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Existence of a sequence satisfying Cioranescu-Murat conditions in homogenization of Dirichlet problems in perforated domains

J. CASADO-DÍAZ

RIASSUNTO: *In un lavoro del 1982, D. Cioranescu e F. Murat hanno considerato il problema soddisfatto dal limite u di una successione u_n di soluzioni di*

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n, \\ u_n = 0 & \text{su } \partial\Omega_n, \end{cases}$$

dove Ω_n è una successione di insiemi aperti che sono contenuti in un fissato insieme aperto limitato Ω. Tale studio richiede di imporre numerose ipotesi sulla successione Ω_n. I risultati di D. Cioranescu e F. Murat sono stati estesi in seguito da N. Labani e C. Picard al caso del p-Laplaciano. Nel presente lavoro, noi dimostriamo che le ipotesi su Ω_n possono essere ridotte a un'unica ipotesi, la seguente: esiste una successione $z_n \in W^{1,p}(\Omega)$ che vale zero su $\Omega \setminus \Omega_n$ e che converge a 1 debolmente in $W^{1,p}(\Omega)$.

Il problema di omogeneizzazione nel caso generale in cui non si fa alcuna ipotesi sulla successione Ω_n è stato risolto da G. Dal Maso e U. Mosco con metodi di Γ-convergenza e recentemente G. Dal Maso e A. Garroni hanno risolto il problema generale con metodi prossimi a quelli usati da D. Cioranescu e F. Murat.

ABSTRACT: *In a paper of 1982, D. Cioranescu and F. Murat considered the problem satisfied by the limit u of the sequence u_n solution of*

$$\begin{cases} -\Delta u_n = f & \text{in } \Omega_n, \\ u_n = 0 & \text{on } \partial\Omega_n, \end{cases}$$

where Ω_n is a sequence of open sets which are contained in a fixed bounded open set Ω. In order to make this, they imposed several hypotheses about the sequence Ω_n. Their results were later extended to the p-Laplacian operator by N. Labani and C. Picard. In *the present paper, we prove that these hypotheses may be reduced to the following one: There exists a sequence $z_n \in W^{1,p}(\Omega)$ which is zero in $\Omega \setminus \Omega_n$ and which converges weakly to 1 in $W^{1,p}(\Omega)$.*

Indeed, G. Dal Maso and U. Mosco have solved the above homogenization problem in the general case in which we do not make any hypothesis about Ω_n using Γ-convergence methods and recently, G. Dal Maso and A. Garroni have also solved this general problem by a method close to the one used by D. Cioranescu and F. Murat.

– Introduction

The contents of this paper is concerned with the study of the homogenization problem

$$\begin{cases} -\Delta_p u_n = f & \text{in } \mathcal{D}'(\Omega_n), \\ u_n \in W_0^{1,p}(\Omega_n), \end{cases} \tag{0.1}$$

where Ω_n denotes a sequence of open sets contained in a fixed bounded open set $\Omega \subset \mathbb{R}^N$, p is a given number with $1 < p < +\infty$, f is an element of $W^{-1,p'}(\Omega)$ and Δ_p is the p-lapacian operator defined by

$$-\Delta_p u = -\operatorname{div} |\nabla u|^{p-2} \nabla u.$$

The solutions u_n of (0.1) are bounded in $W_0^{1,p}(\Omega)$ (we identify u_n with its extension by zero to $\Omega \setminus \Omega_n$) and so, there exists a subsequence which converges weakly to a function u in $W_0^{1,p}(\Omega)$. The homogenization problem is to find the equation satisfied by the function u and also, to give an approximate representation of the gradient of u_n in the strong topology of $L^p(\Omega)$ using the function u and some explicit auxiliary functions (corrector problem).

In the case $p = 2$, this homogenization problem has been solved by D. Cioranescu and F. Murat in [6] (see also [15]) assuming the following hypotheses about the sequence Ω_n:

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There exists a sequence of functions w_n and a distribution μ satisfying

(H1) $$w_n \in H^1(\Omega),$$

(H2) $$w_n = 0 \text{ in } \Omega \setminus \Omega_n,$$

(H3) $$w_n \rightharpoonup 1 \text{ in } H^1(\Omega),$$

(H4) $$\mu \in W^{-1,\infty}(\Omega),$$

(H5) $$\begin{cases} \text{for any sequence } v_n \text{ and any } v \text{ satisfying} \\ v_n \rightharpoonup v \text{ in } H^1(\Omega), \quad v_n = 0 \text{ in } \Omega \setminus \Omega_n, \\ \text{and for any } \varphi \in \mathcal{D}(\Omega), \text{ we have} \\ \displaystyle\int_\Omega \nabla w_n \nabla (\varphi v_n) \to \langle \mu, \varphi v \rangle. \end{cases}$$

It was then proved in [15] that hypothesis (H4) can be weakened in $\mu \in H^{-1}(\Omega)$. These hypotheses are justified in [6] by several examples, the most typical case being when Ω_n is obtained by removing from Ω the union of closed balls of radius $\varepsilon_n^{\frac{N}{N-2}}$ centered at the centers of cubes of size ε_n which cover $\mathbb{R}^N$ periodically. With these assumptions, D. Cioranescu and F. Murat prove that the limit u of the sequence u_n satisfies

(0.2) $$\begin{cases} -\Delta u + \mu u = f \quad \text{in } \mathcal{D}'(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$

Moreover, they prove that when u belongs to $W^{1,\infty}(\Omega)$, $u_n - w_n u$ converges strongly to zero in $H_0^1(\Omega)$. Their method has been generalized by N. Labani and C. Picard to the case of the p-Laplacian in [16].

The goal of the present paper is to prove the existence of a sequence w_n and of a distribution μ satisfying properties similar to (H1),..., (H5) for the p-Laplacian, starting from the only assumption that the w_n satisfy (H1), (H2), (H3) (with $H^1(\Omega)$ replaced by $W^{1,p}(\Omega)$, see Theorem 2.1). In this case, μ is no more in $W^{-1,\infty}(\Omega)$ but only in the set of bounded nonnegative measures vanishing on the sets of p-capacity zero.

We will also generalize the results obtained in [6] to this new context. In particular we obtain an improvement of the corrector result given in

[6], proving that it is enough to have $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in order to have the strong convergence of $u_n - w_n u$ in $W_0^{1,p}(\Omega)$ (see Theorem 4.1).

The method we use here can be extended to the case of general quasi-linear problems, but for our purpose it is enough to study the p-laplacian, because in the general case we will obtain better results reasoning by comparison. This will be carried out in [3] and [4], where we will use the results obtained in the present paper to solve on the other hand general monotone problems, and on the other one quasi-linear problems with a perturbation term, which is quadratic with respect to the gradient, respectively.

Hypothesis (H1), (H2), (H3) mean that $\Omega \setminus \Omega_n$ are small enough. In the limit, Ω_n fills the whole of Ω. Indeed, the general problem in which we do not assume any hypothesis about the sequence Ω_n has been solved by G. Dal Maso and U. Mosco ([9], [10]) in the linear case and by G. Dal Maso and A. Defranceschi [7] in the monotone case, using Γ-convergence methods. To use $\Gamma-$convergence, the problem has to be written as a minimization problem, which is not always possible for a general quasi-linear problem. Also there is no corrector result in these papers, while this is essential for us, in order to apply the comparison method which allows us to study more general equations. On the other hand, G. Dal Maso and A. Garroni [8] have recently used a different argument to study the linear case without any hypotheses about Ω_n. This argument, which is close to the one used in [6] (the main difference lies in the definition of the function w_n), does not need symmetry assumptions and gives a corrector result. The method used in [8] has been extended by G. Dal Maso and F. Murat [11], [12] to the case of monotone operators assuming a homogenity hypothesis for the operator. Using the corrector result which appears in [8] or [11], [12] and the comparison method we are able to solve in [5] the case of general monotone systems without homogeneity hypothesis, and without any hypothesis on the open sets.

1 – Preliminaries and notation

Throughout the present paper:

- Ω denotes a bounded open set contained in $\mathbb{R}^N$.
- $L^p(\Omega, d\mu)$, $1 \leq p < +\infty$, denotes the space of functions with power p integrable in Ω with respect to the measure μ.

- $L^\infty(\Omega, d\mu)$ denotes the space of functions essentialy bounded in Ω with respect to the measure μ.
- If the measure μ is the Lebesgue measure we abbreviate the notation using $L^p(\Omega)$ or $L^\infty(\Omega)$.
- For $1 \le p \le +\infty$ we denote p' the conjugate exponent of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$.
- $\mathcal{D}(\Omega)$ denotes the space of infinitely derivable functions with compact support contained in Ω. The dual space of $\mathcal{D}(\Omega)$ is the space of distributions which will be denoted by $\mathcal{D}'(\Omega)$.
- $W^{1,p}(\Omega)$ denotes the usual Sobolev space of functions of $L^p(\Omega)$ with distributional derivatives in $L^p(\Omega)$.
- $W_0^{1,p}(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. For $1 \le p < +\infty$, the dual space of $W_0^{1,p}(\Omega)$ will be denoted by $W^{-1,p'}(\Omega)$
- ∇ denotes the gradient operator.
- div denotes the divergence operator.
- Δ_p denotes the p-Laplacian operator, i.e. $\Delta_p\, u = \mathrm{div}\, |\nabla\, u|^{p-2}\nabla\, u$.
- χ_S denotes the characteristic function of the set S, i.e. $\chi_S(x) = 1$ if $x \in S$, $\chi_S(x) = 0$ if $x \notin S$.
- $\mathcal{M}_b(\Omega)$ denotes the space of bounded Borel measures in Ω.
- cap(S) denotes the p-capacity of the set $S \subset \Omega$ with respect to Ω (p will be specified by the context), which is defined in the following way:

If S is a compact set, the capacity of S is defined by

$$\mathrm{cap}(S) = \inf\Big\{\int_\Omega |\nabla\, \varphi|^p : \ \varphi \in \mathcal{D}(\Omega),\ \varphi \ge \chi_S\Big\}.$$

If S is an open set, the capacity of S is defined by

$$\mathrm{cap}(S) = \sup\Big\{\mathrm{cap}(K) : \ K \subset S,\ K \text{ compact}\Big\}.$$

If S is an arbitrary set, the capacity of S is defined by

$$\mathrm{cap}(S) = \inf\Big\{\mathrm{cap}(G) : \ S \subset G \subset \Omega,\ G \text{ open}\Big\}.$$

It is well known (see e.g. [14], [13], [19]) that a function of $W^{1,p}(\Omega)$ has a representative which is defined quasi-everywhere, i.e. except on a set of zero p-capacity. In the whole of the present paper we will select this representative for any function $u \in W^{1,p}(\Omega)$.

- $\mathcal{M}_b^p(\Omega)$ denotes the set of nonnegative bounded Borel measures vanishing on the sets of zero capacity. By the above mentioned result, the functions of $W^{1,p}(\Omega)$ are μ-measurable for $\mu \in \mathcal{M}_b^p(\Omega)$. We have

$$W^{1,p}(\Omega) \cap L^\infty(\Omega) \hookrightarrow L^\infty(\Omega, d\mu) \hookrightarrow L^q(\Omega, d\mu) \quad \text{for any } 1 \leq q < +\infty, \tag{1.1}$$

where the last inclusion holds since μ belongs to $\mathcal{M}_b(\Omega)$.

- For $r \geq 0$, $T_r : \mathbb{R} \mapsto \mathbb{R}$ is the truncation function definded by

$$T_r(s) = \begin{cases} r & \text{if } s \geq r \\ s & \text{if } -r \leq s \leq r \\ -r & \text{if } s \leq -r, \end{cases}$$

while $R_r : \mathbb{R} \mapsto \mathbb{R}$ is the function defined by

$$R_r(s) = \begin{cases} 0 & \text{if } |s| \leq \dfrac{r}{2} \\ \dfrac{2}{r}|s| - 1 & \text{if } \dfrac{r}{2} \leq |s| \leq r \\ 1 & \text{if } |s| \geq r. \end{cases}$$

The following properties of the function $\xi \in \mathbb{R}^N \mapsto |\xi|^{p-2}\xi \in \mathbb{R}^N$ will be often used:
For any ξ, $\eta \in \mathbb{R}^N$, we have for $p \geq 2$

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq 2^{2-p}|\xi - \eta|^p, \tag{1.2}$$

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \leq (p-1)\big(|\xi|^{p-2} + |\eta|^{p-2}\big)|\xi - \eta| \tag{1.3}$$

and for $1 \leq p \leq 2$

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq (p-1)\frac{|\xi - \eta|^2}{|\xi|^{2-p} + |\eta|^{2-p}}, \tag{1.4}$$

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \leq 2^{2-p}|\xi - \eta|^{p-1}. \tag{1.5}$$

Inequality (1.4) will be used in the following form: given $u, v \in W^{1,p}(\Omega)$, $1 < p < 2$, then

$$
\begin{aligned}
&\int_\Omega |\nabla (u-v)|^p \le \\
&\le \frac{1}{(p-1)^{\frac{p}{2}}} \int_\Omega \left[(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v)\right]^{\frac{p}{2}} \cdot \\
&\quad \cdot \left[|\nabla u|^{2-p} + |\nabla v|^{2-p}\right]^{\frac{p}{2}} \le \\
&\le \frac{2^{p-1}}{(p-1)^{\frac{p}{2}}} \left[\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v)\right]^{\frac{p}{2}} \cdot \\
&\quad \cdot \left[\int_\Omega (|\nabla u|^p + |\nabla v|^p)\right]^{\frac{2-p}{2}} .
\end{aligned} \tag{1.6}
$$

2 – The main result and its proof

Let us consider a fixed bounded open set $\Omega \subset \mathbb{R}^N$ and a sequence of open sets Ω_n contained in Ω.

In the whole of the present paper, the functions of $W_0^{1,p}(\Omega_n)$ will be always extended by zero outside of Ω_n and therefore considered defined as in the whole of Ω.

Theorem 2.1 establishes the existence of a sequence satisfying properties analogous to those of the sequence w_n defined in [6].

THEOREM 2.1. *Assume that there exists a sequence $z_n \in W^{1,p}(\Omega)$, with $z_n = 0$ in $\Omega \setminus \Omega_n$, which converges weakly in $W^{1,p}(\Omega)$ $(1 < p < \infty)$ to a function z. Assume also that there exists a constant $\rho > 0$ with $z \ge \rho$ quasi-everywhere in Ω. Then, there exists a subsequence of n (which will still denoted by n to simplify the notation), a sequence of functions w_n and a measure μ satisfying*

(P1) $\quad w_n \in W^{1,p}(\Omega)$,

(P2) $\quad w_n = 0$ *in* $\Omega \setminus \Omega_n$,

(P3) $\quad 0 \le w_n \le 1$,

(P4) $\quad w_n \rightharpoonup 1$ *weakly in* $W^{1,p}(\Omega)$ *and strongly in* $W^{1,q}(\Omega)$, $1 \le q < p$, (P5) $\mu \in \mathcal{M}_b^p(\Omega)$,

(P6) $\begin{cases} \textit{for any } \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \textit{ we have} \\ \int_\Omega |\nabla w_n|^p \varphi \to \int_\Omega \varphi \, d\mu, \end{cases}$

(P7) $\begin{cases} \textit{for any } v_n \in W_0^{1,p}(\Omega_n) \textit{ and for any } v \in W_0^{1,p}(\Omega) \textit{ such that} \\ v_n \rightharpoonup v \textit{ in } W_0^{1,p}(\Omega), \textit{ we have} \\ v \in L^1(\Omega, d\mu) \textit{ and } \displaystyle\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n \to \int_\Omega v \, d\mu. \end{cases}$

(P8) $\begin{cases} \textit{for any } v_n \in W^{1,p}(\Omega), \textit{ such that } v_n = 0 \textit{ in } \Omega \setminus \Omega_n \textit{ and} \\ v_n \rightharpoonup 0 \textit{ in } W^{1,p}(\Omega), \textit{ we have} \\ \displaystyle\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n \to 0. \end{cases}$

Moreover, if the properties (P1), (P2), ..., (P8) *hold true for the same subsequence* n *and for some* $\hat{w}_n$ *and* $\hat{\mu}$, *then we have*

$$\begin{cases} \hat{\mu} = \mu \\ \hat{w}_n - w_n \to 0 \textit{ in } W^{1,p}(\Omega) \textit{ strongly.} \end{cases}$$

Remark 2.1. It will be proved below that in Property (P7), v actually belongs to $L^p(\Omega, d\mu)$, (see Theorem 3.1).

Remark 2.2. The sequence w_n will provide us a corrector for the homogenization problem

$$\begin{cases} -\Delta_p u_n = f & \text{in } \mathcal{D}'(\Omega_n), \\ u_n \in W_0^{1,p}(\Omega_n). \end{cases}$$

Remark that the behaviour of w_n is similar to of a sequence $\tilde{w}_n$ which satisfies (μ does not belong in general to $W^{-1,p'}(\Omega)$ and therefore such that $\tilde{w}_n$ does not exist in general)

$$\begin{cases} \tilde{w}_n \in W^{1,p}(\Omega_n), \quad \tilde{w}_n = 0 & \text{in } \Omega \setminus \Omega_n, \\ -\Delta_p \tilde{w}_n = \mu & \text{in } \mathcal{D}'(\Omega_n), \\ \tilde{w}_n \rightharpoonup 1 & \text{in } W^{1,p}(\Omega). \end{cases}$$

Compare this one with the homogenization problem

$$\begin{cases} -\operatorname{div}(A_n \nabla u_n) = f \quad \text{in } \mathcal{D}'(\Omega), \\ u_n \in H_0^1(\Omega) \end{cases}$$

where $A_n \in L^\infty(\Omega)^{N\times N}$ are such that there exist α, $\beta > 0$ with $\alpha I \leq A_n \leq \beta I$ in the sense of the matrices. The idea of L. TARTAR (see [18]) to construct a corrector for this problem is to consider for any i with $1 \leq i \leq N$ a sequence w_n^i such that

$$\begin{cases} w_n^i \in H^1(\Omega), \\ -\operatorname{div}(A_n \nabla w_n^i) = -\operatorname{div}(A_0 \nabla x_i) = -\operatorname{div}(A_0\, e_i) \quad \text{in } \mathcal{D}'(\Omega), \\ w_n^i \rightharpoonup x_i \quad \text{in } H^1(\Omega), \end{cases}$$

where A_0 will be the H-limit of A_n.

PROOF OF THEOREM 2.1. The proof of Theorem 2.1 will be divided in nine steps.

STEP 1: Definition of the subsequence n, of the sequence w_n and of μ; the subsequence w_n satisfies (P1), (P2) and (P3).

PROOF. Define

$$\mathcal{A} = \{\{v_n\} : \; v_n \in W^{1,p}(\Omega) : \; v_n = 0 \text{ in } \Omega \setminus \Omega_n, \; v_n \rightharpoonup 1 \text{ in } W^{1,p}(\Omega)\},$$
$$\alpha = \inf\left\{\liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p : \; \{v_n\} \in \mathcal{A}\right\}.$$

The set $\mathcal{A}$ is not empty since the sequence $v_n = \frac{T_\rho^+(z_n)}{\rho}$ belongs to $\mathcal{A}$. For any $k \in \mathbb{N}$, consider a sequence $\{v_n^k\} \in \mathcal{A}$ such that

$$\liminf_{n\to\infty} \int_\Omega |\nabla v_n^k|^p < \alpha + \frac{1}{k}.$$

Defining $\tilde{v}_n^k$ as $\tilde{v}_n^k = T_1^+(v_n^k)$, we have that $\{\tilde{v}_n^k\} \in \mathcal{A}$, $0 \leq \tilde{v}_n^k \leq 1$, and

$$\liminf_{n\to\infty} \int_\Omega |\nabla \tilde{v}_n^k|^p \leq \liminf_{n\to\infty} \int_\Omega |\nabla v_n^k|^p < \alpha + \frac{1}{k}.$$

Rellich-Kondrachov's compactness and Lebesgue's dominated convergence theorems imply that the embedding $W^{1,p}(\Omega) \cap L^\infty(\Omega) \hookrightarrow L^q(\Omega)$

($1 \leq q < +\infty$) is compact (even if no smoothness is assumed on $\partial\Omega$). Thus $\tilde{v}_n^k$ converges strongly to 1 in $L^p(\Omega)$ as n tends to infinity, for k fixed. It is then possible to define a subsequence n_k of n, which is increasing and tends to infinity, such that

$$\begin{cases} \displaystyle\int_\Omega |\nabla \tilde{v}_{n_k}^k|^p < \alpha + \frac{2}{k}, \\ \| \tilde{v}_{n_k}^k - 1 \|_{L^p(\Omega)} < \dfrac{1}{k}. \end{cases}$$

The subsequence n_k is the sequence which appears in Theorem 2.1. For the sake of simplicity, we will from now on denote it by n. We also define w_n as $w_n = w_{n_k} = \tilde{v}_{n_k}^k$.

The sequence w_n converges weakly to 1 in $W^{1,p}(\Omega)$ and so satisfies (P1), (P2) and (P3). Extracting if necessary a further subsequence we can also assume the existence of a bounded nonnegative Radon measure μ such that $|\nabla w_n|^p$ converges to μ in the weak-$*$ sense of $\mathcal{M}_b(\Omega)$.

Step 2: The sequence w_n satisfies (P8).

Proof. Define the functional $F : W^{1,p}(\Omega) \mapsto \mathbb{R}$ by

$$F(u) = \int_\Omega |\nabla u|^p.$$

This functional is Fréchet differentiable with continuous derivative

$$F'(u)v = p \int_\Omega |\nabla u|^{p-2} \nabla u \, \nabla v, \ \forall\, v \in W^{1,p}(\Omega).$$

Let $v_n \in W^{1,p}(\Omega)$ be a sequence which converges weakly to zero in $W^{1,p}(\Omega)$ and such that $v_n = 0$ in $\Omega \setminus \Omega_n$. For any $\lambda > 0$, by definition of α and of w_n, we have $\liminf_{n\to\infty} F(w_n + \lambda v_n) \geq \alpha$ while $\lim_{n\to\infty} F(w_n) = \alpha$. We deduce that

$$\liminf_{n\to\infty} \frac{F(w_n + \lambda v_n) - F(w_n)}{\lambda} \geq 0, \ \forall\, \lambda > 0. \tag{2.1}$$

Lagrange's theorem implies that there exists some $\theta_n^\lambda \in (0,1)$ such that

$$\frac{F(w_n + \lambda v_n) - F(w_n)}{\lambda} = F'(w_n + \theta_n^\lambda \lambda v_n) v_n.$$

Using the uniform continuity (1.3) or (1.5) of F' on the bounded sets, we have

$$F'(w_n + \theta_n^\lambda \lambda v_n)v_n = F'(w_n)v_n + r_n(\lambda),$$

where

$$|r_n(\lambda)| \leq \begin{cases} C(|\lambda|^{p-1} + |\lambda|) & \text{if } p \geq 2, \\ C|\lambda|^{p-1} & \text{if } 1 \leq p \leq 2 \end{cases}.$$

This implies that

$$\liminf_{n\to\infty} F'(w_n)v_n \geq 0.$$

Since the sequence $-v_n$ satisfies the same conditions as v_n, we also have

$$\limsup_{n\to\infty} F'(w_n)v_n = -\liminf_{n\to\infty} F'(w_n)(-v_n) \leq 0.$$

Therefore $F'(w_n)v_n$ tends to 0, which is (P8).

A more general result is the following:

STEP 3: Let $v_n^k \in W^{1,p}(\Omega)$, such that $v_n^k = 0$ in $\Omega \setminus \Omega_n$ and v_n^k converges weakly to zero in $W^{1,p}(\Omega)$ when n and k tend to infinity, i.e.

$$\lim_{n,k\to\infty} \int_\Omega v_n^k \varphi = \lim_{n,k\to\infty} \int_\Omega \nabla v_n^k \nabla \varphi = 0, \quad \forall \varphi \in W^{1,p'}(\Omega).$$

Then

$$\lim_{n,k\to\infty} \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n^k = 0. \tag{2.2}$$

PROOF. This follows from Step 2 and from the following easy result:

LEMMA 2.1. *For any sequence $h_{n,k} \in \mathbb{R}$, with two indices, and any $l \in \mathbb{R}$ the following assertions are equivalent:*

i) *The double limit $\lim_{n,k\to\infty} h_{n,k}$ exists and* $\lim\limits_{n,k\to\infty} h_{n,k} = l$.

ii) *For any sequence $k_n \in \mathbb{N}$ which converges to infinity, we have* $\lim\limits_{n\to\infty} h_{n,k_n} = l$.

STEP 4: The sequence w_n satisfies (P4).

PROOF. We already know that w_n converges to 1 weakly in $W^{1,p}(\Omega)$ and, using Rellich-Kondrachov's compactness theorem, strongly in $L^q(\Omega)$, for any q with $1 \leq q < p$ (even if no smoothness is assumed on $\partial\Omega$). By Egorov's theorem, w_n converges almost uniformly (at least for a subsequence). Hence, for any $\delta > 0$ there exists a set $A_\delta \subset \Omega$ such that the Lebesgue measure of $\Omega \setminus A_\delta$ is less than δ and such that w_n converges uniformly to 1 in A_δ. For $k \in \mathbb{N}$, define $v_n^k = (\frac{1}{k} + T_{\frac{1}{k}}(w_n - 1))$, which belongs to $W_0^{1,p}(\Omega)$, is zero in $\Omega \setminus \Omega_n$ and converges weakly to zero in $W^{1,p}(\Omega)$ when n, k tends to infinity (use Lemma 2.1). By Step 3, we have

$$\Big| \int_{|w_n - 1| < \frac{1}{k}} |\nabla w_n|^p \Big| = \Big| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n^k \Big| \leq \eta,$$

for every $n \geq n_0(\eta)$, $k \geq k_0(\eta)$. Thus

$$\lim_{k\to\infty} \limsup_{n\to\infty} \int_{|w_n - 1| < \frac{1}{k}} |\nabla w_n|^p = 0.$$

But, for n large enough, $A_\delta \subset \{x : \ |w_n(x) - 1| < \frac{1}{k}\}$, and hence we obtain

$$\int_{A_\delta} |\nabla w_n|^p \to 0, \quad \forall\, \delta > 0.$$

Therefore, ∇w_n converges pointwise to zero (at least for a subsequence) and because ∇w_n converges weakly in $L^p(\Omega)^N$, we obtain the strong convergence in $L^q(\Omega)^N$, for any q, with $1 \leq q < p$, hence (P4).

STEP 5: Property (P5) is satisfied.

PROOF. We will prove that for any Borel set $A \subset \Omega$ of zero capacity, we have $\mu(A) = 0$. By standard properties of Radon measures, it is enough to see that $\mu(K) = 0$ for every compact set K of zero capacity.

When $K \subset \Omega$ is a compact set with zero capacity, for any $k \in \mathbb{N}$, there exists $\varphi_k \in \mathcal{D}(\Omega)$ such that

$$\varphi_k \geq \chi_K, \ 0 \leq \varphi_k \leq 1, \ \| \varphi_k \|_{W_0^{1,p}(\Omega)} \leq \frac{1}{k}.$$

From Step 3 applied to the sequence $v_n^k = w_n \varphi_k$, we obtain that for any $\eta > 0$

$$\Big| \int_\Omega |\nabla w_n|^p \varphi_k + \int_\Omega w_n |\nabla w_n|^{p-2} \nabla w_n \nabla \varphi_k \Big| \le \eta, \quad \forall\, n \ge n_0(\eta),\ k \ge k_0(\eta).$$

Since

$$\Big| \int_\Omega w_n |\nabla w_n|^{p-2} \nabla w_n \nabla \varphi_k \Big| \le \| \nabla w_n \|_{L^p(\Omega)}^{p-1} \| \nabla \varphi_k \|_{L^p(\Omega)} \le \frac{C}{k},$$

we have

$$\int_\Omega |\nabla w_n|^p \varphi_k \le 2\eta, \quad \forall\, n \ge n_0(\eta),\ k \ge k_1(\eta),$$

which by the weak-$*$ convergence of $|\nabla w_n|^p$ to μ in the sense of measures implies

$$\int_\Omega \varphi_k \, d\mu \le 2\eta, \quad \forall k \ge k_1(\eta).$$

Since $\varphi_k \ge \chi_K$ this yields

$$\mu(K) \le 2\eta, \quad \forall \eta > 0.$$

Step 6: Property (P6) is satisfied.

Proof. Let $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. It is well known (see e.g. [13], [19]) that there exists a sequence of functions φ_k satisfying

$$\varphi_k \in \mathcal{D}(\Omega), \varphi_k \text{ is bounded in } L^\infty(\Omega), \varphi_k \to \varphi \ \text{ in } W_0^{1,p}(\Omega) \text{ and } \mu\text{-a.e. in } \Omega.$$

Hence, from Lebesgue's dominated convergence theorem we have $\varphi \in L^\infty(\Omega, d\mu)$ (which is already known by (1.1)) and φ_k converges strongly to φ in $L^1(\Omega, d\mu)$. Thus

$$\begin{aligned} \Big| \int_\Omega |\nabla w_n|^p \varphi - \int_\Omega \varphi \, d\mu \Big| \le & \int_\Omega |\nabla w_n|^p |\varphi - \varphi_k| + \\ & + \Big| \int_\Omega |\nabla w_n|^p \varphi_k - \int_\Omega \varphi_k \, d\mu \Big| + \int_\Omega |\varphi_k - \varphi| \, d\mu, \end{aligned}$$

By the weak-$*$ convergence of $|\nabla w_n|^p$ to μ in $\mathcal{M}_b(\Omega)$, we obtain for k fixed

$$\limsup_{n\to\infty}\Big|\int_\Omega |\nabla w_n|^p\varphi - \int_\Omega \varphi\, d\mu\Big| \le \limsup_{n\to\infty}\int_\Omega |\nabla w_n|^p|\varphi_k-\varphi| + \\ + \int_\Omega |\varphi-\varphi_k|\, d\mu.$$

Taking now the limit in k we find

$$\limsup_{n\to\infty}\Big|\int_\Omega |\nabla w_n|^p\varphi - \int_\Omega \varphi\, d\mu\Big| \le \limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla w_n|^p|\varphi_k-\varphi|.$$

Therefore, in order to prove (P6) we need only to show that the limit in the second member is zero. Step 3 applied to the sequence $v_n^k = w_n|\varphi_k-\varphi|$, gives

$$\lim_{k\to\infty}\limsup_{n\to\infty}\Big|\int_\Omega |\nabla w_n|^p|\varphi_k-\varphi| + \int_\Omega w_n|\nabla w_n|^{p-2}\nabla w_n\nabla\left(|\varphi_k-\varphi|\right)\Big| = 0.$$

But since

$$\Big|\int_\Omega w_n|\nabla w_n|^{p-2}\nabla w_n\nabla(|\varphi_k-\varphi|)\Big| \le \Big(\int_\Omega |\nabla w_n|^p\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |\nabla(|\varphi_k-\varphi|)|^p\Big)^{\frac{1}{p}}$$

and φ_k tends strongly to φ in $W_0^{1,p}(\Omega)$, we have that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\Big|\int_\Omega w_n|\nabla w_n|^{p-2}\nabla w_n\nabla\left(|\varphi_k-\varphi|\right)\Big| = 0$$

and therefore

$$\lim_{k\to\infty}\limsup_{n\to\infty}\Big|\int_\Omega |\nabla w_n|^p|\varphi_k-\varphi|\Big| = 0.$$

Step 7: In this step, we will prove (P7) when v is supposed to belong also to $L^\infty(\Omega)$. More precisely, let v_n and v satisfying

$$\begin{cases} v_n \in W_0^{1,p}(\Omega_n), \\ v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega), \\ v_n \rightharpoonup v \text{ in } W_0^{1,p}(\Omega). \end{cases}$$

Then

$$\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n \to \int_\Omega v \, d\mu. \tag{2.3}$$

Proof. We have

$$\begin{aligned} & v_n - w_n v \rightharpoonup 0 \ \text{ in } W_0^{1,p}(\Omega), \\ & v_n - w_n v \in W_0^{1,p}(\Omega_n), \end{aligned}$$

and so, from (P8) (proved in Step 2) we obtain

$$\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla (v_n - w_n v) \to 0,$$

or developping

$$\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n - \int_\Omega |\nabla w_n|^p v - \int_\Omega w_n |\nabla w_n|^{p-2} \nabla w_n \nabla v \to 0. \tag{2.4}$$

In the third integral, the integrand is pointwise convergent to zero and equi-integrable, so it converges to zero strongly in $L^1(\Omega)$. Property (P6) (proved in Step 6), implies (2.3).

Step 8: Proof of (P7). Let us now prove that if $v_n \in W_0^{1,p}(\Omega_n)$ converges weakly in $W_0^{1,p}(\Omega)$ to v, then

$$v \in L^1(\Omega, d\mu) \ \text{ and } \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n \to \int_\Omega \varphi v \, d\mu, \tag{2.5}$$

which will complete the proof of (P7).

Proof. Using the decomposition $v_n = (v_n)^+ - (v_n)^-$, it is sufficient to prove the result (2.5) for $v_n \geq 0$.

Defining v_n^k by $v_n = T_k(v_n) + v_n^k$ we have

$$\begin{aligned} \Big| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n - \int_\Omega T_k(v) \, d\mu \Big| \leq \Big| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n^k \Big| + \\ + \Big| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla T_k(v_n) - \int_\Omega T_k(v) \, d\mu \Big|. \end{aligned}$$

By Step 7, the second term of the right-hand side tends to zero as n tends to infinity for k fixed, while by Step 3 the first term tends to zero when n and k tend to infinity. This proves that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \Big| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n - \int_\Omega T_k(v)\, d\mu \Big| = 0, \tag{2.6}$$

which implies that $\int_\Omega T_k(v)\, d\mu$ is bounded independently of k. Hence, from the Beppo Levi's monotone convergence theorem, $v \in L^1(\Omega, d\mu)$ and $T_k(v)$ converges strongly to v in $L^1(\Omega, d\mu)$.

To prove (2.5) it is now enough to write

$$\begin{aligned}\limsup_{n\to\infty} \Big| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n - \int_\Omega v\, d\mu \Big| &\leq \lim_{k\to\infty} \int_\Omega |v - T_k(v)|\, d\mu + \\ + \lim_{k\to\infty} \limsup_{n\to\infty} \Big| \int_\Omega |\nabla w_n|^{p-2} \nabla w_n \nabla v_n - \int_\Omega T_k(v)\, d\mu \Big|\end{aligned}$$

which is zero.

Step 9: Uniqueness.

Proof. Let $\hat{w}_n$ be another sequence which together with some $\hat{\mu}$ satisfies properties (P1), (P2),…, (P8). Using Property (P8) with $v_n = w_n - \hat{w}_n$, we have

$$\int_\Omega (|\nabla w_n|^{p-2} \nabla w_n - |\nabla \hat{w}_n|^{p-2} \nabla \hat{w}_n)(\nabla w_n - \nabla \hat{w}_n) \to 0.$$

Inequality (1.2) or (1.6) then gives the strong convergence to zero of $w_n - \hat{w}_n$ in $W^{1,p}(\Omega)$. On the other hand by (P6) and this strong convergence, for any function $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega \varphi\, d\hat{\mu} = \lim_{n\to\infty} \int_\Omega |\nabla \hat{w}_n|^p \varphi = \lim_{n\to\infty} \int_\Omega |\nabla w_n|^p \varphi = \int_\Omega \varphi\, d\mu,$$

i.e. $\mu = \hat{\mu}$. ∎

3 – Semicontinuity

We will now improve the result already obtained in (P7) and to prove that every function $v \in W_0^{1,p}(\Omega)$ which is the weak limit in $W_0^{1,p}(\Omega)$ of a sequence $v_n \in W_0^{1,p}(\Omega_n)$ belongs to $L^p(\Omega, d\mu)$. We will also obtain a semicontinuity result for the energy.

THEOREM 3.1. *Consider a sequence $v_n \in W_0^{1,p}(\Omega_n)$ which converges weakly in $W_0^{1,p}(\Omega)$ to a function v. Then*

$$v \in L^p(\Omega, d\mu) \text{ and } \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p \geq \int_\Omega |\nabla v|^p + \int_\Omega |v|^p d\mu. \tag{3.1}$$

REMARK 3.1 Theorem 3.1 can be deduced in a straightforward way from the Γ-convergence result given in [7], where the result is actually stronger because no hypothesis on the sequence Ω_n is imposed there. This general result can also be obtained by a method close to the present one (see [5]) which also allows one to obtain the corrector result of Theorem 4.1 in a framework where no hypothesis is imposed on the sequence Ω_n.

REMARK 3.2 A consequence of Theorem 3.1 is that for a given $v \in W_0^{1,p}(\Omega)$ which is not zero μ-almost everywhere there does not exist any sequence $v_n \in W_0^{1,p}(\Omega_n)$ which converges strongly in $W_0^{1,p}(\Omega)$ to v.

PROOF. As for Theorem 2.1, the proof of Theorem 3.1 will be divided in several steps which are interesting in themselves and which establish in particular that for $z \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, the sequence $w_n z$ satisfies properties similar to those of w_n.

STEP 1: If $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then we have

$$\int_\Omega |\nabla (w_n z)|^p \to \int_\Omega |\nabla z|^p + \int_\Omega |z|^p \, d\mu. \tag{3.2}$$

PROOF. Write

$$\begin{aligned} \int_\Omega |\nabla (w_n z)|^p &= \int_\Omega |z \nabla w_n + w_n \nabla z|^p = \\ &= \int_\Omega \big[|z \nabla w_n + w_n \nabla z|^p - |z \nabla w_n|^p\big] + \int_\Omega |z \nabla w_n|^p. \end{aligned} \tag{3.3}$$

In the first integral of the right-hand side of (3.3) by Lagrange's theorem we have

$$\begin{aligned}\big||z\,\nabla\, w_n + w_n \nabla\, z|^p - |z\,\nabla\, w_n|^p\big| &\le \\ &\le p\left[|z\,\nabla\, w_n + w_n\,\nabla\, z|^{p-1} + |z\,\nabla\, w_n|^{p-1}\right]|w_n\,\nabla\, z|.\end{aligned}$$

Note that the right-hand side is equi-integrable. Therefore the left hand side, which converges almost everywhere, converges strongly in $L^1(\Omega)$ to $|\nabla\, z|^p$.

For the second integral of the right-hand side of (3.3), we use (P6), obtaining that

$$\int_\Omega |z|^p |\nabla\, w_n|^p \to \int_\Omega |z|^p\, d\mu.$$

This completes the proof of (3.2).

STEP 2: Consider a sequence $v_n \in W_0^{1,p}(\Omega_n)$ which converges weakly in $W_0^{1,p}(\Omega)$ to a function v. Then for any function $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \varphi |\nabla\, w_n|^{p-2} \nabla\, w_n\, \nabla\, v_n \to \int_\Omega v\varphi\, d\mu. \tag{3.4}$$

PROOF. Suppose first that v_n is also bounded in $L^\infty(\Omega)$. Then, using (P7), Rellich-Kondrachov's compactness theorem and the pointwise convergence of $\nabla\, w_n$, we have

$$\begin{cases} \displaystyle\lim_{n\to\infty} \int_\Omega \varphi |\nabla\, w_n|^{p-2} \nabla\, w_n\, \nabla\, v_n = \lim_{n\to\infty} \int_\Omega |\nabla\, w_n|^{p-2} \nabla\, w_n\, \nabla\, (v_n \varphi) \\ \displaystyle - \lim_{n\to\infty} \int_\Omega v_n |\nabla\, w_n|^{p-2} \nabla\, w_n\, \nabla\, \varphi = \int_\Omega v\varphi\, d\mu. \end{cases} \tag{3.5}$$

In the general case, (v_n is not in $L^\infty(\Omega)$), by the above proved, we have that

$$\int_\Omega \varphi |\nabla\, w_n|^{p-2} \nabla\, w_n\, \nabla\, T_k(v_n) \to \int_\Omega T_k(v)\varphi\, d\mu, \quad \forall\, k \in \mathbb{N}$$

and then, by the Lebesgue's dominated convergence theorem, we get

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_\Omega \varphi |\nabla\, w_n|^{p-2} \nabla\, w_n\, \nabla\, T_k(v_n) = \int_\Omega v\varphi\, d\mu.$$

To finish the proof of (3.4) it is then enough to prove that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_\Omega \varphi|\nabla w_n|^{p-2}\nabla w_n\,\nabla(v_n - T_k(v_n)) = 0$$

or, using Hölder's inequality and that $\varphi \in L^\infty(\Omega)$, that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_{|v_n|\geq k}|\nabla w_n|^p = 0. \tag{3.6}$$

Applying (2.2) with $v_n^k = w_n R_k(v_n)$, we obtain

$$\lim_{n,k\to\infty}\Big(\int_\Omega |\nabla w_n|^p R_k(v_n) + \int_\Omega w_n R_k'(v_n)|\nabla w_n|^{p-2}\nabla w_n\,\nabla v_n\Big) = 0. \tag{3.7}$$

But

$$\Big|\int_\Omega w_n R_k'(v_n)|\nabla w_n|^{p-2}\nabla w_n\,\nabla v_n\Big| \leq \frac{2}{k}\Big|\int_{k\geq|v_n|\geq k/2}|\nabla w_n|^{p-2}\nabla w_n\,\nabla v_n\Big|,$$

which, using that w_n and v_n are bounded in $W_0^{1,p}(\Omega)$, implies

$$\lim_{n,k\to\infty}\int_\Omega R_k'(v_n)|\nabla w_n|^{p-2}\nabla w_n\,\nabla v_n = 0$$

and therefore, by (3.7) we have

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla w_n|^p R_k(v_n) = 0.$$

which gives (3.6), by $R_k(v_n) \geq \chi_{\{|v_n|\geq k\}}$.

STEP 3: Let $z \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. Then for any sequence $v_n \in W_0^{1,p}(\Omega_n)$ which converges weakly in $W_0^{1,p}(\Omega)$ to a function v, we have

$$\int_\Omega |\nabla(w_n z)|^{p-2}\nabla(w_n z)\nabla v_n \to \int_\Omega |\nabla z|^{p-2}\nabla z\nabla v + \int_\Omega |z|^{p-2}zv\,d\mu. \tag{3.8}$$

PROOF. As in Step 1, it is easy to see, using (1.3) or (1.5), that

$$|\nabla(w_n z)|^{p-2}\nabla(w_n z) - |z\,\nabla w_n|^{p-2}z\nabla w_n$$

is equi-integrable in $L^{p'}(\Omega)$ and converges pointwise to $|\nabla z|^{p-2}\nabla z$, and thus converges strongly in $L^{p'}(\Omega)^N$ to $|\nabla z|^{p-2}\nabla z$. Therefore

$$\lim_{n\to\infty}\int_\Omega |\nabla (w_n z)|^{p-2}\nabla (w_n z)\,\nabla v_n = \\ = \int_\Omega |\nabla z|^{p-2}\nabla z\,\nabla v + \lim_{n\to\infty}\int_\Omega |z|^{p-2}z|\nabla w_n|^{p-2}\nabla w_n\,\nabla v_n.$$

To obtain (3.8) it is enough to use Step 2 with $\varphi = |z|^{p-2}z$.

Step 4: Proof of (3.1).

Using the convexity inequality

$$|\xi|^p \geq |\eta|^p + p|\eta|^{p-2}\eta(\xi-\eta), \quad \forall\,\xi,\,\eta\in \mathbb{R}^N,$$

we have for $k\in\mathbb{N}$

$$\int_\Omega |\nabla v_n|^p \geq \int_\Omega |\nabla (w_n T_k(v))|^p + \\ + p\int_\Omega |\nabla (w_n T_k(v))|^{p-2}\nabla \big(w_n T_k(v)\big)\big(\nabla v_n - \nabla (w_n T_k(v))\big).$$

Using (3.2) and (3.8) (with $z = T_k(v)$) and then

$$\begin{cases} |\nabla T_k(v)|^{p-2}\nabla T_k(v)(\nabla v - \nabla T_k(v)) = 0 & \text{a.e. in } \Omega \\ |T_k(v)|^{p-2}T_k(v)(v - T_k(v)) \geq 0 & \mu\text{-a.e. in } \Omega \end{cases}$$

we obtain

$$\liminf_{n\to\infty}\int_\Omega |\nabla v_n|^p \geq \int_\Omega |\nabla T_k(v)|^p + \int_\Omega |T_k(v)|^p\,d\mu + \\ + p\int_\Omega |\nabla T_k(v)|^{p-2}\nabla T_k(v)(\nabla v - \nabla T_k(v)) + \\ + p\int_\Omega |T_k(v)|^{p-2}T_k(v)(v - T_k(v))\,d\mu \geq \\ \geq \int_\Omega |\nabla T_k(v)|^p + \int_\Omega |T_k(v)|^p\,d\mu.$$

The Beppo Levi's monotone convergence theorem then implies that v belongs to $L^p(\Omega, d\mu)$ and that $T_k(v)$ converges in $L^p(\Omega, d\mu)$ to v. Using also the convergence of $T_k(v)$ to v in $W_0^{1,p}(\Omega)$ we obtain (3.1). $\square$

4 – Corrector

In the case where the lim inf in (3.1) is actually a limit, and where the inequality is actually an equality, we have the following corrector result, which provides an approximate representation of the gradient of v_n in the strong topology of $L^p(\Omega)^N$.

THEOREM 4.1. *Consider a sequence $v_n \in W_0^{1,p}(\Omega_n)$ which converges weakly in $W_0^{1,p}(\Omega)$ to a function v. Assume that*

$$\lim_{n\to\infty} \int_\Omega |\nabla v_n|^p = \int_\Omega |\nabla v|^p + \int_\Omega |v|^p d\mu. \tag{4.1}$$

Then we have

$$\lim_{k\to\infty} \limsup_{n\to\infty} \int_\Omega |\nabla (v_n - w_n T_k(v))|^p = 0. \tag{4.2}$$

REMARK 4.1 In particular, when $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, (4.2) implies that

$$v_n - w_n v \to 0 \ \text{ in } W_0^{1,p}(\Omega).$$

PROOF.

STEP 1. In this step we do not use hypothesis (4.1). We will prove that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_\Omega |\nabla v_n|^{p-2} \nabla v_n (\nabla v_n - \nabla (w_n T_k(v))) = 0 \tag{4.3}$$

implies that (4.2) holds true.

Indeed, If $p \geq 2$, by (1.2), (3.8) (see Step 4 in the proof of Theorem 3.1) and using that $T_k(v)$ converges strongly to v in $W^{1,p}(\Omega) \cap L^p(\Omega, d\mu)$, we have

$$\begin{cases} \limsup\limits_{k\to\infty} \limsup\limits_{n\to\infty} \displaystyle\int_\Omega |\nabla v_n - \nabla (w_n T_k(v))|^p \leq \\ \leq 2^{p-2}\Big[\limsup\limits_{k\to\infty} \limsup\limits_{n\to\infty} \displaystyle\int_\Omega |\nabla v_n|^{p-2} \nabla v_n (\nabla v_n - \nabla (w_n T_k(v))) - \\ -\lim\limits_{k\to\infty} \lim\limits_{n\to\infty} \displaystyle\int_\Omega |\nabla (w_n T_k(v))|^{p-2} \nabla (w_n T_k(v)) (\nabla v_n - \nabla (w_n T_k(v)))\Big] = \\ = 2^{p-2} \limsup\limits_{k\to\infty} \limsup\limits_{n\to\infty} \displaystyle\int_\Omega |\nabla v_n|^{p-2} \nabla v_n (\nabla v_n - \nabla (w_n T_k(v))) \end{cases} \tag{4.4}$$

If $1 < p < 2$, we use (1.6) which gives

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla\,(v_n - w_n T_k(v))|^p \le \frac{2^{p-1}}{(p-1)^{\frac{p}{2}}}\,\Big[\limsup_{k\to\infty}\limsup_{n\to\infty}$$

$$\int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla(w_n T_k(v))|^{p-2}\nabla(w_n T_k(v)))\,(\nabla v_n - \nabla(w_n T_k(v)))\Big]^{\frac{p}{2}}\cdot$$

$$\cdot\Big[\limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega (|\nabla\, v_n|^p + |\nabla\,(w_n T_k(v))|^p)\Big]^{\frac{2-p}{2}}.$$

By (3.2) and the strong convergence of $T_k(v)$ to v in $W_0^{1,p}(\Omega)\cap L^p(\Omega, d\mu)$ we have

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla\,(w_n T_k(v))|^p = \int_\Omega |\nabla\, v|^p + \int_\Omega |v|^p\, d\mu < +\infty\,.$$

Applying then (3.8) as in (4.4), we get

$$\text{(4.5)}\quad \begin{cases} \limsup\limits_{k\to\infty}\limsup\limits_{n\to\infty}\displaystyle\int_\Omega |\nabla\,(v_n - w_n T_k(u))|^p \le \\ \le C\,\Big[\limsup\limits_{k\to\infty}\limsup\limits_{n\to\infty}\displaystyle\int_\Omega |\nabla\, v_n|^{p-2}\nabla\, v_n(\nabla\, v_n - \nabla\,(w_n T_k(v)))\Big]^{\frac{p}{2}}. \end{cases}$$

In both case $2 \le p < +\infty$ and $1 < p < 2$, we have proved that (4.3) implies (4.2).

STEP 2. Define

$$\mathcal{B} = \big\{\{z_n\} :\ z_n \in W_0^{1,p}(\Omega_n) :\ z_n \rightharpoonup v \text{ in } W_0^{1,p}(\Omega)\big\},$$

and note that (4.1) and Theorem 3.1 imply that the sequence v_n satisfies

$$\lim_{n\to\infty}\int_\Omega |\nabla\, v_n|^p = \min\Big\{\liminf_{n\to\infty}\int_\Omega |\nabla\, z_n|^p :\ \{z_n\}\in\mathcal{B}\Big\}.$$

Then (see Steps 2 and 3 in the proof of Theorem 2.1) v_n satisfies the following property:

$$\text{(4.6)}\quad \begin{cases} \forall\, z_n^k \in W_0^{1,p}(\Omega_n) \text{ such that } z_n^k \rightharpoonup 0 \text{ in } W_0^{1,p}(\Omega) \text{ when } n,k\to\infty, \\ \quad\text{we have} \\ \lim\limits_{n,k\to\infty}\displaystyle\int_\Omega |\nabla\, v_n|^{p-2}\nabla\, v_n \nabla\, z_n^k = 0. \end{cases}$$

Step 3. In order to prove (4.3), we cannot apply directly (4.6) since $v_n - w_n T_k(v)$ is not in general bounded in $W_0^{1,p}(\Omega)$ independently of n and k. To bypass this difficulty, we write

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla v_n - \nabla(w_n T_k(v))) \le$$

$$\le \limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla v_n - \nabla T_k(v_n)) +$$

$$+ \limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla T_k(v_n) - \nabla(w_n T_k(v))).$$

Now, by (4.6),

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla v_n - \nabla T_k(v_n)) \le$$

$$\le \limsup_{k,n\to\infty}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla v_n - \nabla T_k(v_n)) = 0$$

while for k fixed (4.6) with z_n^k independent of k implies

$$\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n(\nabla T_k(v_n) - \nabla(w_n T_k(v))) = 0.$$

This proves (4.3). □

5 – Homogenization

As an application of the results established in the previous sections, let us make a brief study of the homogenization problem for the p-Laplacian in perforated domains. (This result will be used in [3] and [4] to obtain similar homogenization results for more general quasi-linear problems). For a general result without any hypothesis about the sequence $\Omega \setminus \Omega_n$, see [7], [12].

Theorem 5.1. *Consider a sequence Ω_n (whose existence is given in Theorem* 2.1) *for which there exist w_n and μ satisfying* (P1), (P2),... , (P8). *Then the following homogenization result holds: For any* $f \in W^{-1,p'}(\Omega)$, *the solution* u_n *of the problem*

$$\left\{\begin{array}{l} -\Delta_p u_n = f \quad in\ \mathcal{D}'(\Omega_n), \\ u_n \in W_0^{1,p}(\Omega_n), \end{array}\right. \tag{5.1}$$

converges weakly in $W_0^{1,p}(\Omega)$ *to the solution* u *of the problem*

$$\left\{\begin{array}{l} -\Delta_p u + |u|^{p-2}u\mu = f \quad in\ \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega) \cap L^p(\Omega, d\mu), \end{array}\right. \tag{5.2}$$

which is equivalent to the variational formulation

$$\left\{\begin{array}{l} u \in W_0^{1,p}(\Omega) \cap L^p(\Omega, d\mu), \\ \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv\, d\mu = \langle f, v\rangle, \\ \forall\, v \in W_0^{1,p}(\Omega) \cap L^p(\Omega, d\mu). \end{array}\right. \tag{5.3}$$

The sequence u_n *also satisfies*

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p\, d\mu \tag{5.4}$$

and so the corrector result of Theorem 4.1 applies

$$\lim_{k\to\infty} \limsup_{n\to\infty} \int_\Omega |\nabla (u_n - w_n T_k(u))|^p = 0. \tag{5.5}$$

PROOF. Using u_n as test function in (5.1) we prove that the sequence u_n is bounded in $W_0^{1,p}(\Omega)$. We thus can extract a subsequence of u_n which converges weakly in $W_0^{1,p}(\Omega)$ to some u. By Theorem 3.1 u belongs to $W_0^{1,p}(\Omega) \cap L^p(\Omega, d\mu)$. For the sake of simplicity, let us still denote this subsequence by u_n. (Indeed, we will prove that u satisfies (5.3), which has a unique solution, and thus uniqueness will imply the convergence of the whole sequence u_n).

Let us first prove the corrector result (5.5). For this, we use $u_n - w_n T_k(u)$ as test function in (5.1). This gives

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\left(\nabla u_n - \nabla\left(w_n T_k(u)\right)\right) =$$

$$= \lim_{k\to\infty}\limsup_{n\to\infty}\langle f, u_n - w_n T_k(u)\rangle = 0.$$

This is analogous to (4.3), and by the proof of Step 1 of Theorem 4.1 this implies (5.5).

Now, for $\varphi \in \mathcal{D}(\Omega)$ we take $w_n\varphi \in W_0^{1,p}(\Omega_n)$ as test function in (5.1). We obtain

$$\int_\Omega |\nabla u_n|^{p-2}\,\nabla u_n \nabla\left(w_n\varphi\right) = \langle f, w_n\varphi\rangle.$$

The right hand side satisfies

$$\langle f, w_n\varphi\rangle \to \langle f, \varphi\rangle.$$

On the other hand, using the corrector result (5.5), then (3.8) and then that $T_k(u)$ converges strongly to u in $W_0^{1,p}(\Omega)\cap L^p(\Omega, d\mu)$, we have

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\,\nabla\left(w_n\varphi\right) =$$

$$= \lim_{k\to\infty}\lim_{n\to\infty}\int_\Omega |\nabla\left(w_n T_k(u)\right)|^{p-2}\nabla\left(w_n T_k(u)\right)\nabla\left(w_n\varphi\right) =$$

$$= \int_\Omega |\nabla u|^{p-2}\nabla u\,\nabla\varphi + \int_\Omega |u|^{p-2}u\,\varphi.$$

Therefore

$$\int_\Omega |\nabla u|^p\nabla u\,\nabla\varphi + \int_\Omega |u|^p u\varphi\, d\mu = \langle f,\varphi\rangle, \quad \forall\,\varphi\in\mathcal{D}(\Omega). \tag{5.6}$$

By density, (5.6) holds for any $\varphi \in W_0^{1,p}(\Omega)\cap L^p(\Omega, d\mu)$ and so u is the (unique) solution of (5.3). To obtain (5.4), use (5.5) or more directly, take u_n as test function in (5.1). $\square$

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Indirizzo dell'Autore:

Juan Casado-Díaz – Departamento de Ecuaciones Diferenciales y Análisis Numérico – Facultad de Matemáticas – C. Tarfía s/n, 41012 Sevilla, Spain