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Spectral and Distributional Problems for Homogeneous Extensions of Dynamical Systems And the Rate of Mixing of Two–Dimensional Markov Shifts

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Untuk ugama ku dan untuk Mardiana

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Declaration

The material in this thesis is original as far as I am aware except when stated otherwise. Chapter 1 is joint work with Bill Parry and has already appeared in the *Boletim Da Sociedade Brasileira De Matemática, Vol. 23, Ns. 1-2, 137-151, 1992.*

Brief Summary

This thesis consists of four chapters. Chapters 1 and 2 are somewhat related in the sense that they deal with similar dynamical systems. Each chapter comes complete with its own references and notations.

For the convenience of the reader, we provide an introduction and indeed an elongated summary to the whole thesis in Chapter 0.

In Chapter 1, we study how closed orbits of a subshift of finite type lifts to a finite homogeneous extension. In particular, we obtain an asymptotic formula for the number of closed orbits according to how they lift to the extension space. We apply our findings to the case of finite extensions and also to automorphism extensions of hyperbolic toral automorphisms.

Chapter 2 deals with lifting ergodic properties of an arbitrary measure-preserving transformation T to homogeneous extensions of T. Our results extends well known theorems already obtained for the case of compact group extensions of measure-preserving transformations. We also give simplified results to the special case when the base transformation is a Markov shift and the skewing function depends on a finite number of coordinates.

In Chapter 3, we look at the rate of mixing of rectangle sets of two dimensional Markov shifts with respect to the natural shift actions. We show that if one of the matrix defining the Markov measure is aperiodic then this rate is exponentially fast. We provide an example to illustrate what could happen in general.

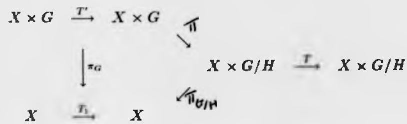
Contents

Chapter 0
Introduction

One of the basic constructions in ergodic theory is the so called skew-product of measure preserving transformations. These mathematical objects are measure preserving transformations of the form $T: X \times Y \to X \times Y$ which is given by $T(x, y) = (T_1(x), S_x(y))$ where $T_1: X \to X$ is a measure-preserving transformation on some probability space $(X, \mathcal{B}, m)$ and $\{S_x : x \in X\}$ is a family of measure–preserving transformation on some other probability space $(Y, \mathcal{A}, \mu)$. Here T is equipped with the product σ–algebra $\mathcal{B} \times \mathcal{A}$ and product measure $m \times \mu$. A thorough discussion on skew–products can be found in Chapter 10 of Cornfeld *et al* [2].

A special form of skew–product is when T is given by $T: X \times G/H \to X \times G/H$ and $T(x, gH) = (T_1 x, \alpha(x) gH)$, i.e., the space Y is taken to be the homogeneous space G/H for some compact group G, some subgroup H (not necessarily normal) of G and the family $\{S_x : x \in X\}$ is replaced by some measurable map $\alpha: X \to G$. In this situation, the space G/H is endowed with the Borel σ algebra together with the restricted Haar measure. This form of skew-product is more generally known (for obvious reasons) as a homogeneous extension of T_1. The main part of this thesis (i.e. Chapters 1 and 2) will then be dealing with some problems relating to these extensions and the associated factor system. In fact, the problems we consider are of two types: distributional problem (see Chapter 1) and spectral problem (see Chapter 2). We remark that when dealing with the distributional problem, then we shall mainly be interested in topological skew-products.

Observe that given a homogeneous extension T, there always exists a group extension $T': X \times G \to X \times G$ given by $T'(x, g) = (T_1(x), \alpha(x) g)$ which is an extension of both T and T_1. This crucial observation will be used on a number of occasions to help us with our work. It will be helpful to have the following picture in mind:



The maps π_G, $\pi_{G/H}$ and π in the above picture are the obvious projection maps.

Distribution Of Closed Orbits

The first problem considered (which is joint work with Bill Parry) and in fact the content of Chapter 1, is to do with counting closed orbits of the base transformation according to how they lift to the extension space. More precisely, in this situation, the base transformation $\sigma: X \to X$ is a subshift of finite type, G is a *finite* group and the skewing-function α is a continuous map from X to G. Note that, this means α is a function of a finite number of coordinates. Given a subgroup H of G, we have the homogeneous extension $T: X \times G/H \to X \times G/H$ which is given by $T(x, gH) = (\sigma x, \alpha(x)gH)$. Then, the main theorem of the first chapter is the dynamical analogue of the Chebotarev Theorem of classical number theory for finite homogeneous extensions of subshifts of finite type. Roughly speaking, the number theoretical result gives us an asymptotic formula for the number of primes according to how they lift to a finite extension field (see Heilbronn's article in Cassels and Fröhlich [1]). We stress that the proof of our theorem relies heavily on the analogous result for group extensions of subshift of finite type (i.e. when $H = \{e\}$, the trivial subgroup) which was obtained earlier by Parry and Pollicott [6] .

Working analogously with the number theoretical result, we first need to classify the σ-closed orbits according to how they lift to the space $X \times G/H$. For group extensions, this was done via the notion of Frobenius classes, which are just conjugacy classes of G. In fact, it is with respect to these Frobenius classes that the asymptotic formulas of Parry and Pollicott applies [6]. We remark that for group extensions, the closed orbits covering a given σ-closed orbit all have equal length. Unfortunately, in our case, this way of classifying the σ-closed orbits is no longer possible. Just as well, since for a general homogeneous extension the statement about the closed orbits covering a given σ-closed orbit all have equal length is no longer valid.

Now, given a σ-closed orbit τ and $\tilde{\tau}_1, \ldots, \tilde{\tau}_m$ are the T-closed orbits that cover τ, then the numbers

$$\frac{\lambda(\tilde{\tau}_1)}{\lambda(\tau)}, \ldots, \frac{\lambda(\tilde{\tau}_m)}{\lambda(\tau)}$$

forms a partition of the integer $|G/H|$. (Here, $\lambda(\ )$ denotes the length of the corresponding closed orbits.) The various partitions of the integer $|G/H|$ being induced by the σ-closed orbits will then be our way of classifying these closed orbits. Note that the existence of T-closed orbits covering a given σ-closed orbit is a consequence of the finiteness of G.

The next step in getting to the main result is to understand how these equivalence classes of σ-closed orbits arises. First observe that by lifting σ-closed orbits to the associated group extension, we can assign to each σ-closed orbit τ, its Frobenius class $[\tau]$. It is shown that, given an equivalence class A_l of σ-closed orbits (with respect to the homogeneous extension) corresponding to some partition l, say, of the integer $|G/H|$, then A_l is a disjoint union of σ-closed orbits corresponding to some distinct Frobenius classes. In fact, the σ-closed orbits that make up this disjoint union are exactly the ones with Frobenius classes $C(g_1)\ldots, C(g_n)$, such that the action of the cyclic group generated by g_i, $< g_i >$, on the coset space G/H also induces the partition l, for each $i = 1, \ldots, n$. By this we mean the 'sizes' of the distinct orbits of the action of $< g_i >$ on G/H also forms the partition l. This being the case, then our main theorem, which is an asymptotic formula for the number of σ-closed orbits in a given class A_l with length $\leq x$, as $x \to \infty$, is just a simple application of the Chebotarev Theorem for group extensions of Parry and Pollicott. That is, we just add up the asymptotic formulas corresponding to the various Frobenius classes that make up A_l (see Theorem 2.3 of Chapter 1).

After establishing the main result, we then apply our findings to specific examples. The first example considered is the so-called finite extensions of subshifts of finite type. These take the form $T: X \times F \to X \times F$, $T(x, i) = (\sigma x, \alpha(x)(i))$, where F is some finite set $\{1, 2, \ldots, k\}$ say, and α is a continuous map from X to S_k, the symmetric group on k-symbols. Since F is a homogeneous space with respect to the group action, then it is easy to see that we can identify F with the coset space S_k/S_{k-1}. Thus we can rewrite T in the 'homogeneous extension form' $T: X \times S_k/S_{k-1} \to X \times S_k/S_{k-1}$ where

$T(x, gS_{k-1}) = (\sigma x, \alpha(x)gS_{k-1})$. It is interesting to note that in this example, given an equivalence class A_l of σ–closed orbits, arising from some partition l, there exists a unique conjugacy class C_l such that A_l is precisely those σ–closed orbits τ with Frobenius class $[\tau]$ equals C_l. This in turn implies that the asymptotics for the homogeneous and the group extension are the same (see §3 of Chapter 1 for details).

We are also able to apply our main theorem to a so-called automorphism extension of the shift. These are skew-products of the form $T: X \times G \to X \times G$, $T(x, g) = (\sigma x, \beta(x)\gamma(g))$ where $\gamma: G \to G$ is an automorphism of the (finite) group G and β is a continuous map from X to G. Then T is also a homogeneous extension. To see this, first note that since G is finite, then there exists a least n such that $\gamma^n = \text{id}$. Now, let $G' = \mathbb{Z}_n \times_\gamma G$ be the semi-direct product group of G by $\mathbb{Z}_n$. Moreover, let H be the subgroup $\mathbb{Z}_n \times \{e\}$ (e = identity of G) of G'. Then we can identify T with the map $T: X \times G'/H \to X \times G'/H$ where now

$$T(x, (0, g)H) = (\sigma x, (0, \beta(x)\gamma(g))H)$$

for all $(x, (0, g)H) \in X \times G'/H$ (see §4 of Chapter 1 for details). In particular, our main theorem also applies to this extension of the shift.

In fact, we can go one step further with these automorphism extensions. For this, let $\bar{A}$ be a hyperbolic automorphism of a finite dimensional torus $\mathbb{T}$. Let G be the set of all points in $\mathbb{T}$ with order m, say. Then G acts on $\mathbb{T}$ and in relation to $\bar{A}$ this action satisfies

$$\bar{A}(x + g) = \bar{A}(x) + \bar{A}(g) \qquad x \in \mathbb{T}, g \in G.$$

Moreover, $\bar{A}$ induces an action A on the G–orbit space $\mathbb{T}/G$ such that $A: \mathbb{T}/G \to \mathbb{T}/G$ is also a hyperbolic toral automorphism. By using similar ideas to the one contained in pg. 137 of Parry and Pollicott [6], one can show that the automorphism extension of the shift considered earlier actually is the 'symbolic model' for our hyperbolic toral automorphisms. More importantly, since the counting function for the shift and the toral automorphism are asymptotic (see Parry and Pollicott [7]), we gather that any asymptotic formula which is true for the automorphism extension of shifts, is also true for 'automorphism extension' of hyperbolic toral automorphisms. In particular, our main result also holds for this latter kind of dynamical system (see §4 of Chapter 1 for details). This concludes the main considerations of Chapter 1.

Lifting of Ergodic Properties

Chapter 2 of this thesis is concerned with the spectral properties of homogeneous extensions of arbitrary measure-preserving transformations. More precisely, the problem considered here is to do with lifting ergodic properties of the base transformation to the homogeneous extension. The ergodic properties that interest us are ergodicity, weak–mixing, being a K-automorphism and being a Bernoulli shift. Suppose we know that the base transformation satisfies some ergodic property then, in general, it is not true that the corresponding homogeneous extension also satisfies the same ergodic property. Thus, the problem is to find conditions for the homogeneous extension to satisfy the same ergodic property as the base transformation.

For (compact) group extensions of measure-preserving transformations, these conditions are well-known. Indeed, these well-known conditions for being ergodic, weakly mixing, a K-automorphism and a Bernoulli shift are all necessary and sufficient. These results are due to Keynes & Newton [4], Parry & Pollicott [6], Thomas [11] and Rudolph [9] respectively. We have included these results and some of their proofs for completeness in §2 of Chapter 2. In fact, all the conditions supplied by them can be written in the form of a certain functional equation. As a sample of these conditions, we quote the result, due to Keynes & Newton [4], which gives a necessary and sufficient condition for the ergodicity of the group extension knowing that the base transformation T_1 is ergodic (see §2 of Chapter 2). First, recall that α is the skewing function of the group extension T'.

> **Theorem A** *Let T_1 be ergodic. Then the group extension T' is ergodic if and only if for any non-trivial unitary representation R of G (of degree d, say), the equation*
>
> $$F(T_1(x)) = R(\alpha(x))F(x) \quad a.e.\ x$$
>
> *has no non-trivial measurable solutions $F: X \to \mathbb{C}^d$.*

This motivates us to find analogous criteria for homogeneous extensions of measure-preserving transformations. Indeed, we show that necessary and sufficient conditions also exist for homogeneous extensions to have the same ergodic properties as the base transformations. As a comparison with Theorem A, we quote our result for the ergodicity of the homogeneous extension knowing that the base transformation T_1 is ergodic (see Theorem 5 of Chapter 2).

Theorem B *Let T_1 be ergodic. Then the homogeneous extension T is ergodic if and only if for any non-trivial irreducible unitary representation R of G (of degree d, say,) satisfying $bR(h) = b$ $\forall h \in H$, for some non-zero $b \in \mathbb{C}^d$, the equation*

$$F(T_1(x)) = R(\alpha(x))F(x) \quad a.e.\ x$$

has no non-trivial measurable solutions $F: X \to \mathbb{C}^d$.

Note that, in contrast to Theorem A, the group representations involved in the above theorem are, roughly speaking, of the type that 'annihilates' H. Indeed, this is true for abelian G since in this case we are interested in representations (or more precisely, characters) χ such that $\chi(h) = 1$, $\forall h \in H$. We also obtained a similar condition to Theorem B for the case of a weakly-mixing T_1 (see Theorem 6 of Chapter 2). As one would expect, the proof of Theorem B and also for the weak-mixing case follows similar arguments used in the case of group extensions.

In the case of a Bernoulli base, the result we provide (see Corollary 2 of Chapter 2) is just a simple rewording of a theorem of Rudolph [9]. Our result says that if the base transformation T_1 is a Bernoulli shift then the homogeneous extension T is also Bernoulli provided T is weak-mixing. Unfortunately, in the case where the base transformation is a K-automorphism, we are only able to solve the problem for the special case of a finite G. This result of ours, which says that T is a K-automorphism if it is weak-mixing, follows from a result of Thomas [11]. Nevertheless, we conjecture that the aforementioned result also holds for arbitrary compact groups G. Work is in progress in this direction.

In the final section of Chapter 2, we specialize our considerations to the case where the base transformation is a Markov shift and the skewing function α is a function of a finite number of coordinates. Note that, by using a standard recoding argument (see, for e.g., Denker *et al* [3]) we can always assume the function α to depend on only two coordinates. Recall that the necessary and sufficient conditions for homogeneous extensions and indeed for group extensions to satisfy some ergodic property involves a certain functional equation (c.f. Theorems A and B). Our objective now is to obtain a simplification relating to the solutions of these functional equations. In fact, in this situation, the results obtained by us implies that solutions F to the functional equation , in general, depends on one less coordinate than that of the skewing function α. In particular, when α depends on two coordinates then the solution F to the functional equation depends on only one coordinate. Hence we can apply this finding to Theorem B, say, to obtain:

> **Theorem C** *Let T_1 be an ergodic Markov shift. Suppose the skewing function α depends on two coordinates. Then the homogeneous extension T is ergodic if and only if for any non-trivial irreducible unitary representation R of G (of degree d, say) satisfying $bR(h) = b$, $\forall h \in H$, for some non-zero $b \in \mathbb{C}^d$, the equation*
>
> $$F(T_1(x)) = R(\alpha(x))F(x) \quad a.e.\ x$$
>
> *has no non-trivial measurable solutions $F: X \to \mathbb{C}^d$ depending on only one coordinate, i.e. $F(x) = F(x_0)$.*

We remark that for abelian G, this kind of simplification was proven earlier by Parry [5] (see also Parry & Tuncel [8]). In fact, the main result of this section (§5, Prop. 4), was shown to us by Bill Parry, for which we are grateful. The above somewhat summarises the main content of Chapter 2.

Higher–Dimensional Markov shifts

Unlike the theory of one-dimensional Markov shifts, the theory of higher-dimensional Markov shifts is filled with anomalies and difficulties (see Schmidt [10] for a brief survey). In the final chapter of this thesis (Chapter 3), we make a very small contribution towards a better understanding of these higher-dimensional Markov shifts.

The problem that interest us in this short chapter is with regard to the rate of mixing rectangle sets in a two-dimensional Markov shift. More precisely, let Y be the finite set $\{1, 2, \ldots, k\}$ and $Y^{\mathbb{Z}^2}$ be the space of all functions $x: \mathbb{Z}^2 \to Y$ endowed with the product σ-algebra B. Let P and Q be two commuting $k \times k$ stochastic matrices such that there exists a probability vector p with $pP = pQ = p$. Moreover, assume that P^0Q^0 is also a zero-one matrix and that $P^0Q^0 = Q^0P^0$. (Here P^0, Q^0 denotes the zero-one matrices which are compatible to P and Q respectively.) Then P, Q defines a probability measure m on $(Y^{\mathbb{Z}^2}, \mathcal{B})$. The probability space $(Y^{\mathbb{Z}^2}, \mathcal{B}, m)$ together with the natural shift actions σ (horizontal shift) and τ (vertical shift) then defines our two dimensional Markov shift. We, remark that this definition is not empty since the corresponding subshift of finite type (defined by P^0 and Q^0) is non empty.

Now, given two rectangle sets A and B in $\mathcal{B}$, the problem is to look at the rate of convergence of the sequence

$$\left(m(A \cap \sigma^{-m}\tau^{-n}B) - m(A)m(B)\right)_{m,n\geq 0}$$

to zero, as $m, n \to \infty$. For one-dimensional Markov shifts, it is by now well-known that when the matrix defining the Markov measure is aperiodic then the analogous sequence in this case converges to zero at an exponential rate. We show that, for two-dimensional Markov shifts, the aforementioned sequence also converges to zero at an exponential rate provided P or Q is aperiodic. The proof of this result is pretty much the same as in the one–dimensional case. In particular, we show that the sequence of matrix entries $P^mQ^n(i,j)$ converges to $p(j)$ at an exponential rate as $m, n \to \infty$, for all $i, j = 1, 2, \ldots, k$ when either P or Q is aperiodic.

It is interesting to note that our original conjecture for this kind of convergence to hold is that $P^sQ^t > 0$ for some integers $s, t > 0$. We give an example of a pair of matrices P, Q satisfying the above properties to illustrate what could happen to the above result if we relax the aperiodicity assumption on P or Q. In particular, this example demolish the previously mentioned conjecture. An immediate corollary to the main result of this chapter is that the Markov shift is strong–mixing. Of course, all the above can be generalised to higher–dimensional Markov shifts.

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Chapter 1

A Chebotarev Theorem For Finite Homegeneous Extensions Of Shifts

by

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Abstract: We derive a Chebotarev Theorem for finite homogeneous extensions of shifts of finite type. These extensions are of the form $\hat{\sigma}: X \times G/H \to X \times G/H$ where $\hat{\sigma}(x, gH) = (\sigma x, \alpha(x)gH)$, for some finite group G and subgroup H. Given a σ-closed orbit τ, the periods of the $\hat{\sigma}$-closed orbits covering τ defines a partition of the integer $|G/H|$. The theorem then gives us an asymptotic formula for the number of closed orbits with respect to the various partitions of the integer $|G/H|$. We apply our theorem to the case of a finite extension and of an automorphism extension of shifts of finite type. We also give a further application to 'automorphism extensions' of hyperbolic toral automorphisms.

0. Introduction

The Chebotarev Theorem for a group extension of a shift of finite type σ gives us an asymptotic formula for the number of σ-closed orbits according to how they lift in the extension space and how they lift is completely determined by their Frobenius classes. It is with respect to these classes that the asymptotic formula applies. This, and indeed many more distribution results for closed orbits of shifts of finite type has been derived by the second author together with Mark Pollicott (see [4] for the entire collection).

Strictly speaking the above mentioned result is for a suspension flow over a shift of finite type. To obtain the appropriate result for the discrete case, all one needs do is use the constant function 1 as the suspension function.

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Our aim in this paper is to study the analogous problem for a more general extension of the shift. In fact,our consideration here were motivated by two examples: a finite extension and a so-called automorphism extension of the shift. It will become apparent that to cater for these examples, the appropriate extension one should consider is a homogeneous extension. i.e., of the form $\tilde{\sigma}: X \times G/H \to X \times G/H$ for some (finite) group G and subgroup H.

We observe that, unlike the group extension case, the lifted closed orbits may not have the same length. This unevenness will then be the basis for classifying the σ-closed orbits. To proceed with the asymptotics with respect to this classification we have to understand how these classes come about. We show that this is equivalent to looking at actions of certain cyclic subgroups on G/H. This is done by resorting to a certain group extension. Thus it is no surprise that our main result is just a direct application of the Chebotarev Theorem for group extensions.

1. Basics facts and Definitions

Let $\{1,2,\ldots,n\}$ be given the discrete topology and A be a $n \times n$ irreducible $0-1$ matrix. Define the set

$$X_A = \{x \in \prod_{-\infty}^{\infty}\{1,2,\ldots,n\} : A(x_i, x_{i+1}) = 1, \forall i \in \mathbb{Z}\}$$

Then X_A is a compact zero dimensional space. Let $\sigma: X_A \to X_A$ be defined by $(\sigma x)_i = x_{i+1}$. Then σ is called a shift of finite type (with transition matrix A). From now on we shall write X for X_A.

Recall that a homeomorphism $T: Y \to Y$ is said to be topologically transitive if T has a dense orbit. Also T is said to be topologically mixing if for any two non-empty open sets U,V in Y, there is an integer N such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

For shifts of finite type, it is well-known that these two notions are equivalent to the requirement that the transition matrix A be irreducible and aperiodic respectively. The topological entropy of σ is $\log \beta$, where β is the maximal positive eigenvalue of A as furnished by the Perron-Frobenius Theorem.

Given a closed orbit (i.e. periodic orbit) τ of σ, we shall denote its least period by $\lambda(\tau)$. Then the zeta function of σ is defined as

$$\zeta_\sigma(z) = \prod_\tau (1 - z^{\lambda(\tau)})^{-1}$$

$$= \exp \sum_\tau \sum_{n=1}^{\infty} \frac{z^{\lambda(\tau)n}}{n} \qquad \text{for } |z| < \beta^{-1}.$$

In fact we have the following well known result of Bowen and Lanford [1].

1.1 Proposition. *Let σ be a shift of finite type with transition matrix A. And let β be the associated maximal positive eigenvalue. Then*

$$\zeta_\sigma(z) = \frac{1}{\det(I - zA)} \qquad \text{for } |z| < \beta^{-1}.$$

An immediate corollary to the above result is

1.2 Corollary. *Let σ be a topologically mixing shift of finite type. Then $\zeta_\sigma(z)$ has a non-zero analytic extension to a disc of radius greater than β^{-1} except for a simple pole at β^{-1}.*

Using this result Parry and Pollicott (see [4] pg 104) deduced the Prime Orbit Theorem for shifts of finite type:

1.3 Theorem. *Let σ be a mixing shift of finite type and let $\pi(x) =$ Card$\{\tau \subset X | \lambda(\tau) \le x\}$. Then*

$$\pi(x) \sim \frac{\beta}{\beta - 1} \cdot \frac{\beta^x}{x} \qquad \text{as } x \to \infty.$$

When σ is not topologically mixing, we can decompose X into a disjoint union of d (= period of A) closed-open sets $X_1, \ldots, X_d$ such that $\sigma(X_i) = X_j$ ($j = i + 1 \bmod d$) and $\sigma^d \big|_{X_i}$ is topologically mixing, $i = 1, \ldots, d$. Hence applying (1.3) to this case we have

1.4 Proposition. *When σ is not topologically mixing, then*

$$\pi(dx) \sim \frac{\beta^d}{\beta^d - 1} \cdot \frac{\beta^{dx}}{x} \qquad \text{as} \quad x \to \infty$$

where d is the period of the transition matrix A.

All groups considered in this paper are assumed to be finite. So let G be such a group and $\alpha: X \to G$ be a function depending on a finite number of coordinates. The group extension $\hat{\sigma}: X \times G \to X \times G$ of σ is then defined by the skew-product $\hat{\sigma}(x,g) = (\sigma x, \alpha(x)g)$. We shall always assume that $\hat{\sigma}$ is topologically transitive. Thus by definition $\hat{\sigma}$ is also a shift of finite type. Letting $\pi: X \times G \to X$ be $\pi(x,g) = x$, we have $\pi\hat{\sigma} = \sigma\pi$.

We are interested in how σ-closed orbits lift into the extension space. To classify these closed orbits, we introduce a free right-action of G on $X \times G$ by $h \cdot (x,g) = (x, gh)$, $h \in G$. This action commutes with $\hat{\sigma}$. Thus given a closed orbit τ of σ with least period $\lambda(\tau)$ and a $\hat{\sigma}$-closed orbit $\hat{\tau}$ covering τ (i.e. $\pi(\hat{\tau}) = \tau$), there exists a unique element $\gamma(\hat{\tau}) \in G$ such that if $p \in \hat{\tau}$, then

$$\gamma(\hat{\tau})p = \hat{\sigma}^{\lambda(\tau)}p.$$

In fact $\gamma(\hat{\tau})$ depends only on $\hat{\tau}$. This group element $\gamma(\hat{\tau})$ is called the *Frobenius element* of $\hat{\tau}$. Moreover if $\hat{\tau}'$ is another $\hat{\sigma}$-closed orbit also covering τ, then since G acts transitively on fibers, there exists an $h \in G$ such that $hp \in \hat{\tau}'$. Thus the Frobenius element $\gamma(\hat{\tau}')$ of $\hat{\tau}'$ satisfies

$$\gamma(\hat{\tau}')hp = \hat{\sigma}^{\lambda(\tau)}hp.$$

Hence $\gamma(\hat{\tau}') = h\gamma(\hat{\tau})h^{-1}$. In other words, the Frobenius elements of the lift of τ are all in the same conjugacy class which is uniquely determined by τ. This conjugacy class is called the Frobenius class of τ and is denoted by $[\tau]$.

Let R_χ be an irreducible representation of G with character χ. The L-function (with respect to $\pi: X \times G \to X$) of χ is defined as

$$L(z,\chi) = \prod_{\tau} \det\left(I - z^{\lambda(\tau)} R_\chi([\tau])\right)^{-1}$$

where the product is taken over all σ-closed orbits. By comparing the above expression with the zeta function of the shift we deduce that $L(z,\chi)$ is non-zero and analytic on $D = \{z \mid |z| < \beta^{-1}\}$. Observe that when $\chi = \chi_0$, the principal character, $L(z,\chi_0) = \zeta_\sigma(z)$. In fact one can show

1.5 Proposition. *Let $\hat{\sigma}$ be a group extension of σ with skewing function $\alpha: X \to G$ depending on a finite number of coordinates. Then*

$$L(z,\chi) = \frac{1}{\det(I - zM_\chi)} \qquad \text{for} \quad |z| < \beta^{-1}$$

for some matrix M_χ closely related to the representation R_χ.

The Chebotarev Theorem of Parry and Pollicott for group extensions is as follows:

1.6 Theorem. *Let $\hat{\sigma}$ be a topologically transitive group extension of a shift of finite type σ. For a conjugacy class C of G, let $\pi_C(x) = \mathrm{Card}\{\tau \subset X : [\tau] = C, \lambda(\tau) \leq x\}$. Then*

a) if $\hat{\sigma}$ is mixing, $\pi_C(x) \sim \frac{|C|}{|G|}\frac{\beta}{\beta - 1} \cdot \frac{\beta^x}{x}$ as $x \to \infty$,

b) if σ is mixing and $\hat{\sigma}$ not mixing with d = period of the transition matrix of $\hat{\sigma}$,

$$\pi_C(x) \sim d\frac{|C|}{|G|}\frac{\beta^d}{\beta^d - 1} \cdot \frac{\beta^x}{x} \qquad \text{as } x \to \infty.$$

We indicate how one proves this result. We shall restrict our attention to the case when $\hat{\sigma}$ is mixing. To capture the σ-closed orbits with a given Frobenius class C, we introduce the following zeta function,

$$\zeta_C(z) = \prod_{\substack{\tau \\ [\tau]=C}} \left(1 - z^{\lambda(\tau)}\right)^{-1}.$$

Let $g \in C$. Then by the orthogonality relation for irreducible characters of G we have

$$\frac{|G|}{|C|}\frac{\zeta_C'(z)}{\zeta_C(z)} = \sum_{\chi \text{ irreducible}} \chi(g^{-1})\frac{L'(z,\chi)}{L(z,\chi)}.$$

Then we identify the poles of $\zeta_C'(z)/\zeta_C(z)$ in a small neighbourhood D' of $\{z \mid |z| \leq \beta^{-1}\}$ and thus calculate their residues. We know that $L(z,\chi_o) = \zeta_\sigma(z)$ has a simple pole at $z = \beta^{-1}$ on the circle $\{z \mid |z| = \beta^{-1}\}$ and if we take D' to be small enough, $z = \beta^{-1}$ is the only pole in this region. To do this we bring in the identity (see Prop. 4 of [5])

$$\zeta_{\hat{\sigma}}(z) = \zeta_\sigma(z) \prod_{\chi \neq \chi_o} L(z,\chi)^{d_\chi}.$$

Since $\hat{\sigma}$ is a mixing shift of finite type with the same topological entropy as σ (because π is $|G|$ to 1) we deduce, via (1.5), that $L(z,\chi)$, $\chi \neq \chi_o$ has a non-zero analytic extension to some neighbourhood D'' of $\{z \mid |z| \leq \beta^{-1}\}$. Thus $\zeta_C'(z)/\zeta_C(z)$ has only one pole in the smaller of the two regions, namely $z = \beta^{-1}$ and its residue is $-|C|/|G|$. Then the proof proceeds analogously with the proof of the Prime Orbit Theorem for shifts of finite type (1.3).

Remarks

1. The argument used in the above discussion comes from [5].

2. Observe that the assumption that the skewing function depends only on a finite number of coordinates plays two roles: Firstly it turns σ into a shift of finite type. Secondly, it also implies a meromorphic (in fact rational) extension of $L(z, \chi)$ to the whole plane.
3. There exists a similar formula for the case when σ is not mixing.

2. The Homogeneous Extension

As before let σ be a shift of finite type and G a finite group together with a map $\alpha: X \to G$ such that α depends on a finite number of coordinates. Let H be an arbitrary subgroup of G. Form the coset space $G/H = \{gH : g \in G\}$. A *homogeneous extension* $\tilde{\sigma}: X \times G/H \to X \times G/H$ of σ is defined by the skew-product $\tilde{\sigma}(x, gH) = (\sigma x, \alpha(x)gH)$. We shall always assume that $\tilde{\sigma}$ is topologically transitive. Let $\tilde{\pi}: X \times G/H \to X$ be such that $\tilde{\pi}(x, gH) = x$. Then $\tilde{\pi}\tilde{\sigma} = \sigma\tilde{\pi}$.

Observe that the group extension $\hat{\sigma}: X \times G \to X \times G$ defined by $\hat{\sigma}(x, g) = (\sigma x, \alpha(x)g)$ is, by using the obvious projection map, an extension of $\tilde{\sigma}$. We have the following multi-commutative diagram:

$$
\begin{array}{ccccccc}
X \times G & \xrightarrow{\hat{\sigma}} & X \times G & & & & \\
 & & & \searrow & & & \\
\Big\downarrow \pi & & & & X \times G/H & \xrightarrow{\tilde{\sigma}} & X \times G/H \\
 & & & \swarrow \tilde{\pi} & & & \\
X & \xrightarrow{\sigma} & X & & & &
\end{array}
$$

Note that we cannot expect $\hat{\sigma}$ to be also topologically transitive. For our purposes, it suffices to note that if $\hat{\sigma}$ is not transitive then, as with all intransitive shifts of finite type, we can decompose $X \times G$ into $\hat{\sigma}$-invariant transitive pieces $X_0, \ldots, X_{s-1}$. Moreover $\hat{\sigma}\,\big|_{X_i}$ is a G_i-invariant extension of X where G_i is the subgroup of G such that $gX_i = X_i$ $\forall g \in G_i$. This follows since the group action commutes with $\hat{\sigma}$ and the fact that σ is transitive. Note that the subgroups G_i, $i = 0, 1, \ldots, s-1$, are conjugate to each other. More importantly, by the transitivity of $\tilde{\sigma}$, we can identify $X \times G/H$ with the H_i-orbit space X_i/H_i where $H_i = H \cap G_i$. Hence by restricting to any X_i if necessary, there is no loss in generality in assuming that $\hat{\sigma}: X \times G \to X \times G$ is topologically transitive.

Recall that a partition of a positive integer k is a collection of positive integers $l_1, l_2, \ldots, l_m$ such that $k \geq l_1 \geq l_2 \geq \ldots \geq l_m \geq 1$ and $l_1 + \ldots + l_m = k$. In this case we write $\underline{l}$ for the m-tuple $(l_1, \ldots, l_m)$.

Let τ be a σ-closed orbit with period $\lambda(\tau)$ and $\tilde{\tau}$ be a $\tilde{\sigma}$-closed orbit with period $\lambda(\tilde{\tau})$ such that $\tilde{\pi}(\tilde{\tau}) = \tau$. Then the degree of $\tilde{\tau}$ over τ is defined by the integer

$$\deg\left(\frac{\tilde{\tau}}{\tau}\right) = \frac{\lambda(\tilde{\tau})}{\lambda(\tau)}$$

Note that this is where the finiteness of G comes in. For in this case the lift of τ in $X \times G/H$ also consists of closed orbits. Moreover if $\tilde{\tau}_1, \ldots, \tilde{\tau}_m$ are the distinct $\tilde{\sigma}$-closed orbits that cover τ then the following basic relation holds:

$$\deg\left(\frac{\tilde{\tau}_1}{\tau}\right) + \ldots + \deg\left(\frac{\tilde{\tau}_m}{\tau}\right) = \frac{|G|}{|H|},$$

so that the above equation gives us a partition of $|G|/|H|$. Thus we say τ induces the partition $\underline{l} = (l_1, \ldots, l_m)$ of the integer $|G|/|H|$ if

$$\underline{l} = \left(deg(\frac{\tilde{\tau}_1}{\tau}), \ldots, deg(\frac{\tilde{\tau}_m}{\tau})\right) \qquad \text{(after reordering if need be).}$$

Let K be another subgroup of G. We can define a left action of $k \in K$ on the coset space G/H by $k \cdot gH = kgH$. Let $K_1, \ldots, K_m$ be the distinct orbits of this action and r_i, $i = 1, \ldots, m$, be their respective 'sizes'. Notice that these r_i's form a partition of $|G|/|H|$. In this case we say K induces the partition $\underline{r} = (r_1, \ldots, r_m)$ of $|G|/|H|$ (after reordering if need be). It is easy to see that if k is conjugate to k' then the respective cyclic subgroups generated by them induces the same partition of $|G|/|H|$.

For each partition $\underline{l}$ of $|G|/|H|$, let $A_{\underline{l}} = \{\tau \subset X : \tau$ induces the partition $\underline{l}\}$. Then we are interested in characterizing those $A_{\underline{l}}$'s that are non-empty. We have

2.1 Proposition. *Let τ be a σ-closed orbit. Then τ induces the partition $\underline{l}$ of $|G|/|H| \iff$ the action of the cyclic group generated by some (and hence all) Frobenius element g associated with τ induces the partition $\underline{l}$ on $|G|/|H|$.*

Proof. Let τ be a σ-closed orbit of period m such that τ induces the partition $\underline{l} = (l_1, \ldots, l_n)$ of $|G|/|H|$. Suppose $\tilde{\tau}_1, \ldots, \tilde{\tau}_n$ be the distinct $\tilde{\tau}$-closed orbits that covers τ. For $x \in \tau$, we write $\alpha_m(x) = \alpha(\sigma^{m-1}(x)) \ldots \alpha(x)$. Then the fiber above x contained in $\tilde{\tau}_i$ consists of the elements

$$(x, a_i H), (x, \alpha_m(x) a_i H), \ldots, (x, (\alpha_m(x))^{l_i - 1} a_i H)$$

for some $(x, a_i H) \in \tilde{\tau}_i$. Note that l_i is the least integer such that

$$(\alpha_m(x))^{l_i} a_i H = a_i H.$$

Thus $l_i = \deg(\tilde{\tau}_i/\tau)$. Let $J_i = \{a_i H, \ldots, (\alpha_m(x))^{l_i - 1} a_i H\}$. Then clearly $J_i \cap J_k = \emptyset$ when $i \neq k$ and $G/H = \bigcup_{i=1}^{n} J_i$.

Let $< \alpha_m(x) >$ be the cyclic group generated by $\alpha_m(x)$. Evidently the J_i's are the distinct and indeed the totality of the orbits of the action of $< \alpha_m(x) >$ on G/H. Thus $< \alpha_m(x) >$ induces the partition $\underline{l}$ on G/H. By the definition of the group extension $\bar{\sigma}: X \times G \to X \times G$ and the right action $g \cdot (x, k) = (x, kg)$ of G on $X \times G$ we deduce that $\alpha_m(x)$ is indeed a Frobenius of τ.

Conversely, let $x \in \tau$, then reversing the previous argument we can construct the fiber above x by considering the distinct orbits of the action of $< \alpha_m(x) >$ on G/H. Then these distinct orbits constitute distinct $\bar{\sigma}$-closed orbits covering τ. In fact this construction is independent of x since if $y \in \tau$ then $\alpha_m(y)$ is conjugate to $\alpha_m(x)$. This completes the proof.□

Remark. Observe that in general, we cannot expect the lifts of τ in $X \times G/H$ to have equal period. For in this case we are dealing with a double coset partitioning of G. In the special case when $H = \{e\}$ (i.e. group extension), we do get equal period since the orbits of the subgroup action are just right cosets. In fact the degree of any $\bar{\tau}$ over τ in this case is then equal to the order of the Frobenius element $\gamma(\bar{\tau})$ of $\bar{\tau}$.

Let $C(g)$ denote the conjugacy class containing g. As an immediate corollary to (2.1), we have

2.2 Corollary. *Let $\underline{l}$ be a partition of $|G/H|$. And let $C_i(\underline{l}) = \{\tau \subset X : [\tau] = C(g_i)\}$ be the distinct classes of σ-closed orbits with Frobenius class $C(g_i)$ respectively such that $< g_i >$ induces the partition $\underline{l}$, $i = 1, \ldots, m$. Then*

$$A_{\underline{l}} = \bigcup_{i=1}^{m} C_i(\underline{l}).$$

For each partition $\underline{l}$, with $A_{\underline{l}} \neq \emptyset$, let $\pi_{\underline{l}}(x) = \text{Card}\{\tau \subset X : \tau \in A_{\underline{l}}, \lambda(\tau) \leq x\}$. Hence by a direct application of the Chebotarev Theorem of Parry and Pollicott, we have, for e.g. the following result for a homogeneous extension.

2.3 Theorem. *Let $\tilde{\sigma}$ be a homogeneous extension of σ where the associated group extension $\hat{\sigma}$ is topologically mixing. Let $\underline{l}$ be a partition of $|G|/|H|$ such that $A_{\underline{l}} \neq \emptyset$. Then*

$$\pi_{\underline{l}}(x) \sim \frac{1}{|G|} \sum_{i=1}^{m} |C(g_i)| \pi(x)$$

where the $C(g_i)$'s are as in (2.2) and

$$\pi(x) \sim \frac{\beta}{\beta - 1} \cdot \frac{\beta^x}{x} \qquad \text{as } x \to \infty.$$

3. Application I: Finite Extensions of shifts of finite type

Let $F = \{1, 2, \ldots, k\}$. A finite (k-point) extension of the shift σ is a skew-product $\tilde{\sigma}: X \times F \to X \times F$ defined by $\tilde{\sigma}(x, i) = (\sigma x, \alpha(x)(i))$ where $\alpha: X \to G$, as usual depends on a finite number of coordinates and G is the symmetric group S_k on k-symbols $\{1, 2, \ldots, k\}$.

Now, let $H = \{h \in G : h(1) = 1\}$. Then $H \cong S_{k-1}$. Also it is clear that the map from F to G/H sending i to gH where $g(1) = i$ is a bijection. Therefore we can identify $X \times F$ with $X \times G/H$ and obtain the homogeneous extension $\tilde{\sigma}: X \times G/H \to X \times G/H$ defined by $\tilde{\sigma}(x, gH) = (\sigma x, \alpha(x)gH)$. Letting $\tilde{\pi}: X \times G/H \to X$ be $\tilde{\pi}(x, gH) = x$ we have $\tilde{\pi}\tilde{\sigma} = \sigma\tilde{\pi}$. We shall assume that $\tilde{\sigma}$ is topologically transitive.

Also we have $\hat{\sigma}: X \times G \to X \times G$ where $\hat{\sigma}(x, g) = (\sigma x, \alpha(x)g)$. So that $\hat{\sigma}$ is a group extension of σ. As mentioned in §2 there is no loss of generality in assuming $\hat{\sigma}$ is also topologically transitive.

Recall that an element g of a symmetric group G is said to have cycle decomposition $\underline{m} = (m_1, m_2, \ldots, m_t)$ if it can be written as the product of disjoint cycles of length $m_1, m_2, \ldots, m_t$ where $m_1 \geq m_2 \geq \ldots \geq m_t$. Recall also that two elements of G are conjugate if and only if they have the same cycle decomposition.

3.1 Proposition. *Let $\underline{l}$ be an arbitrary partition of k. Then $A_{\underline{l}} \neq \emptyset$. Moreover $A_{\underline{l}} = \{\tau \subset X : [\tau] = C_{\underline{l}}\}$ where $C_{\underline{l}}$ is the conjugacy class of G consisting of elements with cycle decomposition $\underline{l}$.*

Proof.

Let $\underline{l} = (l_1, \ldots, l_n)$ be a partition of k. Thus there exists some $g \in G$ such that g has cycle decomposition $\underline{l}$. This follows since each partition $\underline{n}$ of k can be uniquely associated with the conjugacy class $C_{\underline{n}}$ of G consisting of elements with cycle decomposition $\underline{n}$. Now consider the action of the cyclic group $< g >$ generated by g on $F = \{1, 2, \ldots, k\}$. Then using the cycle decomposition form of g, it is clear that this action gives rise to n orbits $O_1, \ldots, O_n$ such that $|O_i| = l_i$, $i = 1, \ldots, n$. In other words $< g >$ induces the partition $\underline{l}$ on F or equivalently on G/H. Since this only depends on the cycle decomposition $\underline{l}$ of g and elements with such a cycle decomposition constitute a whole conjugacy class $C_{\underline{l}}$ we have $A_{\underline{l}} = \{\tau \subset X : [\tau] = C_{\underline{l}}\}$ by (2.2). □

The Cauchy formula (see for e.g. [3]) for the cardinality of $C_{\underline{l}}$ gives us

$$|C_{\underline{l}}| = \frac{k!}{l_{i_1}^{\alpha_{i_1}} \alpha_{i_1}! \ldots l_{i_s}^{\alpha_{i_s}} \alpha_{i_s}!}$$

where the l_{i_j}'s are the distinct components of $(l_1, l_2, \ldots, l_m)$ and α_{i_j} is the number of cycles of length l_{i_j} in the cycle decomposition of $g \in C_{\underline{l}}$. Hence, for a finite extension of a shift of finite type (cf. (2.3)), we have

3.2 Theorem. *Let $\tilde{\sigma}$ be a finite (k-point) extension of a shift of finite type σ where the associated group extension $\hat{\sigma}$ is topologically mixing. For each partition $\underline{l}$ of the integer k, let $\pi_{\underline{l}}(x) = \mathrm{Card}\{\tau \subset X : \tau \in A_{\underline{l}}, \lambda(\tau) \leq x\}$. Then*

$$\pi_{\underline{l}}(x) \sim \frac{1}{l_{i_1}^{\alpha_{i_1}} \alpha_{i_1}! \ldots l_{i_s}^{\alpha_{i_s}} \alpha_{i_s}!} \pi(x)$$

where the l's and α's are as above and

$$\pi(x) \sim \frac{\beta}{\beta - 1} \cdot \frac{\beta^x}{x} \qquad \text{as } x \to \infty.$$

Example. In the case of a 3-point extension $\tilde{\sigma}$ of σ, i.e., $G = S_3$, the σ-closed orbits can lift in the extension space in 3 different ways. These corresponds to the partition (1,1,1), (2,1) and (3). Let us define the density

$D_{\underline{l}}$, of the σ-closed orbits that lift in the extension space corresponding to the partition $\underline{l}$ as

$$D_{\underline{l}} = \lim_{x \to \infty} \frac{\pi_{\underline{l}}(x)}{\pi(x)}$$

Since S_3 has 1 element with cycle decomposition (1,1,1), 3 elements with cycle decomposition (2,1) and 2 elements with cycle decomposition (3), we deduce that the densities $D_{(1,1,1)}, D_{(2,1)}, D_{(3)}$ are 1/6, 1/2 and 1/3 respectively.

We remark that our consideration here was really motivated by an analogous number-theoretic example due to Heilbronn (see pg 227 of [2]). In this example he considered a non-normal cubic field extension K_3/k and in particular was interested in the densities of primes in k according to how they lift into K_3. Roughly speaking, we can say that the primes in k splits in K_3 according to the partitions (1,1,1), (2,1) and (3) (of the number 3). To calculate the densities, Heilbronn then considered the minimal extension K_6 of K_3 that is normal over k and argued that the primes that splits in K_3 according to the partition (1,1,1), (2,1) and (3) corresponds precisely to the primes that splits in K_6 with Frobenius class $C(e), C(2,3), C(1,3,2)$ respectively. Note that the Galois group of K_6/k is S_3. Then applying the Chebotarev Theorem for the normal extension K_6/k, he deduced that (using the above notation) the densities $D_{(1,1,1)}, D_{(2,1)}, D_{(3)}$ are equal to 1/6, 1/2, 1/3 respectively.

4. Application II: Automorphism extensions of shifts of finite type

We now apply our findings of §2 to a so-called automorphism extension of the shift. As always let σ be a shift of finite type and G a finite group. Let $\gamma: G \to G$ be an automorphism of G and $\beta: X \to G$ be a function depending on a finite number of coordinates. An automorphism extension $\tilde{\sigma}$ of the shift is defined as the skew-product $\tilde{\sigma}: X \times G \to X \times G$ where $\tilde{\sigma}(x,g) = (\sigma x, \beta(x)\gamma(g))$. Letting $\tilde{\pi}(x,g) = x$ we have $\tilde{\pi}\tilde{\sigma} = \sigma\tilde{\pi}$. We shall assume that $\tilde{\sigma}$ is topologically transitive. Thus by definition $\tilde{\sigma}$ is also a shift of finite type.

Note that since G is finite there exists a least n such that $\gamma^n = \mathrm{id}$. Now consider the following cyclic extension of $\tilde{\sigma}$. That is $\sigma: \mathbb{Z}_n \times X \times G \to$

$\mathbf{Z}_n \times X \times G$ defined by $\hat{\sigma}(r,(x,g)) = (r+1,(\sigma(x),\beta(x)\gamma(g)))$. Also observe that except possibly for trivial γ, $\hat{\sigma}$ is never mixing. We can rewrite $\hat{\sigma}$ as $\hat{\sigma}: X \times \mathbf{Z}_n \times G \to X \times \mathbf{Z}_n \times G$ and $\hat{\sigma}(x,(r,g)) = (\sigma x,(r+1,\beta(x)\gamma(g)))$. We give the set $\mathbf{Z}_n \times G$ a group structure by defining the product of $(r,g),(s,h) \in \mathbf{Z}_n \times G$ as follows:

$$(r,g)\cdot(s,h) = (r+s, g\gamma^r(h))$$

and denoting the resulting group by $\mathbf{Z}_n \times_\gamma G$. Then $\mathbf{Z}_n \times_\gamma G$ with this operation defined on its elements is known as the semi-direct product of G by $\mathbf{Z}_n$ or a $\mathbf{Z}_n$ cyclic extension of G.

Let $\alpha: X \to \mathbf{Z}_n \times_\gamma G$ be defined as $\alpha(x) = (1,\beta(x))$ and let $G' = \mathbf{Z}_n \times_\gamma G$. Then we can rewrite $\hat{\sigma}$ as $\hat{\sigma}: X \times G' \to X \times G'$ and $\hat{\sigma}(x,k) = (x,\alpha(x)k)$, $k \in G'$, so that we can view $\hat{\sigma}$, as a group extension of σ. We can define a free action of G' on $X \times G'$ by $l(x,k) = (x,kl)$, $k,l \in G'$, and deduce that it commutes with $\hat{\sigma}$. Thus the notion of Frobenius class exists for σ-closed orbits.

Now, let H be the subgroup $\mathbf{Z}_n \times \{e\}$, ($e$ = identity of G) of G'. Consider the action of H on $X \times G'$. Then a typical element of the H-orbit space will take the form $(x,(0,g)H)$ where we write $(x,(0,g)H)$ to mean the set $\{(x,(r,g)) : r \in \mathbf{Z}_n\}$. Moreover it is easy to see that the induced map σ_2 satisfies $\sigma_2(x,(0,g)H) = (\sigma x,(0,\beta(x)\gamma(g))H)$. Hence we can identify $(X \times G, \hat{\sigma})$ with $((X \times G')/H, \sigma_2)$.

In other words we are in the setting of a homogeneous space extension of the shift and thus the result of §2 applies once we have formulated the Chebotarev Theorem for the extension $\pi : X \times G' \to X$. In particular given $(r,g) \in G'$ we would want to look at the action of $<(r,g)>$ on G'/H. Note that this is equivalent to studying the map $T_{(r,g)}: G'/H \to G'/H$ defined by $T_{(r,g)}((s,k)H) = (r,g)(s,k)H$. The following result may simplify the calculations.

4.1 Proposition. *Let $(r,g) \in G'$. Then $T_{(r,g)}$ is conjugate to the map $S_{(r,g)}: G \to G$ defined by $S_{(r,g)}(k) = g\gamma^r(k)$.*

Proof. Recall that a typical element of G'/H takes the form $(s,k)H = \{(s,k)(t,e) : t \in \mathbf{Z}_n\} = \{(s,k) : s \in \mathbf{Z}_n\}$. Hence G'/H can be identified with G via the map $(s,k)H \overset{L}{\mapsto} k$. Also $T_{(r,g)}((s,k)H) = (r,g)(s,k)H = (s, g\gamma^r(k))H$. Thus letting $S_{(r,g)}(k) = g\gamma^r(k)$, $k \in G$, we deduce that $LT_{(r,g)} = S_{(r,g)}L$. The result follows since L, $T_{(r,g)}$ and $S_{(r,g)}$ are bijective maps. □

As usual (a, b) shall denote the h.c.f of a and b.

4.2 Proposition. *Let $\tilde{\tau}$ be a $\tilde{\sigma}$-closed orbit with $\lambda(\tilde{\tau}) = k$ and $\hat{\tau}_1, \ldots, \hat{\tau}_r$ be the $\hat{\sigma}$-closed orbits that cover $\tilde{\tau}$. Then $\lambda(\hat{\tau}_i) = \text{l.c.m}[k, n]$, $i = 1, \ldots, r$ where $r = (k, n)$.*

Proof. Let $x \in \tau$. Then for all $r \in \mathbb{Z}$, $\hat{\sigma}^m(r, x) = (r + m, \tilde{\sigma}^m(x)) = (r, x)$ implies m is a multiple of both n and k. Hence the least period of $(r, x) = \text{l.c.m}[k, n]$. Recall that $\text{l.c.m}[a, b] = ab/(a, b)$. Thus since

$$\sum_{i=1}^{r} \deg \frac{\lambda(\hat{\tau}_i)}{\lambda(\tilde{\tau})} = n,$$

we have $r = (k, n)$. And this completes the proof. □

Let $\zeta_{\hat{\sigma}}(z)$, $\zeta_{\tilde{\sigma}}(z)$ be the zeta functions of $\hat{\sigma}$ and $\tilde{\sigma}$ respectively. Then we have

4.3 Proposition.

$$\zeta_{\hat{\sigma}}(z) = \prod_{i=0}^{n-1} \zeta_{\tilde{\sigma}}(\omega^i z).$$

where ω is a primitive n-th root of unity.

Proof.

First we note that

$$1 + \omega^r + \omega^{2r} + \ldots + \omega^{(n-1)r} = \begin{cases} n, & \text{if } n|r; \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $z \in \mathbb{C}$ and $k \in \mathbb{Z}^+$,

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{z^{km}}{m} \left\{1 + \omega^{km} + \ldots + \omega^{(n-1)km}\right\} &= \sum_{l=1}^{\infty} \frac{(n, k)}{l} z^{\frac{kn}{(n,k)} l} \\ &= \sum_{i=1}^{(n,k)} \sum_{l=1}^{\infty} \frac{z^{\frac{kn}{(n,k)} l}}{l}. \end{aligned}$$

Hence

$$\log(1 - z^k) + \log(1 - (\omega z)^k) + \ldots + \log(1 - (\omega^{n-1} z)^k) = (n, k) \log(1 - z^{\frac{kn}{(n,k)}}).$$

That is

$$(1 - z^k)(1 - (\omega z)^k)\dots(1 - (\omega^{n-1} z)^k) = (1 - z^{\frac{kn}{(n,k)}})^{(n,k)}.$$

Let $\tilde{\tau}$ be a $\tilde{\sigma}$-closed orbit such that $\lambda(\bar{\tau}) = k$. Then by (4.2),

$$\prod_{i=0}^{n-1} \left(1 - (\omega^i z)^{\lambda(\bar{\tau})}\right) = \prod_{\pi(\tilde{\tau}) = \bar{\tau}} \left(1 - z^{\lambda(\tilde{\tau})}\right).$$

The result follows by inverting and taking products over all $\tilde{\sigma}$-closed orbits.

□

4.4 Corollary. *If $\tilde{\sigma}$ is mixing then $\zeta_{\sigma}(z)$ has a non-zero analytic extension to a neighbourhood of $\{z \mid |z| \leq \beta^{-1}\}$ except for simple poles at $(\omega^i \beta)^{-1}$, $i = 0, 1, \dots, n-1$ where ω is a primitive n-th root of unity.*

Proof.

The result follows since $\hat{\sigma}$ is a topologically transitive shift of finite type.

□

Thus for mixing $\tilde{\sigma}$ the second part of (1.6) holds with n = period of the transition matrix of $\hat{\sigma}$. Hence the Chebotarev Theorem for the extension $\pi: X \times \mathbf{Z}_n \times_\gamma G \to X$ is

4.5 Theorem. *Let $\tilde{\sigma}$ be a mixing automorphism extension of σ and $\hat{\sigma}$ be the associated $\mathbf{Z}_n$ cyclic extension. Then, given a conjugacy class C,*

$$\pi_C(x) \sim \frac{|C|}{|G'|} \cdot n \cdot \frac{\beta^n}{\beta^n - 1} \cdot \frac{\beta^x}{x} \qquad \text{as } x \to \infty.$$

We now come to the main result in this section. Since the automorphism extension $\tilde{\sigma}$ can be identified with a homogeneous extension of σ with respect to the subgroup $H = \mathbf{Z}_n \times \{e\}$, we can apply our findings in §2 to the above theorem to obtain

4.6 Theorem. *Let $\tilde{\sigma}$ be a mixing automorphism extension of a shift of finite type σ. If $\underline{l}$ is a partition of $|G'/H|$ such that*

$$\begin{aligned} A_{\underline{l}} &:= \{\tau \subset X : \tau \text{ induces the partition } \underline{l} \text{ on } |G'|/|H|\} \\ &= \bigcup_{i=1}^{m} C_i \end{aligned}$$

where $C_i = \{\tau \subset X : [\tau] = C(r_i, g_i)\}$, $i = 1, \ldots, m$, then

$$\pi_{\tilde{A}}(x) \sim \sum_{i=1}^{m} \frac{|C(r_i, g_i)|}{|G'|} \cdot n \cdot \frac{\beta^n}{\beta^n - 1} \cdot \frac{\beta^x}{x} \qquad \text{as } x \to \infty.$$

There is another situation where a 'homogeneous extension' arises. Let $\tilde{A}$ be a hyperbolic automorphism of a finite dimensional torus $\mathbf{T}$. Let G be the set of all points in $\mathbf{T}$ with order m, say. Then G is an (abelian) group such that $\tilde{A}G = G$. We let G act on the right of $\mathbf{T}$ and using additive notation we have

$$\tilde{A}(x+g) = \tilde{A}(x) + \tilde{A}(g) \qquad x \in \mathbf{T}, g \in G.$$

Then $\tilde{A}$ induces an action A on the G-orbit space $\mathbf{T}/G$ such that $A\colon \mathbf{T}/G \to \mathbf{T}/G$ is also a hyperbolic toral automorphism. If $\tilde{A}^n|_G = \mathrm{Id}$, then working analogously with the automorphism extension, we can define the $\mathbf{Z}_n$-cyclic extension of $\tilde{A}$ and thus is in the setting studied above. In fact, one can show that the automorphism extension of the shift is actually the symbolic model for our toral automorphism (see pg 137 of [5] for the main idea). Furthermore, since the counting functions for the shifts and the toral automorphisms are asymptotic (see [4]), we deduce that the statements of Theorems (4.5) and (4.6) also hold for 'automorphism extensions' of hyperbolic toral automorphisms.

We illustrate the above discussion by the following example:

Example. Let $\tilde{A}$ be the hyperbolic automorphism on the two-dimensional torus $\mathbf{T}$ induced by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Here we take $\mathbf{T}$ to be the unit square on $\mathbf{R}^2$ with respect to addition mod 1 and the appropriate identifications. Let $G = \{(0,0), (0,1/2), (1/2,0), (1/2,1/2)\}$ i.e G consists of elements of $\mathbf{T}$ with order 2. Then $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. Also, one can easily check that $\tilde{A}^3|_G = \mathrm{Id}$. Thus the associated semi-direct product group is $G' = \mathbf{Z}_3 \times_{\tilde{A}} (\mathbf{Z}_2 \times \mathbf{Z}_2)$. Now G' has 4 conjugacy classes:

$$C_1 = \{(0,(0,0))\},$$
$$C_2 = \{(0,(0,1)), (0,(1,1)), (0,(1,0))\},$$
$$C_3 = \{(1,(0,0)), (1,(1,0)), (1,(1,1)), (1,(0,1))\},$$
$$C_4 = \{(2,(0,0)), (2,(1,1)), (2,(1,0)), (2,(0,1))\}.$$

By using (4.1), it is straight-forward to check that C_1 gives rise to the partition (1,1,1,1), C_2 to the partition (2,2) and both C_3, C_4 to the partition (3,1) on $|G'/Z_3 \times \{e\}|$. This implies that, given a closed orbit $\tau \in T/G$, the 'types' of $\tilde{A}$-closed orbits covering τ can only take one of the following forms: There are, depending on the Frobenius class of τ,

i. 4 closed orbits each of degree 1 over τ,

ii. 2 closed orbits each of degree 2 over τ, or

iii. 2 closed orbits, one of degree 3 and one of degree 1 over τ.

Hence the asymptotic formulas for types i. ii. iii. are

$$\pi_{(1,1,1,1)}(x) \sim \frac{1}{12} \cdot 3 \cdot \frac{\beta^3}{\beta^3 - 1} \cdot \frac{\beta^x}{x},$$
$$\pi_{(2,2)}(x) \sim \frac{3}{12} \cdot 3 \cdot \frac{\beta^3}{\beta^3 - 1} \cdot \frac{\beta^x}{x},$$
$$\pi_{(3,1)}(x) \sim \frac{8}{12} \cdot 3 \cdot \frac{\beta^3}{\beta^3 - 1} \cdot \frac{\beta^x}{x} \qquad \text{as } x \to \infty, \text{ respectively.}$$

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Chapter 2
Lifting Ergodic Properties To Homogeneous Extensions And Applications To Markov Shifts

1 Introduction

In a joint paper [12] (see Chapter 1), we study how closed orbits of a shift of finite type (X, σ) lift to a homogeneous extension $\hat{\sigma}: X \times G/H \to X \times G/H$ of X. Here G is a finite group and H is an arbitrary subgroup of G. The result we gave extends that of Parry and Pollicott [17] (see also [18]), which was derived for finite group extensions of shifts of finite type. Crucial to the proof of their theorem is the associated ergodicity and weak-mixing criteria for group extensions. Indeed, these criteria, which are valid for arbitrary compact group extensions of measure-preserving transformations, was utilized by Parry and Pollicott to deduce the analytic properties of a certain L-function and which in turn was used to obtain the necessary asymptotic formulas.

This motivates us to consider the analogous criteria for (compact) homogeneous extensions of measure preserving transformations. In other words, we are interested in lifting ergodicity and weak mixing of the base transformation to the homogeneous extension. Indeed, in this note, we give necessary and sufficient conditions for the ergodicity and weak-mixing of these homogeneous extensions. Similar to group extensions, the criteria we obtained also involves functional equations. The proof of these results are based on similar and in fact well-known results already obtained for the case of compact group extensions. For completeness, we also include the proofs of the analogous results for group extensions.

In addition to ergodicity and weak-mixing, we also take the opportunity to consider the appropriate lifting results in the case where the base transformation is a K-automorphism or a Bernoulli shift. For compact group extensions, the lifting results with respect to these dynamical properties are well-known. Unfortunately, when the base transformation is a K-automorphism, we are only able to solve the problem for the special case of a finite G. In this case , the result we provide follows easily from the analogous result obtained by Thomas [23] for compact group extensions (in fact, (G,τ)-extensions) since homogeneous extensions occur as factors of group extensions. Also, the relevant result for a Bernoulli base is simple since it is implicit in a result of Rudolph [21].

After establishing the above lifting results, we specialize our study to the case where the base transformation is a Markov shift. In particular, we are interested in the case where the associated extension has skewing-function depending on a finite number of coordinates. It is interesting to see how the above results simplify under this additional assumption. We show that, in this case, the solution to the relevant functional equations also involves functions depending on a finite number of coordinates. Indeed, in general, the aforementioned solution depends on one less coordinate than that of the skewing function. The results we obtain generalizes that of Parry [16] and of Adler *et al* [1].

In §2, we give the necessary definitions and basic elementary results. The well known results and their proofs for group extensions is given in §3. In §4, the relevant results for the homogeneous case is provided. Finally, in §5, we restrict our attention to the case of a Markov base and with the aid of several examples, illustrate the simplifications derived in this situation.

2 Preliminaries

In this section, we give the necessary definitions and facts which are needed for later use. The references for this section are Walters [24] and Cornfeld *et al* [2].

Unless otherwise stated, throughout this note, $(X, \mathcal{B}, m)$ will stand for a probability space and $T: X \rightarrow X$ a measure-preserving transformation on $(X, \mathcal{B}, m)$. On certain occasions we will require the probability space $(X, \mathcal{B}, m)$ to also be a Lebesgue space. It is well-known that this extra condition does not impose a serious restriction on the space in question for the property of being Lebesgue includes a lot of interesting dynamical systems. For further information on the theory of Lebesgue space, see Rohlin [20].

We begin with the definitions of the various extensions of T mentioned in the introduction.

2.1. *Compact group extensions.* Let G be a compact group equipped with the Borel σ-algebra $\mathcal{B}(G)$ and the (normalised) Haar measure λ. Also let $\phi: X \to G$ be a measurable map. Then the product space $(X \times G, \mathcal{B} \times \mathcal{B}(G), m \times \lambda)$ together with the map $S: X \times G \to X \times G$ defined by

$$S(x, \gamma) = (Tx, \phi(x)\gamma), \quad (x, \gamma) \in X \times G,$$

is called a compact group extension of T with respect to G. It is not too difficult to see that S is measure-preserving with respect to the Haar extension $m \times \lambda$. Moreover it is clear that S is indeed an extension of T with respect to the obvious projection map $\pi_G: X \times G \to X$. In the literature, S is also called a skew-product of T and G with skewing function ϕ.

Note that G acts on $X \times G$ by invertible measure-preserving transformations T_g, $g \in G$, where $T_g(x, \gamma) = (x, \gamma g)$, $(x, \gamma) \in X \times G$. Let U_{T_g} be the unitary operator on $L^2(X \times G)$ induced by the transformation T_g, $g \in G$. Then the map $G \times L^2(X \times G) \to L^2(X \times G)$ given by $(g, f) \mapsto U_{T_g}(f) = f \circ T_g$ defines a (continuous) action of G on $L^2(X \times G)$. In other words, we have a (continuous) representation of G by unitary operators on the Hilbert space $L^2(X \times G)$. For this reason $X \times G$ is also referred to as a G-space. Observe that the G-action on $X \times G$ commutes with the transformation S. We remark that this observation will be heavily relied upon when proving theorems in the next section.

2.2. *Homogeneous extensions.* Again let G be a compact group. Now let H be an arbitrary closed subgroup of G. Then it is well-known that the left action of G on G/H defined by $g \cdot \gamma H = g\gamma H$, $g \in G$ and $\gamma H \in G/H$, turns G/H into a (compact) homogeneous space. On the Borel σ-algebra $\mathcal{B}(G/H)$, we define the probability measure λ' by

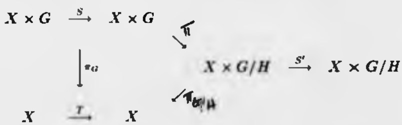
$$\lambda'(A) = \lambda(p^{-1}(A)), \quad A \in \mathcal{B}(G/H),$$

where λ is the (normalised) Haar measure on $(G, \mathcal{B}(G))$ and p is the natural map from G onto G/H. The measure λ' is more commonly known as the restricted Haar measure on $(G/H, \mathcal{B}(G/H))$. Let $\phi: X \to G$ be a measurable map. Then the product space $(X \times G/H, \mathcal{B} \times \mathcal{B}(G/H), m \times \lambda')$ together with the map $S': X \times G/H \to X \times G/H$ defined by

$$S'(x, \gamma H) = (Tx, \phi(x)\gamma H), \quad (x, \gamma H) \in X \times G/H,$$

is called the homogeneous extension of T with respect to G/H. Along similar lines of argument as for the group extension, one can show that S' is measure-preserving with respect to the restricted Haar extension $m \times \lambda'$. Furthermore, it is easy to see that S' is indeed an extension of T with respect to the obvious projection map $\pi_{G/H}: X \times G/H \to X$. Of course, S' is also referred to as a skew-product of T and G/H with skewing function ϕ.

Observe that given a homogeneous extension S' of T with respect to G/H and with skewing function ϕ, there exists a natural compact group extension S of T with respect to G, also with skewing function ϕ such that S is an extension of S'. Here the map connecting S and S' is none other than the map $\pi: X \times G \to X \times G/H$ given by $\pi(x, \gamma) = (x, \gamma H)$, $(x, \gamma) \in X \times G$. It is helpful to have the following commutative picture in mind:



As before, let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, m)$. We now give three examples of homogeneous extensions of T:

Example 1 Let G' be a compact topological group and $\tau: G' \to G'$ be a group automorphism such that $\tau^n = \mathrm{id}$, for some $n \in \mathbb{Z}^+$ (n least). Furthermore, let $\beta: X \to G'$ be a measurable map. Then the map $\tilde{S}: X \times G' \to X \times G'$ which is defined by

$$\tilde{S}(x, \gamma) = (Tx, \beta(x)\tau(\gamma)), \quad (x, \gamma) \in X \times G',$$

is a measure-preserving transformation with respect to the product σ-algebra $\mathcal{B} \times \mathcal{B}(G')$ and Haar extension $m \times \lambda$ (see (2.1)). Observe that the G'-action on $X \times G'$ (i.e. $g \cdot (x, \gamma) = (x, \gamma g)$, $(x, \gamma) \in X \times G'$ and $g \in G'$) satisfies the relation

$$Sg = \tau(g)S, \quad \text{for all } g \in G'.$$

The map S is more commonly known as a (G', τ)-extension or an automorphism extension of T.

Now, form the semi-direct product group $\mathbb{Z}_n \times_\tau G'$ of G' by $\mathbb{Z}_n$ where the product of $(r, \gamma), (s, \gamma') \in \mathbb{Z}_n \times_\tau G'$ is given by the formula

$$(r, \gamma) \cdot (s, \gamma') = (r + s, \gamma \tau^r(\gamma')).$$

Then it is clear that $\mathbb{Z}_n \times_\tau G'$ is also a compact topological group. Let G denote $\mathbb{Z}_n \times_\tau G'$ and H denote the (closed) subgroup $\mathbb{Z}_n \times_\tau \{e\}$ (e = identity of G') of G. We shall write $(0, \gamma)H$ for a typical element of the quotient space G/H. Then it is well-known that we can identify S with the homogeneous extension $S': X \times G/H \to X \times G/H$ where

$$S'(x, (0, \gamma)H) = (Tx, \phi(x)(0, \gamma)H), \quad (x, (0, \gamma)H) \in X \times G/H,$$

and $\phi: X \to G$ is given by $\phi(x) = (1, \beta(x))$. For details, see, for example, Noorani & Parry [12] (see Chapter 1). Hence, we deduce that any (G', τ)-extension of T with $\tau^n = \text{id}$ (some $n \in \mathbb{Z}^+$) reduces to a homogeneous extension of T with respect to the compact semi-direct product group $\mathbb{Z}_n \times_\tau G'$ and subgroup $\mathbb{Z}_n \times_\tau \{e\}$.

Example 2 Let S^{n-1} denotes the $(n-1)$-sphere and let $O(n)$ be the (compact) group of $n \times n$ real orthogonal matrices. Here we identify S^{n-1} with the subset $\{x \in \mathbb{R}^n : \|x\| = 1\}$ of $\mathbb{R}^n$ together with the spherical Lebesgue measure. Furthermore, let $\phi: X \to O(n)$ be a measurable map. Then the map $T': X \times S^{n-1} \to X \times S^{n-1}$ which is defined by

$$T'(x, v) = (Tx, \phi(x)v), \quad (x, v) \in X \times S^{n-1},$$

is a homogeneous extension of T. To see this, recall that the group $O(n)$ acts transitively on S^{n-1}. Then, as with all transitive actions, we can identify S^{n-1} with the quotient space $O(n)/O(n-1)$ where $O(n-1)$ is the closed subgroup of $O(n)$ fixing the element $(1, 0, \ldots, 0) \in S^{n-1}$, say (i.e. $O(n-1)$ is the isotropy group of $(1, 0, \ldots, 0)$). The map T' can then be identified with the map $S': X \times O(n)/O(n-1) \to X \times O(n)/O(n-1)$ where

$$S'(x, \gamma O(n-1)) = (Tx, \phi(x)\gamma O(n-1)),$$

for all $(x, \gamma O(n-1)) \in X \times O(n)/O(n-1)$. Hence T' is indeed a homogeneous extension of T.

Example 3 Let F be the set $\{1, 2, \ldots, n\}$ and S_n be the group of all permutations of F (i.e. the symmetric group on n–symbols). Define the measure ν on F by $\nu(i) = 1/n$, for each $i \in F$ and let $\phi: X \to S_n$ be a measurable map. Then the map $T': X \times F \to X \times F$ which is given by

$$T'(x, i) = (Tx, \phi(x)(i)), \quad (x, i) \in X \times F,$$

is a homogeneous extension of T. In fact, the map T' is more commonly known as a finite (n point) extension of T. To see why T' is a homogeneous extension of T, observe that S_n acts transitively on F. Hence, as is well known, we can identify F with the quotient space S_n/S_{n-1} so that T' is indeed a homogeneous extension of T with respect to this identification (for details, see, for e.g., Noorani & Parry [12] (see Chapter 1)).

We shall need the following standard result from Harmonic Analysis when proving theorems in the next two sections (see, for e.g., Hewitt & Ross [6] Theorem 27.44).

Proposition 1 *Let R be a continuous unitary representation of a compact group G with representation space $\mathcal{H}$. Then $\mathcal{H}$ decomposes into a direct sum of R-invariant closed finite dimensional subspaces $(V_i)_{i \in I}$ such that the restriction of R on V_i is irreducible for each $i \in I$.*

3 Lifting To Compact Group Extensions

As before, let T be a measure preserving transformation of a probability space $(X, \mathcal{B}, m)$ and S a compact group extension of T with skewing function $\phi: X \to G$.

In this section, we gather all the relevant results with respect to lifting ergodic properties of the base transformation T onto the compact group extension S which are needed later . We would like to stress that all the results in this section are well known. We include the proofs concerning ergodicity and weak mixing of the compact group extension S for completeness. We also remark that parts of these proofs go through to the homogeneous case, so that when proving the analogous results in the next section we shall constantly refer to the ones given here.

Recall that a measure-preserving transformation W of a probability space (Y, C, ν) is said to be ergodic if and only if every W-invariant measurable function ($f \circ W = f$ a.e.) on Y is constant a.e. The following result is essentially due to Keynes and Newton.

Theorem 1 ([10]) *Let T be ergodic. Then S is ergodic if and only if for any non-trivial irreducible unitary representation R of G (of degree d, say), the equation*

$$F(Tx) = R(\phi(x))F(x) \qquad a.e.\ x,$$

has no non-trivial measurable solutions $F: X \to \mathbb{C}^d$.

Proof Suppose the equation has a non-trivial measurable solution F (i.e. $F \neq 0$), for some non-trivial irreducible representation R of G. Let d be the dimension of R and define $H: X \times G \to \mathbb{C}^d$ by

$$H(x, \gamma) = R(\gamma^{-1})F(x), \quad (x, \gamma) \in X \times G.$$

Clearly H is measurable and non-constant. Moreover

$$\begin{aligned} H \circ S(x, \gamma) &= H(Tx, \phi(x)\gamma) \\ &= R((\phi(x)\gamma)^{-1})F(Tx) \\ &= R(\gamma^{-1})R(\phi(x)^{-1})F(Tx) \\ &= R(\gamma^{-1})F(x) \\ &= H(x, \gamma) \qquad \text{a.e. } (x, \gamma). \end{aligned}$$

Hence S cannot be ergodic.

Conversely, assume that the functional equation has no non-trivial solutions. For a contradiction, let us suppose that S is not ergodic. We shall construct a non-trivial irreducible representation R of G and a function F satisfying the required equation as follows:

Recall that G acts continuously on $L^2(X \times G)$ by unitary operators U_{T_g}. Now, consider the Hilbert subspace $\mathcal{H} = \{f \in L^2(X \times G) : f \circ S = f \text{ a.e.}\}$.

Then, since $T_g \circ S = S \circ T_g$, for all $g \in G$, we deduce that $\mathcal{H}$ is a G-invariant subspace of $L^2(X \times G)$. Let $\Gamma: G \to U(\mathcal{H})$ be the (continuous) representation induced by the restricted action of G on $\mathcal{H}$. Thus, Proposition 1 now implies that we can choose a non-trivial finite-dimensional subspace V of $\mathcal{H}$ such that $\Gamma_{|_V}$ is irreducible. Note that the non-triviality of V comes from the assumption S is not ergodic.

Let V be of dimension d and $\{f_1, \dots, f_d\}$ be an orthonormal basis for V. Then we have,

$$\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \circ T_g = (a_{ij}(g)) \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \qquad (*)$$

for some $d \times d$ matrix $(a_{ij}(g))$ depending only on $g \in G$. Also since $\{f_1, \dots, f_d\}$ is an orthonormal basis, we see that the matrix $(a_{ij}(g))$ is unitary for each $g \in G$. Now let $R(g) = (a_{ij}(g^{-1}))$, then $R(gh) = R(g)R(h)$. Hence R defines a non-trivial representation of G by $d \times d$ unitary matrices. Moreover, since R is equivalent to $\Gamma_{|_V}$ we deduce that R is also irreducible. Let $\bar{F} = \mathrm{col}\,(f_1, \dots, f_d)$. Then $(*)$ implies $\bar{F}(x, \gamma g) = R(g^{-1})\bar{F}(x, \gamma)$. Also, it is clear that $\bar{F} \circ S(x, \gamma) = \bar{F}(x, \gamma)$ a.e. (x, γ). Moreover, by Fubini's theorem, we can choose a 'good' $\gamma_0 \in G$ such that

$$\bar{F} \circ S(x, \gamma_0) = \bar{F}(x, \gamma_0) \qquad \text{a.e. } x.$$

Observe that

$$\begin{aligned} \bar{F}(x, \gamma_0) &= \bar{F} \circ S(x, \gamma_0) \\ &= \bar{F}(Tx, \phi(x)\gamma_0) \\ &= \bar{F}(Tx, \gamma_0(\gamma_0^{-1}\phi(x)\gamma_0)) \\ &= R(\gamma_0^{-1}\phi(x)^{-1}\gamma_0)\bar{F}(Tx, \gamma_0). \end{aligned}$$

That is

$$R(\gamma_0)\bar{F}(Tx, \gamma_0) = R(\phi(x))R(\gamma_0)\bar{F}(x, \gamma_0) \quad \text{a.e. } x.$$

Letting $F(x) = R(\gamma_0)\bar{F}(x, \gamma_0)$ we have

$$F(Tx) = R(\phi(x))F(x) \qquad \text{a.e. } x,$$

and this gives us the required contradiction. Thus S must be ergodic. □

Remarks

1. One of the first results in this direction was given by Furstenberg [5], where he considered the case where $G = K$. This was then generalised to the case where G is a compact abelian group by Parry [14]. The case when G is an arbitrary compact group was studied by Keynes & Newton [10]. In fact, Keynes & Newton also obtained a criteria for the ergodicity of an arbitrary (G, τ)-extension (i.e. τ need not satisfy $\tau^n = \mathrm{id}$ for some $n \in \mathbb{Z}^+$) of a topological transformation group (X, T) (see [11] for details).

2. Strictly speaking, the result derived by Keynes & Newton was achieved in the setting of a topological transformation group acting on an arbitrary G space. And in this setting the criteria for ergodicity was also given in the form of a functional equation but with an implicit usage of the representation R. In fact, the functional equation they gave involves the so-called R functions (R an irreducible representation of G) (see [10] for details). It is immediate that the same criteria also holds in our case. Furthermore, since we are dealing with the product bundle (with fibre G), the corresponding R functions, which are the functions H and F in the above proof, reduces to a function defined on the base space and thus making the usage of the representation R explicit.

3. For a result similar to that in Theorem 1, but where the criteria is given in terms of the range of the associated cocycle instead of a functional equation, see Zimmer [25].

A measure preserving transformation W on $(Y, \mathcal{C}, \nu)$ is said to be weak mixing if and only if every measurable function f on Y which satisfies $f \circ W = \lambda f$ a.e. is constant a.e. and $\lambda = 1$. We now give the necessary and sufficient conditions for a compact group extension to be weak mixing. We remark that much of the proof of Theorem 1 goes through to this case.

Theorem 2 *Let T be weak-mixing and S be ergodic. Then S is weak-mixing if and only if for any $e^{ia} \neq 1$ and any non-trivial one-dimensional representation χ of G, the equation*

$$F(Tx) = e^{ia}\chi(\phi(x))F(x) \qquad \text{a.e. } x$$

has no non-trivial measurable solution $F\colon X \to \mathbb{C}$.

Proof Suppose S is weak-mixing and the equation has a non-trivial measurable solution F (i.e. $F \neq 0$ a.e) for some non-trivial one-dimensional representation χ of G and some $e^{i\alpha} \neq 1$. Define $\bar{F}: X \times G \to \mathbb{C}$ by $\bar{F}(x,\gamma) = \chi(\gamma^{-1})F(x)$. Then clearly $\bar{F}$ is measurable and non-constant. Also

$$\begin{aligned} \bar{F} \circ S(x,\gamma) &= \bar{F}(Tx, \phi(x)\gamma) \\ &= \chi((\phi(x)\gamma)^{-1})F(Tx) \\ &= \chi(\phi(x)^{-1})\chi(\gamma^{-1})F(Tx) \\ &= e^{i\alpha}\chi(\gamma^{-1})F(x) \\ &= e^{i\alpha}\bar{F}(x,\gamma) \quad \text{a.e. } (x,\gamma). \end{aligned}$$

This contradicts the assumption that S is weak-mixing. Thus, there cannot be a non-trivial F satisfying the above equation.

Conversely, suppose that the equation has no non-trivial solutions. For a contradiction, let us suppose that S is not weak-mixing. Let $\mathcal{H} = \{f \in L^2(X \times G) : f \circ S = e^{i\alpha} f \text{ a.e.}\}$. Then $\mathcal{H}$ is a non-trivial closed subspace of $L^2(X \times G)$ for some $e^{i\alpha} \neq 1$. In particular $\mathcal{H}$ is a Hilbert space. As in the proof of Theorem 1, we have a (continuous) representation Γ of G by unitary operators defined by $\Gamma: G \to U(\mathcal{H})$ where $\Gamma(g) = U_{T_g}$. Invoking Proposition 1, we deduce that there exists a non-trivial finite-dimensional subspace V (of dimension d, say) such that the restriction of Γ on V is irreducible. Let $\{f_1, \ldots, f_d\}$ be an orthonormal basis for V. By definition, we have $f_j \circ S = e^{i\alpha} f_j$, $j = 1, \ldots, d$. Since the zero's of the f_j's have measure zero, we may divide f_j, $j = 1, \ldots, d$, with f_1 say, and apply the ergodicity of S to deduce that $f_j = c_j f_1$ a.e., where c_j are non-zero constants, $j = 1, \ldots, d$.

Thus, if we let $\bar{F} = \operatorname{col}(f_1, \ldots, f_d)$ then $\bar{F}(x,\gamma) = f_1(x,\gamma)c$ where $c = \operatorname{col}(c_1, \ldots, c_d) \neq 0$. Also, as in the proof of Theorem 1, we have, for each $g \in G$,

$$\bar{F}(x,\gamma g) = R(g^{-1})\bar{F}(x,\gamma) \qquad \text{a.e. } (x,\gamma) \qquad (*)$$

where $R: G \to U(\mathbb{C}^d)$ is the irreducible unitary representation of G arising from the matrix representation of the representation $\Gamma_{|_V}$, with respect to the orthonormal basis $\{f_1, \ldots, f_d\}$. Combining this with the fact that $\bar{F}(x,\gamma) = f_1(x,\gamma)c$ we obtain, for each $g \in G$,

$$f_1(x,\gamma g)c = R(g^{-1})f_1(x,\gamma)c \qquad \text{a.e. } (x,\gamma).$$

This last equation then implies that R leaves the one-dimensional subspace generated by the vector c invariant. Hence, since R is irreducible, we deduce that R is necessarily one-dimensional. Thus, by identifying R with its character χ, the equation $(*)$ becomes

$$\tilde{F}(x,\gamma g) = \chi(g^{-1})\tilde{F}(x,\gamma) \qquad \text{a.e. } (x,\gamma),$$

where now $\tilde{F}: X \times G \to \mathbb{C}^1$. Recall that $\tilde{F}$ also satisfy $\tilde{F} \circ S = e^{ia}\tilde{F}$ a.e. Hence, as before, we deduce that

$$\tilde{F}(x,\gamma_0) = e^{-ia}\chi(\phi(x)^{-1})\tilde{F}(Tx,\gamma_0) \qquad \text{a.e. } x$$

for some 'good' $\gamma_0 \in G$. Letting $F(x) = \tilde{F}(x,\gamma_0)$, then gives us the required contradiction. □

Remark The above result was proven by Jones & Parry in [7] for the abelian case. This was then extended to the compact case by Parry & Pollicott in [17] (see also [18]).

We now consider the case when T is a K-automorphism. Recall that an automorphism T of a probability space $(X,\mathcal{B},m)$ is said to be a K-automorphism if it possesses a sub-σ-algebra $\mathcal{A}$ such that $T^{-1}\mathcal{A} \subset \mathcal{A}$, $\cup_n T^n\mathcal{A}$ generates $\mathcal{B}$ and $\cap_n T^{-n}\mathcal{A}$ is the trivial σ-algebra $\mathcal{N}$. The following result is due to Thomas.

Theorem 3 ([23]) *Let T be a K-automorphism of a Lebesgue probability space $(X,\mathcal{B},m)$ and G a compact separable group. Then either S is a K-automorphism or S is not weak-mixing.*

Remark In the case when G is compact abelian, the above result was proven earlier by Parry [15]. The above theorem of Thomas, which generalises that of Parry, was in fact proven in the setting of a (G,τ)-extension of T (see [23] for details).

Before giving the corresponding result for a Bernoulli shift T, we shall need the notion of an isometric C-extension. Of course, by a Bernoulli shift we mean a probability space $(X,\mathcal{B},m)$ which is an infinite direct product $\prod_{-\infty}^{\infty}(Y,\mathcal{C},\nu)$ of a fix probability (state) space $(Y,\mathcal{C},\nu)$, together with the shift map T.

Let C be a compact homogeneous metric space and let G' be the compact metric group of all isometries of C. Then S is an isometric C-extension of T if S is a skew product $S: X \times C \to X \times C$ where

$$S(x,c) = (Tx,\phi(x)(c)), \quad (x,c) \in X \times C,$$

and $\phi: X \to G'$ is a measurable map. When T is Bernoulli, we have the following result of Rudolph:

Theorem 4 ([21]) *Let T be a finite entropy Bernoulli shift. Then an isometric C extension S of T either is itself Bernoulli or is not weak mixing.*

Since a compact metric group G is a homogeneous space (with respect to the left translations, say) and since the original metric on G can be replaced, if need be, with a left invariant metric ($d(gx, gy) = d(x, y)\ \forall x, y, g \in G$) so that the left translations are indeed isometries of G, we have, as an immediate consequence of the above theorem:

Corollary 1 *Let T be a finite entropy Bernoulli shift and G a compact metric group. Then a compact group extension S of T either is itself Bernoulli or is not weak mixing.*

It is interesting to note that the above result of Rudolph (Theorem 4) has recently been extended to the case where T is a $\mathbb{Z}^d$ Bernoulli shift (see Kammeyer [9]).

4 Lifting To Homogeneous Extensions

Our aim in this section is to obtain similar results to the ones given in the previous section for homogeneous extensions of a measure preserving transformation. In particular, let G be a compact group and T be a measure preserving transformation of a probability space $(X, \mathcal{B}, m)$. Let S' be a homogeneous extension of T with respect to some closed subgroup H of G and let ϕ be the skewing function (see §2 for details). Our first result is concerned with the ergodicity of S'.

Theorem 5 *Let T be ergodic. Then S' is ergodic if and only if for any non-trivial irreducible unitary representation R of G (of degree d, say) satisfying $bR(h) = b\ \ \forall h \in H$, for some non-zero $b \in \mathbb{C}^d$, the equation*

$$F(Tx) = R(\phi(x))F(x) \qquad a.e.\ x,$$

has no non trivial measurable solution $F: X \to \mathbb{C}^d$.

Proof Suppose S' is ergodic and the equation has a non trivial solution. That is, there exists some non trivial F, R and $b \in \mathbb{C}^d$ satisfying the above equation. Define $K: X \times G/H \to \mathbf{C}$ by $K(x, \gamma H) = bR(\gamma^{-1})F(x)$. Then K is well-defined and measurable. Also

$$\begin{aligned} K \circ S'(x, \gamma H) &= K(Tx, \phi(x)\gamma H) \\ &= bR((\phi(x)\gamma)^{-1})F(Tx) \\ &= bR(\gamma^{-1})R(\phi(x)^{-1})F(Tx) \\ &= bR(\gamma^{-1})F(x) \\ &= K(x, \gamma H) \qquad \text{a.e.} \quad (x, \gamma H). \end{aligned}$$

Since S' is ergodic and K is non-constant, we have a contradiction. Thus the equation cannot have non-trivial solutions.

Conversely, let us assume that the functional equation has no non-trivial solutions and for a contradiction, suppose that S' is not ergodic. Let S be the associated group extension (see §2). As before, let $\mathcal{H}$ be the Hilbert space consisting of S-invariant functions. By lifting non-constant S'-invariant functions onto the group extension we deduce that there exists some non-constant $k \in \mathcal{H}$ such that $k = k' \circ \pi_H$ for some S'-invariant function k'. Furthermore, it is easy to check that k is also H-invariant, i.e. $k \circ T_h = k \, \forall h \in H$.

As in the proof of Theorem 1, the G-action on $\mathcal{H}$ gives us a decomposition of $\mathcal{H}$ into a direct sum of finite-dimensional subspaces V_i such that G acts irreducibly on each V_i. Hence, we can uniquely write

$$k = \sum_{V_i} f^{V_i}, \qquad f^{V_i} \in V_i.$$

Furthermore, since k is H-invariant, we deduce that

$$k = k \circ T_h = \sum_{V_i} f^{V_i} \circ T_h \qquad \text{for all } h \in H,$$

so that $f^{V_i} \circ T_h = f^{V_i}$ for all V_i and $h \in H$. And in particular, there exists some subspace V such that $f^V \circ T_h = f^V$ for all $h \in H$ with f^V non-constant. Now let $\{f_1, \ldots, f_d\}$ be an orthonormal basis for V where d is the dimension of V. As in the proof of Theorem 1, we obtain a functional equation

$$F(Tx) = R(\phi(x))F(x) \ \text{ a.e. } x \qquad (1)$$

where $F: X \to \mathbf{C}^d$ is a non-trivial measurable function and R is the irreducible unitary representation of G arising from the restricted action of G onto V with respect to the orthonormal basis $\{f_1, \ldots, f_d\}$.

Also we have $f^V = b_1 f_1 + \ldots + b_d f_d$ for some $b = (b_1, \ldots, b_d) \neq 0$. Moreover, by writing $f^V = b\,\mathrm{col}(f_1, \ldots, f_d)$ and noting that $\mathrm{col}(f_1, \ldots, f_d) \circ T_g = R(g^{-1})\,\mathrm{col}(f_1, \ldots, f_d) \forall g \in G$, we obtain

$$f^V \circ T_h = b\,R(h^{-1})\,\mathrm{col}(f_1, \ldots, f_d)$$

for all $h \in H$. Thus, since $f^V \circ T_h = f^V \quad \forall h \in H$ and $\{f_1, \ldots, f_d\}$ is an orthonormal basis, we conclude that

$$bR(h) = b \quad \text{for all} \quad h \in H. \tag{2}$$

Combining (1) and (2) then gives us the required contradiction. □

We illustrate how one can apply the above result to the following situation:

Example Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, m)$. Let $G' = \{e, x, y, z\}$ be the Klein-4 group and $\tau\colon G' \to G'$ be given by $\tau(e) = e, \tau(x) = z, \tau(y) = x, \tau(z) = y$. Then τ is an automorphism of G' such that $\tau^3 = \mathrm{id}$. Suppose S is a (G, τ)-extension of T where $S(x, \gamma) = (Tx, \beta(x)\tau(\gamma))$, $(x, \gamma) \in X \times G'$, for some measurable map $\beta\colon X \to G'$ (see §2 for details). As before, let $G = \mathbb{Z}_3 \times_\tau G'$ be the associated semi-direct product group and let H be the subgroup $\mathbb{Z}_3 \times_\tau \{e\}$. Then one can easily check that there is only one (non-trivial) irreducible representation (up to equivalence class) R of G such that $bR(h) = b\ \forall h \in H$, for some non-zero $b \in \mathbb{C}^d$ ($d =$ degree of R). In fact, R has degree 3 and on H, R satisfies

$$R((0,e)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R((1,e)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R((2,e)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, any (non-zero) $b = (b_1, b_2, b_3) \in \mathbb{C}^3$ with $b_1 = b_2 = b_3$ satisfy $bR(h) = b$, $\forall h \in H$. Now, assume T is ergodic. Then, by Theorem 5, we have

> *S is ergodic if and only if there does not exists a non-trivial measurable function $F\colon X \to \mathbb{C}^3$ such that*
>
> $$F(Tx) = R((1, \beta(x)))F(x) \quad \text{a.e. } x.$$

We remark that this example occur as a 'symbolic model' for the hyperbolic toral automorphism $\tilde{A}$ on the two-dimensional torus $\mathbb{T}$ induced by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Here we take G' to be the group of all elements of $\mathbb{T}$ of order two, so that we have

$$\tilde{A}(x+g) = \tilde{A}(x) + \tilde{A}(g), \quad x \in \mathbb{T}, g \in G'.$$

Then, $\tilde{A}$ induces the toral automorphism A on the G–orbit space $\mathbb{T}/G'$ such that A is also hyperbolic. Of course, in this case, the symbolic model for the automorphism A, which correspond to the map $T: X \to X$ in the above discussion, is a subshift of finite type (see Noorani & Parry [12] (see Chapter 1) for further details).

As in §2, let S be the compact group extension associated with S'. The weak-mixing criteria for the homogeneous extension S' is given by the following theorem.

Theorem 6 *Suppose T is weak-mixing and S is ergodic. Then S' is weak-mixing if and only if for any $e^{i\alpha} \neq 1$ and any non-trivial one-dimensional representation χ of G, satisfying $\chi(h) = 1$ $\forall h \in H$, the equation*

$$F(Tx) = e^{i\alpha}\chi(\phi(x))F(x) \qquad a.e.\ x$$

has no non-trivial measurable solution $F: X \to \mathbb{C}$.

Proof Let S' be weak-mixing and suppose that the equation has a non-trivial measurable solution $F: X \to \mathbb{C}$ for some non-trivial one-dimensional representation χ and $e^{i\alpha} \neq 1$. Define $K: X \times G/H \to \mathbb{C}$ by $K(x, \gamma H) = \chi(\gamma^{-1})F(x)$. Then

$$\begin{aligned} K \circ S'(x,\gamma H) &= K(Tx, \phi(x)\gamma H) \\ &= \chi((\phi(x)\gamma)^{-1})F(Tx) \\ &= \chi(\gamma^{-1})\chi(\phi(x)^{-1})F(Tx) \\ &= e^{i\alpha}\chi(\gamma^{-1})F(x) \\ &= e^{i\alpha}K(x,\gamma H) \quad \text{a.e. } (x,\gamma H). \end{aligned}$$

This contradicts the assumption that S' is weak-mixing. Thus the equation cannot have non-trivial solutions.

Conversely, let us suppose that the equation has no non-trivial solutions and for a contradiction, suppose that S' is not weak-mixing. Thus since S' is assumed to be ergodic there exists $e^{i\alpha} \neq 1$ and non-constant $g: X \times G/H \to \mathbb{C}$ such that $g \circ S' = e^{i\alpha} g$ a.e. By lifting such g's onto the associated group extension $S: X \times G \to X \times G$, we deduce that the Hilbert space

$$\mathcal{H} = \{f \in L^2(X \times G) : f \circ S = e^{i\alpha} f\}$$

contains a non-constant function k such that k is also H-invariant, i.e. $k \circ T_h = k \quad \forall h \in H$. The rest of the proof then follows closely the proofs of theorems 2 and 5. □

We now consider the case when T is a K-automorphism. Unfortunately in this case, we are only able to obtain the appropriate result for the special case when G is a finite group. Before giving the relevant result in this case, we shall need the following observation: Recall that given a homogeneous extension $S': X \times G/H \to X \times G/H$ of T, there exists a group extension $S: X \times G \to X \times G$, which is an extension of both T and S'. Now, suppose S' is ergodic (so that T is ergodic) and S is not ergodic. Then we can decompose $X \times G$ into finitely many ergodic pieces $Y_1, \ldots, Y_n$. For each Y_i, let G_i be the subgroup of G such that $gY_i = Y_i$, for all g in G_i. Then the G_i-orbit space Y_i/G_i together with the map induced by $S_{|Y_i}$ on Y_i/G_i can be identified with the map $T: X \to X$. All the above follows from the ergodicity of T. Now, let $H_i = H \cap G_i$, $i = 1, \ldots, n$ and consider the H_i-orbit space Y_i/H_i together with the associated induced map. The assumption that S' is ergodic then implies that we can identify Y_i/H_i (together with the induced map) with $S': X \times G/H \to X \times G/H$. In other words, under the assumption that S' is ergodic, we have $S_{|Y_i}$ as an extension of both T and S', for each $i = 1, \ldots, n$. Thus, as far as the homogeneous extension S' is concerned there is no loss in generality if we also assume that the group extension S is ergodic. For if S is not ergodic, then we can always restrict our attention to any one of the ergodic pieces Y_i and then apply the appropriate identification.

Proposition 2 *Let T be a K-automorphism of a Lebesgue space $(X, \mathcal{B}, m)$ and G a finite group. Then a homogeneous extension S' of T either is itself a K-automorphism or is not weak-mixing.*

Proof Let T be a K-automorphism and suppose that the homogeneous extension S' is weak-mixing. Then by the above observation we may assume that the associated group extension S is ergodic. If S is also weak-mixing then, by Theorem 3, we deduce that S' is also a K-automorphism. This follows from the fact that factors of K-automorphisms are also K-automorphisms. We are now left with the case where S is ergodic but not weak-mixing. By using Theorem 2 and the fact that G is finite, we deduce that the eigenvalues of S forms a finite cyclic group generated by some $\omega \in \mathbb{C}$, where ω is a primitive d-th root of unity and $d > 1$. This means that the space $X \times G$ decomposes into a disjoint union of d pieces $X_1, \ldots, X_d$ such that $S(X_i) = X_j$ $(j = i+1 \bmod d)$ and $S^d|_{X_i}$ is weak-mixing, for $i = 1, \ldots, d$. Moreover, by applying arguments similar to the one given in the above observation to the map S^d, we deduce that $S^d|_{X_i}$ is an extension of both T^d and $(S')^d$, for each $i = 1, \ldots, d$. This is clear since both T^d and $(S')^d$ are ergodic. Hence, by applying Theorem 3 to $S^d|_{X_i}$ (any i) and noting that T^d is a K-automorphism, we gather that $S^d|_{X_i}$ is a K-automorphism. Thus $(S')^d$ is also a K-automorphism. The proof is completed since the fact that $(S')^d$ is a K-automorphism then implies S' is a K-automorphism. □

We remark that when S' is a (G, τ)-extension of a K-automorphism, Thomas [23] showed that, irrespective of $\tau^n = \mathrm{id}$ or not (for some $n \in \mathbb{Z}^+$), if S' is weak-mixing then it is a K-automorphism. As mentioned earlier, we have been unable to obtain the analogue of the above proposition for a general compact group G. Nevertheless, by taking the above discussions as our support, we would like to make the following conjecture:

Conjecture 1 *Let T be a K-automorphism of a Lebesgue space $(X, \mathcal{B}, m)$ and G a compact separable group. Then a homogeneous extension S' of T either is itself a K-automorphism or is not weak-mixing.*

Similar to Corollary 1, the result for Bernoulli T, is yet another consequence of the theorem of Rudolph (Theorem 4).

Corollary 2 *Let T be a finite entropy Bernoulli shift and G a compact metric group. Then a homogeneous extension S' of T either is itself Bernoulli or is not weak-mixing .*

Proof Let d be a metric on G. Without loss of generality, we may assume that d is translation invariant (i.e. $d(gx, gy) = d(x, y) = d(xh, yh)$ for all

$x, y, g, h \in G$) for if not, then we could replace d with the metric d' which is given by $d'(x,y) = \int(\int d(gxh, gyh)\, d\lambda(h))\, d\lambda(g)$. Set

$$d'(xH, yH) = \inf_{h,h' \in H} d(xh, yh'), \quad xH, yH \in G/H.$$

Then it is not too difficult to see that d' is a metric on G/H which generates the quotient topology (see, for e.g., Dieudonné [4]). In fact, $d'(xH, yH) = \inf_{h \in H} d(x, yh)$. Thus, since d' is translation invariant, we deduce that the left action of G on G/H ($g \cdot xH = gxH$, $g \in G, xH \in G/H$) is indeed by isometries. The result then follows by applying Theorem 4 to the case where $C = G/H$ and noting that G/H is compact. □

We would like to add that in an earlier paper [22], Rudolph obtained a result for finite extensions of a Bernoulli shift in terms of the existence of rotation factors. More precisely, he showed that a finite extension S' of a Bernoulli shift is either Bernoulli or has a finite rotation factor. Of course, the existence of a finite rotation factor then implies S' is not weakly mixing.

5 Applications to Markov Shifts

In this section, we specialize our study to the case where $(X, \mathcal{B}, m)$ is a Markov shift and T is the shift map σ. Recall that a Markov shift is defined as follows:

Let $P = (p_{ij})$ be an irreducible $k \times k$ stochastic matrix, i.e., $p_{ij} \geq 0$, $\sum_{j=0}^{k-1} p_{ij} = 1$ and for each i, j there exists some n such that $P^n(i,j) > 0$. Then there exists a unique $p = (p_0, \ldots, p_{k-1})$ such that $p_i > 0$, $\sum_{i=0}^{k-1} p_i = 1$ and $pP = p$. Let

$$X = \prod_{-\infty}^{\infty} \{0, 1, \ldots, k-1\} = \left\{x = (x_n)_{-\infty}^{\infty} : x_n \in \{0, 1, \ldots, k-1\}\right\}$$

and $\sigma : X \to X$ be defined by $\sigma(x_n) = (x_{n+1})$. Equip X with the σ-algebra $\mathcal{B}$ generated by the cylinder sets. A cylinder set is any set of the form

$${}_s[i_0, i_1, \ldots, i_{t-s}]_t = \{x \in X : x_s = i_0, x_{s+1} = i_1, \ldots, x_t = i_{t-s}\}$$

where $s \leq t$, $s, t \in \mathbb{Z}$. For each cylinder ${}_s[i_0, i_1, \ldots, i_{t-s}]_t$ define

$$m\, {}_s[i_0, i_1, \ldots, i_{t-s}]_t = p_{i_0} P_{i_0 i_1} \cdots P_{i_{t-s-1} i_{t-s}}$$

Then m extends uniquely to a probability measure on $\mathcal{B}$ such that σ preserves m. The probability space $(X, \mathcal{B}, m)$ together with the shift map σ is called an (ergodic two-sided) Markov shift with transition probability (p, P) and state space $\{0, 1, \ldots, k-1\}$. Note that when the matrix P has identical rows, then σ is a Bernoulli shift with state space $\{0, 1, \ldots, k-1\}$.

As mentioned in the introduction, we are interested in how the various lifting results obtained in the previous sections changes when we allow the skewing-function to depend only on a finite number of coordinates. To motivate our study, we would like to recall an old result of Kakutani:

Proposition 3 ([8]) *Let $\sigma: X \to X$ be a Bernoulli shift with state space $\{0, 1, \ldots, k-1\}$. Also let $\Lambda = \{\phi_0, \ldots, \phi_{k-1}\}$ be a (not necessary distinct) family of isomorphisms of a probability space $(Y, \mathcal{C}, \nu)$. Then the skew-product $\hat{\sigma}: X \times Y \to X \times Y$ which is defined by*

$$\hat{\sigma}(x, y) = (\sigma x, \phi_{x_0}(y))$$

is ergodic if and only if the family Λ is ergodic.

By an ergodic family of isomorphisms $\Lambda = \{\phi_0, \ldots, \phi_{k-1}\}$ we mean, any measurable set which is invariant under each $\phi_0, \phi_1, \ldots, \phi_{k-1}$ has measure zero or one (see [8] for further details). Observe that the skewing-function in the above proposition is a function of one coordinate. Also observe that, when applied to group extensions of Bernoulli shifts, the above result, in contrast with Theorem 1, does not involves any functional equation or group representations. It is interesting to see if the above proposition also holds for arbitrary Markov shifts. But, as the following example shows, this is not the case.

Example Let $X = \prod_{-\infty}^{\infty}\{0, 1\}$ be the Bernoulli $(\frac{1}{2}, \frac{1}{2})$-shift. Let K be the unit circle in the complex plane and α be a fixed irrational number. Define the function f on X by $f(x) = x_0$ (i.e. $f(x) = 0$ or 1). Then the skew-product $\hat{\sigma}: X \times K \to X \times K$ which is given by

$$\hat{\sigma}(x, z) = (\sigma x, e^{2\pi i (f(\sigma x) - f(x))\alpha} z), \quad (x, z) \in X \times K,$$

is never ergodic (see Theorem 1). Set $\phi_{x_0 x_1}(z) = e^{2\pi i (x_1 - x_0)\alpha} z$ so that we have the family $\Lambda = \{\phi_{00} (= \phi_{11}), \phi_{01}, \phi_{10}\}$ of rotations of K, as x_0, x_1 varies over 0 and 1. By a suitable recoding (see, for e.g. Denker *et al* [3]), we can consider

X as a 2 step Markov shift with alphabets 00, 01, 10 and 11. However, the family A of multiplications by $1, e^{2\pi i\alpha}, e^{-2\pi i\alpha}$ is ergodic, when α is irrational.

Observe that the above counter example tells us that Kakutani's result is not even true for mixing Markovs shifts. In particular, this negative answer then forces us to concentrate on the related functional equations and try to simplify it under the assumption that the skewing function depends only on a finite number of coordinates. Later on we shall give an analogue of Proposition 3 (i.e. when the skewing function depends only on one coordinate) which is valid for arbitrary Markov shifts. We shall need the following easily proven lemma.

Lemma 1 *Let V be a closed subspace of a Hilbert space $\mathcal{H}$. And let P be the orthogonal projection of $\mathcal{H}$ onto V. Then*

$$< x, Px > = < x, x > \text{ if and only if } Px = x.$$

We are indebted to Prof. W. Parry for showing us the proof of the following result. For the relevant properties of the conditional expectation operator and the Increasing Martingle Theorem, see Parry [13].

Proposition 4 *Let $T: X \to X$ be an automorphism of a probability space $(X, \mathcal{B}, m)$ and $\mathcal{A}$ an invariant sub-σ-algebra of $\mathcal{B}$ such that $T^n\mathcal{A} \uparrow \mathcal{B}$ (i.e. $T^{-1}\mathcal{A} \subset \mathcal{A}$ and $\cup_{n>0} T^n\mathcal{A}$ generates $\mathcal{B}$). Suppose U, V, F are $\mathcal{B}$ measurable functions from X into $U(d)$, the group of $d \times d$ unitary matrices, such that*

$$F \circ T(x) \cdot V(x) = U(x) \cdot F(x) \qquad a.e.\ x.$$

Then F is $\mathcal{A}$ measurable if both U, V are $\mathcal{A}$-measurable.

Proof We shall identify $U(d)$ with a subset of $\mathbb{C}^{d^2}$. Note that since U, V, F are bounded functions we deduce that $U, V, F \in L^2(X, \mathcal{B}, m, \mathbb{C}^{d^2})$. Let $E(\cdot|\mathcal{A})$ be the conditional expectation operator with respect to the sub-σ-algebra $\mathcal{A}$. Then the equation $F \circ T \cdot V = U \cdot F$ a.e. implies

$$E(F \circ T \cdot V|\mathcal{A}) = E(U \cdot F|\mathcal{A}) \text{ a.e.}$$

This in turn implies $E(F \circ T|\mathcal{A}) \cdot V = U \cdot E(F|\mathcal{A})$ a.e., for U, V are assumed to be $\mathcal{A}$ measurable. Also, since T is invertible, the conditional expectation operator satisfy

$$E(F \circ T|\mathcal{A}) = E(F|T\mathcal{A}) \circ T \text{ a.e.}$$

Hence, we obtain $E(F|T\mathcal{A}) \circ T \cdot V = U \cdot E(F|\mathcal{A})$, so that

$$F(x)^{-1} \cdot E(F|\mathcal{A})(x) = V(x)^{-1} \cdot (F \circ T(x))^{-1} \cdot E(F|T\mathcal{A}) \circ T(x) \cdot V(x) \quad \text{a.e. } x.$$

Thus

$$\text{Trace } F^{-1} \cdot E(F|\mathcal{A}) = \text{Trace } (F^{-1} \circ T) \cdot E(F|T\mathcal{A}) \circ T \quad \text{a.e.},$$

and

$$\int \text{Trace } F^{-1} \cdot E(F|\mathcal{A}) \, dm = \int \text{Trace } F^{-1} \cdot E(F|T\mathcal{A}) \, dm$$

since T is m-invariant. Also since F is unitary-valued, we have $F(x)^{-1} = F(x)^t$ for all $x \in X$. A straight forward calculation then gives us

$$\int \sum_{i,j} \bar{F}_{ji} E(F_{ji}|\mathcal{A}) \, dm = \int \sum_{i,j} \bar{F}_{ji} E(F_{ji}|T\mathcal{A}) \, dm$$

where the F_{ij}'s are the coordinate functions of F. Moreover, by considering the operator $E(\cdot|T\mathcal{A})$ and using the assumption $T\mathcal{A} \supset \mathcal{A}$, we deduce that the previous equation is equivalent to the equation

$$\int \sum_{i,j} E(\bar{F}_{ji}|T\mathcal{A}) E(F_{ji}|\mathcal{A}) \, dm = \int \sum_{i,j} E(\bar{F}_{ji}|T\mathcal{A}) E(F_{ji}|T\mathcal{A}) \, dm.$$

Also, observe that $T\mathcal{A} \supset \mathcal{A}$ implies $E(F_{ji}|\mathcal{A}) = E(\, E(F_{ji}|T\mathcal{A})\,|\mathcal{A})$ a.e., for each i, j. Thus, by applying Lemma 1 to the above integral equation with $P = E(\cdot|\mathcal{A})$, and noting that P is an orthogonal projection, we obtain

$$E(F_{ji}|\mathcal{A}) = E(F_{ji}|T\mathcal{A}) \quad \text{a.e., for each } i, j.$$

By repeating the above argument for $T^{n+1}\mathcal{A} \supset T^n\mathcal{A}$, $n \geq 0$, we deduce that

$$E(F_{ij}|\mathcal{A}) = E(F_{ij}|T^n\mathcal{A}) \quad \text{a.e., for all } n \geq 0.$$

Now, the increasing martingle theorem gives us , $E(F_{ij}|T^n\mathcal{A}) \to E(F_{ij}|\mathcal{B})$ a.e., for each i, j. So that, for each i, j, $E(F_{ij}|\mathcal{A}) = E(F_{ij}|\mathcal{B}) = F_{ij}$ a.e. This means each F_{ij} is $\mathcal{A}$-measurable. Thus F is $\mathcal{A}$-measurable. □

We also have

Proposition 5 *Let $T: X \to X$ be an automorphism of a probability space $(X, \mathcal{B}, m)$ and $\mathcal{A}$ an invariant sub-σ-algebra of $\mathcal{B}$ such that $T^n\mathcal{A} \uparrow \mathcal{B}$. Let $U: X \to U(d)$ and $F: X \to \mathbb{C}^d$ be $\mathcal{B}$-measurable functions such that they satisfy*

$$F(T(x)) = U(x) \cdot F(x) \qquad a.e.\ x.$$

Then F is $\mathcal{A}$-measurable if U is $\mathcal{A}$-measurable.

Proof We shall identify $U(d)$ with a subset of $\mathbb{C}^{d^2}$. First, we assume that $F \in L^2(X, \mathcal{B}, \mathbb{C}^d)$, so that $U \cdot F \in L^2(X, \mathcal{B}, \mathbb{C}^d)$. Let $E(\cdot|\mathcal{A})$ be the conditional expectation operator with respect to the sub–σ–algebra $\mathcal{A}$. Then, from the above equation we obtain $E(F \circ T|\mathcal{A}) = E(U \cdot F|\mathcal{A})$ a.e. That is

$$E(F|T\mathcal{A}) \circ T = U \cdot E(F|\mathcal{A}) \text{ a.e.}$$

since U is $\mathcal{A}$–measurable and T is invertible. Let $<,>$ be the standard inner–product on $L^2(X, \mathbb{C}^d)$. Then

$$< E(F|T\mathcal{A}) \circ T, F \circ T > = < U \cdot E(F|\mathcal{A}), U \cdot F > \quad \text{a.e.,}$$

implies

$$< E(F|T\mathcal{A}), F > \circ T \ = < E(F|\mathcal{A}), F > \quad \text{a.e.}$$

since U is unitary–valued. Therefore,

$$\int \sum_{i=1}^{d} \bar{F}_i E(F_i|T\mathcal{A}) \circ T \, dm = \int \sum_{i=1}^{d} \bar{F}_i E(F_i|\mathcal{A}) \, dm$$

where F_i, $i = 1, \ldots, d$ are the coordinate functions of F. Then, by imitating the proof of the above proposition we deduce that F is indeed $\mathcal{A}$–measurable.

Now, suppose that F is only $\mathcal{B}$–measurable. Let $|| \; ||$ be the standard norm on $\mathbb{C}^d$ and let $B_n = \{x \in X : ||F(x)|| \leq n\}$. Since U is norm–preserving we deduce that $||F|| \circ T = ||F||$ a.e., so that the set B_n is T–invariant for each $n \geq 1$. Let $F_n = \chi_{B_n} F$ for each $n \geq 1$. Then $F_n \circ T = U \cdot F_n$, for all $n \geq 1$. Thus, by the above F_n is $\mathcal{A}$–measurable. It is clear that $F_n \to F$ a.e. Hence F is $\mathcal{A}$–measurable. □

The above result generalises that of Parry [16] (see also (2.37) of Parry & Tuncel [19]).

We shall need the following result whose proof depends on Proposition 5 and follows closely the arguments given in (2.38) of Parry and Tuncel [19]. First, recall that a generator α of an automorphism T of a probability space $(X, \mathcal{B}, m)$ is a countable measurable partition of the space X such that

$$\bigvee_{i=-\infty}^{\infty} T^i \mathcal{A}(\alpha) = \mathcal{B} \qquad \text{(i.e. } \bigcup_{i=-\infty}^{\infty} T^i \mathcal{A}(\alpha) \text{ generates } \mathcal{B}\text{).}$$

Here $\mathcal{A}(\alpha)$ is the sub–σ–algebra generated by α.

Proposition 6 *Let T be an automorphism of a probability space $(X, \mathcal{B}, m)$ with generator α. Let $U: X \to U(d)$ and $F: X \to \mathbb{C}^d$ satisfy $F \circ T = U \cdot F$ a.e. If U is*

$$\mathcal{A}(\alpha) \vee T^{-1}\mathcal{A}(\alpha) \vee \ldots \vee T^{-n}\mathcal{A}(\alpha)$$

-measurable then F is measurable with respect to the sub-σ-algebra

$$\left(\bigvee_{i=0}^{\infty} T^{-i}\mathcal{A}(\alpha)\right) \bigcap \left(\bigvee_{i=-(n-1)}^{\infty} T^{i}\mathcal{A}(\alpha)\right), \quad \text{for } n = 1, 2, \ldots .$$

Corollary 3 *Let (X, σ) be a Markov shift. Suppose $U: X \to U(d)$ and $F: X \to \mathbb{C}^d$ satisfy $F \circ \sigma = U \cdot F$ a.e. If U is a function of $n+1$ coordinates (i.e. $U(x) = U(x_0 x_1 \ldots x_n)$ then F is a function of n coordinates (i.e. $F(x) = F(x_0 x_1 \ldots x_{n-1})$), $n = 1, 2, \ldots$.*

Proof Let α be the state partition of X, i.e., α consists of the cylinders $_0[i]_0$, where i is an element of the state space of X. Then it is well-known that α is a generator for (X, σ). The result then follows from the previous proposition, since for Markov shifts

$$\left(\bigvee_{i=0}^{\infty} T^{-i}\mathcal{A}(\alpha)\right) \bigcap \left(\bigvee_{i=-(n-1)}^{\infty} T^{i}\mathcal{A}(\alpha)\right) = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{A}(\alpha)$$

for $n = 1, 2, \ldots$ (see (2.41) of [19]). □

Remarks

1. The above result generalises Proposition 3 of Parry [16].
2. Note that in the special case when U is a function of one coordinate then F is also a function of one coordinate. We would like to add that in this case, there is a much shorter proof of Corollary 3 which only uses Proposition 4.
3. By imitating the proofs of the above results, one can also show that the analogue of Proposition 6 and Corollary 3 are also true for Markov shifts (X, σ) with unitary-valued functions U, V, F defined on X such that they satisfy $F \circ \sigma \cdot V = U \cdot F$ a.e. This result then generalises that of Adler *et al* [1].

We now give two examples to illustrate how one could apply Corollary 3 to the results obtained in the previous sections. As promised, the first example gives us an analogue of Kakutani's result for group extensions of arbitrary Markov shifts with skewing function depending only on one coordinate.

Example 1 Let (X,σ) be an ergodic Markov shift and let G be a compact group with a map $\phi: X \to G$ such that $\phi(x) = \phi(x_0 x_1)$, $\forall x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in X$, i.e., ϕ is a function of two coordinates. Furthermore, let $S: X \times G \to X \times G$ be the group extension of σ with skewing function ϕ. Then by combining Theorem 1 and Corollary 3, we obtain the following criteria for the ergodicity of S:

> *S is ergodic if and only if the only non-trivial solutions to the equation $F(x_1) = R(\phi(x_0 x_1))F(x_0)$ a.e. x, is when R is the trivial representation of the group G and F is constant a.e.*

Observe that the function F is a function of one coordinate. In the counterexample to Kakutani's result, the function F in the above functional equation may be chosen to be $F(x) = e^{2\pi i x_0 \alpha n}$ for each representation $R(z) = z^n$ of the unit circle K. In particular, this confirms the non ergodicity of σ. Note that the above criteria is also valid for the case when ϕ is a function of only one coordinate. This then gives us the analogue of Kakutani's result for Markov shifts (c.f. Prop. 3).

The following example is the analogue of the above criteria for homogeneous extensions.

Example 2 Let (X,σ) be an ergodic Markov shift and G be a compact group. Let H be a closed subgroup of G. Furthermore, let $S': X \times G/H \to X \times G/H$ be the homogeneous extension of σ with skewing function $\phi: X \to G$ depending on two coordinates, i.e., $\phi(x) = \phi(x_0 x_1)$. Then, by combining Theorem 5 and Corollary 3, we have:

> *S' is ergodic if and only if for any non-trivial irreducible unitary representation R of G satisfying $bR(h) = b$, $\forall h \in H$, for some non-zero $b \in \mathbb{C}^d$ ($d=$ degree of R), the equation*

$$F(x_1) = R(\phi(x_0 x_1))F(x_0) \quad a.e.\ x$$

has no non-trivial measurable solutions $F: X \to \mathbb{C}^d$ depending on one coordinate.

We can apply this criteria to the following situation. Suppose K is the unit circle in the complex plane and H is the subgroup $\{1, \omega, \ldots, \omega^{p-1}\}$, where ω is a primitive p-th root of unity. Furthermore, let $X = \prod_{-\infty}^{\infty}\{0,1\}$ be equipped with some ergodic Markov measure. Then, for irrational α, the homogeneous extension $S': X \times K/H \to X \times K/H$ where $S'(x, zH) = (\sigma x, e^{2\pi i(x_1 - x_0)\alpha} zH)$ is not ergodic. To see this, let χ be the representation of K satisfying $\chi(z) = z^p$, $\forall z \in K$. Then $\chi(h) = 1, \forall h \in H$. Moreover, the function $F: X \to \mathbb{C}$, defined by $F(x) = e^{2\pi x_0 \alpha p}$ satisfy

$$F(x_1) = \chi(\phi(x_0 x_1))F(x_0).$$

Thus S' is not ergodic by the above criteria.

As a final remark, we would like to add that, as far as Markov shifts are concerned, the condition for the associated group or homogeneous extensions to be ergodic, weak mixing, a K-automorphism or a Bernoulli shift (i.e. existence or non-existence of solutions to the relevant functional equations) is independent of the transition probability measure (p, P) in the following sense: if Q is another stochastic matrix which is compatible with P (i.e. $P(i,j) = 0 \iff Q(i,j) = 0$) then the same condition also holds when the base space X is endowed with the transition probability measure (q, Q) (for some unique probability vector q). This is clear since the measures induced by compatible stochastic matrices provide X with the same ergodic properties and since any solution (if it exists) to the relevant functional equation of a given extension with respect to one measure is also a solution to the functional equation of the same extension with respect to any other compatible measure.

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Chapter 3
A Note On The Rate Of Mixing Of Two–Dimensional Markov Shifts

1 Motivation

A standard result in the ergodic theory of one-dimensional Markov shift is as follows: Let $(T, X, \mathcal{C}, \mu)$ be a (one-dimensional) Markov shift where the Markov measure is given by some transition probability (p, P). Suppose A and B are arbitrary cylinder sets in $\mathcal{C}$. Then the sequence $(\mu(A \cap T^{-n}B))_{n \geq 0}$ converges to $\mu(A)\mu(B)$ at an exponential rate as n tends to infinity, when the matrix P is aperiodic (i.e., there exists some integer $N > 0$ such that all the entries of P^N are strictly positive). We remark that this result follows from the crucial matrix fact that when P is aperiodic, then the sequence $(P^n(i,j))_{n \geq 0}$ converges exponentially fast to $p(j)$ as n tends to infinity, for all i, j. Note that an immediate corollary to the above result is that T is strong mixing.

Our purpose in this short note is to generalize the aforementioned results to the case of a two-dimensional Markov shift. Observe that, the dynamical system in question consists of two commuting (invertible) measure-preserving transformations acting on the measurable space of functions from $\mathbb{Z}^2$ to some fixed finite set together with the Markov measure. Here the Markov measure is defined by two commuting stochastic matrices P and Q such that they share a common stationary probability vector p (see later for details). Working analogously with the one-dimensional case, we need to look at the rate of convergence of the sequence $(P^m Q^n)(i,j)_{m,n \geq 0}$ to $p(j)$ as m, n tends to infinity, for all i, j. We show that if either P or Q is aperiodic then the convergence rate of the aforementioned sequence is exponentially fast. This in turn implies the exponential convergence of measures on rectangle sets for the corresponding two-dimensional Markov shift (see Theorem 1). We indicate by an example what could happen if we relax the aperiodicity assumption on either P or Q. An immediate corollary to the above is that the two-dimensional Markov shift is strong-mixing.

2 Definitions and Results

Let Y be the finite set $\{1, 2, \ldots, k\}$ equipped with the σ–algebra 2^Y. The measurable space $(Y^{\mathbb{Z}^2}, \mathcal{B})$ is defined to be the space of all functions $x: \mathbb{Z}^2 \to Y$ endowed with the product σ–algebra $\mathcal{B}$. Recall that this means, $\mathcal{B}$ is the smallest σ–algebra such that the collection of all projection maps $\pi_F: Y^{\mathbb{Z}^2} \to Y^F$ which is given by $\pi_F(x) = x_{|_F}$ for each *finite* subset F of $\mathbb{Z}^2$, is measurable. Of course, the set Y^F here is equipped with the product σ–algebra $\prod_{\mathbf{c}\in F} 2^Y$. Given $x \in Y^{\mathbb{Z}^2}$, then we shall write $x_{\mathbf{c}}$ for the value of the function x at $\mathbf{c} \in \mathbb{Z}^2$.

We shall be interested in the following subsets of $Y^{\mathbb{Z}^2}$. Firstly, let F be the set $\{\mathbf{c} = (c_1, c_2) \in \mathbb{Z}^2 : a_t \le c_t \le a_t + u_t, t = 1, 2\}$ for some given $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}^2$, $\mathbf{u} = (u_1, u_2) \in (\mathbb{Z}^+)^2$. Then, an (elementary) rectangle $R_{\mathbf{a},\mathbf{u}}$ is any subset of $Y^{\mathbb{Z}^2}$ which takes the form

$$R_{\mathbf{a},\mathbf{u}} = \{x \in Y^{\mathbb{Z}^2} : x_{(c_1,c_2)} = i_{(c_1,c_2)}, \forall a_t \le c_t \le a_t + u_t, t = 1, 2\}$$

for some fix elements $i_{\mathbf{c}}$ of Y, for each $\mathbf{c} \in F$. It is clear that such subsets are measurable. Moreover, it is not difficult to see that the collection of such rectangles generates the product σ–algebra $\mathcal{B}$.

We shall now move on to the notion of a Markov measure on $(Y^{\mathbb{Z}^2}, \mathcal{B})$. For this, assume that we are given two $k \times k$–matrices P and Q satisfying the following three properties:

1. P, Q are stochastic matrices such that $PQ = QP$.

2. There exists a probability vector $p = (p(1), \ldots, p(k))$ such that $pP = p$ and $pQ = p$.

3. If P^0, Q^0 denotes the 0–1 matrices which are compatible with P and Q respectively, then we require $P^0Q^0 = Q^0P^0$ and P^0Q^0 is also a 0–1 matrix.

Let

$$R_{\mathbf{a},\mathbf{u}} = \{x \in Y^{\mathbb{Z}^2} : x_{(c_1,c_2)} = i_{(c_1,c_2)}, \forall a_t \leq c_t \leq a_t + u_t, t = 1,2\}$$

be a rectangle, for some $\mathbf{a} = (a_1, a_2) \in \mathbb{Z}^2$ and $\mathbf{u} = (u_1, u_2) \in (\mathbb{Z}^+)^2$. We shall call $R_{\mathbf{a},\mathbf{u}}$ an allowable rectangle if, in addition,

$$P^{0}(x_{(c_1,c_2)}, x_{(c_1+1,c_2)}) = Q^{0}(x_{(c_1,c_2)}, x_{(c_1,c_2+1)}) = 1$$

for all $x \in R_{\mathbf{a},\mathbf{u}}$ and $a_t \leq c_t \leq a_t + u_t - 1$, $t = 1,2$. We are now ready to define the Markov measure m on $(Y^{\mathbb{Z}^2}, \mathcal{B})$ associated with the matrices P and Q. Let $R_{\mathbf{a},\mathbf{u}} = \{x \in Y^{\mathbb{Z}^2} : x_{(c_1,c_2)} = i_{(c_1,c_2)}, \forall a_t \leq c_t \leq a_t + u_t, t = 1,2\}$ be an allowable rectangle. Then the measure of $R_{\mathbf{a},\mathbf{u}}$ is taken to be

$$\begin{aligned} m(R_{\mathbf{a},\mathbf{u}}) &= p(i_{(a_1,a_2)}) \prod_{e=0}^{u_1-1} P(i_{(a_1+e,a_2)}, i_{(a_1+e+1,a_2)}) \times \\ &\quad \prod_{f=0}^{u_2-1} Q(i_{(a_1+u_1,a_2+f)}, i_{(a_1+u_1,a_2+f+1)}) \end{aligned}$$

For non-allowable rectangles R, we take $m(R)$ to be zero. By using the Kolmogorov consistency theorem (see, for e.g., Parthasarathy [3]), m extends uniquely to a probability measure on the product σ-algebra $\mathcal{B}$. In analogy with the one dimensional case, we shall call this measure m the Markov measure defined by the matrices P and Q.

We shall define the horizontal shift $\sigma: Y^{\mathbb{Z}^2} \to Y^{\mathbb{Z}^2}$ and the vertical shift $\tau: Y^{\mathbb{Z}^2} \to Y^{\mathbb{Z}^2}$ by

$$(\sigma x)_{(c_1,c_2)} = x_{(c_1+1,c_2)} \text{ and } (\tau x)_{(c_1,c_2)} = x_{(c_1,c_2+1)}$$

for all $x \in Y^{\mathbb{Z}^2}$ and $(c_1, c_2) \in \mathbb{Z}^2$. Then, it is clear that σ and τ commutes. Moreover, since each σ and τ preserves the measure m on the algebra $\mathcal{A}$ of finite disjoint union of rectangles then they are measure preserving on the smallest σ-algebra containing $\mathcal{A}$, which is, of course, the product σ-algebra $\mathcal{B}$. Thus σ and τ are two commuting measure-preserving automorphisms acting on $(Y^{\mathbb{Z}^2}, \mathcal{B}, m)$. We shall call the resulting (invertible) measure-preserving dynamical system $(Y^{\mathbb{Z}^2}, \mathcal{B}, m, \sigma, \tau)$ a (two-dimensional) Markov shift with transition probability (p, P, Q).

Remarks

1. By working on rectangle sets, it can be shown that the assumption that

P^0Q^0 is also a 0–1 matrix is needed to check consistency of the Markov measure (c.f. Kolmogorov's theorem).

2. A second implication of the 0–1 assumption on the matrix P^0Q^0 is that for allowable rectangles $R_{\mathbf{a},\mathbf{u}}$, $m(R_{\mathbf{a},\mathbf{u}})$ is also given by

$$\begin{aligned} m(R_{\mathbf{a},\mathbf{u}}) &= p(i_{(a_1,a_2)}) \prod_{e'=0}^{u_2-1} P(i_{(a_1,a_2+e')}, i_{(a_1,a_2+e'+1)}) \times \\ & \prod_{f'=0}^{u_1-1} Q(i_{(a_1+f',a_2+u_2)}, i_{(a_1+f'+1,a_2+u_2)}) \end{aligned}$$

3. Suppose we give the set Y the discrete topology. Then the Markov measure m is supported by the subshift of finite type

$$X = \{x \in Y^{\mathbb{Z}^2} : P^0(x_{(e,f)}, x_{(e+1,f)}) = Q^0(x_{(e,f)}, x_{(e,f+1)}) = 1, \forall e, f \in \mathbb{Z}\}$$

when we assume that the stationary probability vector p is a strictly positive vector. Note that the fact that X is non–empty follows from the commuting assumption on P^0 and Q^0.

Using well–known methods from the theory of one–dimensional Markov shifts, we prove:

Lemma 1 *Suppose P and Q are two commuting $k \times k$-stochastic matrices such that there exists some probability vector $p = (p(1), \ldots, p(k))$ satisfying $pP = p = pQ$. If either P or Q is aperiodic then the sequence $(P^mQ^n(i,j))_{m,n>0}$ converges to $p(j)$ at an exponential rate as m, n tends to infinity, for all $i, j = 1, \ldots, k$, i.e., there exists constants $C > 0$, $0 \leq \alpha, \beta \leq 1$ with $\alpha\beta < 1$ such that*

$$| P^mQ^n(i,j) - p(j) | \leq C\alpha^m\beta^n \qquad \text{for all } m, n \geq 0$$

and for all $i, j = 1, \ldots, k$.

Proof Without loss on generality, we shall assume that P is aperiodic. Thus by the Perron–Frobenius theorem (see, for instance, Seneta [4]), the dominant eigenvalue 1 is a simple eigenvalue for P. Now, let V be the subspace $\{v \in \mathbb{C}^k :< p, v >= 0\}$. (Here $<,>$ denotes the standard inner product in $\mathbb{C}^k$). Then, it is easy to see that P leaves V invariant, i.e., $PV \subset V$.

Moreover, if we denote the vector $(1, 1, \ldots, 1)$ by $\mathbf{1}$, then each $w \in \mathbf{C}^k$ can be uniquely written as

$$w = w - < p, w > \cdot\mathbf{1} + < p, w > \cdot\mathbf{1}$$

such that $w - < p, w > \cdot\mathbf{1} \in V$ and $< p, w > \cdot\mathbf{1} \in U$, where U is the one dimensional subspace generated by the vector $\mathbf{1}$. Thus we have,

$$\mathbf{C}^k = V \oplus U.$$

Hence, by virtue of the simplicity of the eigenvalue 1 of P, we deduce that the spectral radius of $P_{|_V}$ is strictly less than 1. The spectral radius formula then implies there exists some $0 \leq \alpha < 1$ such that

$$\| P^m_{|_V} \| \leq C_1 \alpha^m$$

for all $m \geq 0$ and some constant $C_1 \geq 0$. Now, if Q also has 1 as a simple eigenvalue then using the same argument as above, gives us

$$\| Q^n_{|_V} \| \leq C_2 \beta^n$$

for all $n \geq 0$ and some constants $0 \leq \beta < 1$, $C_2 > 0$. Thus, in this case, we have

$$\| P^m_{|_V} Q^n_{|_V} \| \leq C \alpha^m \beta^n$$

for all $m, n \geq 0$ and some constant $C > 0$.

On the other hand, when the eigenvalue 1 of Q is no longer simple we may have that the spectral radius of $Q_{|_V}$ equals 1. In this case, it suffices to note that $Q^n_{|_V}$ is bounded so that

$$\| Q^n_{|_V} \| \leq C_3$$

for all $n \geq 0$ and for some constant $C_3 > 0$. This in turn implies that

$$\| P^m_{|_V} Q^n_{|_V} \| \leq C' \alpha^m$$

for all $m, n \geq 0$, and some constant $C' > 0$. Hence, in either case, we have

$$\| P^m_{|_V} Q^n_{|_V} \| \leq C \alpha^m \beta^n \qquad \forall m, n \geq 0$$

and for some constants $C > 0$, $0 \leq \alpha, \beta \leq 1$ with $\alpha\beta < 1$. From this last inequality we deduce that

$$\| P^m Q^n v \| \leq C \alpha^m \beta^n \| v \| \qquad \forall m, n \geq 0, \ \forall v \in V.$$

Recall that given $w \in \mathbb{C}^k$, then $w - \langle p, w \rangle \cdot \mathbf{1} \in V$. Thus, since $P^m Q^n$ is stochastic for all $m, n \geq 0$, we have

$$\| P^m Q^n w - \langle p, w \rangle \cdot \mathbf{1} \| \leq C\alpha^m \beta^n \| w - \langle p, w \rangle \cdot \mathbf{1} \|$$

for all $m, n \geq 0$ and all $w \in \mathbb{C}^k$. Furthermore, by taking $w = (0, \ldots, 0, 1, 0, \ldots, 0)$, the j-th unit vector, it is easy to see that

$$|P^m Q^n(i,j) - p(j)| \leq \| P^m Q^n w - \langle p, w \rangle \cdot \mathbf{1} \|$$

for all $m, n \geq 0$ and $i, j = 1, 2, \ldots, k$. Thus

$$|P^m Q^n(i,j) - p(j)| \leq C'\alpha^m \beta^n \qquad \forall m, n \geq 0,$$

and for all $i, j = 1, 2, \ldots, k$ where $C' = C \| w - \langle p, w \rangle \cdot \mathbf{1} \| > 0$. And this gives us the required result. □

Observe that the essential ingredient in the above proof is the fact that either P or Q has a simple dominant eigenvalue 1 and the rest of the spectrum having modulus strictly less than 1. Of course, a similar observation also holds in the case of a one–dimensional Markov shift. We will see later that this assumption is not necessary.

The following example of matrices illustrates what could go wrong if the hypothesis in the above lemma is relaxed. We remark that the matrices we are about to give are just the 'stochastic version' of an example due to Markley and Paul [2], which was used by them to study some topological dynamical questions with regards to higher–dimensional subshifts of finite type.

Example Let P and Q be the matrices

$$\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \quad \text{respectively.}$$

Then, it is easy to see that neither P nor Q is aperiodic and that $PQ = QP$ with $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ satisfying $pP = p = pQ$. Observe that since PQ is aperiodic then such a p is unique. This follows from the Perron-Frobenius theorem. Also, observe that the corresponding 0–1 matrix P^0 and Q^0 commutes and that P^0Q^0 is a 0–1 matrix. One can easily check that the characteristic polynomials of the matrices P, Q, PQ are given by

$$C_P(\lambda) = C_Q(\lambda) = (1-\lambda)^2(-\frac{1}{2}-\lambda)^2 \text{ and } C_{PQ}(\lambda) = (1-\lambda)(-\frac{1}{2}-\lambda)^2(\frac{1}{4}-\lambda)$$

respectively and that each of the matrices are diagonalizable. Recall that our aim is to look at the rate of convergence of the sequence $(P^mQ^n(i,j))_{m,n\geq 0}$ to 1/4, for each $i, j = 1, 2, 3, 4$. Note that in this situation, we can no longer use the method as in the proof of Lemma 1 for the eigenvalue 1 of P and Q is not simple. Nevertheless, in this specific example we can call upon the so called spectral decomposition method to look at the rate of convergence of the relevant sequences. For this, let R be the (non-singular) matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then

$$P = R\,\text{diag}(1, 1, -\frac{1}{2}, -\frac{1}{2})\,R^{-1} \quad \text{and} \quad Q = R\,\text{diag}(1, -\frac{1}{2}, 1, -\frac{1}{2})\,R^{-1}$$

so that

$$PQ = R\,\text{diag}(1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{4})\,R^{-1}.$$

Now, let E_1, E_2, E_3, E_4 be the matrices

$$R\,\text{diag}(1,0,0,0)\,R^{-1},\ R\,\text{diag}(0,1,0,0)\,R^{-1},\ R\,\text{diag}(0,0,1,0)\,R^{-1} \text{ and } R\,\text{diag}(0,0,0,1)\,R^{-1}$$

respectively. In particular,

$$E_1 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, E_2 = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix},$$

$$E_3 = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, E_4 = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

More importantly, we can write

$$P = E_1 + E_2 + (-\frac{1}{2})E_3 + (-\frac{1}{2})E_4$$

and

$$Q = E_1 + (-\frac{1}{2})E_2 + E_3 + (-\frac{1}{2})E_4.$$

Furthermore, since $E_iE_j = 0$ when $i \neq j$ and $E_i^2 = E_i$, we deduce that

$$P^m = E_1 + E_2 + (-\frac{1}{2})^m E_3 + (-\frac{1}{2})^m E_4$$

and

$$Q^n = E_1 + (-\frac{1}{2})^n E_2 + E_3 + (-\frac{1}{2})^n E_4$$

so that

$$P^mQ^n = E_1 + (-\frac{1}{2})^n E_2 + (-\frac{1}{2})^m E_3 + (-\frac{1}{2})^{m+n} E_4 \qquad \forall m, n \geq 0.$$

Thus, the (i,j)th-entry of P^mQ^n satisfy

$$P^mQ^n(i,j) - \frac{1}{4} = (-\frac{1}{2})^n E_2(i,j) + (-\frac{1}{2})^m E_3(i,j) + (-\frac{1}{2})^{m+n} E_4(i,j)$$

for each $i,j = 1,2,3,4$. It is clear that the sequence arising from the right-hand side of the above equality does not converge to zero at an exponential rate. Hence, this implies neither does the sequence $(P^mQ^n(i,j) - \frac{1}{4})_{m,n \geq 0}$ for all $i,j = 1,2,3,4$. □

Remarks

1. Recall that earlier on we mentioned that the validity of Lemma 1 relies on the simplicity of the dominant eigenvalue 1 of P or Q. This assumption is not necessary since in the case where all the non-zero entries of P and Q in the above example takes on the value 1/2, then $P^mQ^n(i,j) = 1/4$ for all $i,j = 1,2,3,4$ and for all $m,n > 0$. Moreover, since the stationary probability vector p is given by $(1/4, 1/4, 1/4, 1/4)$ then the sequence $(P^mQ^n(i,j))_{m,n>0}$ converges to 1/4 at an exponential rate trivially, for all i,j. Here, the eigenvalue 1 of both P and Q have multiplicity 2 and the rest of the eigenvalues are zero.

2. Observe that the above method of looking at the spectral decomposition of two diagonalizable commuting matrices in order to find the rate of convergence of the relevant sequences is quite general. This relies on the fact

that given any two commuting matrices P and Q then it is always possible to simultaneously 'diagonalize' them, i.e., there exists a matrix R such that

$$R^{-1}PR \text{ and } R^{-1}QR$$

are the Jordan canonical forms of P and Q respectively (see, for e.g., Jacobson [1]).

The following theorem is the main result of this note.

Theorem 1 *Let $(Y^{\mathbf{Z}^2}, \mathcal{B}, m, \sigma, \tau)$ be a Markov shift with transition probability (p, P, Q). Suppose either P or Q is aperiodic. Then given any rectangles $A, B \in \mathcal{B}$, there exists an integer $N > 0$ and constants $C > 0, 0 \leq \alpha, \beta \leq 1$ with $\alpha\beta < 1$ such that*

$$|m(A \cap \sigma^{-m}\tau^{-n}B) - m(A)m(B)| \leq C\alpha^m\beta^n$$

for all integers $m, n \geq N$.

Proof Let $A, B \in \mathcal{B}$ be two arbitrary rectangles. Then, by definition, there exists $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbf{Z}^2$, $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in (\mathbf{Z}^+)^2$ such that $A = R_{\mathbf{a},\mathbf{u}}$ and $B = R_{\mathbf{b},\mathbf{v}}$ where

$$R_{\mathbf{a},\mathbf{u}} = \{x \in Y^{\mathbf{Z}^2} : x_{(c_1,c_2)} = i_{(c_1,c_2)}, \forall a_t \leq c_t \leq a_t + u_t, t = 1, 2\}$$

and

$$R_{\mathbf{b},\mathbf{v}} = \{x \in Y^{\mathbf{Z}^2} : x_{(d_1,d_2)} = i'_{(d_1,d_2)}, \forall b_t \leq d_t \leq b_t + v_t, t = 1, 2\}.$$

Hence

$$\begin{aligned} A \cap \sigma^{-m}\tau^{-n}B = \{x \in Y^{\mathbf{Z}^2} &: x_{(c_1,c_2)} = i_{(c_1,c_2)}, \forall a_t \leq c_t \leq a_t + u_t \\ &\text{and } x_{(d_1+m, d_2+n)} = i'_{(d_1,d_2)}, \forall b_t \leq d_t \leq b_t + v_t, t = 1, 2\} \end{aligned}$$

Now, let $m > a_1 + u_1 - b_1$ and $n > a_2 + u_2 - b_2$. Then, in particular, $A \cap \sigma^{-m}\tau^{-n}B$ is a finite disjoint union of elementary rectangles $R_1, R_2, \ldots, R_k$, say. If either A or B is non-allowable, then each of the R_i's are non-allowable. Thus, in this case, we have $m(A \cap \sigma^{-m}\tau^{-n}B) = \sum_{i=1}^{k} m(R_i) = 0$. Moreover, since $m(A)m(B)$ is also zero, then the required result holds trivially in this case. We are now left with the case when the elementary rectangles A and B

are both allowable. Then, by the assumption P^0Q^0 is also a 0–1 matrix (see remark at the beginning of this section), it is straight forward (but tedious) to check that

$$m(A \cap \sigma^{-m}\tau^{-n}B) = p(i_{(a_1,a_2)}) \prod_{e=0}^{u_1-1} P(i_{(a_1+e,a_2)}, i_{(a_1+e+1,a_2)}) \times$$
$$\prod_{f=0}^{u_2-1} Q(i_{(a_1+u_1,a_2+f)}, i_{(a_1+u_1,a_2+f+1)}) P^{m'} Q^{n'} (i_{(a_1+u_1,a_2+u_2)}, i'_{(b_1,b_2)})$$
$$\prod_{e=0}^{v_1-1} P(i'_{(b_1+e,b_2)}, i'_{(b_1+e+1,b_2)}) \prod_{f=0}^{v_2-1} Q(i'_{(b_1+v_1,b_2+f)}, i'_{(b_1+v_1,b_2+f+1)})$$

where $m' = b_1 + m - (a_1 + u_1) > 0$, $n' = b_2 + n - (a_2 + u_2) > 0$. Observe that since one of P or Q is aperiodic, then the stationary probability vector p is strictly positive. Hence

$$m(A \cap \sigma^{-m}\tau^{-n}B) = \frac{m(A)m(B)}{p(i'_{(b_1,b_2)})} P^{m'} Q^{n'} (i_{(a_1+u_1,a_2+u_2)}, i'_{(b_1,b_2)}).$$

So that

$$\left| \frac{m(A \cap \sigma^{-m}\tau^{-n}B)}{m(A)m(B)} - 1 \right| = \frac{1}{p(i'_{(b_1,b_2)})} \left| P^{m'} Q^{n'} (i_{(a_1+u_1,a_2+u_2)}, i'_{(b_1,b_2)}) - p(i'_{(b_1,b_2)}) \right|$$

Thus, by combining the previous line and Lemma 1, we gather that there exists $0 \le \alpha, \beta \le 1$ with $\alpha\beta < 1$ such that

$$|m(A \cap \sigma^{-m}\tau^{-n}B) - m(A)m(B)| \le C\alpha^{m'}\beta^{n'}$$

for all $m', n' > 0$ and for some constant $C > 0$. Finally, by taking $N = \max((a_1 + u_1) - b_1, (a_2 + u_2) - b_2, 1)$, we deduce that

$$|m(A \cap \sigma^{-m}\tau^{-n}B) - m(A)m(B)| \le C'\alpha^{m}\beta^{n}$$

for all $m, n \ge N$ and some constant $C' > 0$. This then gives us the required result. □

Let T_1, T_2 be two commuting measure preserving transformations acting on the probability space $(Z, \mathcal{D}, \nu)$. Then the resulting dynamical system is said to be strong mixing if

$$\lim_{m,n \to \infty} m(A \cap T_1^{-m} T_2^{-n} B) = m(A)m(B)$$

for all $A, B \in \mathcal{D}$.

Recall that for Markov shifts, disjoint unions of rectangles forms an algebra that generates the product σ–algebra. Hence, by using a standard approximation theorem (see, for e.g., Walters [5]), we have an immediate corollary to Theorem 1.

Corollary 1 *Let $(Y^{\mathbb{Z}^2}, \mathcal{B}, m, \sigma, \tau)$ be a Markov shift with transition probability (p, P, Q). If either P or Q is aperiodic then the Markov shift is strong–mixing.*

Observe that if P, say, is aperiodic and Q is the identity matrix then we can identify the two–dimensional Markov shift with the one–dimensional Markov shift with transition probability (p, P). Thus we can retrieve the well–known mixing result for one–dimensional Markov shifts from the above corollary.

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