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Floquet stability analysis of Ott–Grebogi–Yorke and difference control

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Abstract. Stabilization of instable periodic orbits of nonlinear dynamical systems has been a widely explored field theoretically and in applications. The techniques can be grouped into time-continuous control schemes based on Pyragas, and the two Poincaré-based chaos control schemes, Ott–Gebogi–Yorke (OGY) and difference control. Here, a new stability analysis of these two Poincaré-based chaos control schemes is given by means of the Floquet theory. This approach allows to calculate exactly the stability restrictions occurring for small measurement delays and for an impulse length shorter than the length of the orbit. This is of practical experimental relevance; to avoid a selection of the relative impulse length by trial and error, it is advised to investigate whether the used control scheme itself shows systematic limitations on the choice of the impulse length. To investigate this point, a Floquet analysis is performed. For OGY control the influence of the impulse length is marginal. As an unexpected result, difference control fails when the impulse length is taken longer than a maximal value that is approximately one half of the orbit length for small Ljapunov numbers and decreases with the Ljapunov number.

Contents

1. Introduction

Controlling chaos, or stabilization of instable periodic orbits of chaotic systems, has matured to a field of large interest in theory and experiment [1]–[18]. Most of these techniques are applied to dissipative systems, but with respect to applications such as suppression of transport in plasmas, there has been recent interest also in controlling chaos in Hamiltonian systems [19]–[25] and in the stability of periodic orbits in multidimensional systems [26]. Control of chaos in general, however, does not rely on the existence of a Hamiltonian, so the system can be given as any dynamical system described by a set of differential equations, or can be an experimental system. The aim of chaos control is the stabilization of unstable periodic orbits (UPOs), or modes in spatial systems. The central idea of the Hübler and Ott–Grebogi–Yorke (OGY) approaches to the control of chaos is to utilize the sensitive dependence on initial conditions, be it *system variables* [1], or *system parameters* [2] to steer the trajectory on an (otherwise unstable) periodic orbit of the system. To accomplish this task, a feedback is applied to the system at each crossing of a suitably chosen Poincaré plane, where the feedback is set proportional to the actual deviation of the desired trajectory. Contrary to Pyragas control [3], where control is calculated quasi continuously at a high sampling rate, here the approach is to stabilize by a feedback calculated at each Poincaré section, which reduces the control problem to stabilization of an unstable fixed point of an iterated map. The feedback can be chosen proportional to the distance to the desired fixed point (OGY scheme), or proportional to the *difference* in phase space position between the actual and last but one Poincaré section. This latter method, *difference control* [4], or Bielawski–Derozier–Glorieux control, being a time-discrete counterpart of the Pyragas approach [3], allows for stabilization of inaccurately known fixed points, and can be extended by a memory term [27, 28] to improve stability and to allow for tracking [7] of drifting fixed points [27].

In this paper, the stability of perturbations $x(t)$ around an UPO being subject to a Poincaré-based control scheme is analyzed by means of the Floquet theory [29]. This approach allows to investigate viewpoints that have not been accessible by considering only the iteration dynamics between the Poincaré sections, such as measurement delays and variable impulse lengths.

The impulse length is, both in OGY and difference control, usually a fixed (quasi invisible) parameter; and the iterated dynamics is uniquely defined only as long as this impulse length is not varied. The influence of the impulse length has not been a point of consideration before; if mentioned at all, usually a relative length of approximately $1/3$ is chosen without any reported sensitivity. Whereas for the Pyragas control method (in which the delayed state feedback enforces a time-continuous description) a Floquet stability analysis is known [30], here the focus is on the time-discrete control.

1.1. Floquet stability analysis

The linearized differential equations of both schemes are invariant under translation in time, $t \to t+T$. Hereby, we assume that the system under control is not explicitly time-dependent. According to the theory of delay-differential equations [29], a stability condition can be derived from a simple eigenvalue analysis of Floquet modes. The Floquet ansatz expands the solutions after periodic solutions $u(t+T) = u(t)$ according to

$$x(t) = \mathrm{e}^{\gamma t} u_\gamma(t). \tag{1}$$

The necessary condition on the Floquet multiplier $\mathrm{e}^{\gamma T}$ of an orbit of duration T for stability of the solution is $\mathrm{Re}\gamma < 0$; and $x(t) \equiv 0$ refers to motion along the orbit. In Poincaré-based control, the effective motion can be transformed into the unstable eigenspace, see e.g. appendix A in [28]; the stability is governed by the motion therein. For the case of one unstable Lyapunov exponent, this subspace is one-dimensional.

1.2. OGY control

The method proposed by OGY [2] applies a control amplitude $r(t) = \varepsilon(t)(x(t_\times) - x^*)$, in the vicinity of a fixed point x^*. Here, $\varepsilon(t)$ is a (possibly time-dependent) feedback gain parameter, and $x(t_\times)$ is the position of the last Poincare crossing. Without loss of generality, we can place the fixed point at $x = 0$, so that the OGY feedback scheme becomes $r(t) = \varepsilon(t)x(t_\times)$, where $t_\times \equiv t - (t \bmod T)$ is the time of the last Poincaré crossing. Now one considers the linearized motion in vicinity of an UPO (which is a stable periodic orbit of a successfully controlled system). The Poincaré crossing reduces the dimensionality from N to $N-1$ dimensions, in the lowest-dimensional case from 3 to 2. In this case, it is sufficient to consider a linearized one-dimensional time-continuous motion around the orbit, $\dot{x}(t) = \lambda x(t) + \mu r(t)$, which now can be complex-valued to account for flip motion around the orbit [30], i.e. one has the dynamical system

$$\dot{x}(t) = \lambda x(t) + \mu\varepsilon x(t - (t \bmod T)). \tag{2}$$

Without control ($r(t) = 0$), the time evolution of this system is simply $x(t) = \mathrm{e}^{\lambda t}$ and the Ljapunov exponent of the uncontrolled system is $\mathrm{Re}\lambda$. Here it must be emphasized that assuming constant λ and μ is a quite crude approximation, so only qualitative results can be concluded. Now we see—a central observation—that no delay-differential equation [29] is obtained: As the 'delay' term always refers to the last Poincaré crossing, this type of dynamics can be integrated piecewise. In the first time interval between $t = 0$ and $t = T$ the differential equation reads

$$\forall_{0<t<T}, \quad \dot{x}(t) = \lambda x(t) + \mu\varepsilon x(0).$$

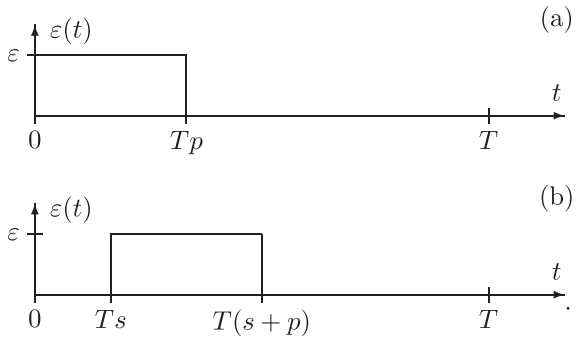

Figure 1. Impulse shapes considered for $\varepsilon(t)$, resulting in the Floquet multipliers (7) for a finite impulse length (a), and (10) if one also adds an additional delay of duration s (b).

Integration of this differential equation yields

$$x(t) = \left(\left(1 + \frac{\mu\varepsilon}{\lambda}\right)\mathrm{e}^{\lambda t} - \frac{\mu\varepsilon}{\lambda}\right)x(0).$$

This gives us an iterated dynamics (here, we label the beginning of the time period again with t)

$$x(t+T) = \left(\left(1 + \frac{\mu\varepsilon}{\lambda}\right)\mathrm{e}^{\lambda T} - \frac{\mu\varepsilon}{\lambda}\right)x(t).$$

This equation allows to determine the Floquet modes $x(t+T) = \mathrm{e}^{\gamma T}x(t)$ by inspection. The Floquet multiplier $\mathrm{e}^{\gamma T}$ of an orbit, assuming an impulse duration of full orbit length, is therefore given by

$$\mathrm{e}^{\gamma T} = \left(1 + \frac{\mu\varepsilon}{\lambda}\right)\mathrm{e}^{\lambda T} - \frac{\mu\varepsilon}{\lambda}. \tag{3}$$

The remainder of the paper investigates, both for OGY and difference controls, how the Floquet multiplier is modified for different impulse lengths.

2. Influence of impulse length: OGY case

The time-discrete viewpoint now allows the influence of timing questions on control to be investigated. First, we consider the case that the control impulse is applied timely in the Poincaré section, but only for a finite period pT within the orbit period ($0 < p < 1$) (see figure 1(a)).

This situation is described by the differential equation

$$\dot{x}(t) = \lambda x(t) + \mu\varepsilon x(t - (t \bmod T))\Theta((t \bmod T) - p), \tag{4}$$

here Θ is a step function ($\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ elsewhere). In the first time interval between $t = 0$ and $t = pT$ the differential equation reads $\dot{x}(t) = \lambda x(t) + \mu\varepsilon x(0)$. Integration of this differential equation yields

$$\forall_{0<t\leqslant pT}, \quad x(t) = \left(\left(1 + \frac{\mu\varepsilon}{\lambda}\right)\mathrm{e}^{\lambda t} - \frac{\mu\varepsilon}{\lambda}\right)x(0). \tag{5}$$

In the second interval between $t = pT$ and $t = T$, the differential equation is the same as without control, $\dot{x}(t) = \lambda x(t)$. From this one has immediately

$$\forall_{pT<t<T} \qquad x(t) = \mathrm{e}^{\lambda(t-pT)}x(pT) \tag{6}$$

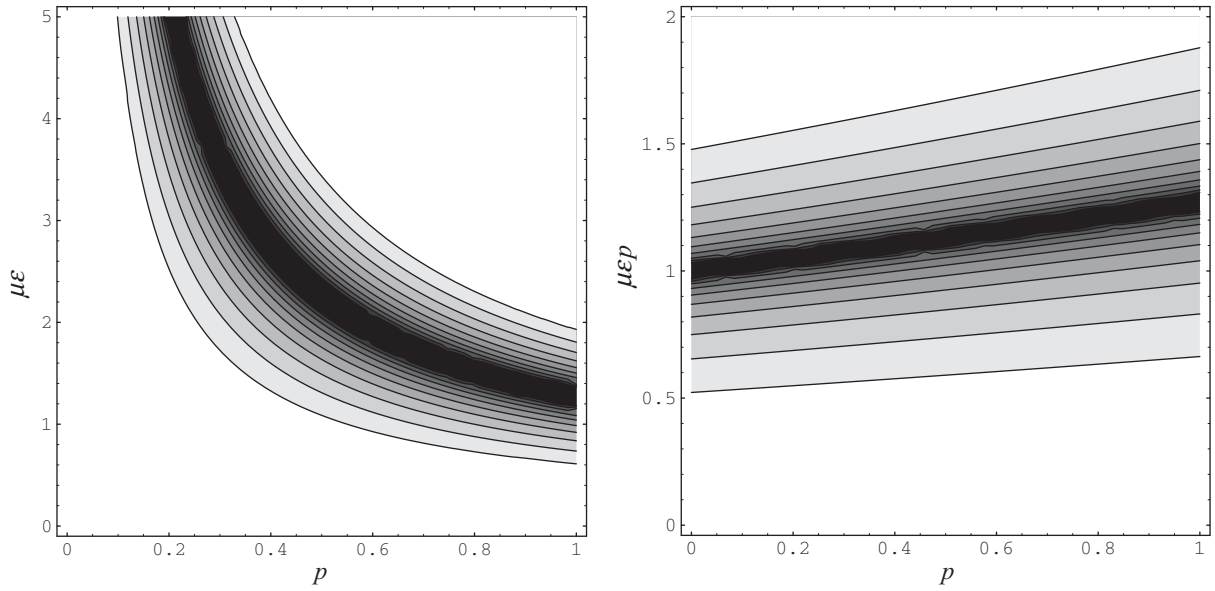

Figure 2. Left panel: dependence of OGY control on the duration $1 \leqslant pT \leqslant T$ and strength of control. Control is possible within the shaded areas, where darker shadings refer to smaller $\mathrm{Re}\gamma$, including an optimal line in between where $\mathrm{Re}\gamma \to -\infty$. In the white areas, $\mathrm{Re}\gamma$ is positive, there the system with applied control becomes unstable. Right panel: plot in the $(p, \mu\varepsilon p)$ plane, the product $\mu\varepsilon p$ shows only a weak dependency on p. This supports that the linear approximation (9) is a good approximation for (7).

and the Floquet multiplier of an orbit is given by

$$\mathrm{e}^{\gamma T} = \mathrm{e}^{\lambda T}\left(1 + \frac{\mu\varepsilon}{\lambda}(1 - \mathrm{e}^{-\lambda p T})\right). \tag{7}$$

The consequences are shown in figure 2. One finds that in zero order the 'strength' of control is given by the product $pT\mu\varepsilon$; in fact, there is a weak linear correction in p. This analysis reproduces well the experimental results of Mausbach [33]. For $\lambda pT \leqslant 1$ one has

$$\mathrm{e}^{\gamma T} = \mathrm{e}^{\lambda T}(1 + \mu\varepsilon pT - \tfrac{1}{2}\mu\varepsilon\lambda p^2 T^2 + o(p^3)), \tag{8}$$

i.e. the condition of a constant strength of control reads

$$\mu\varepsilon pT = \frac{1}{1 - (\lambda T/2)p} = 1 + \frac{\lambda T}{2}p + o(p^2). \tag{9}$$

The result is: apart from a weak linear correction for OGY control the length of the impulse can be chosen arbitrarily, and the 'strength' of control in zero order is given by the time integral over the control impulse.

For the case where the system can only be measured delayed—by a delay time of the orbit length and longer—stability borders [32, 33] and improved control schemes have been given in [28, 34] and successfully applied experimentally [27, 35, 36]. In experimental situations one often has the intermediate case that there is a measurement delay sT that is not neglectable, but is within the orbit length. To keep the case general, we again consider a finite impulse length

pT with $0 < p < 1$ and $0 < (s+p) < 1$, see figure 1(b). Again we can integrate piecewise, and the Floquet multiplier is given by

$$e^{\gamma T} = e^{\lambda T}\left(1 + \frac{\mu\varepsilon}{\lambda} e^{-\lambda s T}(1 - e^{-\lambda p T})\right) \tag{10}$$

and (7) is included as the special case of $s = 0$.

3. Impulse length influence: difference control

Now we analyze the difference control scheme [4]

$$r(t) = \varepsilon(x(t_\times) - x(t_{\times\times})), \tag{11}$$

where $t_\times$ and $t_{\times\times}$ denote the times of the last and last but one Poincaré crossing, respectively. Again the starting point is the linearized equation of motion around the periodic orbit when control is applied. For difference control now there is a dependency on two past time steps,

$$\dot{x}(t) = \lambda x(t) + \mu\varepsilon x(t - (t \bmod T)) - \mu\varepsilon x(t - T - (t \bmod T)). \tag{12}$$

Although the right-hand side of (12) depends on x at three different times, it can be nevertheless integrated exactly, which is mainly due to the fact that the two past times (of the two last Poincaré crossings) have a fixed time difference being equal to the orbit length. This allows not only for an exact solution, but also offers a correspondence to the time-discrete dynamics and the matrix picture used in time-delayed coordinates.

Now also for difference control the experimentally more common situation of a finite but small measurement delay sT is considered, together with a finite impulse length sT (here $0 < p < 1$ and $0 < (s+p) < 1$). An analogous calculation [34, 37] as for the OGY case can also be performed as follows. Piecewise integration in the first interval $0 < t < T \cdot s$, where the differential equation reads $\dot{x}(t) = \lambda x(t)$, yields again $x(T \cdot s) = e^{\lambda T \cdot s} x(0)$. In the second interval $T \cdot s < t < T \cdot (s+p)$ the differential equation reads

$$\dot{x}(t) = \lambda x(t) + \mu\varepsilon(x(0) - x(-T)).$$

Integration yields

$$\forall_{T\cdot s < t < T\cdot(s+p)} x(t) = -\frac{\mu\varepsilon}{\lambda}(x(0) - x(-T)) + \left(\frac{\mu\varepsilon}{\lambda}(x(0) - x(-T)) + e^{\lambda s T} x(0)\right) e^{\lambda(t - sT)},$$

$$x(T \cdot (s+p)) = -\frac{\mu\varepsilon}{\lambda}(x(0) - x(-T))\frac{\mu\varepsilon}{\lambda}(x(0) - x(-T)) + e^{\lambda p T} + e^{\lambda(s+p)T} x(0).$$

For the third interval again there is no control, thus

$$\forall_{T\cdot(s+p) < t < T}, \quad x(t) = e^{\lambda(t - (s+p)T)} x(T \cdot (s+p)).$$

Collecting together, we find for $x(T)$

$$x(T) = x(0)e^{\lambda T}\left(1 + \frac{\mu\varepsilon}{\lambda} e^{-\lambda s T}(1 - e^{-\lambda p T})\right) - x(-T)e^{\lambda T}\frac{\mu\varepsilon}{\lambda} e^{-\lambda s T}(1 - e^{-\lambda p T}) \tag{13}$$

or, in time-delayed coordinates of the last and last but one Poincaré crossing

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} e^{\lambda T}\left(1 + \dfrac{\mu\varepsilon(1 - e^{-\lambda p T})}{\lambda e^{\lambda s T}}\right) & -e^{\lambda T}\dfrac{\mu\varepsilon(1 - e^{-\lambda p T})}{\lambda e^{\lambda s T}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}.$$

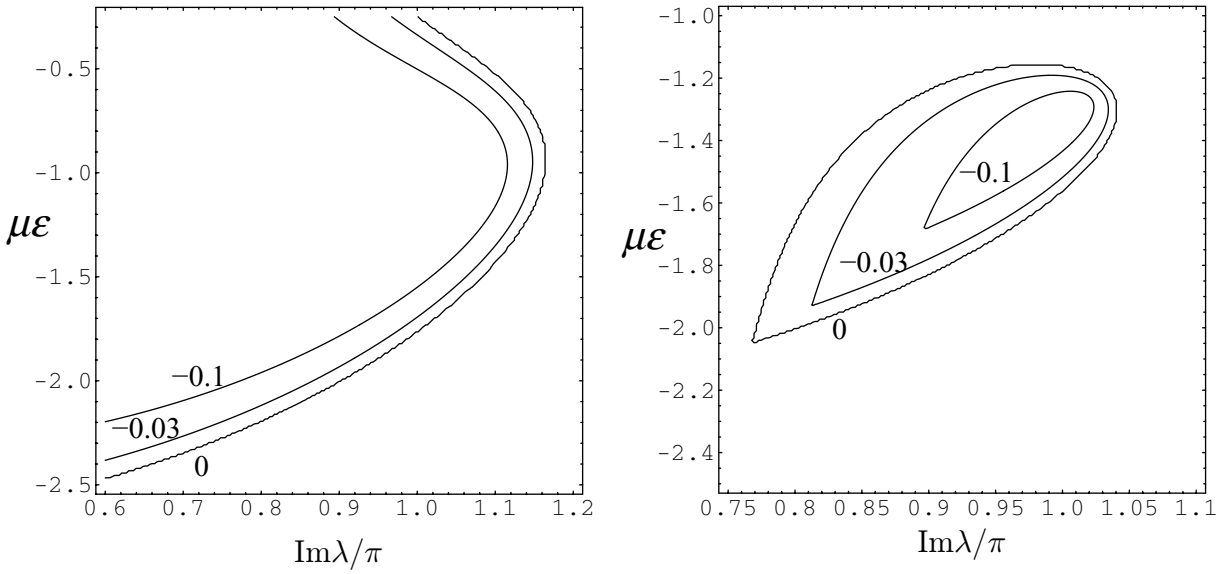

Figure 3. Time-continuous stability analysis of difference control with consideration of the length of the control impulse: contour plot of the real part of the Floquet exponent in (Imλ,$\mu\varepsilon$) space for $T = 1$, $s = 0$, $p = 0.3$, Re$\lambda = 0.2$ (left panel) and Re$\lambda = 0.6$ (right panel). For Im$\lambda = \pi$ one finds (above Re$\lambda =$ 0.52) an island of stability, which completely disappears for $p > p_{\max}$ (figure 5). The Floquet analysis shows that the impulse length is of fundamental importance for the dynamical behavior and stability of difference control, contrary to the situation for OGY control. The contour lines show the real part of the Floquet multiplier for 0 (outer), -0.03 and -0.1 (inner contour).

If we identify with the coefficients of the time-discrete case, $\lambda_{\mathrm{d}} = \mathrm{e}^{\lambda T}$ and $\mu_{\mathrm{d}}\varepsilon_{\mathrm{d}} = \mathrm{e}^{-\lambda s T}(1 - \mathrm{e}^{\lambda p T})\frac{\mu\varepsilon}{\lambda}$, the dynamics in the Poincaré iteration $t = nT$ becomes identical with the pure discrete description; this again illustrates the power of the concept of the Poincaré map. In principle, due to the low degree of the characteristic polynomial, one could explicitly diagonalize the iteration matrix, allowing for a closed expression for the nth power of the iteration matrix. However, for the stability analysis only the eigenvalues are needed. For the Floquet multiplier one has

$$\mathrm{e}^{2\gamma T} = \mathrm{e}^{\gamma T}\mathrm{e}^{\lambda T}\left(1 + \frac{\mu\varepsilon}{\lambda}\mathrm{e}^{-\lambda s T}(1 - \mathrm{e}^{-\lambda p T})\right) - \mathrm{e}^{\lambda T}\frac{\mu\varepsilon}{\lambda}\mathrm{e}^{-\lambda s T}(1 - \mathrm{e}^{-\lambda p T}). \tag{14}$$

This quadratic equation yields two Floquet multipliers,

$$\mathrm{e}^{\gamma T} = \frac{1}{2}\mathrm{e}^{\lambda T}\left(1 + \frac{\mu\varepsilon}{\lambda}\mathrm{e}^{-\lambda s T}(1 - \mathrm{e}^{-\lambda p T})\right) \pm \frac{1}{2}\sqrt{\left(\mathrm{e}^{\lambda T}\left(1 + \frac{\mu\varepsilon}{\lambda}\mathrm{e}^{-\lambda s T}(1 - \mathrm{e}^{-\lambda p T})\right)\right)^2 + 4\mathrm{e}^{\lambda T}\frac{\mu\varepsilon}{\lambda}\mathrm{e}^{-\lambda s T}(1 - \mathrm{e}^{-\lambda p T})}.$$

Figures 3 and 4 show, by example of Re$\lambda = 0.2$, (resp. Re$\lambda = 0.6$) that at Im$\lambda = \pi$ for $p < p_{\max}(\mathrm{Re}\lambda = 0.2) \simeq 0.505401$ (resp. $p < p_{\max}(\mathrm{Re}\lambda = 0.6) \simeq 0.367$) there exists an island of stability, whose width decreases to zero for $p \to p_{\max}$. Here, explicitly the influence of the impulse length can be seen. The maximal value of $p_{\max}$ is shown in figure 5.

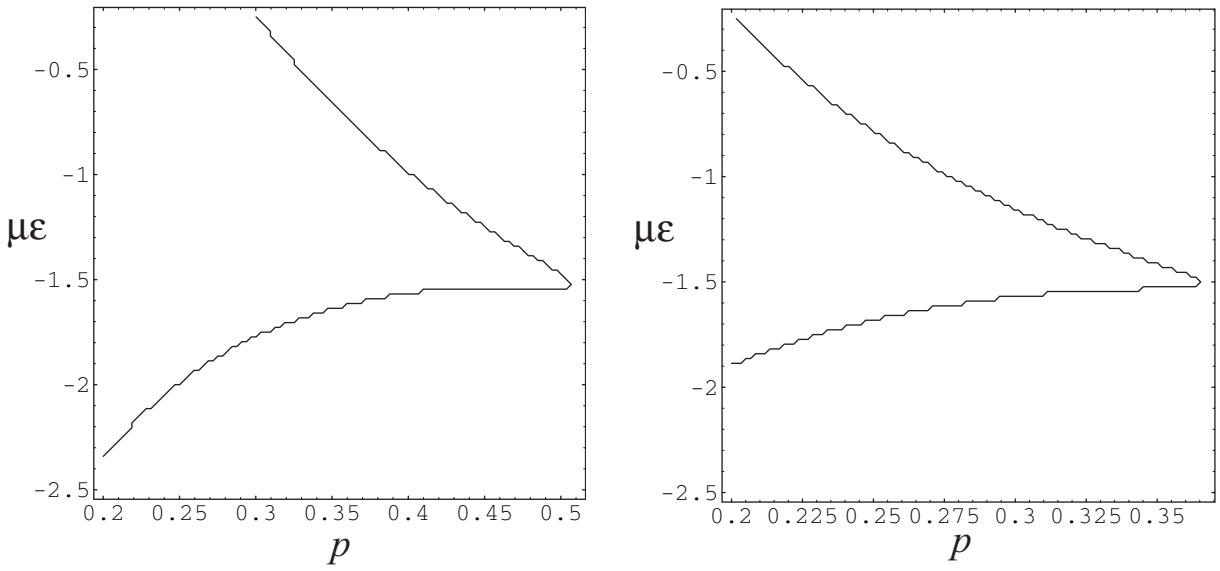

Figure 4. Time-continuous stability analysis of difference control with consideration of the length of the control impulse: stability area for fixed twist $\mathrm{Im}\lambda = \pi$ and independence of amplitude $\mu\varepsilon$ and relative impulse length p, for $T = 1$, $s = 0$, $\mathrm{Re}\lambda = 0.2$ (left panel) and $\mathrm{Re}\lambda = 0.6$ (right panel).

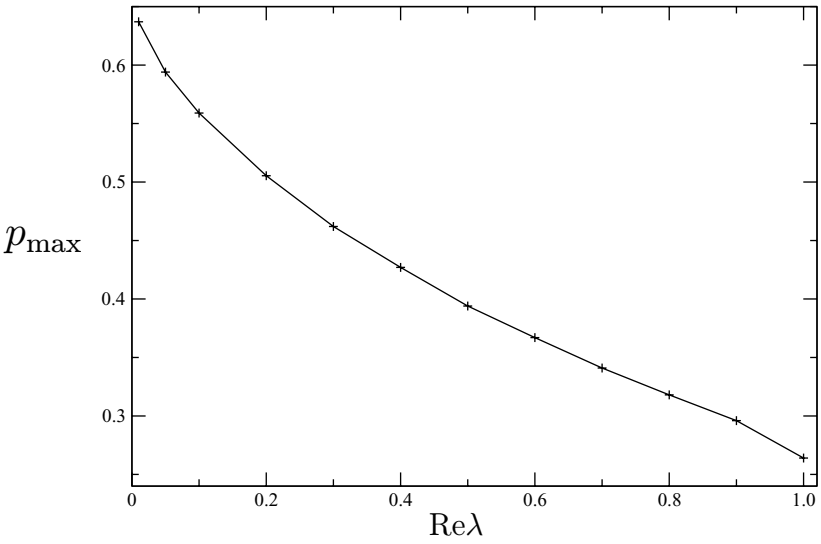

Figure 5. Time-continuous stability analysis of difference control: maximal impulse length p for different values of $\mathrm{Re}\lambda$ in the range from 0.01 to 1.0.

4. Discussion of the time-continuous model: relaxing the assumption of a constant local Ljapunov exponent

The quantitative analysis given above was based on the model assumption that one has a constant local Ljapunov exponent around the orbit—a condition that will almost never be fulfilled exactly for a typical orbit of a chaotic system. To test whether this assumption is

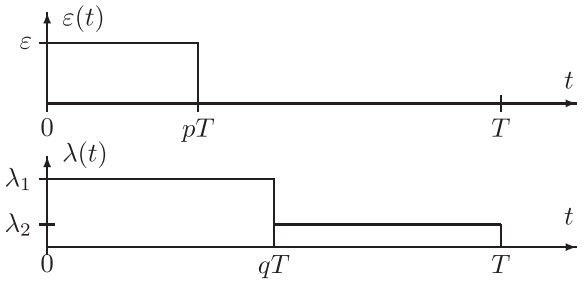

Figure 6. Impulse shapes $\varepsilon(t)$ and time-varying $\lambda(t)$: case of a short impulse $p < q$.

crucial, the stability can be investigated for OGY control by assuming λ to be time-dependent along the orbit. For simplicity, we consider the exemplaric case shown in figure 6 that $\lambda(t) = \lambda_1$ for $0 < t < qT$ and $\lambda(t) = \lambda_2$ for $qT < t < T$, generalizing the case $q = 1/2$ already sketched in [37].

4.1. The case $p < q$ of a short impulse

If we assume $p < q$, we have for the OGY case,

$$\begin{aligned} x(pT) &= (\mathrm{e}^{\lambda_1 pT}(1 + (\mu\varepsilon/\lambda_1)) - (\mu\varepsilon/\lambda_1))x(0), \\ x(qT) &= \mathrm{e}^{\lambda_1(qT-pT)}x(pT), \\ x(T) &= \mathrm{e}^{\lambda_2(T-qT)}x(qT). \end{aligned} \tag{15}$$

Using $\bar{\lambda} := q\lambda_1 + (1-q)\lambda_2$, the Floquet multiplier reads now

$$\mathrm{e}^{\gamma T} = \mathrm{e}^{\bar{\lambda}T}\left(1 + \frac{\mu\varepsilon}{\lambda_1}(1 - \mathrm{e}^{-\lambda_1 pT})\right) \tag{16}$$

in contrast to equation (7). Again in zero order the 'strength' of control is given by the product $p\mu\varepsilon$; in first order $\lambda_1 pT \leqslant 1$ again the weak linear dependence on p applies,

$$\mathrm{e}^{\gamma T} = \mathrm{e}^{\bar{\lambda}T}\left(1 + \mu\varepsilon pT\left(1 - \frac{1}{2}\lambda_1 pT + o(p^2)\right)\right), \tag{17}$$

i.e. for a constant 'strength' of control, one has to fulfill

$$\mu\varepsilon pT = \frac{1}{1 - (\lambda_1 T/2)p} + o(p^2) = 1 + \frac{\lambda_1 T}{2}p + o(p^2). \tag{18}$$

Thus, the value of λ_1, i.e. here, the deviation of $\lambda(t)$ from its average value $\bar{\lambda}$ during the control impulse, only contributes in first order. More complicated cases can be tackled likewise, giving corrections for the quantitative p-dependence of the optimal control gain ε, but preserving the qualitative behavior discussed above.

4.2. The case $p > q$ of a long impulse

While a short impulse is the experimentally more feasible case, for completeness, also the $p > q$ shown in figure 7 can be investigated in this manner. Here, we have two time intervals where the control is active; and in the second one the initial condition $x(pT)$ has to be distinguished

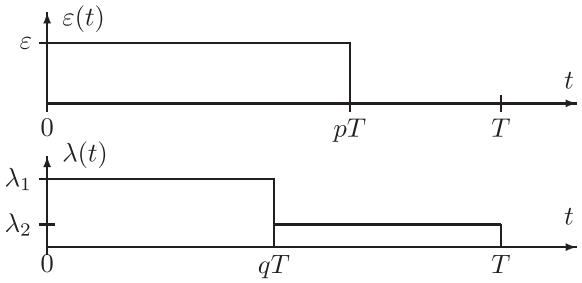

Figure 7. Impulse shapes $\varepsilon(t)$ and time-varying $\lambda(t)$: case of a long impulse $p > q$.

from the position $x(0)$ at the last Poincaré crossing, from which the control value is calculated and which determines the inhomogenity of the ODE.

Integration over the three time intervals yields

$$
\begin{aligned}
x(qT) &= \mathrm{e}^{\lambda_1 qT} x(0)\left(1+\frac{\mu\varepsilon}{\lambda_1}\right) - x(0)\frac{\mu\varepsilon}{\lambda_1},\\
x(pT) &= \mathrm{e}^{\lambda_2(pT-qT)}\left(x(qT)+x(0)\frac{\mu\varepsilon}{\lambda_2}\right) - x(0)\frac{\mu\varepsilon}{\lambda_2},\\
x(T) &= \mathrm{e}^{\lambda_2(T-pT)} x(pT),\\
x(pT) &= \mathrm{e}^{-\lambda_2(p-q)T}\underbrace{x(0)\left[\mathrm{e}^{\lambda_1 qT}\left(1+\frac{\mu\varepsilon}{\lambda_1}\right)\frac{\mu\varepsilon}{\lambda_1}\right]}_{x(qT)} + x(0)\frac{\mu\varepsilon}{\lambda_2}(\mathrm{e}^{-\lambda_2(p-q)T}-1),\\
x(T) &= x(0)\mathrm{e}^{\lambda_2(1-q)T} x(0)\left[\mathrm{e}^{\lambda_1 qT} x(0)\left(1+\frac{\mu\varepsilon}{\lambda_1}\right) - x(0)\frac{\mu\varepsilon}{\lambda_1}\right]\\
&\quad + x(0)\frac{\mu\varepsilon}{\lambda_2}(\mathrm{e}^{\lambda_2(1-q)T} - \mathrm{e}^{\lambda_2(1-pT)}),
\end{aligned}
$$

so that we arrive at

$$
x(T) = x(0)\mathrm{e}^{\bar{\lambda}T}\left[1+\frac{\mu\varepsilon}{\lambda_1}(1-\mathrm{e}^{-\lambda_1 qT}) + \frac{\mu\varepsilon}{\lambda_2}(1-\mathrm{e}^{-\lambda_2(p-q)T})\mathrm{e}^{-\lambda_1 qT}\right]. \tag{19}
$$

4.2.1. Weakly nonlinear approximation for $q = 1/2$. As $p > q$, and q is a fixed value for the given system, a discussion of $p\lambda T \ll 1$ can no longer be based on the $p \to 0$ case. As p is of order 1, an expansion as above is meaningful only for the case where $\lambda T \ll 1$, i.e. we derive an approximation for those UPOs which have an only marginally positive Floquet multiplier. For $q = 1/2$, we now explicitly discuss this 'weakly nonlinear' case $\lambda_1 T \ll 1$, $\lambda_2 T \ll 1$,

$$
\begin{aligned}
x(T) &= x(0)\mathrm{e}^{\bar{\lambda}T}\left[1+\frac{\mu\varepsilon}{\lambda_1}\left(-\lambda_1\frac{T}{2}+\frac{\lambda_1^2 T^2}{8}-\frac{\lambda_1^3 T^3}{48}\right)\right.\\
&\quad \left.+\frac{\mu\varepsilon}{\lambda_2}\left(-\lambda_2 T\left(p-\frac{1}{2}\right)+\frac{\lambda_2^2 T^2(p-1/2)^2}{2}\right)\left(-\frac{\lambda_1 T}{2}+\frac{\lambda_1^2 T^2}{8}\right)\right]\\
&= x(0)\mathrm{e}^{\bar{\lambda}T}\left[1+\mu\varepsilon T\left(\left(-\frac{1}{2}+\frac{\lambda_1 T}{8}-\frac{\lambda_1^2 T^2}{48}\right)\right.\right.
\end{aligned}
$$

$$+\left(-\left(p-\frac{1}{2}\right)+\frac{\lambda_2 T(p-(1/2))^2}{2}\right)\left(-\frac{1}{2}+\frac{\lambda_1 T}{8}\right)\lambda_1 T\Bigg)\Bigg]$$
$$=x(0)\mathrm{e}^{\bar{\lambda}T}\left[1-\frac{\mu\varepsilon T}{2}\left(1-\lambda_1 T\left(p-\frac{1}{4}\right)+o(\lambda_1^2,\lambda_1\lambda_2,\lambda_2^2)\right)\right],$$

that is, control is kept constant in lowest order for

$$\frac{\mu\varepsilon T}{2}\stackrel{!}{=}\frac{1}{1-\lambda_1 T(p-(1/4))}\simeq 1+\lambda_1 T(p-(1/4)) \tag{20}$$

for orbits sharing the same value of $\bar{\lambda}$.

4.3. Difference control and nonconstant local Ljapunov exponent

For completeness, the case of difference control is now also considered. As has been shown before, for the case of a constant Ljapunov exponent, impulse lengths of $p>\frac{1}{2}$ do not lead to stable control; therefore the case $p>q$ is completely irrelevant, and only the case $p<q$ has to be considered. In the first interval, λ_1 is active and integration yields

$$x(pT)=\mathrm{e}^{\lambda_1 pT}\left[x(0)\left(1+\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1)\right)-x(-T)\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1)\right] \tag{21}$$

and the subsequent intervals have the control switched off,

$$x(T)=\mathrm{e}^{\lambda_2(1-q)T}x(qT)=\mathrm{e}^{\lambda_2(1-q)T}\mathrm{e}^{\lambda_1(q-p)T}x(pT) \tag{22}$$
$$=\mathrm{e}^{\bar{\lambda}T}\left[x(0)\left(1+\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1)\right)-x(-T)\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1)\right]. \tag{23}$$

Again we use the average value $\bar{\lambda}=\lambda_1 q+\lambda_2(1-q)$ to simplify the expressions, and the coordinates in the Poincaré countings $x_{n+1}=x(T)$, $x_n=x(0)$ and $x_{n-1}=x(-T)$. We have

$$\begin{pmatrix}x_{n+1}\\x_n\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{\bar{\lambda}T}\left(1+\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1)\right) & -\mathrm{e}^{\bar{\lambda}T}\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1)\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_n\\x_{n-1}\end{pmatrix}, \tag{24}$$

which leads to the characteristic equation for the two Floquet multipliers $\mathrm{e}^{\gamma T}$

$$\mathrm{e}^{2\gamma T}=\mathrm{e}^{\gamma T}\mathrm{e}^{\bar{\lambda}T}\left(1+\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1)\right)-\mathrm{e}^{\bar{\lambda}T}\frac{\mu\varepsilon}{\lambda_1}(\mathrm{e}^{-\lambda_1 pT}-1). \tag{25}$$

This generalizes the discussion of difference control to the case of nonconstant λ.

5. Conclusions and outlook

To summarize, a new time-continuous stability analysis of Poincaré-based control methods was introduced. This general and novel approach allows to investigate timing questions of Poincaré-based control schemes that cannot be analyzed within the picture of the Poincaré iteration. For both OGY and difference control it has been possible for a homogeneous case to integrate the dynamics exactly. While for OGY control the impulse length turns out not to be crucial, for difference control it is, and the impulse has to be shorter than a critical fraction of the period, which is of the order of half of the period and decreases for larger Ljapunov exponents.

Such timing dependence is not completely uncommon in feedback systems with delay; in time-continuous feedback control, a half-period feedback resulted in an enlarged stability range [38].

Techniques of chaos control, be it Poincaré-based, following the Pyragas technique, or open-loop [39], [40]–[43] have been of great interest not only in technical systems, but also in biological, especially neural systems and excitable media [44, 45]. Besides the approach of controlling pathological neural subsystems directly, the intact brain already bears implementations of feedback control [46], implying that the failure of the respective circuits eventually results in migraine or stroke. Also the human gait system, like virtually any perception-motor system, performs control of a bio-mechanical system with delays; and disturbances of the delay loops as well as the cortical control may result in tremor and related movement disorders [47]–[50]. In the thalamocortical system, a designated impulse shape, formed by the so-called slow waves that emerge in the cortex during S2 sleep, has been shown to act as an open-loop controller of thalamic oscillator networks [51]. This offers further possibilities to influence human sleep in the case of sleep disturbances: recent control techniques by transcranial electrical or magnetic stimulation [52]–[55] have been demonstrated to influence human sleep as well as to affect memory consolidation during sleep. In most of these techniques, the impulse shape and relative duration of the control impulse has significant impact on the results, thus different control goals may become accessible within the same setup. For systematic understanding of how such control techniques influence the brain, detailed models are of similar importance to the methodical understanding of the theoretically possible control methods.

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