(r)$, is labelled by reading the relator, r, of length s in which it is contained. We read the relator clockwise from the central vertex. Note that if we read anticlockwise, we would get the free inverse of r.

Define the side-pairings by $a : e(a^{-1}) \mapsto e(a)$ (see Figures 5.12, 5.13 and 5.14). We will choose the side pairings to be orientation preserving. In the case when a has order 2, the side pairing transformation is to rotate through π about the centre of the edge $e(a)$. If $e(a)$ and $e(a^{-1})$ have a vertex in common, we take the side-pairing transformation to be a rotation through $2\pi/s$ about the common vertex. (This happens if and only if a has order

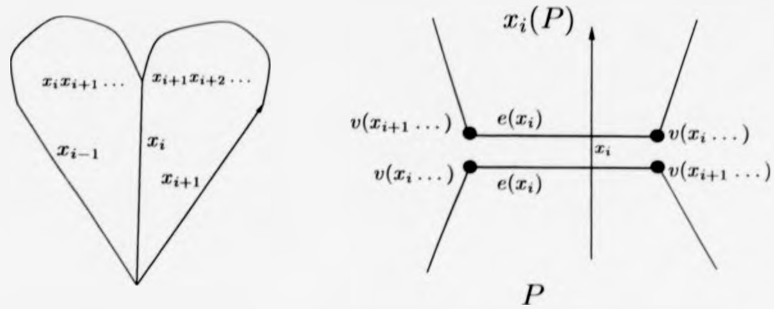


Figure 5.12. The side pairing for a generator x_i of order 2 (rotate through the centre of the edge $e(x_i)$)

s.) Otherwise, we take the side-pairing transformation to be a translation. The three different types of side-pairing are illustrated in Figures 5.12, 5.13 and 5.14. The pictures could be mirror images of the ones shown, in which case each relator (and therefore vertex label) gets replaced by its free inverse. Each relator has a vertex labelled either by itself or by its free inverse. In Figure 5.14, when $j = 1$, $b_{j-1,x} = a_x$.

Clearly, the reflection relations give the relations $a_i^2 = e$, $(k + 1 \leqslant i \leqslant d - 1)$ and $b_{j,i}^2 = e$ $(1 \leqslant j \leqslant l - 1, 2 \leqslant i \leqslant d - 1$ or $j = l - 1,\ i = d)$.

We now examine the cycle relations.

Consider a vertex of the form $v(a_i^s)$ or $v((a_i^{-1})^s)$. Observe that $a_i : v(a_i^s) \mapsto v(a_i^s)$ and $a_i^{-1} : v((a_i^{-1})^s) \mapsto v((a_i^{-1})^s)$. Working clockwise around such a vertex, we get the cycle relation $a_i^s = e$ or $(a_i^{-1})^s = e$.

For $0 \leqslant j < l$, let $r = b_{j,1} b_{j,2} \cdots b_{j,d}$ $(b_{0,x} = a_x)$, Recall (Definition 5.2.8)

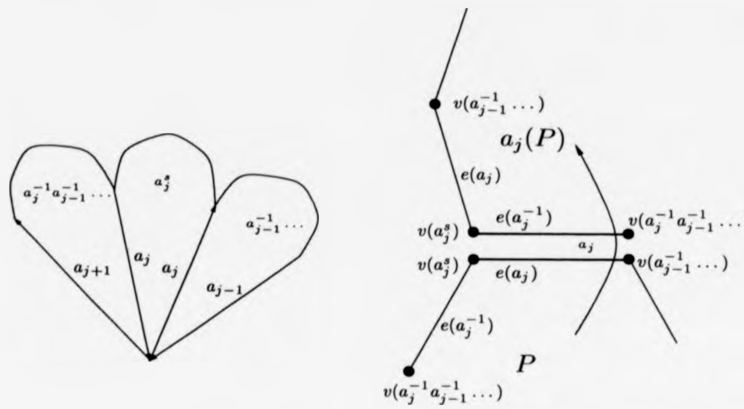


Figure 5.13. The side pairing of a generator a_j of order s (rotate through the vertex $v(a_j^s)$)

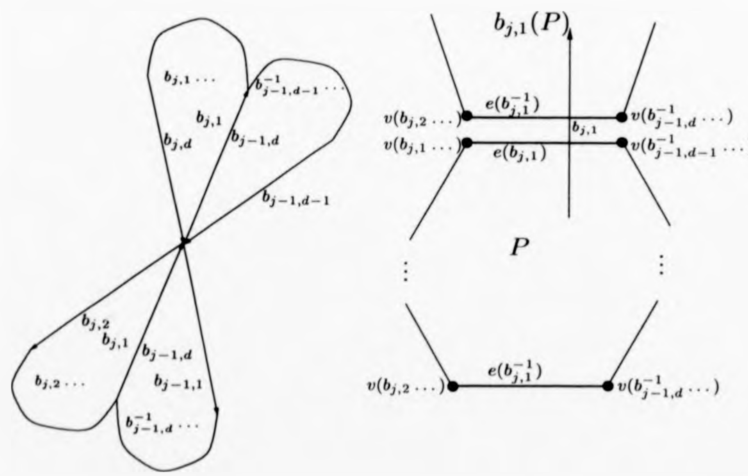


Figure 5.14. The side pairing of an infinite order generator $b_{j,1}$ (translate)

That r_i denotes a cyclic conjugate of r. Observe that $b_{j,i} : v(r_i) \mapsto v(r_{i+1})$. Working clockwise around the vertex $v(r)$, we get the relation $b_{j,1}b_{j,2}\cdots b_{j,d}$. Similarly, working about r^{-1}, we get the relation $b_{j,d}^{-1}b_{j,d-1}^{-1}\cdots b_{j,1}^{-1}$.

These are the only types of vertex which appear. Combining the reflection and cycle relations, Poincaré's polygon theorem (Theorem 5.3.4) tells us that the group

$$\left\langle \begin{array}{c|c} a_1,\ldots,a_d, & a_1^s,\ldots,a_k^s,a_{k+1}^2,\ldots,a_{d-1}^2,(a_1a_2\cdots a_d)^{s/d} \\ b_{1,1},\ldots,b_{1,d}, & b_{1,2}^2,\ldots,b_{1,d-1}^2,(b_{1,1}b_{1,2}\cdots b_{1,d})^{s/d},a_d=b_{1,1} \\ b_{2,1},\ldots,b_{2,d}, & b_{2,2}^2,\ldots,b_{2,d-1}^2,(b_{2,1}b_{2,2}\cdots b_{2,d})^{s/d},b_{1,d}=b_{2,1} \\ \vdots & \vdots \\ b_{l-1,1},\ldots,b_{l-1,d} & b_{l-1,2}^2,\ldots,b_{l-1,d}^2,(b_{l-1,1}b_{l-1,2}\cdots b_{l-1,d})^{s/d} \end{array} \right\rangle$$

has as its Cayley graph the dual to the tiling of the dual polygon. Thus we have a presentation of a group whose Cayley graph is s-gons meeting $v = ld + k$ to a vertex.

We use $B(s, v, d)$ to denote the group made by this construction.

In the case when $d = s$ and $l > 1$, the subgroup generated by the $(d+k)$-block is $\mathbb{Z}_s * \cdots * \mathbb{Z}_s * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$; the free product of k copies of $\mathbb{Z}_s$ and $d-k-1$ copies of $\mathbb{Z}_2$. The subgroup generated by a $(d-1,1)$-block is $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2 * \mathbb{Z}$; the free product of $d-2$ copies of $\mathbb{Z}_2$ and 1 copy of $\mathbb{Z}$. The subgroup generated by an end d-block is $\mathbb{Z}_2 * \cdots * \mathbb{Z}_2$; the free product of $d-1$ copies of $\mathbb{Z}_2$. The relators $b_{i,d} = b_{i+1,1}$ and $a_d = b_{1,1}$ show that we amalgamate these subgroups over $\mathbb{Z}$. The copy of $\mathbb{Z}$ which we amalgamate over is, in each case, either the product of the other generators in order, using each generator exactly once, or the generator of the $\mathbb{Z}$ part of the free product.

5.3.4 The Case When s is Odd

Proposition 5.3.7: *Let s be odd and d be the smallest divisor of s which is strictly greater than 1. If $v \geqslant d$, then $\mathcal{G}(s;v)$ is the Cayley graph of a group.*

Proof: Use the above construction with d as the smallest factor of s. ❑

Note that we have several choices as to how to perform such a construction. We could replace any $(d-1,1)$-block with a $(d-j,j)$-block (for any $1 \leqslant j < d$) as long as we glue adjacent blocks along the torsion free generators. The k generators of order s could, in fact, be *any* of the order 2 generators—as long as there were precisely k of them. Also, we have the choice of any divisor d of s such that $3 \leqslant d \leqslant v$.

Combining Propositions 5.3.5, 5.3.7 and 5.3.3 we get:

Theorem 5.3.8:
Let $s, v \in \mathbb{N}$ with $v \geqslant 3$. Let d be the smallest divisor of s which is strictly greater than 1. Then $\mathcal{G}(s;v)$ is the Cayley graph of a group if and only if $v \geqslant d$. ❑

Remark: This result was proved independently in [CK96].

5.3.5 Some Other Examples

Example 5.3.9: Suppose that $v|s$, so that $s = pv$ for some p. Then $\mathcal{G}(pv;v)$

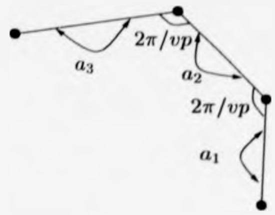


Figure 5.15. The side pairing for $D(pv, v)$.

is the Cayley graph of the group

$$D(pv, v) = \langle a_1, \ldots, a_v \mid a_i^2, (a_1 a_2 \cdots a_v)^p \rangle.$$

Proof: Consider the v-gon with angles $\frac{2\pi}{vp}$ and side pairing given by rotations by π about the centre of each edge (Figure 5.15). Then the group generated by these side pairing transformations is $D(pv, v)$ and the Cayley graph is pv-gons meeting v to a vertex. ❑

Observation 5.3.10: *Note that this is the block construction (Section 5.3.3) with $d = v$, $l = 1$ and $k = 0$.*

The group $D(3, 3)$ is $\langle a, b, c \mid a^2, b^2, c^2, abc \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ (see Graph 5).

Example 5.3.11: If s is odd then $\mathcal{G}(s; 2s)$ is the Cayley graph of the group

$$F(s, 2s) = \langle a_1, \ldots, a_s \mid a_1 a_2 \cdots a_s, a_s a_{s-1} \cdots a_1 \rangle.$$

Proof: Consider the $2s$-gon with angles $2\pi/s$ with the 'opposite' side pairing. If we label the vertices $v_1, \ldots, v_{2s}$, working clockwise from a vertex, then the

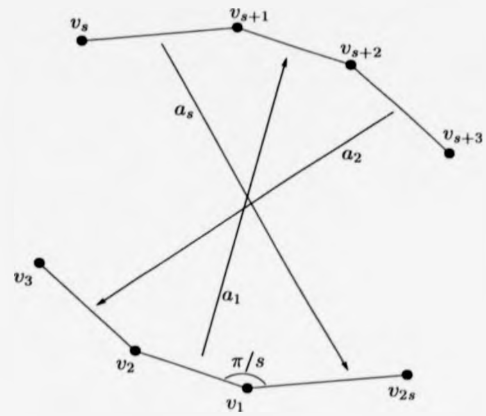


Figure 5.16. The side pairing for $F(s, 2s)$ where s is odd.

edges are (v_i, v_{i+1}) (cyclically) and the side pairings are (see Figure 5.16)

$$\begin{array}{ll} a_i : (v_i, v_{i+1}) \mapsto (v_{s+i+1}, v_{s+i}) & \text{if } i \text{ is odd} \\ a_i : (v_{s+i}, v_{s+i+1}) \mapsto (v_{i+1}, v_i) & \text{if } i \text{ is even.} \end{array}$$

We obtain the cycle relation $a_1 a_2 \cdots a_s = 1$ about the odd numbered vertices and the r cycle relation $a_s a_{s-1} \cdots a_1 = 1$ about the even numbered vertices (Figure 5.17). Thus, we have that the group $F(s, 2s) = \langle a_1, \ldots, a_s \mid a_1 a_2 \cdots a_s, a_s a_{s-1} \cdots a_1 \rangle$ has $\mathcal{G}(s; 2s)$ as its Cayley graph. ❑

The group $F(3, 6)$ is $\langle a, b, c \mid abc, cba \rangle = \langle a, b \mid ab = ba \rangle = \mathbb{Z} \times \mathbb{Z}$.

The block of $a_1 \cdots a_s$ and the block of $a_s \cdots a_1$ are both $(1, 1, \ldots, 1) -$ *blocks* (where there are s 1's). The blocks are interwoven.

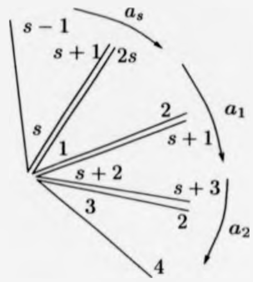


Figure 5.17. The relators for $F(s, 2s)$ where s is odd.

5.4 Orbifolds

A surface can be thought of as a the quotient of a properly-discontinuous, free action of a group on 2-dimensional (spherical, Euclidean or hyperbolic) space. We can reconstruct the group action from the quotient. When the group action is properly discontinuous but not free, we need some additional structure (on the fixed points) to recover the group action. The quotient space together with this additional structure is called an *orbifold*.

We describe the orbifolds associated to the group actions given in Section 5.3. In particular, we give their genus and describe the singular points. Recall ([Thu90, Proposition 5.4.2]) that there are 3 types of singular locus:

Mirror lines; the orbifold is locally $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflection.

Cone point of order n; the orbifold is locally cone shaped, $\mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts by rotation through $2\pi/n$.

Corner reflections of order n; the orbifold is locally $\mathbb{R}^2/D_{2n}$. The action is generated by reflections in two lines which meet at an angle of π/n.

We use $E(s,v)$, $B(s,v,d)$, $D(pv,v)$ and $F(s,2s)$ to denote the groups constructed in Section 5.3 together with their actions generated by the side transformations on the Poincaré tilings of the appropriate 2-dimensional space. We use $Q(G)$ to denote the orbifold of the group G with its associated action.

5.4.1 The Orbifold Associated to $\mathbf{E(s,v)}$

Recall that the group $E(s,v)$ acts on the tiling by v-gons with interior angles $2\pi/s$. The action is generated by reflections in the edges of this polygon. The orbifold $Q(E(s,v))$ is a v-gon. All edges are mirror lines. Each vertex is a corner reflection of order $s/2$. Topologically, the orbifold is a closed disk.

5.4.2 The Orbifold Associated to $\mathbf{B(s,v,d)}$

The group $B(s,v,d)$ acts on the tiling by v-gons with interior angles $2\pi/s$. Recall the block diagram (Figure 5.10). The block diagram separates naturally into slabs of size $d+k$, $d-1$, d or 1. Each vertex of the tiling lies in one of these slabs. An edge either lies within a slab or joins two slabs. Recall the labelling of vertices and edges:

Observation 5.4.1: *An edge lies entirely within a slab if and only if it is labelled by a torsion generator (that is, a generator of order s or 2).*

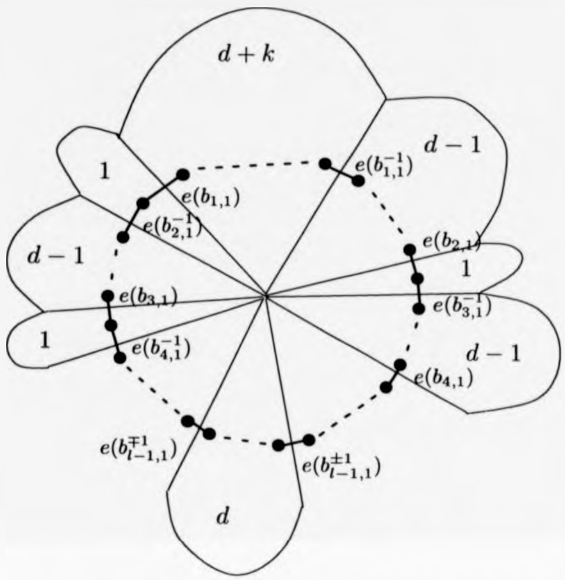


Figure 5.18. The block diagram revisited

Observation 5.4.2: *The edges labelled by non-torsion generators join two slabs. The non-torsion generators are labelled by* $b_{j,1}$ $(1 \leqslant j \leqslant l-1)$ *(if* $l = 1$, *then there are no non-torsion generators). If we read the edges corresponding to non-torsion generators clockwise from the* $(d+k)$*-block, we get the sequence* $e(b_{1,1}^{-1})$, $e(b_{2,1}), \ldots, e(b_{l-1,1}^{\pm 1})$, $e(b_{l-1,1}^{\mp 1}), \ldots, e(b_{2,1}^{-1}), e(b_{1,1})$ *(see Figure 5.18).*

Consider a vertex of a relator of the form a_i^s. In the quotient orbifold, this vertex becomes a cone point of order s (the action of the stabiliser of this vertex $(\mathbb{Z}_s = \langle a_i \rangle)$ on a neighbourhood of this vertex is rotation through $2\pi/s$). The edges $e(a_i)$ and $e(a_i^{-1})$ become identified (Figure 5.19).

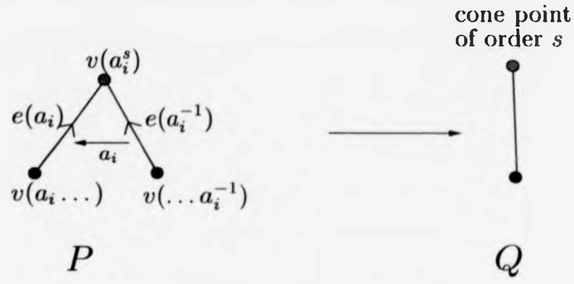


Figure 5.19. A cone point of order s

Consider $e(x_i)$ where x_i is a generator of order 2. There is a fixed point at the centre c_i of the edge $e(x_i)$. The action of the stabiliser of c_i ($\mathbb{Z}_2 = \langle x_i \rangle$) is a rotation through π. So, in the quotient orbifold, we get a cone point of order 2. The 2 half edges of $e(x_i)$ are identified (Figure 5.20).

Fix the integer j, $0 \leqslant j \leqslant l-1$. Consider the vertices $v(b_{j,1}b_{j,2}\cdots b_{j,d})$, $v(b_{j,2}\cdots b_{j,d}b_{j,1}), \dots, v(b_{j,d}b_{j,1}\cdots b_{j,d-1})$ (again, $b_{0,i} = a_i$). These form a single orbit. There are d 'slices', each with an angle of $2\pi/s$. Therefore, in Q, it becomes a cone point of order s/d.

Observation 5.4.3: *In the quotient orbifold $Q(B(s,v,d))$; the $(d+k)$-slab becomes a vertex with k cone points of order s and $d-k-1$ cone points of order 2 attached, each $(d-1)$-slab becomes a vertex with $d-2$ cone points of order 2 attached, the d-slab becomes a vertex with $d-1$ cone points of order 2 attached. Each of these vertices is a cone point of order s/d. The edges joining these vertices (labelled by the non-torsion generators) are identified cyclically so that $Q(B(s,v,d))$ is topologically a sphere. (See Figure 5.21.)*

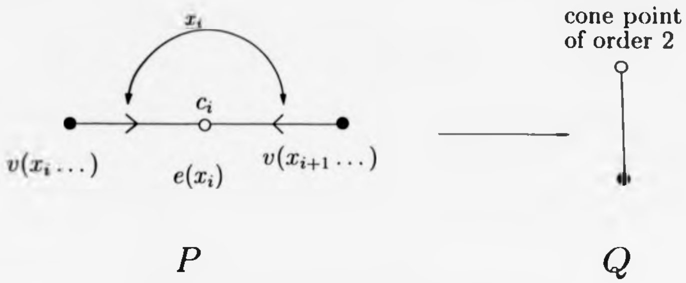


Figure 5.20. A cone point of order 2

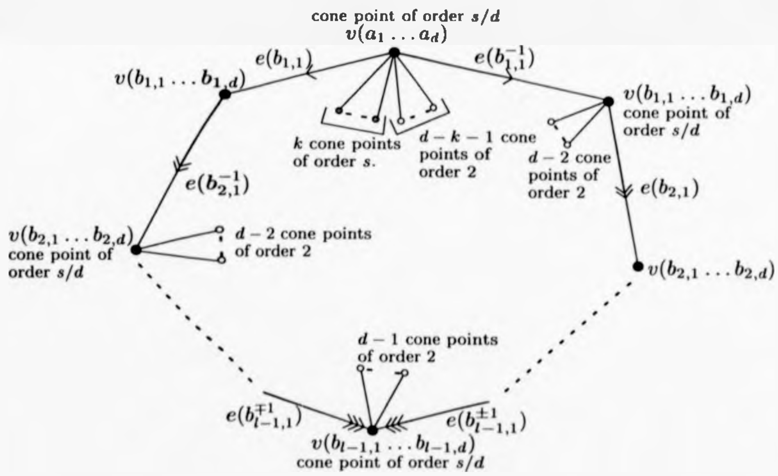


Figure 5.21. The orbifold $Q(B(s, v, d))$

Proposition 5.4.4: *The orbifold associated to the action of $B(s, v, d)$ ($s = ld + k$, ($l \geqslant 1$, $0 \leqslant k < d$)) on the tiling by v-gons with interior angles $2\pi/s$ is a topological sphere with k cone points of order s, l cone points of order s/d and $(d - k - 1 + (l - 1) \times (d - 2) + 1) = l(d - 2) - k + 2$ cone points of order 2.* ❑

5.4.3 The Orbifold Associated to D(pv, v)

The group $D(pv, v)$ acts on the tiling by v-gons with interior angles $2\pi/pv$ (Figure 5.15). The action is generated by rotations through π about the centre of each edge. The centre of each edge is a cone point of order 2. The vertices form a single orbit, which is a cone point of order p in the quotient orbifold. The orbifold $Q(D(pv, v))$ is a topological sphere with v cone points of order 2 and 1 cone point of order p. This also follows from Observation 5.3.10 and Proposition 5.4.4.

5.4.4 The Orbifold Associated to F(s, 2s)

The group $F(s, 2s)$ acts on the tiling by $2s$-gons with interior angles $2\pi/s$ (see Figure 5.16 on page 144). There are no fixed points and so the quotient orbifold is a surface.

Proposition 5.4.5: *The quotient surface $Q(F(s, 2s))$ is a torus of genus* $\dfrac{s-1}{2}$.

Proof: The side identifications needed to obtain $Q(F(s, 2s))$ are shown in

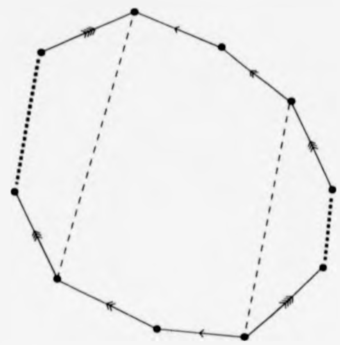


Figure 5.22. Identifications to obtain $Q(F(s, 2s))$

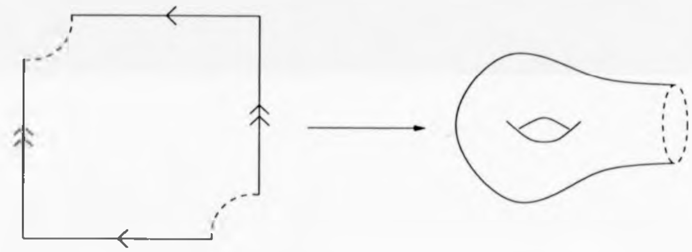


Figure 5.23. The first piece of $Q(F(s, 2s))$

Figure 5.22. We proceed inductively.

The surface $Q(F(3,6))$ is the torus. This follows from the fact that $F(3,6) \cong \mathbb{Z} \times \mathbb{Z}$. It is also easy to check directly from the identifications.

For the inductive step, cut out the first two side identifications ($\longrightarrow$ and $\twoheadrightarrow$) along the dashed line as shown in Figure 5.22. The surface splits into two pieces. The first piece is a torus with a disk removed (see

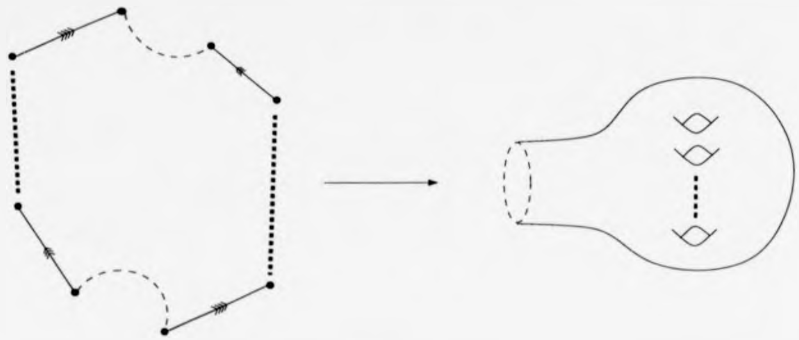


Figure 5.24. The second piece of $Q(F(s, 2s))$

Figure 5.23). The second piece is $Q(F(s-2, 2(s-2)))$ with a disk removed (see Figure 5.24). By the inductive hypothesis, this is a torus of genus $\frac{s-3}{2}$ with a disk removed. The space $Q(F(s, 2s))$ is obtained by gluing these two spaces along the boundary of the removed disk. Thus $Q(F(s, 2s))$ is a torus of genus $1 + \frac{s-3}{2} = \frac{s-1}{2}$ as required. ❑

5.5 Semi-Regular Tilings

A semi-regular tiling of 2-dimensional space is a tiling by regular polygons which are not all of the same type. We concentrate on 2 particular classes of semi-regular tilings; Archimedean solids and quasi-regular tilings.

5.5.1 Archimedean Solids

An Archimedean solid, or semi-regular polyhedron, is a body inscribed in a sphere, bounded by regular polygons which are not all of the same type, and such that its symmetry group acts transitively on the vertices. A tiling of the sphere can be obtained from an Archimedean solid by projecting the edges from the centre of the solid onto the sphere which contains its vertices. Combinatorially, the tiling and the Archimedean solid are identical. It is well known that there are 13 Archimedean solids. We classify which of these 13 polyhedra (shown in Figure 5.25) have the Cayley graph of a group as their 1-skeleton.

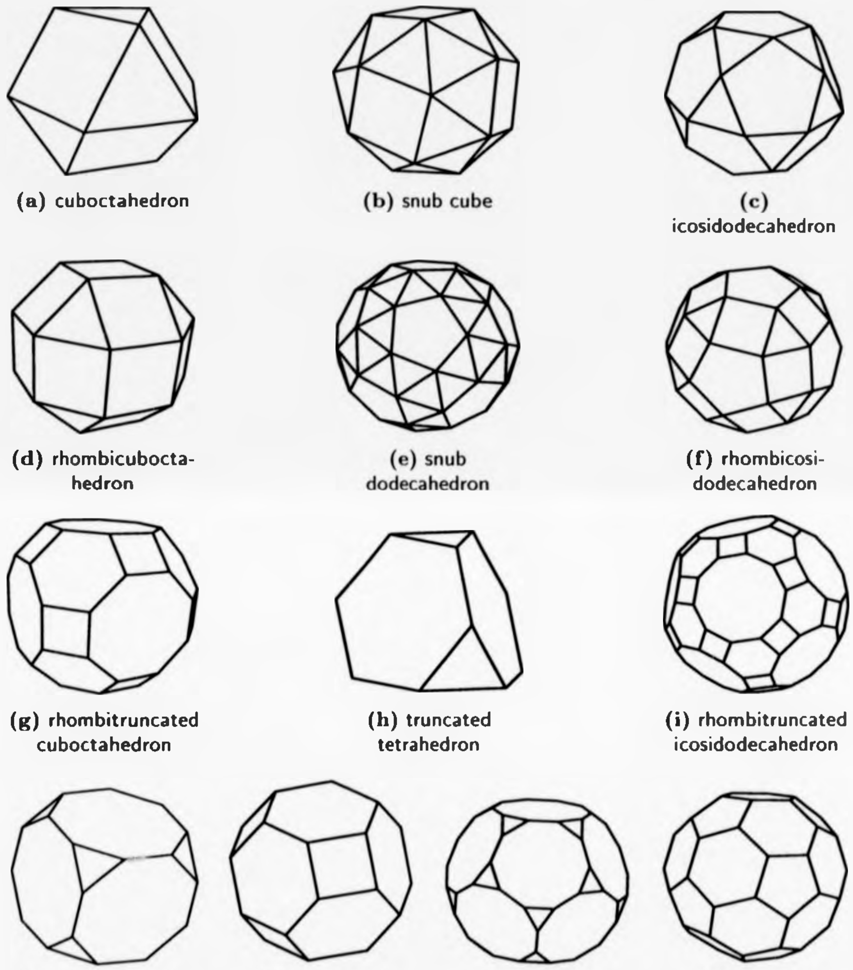
The arguments use the machinery introduced in Sections 5.2.4–5.2.6. In particular, we use relator types to restrict the orders of the generators.

Cuboctahedron (Figure 5.25(a)): The cuboctahedron has 2 triangles and 2 squares at each vertex and has valency 4. These are the only simple loops of lengths 3 and 4. Therefore, there must be 2 generators, a, b, each having order 3. There are not any generators of order 2 or 4, therefore, by Corollary 5.2.15, there cannot be a relator class of length 4 and type 1. Therefore, there must be one relator class of length 4 and it must have type 2. The group

$$A_4 = \langle a, b \mid a^3, b^3, (ab)^2 \rangle$$

is the only group with the cuboctahedron as its Cayley graph (compare with Graph 13).

Snub cube (Figure 5.25(b)): The snub cube has 4 triangles at each vertex



(a) cuboctahedron

(b) snub cube

(c) icosidodecahedron

(d) rhombicuboctahedron

(e) snub dodecahedron

(f) rhombicosidodecahedron

(g) rhombitruncated cuboctahedron

(h) truncated tetrahedron

(i) rhombitruncated icosidodecahedron

(j) truncated cube

(k) truncated octahedron

(l) truncated dodecahedron

(m) truncated icosahedron

Figure 5.25. The Archimedean solids. These pictures are screen snapshots from geomview [LMP96].

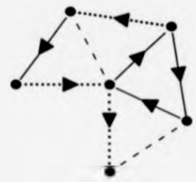


Figure 5.26. An unflanked 4-block

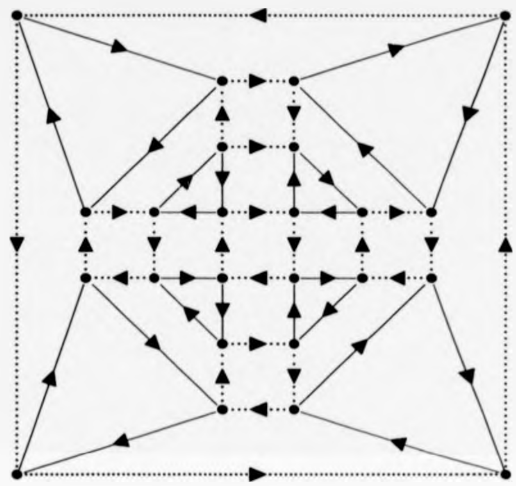
and has valency 5. The triangles are the only simple loops of length 3 (there are 7 simple loops of length 4; 1 of them is a square, the other 6 are boundaries of adjacent triangles). Therefore we must have exactly one generator, a, of order 3. The generator a must be part of another relator of length 3, abc say. We cannot have $b = c$ or $b^2 = c^2 = 1$ because then we would have an octahedron subgraph (compare with Graphs 11 and 12). The snub cube has valency 5, so, without loss of generality, c is an involution. The block of the set of relators $R(a^3) \cup R(c^2) \cup R(abc)$ (where $R(r)$ denotes the relator class of r) is an unflanked 4-block (see Figure 5.26). The relator class of $abab$ has type 2 and the relator class of $aacb^{-1}$ has type 4. Therefore, the remaining relator class of length 4 (the one which goes round the square) must have type 1; it must be b^4. Thus we have that the group

$$S_4 = \langle a, b, c \mid a^3, b^4, c^2, abc \rangle$$

is the only group which has the snub cube as its Cayley graph (compare with Graph 14). A similar argument can be used for the snub dodecahedron.

Icosidodecahedron (Figure 5.25(c)): The icosidodecahedron is not the Cayley graph of a group. Suppose otherwise. There are 2 triangles at each vertex; thus 2 generators of order 3. There are 5 pentagons at each vertex;

Cayley Graph 14. $S_4 = \langle a, b \mid a^3, b^4, (ab)^2 \rangle$



thus 2 generators of order 5. This contradicts the fact that the icosidodecahedron has valency 4. (Compare with Proposition 5.5.3.)

Rhombicuboctahedron (Figure 5.25(d)): The rhombicuboctahedron has 3 squares and 1 triangle at each vertex. These are the only simple loops of lengths 3 and 4. Thus we have a generator of order 3. The generator of order 3 must appear in a relator class of length 4, which, by Corollary 5.2.15, must have type 2. Therefore, the only possibility is

$$S_4 = \langle a, b \mid a^3, b^4, (ab)^2 \rangle,$$

(see Graph 14). A similar argument applies to the rhombicosidodecahedron.

We list the remaining cases (Isom(X) denotes the group of isometries of X

and $\mathrm{Isom}^+(X)$ denotes the group of orientation preserving isometries of X):

Snub dodecahedron (Figure 5.25(e)):

$$A_5 = \langle a, b, c \mid a^3, b^5, c^2, abc \rangle.$$

Rhombicosidodecahedron (Figure 5.25(f)):

$$A_5 = \langle a, b \mid a^3, b^5, (ab)^2 \rangle.$$

Rhombitruncated cuboctahedron (Figure 5.25(g)):

$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^3, (bc)^4 \rangle = \mathrm{Isom(cube)}$$
$$= \mathrm{Isom(octahedron)} = \mathrm{Isom(cuboctahedron)}.$$

Truncated tetrahedron (Figure 5.25(h)):

$$A_4 = \langle a, b \mid a^3, b^2, (ab)^3 \rangle = \mathrm{Isom}^+(\mathrm{tetrahedron}).$$

Rhombitruncated icosidodecahedron (Figure 5.25(i)):

$$\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^3, (bc)^5 \rangle = \mathrm{Isom(icosahedron)}$$
$$= \mathrm{Isom(dodecahedron)} = \mathrm{Isom(icosidodecahedron)}.$$

Truncated cube (Figure 5.25(j)):

$$S_4 = \langle a, b \mid a^2, b^3, (ab)^4 \rangle = \mathrm{Isom}^+(\mathrm{cube}).$$

Truncated octahedron (Figure 5.25(k)):

$$S_4 = \langle a, b \mid a^2, b^4, (ab)^3 \rangle = \mathrm{Isom}^+(\mathrm{octahedron})$$
$$\cong \langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (ac)^3, (bc)^3 \rangle = \mathrm{Isom(tetrahedron)}.$$

Truncated dodecahedron (Figure 5.25(l)):

$$A_5 = \langle a, b \mid a^2, b^3, (ab)^5 \rangle = \mathrm{Isom}^+(\mathrm{dodecahedron}).$$

Truncated icosahedron (Figure 5.25(m)):

$$A_5 = \langle a, b \mid a^2, b^5, (ab)^3 \rangle = \mathrm{Isom}^+(\text{icosahedron}).$$

In each case, the group given is the unique group which has a Cayley graph which is the 1-skeleton of the desired polyhedron. Further, with the exception of the truncated octahedron, the choice of generators is unique up to automorphism and replacement by inverses.

The generating set which has the rhombitruncated cuboctahedron as Cayley graph is unique because there have to be relator classes of type 1 of lengths 4, 6 and 8 with edges in common. There cannot be a generator of order 4 because it would have to be part of a relator class of length 6 and type 1, which would contradict Corollary 5.2.15. Similarly, there cannot be generators of order 6 or 8. The same reasoning applies to the rhombitruncated icosidodecahedron.

The truncated tetrahedron, cube, dodecahedron and icosahedron all must have one generator of order 3 (order 5 in the case of the truncated icosahedron) and one generator of order 2. The rest is decided from there. For the truncated octahedron, there are 2 ways of making the single square. The remaining generator has order 2 and the rest is decided.

Theorem 5.5.1: *With the exception of the icosidodecahedron, the 1-skeleton of an Archimedean solid is the Cayley graph of a group. There is a unique group which has the 1-skeleton of a given Archimedean solid as a Cayley graph.* ❑

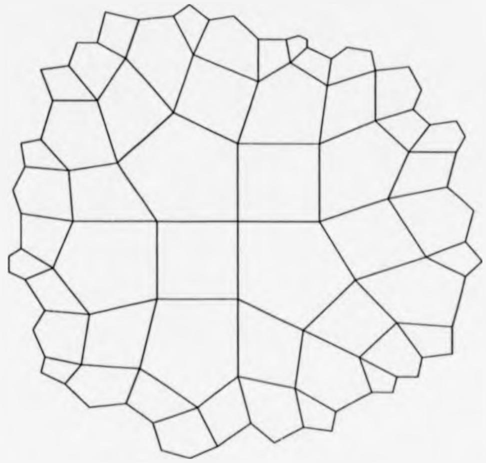


Figure 5.27. A quasi-regular tiling by squares and pentagons

5.5.2 Quasiregular Tilings

A *quasiregular* tiling is a tiling by 2 types of regular polygon, s-gons and t-gons, with v of each at each vertex, and such that each s-gon is adjacent only to t-gons and each t-gon is adjacent only to s-gons (see, for example, Figures 5.27 and 5.28) . We refer to the 1-skeleton of such a tiling as $\mathcal{Q}(s,t;v)$. Note that $\mathcal{Q}(s,t;v) = \mathcal{Q}(t,s;v)$. The 1-skeleton of the cuboctahedron is $\mathcal{Q}(3,4;2)$, the 1-skeleton of the icosidodecahedron is $\mathcal{Q}(3,5;2)$.

Observation 5.5.2: *The only simple loops of $\mathcal{Q}(s,t;v)$ of length s are s-gons.*

Throughout this section, $n_s = \min\{v \mod d \mid d|s\}$, d_s is a divisor of s such

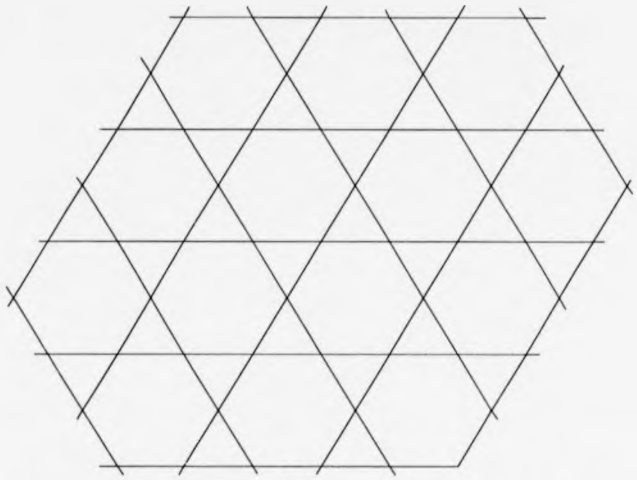


Figure 5.28. A quasi-regular tiling by triangles and hexagons

that $n_s = v \mod d_s$, $n_t = \min\{v \mod d \mid d|t\}$ and d_t is a divisor of t such that $n_t = v \mod d_t$.

The first result is the analogue of Proposition 5.3.3:

Proposition 5.5.3: *If $n_s + n_t > v$ then $Q(s,t;v)$ is not the Cayley graph of a group.*

Proof: First note that if s and t are both even, then $n_s + n_t \leqslant 2 \leqslant v$. So we may assume that s is odd. Suppose that $Q(s,t;v)$ is the Cayley graph of a group.

Claim: *There are no generators of order 2.*

Proof: Suppose that a is a generator of order 2. Then a is part of a relator, $r = ab_2b_3\cdots b_s$, of length s. Now, $ar^{-1}a = ab_s^{-1}\cdots b_2^{-1}$. So, if $ar^{-1}a \equiv r$, then $b_2 = b_s^{-1}$, $b_3 = b_{s-1}^{-1}, \ldots, b_k = b_{k+1}^{-1}$, where $k = (s-1)/2$, which contradicts the fact that r is freely reduced. Therefore $ar^{-1}a$ and r are distinct relators of length s which have an edge in common. Thus the Cayley graph has a pair of adjacent s-gons. This cannot happen. □

By Lemmas 5.2.10, 5.2.13, Proposition 5.2.12, Observation 5.5.2 and the above Claim, there must be at least n_s generators of order s and at least n_t generators of order t. Therefore $Q(s,t;v)$ has valency at least $2(n_s+n_t) > 2v$, which is a contradiction. ❑

Proposition 5.5.4: *Suppose that $n_s = v$ and that $v \nmid t$, then $Q(s,t;v)$ is not the Cayley graph of a group*

Proof: Suppose that $Q(s,t;v)$ is the Cayley graph of a group and that $n_s = v$. Then s is odd and there must be $n_s = v$ generators, each having order s. Since $v \nmid t$, Lemma 5.2.10 implies that there cannot be relators of length t and type v. Therefore, there is a relator of length t which involves m generators, where $m < v$. This gives us a flanked $2k$-block, for some $k < m$ $(k|m)$. This cannot happen if the Cayley graph is $Q(s,t;v)$. ❑

Proposition 5.5.5: *If s and t are both even then $Q(s,t;v)$ is the Cayley graph of the group*

$$E(s,t;v) = \langle a_1,\ldots,a_v,\ b_1,\ldots,b_v \mid a_i^2, b_i^2, (a_ib_i)^{s/2}, (b_ia_{i+1})^{t/2}\rangle.$$

If $v|s$ *then* $Q(s,t;v)$ *is the Cayley graph of the group*

$$D(s,t;v) = \langle a_1, \ldots, a_v \mid a_i^t, (a_1 \cdots a_v)^{s/v}.$$

If $v = 2$ *and* s *is even, then* $Q(s,t;v)$ *is the Cayley graph of the group*

$$T(s,t;2) = \langle a, b \mid a^t, b^t, (ab)^{s/2} \rangle.$$

The group $T(4,3;2) = A_4$ has the 1-skeleton of the cuboctahedron as its Cayley graph.

Conjecture 5.5.6: *If* $Q(s,t;v)$ *satisfies neither the conditions for Proposition 5.5.3 nor the conditions for Proposition 5.5.4 then* $Q(s,t;v)$ *is the Cayley graph of a group.*

Remark: One method of construction is to take n_s generators of order s, n_t generators of order t and $v - n_s - n_t$ generators of infinite order. We make relators of length s (a string of d_s distinct symbols raised to the power s/d_s) out of the generators which do not have order s and relators of length t out of the generators which do not have order t. We try to choose these relators so that they fit together nicely to form $Q(s,t;v)$ as the Cayley graph. This method has worked for every example that I have tried, but there does not seem to be a general method for choosing the relators. For example, $Q(5,7;8)$ is the Cayley graph of the group

$$\left\langle \begin{array}{c} a, b, c \\ d \\ e, f, g, h \end{array} \middle| \begin{array}{c} a^5, b^5, c^5 \\ d^7 \\ defgh, abcghfe \end{array} \right\rangle$$

and $\mathcal{Q}(5,7;12)$ is the Cayley graph of the group

$$\left\langle \begin{array}{c|c} a, b & a^5, b^5 \\ c, d, e, f, g & c^7, d^7, e^7, f^7, g^7 \\ h, i, j, k, l & cdefh, gijkl^{-1}, ablkjih \end{array} \right\rangle.$$

(The appearance of l^{-1}, as opposed to l, in one of the relators in the second presentation is necessary.)

Bibliography

[ABC$^+$91] Alonzo, Brady, Cooper, Delzant, Ferlini, Lustig, Mihalik, Shapiro, and Short, *Notes on word hyperbolic groups*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publishing, River Edge, NJ, 1991, Edited by H. Short, pp. 3–63.

[Art65] E. Artin, *Ein mechanishes System mit quasi ergodischen Bahnen*, Collected papers (S. Lang and J. T. Tate, eds.), Addison Wesley, 1965, pp. 499–501.

[Aut] M. Autenrieth, *Rationality of a zeta function of Gromov.*

[BEH] S. E. Billington, D. B. A. Epstein, and D. F. Holt, *Geodesics in word-hyperbolic groups*, In preparation.

[BG95] O. V. Bogopolskiĭ and V. N. Gerasimov, *Finite subgroups of hyperbolic groups*, Algebra i Logika **34** (1995), no. 6, 619–622, 728.

[Bil96] S. E. Billington, *Growth of finite state automata*, Master's thesis, University of Warwick, 1996.

[BMS94] G. Baumslag, C. F. Miller, and H. Short, *Unsolvable problems about small cancellation and word hyperbolic groups*, Bull. London Math. Soc. **26** (1994), no. 1, 97–101.

[Boo57] W. W. Boone, *Certain simple, unsolvable problems of group theory. V, VI*, Nederl. Akad. Wetensch. Proc. Ser. A. **60** (1957), 22–27, 227–232.

[CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, Berlin, 1990.

[Cha94] Christophe Champetier, *Petite simplification dans les groupes hyperboliques*, Ann. Fac. Sci. Toulouse Math. (6) **3** (1994), no. 2, 161–221.

[CK96] T. Chaboud and C. Kenyon, *Planar Cayley graphs with regular dual*, Internat. J. Algebra Comput. **6** (1996), no. 5, 553–561.

[Clow] A. D. M. Clow, Ph.D. thesis, University of Warwick.

[Cor92] Jon Michael Corson, *Complexes of groups*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 199–224.

[CP93] M. Coornaert and A. Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Lecture Notes in Mathematics, vol. 1539, Springer-Verlag, Berlin, 1993.

[CP98] M. Coornaert and A. Papadopoulos, *Horofunctions and symbolic dynamics on Gromov hyperbolic groups*, Preprint 021, IRMA Strasbourg, 1998.

[CP99] M. Coornaert and A. Papadopoulos, *Symbolic coding for the geodesic flow associated to a word hyperbolic group*, Preprint 018, IRMA Strasbourg, 1999.

[Del91] T. Delzant, *Sous-groupes à deux générateurs des groupes hyperboliques*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publishing, River Edge, NJ, 1991, pp. 177–189.

[ECH⁺92] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, *Word processing in groups*, A. K. Peters, Natick, MA, 1992.

[EH] D. B. A. Epstein and D. F. Holt, *Computation in word-hyperbolic groups*, In preparation.

[EHR91] D. B. A. Epstein, D. F. Holt, and S. E. Rees, *The use of Knuth-Bendix methods to solve the word problem in automatic groups*, J. Symbolic Comput. **12** (1991), no. 4-5, 397–414, Computational group theory, Part 2.

[EP94] D. B. A. Epstein and C. Petronio, *An exposition of Poincaré's polyhedron theorem*, Enseign. Math. (2) **40** (1994), no. 1-2, 113–170.

[Fri87] D. Fried, *Finitely presented dynamical systems*, Ergodic Theory Dynamical Systems **7** (1987), no. 4, 489–507.

[GdlH90] É. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1990, Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[Geo] Ross Geoghegan, *Topological methods in group theory*, Book in preparation.

[Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. Gersten, ed.), Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.

[Hae91] A. Haefliger, *Complexes of groups and orbihedra*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publishing, River Edge, NJ, 1991, pp. 504–540.

[Hae92] A. Haefliger, *Extension of complexes of groups*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 275–311.

[Hol95] D. F. Holt, *Knuth-Bendix in Monoids and Automatic Groups*, Computer package, University of Warwick, 1995.

[Koe29] P. Koebe, *Riemannische Manigfaltigkeiten und nichteuklidische Raumformen*, Sitzungberichte der Preussichen Akad. der Wissenschaften (1929), 414–457.

[LMP96] S Levy, T Munzner, and M Phillips, *Geomview version 1.6.1*, Computer package, The Geometry Center, University of Minnesota, 1996.

[Mac71] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1971.

[Mor21] M. Morse, *A one-to-one representaion of geodesics on a surface of negative curvature*, Amer. J. Math **43** (1921), 33–51.

[Nov55] P. S. Novikov, *Ob algoritmičeskoĭ nerazrešimosti problemy toždestva slov v teorii grupp*, Izdat. Akad. Nauk SSSR, Moscow, 1955, Trudy Mat. Inst. im. Steklov. no. 44.

[Poi82] Henri Poincaré, *Théorie des groupes fuchsienne*, Acta. Math. 1 (1882), 1–62.

[Rip82] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math. Soc. **14** (1982), no. 1, 45–47.

[Seg68] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. No. **34** (1968), 105–112.

[Ser86] C. M. Series, *Geometrical Markov coding of geodesics on surfaces of constant negative curvature*, Ergodic Theory Dynamical Systems **6** (1986), no. 4, 601–625.

[Thu90] W. P. Thurston, *Three-dimensional geometry and topology*, Draft, The Geometry Center, University of Minnesota, 1990.