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Structure of words with short 2-length in a free product of groups.

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Abstract

Howie and Duncan observed that a word in a free product with length at least two, which is not a proper power and involves no letter of order two can be decomposed as a product of two cyclic subwords each of which is uniquely positioned. Using this property, they proved various important results about a one-relator product of groups with such word as the relator. In this paper, we show that similar results hold in a more general setting where we allow a certain number of elements of order two.

Keywords: One-relator product, unique position, pictures, 2-length
2000 MSC: 20E06, 20F06, 20F10

1. Introduction

Let R be a cyclically reduced word which is not a proper power and has length at least two in the free group $F = F(X)$. In [12], Weinbaum showed that some cyclic conjugate of R has a decomposition of the form UV, where U and V are non-empty cyclic subwords of R, each of which is uniquely positioned in R i.e occurs exactly once as a cyclic subword of R. Weinbaum also conjectured that U and V can be chosen so that neither is a cyclic subword of R^{-1}. A stronger version of Weinbaum's conjecture was proved by Duncan and Howie [4]. In this paper, a cyclic subword is uniquely positioned if it is non-empty, occurs exactly once as a subword of R and does not occur as a subword of R^{-1}.

Throughout this paper G_1 and G_2 are nontrivial groups and R is a cyclically reduced word in the free product $G_1 * G_2$, which is not a proper power and has length at least two. Before we can continue, we need to define the notion of *n-length*

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of a word. We do this in the special case when $n = 2$ and the word is R, but of course the definition can be generalized to any integer $n > 1$ and any word in a group.

For each letter a of order 2 involved in R, let $D(a)$ denote the number of times it occurs in R. In other words suppose R has free product length of $2k$ for some integer $k > 0$. Then without loss of generality, R has an expression of the form

$$R = \prod_{i=1}^{k} a_i b_i,$$

with $a_i \in G_1$ and $b_i \in G_2$. If $a^2 = 1$, then we define $D(a)$ to be the cardinality of the set $\{i \in \{1, 2, \cdots, k\} \mid a_i = a \textbf{ or } b_i = a\}$. Denote by $\mathbf{S}_R$ the symmetrized closure of R in $G_1 * G_2$ i.e the smallest subset of $G_1 * G_2$ containing R which is closed under cyclic permutations and inversion. Since $D(a)$ is unchanged by replacing R with any other element in $\mathbf{S}_R$, we make the following definition.

Definition 1. The 2-*length* of $\mathbf{S}_R$ denoted by $D_2(\mathbf{S}_R)$, is the maximum $D(a)$, such that a is a letter of order 2 involved in R.

In this paper, we will be mostly concerned with the element R' in $\mathbf{S}_R$ of the form

$$R' = \prod_{i=1}^{D_2(\mathbf{S}_R)} aM_i,$$

where $D(a) = D_2(\mathbf{S}_R)$ and M_i is a word $G_1 * G_2$. It follows that each M_i has odd length (as a reduced but not cyclically reduced word in the free product) and does not involve the letter a. When we use the notation " $=$" for words, it will mean identical equality. We will use $\ell()$ to denote the length operator of a reduced free product word which is not necessarily cyclically reduced.

As mentioned in the abstract, the authors of [4] observed that in the case of $D_2(\mathbf{S}_R) = 0$, the word R can be decomposed as a product of two uniquely positioned subwords. Using this property, they showed that every reduced picture over a one-relator product with relator R^m, $m \geq 3$ satisfies the small cancellation condition $C(6)$, from which important results about the group were proved. One of such results is the Freiheitssatz for one-relator products which states that G_1, G_2 embed in $G = (G_1 * G_2)/N(R^m), m \geq 3$ under the natural homomorphisms. In this paper, we work in a slightly more general setting where $D_2(\mathbf{S}_R) \leq 2$. In this setting, it is no longer always possible that R has a decomposition into two uniquely positioned subwords. Nonetheless, it can be shown that R has a certain structure which allows one to obtain similar results as in [4]. This structure is captured in the following theorem which is our main result.

Theorem 1. *Let R be a word in a free product of length at least 2 and which is not a proper power. Suppose that $D_2(\boldsymbol{S}_R) \leq 2$. Then either a cyclic conjugate of R has a decomposition of the form UV such that U and V are uniquely positioned or one of the following holds:*

(a) A cyclic conjugate of R has the form $aXbX^{-1}$, for some word X and some letters a, b satisfying $a^2 = b^2 = 1$.

(b) A cyclic conjugate of R has the form $aXbX^{-1}$, for some word X and some letters a, b satisfying $a^2 = 1 \neq b^2$.

In Theorem 1, the requirement that $D_2(\mathbf{S}_R) \leq 2$ is optimal in the sense that there is no hope to obtain such result when $D_2(\mathbf{S}_R) > 2$. To see why this is true, consider the word $S = \prod_{i=1}^{n} ab_i$, with $a \in G_1$ and $b_i \in G_2$, $i = 1, 2, \cdots, n$. Suppose that $b_i \neq b_j$ for $i \neq j$ and $a^2 = b_i^2 = 1$ for $i = 1, 2, \cdots, n$. It is easy to verify that $D_2(\mathbf{S}_R) = n$ and Theorem 1 fails for $n > 2$. In other words, neither does S have a decomposition into two uniquely positioned subwords, nor does it have a decomposition of the form $aXbX^{-1}$ such that $a^2 = 1$.

In [3] (see also [13]), the term "exceptional" was used for a one-relator product with relator of the form $aXbX^{-1}$, for some word X and letters a, b (up to cyclic permutation). In particular if p, q are the orders of a, b respectively, then the one-relator product G is said to be of type $E(p, q, m)$. When X is empty, G is the triangle group of type (p, q, m). Hence, G is said to be induced by the (generalized) triangle group of type (p, q, m) if it is of type $E(p, q, m)$. For us, the term *exceptional* is used for the subcase of $E(p, q, m)$ when $p \neq q$, $2 \in \{p, q\}$ and $D_2(\mathbf{S}_R) \leq 2$. In other words, call R *non-exceptional* if it satisfies part (a) of Theorem 1, and *exceptional* otherwise.

There is an already developed theory for one-relator products of type $E(p, q, m)$ (see [9, 1, 2]). Hence by Theorem 1, we can apply this theory in our setting. In the non-exceptional case, the extra structure that a, b are both letters of order 2 (as opposed to just one of them in the exceptional case), allows us to do more. In particular we have the following result.

Theorem 2. *Let R be a cyclically reduced word in the free product $G_1 * G_2$ such that $D_2(\boldsymbol{S}_R) \leq 2$. Suppose that R is non-exceptional. Then a non-trivial reduced picture on D^2 over $G = (G_1 * G_2)/N(R^m), m \geq 3$ satisfies $C(6)$.*

The rest of the paper is arranged as follows. We begin in Section section 2 by providing some literature on related results. We also recall only the basic ideas about pictures. In Section section 3 we prove a number of lemmas about word combinatorics and pictures. These lemmas are then used to deduce Theorems [1– 2]. In Section section 4 we deduce a number of applications of our results.

2. Preliminaries

Let G_1 and G_2 are nontrivial groups and $w \in G_1 * G_2$, a reduced word of length at least two. Let G be the quotient of the free product $G_1 * G_2$ by the normal closure of w, denoted $N(w)$. Then G is called a one-relator product and denoted by

$$G = (G_1 * G_2)/N(w).$$

We refer to G_1, G_2 as the factors of G, and w as the relator. For us, $w = R^m$, $m \geq 3$, and R is a cyclically reduced word which is not proper power and has length at least two. When $m \geq 4$, a number of results were proved in [6, 7, 8], about G. These results were also proved in [4] when $m = 3$, but not without the extra condition that R involves no letter of order 2. We also mention that the case when $m = 2$ is largely open. For partial results in this case see [5, 1, 2]. The aim of this paper is to extend the result in [4] by allowing to an extent letters of order 2 in R. In [4] it was shown that R^m satisfies the small cancellation condition $C(2m)$ when $D_2(\mathbf{S}_R) = 0$, which is essentially an observation in [11]. A general exposition on small cancellation theory can be found in [10]. We show that the same result holds in a more general setting, using the idea of *pictures* Pictures can be seen as duals of van Kampen diagrams and have been widely used to prove results about one-relator groups and one-relator products. Below, we recall only basic concepts on pictures over a one-relator product. For more details, the reader can see [6, 7, 8, 4, 2].

2.1. Pictures

Let G the one-relator product described above. A *picture* Γ over G on an oriented surface Σ is made up of the following data:

(a) a finite collection of pairwise disjoint closed discs in the interior of Σ called *vertices*;

(b) a finite collection of disjoint closed arcs called *edges*, each of which is a simple closed arc in the interior of Σ meeting no vertex of Γ or a simple arc joining two vertices (possibly the same one) on Γ or a simple arc joining a vertex to the boundary $\partial\Sigma$ of Σ or a simple arc joining $\partial\Sigma$ to $\partial\Sigma$;

(c) a collection of *labels* (i.e elements in $G_1 \cup G_2$), one for each corner of each *region* (i.e connected component of the complement in Σ of the union of vertices and arcs of Γ) at a vertex and one along each component of the intersection of the region with $\partial\Sigma$. For each vertex, the label around it spells out the word $R^{\pm m}$ (up to cyclic permutation) in the clockwise order as a cyclically reduced word in $G_1 * G_2$. We call a vertex *positive* or *negative* depending on whether the label around it is R^m or R^{-m} respectively. The labels in all corners of any given region must all be non-trivial elements of the same factor group, G_1 or G_2. A G_1-region is one in which the labels come from G_1. Similarly, a G_2-region is one in which the labels come from G_2. Each arc is required to separate a G_1-region from a G_2-region. This is compatible with the alignment of regions around a vertex, where the labels spell a cyclically reduced word, so must come alternately from G_1 and G_2.

For us Σ will either be the 2-sphere S^2 or 2-disc D^2. A picture on Σ is called *spherical* if either $\Sigma = S^2$ or $\Sigma = D^2$ but with no arcs connected to ∂D^2. If Γ is not spherical, ∂D^2 is one of the boundary components of a non-simply connected region (provided, of course, that Γ contains at least one vertex or arc), which is called the *exterior* region. All other regions are called *interior* regions.

We shall be interested mainly in *connected* pictures. A picture is *connected* if the union of its vertices and arcs is connected. In particular, no arc of a connected picture is a closed arc or joins two points of $\partial\Sigma$, unless the picture consists only of that arc. In a connected picture, all interior regions $\triangle$ are simply-connected, i.e topological discs. Just as in the case of vertices, the label around each region – read *anticlockwise* – gives a word, which is required to be trivial in G_1 or G_2.

A vertex is called *exterior* if it is possible to join it to the *exterior* region by some arc without intersecting any arc of Γ, and *interior* otherwise. For simplicity we will indeed assume from this point that our Σ is either S^2 or D^2. It follows that reading the label round any *interior* region spells a word which is trivial in G_1 or G_2. The *boundary label* of Γ on D^2 is a word obtained by reading the *labels* on ∂D^2 in an *anticlockwise* direction. This word (which may be assumed to cyclically reduced in $G_1 * G_2$) represents the identity element in G. In the case where Γ is spherical, we may assume (by capping off $\partial\Sigma$ if necessary) that the underlining surface is $\Sigma = S^2$. We then define the *boundary label* of Γ to be the label of the *exterior* region, which may be non-trivial in G_1 or G_2. Note that this is uniquely defined since Γ is connected. For non-connected pictures the exterior region may in general have more than one boundary component.

Two distinct vertices of a picture are said to *cancel* along an arc e if they are joined by e and if their labels, read from the endpoints of e, are mutually inverse words in $G_1 * G_2$. Such vertices can be removed from a picture via a sequence of *bridge moves* (see Figure 1 and [4] for more details), followed by deletion of a *dipole* without changing the boundary label. A *dipole* is a connected spherical sub-picture that contains precisely two vertices, does not meet $\partial\Sigma$, and such that none of its interior regions contain other components of Γ. This gives an alternative picture with the same boundary label and two fewer vertices.

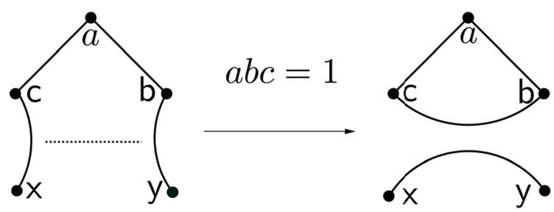

Figure 1: *Diagram showing bridge-move.*

We say that a picture Γ is *reduced* if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. Any cyclically reduced word in $G_1 * G_2$ representing the identity element of G occurs as the boundary label of some reduced picture on D^2.

Definition 2. Two arcs of Γ are said to be *parallel* if they are the only two arcs in the boundary of some simply-connected region $\triangle$ of Γ.

We will also use the term *parallel* to denote the equivalence relation generated by this relation, and refer to any of the corresponding equivalence classes as a *class of ω parallel arcs* or *ω-zone*. Given a *ω-zone* with $\omega > 1$ joining vertices u and v of Γ, consider the $\omega - 1$ two-sided regions separating these arcs. Each such region has a corner label x_u at u and a corner label x_v at v, and the picture axioms imply that $x_u x_v = 1$ in G_1 or G_2. The $\omega - 1$ corner labels at v spell a cyclic subword s of length $\omega - 1$ of the label of v. Similarly the corner labels at u spell out a cyclic subword t of length $\omega - 1$ of the label of u. Moreover, $s = t^{-1}$. If we assume that Γ is reduced, then u and v do not cancel. In the spirit of small-cancellation-theory, we refer to t and s as *pieces*.

As in graphs, the *degree* of a vertex in Γ is the number of *zones* incident on it. For a region, the *degree* is the number corners it has. For some positive integer p, we say that a vertex v of Γ satisfies the (local) $C(p)$ condition if it is joined to at least p *zones*. We say that Γ satisfies $C(p)$ if every interior vertex satisfies $C(p)$.

3. Technical results

Let G be the quotient of $G_1 * G_2$ by $N(R^m)$ for some natural number $m \geq 3$ and cyclically reduced word $R \in G_1 * G_2$ of length at least two. The aim of this section is to give a number of results on the structure of R when $D_2(\mathbf{S}_R) \leq 2$, from which Theorem 1 follows. It is assumed that no element of $\mathbf{S}_R$ has the form UV, where U and V are both uniquely positioned. In particular if $D(a) \geq 2$, there exists at most one $i \in \{1, 2, \cdots, D(a)\}$ such that M_i is uniquely positioned in the decomposition $R = \prod_{i=1}^{D(a)} aM_i$.

We now proceed to state and prove a number of lemmas which will be used to prove Theorem 1.

Lemma 3. *Let $G = (G_1 * G_2)/N(R^m), m \geq 3$ with $D_2(\boldsymbol{S}_R) = 1$. Then R has a cyclic conjugate of the form aM or $aXbX^{-1}$, where a, b are letters of order 2 and M does not involve any letter of order 2.*

Proof. Since $D_2(\mathbf{S}_R) = 1$, we can assume without loss of generality that $R = aM$, where $a \in G_1 \cup G_2$ is of order 2 and M is a word in $G_1 * G_2$ which does not involve a. We now proceed to show that either M does not involve any letter of order 2 or M can be decomposed in the form XbX^{-1}, where $b \in G_1 \cup G_2$ is a letter of order 2 and X is a (possibly empty) word in $G_1 * G_2$.

Suppose by contradiction that M has a decomposition of the form XbY with $b^2 = 1$ and $X \neq Y^{-1}$. Without loss of generality we can assume that $0 \leq \ell(X) \leq \ell(Y)$. If $\ell(X) = \ell(Y) > 0$, then both aX and bY are uniquely positioned which is a contradiction. There is nothing to prove if $\ell(X) = \ell(Y) = 0$. Also if $\ell(X) = 0 \neq \ell(Y)$, we get a contradiction since ab and Y will be uniquely positioned. Hence the inequality $0 < \ell(X) < \ell(Y)$ holds.

Suppose that $X^2 = 1 = Y^2$. Then by setting $X = X_1 p X_1^{-1}$ and $Y = Y_1^{-1} q Y_1$, where X_1, Y_1 are (possibly empty) words in $G_1 * G_2$ and p, q are distinct letters of order 2 in $G_1 \cup G_2$, we can replace R with

$$R' = pX'qY',$$

where $X' = (Y_1 b X_1)^{-1}$ and $Y' = Y_1 a X_1$. Since $a \neq b$, we have that $X' \neq Y'^{-1}$. Given that $\ell(X') = \ell(Y')$, we easily conclude that pX' and qY' are uniquely positioned. This is a contradiction.

Suppose that $X^2 = 1 \neq Y^2$. By the assumption that $D_2(\mathbf{S}_R) = 1$, we know that X can not be equal to a segment of Y. Hence aX and bY are both uniquely positioned.

This is a contradiction. Similarly, suppose that $X^2 \neq 1 = Y^2$. Since $\ell(X) < \ell(Y)$ and $D_2(\mathbf{S}_R) = 1$, we have that both bY and Ya are uniquely positioned. Hence, neither aX nor Xb is uniquely positioned. This means that X^{-1} is identically equal to an initial and a terminal segment of Y. Therefore, $X^2 = 1$. This is a contradiction.

Finally if $X^2 \neq 1 \neq Y^2$, then aXb and Y are both uniquely positioned. This contradiction completes the proof.

Lemma 4. *Let $G = (G_1 * G_2)/N(R^m), m \geq 3$ with $D_2(\boldsymbol{S}_R) = 2$. Then R has a cyclic conjugate of the form $aXbX^{-1}$ where a is a letter of order 2.*

Proof. Since $D_2(\mathbf{S}_R) = 2$, we can assume without loss of generality that

$$R = aM_1aM_2,$$

where $M_1, M_2 \in G_1 * G_2$, and neither involves the letter a. By assumption M_1 and M_2 can not both be uniquely positioned. Let us assume that M_1 is not uniquely positioned. If $M_1^2 = 1$ and $M_2^2 = 1$ hold simultaneously, then by replacing R with a cyclic conjugate, it can be shown that R has the desired form.

Suppose that $\ell(M_1) = \ell(M_2)$. We can not have $M_1 = M_2$ since R is not a proper power. Also if $M_1 = M_2^{-1}$, then there is nothing to prove. Since M_1 is not uniquely positioned, we must have that $M_1^2 = 1$. Similarly, if M_2 is not uniquely positioned, then $M_2^2 = 1$, contradicting the above assumption. Hence we may assume M_2 is uniquely positioned. If $\ell(M_1) = 1$, then there is nothing to prove since M_1 has order 2 and so R has the desired form. Hence we assume that $\ell(M_1) = \ell(M_2) \geq 3$. Let $M_1 = XpX^{-1}$ and $M_2 = YqZ$, with $p, q \in G_1 \cup G_2$, $p^2 = 1$, $\ell(Y) = \ell(Z)$ and $Y \neq Z^{-1}$ (as otherwise there is nothing to prove). Then

$$R = aXpX^{-1}aYqZ.$$

Set $U = aYq, U' = qZa, V = ZaXpX^{-1}$ and $V' = XpX^{-1}aY$. Clearly, $V^2 \neq 1 \neq V'^2$ since $D(a) = 2$. Also since $Y \neq Z^{-1}$, it follows that V and V' are both uniquely positioned. Hence neither U nor U' is uniquely positioned. It is easy to see that this means that $U^2 = 1$ or $U'^2 = 1$ or $U' = U^{\pm 1}$. However, any such occurrence will imply that $a = q$ or $Y = Z^{-1}$. This is a contradiction.

Now suppose that $\ell(M_i) \neq \ell(M_j)$, where $i, j \in \{1, 2\}$ with $i \neq j$. Note that it is not possible to have that $M_i^2 \neq 1 \neq M_j^2$ as that will imply that aM_ia and M_j are both uniquely positioned, assuming $\ell(M_i) < \ell(M_j)$. Suppose that $M_i^2 = 1$. Let $M_i = XpX^{-1}$ and $M_j = YqZ$, with $p, q \in G_1 \cup G_2$, $p^2 = 1$, $\ell(Y) = \ell(Z)$ and

$Y \neq Z^{-1}$. We claim that exactly one of aY or Za is uniquely positioned. This is because if both are uniquely positioned, then there is nothing to prove. Also if neither is uniquely positioned, then $Y = Z^{-1}$. In both cases we get a contradiction. By symmetry we assume that aY is uniquely positioned, and hence $qZaM_i$ is not. This leads to a contradiction when $\ell(Y) \geq \ell(M_i)$ since that will mean $Y = Z^{-1}$. Suppose then that $\ell(Y) < \ell(M_i)$. This implies that either M_i is an initial segment of M_j or M_i^{-1} is a terminal segment of M_j. As $M_i^2 = 1$, it follows that M_i is either an initial or terminal segment of M_j. Hence, these exists some $W \in G_1 * G_2$, satisfying $\ell(W) = 2n$ for some integer $n > 0$, such that either $M_j = M_iW$ or $M_j = WM_i$. Next, replace R by

$$R' = pMpN,$$

where $M = X^{-1}aX$ and $N = X^{-1}WaX$ or $N = X^{-1}aWX$. We consider first the case when $N = X^{-1}WaX$. In this case, the initial segment $X^{-1}W$ of N has length $\ell(X^{-1}W) \geq \ell(X) + 2$. Since $D_2(\mathbf{S}_R) = 2$, $X^{-1}W$ neither involves a nor p. It follows that $aXpX^{-1}aXp$ is uniquely positioned. Hence, $X^{-1}W$ is not uniquely positioned. The length condition on $X^{-1}W$ implies that $(X^{-1}W)^2 = 1$. Again since $D_2(\mathbf{S}_R) = 2$, X does not involve a letter of order 2. So $W = SxS^{-1}X$, for some (possibly empty) word S and some letter x of order 2. Hence

$$R' = pX^{-1}aXpX^{-1}SxS^{-1}XaX.$$

Consider the cyclic subwords $W_1 = S^{-1}XaXpX^{-1}aX$ and $W_2 = pX^{-1}Sx$. Clearly, $W_1^2 \neq 1$ as otherwise S is empty and more importantly $X^2 = 1$, which is a contradiction. Also, $W_2^2 \neq 1$ since $p \neq x$. In fact, it is easy to see that both W_1 and W_2 are uniquely positioned. This is a contradiction. Similar argument works when $N = X^{-1}aWX$ by replacing W_1 and W_2 with their inverses. This completes the proof.

The following lemma gives a necessary and sufficient condition under which the word R has a decomposition into a pair of uniquely positioned subwords when $D_2(\mathbf{S}_R) = 1$.

Lemma 5. *Let r be a cyclically reduced word which is not a proper power in the free product $G_1 * G_2$ such that $D_2(\boldsymbol{S}_r) = 1$. Then, r has a decomposition into two uniquely positioned subwords if and only if $\ell(r) > 2$ and there exists $r' \in \boldsymbol{S}_r$ such that $r' = aXxYyX^{-1}$ with $X, Y, x, y, a \in G_1 * G_2$, $\ell(Y) \geq 1$, $\ell(x) = \ell(y) = \ell(a) = 1$, $x \neq y^{-1}$ and $a^2 = 1$.*

Proof. Suppose that r has a decomposition into two uniquely positioned subwords U and V. Since $D(\mathbf{S}_r) = 1$, we have that $\ell(r) > 2$. Without loss of generality, it follows that a cyclic conjugate of r has the form

$$r' = aU_2VU_1,$$

where $U = U_1aU_2$ and $a^2 = 1$. Hence $U_2VU_1 = XYX^{-1}$ for some words $X, Y \in G_1 * G_2$, where X is possibly empty. Since U and V are uniquely positioned in r, we conclude that $\ell(Y) \geq 3$ and the first and last letters of Y are not inverses. The result follows.

For the other direction, suppose that $r' = aXxYyX^{-1}$ with $X, Y, x, y, a \in G_1 * G_2$, and satisfying $\ell(x) = \ell(y) = \ell(a) = 1$, $x \neq y^{-1}$ and $a^2 = 1$. Then aXx is clearly uniquely positioned in r since $x \neq y^{-1}$. For the same reason, we deduce from part(a) of Theorem 1 that $XxYyX^{-1}$ has no element of order two. In particular, this means that YyX^{-1} and its inverse do not intersect (in an initial or terminal segment). We claim that this means that YyX^{-1} is also uniquely positioned. We prove this by contradiction by assuming that YyX^{-1} is not uniquely positioned and showing that $XxYyX^{-1}$ contains an element of order two.

Let $XxYyX^{-1} = x_1x_2\cdots x_n$, with $X = x_1x_2\cdots x_p$. Suppose that YyX^{-1} is not uniquely positioned. Then, $(YyX^{-1})^{\pm 1}$ is identically equal to some segment of $XxYyX^{-1}$. This segment must intersect YyX^{-1}. By the above discussion, we have that YyX^{-1} is identically equal to the segment

$$x_kx_{k+1}\cdots x_{\ell(YyX^{-1})-1},$$

with $k \leq p$. Hence, we have that the terminal segment of $XxYyX^{-1}$ of length $n+1-k$ has period $\lambda = p+2-k$. Consider the initial segment of this periodic segement given by

$$W_k = x_kx_{k+1}\cdots x_{n+k-(p+2)}.$$

In particular W_k is of length $n-(p+1)$. Note that $X^{-1} = x_p^{-1}x_{p-1}^{-1}\cdots x_1^{-1} = x_{n+1-p}x_{n+2-p}\cdots x_n$. If $x_i = x_i^{-1}$ for some $k \leq i \leq p$, then we are done. Suppose not. If x_p (alternatively x_k) is identified with x_i^{-1} for some $k \leq i \leq p$, then $x_{\frac{p+i}{2}} = x^{-1}_{\frac{p+i}{2}}$ (alternatively $x_{\frac{k+i}{2}} = x^{-1}_{\frac{k+i}{2}}$). This is a contradiction. Otherwise, both x_k and x_p are identified with x_i^{-1} and x_j^{-1} respectively, where $1 \leq j \leq i < k-1$ (since we are in a free product). In fact, $j = i+k-p < 2k-1-p$. Choose j such that under this periodicity, x_j^{-1} is the letter that provides the first identification with x_p. We claim that $j+\lambda$ lies between k and p. To verify this claim, it is enough to show that

$p \geq j + \lambda$. We have that $j + \lambda < 2k - 1 - p + \lambda = k + 1$. Therefore, $j + \lambda \leq k \leq p$. Hence $x_p = x_{j+\lambda}^{-1}$ and $j + \lambda \leq p$. By the choice of j, we must have that $k \leq j + \lambda \leq p$. This is a contradiction. Hence YyX^{-1} is uniquely positioned. This completes the proof.

By combining Lemmas [3–5], we obtain Theorem 1 as follows.

Proof of Theorem 1. By Lemmas 3 and 4, we can assume that R has the form aM, where M is some word and a is the unique letter of order two involved in R. Express M in the form XbY, for some (possibly empty) words X, Y of equal lengths, and letter b. If $X = Y^{-1}$, then R is exceptional, so we are done. On the other hand if $X \neq Y^{-1}$, then by Lemma 5, R has a decomposition into two uniquely positioned subwords. This contradiction completes the proof.

Lemma 6. *Let Γ be a reduced picture over $G = (G_1 * G_2)/N(R^m), m \geq 3$ on D^2 such that $R = aXbX^{-1}$ for some letters a, b. If X involves neither a nor b, then Γ is empty or it satisfies $C(6)$.*

Proof. Suppose that Γ is a non-empty picture over G on D^2 which is reduced. Suppose also that Γ contains some interior vertex v of degree less than six. Then v is connected to another vertex u by a zone containing $(aX)^{\pm 1}$ or $(bX^{-1})^{\pm 1}$. Using this zone, we can do bridge moves so that u and v form a dipole. This contradicts the assumption that Γ was reduced.

As a corollary we obtain Theorem 2.

Corollary 7. *(Theorem 2) Let R be a cyclically reduced word in the free product $G_1 * G_2$ such that $D_2(\boldsymbol{S}_R) \leq 2$. Suppose that R is non-exceptional. Then a non-trivial reduced picture on D^2 over $G = (G_1 * G_2)/N(R^m), m \geq 3$ satisfies $C(6)$.*

Proof. If R has a decomposition into two uniqely positioned subwords, then the result follows from [[4] Lemma 3.1]. Otherwise the result follows from Theorem 1 and Lemma 6.

4. Applications

In this section we deduce a number of applications of our results. But first, we recall the setting.

Let G_1 and G_2 be non-trivial groups and R is a cyclically reduced word in $G_1 * G_2$ which is not a proper power and has length at least 2. In addition, we also require that no letter of order two involved in R appears more than twice i.e $D_2(\mathbf{S}_R) \leq 2$. For a natural number $m \geq 3$, the object of study is the group G, which is the quotient of $G_1 * G_2$ by the normal closure of R^m. Using Theorem 1, R can be classified as exceptional and non-exceptional as described in Section section 1. We mention applications of our results in each of the two cases beginning with the non-exceptional case.

Theorem 8. *Suppose that G is as above and R is non-exceptional. Then the following hold.*

(a) ***Freiheitssatz.*** *The natural homomorphisms $G_1 \to G$ and $G_2 \to G$ are injective.*

(b) ***Weinbaum's Theorem.*** *No non-empty proper subword of R^m represents the identity element of G.*

(c) ***Word problem.*** *If G_1 and G_2 are given by a recursive presentation with soluble word problem, then so is G. Moreover, the generalized word problem for G_1 and G_2 in G is soluble with respect to these presentations.*

(d) ***The Identity Theorem.*** *$N(R^m)/[N(R^m), N(R^m)] = \mathbb{Z}G/(1-R)\mathbb{Z}G$ as a (right) $\mathbb{Z}G$-module, where $\mathbb{Z}$ is the integers.*

Corollary 9. *There are natural isomorphisms for all $k > 3$;*

$$H^k(G;-) \longrightarrow H^k(G_1;-) \times H^k(G_2;-) \times H^k(\mathbb{Z}_m;-),$$
$$H_k(G;-) \longleftarrow H_k(G_1;-) \oplus H_k(G_2;-) \oplus H_k(\mathbb{Z}_m;-);$$

a natural epimorphism

$$H^2(G;-) \longrightarrow H^2(G_1;-) \times H^2(G_2;-) \times H^2(\mathbb{Z}_m;-),$$

and a natural monomorphism

$$H_2(G;-) \longleftarrow H_2(G_1;-) \oplus H_2(G_2;-) \oplus H_2(\mathbb{Z}_m;-).$$

These are defined on the category of $\mathbb{Z}G$-modules, $\mathbb{Z}_m$ is the cyclic subgroup of order m generated by R, and all these maps are induced by restriction on each factor.

Next we consider the exceptional case. Recall that R has the form $aXbX^{-1}$, for some word X and some letters a, b satisfying $a^2 = 1 \neq b^2$. Let $A := \langle a \rangle$ and $X^{-1}BX := \langle b \rangle$ be the cyclic subgroups of G_1 or G_2 generated by a and b respectively.

$$
\begin{tikzcd}
A * B \arrow[r] \arrow[d] & H \arrow[d] \\
G_1 * G_2 \arrow[r] & G
\end{tikzcd}
$$

Figure 2: *Push-out diagram.*

Let H be the quotient of $(A*B)$ by $N(R^m)$. Note that G can be realized as a push-out of groups as shown in Figure 2.

This pushout representation of G is referred to as a generalized triangle group description of G. In order for the results in [9] to hold in our case, we require it to be *maximal* in the sense of [1]. Another technical requirement is that (a, b) be *admissible*: whenever both a and b belong to same factor, say G_1, then either the subgroup of G_1 generated by $\{a, b\}$ is cyclic or $\langle a\rangle \cap \langle b\rangle = 1$. It is very easy to verify that these conditions are satisfied in our setting. Hence the results in [9] hold, and so we state them without proof.

Theorem 10. *Suppose that G is as above and R is exceptional. Then the following hold.*

(a) ***Freiheitssatz.*** *The natural homomorphisms $G_1 \to G$, $G_2 \to G$ and $H \to G$ are all injective.*

(b) ***Weinbaum's Theorem.*** *No non-empty proper subword of R^m represents the identity element of G.*

(c) ***Membership problem.*** *Assume that the membership problems for $\langle a\rangle$ and $\langle b\rangle$ in $G_1 * G_2$ are solvable. Then the word problem for G is also soluble.*

(d) ***Mayer-Vietoris.*** *The pushout of groups in Figure 2 is* geometrically Mayer-Vietoris *in the sense of [9]. In particular it gives rise to Mayer-Vietoris sequences*

$$\cdots \to H_{k+1}(G, M) \to H_k(A * B, M) \to$$
$$H_k(G_1 * G_2, M) \oplus H_k(H, M) \to H_k(G, M) \to \cdots$$

and

$$\cdots \to H^k(G, M) \to H^k(G_1 * G_2, M) \oplus H^k(H, M)$$
$$\to H^k(A * B, M) \to H^{k+1}(G, M) \to \cdots$$

for any $\mathbb{Z}G$-module M.

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[1] Ihechukwu Chinyere and James Howie. On one-relator products induced by generalized triangle groups i. *Communications in Algebra*, 46(3):1138–1154, 2018.

[2] Ihechukwu Chinyere and James Howie. On one-relator products induced by generalized triangle groups ii. *Communications in Algebra*, 46(4):1464–1475, 2018.

[3] AJ Duncan and James Howie. Spelling theorems and cohen-lyndon theorems for one-relator products. *Journal of Pure and Applied Algebra*, 92(2):123–136, 1994.

[4] Andrew J Duncan and James Howie. Weinbaum's conjecture on unique subwords of nonperiodic words. *Proceedings of the American Mathematical Society*, 115(4):947–954, 1992.

[5] Benjamin Fine, James Howie, and Gerhard Rosenberger. One-relator quotients and free products of cyclics. *Proceedings of the American Mathematical Society*, 102(2):249–254, 1988.

[6] James Howie. The quotient of a free product of groups by a single high-powered relator. i. pictures. fifth and higher powers. *Proceedings of the London Mathematical Society*, 3(3):507–540, 1989.

[7] James Howie. The quotient of a free product of groups by a single high-powered relator. ii. fourth powers. *Proceedings of the London Mathematical Society*, 3(1):33–62, 1990.

[8] James Howie. The quotient of a free product of groups by a single high-powered relator. iii. the word problem. *Proceedings of the London Mathematical Society*, 3(3):590–606, 1991.

[9] James Howie and Robert Shwartz. One-relator products induced from generalized triangle groups. 2004.

[10] Roger C Lyndon and Paul E Schupp. *Combinatorial group theory*. Springer, 2015.

[11] Stephen J Pride. Small cancellation conditions satisfied by one-relator groups. *Mathematische Zeitschrift*, 184(2):283–286, 1983.

[12] CM Weinbaum. Unique subwords in nonperiodic words. *Proceedings of the American Mathematical Society*, 109(3):615–619, 1990.

[13] Gerald Williams. Euler characteristics for one-relator products of groups. *Bulletin of the London Mathematical Society*, 39(4):641–652, 2007.