NEAR-INFINITY CONCENTRATED NORMS AND THE FIXED POINT PROPERTY FOR NONEXPANSIVE MAPS ON CLOSED, BOUNDED, CONVEX SETS

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Abstract. In this paper we define the concept of a near-infinity concentrated norm on a Banach space X with a boundedly complete Schauder basis. When $\|\cdot\|$ is such a norm, we prove that $(X, \|\cdot\|)$ has the fixed point property (FPP); that is, every nonexpansive self-mapping defined on a closed, bounded, convex subset has a fixed point. In particular, P.K. Lin's norm in ℓ_1 [14] and the norm $\nu_p(\cdot)$ (with $p = (p_n)$ and $\lim_n p_n = 1$) introduced in [3] are examples of near-infinity concentrated norms. When $\nu_p(\cdot)$ is equivalent to the ℓ_1-norm, it was an open problem as to whether $(\ell_1, \nu_p(\cdot))$ had the FPP. We prove that the norm $\nu_p(\cdot)$ always generates a nonreflexive Banach space $X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \dots))$ satisfying the FPP, regardless of whether $\nu_p(\cdot)$ is equivalent to the ℓ_1-norm. We also obtain some stability results.

1. Introduction and Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and C a subset of X. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. The Banach space X endowed with the norm $\|\cdot\|$ has the fixed point property (FPP) if every nonexpansive mapping defined from a closed bounded convex subset C of X into itself has a fixed point. This property is not preserved by isomorphism, that is, it strongly depends on the underlying norm [14] . There is a wide literature relating geometric properties of reflexive Banach spaces with the fulfilment of the fixed point property (see, for instance, the monographs [9], [13] and the references therein).

The Banach space ℓ_1 endowed with its standard norm $\|\cdot\|_1$ is a classical example of a nonreflexive Banach space that fails to have the FPP. It is possible to "perturb" this $(\ell_1, \|\cdot\|_1)$-example to obtain other Banach spaces that fail to have the FPP. One such class of Banach spaces are those that contain asymptotically isometric copies of ℓ_1.

Recall that a Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy (a.i.c.) of ℓ_1 if there are a sequence $(x_n) \subset X$ and a decreasing sequence $(\epsilon_n) \subset (0, 1)$ with $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty} (1 - \epsilon_n)|t_n| \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le \sum_{n=1}^{\infty} |t_n|$$

1991 *Mathematics Subject Classification.* 46B03, 47H09, 47H10.

Key words and phrases. fixed point property; nonexpansive mappings; renorming theory.

The third author is partially funded by CONACYT grants 243722 and 613207. The fourth author is partially supported by MCIN, Grant MTM2015-65242-C2-1-P and Andalusian Regional Government Grant FQM-127. She is also grateful to the Department of Mathematics of the University of Pittsburgh for its support while preparing this manuscript. The fifth author thanks the University of Newcastle for its financial support during part of the preparation of this paper; and Brailey, Gail and the Mathematics Department for their hospitality.

for every $(t_n) \in \ell_1$. It was proved in [4] that if a Banach space contains an a.i.c. of ℓ_1 then it fails to have the FPP. It turns out that there exist equivalent norms on ℓ_1 which fail to contain an a.i.c. of ℓ_1. Let us state some examples:

- The so-called P.K. Lin norm, defined as
$$|||x|||_L := \sup_{k\geq 1} \gamma_k \sum_{n=k}^{\infty} |x(n)|; \qquad x = \sum_{n=1}^{\infty} x(n)e_n$$
where (γ_k) is a nondecreasing sequence in $(0,1)$ with $\lim_k \gamma_k = 1$. In [5] it was proved that $(\ell_1, |||\cdot|||_L)$ fails to contain an a.i.c. of ℓ_1. Later on, P.K. Lin [14] proved that $(\ell_1, |||\cdot|||_L)$ has the FPP for $\gamma_k := \frac{8^k}{1+8^k}$. This condition was extended to every sequence (γ_k) with $\lim_k \gamma_k = 1$ (see [7] and [11]). P. K. Lin's result opened new avenues of research in the fixed point theory of nonexpansive mappings, since he settled negatively the long-standing open question: "Does the fixed point property imply reflexivity?" Since then, many other articles have appeared obtaining sufficient conditions that imply the FPP for equivalent norms on ℓ_1 (see for instance [2, 6, 7, 8, 10, 11, 12, 15]).
- Fix a nonincreasing sequence $p = (p_n)_n \subset (1, +\infty)$ with $\lim_n p_n = 1$. In the sequence space c_{00} of all real sequences with finitely many non-null coordinates, we define the norm $\nu_p(x) = \lim_n \nu_n(p, x)$ where
$$\nu_1(p,x) := |x_1|, \qquad \nu_{n+1}(p,x) := \left(|x_1|^{p_1} + \nu_n(Sp, Sx)^{p_1}\right)^{1/p_1},$$
with $x = (x_1, x_2, ...)$ and $Sz := (z_2, z_3, ...)$ when $z = (z_1, z_2, ...)$. The completion of c_{00} with the $\nu_p(\cdot)$ norm gives us a Banach space X with a *boundedly complete* Schauder basis (e_n). Also, X is the set of all real sequences $x = (x_n)$ for which $\nu_p(x) := \sup_n \nu_n(p,x) = \lim_n \nu_n(p,x) < \infty$; which we summarize by writing $X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} \ldots))$.
  Let $q = (q_n)$ be the sequence satisfying $\frac{1}{p_n} + \frac{1}{q_n} = 1$ for every $n \in \mathbb{N}$. Whenever the sequence (p_n) converges to 1 quickly enough, the norm $\nu_p(\cdot)$ provides an equivalent norm in ℓ_1; that is, $(X, \nu_p(\cdot))$ and ℓ_1 are isomorphic Banach spaces. In fact, it was proved in [3, Proposition 1] that $\nu_p(\cdot)$ is equivalent to the ℓ_1 norm if and only if there exists some $\delta > 0$ so that $q_n \geq \delta \log n$ for all $n \in \mathbb{N}$. It is also known that $(\ell_1, \nu_p(\cdot))$ fails to contain asymptotically isometric copies of ℓ_1 [3, Theorem 1]. However, unlike P.K. Lin's norm, it was unknown whether ℓ_1 with the norm $\nu_p(\cdot)$ had the fixed point property.

In what follows, we enlarge the class of norms on ℓ_1 satisfying the FPP and we include, as a particular case, the norm $\nu_p(\cdot)$ defined in [3]. We will extend our result to a more general framework. For instance, we will prove the fulfilment of the FPP for $(X, \nu_p(\cdot))$ even when this norm fails to be an ℓ_1-norm.

Furthermore, we obtain stability of the fixed point property for certain norms along rays emanating from near-infinity concentrated norms.

2. Near-infinity concentrated norms and the FPP

Throughout this paper, let X denote a Banach space with a Schauder basis $\{e_n\}_n$. Given $x = \sum_{n=1}^{\infty} x(n)e_n \in X$, we denote by $supp(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$, $Q_k(x) = \sum_{n=k}^{\infty} x(n)e_n$ and $P_k(x) = \sum_{n=1}^{k-1} x(n)e_n$ $(P_1 = 0)$. The basis is said to be premonotone for the norm $\|\cdot\|$ when $\|Q_k\| \leq 1$ for every $k \in \mathbb{N}$.

Given $k \in \mathbb{N}$ and $x \in X$, we write $k \leq x$ whenever $k \leq \min\{supp(x)\}$ and $k < x$ whenever $k < \min\{supp(x)\}$. We say that (y_n) is a block basic sequence for $\{e_n\}_n$ if it is bounded and there exist positive integers $p_1 \leq q_1 < p_2 \leq q_2 < ...$ such that y_n belongs to the span of $\{e_{p_n}, \cdots, e_{q_n}\}$ for every $n \in \mathbb{N}$.

The Schauder basis is said to be boundedly complete if $\sup_n \left\| \sum_{i=1}^n t_i e_i \right\| < +\infty$ implies that $\sum_{i=1}^{\infty} t_i e_i \in X$. When the Schauder basis (e_n) is boundedly complete, the Banach space X is isomorphic to a dual space Z^*, where Z is the closed subspace spanned by the biorthogonal functionals (e_n^*) in X^*. In this case, we can consider in X the weak* topology $\sigma(X,Z)$, for which the convergence coincides with the coordinate-to-coordinate convergence for norm-bounded sequences. Moreover, the closed unit ball is $\sigma(X,Z)$-sequentially compact and therefore every bounded sequence in X has a subsequence which converges coordinatewise (see for instance Theorem 3.2.10 in [1]). In what follows the weak* topology always refers to the $\sigma(X,Z)$ topology for Banach spaces with boundedly complete Schauder basis. In the case where $X = \ell_1$ endowed with the standard Schauder basis, this w^*-topology coincides with the $\sigma(\ell_1, c_0)$ topology.

Definition 2.1. [2] *A norm $|||\cdot|||$ on a Banach space X with a Schauder basis $\{e_n\}$ is said to be a sequentially separating norm if for every $\epsilon > 0$ there exists some $k \in \mathbb{N}$ such that*

$$|||x||| + \limsup_n |||x_n||| \leq (1+\epsilon) \limsup_n |||x + x_n|||$$

whenever $k \leq x$ and $(x_n)_n$ is a block basic sequence of $\{e_n\}_n$ in X.

Definition 2.2. *Let X be a Banach space with a Schauder basis $\{e_n\}_n$ and let $|||\cdot|||$ be a norm on X. This norm is called near-infinity concentrated (n.i.c.) if it has the following properties:*

(1) *It is a sequentially separating norm.*

(2) *It is premonotone.*

(3) *There exist $R_0 > 5$ and $M \in [0,1)$ such that for every $k \in \mathbb{N}$, there exists a function $F_k : (0,+\infty) \to [0,+\infty)$ satisfying the following conditions:*

   (a) $\lim_{\lambda \to 0^+} \dfrac{F_k(\lambda)}{\lambda} \leq \dfrac{M}{R_0}$.

   (b) *For every bounded pointwise-null sequence (x_n) with $\liminf_n |||x_n||| \geq 1$, for all $\lambda \in (0,+\infty)$, and for every $z \in X$ with $Q_k(z) = 0$ and $|||z||| \leq R_0$,*

$$\limsup_n |||x_n + \lambda z||| \leq \limsup_n |||x_n||| + F_k(\lambda)\, |||z||| \,.$$

Remark 2.3. *Observe that Property (3) can be re-written as: There exists $K \geq 0$ such that for every $k \in \mathbb{N}$, there exists a function $F_k : (0,+\infty) \to [0,+\infty)$ satisfying (a)' and (b); where condition (a)' is: $\lim_{\lambda \to 0^+} \dfrac{F_k(\lambda)}{\lambda} \leq\ K < \dfrac{1}{5}$.*

Given K, we may take $M := 1 - \frac{1-5K}{K+1} = \frac{6\,K}{K+1}$ and $R_0 := 5 + \frac{1-5K}{K+1} = \frac{6}{K+1}$.

Note that if $|||\cdot|||$ is an equivalent norm on ℓ_1 satisfying

$$a_k \|Q_k(x)\|_1 \leq |||Q_k(x)||| \leq b_k \|Q_k(x)\|_1 \ , \text{ for all } x \in \ell_1 \ ,$$

for every $k \in \mathbb{N}$, with $0 < a_k \leq b_k$ and $\lim_k b_k/a_k = 1$, then it is clear that $|||\cdot|||$ is a sequentially separating norm. Nevertheless, there exist some equivalent norms on ℓ_1 which do not satisify this condition but they are still sequentially separating [2, Example 3.2]. Furthermore, there exist Banach spaces with sequentially separating norms that are not isomorphic to ℓ_1, although the existence of such a norm implies that the Banach space X is "similar" to ℓ_1, in the sense that it has the Schur property, and so is hereditarily ℓ_1 [2, Corollary 7.4]. Recall that a Banach space X is hereditarily ℓ_1 if each infinite dimensional closed subspace of X contains a further subspace isomorphic to ℓ_1. This implies, in particular, that if a Banach space with an unconditional Schauder basis has a sequentially separating norm, then the basis is boundedly complete, since otherwise X would contain an isomorphic copy of c_0 (see for instance [1, Theorem 3.3.2]).

Also note that in Definition 2.2, Property (3)(b), if (x_n) is an arbitrary sequence of "bump functions sliding towards infinity", each with their $|||\cdot|||$-norm asymptotically no less than 1, then

$$\frac{\limsup_n |||x_n + \lambda z||| - \limsup_n |||x_n|||}{\lambda}$$

is smaller than one would expect from just the triangle inequality: for all $z = P_k(z)$ with $|||z||| \leq R_0$, for all λ positive and very small, the "upper asymptotic value" of the norm of x_n is changed less than expected when we perturb each, x_n by λz, since $F_k(\lambda)\, R_0/\lambda$ is approximately bounded by $M < 1$. In this sense, $|||\cdot|||$ is "near-infinity concentrated". Moreover, this third property prevents X from containing an asymptotically isometric copy of ℓ_1, which we will now prove.

Lemma 2.4. *Let X be a Banach space with a boundedly complete Schauder basis. If $|||\cdot|||$ is an equivalent norm in X satisfying property (3) in Definition 2.2, then $(X, |||\cdot|||)$ fails to have an a.i.c. of ℓ_1.*

Proof. Assume to the contrary that there exists a basic sequence (x_n) in X generating an a.i.c. of ℓ_1, that is, there is a decreasing sequence $(\epsilon_n) \subset (0,1)$ with $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \leq \left|\left|\left|\sum_{n=1}^{\infty} t_n x_n\right|\right|\right| \leq \sum_{n=1}^{\infty}|t_n| \ .$$

By extracting a subsequence, we can assume that (x_n) is w^*-convergent and, by replacing (x_n) by $((x_{2n} - x_{2n-1})/2))$, that it is w^*-convergent to the null vector. Finally, using the sliding hump method and the fact that asymptotically isometric copies are stable by adding norm-null sequences, we can assume that the sequence (x_n) generating the a.i.c. of ℓ_1 is a disjointly supported w^*-null sequence.

Take $R_0 > 5$ and $M \in [0,1)$ as in (3) of Definition 2.2. By omitting the first few terms of the sequence $(x_n)_n$, we can also assume that $\epsilon_1 < (R_0 - M)/R_0$.

From the previous inequalities $|||x_n||| \leq 1$ for every $n \in \mathbb{N}$ and $\lim_n |||x_n||| = 1$. Let $k := 1 + \max\{supp(x_1)\}$. Since $|||\cdot|||$ satisfies property (3) of a near-infinity concentrated norm, there exists a function $F_k(\lambda)$ such that $\lim_{\lambda\to 0^+} F_k(\lambda)/\lambda \leq \frac{M}{R_0}$, and for every $\lambda > 0$

$$\limsup_n |||x_n + \lambda R_0 x_1||| \leq \limsup_n |||x_n||| + F_k(\lambda) R_0 |||x_1||| \leq 1 + F_k(\lambda) R_0.$$

On the other hand, for every $n \geq 2$,

$$1 - \epsilon_n + \lambda R_0(1-\epsilon_1) \leq |||x_n + \lambda R_0 x_1|||.$$

Letting n tend to infinity, we see that

$$1 + \lambda R_0(1-\epsilon_1) \leq 1 + F_k(\lambda) R_0$$

and so $\lambda(1-\epsilon_1) \leq F_k(\lambda)$ for every $\lambda > 0$. Letting $\lambda \to 0$, we get that $(1-\epsilon_1) \leq \lim_{\lambda\to 0^+} \frac{F_k(\lambda)}{\lambda} \leq \frac{M}{R_0}$, which implies that $R_0\,(1-\epsilon_1) \leq M$, and this is a contradiction. □

Before stating our main result, we recall some standard arguments used to prove the FPP (see for instance [14] or [11]):

Let C be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a nonexpansive mapping. Using Banach's Contraction Mapping Theorem, we can always find a sequence $(x_n) \subset C$ such that $\lim_n \|x_n - Tx_n\| = 0$. Such sequences are called approximate fixed point sequences (a.f.p.s.). In fact, if (x_n) is an a.f.p.s. and $r > 0$, the set

$$\{x \in C : \limsup_n \|x_n - x\| \leq r\}$$

is either empty, or a non-empty closed convex T-invariant subset of C, in which we can find new approximate fixed point sequences.

In a dual Banach space X with separable predual Y, every a.f.p.s. has a subsequence which is w^*-convergent. For example, if X is a Banach space with a boundedly complete Schauder basis $\{e_n\}_n$, and corresponding biorthogonal functionals $\{f_n\}_n \subset X^*$, then for $Y :=$ the closed linear span of $\{f_n\}_n$ in X^*, Y^* is isomorphic to X and every a.f.p.s. in X has a subsequence that is $\sigma(X,Y)$-convergent. Therefore, we will subsequently assume that approximate fixed point sequences in bounded subsets of X are w^*-convergent.

Using Cantor's theorem (see [14] or [11, Lemma 1]), the above argument lets us deduce that if $T : C \to C$ is a fixed point free nonexpansive mapping, there exist some $a > 0$ and a closed convex T-invariant subset, denoted again by C, such that $\limsup_n \|y_n - y\| > a$ whenever $(y_n) \subset C$ is an a.f.p.s. and $y = w^*\text{-}\lim y_n$.

Note that from Definition 2.1 it is not difficult to check the following [2]:

Lemma 2.5. *A norm $|||\cdot|||$ in a Banach space X with a Schauder basis $\{e_n\}_n$ is sequentially separating if and only if $\lim_k S_k(X, |||\cdot|||) = 1$, with*

$$S_k(X, |||\cdot|||) := \sup\left\{\frac{|||x||| + \limsup_n |||x_n|||}{\limsup_n |||x + x_n|||}\right\},$$

where the supremum is taken over all vectors $x \in X$ with $k \leq x$ and all block basic sequences of $\{e_n\}_n$.

If we fix the norm $|||\cdot|||$ in the Banach space X, we will use S_k to denote $S_k(X, |||\cdot|||)$.

Lemma 2.6. *Let $(X, |||\cdot|||)$ be a Banach space with a boundedly complete Schauder basis $\{e_n\}_n$ such that $|||\cdot|||$ is premonotone and sequentially separating. The following holds: if (x_n), (y_n) are two sequences in X that are w^*-convergent to x and y respectively, then*

$$\limsup_m \limsup_n |||x_n - y_m||| \geq \limsup_n |||x_n - x||| + \limsup_m |||y_m - y|||.$$

Proof. Let $k \in \mathbb{N}$ and $\delta > 0$ be given. Choose a subsequence (x_{n_ℓ}) of (x_n) such that

$$\limsup_n |||Q_k(x_n - x)||| = \lim_\ell |||Q_k(x_{n_\ell} - x)|||\,.$$

Fix $m \in \mathbb{N}$. Then

$$\begin{aligned}\limsup_n |||x_n - y_m||| &\geq \limsup_n |||Q_k(x_n - y_m)|||\\ &= \limsup_n |||Q_k(x_n - x) - Q_k(y_m - x)|||\\ &\geq \limsup_\ell |||Q_k(x_{n_\ell} - x) - Q_k(y_m - x)|||\,.\end{aligned}$$

By the Bessaga-Pełczynski Selection Principle (see, for example, [1, p. 14]), passing to a further subsequence if necessary, we may choose a block basic sequence (u_ℓ) of (e_n) such that $Q_k(u_\ell) = u_\ell$ and $|||u_\ell - Q_k(x_{n_\ell} - x)||| < \delta$. Then

$$\begin{aligned}
\limsup_n |||x_n - y_m||| &\geq \limsup_\ell |||u_\ell - Q_k(y_m - x)||| - \delta \\
&\geq \frac{1}{S_k}\left(|||Q_k(y_m - x)||| + \limsup_\ell |||u_\ell||| \right) - \delta \\
&\geq \frac{1}{S_k}\left(|||Q_k(y_m - x)||| + \lim_\ell |||Q_k(x_{n_\ell} - x)||| \right) - 2\delta \\
&= \frac{1}{S_k}\left(|||Q_k(y_m - x)||| + \limsup_n |||Q_k(x_n - x)||| \right) - 2\delta \,.
\end{aligned}$$

Letting m tend to ∞; noting that $S_k \geq 1$; and using a perturbation argument similar to the one above then yields:

$$\begin{aligned}
\limsup_m \ \limsup_n |||x_n - y_m||| &\geq \frac{1}{S_k}\left(\limsup_m \ |||Q_k(y_m - x)||| + \limsup_n |||Q_k(x_n - x)||| \right) - 2\delta \\
&= \frac{1}{S_k}\left(\limsup_n |||x_n - x||| + \limsup_m \ |||Q_k(y_m - y) + Q_k(y - x)||| \right) - 2\delta \\
&\geq \frac{1}{S_k} \limsup_n |||x_n - x||| + \frac{1}{S_k^2}\left(|||Q_k(y - x)||| + \limsup_m |||Q_k(y_m - y)||| \right) - 4\delta \\
&\geq \frac{1}{S_k} \limsup_n |||x_n - x||| + \frac{1}{S_k^2} \limsup_m |||Q_k(y_m - y)||| - 4\delta \\
&= \frac{1}{S_k} \limsup_n |||x_n - x||| + \frac{1}{S_k^2} \limsup_m |||y_m - y||| - 4\delta \,.
\end{aligned}$$

In the above calculation, we used the fact that $\limsup_n |||Q_k(x_n - x)||| = \limsup_n |||x_n - x|||$ and $\limsup_m |||Q_k(y_m - y)||| = \limsup_m |||y_m - y|||$ for every $k \in \mathbb{N}$.

Since the above inequalities hold for every $k \in \mathbb{N}$, letting k tend to infinity gives

$$\limsup_m \ \limsup_n |||x_n - y_m||| \geq \limsup_n |||x_n - x||| + \limsup_m |||y_m - y||| - 4\delta \,,$$

for every $\delta > 0$. Since $\delta > 0$ is arbitrary, we obtain the desired inequality.

□

Theorem 2.7. *Let X be a Banach space with a boundedly complete Schauder basis and let $|||\cdot|||$ be a near-infinity concentrated (n.i.c.) norm on X. Then $(X, |||\cdot|||)$ has the FPP, that is, every nonexpansive self-map on a closed bounded convex subset of X has a fixed point.*

Proof. Assume, to the contrary, that there exists a closed bounded convex subset C of X and $T : C \to C$ a nonexpansive mapping such that

$$b = \inf\{\limsup_n |||y_n - y||| : (y_n) \subset C \text{ is an a.f.p.s. and } y_n \overset{w^*}{\to} y\} > 0.$$

Without loss of generality we can assume that $b = 1$. We proceed as follows.

Fix some $0 < \epsilon_1 < \min\{\frac{1}{4}(1 - \frac{M+1}{2}), \frac{1}{10}(R_0 - 5)\}$, where $M \in [0, 1)$ and $R_0 > 5$ are the constants given by condition (3) in Definition 2.2.

Consider an a.f.p.s. (x_n) in C such that $x_n \overset{w^*}{\to} x_0 \in X$ and $\limsup_n |||x_n - x_0||| < 1 + \epsilon_1$. Again, without loss of generality, we can assume that $x_0 = 0$ so that $\limsup_n |||x_n||| < 1 + \epsilon_1$. Define the set

$$D := \left\{ z \in C : \limsup_n |||x_n - z||| \leq 2(1 + \epsilon_1) \right\} .$$

Then D is a closed convex T-invariant subset of C. Moreover, using the triangle inequality,

$$\limsup_m \limsup_n |||x_n - x_m||| \leq 2 \limsup_n |||x_n||| < 2(1+\epsilon_1)\,;$$

so D is not empty and we can assume that $x_n \in D$ for n large enough. Define

$$c := \inf\left\{ \limsup_n |||y_n - y||| : (y_n) \subset D \text{ is an a.f.p.s. and } y_n \overset{w^*}{\to} y \right\} .$$

Notice that $1 \leq c$.

To simplify the notation, we define

$$A^*(D) := \left\{ y \in X : \ \exists\ (y_n) \subset D \text{ an a.f.p.s. such that } w^*\text{-}\lim_n y_n = y \right\}.$$

We now prove that $\sup_{y\in A^*(D)} |||y||| \leq 4 + 4\epsilon_1$:

Indeed, let $(y_n) \subset D$ be an a.f.p.s. with w^*-$\lim y_n = y$. In particular, $\limsup_n |||x_n - y_m||| \leq 2(1+\epsilon_1)$ for every $m \in \mathbb{N}$. Using the triangle inequality and Lemma 2.6 (with $x = 0$):

$$|||y||| \leq \limsup_m |||y - y_m||| + \limsup_n |||x_n||| + \limsup_m \limsup_n |||x_n - y_m||| \leq 4(1+\epsilon_1).$$

Next we show that $\sup_{y\in A^*(D)} |||Q_k(y)||| \leq \mu_k := 2S_k^2(1+\epsilon_1) - S_k - 1$ for every $k \in \mathbb{N}$. Using the proof of Lemma 2.6 we deduce that:

$$2(1+\epsilon_1) \geq \limsup_m \limsup_n |||x_n - y_m||| \geq$$

$$\frac{1}{S_k} \limsup_n |||x_n||| + \frac{1}{S_k^2}\left[\limsup_m |||y_m - y||| + |||Q_k(y)||| \right] \geq \frac{1}{S_k} + \frac{1}{S_k^2}\left[1 + |||Q_k(y)|||\right],$$

which implies that

$$|||Q_k(y)||| \leq 2S_k^2(1+\epsilon_1) - S_k - 1$$

for all $y \in A^*(D)$.

Choose $x := x_\nu$ with $\nu \in \mathbb{N}$ large enough so that $x \in D$ and $|||x||| < 1 + \epsilon_1$. Since the norm satisfies condition (1) in Definition 2.2, we know that $\lim_k S_k = 1$ and therefore $\lim_k \mu_k = 2\epsilon_1$. Take $k_1 \in \mathbb{N}$ so that

$$|||Q_k(x)||| < \epsilon_1, \quad \text{and} \quad \mu_k < 3\epsilon_1$$

if $k \geq k_1$. In particular, this implies that

$$|||Q_{k_1}(y-x)||| \leq |||Q_{k_1}(y)||| + |||Q_{k_1}(x)||| \leq 3\epsilon_1 + \epsilon_1 = 4\epsilon_1 < 1,$$

and

$$\begin{aligned} |||P_{k_1}(y-x)||| &\leq |||x-y||| + |||Q_{k_1}(y-x)||| \\ &\leq |||x||| + |||y||| + 4\epsilon_1 \\ &\leq 1 + \epsilon_1 + 4 + 4\epsilon_1 + 4\epsilon_1 = 5 + 9\epsilon_1 < R_0 \end{aligned}$$

for every $y \in A^*(D)$.

Given $k_1 \in \mathbb{N}$ as before, there exists a corresponding function $F(\lambda) := F_{k_1}(\lambda)$ satisfying property (3) in Definition 2.2. Since $\lim_{\lambda\to 0^+} \frac{F(\lambda)}{\lambda} \leq \frac{M}{R_0}$, take $\lambda \in (0,1)$ such that

$$\frac{F(\lambda)}{\lambda} < \frac{M+1}{2R_0} < \frac{1-4\epsilon_1}{R_0} \leq \frac{c - 4\epsilon_1}{R_0},$$

which implies that

$$(2-\lambda)c + F(\lambda)R_0 + \lambda 4\epsilon_1 < 2c.$$

Now, choose $\epsilon_2 > 0$ with

$$(2-\lambda)(c+\epsilon_2) + F(\lambda)R_0 + \lambda 4\epsilon_1 < 2c.$$

Choose $(y_n) \subset D$ an a.f.p.s. with $w^*\text{-}\lim_n y_n = y$ and such that

$$\limsup_n |||y_n - y||| \leq c + \epsilon_2.$$

By passing to a subsequence, we may also suppose that

$$\liminf_n |||y_n - y||| = \limsup_n |||y_n - y||| \geq c \geq 1.$$

Notice that, for every $m \in \mathbb{N}$, the vectors $(1-\lambda)y_m + \lambda x \in D$. We claim that

$$(**) \qquad \limsup_m \limsup_n |||y_n - [(1-\lambda)y_m + \lambda x]\,||| < 2c.$$

Assume that $(**)$ holds. Then we can find some $m \in \mathbb{N}$ such that

$$\limsup_n |||y_n - [(1-\lambda)y_m + \lambda x]\,||| < 2c.$$

This implies that for some $r \in (0, 2c)$ the set

$$G := \{z \in D : \limsup_n |||y_n - z||| \leq r\}$$

is a nonempty closed convex T-invariant subset of D, and therefore it contains an a.f.p.s. (z_s), which tends to some $z \in X$ with respect to the w^*-topology. In this case, using the definition of c, Lemma 2.6, and that each $z_s \in G$, we have

$$2c \leq \limsup_s |||z_s - z||| + \limsup_n |||y_n - y||| \leq \limsup_s \limsup_n |||y_n - z_s||| \leq r\,,$$

which is a contradiction.

We finish by proving the claim $(**)$. Noting that $\liminf_n |||y_n - y||| \geq 1$ and by property (3) in Definition 2.2, we have:

$$\begin{aligned}\limsup_n |||(y_n - y) + \lambda P_{k_1}(y-x)||| &\leq \limsup_n |||y_n - y||| + F(\lambda)|||P_{k_1}(y-x)||| \\ &\leq c + \epsilon_2 + F(\lambda)R_0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_m \limsup_n |||y_n - [(1-\lambda)y_m + \lambda x]\,||| \\ &= \limsup_m \limsup_n |||y_n - y + y - (1-\lambda)y_m - \lambda x||| \\ &= \limsup_m \limsup_n |||y_n - y + (1-\lambda)y + \lambda y - (1-\lambda)y_m - \lambda x||| \\ &\leq \limsup_m \limsup_n [(1-\lambda)|||y_m - y||| + |||(y_n - y) + \lambda(y-x)|||] \\ &\leq (1-\lambda)\limsup_m |||y_m - y||| + \limsup_n |||(y_n - y) + \lambda P_{k_1}(y-x)||| + \lambda|||Q_{k_1}(y-x)||| \\ &\leq (1-\lambda)(c+\epsilon_2) + c + \epsilon_2 + F(\lambda)R_0 + \lambda 4\epsilon_1 \\ &\leq (2-\lambda)(c+\epsilon_2) + F(\lambda)R_0 + \lambda 4\epsilon_1 \\ &< 2c\,,\end{aligned}$$

which proves $(**)$, and completes the proof of the theorem.

□

3. Norms with the Fixed Point Property

Throughout this section, we will study several examples of norms which are near-infinity concentrated norms and therefore they satisfy the FPP according to Theorem 2.7. As a particular case of a more general result, we will deduce that $(\ell_1, \nu_p(\cdot))$ has the FPP whenever $\nu_p(\cdot)$ is a renorming of ℓ_1.

We will start by proving that P.K. Lin's norm is an example of a near-infinity concentrated norm. We will deduce this assertion from the following lemma.

Lemma 3.1. *Let $(X, |\cdot|)$ be a Banach space with a Schauder basis and assume that $|\cdot|$ satisfies properties (1) and (2) in Definition 2.1. If (γ_k) is a nondecreasing sequence in $(0,1)$ converging to 1, then the norm defined as*

$$|x|_1 := \sup_k \gamma_k |Q_k(x)|\ , \ \textit{for all } x \in X\ ,$$

is a near-infinity concentrated norm on X that is equivalent to $|\cdot|$.

Proof. Notice that $\gamma_k|Q_k(x)| \leq |Q_k(x)|_1 \leq |Q_k(x)|$ for every $k \in \mathbb{N}$ and $x \in X$, which implies that $|\cdot|_1$ is a sequentially separating norm whenever $|\cdot|$ satisfies the same property. It is also easy to check that $|\cdot|_1$ satisfies (2) in Definition 2.2. It remains to prove condition (3). Fix some $k \in \mathbb{N}$ and $R > 0$. Let (x_n) be a bounded pointwise-null sequence in X with $\liminf_n |x|_1 \geq 1$. Without loss of generality we can assume that $Q_l(x_n) = Q_k(x_n)$ for every $l \leq k$. Moreover, it is not difficult to check that $\limsup_n |x_n| = \limsup_n |x_n|_1$. For every $z \in X$ with $Q_k(z) = 0$, $|z|_1 \leq R$ and for every $\lambda > 0$ we have

$$\begin{aligned} |x_n + \lambda z|_1 &= \sup_l \gamma_l |Q_l(x_n + \lambda z)| = \sup_l \gamma_l |Q_l(x_n) + \lambda Q_l(z)| \\ &= \max\left\{ \max_{1\leq l\leq k-1} \gamma_l |Q_l(x_n) + \lambda Q_l(z)|, \sup_{l\geq k} \gamma_l |Q_l(x_n)| \right\} \\ &= \max\left\{ \max_{1\leq l\leq k-1} \gamma_l |Q_l(x_n) + \lambda Q_l(z)|, |x_n|_1 \right\} \\ &\leq \max\{\gamma_{k-1}|x_n| + \lambda |z|_1, |x_n|_1\} \end{aligned}$$

Taking limits when n goes to infinity:

$$\limsup_n |x_n + \lambda z|_1 \leq \max\{\gamma_{k-1} \limsup_n |x_n|_1 + \lambda |z|_1, \limsup_n |x_n|_1\}.$$

From above, $\limsup_n |x_n|_1 \geq 1$ and $|z|_1 \leq R$; and so

$$\limsup_n |x_n + \lambda z|_1 \leq \limsup_n |x_n|_1 + F_k(\lambda)|z|_1 \ ,$$

where $F_k(\lambda) := 0$ if $\lambda \leq (1-\gamma_{k-1})/R$, and $F_k(\lambda) := \lambda$ otherwise. Taking $M = 0$ and any $R > 5$ in Definition 2.2(3), we see that $|\cdot|_1$ is a near infinity concentrated norm. □

If we let $|\cdot| := \|\cdot\|_1$ in ℓ_1 we obtain that $|\cdot|_1$ coincides with P.K. Lin's norm $|||\cdot|||_L$. Moreover, given a norm $|\cdot|_0 := |\cdot|$ satisfying (1) and (2) in Definition 2.2 and defining in a recursive way the equivalent norms

$$|\cdot|_n = \sup_k \gamma_k |Q_k(\cdot)|_{n-1}$$

for every $n \in \mathbb{N}$, we can construct sequences of near-infinity concentrated norms. All of these norms $|\cdot|_n$ $(n \geq 1)$ satisfy the FPP when the basis is boundedly complete, according to Theorem 2.7.

Lemma 3.2. *Assume that $(p_n) \subset (1, +\infty)$ is a nonincreasing sequence with $\lim_n p_n = 1$. Then the norm $\nu_p(\cdot)$ is a near-infinity concentrated norm in the Banach space X, defined as the completion of c_{00} with the norm $\nu_p(\cdot)$.*

Proof. Let us start by proving that $\nu_p(\cdot)$ is a sequentially separating norm, that is, $\nu_p(\cdot)$ satisfies property (1) in Definition 2.2. By Lemma 2.5, it suffices to check that $\lim_k S_k(X, \nu_p(\cdot)) = 1$.

Fix $k \in \mathbb{N}$. First note that if $x = \sum_{i=k}^{l} x(i)e_i$ with $k \leq l$ and $l < y$ then

$$\nu_p(x+y) = \nu_p\left(\sum_{i=k}^{l} x(i)e_i + \nu_p(y)e_{l+1}\right) .$$

On the other hand, it is not difficult to check that if $1 < p \leq q$ and $a, b, c \geq 0$ then:

$$\|(a, \|(b,c)\|_p)\|_q \geq \|(a, \|(b,c)\|_q)\|_q = \left(a^q + \|(b,c)\|_q^q\right)^{1/q} = (a^q + b^q + c^q)^{1/q} = \\ = \|(\|(a,b)\|_q, c)\|_q \, .$$

Using now the definition of the norm $\nu_p(\cdot)$ and taking into account that

$$\cdots p_n \leq p_{n-1} \leq \cdots p_2 \leq p_1,$$

it can recursively be checked that for $x = \sum_{i=k}^{l} x(i)e_i$ with $k \leq l$ and $l < y$

$$\begin{aligned} \nu_p(x+y) &= \nu_p\left(\textstyle\sum_{i=k}^{l} x(i)e_i + \nu_p(y)e_{l+1}\right) \\ &\geq \|(\nu_p(x), \nu_p(y))\|_{p_k} \\ &\geq C_k(\nu_p(x) + \nu_p(y)) \qquad (\dagger) \end{aligned}$$

where $C_k := 2^{-1+1/p_k}$.

Let (x_n) be a block basic sequence in X and let $x \in X$ with $Q_k(x) = x$. If $\epsilon > 0$ is given, choose $k' > k$ such that

$$\nu_p(Q_{k'}(x)) < \epsilon\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right).$$

Without loss of generality, we can choose n large enough so that $k' < x_n$. Applying ($\dagger$) to the vectors $x_0 := x - Q_{k'}(x)$, $x_1 := Q_{k'}(x)$, and x_n, we obtain:

$$\begin{aligned} \limsup_n \nu_p(x + x_n) &\geq \limsup_n \nu_p(x_0 + x_n) - \nu_p(x_1) \\ &\geq \limsup_n C_k\left(\nu_p(x_0) + \nu_p(x_n)\right) - \nu_p(x_1) \\ &\geq C_k\left(\nu_p(x) - \nu_p(x_1) + \limsup_n \nu_p(x_n)\right) - \nu_p(x_1) \\ &= C_k\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) - (C_k+1)\nu_p(x_1) \\ &\geq C_k\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) - (C_k+1)\epsilon\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) \\ &= \left(C_k - \epsilon(C_k+1)\right)\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup_n \nu_p(x + x_n) \geq C_k\left(\nu_p(x) + \limsup_n \nu_p(x_n)\right).$$

Then, by the definition of the coefficient $S_k(X, \nu_p(\cdot))$, we deduce that

$$S_k(X, \nu_p(\cdot)) \leq \frac{1}{C_k}.$$

Taking limits as k goes to infinity and using Lemma 2.5 shows that $\nu_p(\cdot)$ is a sequentially separating norm.

It is clear that $\nu_p(\cdot)$ satisfies (2) in Definition 2.2. Therefore, it remains to prove (3) in Definition 2.2.

Fix $k \in \mathbb{N}$. Consider the equivalent finite dimensional Banach spaces $(\mathbb{R}^k, \|\cdot\|_{p_k})$ and $(\mathbb{R}^k, \nu_p(\cdot))$. Take some constant $L_k > 0$ such that $\|x\|_{p_k} \leq L_k \nu_p(x)$ for every $x \in \mathbb{R}^k$.

Let (x_n) be a bounded pointwise-null sequence with $\liminf_n \nu_p(x_n) \geq 1$, and $z \in X$ with $Q_k(z) = 0$ and $\nu_p(z) \leq R$ for some $R > 0$. We can assume, without loss of generality, that $P_k(x_n) = 0$ for every $n \in \mathbb{N}$.

Having in mind that $(p_n)_n$ is a nonincreasing sequence, it is not difficult to check that for all $\lambda \in (0, +\infty)$,

$$\begin{aligned} \nu_p(x_n + \lambda z) &\leq \left(\lambda^{p_k}|z(1)|^{p_k} + \cdots \lambda^{p_k}|z(k-1)|^{p_k} + \nu_p(x_n)^{p_k}\right)^{1/p_k} \\ &= \nu_p(x_n)\left[\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k}}\left(|z(1)|^{p_k} + \cdots |z(k-1)|^{p_k}\right) + 1\right]^{1/p_k} \\ &= \nu_p(x_n)\left[\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k}}\|z\|_{p_k}^{p_k} + 1\right]^{1/p_k}. \end{aligned}$$

It is easy to check that $(1+v)^\alpha \le 1+\alpha v$ if $0<v$ and $0<\alpha<1$. Therefore

$$\begin{aligned}\nu_p(x_n+\lambda z) &\le \nu_p(x_n)\left[\frac{1}{p_k}\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k}}\|z\|_{p_k}^{p_k}+1\right]\\ &= \nu_p(x_n)+\frac{1}{p_k}\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k-1}}\|z\|_{p_k}^{p_k}\\ &\le \nu_p(x_n)+\frac{1}{p_k}\frac{\lambda^{p_k}}{\nu_p(x_n)^{p_k-1}}L_k^{p_k}\nu_p(z)^{p_k}.\end{aligned}$$

Then, for every $\lambda>0$,

$$\begin{aligned}\limsup_n \nu_p(x_n+\lambda z) &\le \limsup_n \nu_p(x_n)+\frac{1}{p_k}\frac{\lambda^{p_k}}{\liminf_n \nu_p(x_n)^{p_k-1}}L_k^{p_k}\nu_p(z)^{p_k-1}\nu_p(z)\\ &\le \limsup_n \nu_p(x_n)+\frac{1}{p_k}\lambda^{p_k}L_k^{p_k}R^{p_k-1}\nu_p(z).\end{aligned}$$

Define

$$F_k(\lambda) := \frac{1}{p_k}\lambda^{p_k}L_k^{p_k}R^{p_k-1}\ ,$$

for every $\lambda>0$. Since, $\lim_{\lambda\to 0^+}F_k(\lambda)/\lambda = 0$, we can take $M=0$ in property (3) of Definition 2.2 and $\nu_p(\cdot)$ is a near-infinity concentrated norm whenever $\lim_n p_n = 1$. □

Corollary 3.3. *For every nonincreasing sequence $(p_n)\subset(1,+\infty)$ with $\lim_n p_n=1$, the Banach space $(X,\nu_p(\cdot))$ satisfies the FPP. In particular, if the sequence of conjugates of (p_n) satisfies that $q_n \ge \delta\log n$ for all $n\in\mathbb{N}$ and some $\delta>0$, the norm $\nu_p(\cdot)$ is an equivalent norm in ℓ_1 with the FPP.*

Remark 3.4. We would like to point out that there exist some norms verifying the FPP which are not near-infinity concentrated. For instance we can consider the Banach space ℓ_1 and the equivalent norm $|x|_1 = |||P_A(x)|||_L + |||P_B(x)|||_L$ where $A=\{2n : n\ge 1\}$, $B=\{2n-1 : n\ge 1\}$, P_A, P_B denote the corresponding projections onto the subspaces $[e_n : n\in A]$ and $[e_n : n\in B]$ respectively and $|||\cdot|||_L$ denotes P.K. Lin's norm as usual. It is easy to check that $(\ell_1,|\cdot|_1)$ is isometric to the space $(\ell_1,|||\cdot|||_L)\oplus_1(\ell_1,|||\cdot|||_L)$, which has the FPP according to [6]. However, $|\cdot|_1$ does not satisfies condition (3) in Definition 2.2.

A similar situation occurs when we consider the ℓ_1-renorming $|x|_1 = \|x\|_1 + |||x|||_L$. From [10] we know that $(\ell_1, \|x\|_1 + |||x|||_L)$ verifies the FPP. However, condition (3) in Definition 2.2 also fails in this example.

Nevertheless, by means of Theorem 2.7, we can obtain a stability result in the following sense.

Theorem 3.5. *Let $|||\cdot|||$ be a near-infinity concentrated norm on a Banach space X with a Schauder basis $\{e_n\}_n$, and let $\|\cdot\|$ be an equivalent norm satisfying conditions (1) and (2) in Definition 2.2. Then there exists $r_0>0$ such that the norm $|\cdot| = |||\cdot||| + r\|\cdot\|$ is also near-infinity concentrated for every $0\le r\le r_0$.*

Moreover, if $\{e_n\}_n$ is boundedly complete, then the spaces $(X,|\cdot|)$ have the FPP for every $0\le r\le r_0$.

Proof. Since $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms, there exist $0<a\le b$ such that $a|||x|||\le\|x\|\le b|||x|||$ for every $x\in X$. Therefore, for all $x\in X$,

$$(1+ar)|||x|||\le|x|\le(1+rb)|||x||| \text{ and } \left(\frac{1}{b}+\frac{ra}{b}\right)\|x\|\le|x|\le\left(\frac{1}{a}+\frac{rb}{a}\right)\|x\|\,.$$

It is easy to check that $|\cdot|$ satisfies conditions (1) and (2). Let us prove (3). By hypotheses, there exists some $R_0>5$ and $M\in[0,1)$ satisfying Definition 2.2.3 for the $|||\cdot|||$ norm. Take some $M'\in(M,1)$. Let (x_n) be a bounded pointwise-null sequence with $\liminf_n |x_n|\ge 1$ and let $z\in X$ with $Q_k(z)=0$ and $|z|\le R_0$. In this case, $\liminf_n |||x_n||| \geq 1/(1+rb)$ and $|||z||| \leq R_0/(1+ar)$. Fix $\lambda \in (0,\infty)$. Then there exists a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\Gamma := \limsup_n |x_n + \lambda z| = \lim_j |x_{n_j} + \lambda z| .$$

Taking further subsequences if necessary, we may assume without loss of generality, that the limits

$$\lim_j |||x_{n_j}||| \ , \ \lim_j \|x_{n_j}\| \ , \ \lim_j |||x_{n_j} + \lambda z||| \text{ and } \lim_j \|x_{n_j} + \lambda z\|$$

all exist in $[0,\infty)$. Therefore,

$$\begin{aligned}
\Gamma &= r\lim_j \|x_{n_j} + \lambda z\| + \lim_j |||x_{n_j} + \lambda z||| \\
&\leq r\lim_j \|x_{n_j}\| + \lambda r\|z\| \\
&\quad + \frac{1}{1+rb}\lim_j |||(1+rb)x_{n_j} + \lambda\frac{1+rb}{1+ar}(1+ar)z||| \\
&\leq r\lim_j \|x_{n_j}\| + \lambda r\|z\| \\
&\quad + \limsup_j |||x_{n_j}||| + F_k\left(\frac{\lambda(1+rb)}{1+ar}\right)\frac{1+ar}{1+rb}|||z||| \\
&= \lim_j |x_{n_j}| + \lambda r\|z\| + F_k\left(\frac{\lambda(1+rb)}{1+ar}\right)\frac{1+ar}{1+rb}|||z||| \\
&\leq \limsup_n |x_n| + \lambda\frac{r}{(\frac{1}{b}+\frac{ra}{b})}|z| + G(\lambda)\frac{1}{1+ar}|z| \\
&= \limsup_n |x_n| + \left[\lambda\frac{r}{(\frac{1}{b}+\frac{ra}{b})} + G(\lambda)\frac{1}{1+ar}\right]|z|,
\end{aligned}$$

where $G(\lambda) := F_k\left(\lambda\frac{(1+rb)}{(1+ar)}\right)\left(\frac{1+rb}{1+ar}\right)^{-1}$. Define the corresponding "F_k-type" function for the $|\cdot|$ norm by

$$F_k'(\lambda) := \lambda\frac{r}{(\frac{1}{b}+\frac{ra}{b})} + G(\lambda)\frac{1}{1+ar} \ , \ \text{for all } \lambda \in (0,\infty) .$$

Since $\lim_{\lambda\to 0^+} G(\lambda)/\lambda \leq \frac{M}{R_0}$, we have that

$$\lim_{\lambda\to 0^+}\frac{F_k'(\lambda)}{\lambda} \leq \frac{r}{(\frac{1}{b}+\frac{ra}{b})} + \frac{1}{1+ar}\frac{M}{R_0} := g(r)$$

Notice that $\lim_{r\to 0} g(r) = M/R_0$, which implies that there exists some $r_0 > 0$ and $M' \in (M,1)$ (depending on the constants a, b) such that for every $0 \leq r \leq r_0$, $g(r) \leq M'/R_0$; and so $|||\cdot||| + r\|\cdot\|$ is a near-infinity concentrated norm.

The rest of the theorem follows by applying Theorem 2.7. □

If we proceed as in the previous proof, using the above arguments and the fact that $\lim_{\lambda\to 0^+}\frac{F_k(\lambda)}{\lambda} = 0$ for both P.K. Lin's norm and the $\nu_p(\cdot)$ norm, it is not difficult to check that, in the case where $\nu_p(\cdot)$ is equivalent to the $\|\cdot\|_1$-norm, $\nu_p(\cdot) + \lambda|||\cdot|||_L$ is a renorming in ℓ_1 which is also near-infinity concentrated for every $\lambda > 0$. Therefore we can also deduce (see also [2, Section 4]):

Corollary 3.6. *Let (p_n) be a nonincreasing sequence in $(1,+\infty)$ such that there exists some $\delta > 0$ so that $q_n \geq \delta \log n$ for all $n \in \mathbb{N}$. Then $(\ell_1, \nu_p(\cdot) + \lambda|||\cdot|||_L)$ has the FPP for every $\lambda \geq 0$.*

Acknowledgment: The authors would like to thank two anonymous reviewers for their constructive and valuable comments, which helped us to improve the quality of this manuscript.

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