Stability Analysis of Lur'e Systems With Additive Delay Components Via A Relaxed Matrix Inequality

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Abstract

This paper is concerned with the stability analysis of Lur'e systems with sector-bounded nonlinearity and two additive time-varying delay components. In order to accurately understand the effect of time delays on the system stability, the extended matrix inequality for estimating the derivative of the Lyapunov-Krasovskii functionals (LKFs) is employed to achieve the conservatism reduction of stability criteria. It reduces estimation gap of the popular reciprocally convex combination lemma (RCCL). Combining the extended matrix inequality and two types of LKFs lead to several stability criteria, which are less conservative than the RCCL-based criteria under the same LKFs. Finally, the advantages of the proposed criteria are demonstrated through two examples.

Keywords: Lur'e system, additive time-varying delays, stability, matrix inequality, linear matrix inequality

1. Introduction

For considering the trade-off between the accurate modeling ability of nonlinear systems and the easy-to-analyze characteristic of linear systems, Lur'e systems with linear and sector-bounded nonlinear elements provide an effective way to model the practical systems, such as neural networks, Chua's circuits, quadruple-tank process systems [1–6]. The stability is the basic requirement for the systems, while this requirement may not be guaranteed due to the existence of time delays in communication channels [7–9]. Therefore, understanding the effect of delays on system stability is an important issue and has become a hot topic in the past few decades [10]. Main attention of those researches has been paid to the systems with single delay, which is combined all possible delays arising in the total communication channel. While signals transmitted in many systems may experience several different channels (for example, signals from sensors to controllers and from controllers to actuators) and successive delays with different properties could be induced [11]. Hence, it is also an important issue to analyze the stability of the systems with additive delay components.

The researches of the systems with additive delays were started from the continuous linear systems [11]. For the Lur'e systems, the related researches have mainly focused on different neural networks. In [12], the neural network with two additive delay components was proposed firstly to model different properties of delays. After that, many results for the analysis and design of similar models were reported. By constructing a Lyapunov-Krasovskii functional (LKF) and using a convex polyhedron method to estimate its derivative, two stability criteria were established in [13]. Tian *et al.* gave improved stability criteria by combining several useful techniques [14]. In [15], two stability criteria were developed through the free-weighting-matrix (FWM) approach and the Jensen inequality, respectively, and the relationship between them was discussed. In [16], the reciprocally convex combination lemma and the convex polyhedron method respectively led to two stability criteria with different computation burdens. Several techniques for the stability analysis of neural networks were reviewed and compared with each other in [17]. The LKFs with more general form, including triple and quadruple integral terms, were constructed to obtain stability criteria for neural networks with additive delays in [18] and [21]. In [22], the Wirtinger-based integral inequality, together with an augmented LKF, was applied to develop robust stability criteria for uncertain neural networks. The dynamic delay interval based LKF was developed in [23] to greatly reduce the conservatism of the resulting stability criteria. Very recently, the free-matrix-based integral inequalities developed in [26] and [27] were respectively extended to the stability/dissipativity analysis of neural networks with additive delays in [28] and the robust stability analysis of neural networks with Markov jump parameters [5].

Since the time delays arising in the communication channels are usually time-varying, the popular investigation framework for this type of time-delay system is to combine the LKF with the linear matrix inequality (LMI) [29, 30]. The aforementioned results are all obtained in this framework and the efforts were devoted to reduce the conservatism of the obtained stability criteria. During the estimation of derivative of the LKFs, bounding integral terms and eliminating time-varying delays are two key issues related to the conservatism of criteria [31]. Up to now, most researches, including but not limited to the ones reviewed above, have focused on bounding the integral terms via various integral inequalities, such as Jensen inequality [14–20], FWM-based approach [12–14, 17, 28], and Wirtinger-based inequality [22–24]. The elimination of the time-varying delays is usually achieved through the direct enlargement [12, 18], the convex combination technique [13–15, 17] and/or the reciprocally convex combination lemma [14, 16, 17, 22, 23, 25]. Very recently, a relaxed integral inequality was reported in our work [32], which combines the bounding of integral terms and the elimination of time-varying delays in the denominator so as to achieve the decrease of estimation gap. Moreover, an extended matrix inequality was developed in recent work [40], and similar results were obtained compared with [32]. It is expected that the stability criterion of the systems with additive delays will also be improved by following this idea. This motivates the current research.

This paper investigates the stability of Lur'e type nonlinear systems with two additive delay components and aims to develop stability criteria with less conservatism. To achieve the objective, two types of LFKs are summarized and combined with the extended matrix inequality. The comparison of the results shows the advantages of the proposed methods.

The remainder of the paper is organized as follows. Section II gives problem formulation. In Section III, several stability criteria are established. Two examples are studied to demonstrate the benefits of the proposed criteria in Section IV. Conclusions are presented in Section V.

Notations: Throughout this paper, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ (≥ 0) means that P is a real symmetric and positive-definite (semi-positive-definite) matrix; $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix; $\mathrm{col}\{x, y\} = [x^T, y^T]^T$; symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\} = X + X^T$. Matrices, if the dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation and Preliminaries

Consider the following Lur'e system with additive time-varying delay components:

$$\begin{cases} \dot{x}(t)=A_0x(t)+A_1x(t-d_1(t))+A_2x(t-d_1(t)-d_2(t)) \\ \qquad +W_0f(Wx(t)) + W_1f(Wx(t-d_1(t))) \\ \qquad +W_2f(Wx(t-d_1(t)-d_2(t))) \\ x(t)=\phi(t),\ t \in [-d, 0] \end{cases} \tag{1}$$

where $x(t)=[x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T$ is the state vector; W, W_i, and A_i, $i = 0, 1, 2$ are constant matrices with appropriate dimensions; $d_1(t)$ and $d_2(t)$ are time-varying delays satisfying

$$0 \leq d_1(t) \leq d_1, |\dot{d}_1(t)| \leq \mu_1 \tag{2}$$

and

$$0 \leq d_2(t) \leq d_2, |\dot{d}_2(t)| \leq \mu_2 \tag{3}$$

where d_i and μ_i, $i = 1, 2$, are constant, let $d(t) = d_1(t) + d_2(t)$ and $d = d_1 + d_2$; the initial function $\phi(t)$ is a continuous differentiable vector-valued function defined on the interval $[-d, 0]$; and $f(\cdot)=[f_1(\cdot)\ f_2(\cdot)\ \cdots\ f_n(\cdot)]^T$ is the nonlinear function and assumed to satisfy the following restriction [42]:

$$f_i(0) = 0$$

and

$$\sigma_i^- \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le \sigma_i^+, \quad s_1 \ne s_2 \tag{4}$$

and

$$\delta_i^- \le \frac{f_i(s)}{s} \le \delta_i^+, \quad s \ne 0 \tag{5}$$

where σ_i^-, σ_i^+, δ_i^-, and δ_i^+ are known real constants. Let $\Sigma_1 = \mathrm{diag}\{\sigma_1^+, \cdots, \sigma_n^+\}$, $\Sigma_2 = \mathrm{diag}\{\sigma_1^-, \cdots, \sigma_n^-\}$, $\Delta_1 = \mathrm{diag}\{\delta_1^+, \cdots, \delta_n^+\}$, and $\Delta_2 = \mathrm{diag}\{\delta_1^-, \cdots, \delta_n^-\}$.

Remark 1. *System (1) gives a general form of Lur'e systems and covers many widely studied systems (such as neural networks, genetical regulation networks, Chua's circuits, etc.) as special cases and it can be converted to each of them via the choice of related matrices, W, W_i, and A_i, $i = 0, 1, 2$.*

This paper aims to develop new stability criteria with less conservatism to understand the effect of time delays on the stability of system (1). Specifically, the key problem concerned for achieving this aim is to develop a new technique to estimate the following two integral terms:

$$\mathcal{S}(t) = \int_{\alpha_3(t)}^{\alpha_2(t)} \dot{x}^T(s) R \dot{x}(s) ds + \int_{\alpha_2(t)}^{\alpha_1(t)} \dot{x}^T(s) R \dot{x}(s) ds \tag{6}$$

where $R > 0$ is the symmetric matrix, $\alpha_i(t), i = 1, 2, 3$ are time-varying, $\alpha_2(t)$ includes the time-varying delays, and $\alpha_1(t)$ and $\alpha_3(t)$ satisfy $\alpha_1(t) - \alpha_3(t) = c$ with c being constant.

The Wirtinger-based integral inequality to be used is given in the following lemma.

Lemma 1. *[34] For symmetric matrix $R > 0$, scalars a and b with $a < b$, and vector ω such that the integration concerned is well defined, the following inequality holds*

$$(b - a) \int_a^b \dot{\omega}^T(s) R \dot{\omega}(s) ds \ge \chi_1^T R \chi_1 + 3\chi_2^T R \chi_2 \tag{7}$$

where $\chi_1 = \omega(b) - \omega(a)$ and $\chi_2 = \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(s) ds$.

For comparison study, the simple form of reciprocally convex combination lemma [35] is summarized as the following matrix inequality (named as RCMI in the subsequent sections):

Lemma 2. *[36] For a real scalar $\alpha \in (0, 1)$, symmetric matrices $R_1 > 0$ and $R_2 > 0$, and any matrix S satisfying $\begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \ge 0$, the following matrix inequality holds*

$$\begin{bmatrix} \frac{1}{\alpha} R_1 & 0 \\ 0 & \frac{1}{1-\alpha} R_2 \end{bmatrix} \ge \begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \tag{8}$$

An extended matrix inequality (named as ERCMI in the following sections) shown as following:

Lemma 3. *[40] For a real scalar $\alpha \in (0, 1)$, symmetric matrices $R_1 > 0$ and $R_2 > 0$, and any matrices S_1 and S_2, the following matrix inequality holds*

$$\begin{bmatrix} \frac{1}{\alpha} R_1 & 0 \\ 0 & \frac{1}{1-\alpha} R_2 \end{bmatrix} \ge \begin{bmatrix} R_1 + (1-\alpha) T_1 & (1-\alpha) S_1 + \alpha S_2 \\ * & R_2 + \alpha T_2 \end{bmatrix} \tag{9}$$

where $T_1 = R_1 - S_2 R_2^{-1} S_2^T$ and $T_2 = R_2 - S_1^T R_1^{-1} S_1$,

Remark 2. *Currently, an improved RCMI was developed in [41] and is shown below:*

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \geq \begin{bmatrix} R+(1-\alpha)T_3 & S \\ * & R+\alpha T_4 \end{bmatrix} \tag{10}$$

where, $T_3 = R - SR^{-1}S^T$ *and* $T_4 = R - S^T R^{-1} S$. *Clearly, the ERCMI is of less conservatism compared with inequality (10) and the RCMI:*

- *On one hand, the ERCMI includes the inequality (10) and the RCMI as a special case, respectively. By setting* $R_1 = R_2$, $S_1 = S_2$ *or* $S_1 = S_2$, $T_1 = T_2 = 0$, *the ERCMI can be rewritten as the inequality (10) or the popular RCMI.*
- *On the other hand, the restriction* $\begin{bmatrix} R & S \\ * & R \end{bmatrix} \geq 0$ *of the popular RCMI is relaxed in inequality (9). Therefore, extra freedom can be provided by the ERCMI which possibly reduces the conservatism.*

3. Main Results

In this section, several stability criteria are established by combining the RCMI and ERCMI with two types of LKFs.

3.1. Stability criteria based on the first LKF

Box I

$$d_3(t) = d_1(t) + d_2, \quad f(s) = f(Wx(s))$$

$$v_1(t) = \int_{t-d_1(t)}^{t} \frac{x(s)}{d_1(t)} ds, \quad v_2(t) = \int_{t-d(t)}^{t-d_1(t)} \frac{x(s)}{d_2(t)} ds, \quad v_3(t) = \int_{t-d_3(t)}^{t-d(t)} \frac{x(s)}{d_2 - d_2(t)} ds, \quad v_4(t) = \int_{t-d}^{t-d_3(t)} \frac{x(s)}{d_1 - d_1(t)} ds$$

$$\xi_1(t) = \mathrm{col}\{x(t),\ d_1(t)v_1(t),\ d_2(t)v_2(t), (d_2 - d_2(t))v_3(t),\ (d_1 - d_1(t))v_4(t)\}, \quad \xi_2(t) = \mathrm{col}\{x(t), f(t)\}$$

$$\zeta_a(t) = \mathrm{col}\{x(t), x(t-d_1(t)), x(t-d(t)), x(t-d_3(t)), x(t-d), v_1(t), v_2(t), v_3(t), v_4(t), f(t), f(t-d_1(t)), f(t-d(t))\}$$

As mentioned in [31], the integral terms are always included in the LKFs and their integration ranges (upper/lower limits of integration) are linked to the conservatism of the obtained criterion. For system (1), many time instants can be used as the limits of integration, such as t, $t - d_1(t)$, $t - d_1$, $t - d_2(t)$, $t - d_2$, $t - d(t)$, $t - d_1(t) - d_2$, $t - d_2(t) - d_1$, and $t - d$. The relationships among them are given as follows:

$$\begin{aligned} &t > t - d_1(t) > t - d_1 > t - d_2(t) - d_1 > t - d \\ &t > t - d_2(t) > t - d_2 > t - d_1(t) - d_2 > t - d \\ &t > t - d_2(t) > t - d(t) > t - d_2(t) - d_1 > t - d \\ &t > t - d_1(t) > t - d(t) > t - d_1(t) - d_2 > t - d \end{aligned} \tag{11}$$

As mentioned in [37], the LKF including integral terms with tighter integration ranges has the potential to reduce the conservatism. Moreover, system (1) includes the state at instant $t - d(t)$. Then, the relationship shown in (11) is used to construct single integral terms of the LKFs. That is, the first type of LKF is given as follows:

$$V_a(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \tag{12}$$

where

$$
\begin{aligned}
V_1(t) &= \xi_1^T(t)P\xi_1(t)\\
V_2(t) &= \int_{t-d_1(t)}^{t} \xi_2^T(s)Q_1\xi_2(s)ds + \int_{t-d(t)}^{t-d_1(t)} \xi_2^T(s)Q_2\xi_2(s)ds\\
&\quad + \int_{t-d_3(t)}^{t-d(t)} x^T(s)Q_3x(s)ds + \int_{t-d}^{t-d_3(t)} x^T(s)Q_4x(s)ds\\
V_3(t) &= 2\sum_{i=1}^{n}\int_0^{W_{2i}x} [\lambda_{1i}(\delta_i^+ s - f_i(s)) + \lambda_{2i}(f_i(s) - \delta_i^- s)]ds\\
V_4(t) &= \int_{-d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta
\end{aligned}
$$

and $P > 0$, $Q_i > 0, i = 1,2,3,4$, and $R > 0$ are the symmetric matrices with approximate dimension; and $L_i = \text{diag}\{\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{in}\} > 0, i = 1,2$ are the symmetric diagonal matrices; and other notations are listed in Box I.

Based on LKF (12) and Wirtinger-based inequality (7), two stability conditions of system (1) are obtained through inequalities (8) and (9), respectively, and summarized in the following theorem.

Theorem 1. *For given integers h_1, h_2, μ_1, and μ_2, system (1) with time-varying delay satisfying (2) and (3) is asymptotically stable, if the one of the following conditions holds*

C1. (9)-based condition: if there exist positive-definite symmetric matrices $P \in \mathcal{R}^{5n\times 5n}$, $Q_i \in \mathcal{R}^{2n\times 2n}, i = 1,2$, $Q_j \in \mathcal{R}^{n\times n}, j = 3,4$, and $R \in \mathcal{R}^{n\times n}$, positive-definite diagonal matrices $L_i \in \mathcal{R}^{n\times n}, i = 1,2$, $H_j \in \mathcal{R}^{n\times n}$, and $G_j \in \mathcal{R}^{n\times n}$, $j = 1,2,3$, and any matrices $X_i, Y_i \in \mathcal{R}^{2n\times 2n}, i = 1,2$, such that the following holds for $(\dot{d}_1(t), \dot{d}_2(t)) \in \mathbf{\Omega_2}$ (Note that $\Psi_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))$ is simplified as Ψ_1):

$$
\begin{bmatrix} \Psi_1|_{d_1(t)=0,d_2(t)=0} & E_1^TX_2 & E_2^TY_2\\ * & -d_1\tilde{R} & 0\\ * & * & -d_2\tilde{R}\end{bmatrix} < 0 \tag{13}
$$

$$
\begin{bmatrix} \Psi_1|_{d_1(t)=0,d_2(t)=d_2} & E_1^TX_2 & E_3^TY_1^T\\ * & -d_1\tilde{R} & 0\\ * & * & -d_2\tilde{R}\end{bmatrix} < 0 \tag{14}
$$

$$
\begin{bmatrix} \Psi_1|_{d_1(t)=d_1,d_2(t)=0} & E_4^TX_1^T & E_2^TY_2\\ * & -d_1\tilde{R} & 0\\ * & * & -d_2\tilde{R}\end{bmatrix} < 0 \tag{15}
$$

$$
\begin{bmatrix} \Psi_1|_{d_1(t)=d_1,d_2(t)=d_2} & E_4^TX_1^T & E_3^TY_1^T\\ * & -d_1\tilde{R} & 0\\ * & * & -d_2\tilde{R}\end{bmatrix} < 0 \tag{16}
$$

C2. (8)-based condition: if there exist positive-definite symmetric matrices $P \in \mathcal{R}^{5n\times 5n}$, $Q_i \in \mathcal{R}^{2n\times 2n}, i = 1,2$, $Q_j \in \mathcal{R}^{n\times n}, j = 3,4$, and $R \in \mathcal{R}^{n\times n}$, positive-definite diagonal matrices $L_i \in \mathcal{R}^{n\times n}, i = 1,2$, $H_j \in \mathcal{R}^{n\times n}$, and $G_j \in \mathcal{R}^{n\times n}$, $j = 1,2,3$, and any matrices $X, Y \in \mathcal{R}^{2n\times 2n}$, such that the following holds for $(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) \in \mathbf{\Omega}$:

$$
\begin{bmatrix} \tilde{R} & X\\ * & \tilde{R}\end{bmatrix} > 0 \tag{17}
$$

$$
\begin{bmatrix} \tilde{R} & Y\\ * & \tilde{R}\end{bmatrix} > 0 \tag{18}
$$

$$
\Psi_2(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0 \tag{19}
$$

where the related notations are given in Box II.

Box II: notations used in Theorem 1

$$\mathbf{\Omega} = \Big\{(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))\Big| d_1(t) \in \{0, d_1\}, d_2(t) \in \{0, d_2\}, \dot{d}_1(t) \in \{-\mu_1, \mu_1\}, \dot{d}_2(t) \in \{-\mu_2, \mu_2\}\Big\}$$

$$\mathbf{\Omega_1} = \Big\{(d_1(t), d_2(t))\Big| d_1(t) \in \{0, d_1\}, d_2(t) \in \{0, d_2\}\Big\}, \quad \mathbf{\Omega_2} = \Big\{(\dot{d}_1(t), \dot{d}_2(t))\Big| \dot{d}_1(t) \in \{-\mu_1, \mu_1\}, \dot{d}_2(t) \in \{-\mu_2, \mu_2\}\Big\}$$

$$\Psi_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))$$

$$= \begin{cases} \sum_{i=1}^{4} \Phi_i + F_1^T P F_2 + F_2^T P F_1 + e_s^T (dR) e_s - \frac{1}{d_1} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R} & X_1 \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix} - \frac{1}{d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R} & Y_1 \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{if } d_1(t) = 0, d_2(t) = 0 \\ \sum_{i=1}^{4} \Phi_i + F_1^T P F_2 + F_2^T P F_1 + e_s^T (dR) e_s - \frac{1}{d_1} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & X_2 \\ * & 2\tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix} - \frac{1}{d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R} & Y_1 \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{if } d_1(t) = d_1, d_2(t) = 0 \\ \sum_{i=1}^{4} \Phi_i + F_1^T P F_2 + F_2^T P F_1 + e_s^T (dR) e_s - \frac{1}{d_1} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R} & X_1 \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix} - \frac{1}{d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & Y_2 \\ * & 2\tilde{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{if } d_1(t) = 0, d_2(t) = d_2 \\ \sum_{i=1}^{4} \Phi_i + F_1^T P F_2 + F_2^T P F_1 + e_s^T (dR) e_s - \frac{1}{d_1} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & X_2 \\ * & 2\tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix} - \frac{1}{d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & Y_2 \\ * & 2\tilde{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{if } d_1(t) = d_1, d_2(t) = d_2 \end{cases}$$

$$\Psi_2(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) = \sum_{i=1}^{4} \Phi_i + F_1^T P F_2 + F_2^T P F_1 + e_s^T (dR) e_s - \frac{1}{d_1} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & X \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix} - \frac{1}{d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & Y \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}$$

$$\Phi_1 = \left[\begin{array}{c} e_1 \\ e_{10} \\ \hline e_5 \end{array}\right]^T \left[\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & -Q_4 \end{array}\right] \left[\begin{array}{c} e_1 \\ e_{10} \\ \hline e_5 \end{array}\right] - (1 - \dot{d}_1(t)) \left[\begin{array}{c} e_2 \\ e_{11} \\ \hline e_4 \end{array}\right]^T \left[\begin{array}{c|c} Q_1 - Q_2 & 0 \\ \hline 0 & Q_3 - Q_4 \end{array}\right] \left[\begin{array}{c} e_2 \\ e_{11} \\ \hline e_4 \end{array}\right] - (1 - \dot{d}(t)) \begin{bmatrix} e_3 \\ e_{12} \end{bmatrix}^T \left(Q_2 - \begin{bmatrix} Q_3 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} e_3 \\ e_{12} \end{bmatrix}$$

$$\Phi_2 = Sym\Big\{[(\Delta_1 W e_1 - e_{10})^T L_1 + (e_{10} - \Delta_2 W e_1)^T L_2] W e_s\Big\}, \ \Phi_3 = Sym\left\{ \sum_{i=1}^{3} [\Delta_1 W e_i - e_{i+9}]^T H_i [e_{i+9} - \Delta_2 W e_i] \right\}$$

$$\Phi_4 = Sym \left\{ \begin{array}{l} \sum_{i=1}^{2} [\Sigma_1 W(e_i - e_{i+1}) - (e_{i+9} - e_{i+10})]^T G_i [(e_{i+9} - e_{i+10}) - \Sigma_2 W(e_i - e_{i+1})] \\ +[\Sigma_1 W(e_1 - e_3) - (e_{10} - e_{12})]^T G_3 [(e_{10} - e_{12}) - \Sigma_2 W(e_1 - e_3)] \end{array} \right\}$$

$$F_1 = \mathrm{col}\{e_1, d_1(t) e_6, d_2(t) e_7, (d_2 - d_2(t)) e_8, (d_1 - d_1(t)) e_9\}$$

$$F_2 = \mathrm{col}\Big\{e_s, e_1 - (1 - \dot{d}_1(t)) e_2, (1 - \dot{d}_1(t)) e_2 - (1 - \dot{d}(t)) e_3, (1 - \dot{d}(t)) e_3 - (1 - \dot{d}_1(t)) e_4, (1 - \dot{d}_1(t)) e_4 - e_5\Big\}$$

$$E_i = \begin{bmatrix} e_i - e_{i+1} \\ e_i + e_{i+1} - 2e_{i+5} \end{bmatrix}, i = 1, 2, 3, \quad \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix}$$

$$e_i = [0_{n \times (i-1)n} \ \ I_{n \times n} \ \ 0_{n \times (12-i)n}], \ i = 1, 2, \cdots, 12, \quad e_s = A_0 e_1 + A_1 e_2 + A_2 e_3 + W_0 e_{10} + W_1 e_{11} + W_2 e_{12}$$

Proof: The condition that P, Q_i, $i = 1, 2, 3, 4$, R, and L_j, $j = 1, 2$, are positive-definite matrices leads that the LKF satisfies $V_a(t) \geq \epsilon \|x(t)\|^2$ with $\epsilon > 0$.

The derivative of $V_1(t)$ along the solution of system (1) can be obtained as

$$\dot{V}_1(t) = 2 \begin{bmatrix} x(t) \\ d_1(t) v_1(t) \\ d_2(t) v_2(t) \\ (d_2 - d_2(t)) v_3(t) \\ (d_1 - d_1(t)) v_4(t) \end{bmatrix}^T P$$

$$\times \begin{bmatrix} \dot{x}(t) \\ x(t) - (1-\dot{d}_1(t))x(t-d_1(t)) \\ (1-\dot{d}_1(t))x(t-d_1(t)) - (1-\dot{d}(t))x(t-d(t)) \\ (1-\dot{d}(t))x(t-d(t)) - (1-\dot{d}_1(t))x(t-d_3(t)) \\ (1-\dot{d}_1(t))x(t-d_3(t)) - x(t-d) \end{bmatrix}$$

$$= \zeta_a^T(t)(F_1^T PF_2 + F_2^T PF_1)\zeta_a(t) \tag{20}$$

where F_1 and F_2 are defined in Box II.

The derivative of $V_2(t)$ along the solution of system (1) can be obtained as

$$\begin{aligned}
\dot{V}_2(t) = \ & \xi_2^T(t)Q_1\xi_2(t) - x^T(t-d)Q_4x(t-d) \\
& - (1-\dot{d}_1(t))\xi_2^T(t-d_1(t))(Q_1 - Q_2)\xi_2(t-d_1(t)) \\
& - (1-\dot{d}(t))\xi_2^T(t-d(t))Q_2\xi_2(t-d(t)) \\
& - (1-\dot{d}(t))x^T(t-d(t))Q_3x(t-d(t)) \\
& - (1-\dot{d}_1(t))x^T(t-d_3(t))(Q_3 - Q_4)x(t-d_3(t)) \\
= \ & \zeta_a^T(t)\Phi_1\zeta_a(t)
\end{aligned} \tag{21}$$

where Φ_1 is defined in Box II.

The derivative of $V_3(t)$ along the solution of system (1) can be obtained as

$$\begin{aligned}
\dot{V}_3(t) &= 2\{[\Delta_1 Wx(t) - f(t)]^T L_1 + [f(t) - \Delta_2 Wx(t)]^T L_2\} W\dot{x}(t) \\
&= \zeta_a^T(t)\Phi_2\zeta_a(t)
\end{aligned} \tag{22}$$

where Φ_2 is defined in Box II.

The derivative of $V_4(t)$ along the solution of system (1) can be obtained as

$$\dot{V}_4(t) = d\dot{x}^T(t)R\dot{x}(t) - \int_{t-d}^{t} \dot{x}^T(s)R\dot{x}(s)ds \tag{23}$$

Under the assumption on the nonlinear function, (4) and (5), the following inequalities hold:

$$h_i(s) = 2\,[\Delta_1 Wx(s) - f(s)]^T\, H_i\, [f(s) - \Delta_2 Wx(s)] \geq 0$$

$$\begin{aligned}
g_j(s_1, s_2) = \ & 2\,[\Sigma_1 W(x(s_1) - x(s_2)) - (f(s_1) - f(s_2))]^T\, G_j \\
& \times [(f(s_1) - f(s_2)) - \Sigma_2 W(x(s_1) - x(s_2))] \geq 0
\end{aligned}$$

where $H_i = \mathrm{diag}\{h_{1i}, h_{2i}, \cdots, h_{ni}\} \geq 0, i = 1, 2, 3$ and $G_j = \mathrm{diag}\{g_{1j}, g_{2j}, \cdots, g_{nj}\} \geq 0, j = 1, 2, 3$. Thus, the following inequalities hold:

$$\begin{aligned}
& h_1(t) + h_2(t-d_1(t)) + h_3(t-d(t)) \\
& \quad = \zeta_a^T(t)\Phi_3\zeta_a(t) \geq 0
\end{aligned} \tag{24}$$

$$\begin{aligned}
& g_1(t, t-d_1(t)) + g_2(t-d_1(t), t-d(t)) + g_3(t, t-d(t)) \\
& \quad = \zeta_a^T(t)\Phi_4\zeta_a(t) \geq 0
\end{aligned} \tag{25}$$

where Φ_3 and Φ_4 are defined in Box II.

Based on (11), the R-dependent integral term in (23) can be divided into the following four parts:

$$\dot{V}_{4a}(t) := \int_{t-d_1(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds + \int_{t-d}^{t-d_3(t)} \dot{x}^T(s)R\dot{x}(s)ds$$

$$\dot{V}_{4b}(t) := \int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(s)R\dot{x}(s)ds + \int_{t-d_3(t)}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds$$

Based on Wirtinger-based inequality, the following holds

$$\int_{t-d_1(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{d_1(t)}\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix}^T\begin{bmatrix}R & 0\\ 0 & 3R\end{bmatrix}\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix}$$
$$= \frac{1}{d_1(t)}\zeta_a^T(t)E_1^T\tilde{R}E_1\zeta_a(t)$$

where

$$\begin{bmatrix}\eta_1(t)\\ \eta_2(t)\end{bmatrix} = \begin{bmatrix}x(t)-x(t-d_1(t))\\ x(t)+x(t-d_1(t))-2v_1(t)\end{bmatrix} = E_1\zeta_a(t)$$

Similarly, it follows from Wirtinger-based inequality that

$$\int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{d_2(t)}\zeta_a^T(t)E_2^T\tilde{R}E_2\zeta_a(t)$$
$$\int_{t-d_3(t)}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{d_2-d_2(t)}\zeta_a^T(t)E_3^T\tilde{R}E_3\zeta_a(t)$$
$$\int_{t-d}^{t-d_3(t)} \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{d_1-d_1(t)}\zeta_a^T(t)E_4^T\tilde{R}E_4\zeta_a(t)$$

Thus, $\dot{V}_{4a}(t)$ and $\dot{V}_{4b}(t)$ can be rewritten as

$$\dot{V}_{4a}(t) \geq \zeta_a^T(t)\begin{bmatrix}E_1\\ E_4\end{bmatrix}^T\begin{bmatrix}\frac{1}{d_1(t)}\tilde{R} & 0\\ 0 & \frac{1}{d_1-d_1(t)}\tilde{R}\end{bmatrix}\begin{bmatrix}E_1\\ E_4\end{bmatrix}\zeta_a(t) \tag{26}$$
$$\dot{V}_{4b}(t) \geq \zeta_a^T(t)\begin{bmatrix}E_2\\ E_3\end{bmatrix}^T\begin{bmatrix}\frac{1}{d_2(t)}\tilde{R} & 0\\ 0 & \frac{1}{d_2-d_2(t)}\tilde{R}\end{bmatrix}\begin{bmatrix}E_2\\ E_3\end{bmatrix}\zeta_a(t) \tag{27}$$

Then, applying matrix inequalities (8) and (9) to estimate above two terms, respectively, leads to two conditions of Theorem 1.

Case I: For any matrices X_i and $Y_i, i = 1, 2$, using (9) to estimate $\dot{V}_{4a}(t)$ and $\dot{V}_{4b}(t)$ yields

$$\dot{V}_{4a}(t)+\dot{V}_{4b}(t) \geq \zeta_a^T(t)\left\{\frac{1}{d_1}\tilde{\Phi}_{6a}+\frac{1}{d_2}\tilde{\Phi}_{6b}\right\}\zeta_a(t) \tag{28}$$

where

$$\tilde{\Phi}_{6a} = \begin{bmatrix}E_1\\ E_4\end{bmatrix}^T\begin{bmatrix}\tilde{R}+\frac{(d_1-d_1(t))}{d_1}T_1 & \frac{(d_1-d_1(t))}{d_1}X_1+\frac{d_1(t)}{d_1}X_2\\ * & \tilde{R}+\frac{d_1(t)}{d_1}T_2\end{bmatrix}\begin{bmatrix}E_1\\ E_4\end{bmatrix}$$
$$\tilde{\Phi}_{6b} = \begin{bmatrix}E_2\\ E_3\end{bmatrix}^T\begin{bmatrix}\tilde{R}+\frac{(d_2-d_2(t))}{d_2}T_3 & \frac{(d_2-d_2(t))}{d_2}Y_1+\frac{d_2(t)}{d_2}Y_2\\ * & \tilde{R}+\frac{d_2(t)}{d_2}T_4\end{bmatrix}\begin{bmatrix}E_2\\ E_3\end{bmatrix}$$
$$T_1 = \tilde{R}-X_2\tilde{R}^{-1}X_2^T,\quad T_2 = \tilde{R}-X_1^T\tilde{R}^{-1}X_1$$
$$T_3 = \tilde{R}-Y_2\tilde{R}^{-1}Y_2^T,\quad T_4 = \tilde{R}-Y_1^T\tilde{R}^{-1}Y_1$$

By combining (20)-(28), the derivative of $V_a(t)$ is given as

$$\dot{V}(t) \leq \zeta_a^T(t)\tilde{\Psi}_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))\zeta_a(t)$$

where

$$\begin{aligned}&\tilde{\Psi}_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))\\ &= \sum_{i=1}^{4} \Phi_i + F_1^T P F_2 + F_2^T P F_1 + e_s^T (dR) e_s - \frac{\tilde{\Phi}_{6a}}{d_1} - \frac{\tilde{\Phi}_{6b}}{d_2}\end{aligned}$$

It can be easily found that $\tilde{\Psi}_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))$ can be represented as the following form:

$$\begin{aligned}&\tilde{\Psi}_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))\\ &= \Gamma_1 + \dot{d}_1(t)\Gamma_2 + \dot{d}_2(t)\Gamma_3 + d_1(t)[\Gamma_4 + \dot{d}_1(t)\Gamma_5 + \dot{d}_2(t)\Gamma_6]\\ &\quad + d_2(t)[\Gamma_7 + \dot{d}_1(t)\Gamma_8 + \dot{d}_2(t)\Gamma_9]\end{aligned}$$

where $\Gamma_i, i = 1, 2, \cdots, 9$ are time-independent matrix-combinations. Similar to the idea of [17] [Eqs. (59)-(62) therein], the following holds for time delays satisfying (2):

$$\tilde{\Psi}_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0 \tag{29}$$

when it holds for $(d_1(t), d_2(t)) \in \mathbf{\Omega_1}$, i.e.,

$$\begin{aligned}\tilde{\Psi}_1(0, 0, \dot{d}_1(t), \dot{d}_2(t)) < 0\\ \tilde{\Psi}_1(0, d_2, \dot{d}_1(t), \dot{d}_2(t)) < 0\\ \tilde{\Psi}_1(d_1, 0, \dot{d}_1(t), \dot{d}_2(t)) < 0\\ \tilde{\Psi}_1(d_1, d_2, \dot{d}_1(t), \dot{d}_2(t)) < 0\end{aligned}$$

Similarly, it is also easy to find that (29) holds for all time-varying delays satisfying (2) and (3) when above four inequalities hold for $(\dot{d}_1(t), \dot{d}_2(t)) \in \mathbf{\Omega_2}$.

In (28), $\tilde{\Phi}_{6a}$ and $\tilde{\Phi}_{6b}$ can be rewritten as

$$\tilde{\Phi}_{6a} = \begin{cases} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R} - X_2\tilde{R}^{-1}X_2^T & X_1 \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}, & if\ d_1(t) = 0 \\ \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & X_2 \\ * & 2\tilde{R} - X_1^T\tilde{R}^{-1}X_1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix}, & if\ d_1(t) = d_1 \end{cases}$$

$$\tilde{\Phi}_{6b} = \begin{cases} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} 2\tilde{R} - Y_2\tilde{R}^{-1}Y_2^T & Y_1 \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & if\ d_2(t) = 0 \\ \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & Y_2 \\ * & 2\tilde{R} - Y_1^T\tilde{R}^{-1}Y_1 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & if\ d_2(t) = d_2 \end{cases}$$

Based on Schur complement, for the case of $d_1(t) = 0$, $d_2(t) = 0$, the condition $\tilde{\Psi}_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0$ is equivalent to inequality (13), i.e.,

$$(13) \Leftrightarrow \tilde{\Psi}_1(0, 0, \dot{d}_1(t), \dot{d}_2(t)) < 0 \tag{30}$$

Similarly, the follow relationships are true:

$$(14) \Leftrightarrow \tilde{\Psi}_1(0, d_2, \dot{d}_1(t), \dot{d}_2(t)) < 0 \tag{31}$$

$$(15) \Leftrightarrow \tilde{\Psi}_1(d_1, 0, \dot{d}_1(t), \dot{d}_2(t)) < 0 \tag{32}$$

$$(16) \Leftrightarrow \tilde{\Psi}_1(d_1, d_2, \dot{d}_1(t), \dot{d}_2(t)) < 0 \tag{33}$$

Combining (30)-(33) yields

$$(13) - (16) \Leftrightarrow \tilde{\Psi}_1(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0$$

which implies $\dot{V}_a(t) \leq -\epsilon\|x(t)\|^2$ for a sufficient small scalar $\epsilon > 0$. Therefore, when (13)-(16) hold for $(\dot{d}_1(t), \dot{d}_2(t)) \in \mathbf{\Omega_2}$, system (1) is asymptotically stable.

Case II: When (17) and (18) hold, using RCMI (8) to estimate $\dot{V}_{4a}(t)$ and $\dot{V}_{4b}(t)$ yields

$$\begin{aligned} &\dot{V}_{4a}(t) + \dot{V}_{4b}(t) \\ &\geq \zeta_a^T(t)\left\{\begin{bmatrix} E_1 \\ E_4 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & X \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_1 \\ E_4 \end{bmatrix} + \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{R} & Y \\ * & \tilde{R} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}\right\}\zeta_a(t) \end{aligned} \tag{34}$$

Then, combining (20)-(27) and (34) leads to

$$\dot{V}_a(t) \leq \zeta_a^T(t)\Psi_2(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))\zeta_a(t)$$

where $\Psi_2(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))$ is defined in Box II.

Similar to the proof of Theorem 1.C1, the following holds for all time-varying delays satisfying (2) and (3):

$$\Psi_2(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0$$

when it holds for $(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) \in \mathbf{\Omega}$. Therefore, when (17)-(19) hold, $\dot{V}_a(t) \leq -\epsilon\|x(t)\|^2$ for a sufficient small scalar $\epsilon > 0$, which shows the asymptotically stable of system (1). The proof of Theorem 1 completes. ■

Remark 3. *In the above proof, the only difference for developing two conditions is that different matrix inequalities are used to estimate the single integral terms ($\dot{V}_{4a}(t)$ and $\dot{V}_{4b}(t)$ in (26) and (27)). Thus, the advantages of the ERCMI (9) compared with the popular inequality (8) can be found through the comparison of the results provided by those criteria. Furthermore, by following the same lines in [32] [proof of Theorem 2 therein], it can be proved in theoretical that Theorem 1.C1 is less conservative than Theorem 1.C2.*

3.2. Stability criteria based on the second LKF

Very recently, a dynamic delay interval method was developed in [23] to improve the stability criteria of system (1). The main characteristic of this method is that the upper/lower bounds of double integral terms of the LKF are extended into the time-varying delays combination, instead of their bounds used in most literature. Based on this type of LKF, this subsection develops enhanced stability criteria by replacing the RCMI used in [23] with the ERCMI. For using this method, set $A_1 = W_1 = 0$.

Construct the following LKF candidate:

$$V_b(t) = \bar{V}_1(t) + \bar{V}_2(t) + V_3(t) + \bar{V}_4(t) \tag{35}$$

where $V_3(t)$ is defined in (12), and

$$\begin{aligned} \bar{V}_1(t) =& \xi_3^T(t)\bar{P}\xi_3(t) \\ \bar{V}_2(t) =& \int_{t-a(t)}^{t} \xi_2^T(s)R_1\xi_2(s)ds + \int_{t-d(t)}^{t-a(t)} \xi_2^T(s)R_2\xi_2(s)ds \\ &+ \int_{t-b(t)}^{t} \xi_2^T(s)R_3\xi_2(s)ds \end{aligned}$$

$$\begin{aligned} V_3(t) =& 2\sum_{i=1}^{n}\int_0^{W_{2i}x} [\lambda_{1i}(\delta_i^+ s - f_i(s)) + \lambda_{2i}(f_i(s) - \delta_i^- s)]ds \\ \bar{V}_4(t) =& \int_{-a(t)}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z_1\dot{x}(s)dsd\theta \\ &+ \int_{-b(t)}^{-a(t)}\int_{t+\theta}^{t} \dot{x}^T(s)Z_2\dot{x}(s)dsd\theta \end{aligned}$$

and $\bar{P} > 0$, $R_i > 0, i = 1, 2, 3$, and $Z_i > 0, i = 1, 2$ are the symmetric matrices with approximate dimension; and other notations are listed in Box III (Note that the notations given in Box I are not listed in Box III).

Remark 4. *Worth mentioning, if the LKF $V_b(t)$ was employed, some integral terms multiplied by $-(1-\dot{a}(t))$ would be produced. For possibly bounded by integral inequality method, the variable $(1-\dot{a}(t))$ is required to be greater than 0. Therefore, some restrictions on α, β, μ_1, and μ_2 are needed and the detailed descriptions are expressed in Box III.*

Box III

$$a(t) = \alpha d_1(t)+\beta d_2(t),\ b(t)=d-\alpha(d_1-d_1(t))-\beta(d_2-d_2(t)),\ (\alpha,\beta)\in\begin{cases}\mathcal{N}=\{(\alpha,\beta)\big|\alpha\in[0,1],\beta\in[0,1]\}, & if\ \mu_1+\mu_2<1\\ \mathcal{N}\cap\{(\alpha,\beta)|\alpha\mu_1+\beta\mu_2<1\}, & if\ \mu_1+\mu_2\geq 1\end{cases}$$

$$f(s) = f(Wx(s)), v_5(t) = \int_{t-a(t)}^{t}\frac{x(s)}{a(t)}ds,\ \ v_6(t) = \int_{t-d(t)}^{t-a(t)}\frac{x(s)}{d(t)-a(t)}ds,\ \ v_7(t)=\int_{t-b(t)}^{t-d(t)}\frac{x(s)}{b(t)-d(t)}ds$$

$$\xi_3(t) = \mathrm{col}\{x(t),\ a(t)v_5(t),\ (d(t)-a(t))v_6(t), (b(t)-d(t))v_7(t)\},\quad \xi_2(t) = \mathrm{col}\{x(t), f(t)\}$$

$$\zeta_b(t) = \mathrm{col}\{x(t), x(t-a(t)), x(t-d(t)), x(t-b(t)), v_5(t), v_6(t), v_7(t), f(t), f(t-a(t)), f(t-d(t)), f(t-b(t))\}$$

Based on LKF (35) and Wirtinger-based inequality (7), two stability conditions of system (1) are obtained through matrix inequalities (8) and (9), respectively, and summarized in the following theorem.

Theorem 2. *For given integers h_1, h_2, μ_1, μ_2, α, and β, system (1) with time-varying delays satisfying (2) and (3) is asymptotically stable, if the one of the following conditions holds*

C1. ERCMI-based condition: if there exist positive-definite symmetric matrices $\bar{P}\in\mathcal{R}^{4n\times4n}$, $R_i\in\mathcal{R}^{2n\times2n}, i=1,2,3,$ and $Z_i\in\mathcal{R}^{n\times n}, i=1,2,$ positive-definite diagonal matrices $L_i\in\mathcal{R}^{n\times n}, i=1,2,$ $H_j\in\mathcal{R}^{n\times n}$, and $G_j\in\mathcal{R}^{n\times n}$, $j=1,2,3,4,$ and any matrices $X_i\in\mathcal{R}^{2n\times2n}, i=1,2,$ such that the following holds for $(\dot{d}_1(t),\dot{d}_2(t))\in\mathbf{\Omega_2}$ (Note that $\Psi_3(d_1(t),d_2(t),\dot{d}_1(t),\dot{d}_2(t))$ is simplified as Ψ_3):

$$\begin{bmatrix}\Psi_3|_{d_1(t)=0,d_2(t)=0} & (1-\dot{a}(t))E_2^TX_2\\ * & -(1-\dot{a}(t))\rho\tilde{Z}_2\end{bmatrix}<0 \tag{36}$$

$$\begin{bmatrix}\Psi_3\Big|_{\begin{cases}d_1(t)=0\\ d_2(t)=d_2\end{cases}} & (1-\dot{a}(t))E_2^TX_2 & (1-\dot{a}(t))E_3^TX_1^T\\ * & \frac{(\dot{a}(t)-1)\rho}{\tau_\alpha}\tilde{Z}_2 & 0\\ * & * & \frac{(\dot{a}(t)-1)\rho}{\tau_\beta}\tilde{Z}_2\end{bmatrix}<0 \tag{37}$$

$$\begin{bmatrix}\Psi_3\Big|_{\begin{cases}d_1(t)=d_1\\ d_2(t)=0\end{cases}} & (1-\dot{a}(t))E_2^TX_2 & (1-\dot{a}(t))E_3^TX_1^T\\ * & \frac{(\dot{a}(t)-1)\rho}{\tau_\beta}\tilde{Z}_2 & 0\\ * & * & \frac{(\dot{a}(t)-1)\rho}{\tau_\alpha}\tilde{Z}_2\end{bmatrix}<0 \tag{38}$$

$$\begin{bmatrix}\Psi_3|_{d_1(t)=d_1,d_2(t)=d_2} & (1-\dot{a}(t))E_3^TX_1^T\\ * & -(1-\dot{a}(t))\rho\tilde{Z}_2\end{bmatrix}<0 \tag{39}$$

C2. RCMI-based condition: if there exist positive-definite symmetric matrices $\bar{P}\in\mathcal{R}^{4n\times4n}$, $R_i\in\mathcal{R}^{2n\times2n}, i=1,2,3,$ and $Z_i\in\mathcal{R}^{n\times n}, i=1,2,$ positive-definite diagonal matrices $L_i\in\mathcal{R}^{n\times n}, i=1,2,$ $H_j\in\mathcal{R}^{n\times n}$, and $G_j\in\mathcal{R}^{n\times n}$, $j=1,2,3,4,$ and any matrices $X\in\mathcal{R}^{2n\times2n}$, such that the following holds for $(d_1(t),d_2(t),\dot{d}_1(t),\dot{d}_2(t))\in\mathbf{\Omega}$:

$$\begin{bmatrix}\tilde{Z}_2 & X\\ * & \tilde{Z}_2\end{bmatrix}>0 \tag{40}$$

$$\Psi_4(d_1(t),d_2(t),\dot{d}_1(t),\dot{d}_2(t))<0 \tag{41}$$

Box IV: notations used in Theorem 2

$$\Psi_3(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t))$$

$$= \begin{cases} \sum\limits_{i=1,3,4,5} \bar{\Phi}_i + \Phi_2 - \frac{(1-\dot{a}(t))}{(1-\alpha)d_1+(1-\beta)d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} 2\tilde{Z}_2 & X_1 \\ * & \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{when } d_1(t) = 0, d_2(t) = 0 \\ \sum\limits_{i=1,3,4,5} \bar{\Phi}_i + \Phi_2 - \frac{(1-\dot{a}(t))}{(1-\alpha)d_1+(1-\beta)d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} (1+\tau_\beta)\tilde{Z}_2 & \tau_\beta X_1 + \tau_\alpha X_2 \\ * & (1+\tau_\alpha)\tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{when } d_1(t) = d_1, d_2(t) = 0 \\ \sum\limits_{i=1,3,4,5} \bar{\Phi}_i + \Phi_2 - \frac{(1-\dot{a}(t))}{(1-\alpha)d_1+(1-\beta)d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} (1+\tau_\alpha)\tilde{Z}_2 & \tau_\alpha X_1 + \tau_\beta X_2 \\ * & (1+\tau_\beta)\tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{when } d_1(t) = 0, d_2(t) = d_2 \\ \sum\limits_{i=1,3,4,5} \bar{\Phi}_i + \Phi_2 - \frac{(1-\dot{a}(t))}{(1-\alpha)d_1+(1-\beta)d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{Z}_2 & X_2 \\ * & 2\tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, & \text{when } d_1(t) = d_1, d_2(t) = d_2 \end{cases}$$

$$\Psi_4(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) = \sum_{i=1,3,4,5} \bar{\Phi}_i + \Phi_2 - \frac{(1-\dot{a}(t))}{(1-\alpha)d_1 + (1-\beta)d_2} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{Z}_2 & X \\ * & \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}$$

$$\bar{\Phi}_1 = \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}^T (R_1 + R_3) \begin{bmatrix} e_1 \\ e_8 \end{bmatrix} - (1-\dot{a}(t)) \begin{bmatrix} e_2 \\ e_9 \end{bmatrix}^T (R_1 - R_2) \begin{bmatrix} e_2 \\ e_9 \end{bmatrix} - (1-\dot{d}(t)) \begin{bmatrix} e_3 \\ e_{10} \end{bmatrix}^T R_2 \begin{bmatrix} e_3 \\ e_{10} \end{bmatrix} - (1-\dot{b}(t)) \begin{bmatrix} e_4 \\ e_{11} \end{bmatrix}^T R_3 \begin{bmatrix} e_4 \\ e_{11} \end{bmatrix}$$

$$\Phi_2 = Sym\Big\{[(\Delta_1 We_1 - e_8)^T L_1 + (e_8 - \Delta_2 We_1)^T L_2] We_s\Big\}, \quad \bar{\Phi}_3 = Sym\left\{\sum_{i=1}^{4} [\Delta_1 We_i - e_{i+7}]^T \bar{H}_i [e_{i+7} - \Delta_2 We_i]\right\}$$

$$\bar{\Phi}_4 = Sym\left\{ \begin{array}{l} \sum\limits_{i=1}^{3} [\Sigma_1 W(e_i - e_{i+1}) - (e_{i+7} - e_{i+8})]^T \bar{G}_i [(e_{i+7} - e_{i+8}) - \Sigma_2 W(e_i - e_{i+1})] \\ +[\Sigma_1 W(e_1 - e_3) - (e_8 - e_{10})]^T \bar{G}_4 [(e_8 - e_{10}) - \Sigma_2 W(e_1 - e_3)] \end{array} \right\}$$

$$\bar{\Phi}_5 = F_3^T \bar{P} F_4 + F_4^T \bar{P} F_3 + e_s^T [a(t) Z_1 + (b(t) - a(t)) Z_2] e_s - (1 - \dot{a}(t)) \frac{E_1^T \tilde{Z}_1 E_1}{\alpha d_1 + \beta d_2}$$

$$F_3 = \mathrm{col}\{e_1,\ a(t)e_5,\ (d(t) - a(t))e_6,\ (b(t) - d(t))e_7\}$$

$$F_4 = \mathrm{col}\Big\{e_s,\ e_1 - (1 - \dot{a}(t))e_2,\ (1 - \dot{a}(t))e_2 - (1 - \dot{d}(t))e_3,\ (1 - \dot{d}(t))e_3 - (1 - \dot{b}(t))e_4\Big\}$$

$$E_i = \begin{bmatrix} e_i - e_{i+1} \\ e_i + e_{i+1} - 2e_{i+4} \end{bmatrix}, i = 1, 2, 3, \quad \tilde{Z}_j = \begin{bmatrix} Z_j & 0 \\ 0 & 3Z_j \end{bmatrix}, j = 1, 2$$

$$e_i = [0_{n \times (i-1)n} \ \ I_{n \times n} \ \ 0_{n \times (11-i)n}],\ i = 1, 2, \cdots, 11, \quad e_s = A_0 e_1 + A_2 e_3 + W_0 e_8 + W_2 e_{10}$$

$$\tau_\alpha = \frac{(1-\alpha)d_1}{(1-\alpha)d_1 + (1-\beta)d_2}, \quad \tau_\beta = \frac{(1-\beta)d_2}{(1-\alpha)d_1 + (1-\beta)d_2}, \quad \rho = (1-\alpha)d_1 + (1-\beta)d_2$$

where the related notations are given in Box IV.

Proof: The condition that $\bar{P}$, R_i, $i = 1, 2, 3$, Z_j, and L_j, $j = 1, 2$, are positive-definite matrices leads that the LKF satisfies $V_b(t) \geq \epsilon \|x(t)\|^2$ with $\epsilon > 0$.

The derivative of $\bar{V}_1(t)$ along the solution of system (1) can be obtained as

$$\dot{\bar{V}}_1(t) = 2\xi_3^T(t) \bar{P} \dot{\xi}_3(t) = \zeta_b^T(t) (F_3^T \bar{P} F_4 + F_4^T \bar{P} F_3) \zeta_b(t) \tag{42}$$

where F_3 and F_4 are defined in Box IV.

The derivative of $\bar{V}_2(t)$ along the solution of system (1) can be obtained as

$$\dot{\bar{V}}_2(t) = \zeta_b^T(t)\bar{\Phi}_1\zeta_b(t) \tag{43}$$

where $\bar{\Phi}_1$ is defined in Box IV. (Note that $\dot{a}(t) = \dot{b}(t)$).

The derivative of $V_3(t)$ along the solution of system (1) can be obtained as

$$\begin{aligned}\dot{V}_3(t) &= 2\{[\Delta_1 Wx(t) - f(t)]^T L_1 + [f(t) - \Delta_2 Wx(t)]^T L_2\} W\dot{x}(t) \\ &= \zeta_a^T(t)\Phi_2\zeta_a(t) \end{aligned}\tag{44}$$

where Φ_2 is defined in Box IV.

The derivative of $\bar{V}_4(t)$ along the solution of system (1) can be obtained as [23]

$$\begin{aligned}\dot{\bar{V}}_4(t) =& \dot{x}^T(t)[a(t)Z_1 + (b(t) - a(t))Z_2]\dot{x}(t) - (1 - \dot{a}(t)) \\ &\times\left[\int_{t-a(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds + \int_{t-b(t)}^{t-a(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds\right]\end{aligned}\tag{45}$$

Under the assumption on the nonlinear function, (4) and (5), the following inequalities hold:

$$\begin{aligned} h_i(s) =& 2\left[\Delta_1 Wx(s) - f(s)\right]^T H_i \left[f(s) - \Delta_2 Wx(s)\right] \geq 0 \\ g_j(s_1, s_2) =& 2\left[\Sigma_1 W(x(s_1) - x(s_2)) - (f(s_1) - f(s_2))\right]^T G_j \\ &\times \left[(f(s_1) - f(s_2)) - \Sigma_2 W(x(s_1) - x(s_2))\right] \geq 0 \end{aligned}$$

where $H_i = \text{diag}\{h_{1i}, h_{2i}, \cdots, h_{ni}\} \geq 0, i = 1, 2, 3, 4$ and $G_j = \text{diag}\{g_{1j}, g_{2j}, \cdots, g_{nj}\} \geq 0, j = 1, 2, 3$. Thus, the following inequalities hold:

$$\begin{aligned} & h_1(t) + h_2(t - a(t)) + h_3(t - d(t)) + h_4(t - b(t)) \\ &\quad = \zeta_a^T(t)\bar{\Phi}_3\zeta_a(t) \geq 0 \end{aligned}\tag{46}$$

$$\begin{aligned} & g_1(t, t - a(t)) + g_2(t - a(t), t - d(t)) \\ &\quad + g_3(t - d(t), t - b(t)) + g_4(t, t - d(t)) \\ &\quad = \zeta_a^T(t)\bar{\Phi}_4\zeta_a(t) \geq 0 \end{aligned}\tag{47}$$

where $\bar{\Phi}_3$ and $\bar{\Phi}_4$ are defined in Box IV.

The Z_1-dependent term in (45) is estimated as [23]:

$$\int_{t-a(t)}^{t} \dot{x}^T(s)Z_1\dot{x}(s)ds \geq \zeta_b^T(t)\left(\frac{E_1^T\tilde{Z}_1E_1}{\alpha d_1 + \beta d_2}\right)\zeta_b(t) \tag{48}$$

For the Z_2-dependent term in (45), it follows from Wirtinger-based inequality that

$$\begin{aligned} &\int_{t-b(t)}^{t-a(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \\ &\quad = \int_{t-d(t)}^{t-a(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds + \int_{t-b(t)}^{t-d(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \\ &\quad \geq \zeta_b^T(t)\begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \frac{1}{d(t)-a(t)}\tilde{Z}_2 & 0 \\ 0 & \frac{1}{b(t)-d(t)}\tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}\zeta_b(t) \end{aligned}\tag{49}$$

Then, applying two matrix inequalities to estimate above term, respectively, leads to two conditions of Theorem 2.

Case I: By using (9), Z_2-dependent integral term in (45) can be estimated as

$$\int_{t-b(t)}^{t-a(t)} \dot{x}^T(s)Z_2\dot{x}(s)ds \geq \frac{\zeta_b^T(t)\tilde{\Phi}_6(a(t), d(t), b(t))\zeta_b(t)}{(1-\alpha)d_1 + (1-\beta)d_2} \tag{50}$$

where

$$\begin{aligned}
&\tilde{\Phi}_6(a(t),d(t),b(t))\\
&=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\tilde{Z}_2+\frac{b(t)-d(t)}{b(t)-a(t)}\bar{T}_1 & \frac{b(t)-d(t)}{b(t)-a(t)}X_1+\frac{d(t)-a(t)}{b(t)-a(t)}X_2\\ * & \tilde{Z}_2+\frac{d(t)-a(t)}{b(t)-a(t)}\bar{T}_2\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}\\
&=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\tilde{Z}_2+(1-\tau(t))\bar{T}_1 & (1-\tau(t))X_1+\tau(t)X_2\\ * & \tilde{Z}_2+\tau(t)\bar{T}_2\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}\\
&\tau(t)=\frac{(1-\alpha)d_1(t)+(1-\beta)d_2(t)}{(1-\alpha)d_1+(1-\beta)d_2}\\
&\bar{T}_1=\tilde{Z}_2-X_2\tilde{Z}_2^{-1}X_2^T,\;\; \bar{T}_2=\tilde{Z}_2-X_1^T\tilde{Z}_2^{-1}X_1
\end{aligned}$$

Then, combining (44) and (42)-(50) yields

$$\dot{V}_b(t)\leq\zeta_b^T(t)\tilde{\Psi}_3(a(t),b(t),\dot{a}(t),\dot{d}(t))\zeta_b(t)$$

where

$$\tilde{\Psi}_3(a(t),b(t),\dot{a}(t),\dot{d}(t))=\sum_{i=1,3,4,5}\bar{\Phi}_i+\Phi_2-\frac{(1-\dot{a}(t))}{\rho}\tilde{\Phi}_6$$

Similar to the idea of [17] [Eqs. (59)-(62) therein], the following holds for time-varying delays satisfying (2):

$$\tilde{\Psi}_3(d_1(t),d_2(t),\dot{d}_1(t),\dot{d}_2(t))<0 \tag{51}$$

when it holds for $(d_1(t),d_2(t))\in\mathbf{\Omega_1}$, i.e.,

$$\begin{aligned}
\tilde{\Psi}_3(0,0,\dot{d}_1(t),\dot{d}_2(t))<0\\
\tilde{\Psi}_3(0,d_2,\dot{d}_1(t),\dot{d}_2(t))<0\\
\tilde{\Psi}_3(d_1,0,\dot{d}_1(t),\dot{d}_2(t))<0\\
\tilde{\Psi}_3(d_1,d_2,\dot{d}_1(t),\dot{d}_2(t))<0
\end{aligned}$$

And (51) holds for all time-varying delays satisfying (2) and (3) when above four inequalities hold for $(\dot{d}_1(t),\dot{d}_2(t))\in\mathbf{\Omega_2}$.

When $d_1(t)=0, d_2(t)=0$, $\tilde{\Phi}_6(a(t),d(t),b(t))$ is given as

$$\tilde{\Phi}_6=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}2\tilde{Z}_2-X_2\tilde{Z}_2^{-1}X_2^T & X_1\\ * & \tilde{Z}_2\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}$$

Based on Schur complement, $\tilde{\Psi}_3(0,0,\dot{d}_1(t),\dot{d}_2(t))<0$ is equivalent to (36), i.e.,

$$(36)\Leftrightarrow\tilde{\Psi}_3(0,0,\dot{d}_1(t),\dot{d}_2(t))<0$$

When $d_1(t)=0, d_2(t)=d_2$, $\tilde{\Phi}_6(a(t),d(t),b(t))$ is obtained as

$$\tilde{\Phi}_6=\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}(1+\tau_\alpha)\tilde{Z}_2 & \tau_\alpha X_1+\tau_\beta X_2\\ * & (1+\tau_\beta)\tilde{Z}_2\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}-$$

$$\begin{bmatrix}E_2\\E_3\end{bmatrix}^T\begin{bmatrix}\tau_\alpha X_2\tilde{Z}_2^{-1}X_2^T & 0\\ * & \tau_\beta X_1^T\tilde{Z}_2^{-1}X_1\end{bmatrix}\begin{bmatrix}E_2\\E_3\end{bmatrix}$$

where

$$\tau_\alpha = \frac{(1-\alpha)d_1}{(1-\alpha)d_1+(1-\beta)d_2}, \quad \tau_\beta = \frac{(1-\beta)d_2}{(1-\alpha)d_1+(1-\beta)d_2}$$

Based on Schur complement, $\tilde{\Psi}_3(0, d_2, \dot{d}_1(t), \dot{d}_2(t)) < 0$ is equivalent to (37), i.e.,

$$(37) \Leftrightarrow \tilde{\Psi}_3(0, d_2, \dot{d}_1(t), \dot{d}_2(t)) < 0$$

Similarly, the following relationships can be found

$$(38) \Leftrightarrow \tilde{\Psi}_3(d_1, 0, \dot{d}_1(t), \dot{d}_2(t)) < 0$$
$$(39) \Leftrightarrow \tilde{\Psi}_3(d_1, d_2, \dot{d}_1(t), \dot{d}_2(t)) < 0$$

Thus,

$$(36) - (39) \Leftrightarrow \tilde{\Psi}_3(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0$$

which implies $\dot{V}_b(t) \leq -\epsilon\|x(t)\|^2$ for a sufficient small scalar $\epsilon > 0$. Therefore, when (36)-(39) hold for $(\dot{d}_1(t), \dot{d}_2(t)) \in \mathbf{\Omega_2}$, system (1) is asymptotically stable.

Case II: When (40) holds, applying RCMI (8) to estimate Z_2-dependent integral term in (45) yields

$$\int_{t-a(t)}^{t-b(t)} \dot{x}^T(s) Z_2 \dot{x}(s) ds \geq \frac{\zeta_b^T(t) \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{Z}_2 & X \\ * & \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \zeta_b(t)}{(1-\alpha)d_1+(1-\beta)d_2} \tag{52}$$

Therefore, by combining (44), (42)-(48), and (52), the derivative of $V_b(t)$ is obtained as

$$\dot{V}_b(t) \leq \zeta_b^T(t) \Psi_4(a(t), d(t), b(t), \dot{a}(t), \dot{d}(t)) \zeta_b(t)$$

where $\Psi_4(a(t), d(t), b(t), \dot{a}(t), \dot{d}(t))$ is defined in Box IV.

Similar to the proof of Theorem 1.C1, the following holds for all time-varying delays satisfying (2) and (3):

$$\Psi_4(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0 \tag{53}$$

when it holds for $(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) \in \mathbf{\Omega}$. Therefore, when (40) and (41) hold, $\dot{V}_b(t) \leq -\epsilon\|x(t)\|^2$ for a sufficient small scalar $\epsilon > 0$, which shows the asymptotically stable of system (1). This completes the proof Theorem 2. ■

Remark 5. *In the above proof, the LKF used has included more information of time-varying delays themselves, instead of their bounds, and two preset scalars, α and β, have divided the time-varying delays into several parts, both of which will improve the results, as shown in [23]. However, it is required that $1 - \dot{a}(t) > 0$ during the estimation of Z_i-dependent terms in the procedure of proof. This requirement reduces the available choice range of α and β for the case of $\mu_1 + \mu_2 > 1$ such that the feasibility of LMI-based conditions will greatly decrease, which will be shown in the example studies section.*

Remark 6. *Except for the two types of LKFs utilized in this paper, some other LKF methods, such as multipe integral approach [43] and relaxed condition LKF method [44, 45], were proposed for the stability analysis of linear time-delay systems. Very recently, the flexible terminal method was employed to develop less conservative stability criteria for recurrent neural networks with time-varying delay [46]. The effectiveness of the above methods were proved in respective literature, it can be predicted that the applications of these methods to the stability analysis of Lur'e systems with additive components are also able to give further improved results. And this work will be done in the future.*

4. Case study

In this section, several examples are used to demonstrate the advantages of the proposed criteria.

Example 1. Consider the following neural network [17]:

$$\dot{u}(t) = Au(t) + Bg(u(t)) + Cg(u(t - d_1(t) - d_2(t))) + J \tag{54}$$

with the following parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \; C = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}$$
$$g(u) = \begin{bmatrix} 0.4\tanh(u_1) \\ 0.8\tanh(u_2) \end{bmatrix}$$

This example is used to compare the proposed criteria with the ones of literature. As discussed in [17], by using transformation $x(t) = u(t) - u^*$ with u^* being the equilibrium point of (54), system (54) can be rewritten as (1) with $A_0 = A$, $A_1 = A_2 = 0$, $W_0 = B$, $W_1 = 0$, $W_2 = C$, $W = I$, $\Sigma_1 = \Delta_1 = diag\{0.4, 0.8\}$, and $\Sigma_2 = \Delta_2 = diag\{0, 0\}$. For three cases: $\mu_1 = 0.7$ and $\mu_2 \in \{0.1, 0.2, 0.7\}$, the maximal upper bounds of d_2 with respect to $d_1 \in \{0.8, 1.0, 1.2\}$ calculated by the different criteria are listed in Table 1, where '—' indicates the results not reported in literature, and the number of decision variables are also given. The results of Theorem 2 listed are chose from all values calculated by setting two scalars $(\alpha,\beta) \in \{(\alpha,\beta)|\alpha \in \{0.1, 0.2, \cdots, 0.9\}, \beta \in \{0.1, 0.2, \cdots, 0.9\}\}$.

Table 1: The maximal upper bounds of d_2 for various d_1 and μ_i (Example 1).

Criteria	$d_1, \mu_1 = 0.7, \mu_2 = 0.1$			$d_1, \mu_1 = 0.7, \mu_2 = 0.2$			$d_1, \mu_1 = 0.7, \mu_2 = 0.7$			No. of variables
	0.8	1.0	1.2	0.8	1.0	1.2	0.8	1.0	1.2	
Theorem 1 [14]	2.016	1.820	1.619	0.870	0.671	0.471	—	—	—	$71n^2 + 10n$
Theorem 2 [16]	1.952	1.799	1.644	1.136	0.945	0.720	—	—	—	$325.5n^2 + 40.5n$
Theorem 1 [17]	1.966	1.835	1.680	1.129	0.960	0.774	—	—	—	$47.5n^2 + 23.5n$
Theorem 3.1 [21]	2.854	2.485	2.457	1.985	1.888	1.620	—	—	—	$228.5n^2 + 27.5n$
Corollary 1 [28]	3.121	2.911	2.715	1.837	1.636	1.437	—	—	—	$65n^2 + 21n$
Theorem 1 [23]	18.134	16.335	14.537	8.447	6.647	4.848	1.080	0.879	0.695	$16n^2 + 14n$
Theorem 1.C2	3.004	2.661	2.377	1.658	1.504	1.360	1.521	1.394	1.273	$26n^2 + 14n$
Theorem 1.C1	3.211	2.883	2.596	1.738	1.582	1.435	1.584	1.452	1.329	$34n^2 + 14n$
Theorem 2.C2	19.617	17.821	16.025	10.827	9.028	7.229	1.583	1.237	1.031	$19n^2 + 16n$
Theorem 2.C1	19.962	18.164	16.367	11.111	9.312	7.515	1.617	1.305	1.085	$23n^2 + 16n$

The results show that the proposed criteria not only provide less conservative results but also require the less number of decision variables in comparison with the ones developed through FWM approach [14, 17, 28], the convex polyhedron method [16], and the LKF with triple and quadruple integral terms [21]. Moreover, the results provided by two conditions of Theorems 1 and 2 show that the first condition provides less conservative results, which shows the advantages of the ERCMI compared with the popular RCMI, as summarized in Remark 2. Compared with Theorem 1, Theorem 2 greatly improves the existing results for the case of $\mu_1 + \mu_2 < 1$ (i.e., $\mu_1 = 0.7, \mu_2 \in \{0.1, 0.2\}$) while the improvement become small for the case of $\mu_1 + \mu_2 > 1$ (i.e., $\mu_1 = \mu_2 = 0.7$). The reason has been analyzed in Remark 5.

Simulation tests for three cases are carried out: *Case I:* External input: $J = [0.1, 0.6]^T$; Delays: $d_1(t) = \frac{4}{5}\sin(\frac{7}{4}t) + \frac{4}{5}$, $d_2(t) = \frac{19.962}{2}\cos(\frac{0.2}{19.962}t) + \frac{19.962}{2}$; and initial values: $u(0) = [0.2, 0.5]^T$; *Case II:* External input: $J = [0.4, 0.1]^T$; Delays: $d_1(t) = \sin^2(\frac{7}{20}t)$, $d_2(t) = 9.312\cos^2(\frac{0.05}{9.312}t)$; and initial values: $u(0) = [0.1, 0.6]^T$; *Case III:* External input: $J = [0.1, 0.5]^T$; $d_1(t) = 0.12(3\sin(\frac{7}{6}t) + 4\cos(\frac{7}{6}t) + 5)$, $d_2(t) = 0.1085(4\sin(\frac{1.4}{1.085}t) + 3\cos(\frac{1.4}{1.085}t) + 5)$; and initial values: $u(0) = [0.4, 0.2]^T$. From Table 1, the neural networks for the above cases are stable. The responses for three cases are shown in Fig. 1. The result shows that the neural network with given conditions is stable at its equilibrium point. Thus, the results are in accord with the results listed in table.

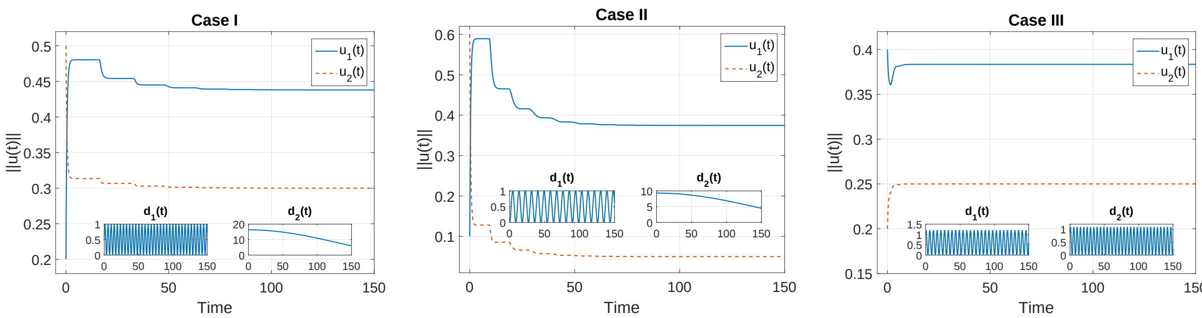

Figure 1: State trajectories of the neural networks.

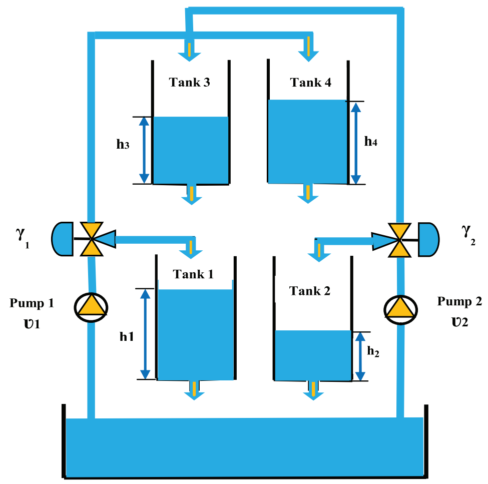

Figure 2: Schematic representation of the quadruple-tank process

Example 2. Consider the qudruple-tank process shown in Fig. 2. The objective is to control the water level in the two lower tanks using two pumps [38]. If the transport delays due to the flow of water before reaching the target tanks are taken into account, then the linearized state-space equation can be given as [39]:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_1(t)) + B_1 u(t-\tau_2(t)) + B_2 u(t-\tau_3(t))$$

where $x(t) = col\{h_1, h_2, h_3, h_4\}$ with $h_i, i = 1, 2, 3, 4$ being the water level of tank i; $\tau_1(t)$ is the transport delay of water-flow from outlet of Tank 3/4 to Tank 1/2, $\tau_2(t)$ is the transport delay of water-flow from Pump 1/2 to Tank 1/2, and $\tau_3(t)$ is the transport delay of water-flow from Pump 1/2 to Tank 4/3; the state-feedback controller is designed as $u(t) = Kx(t)$; γ_1 (γ_2) in figure is the ratio of water diverted to Tank 1 (Tank 2) rather than to Tank 4 (Tank 3); and

$$A_0 = \text{diag}\{-0.0021, -0.0021, -0.0424, -0.0424\}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0.1113\gamma_1 & 0 \\ 0 & 0.1042\gamma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.1042(1-\gamma_2) \\ 0.1113(1-\gamma_1) & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.1609 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix}$$

Due to the limited area of the hose and the capacity of the pumps, the output of the controller has a threshold value. Thus, the state-feedback control law is rewritten as [5]

$$u(t) = Kg(x(t)) \tag{55}$$

where

$$f(x(t)) = col\{f_1(x_1(t)), f_2(x_2(t)), \cdots, f_4(x_4(t))\}$$
$$f_i(x_i(t)) = 0.01(|x_i(t) + 1| - |x_i(t) - 1|),\ i = 1, 2, 3, 4$$

For simplifying analysis, let $\tau_1(t) = \tau_2(t) = d_1(t)$ and $\tau_3(t) = d_1(t) + d_2(t)$. Then, the closed-loop system can be represented as system (1) with $A_0 = A_0$, $A_1 = A_1$, $A_2 = 0$, $W_0 = 0$, $W_1 = B_1K$, $W_2 = B_2K$, $W = I$, $\Sigma_1 = \Delta_1 = diag\{0.02, 0.02, 0.02, 0.02\}$, and $\Sigma_2 = \Delta_2 = diag\{0, 0, 0, 0\}$.

Let $\mu_1 = \mu_2 = 10E + 5$ to cover more types of time-varying delays. The maximal upper bounds of d_2 with respect to $d_1 \in \{1.0, 1.5, 2.0, 2.5\}$ calculated by Theorem 1 are listed in Table 2. The results show that the maximal upper bounds of d_2 increase as the increase of $\gamma_i, i = 1, 2$.

Table 2: The maximal upper bounds of d_2 for various d_1 calculated by Theorem 1.C1 (Example 2).

γ_1/γ_2	d_1			
	1.0	1.5	2.0	2.5
0.1/0.1	2.09	1.15	0.71	0.57
0.333/0.307	2.37	1.15	0.71	0.57

5. Conclusions

This paper has investigated the stability of Lur'e systems with two additive delay components. The extended matrix inequality (named as ERCMI) has been employed to reduce the estimation gap of popular reciprocally convex combination lemma. As a result, several stability criteria with less conservatism have been established by using the ERCMI, together with two types of LKFs. Finally, the advantages of the extended matrix inequality and the corresponding criteria have been demonstrated via two examples.

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