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Optimality and Duality on Riemannian Manifolds

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Abstract. Our goal in this paper is to translate results on function classes that are characterized by the property that all the Karush-Kuhn-Tucker points are efficient solutions, obtained in Euclidean spaces to Riemannian manifolds. We give two new characterizations, one for the scalar case and another for the vectorial case, unknown in this subject literature. We also obtain duality results and give examples to illustrate it.

1. Introduction

At the beginning of the 19th century, the non-Euclidean geometries arose having as a main characteristic the replacement of straight lines by geodesics. The minimization of functions on a Riemannian manifold is, at least locally, equivalent to the smoothly constrained optimization problem on a Euclidean space, due to the fact that every C^∞ Riemannian manifold can be isometrically imbedded in a Euclidean space. It is well known that solving the nonconvex constrained problem in $\mathbb{R}^n$ with the Euclidean metric is equivalent to solve the unconstrained convex minimization problem in the Hadamard manifold feasible set with the affine metric (see [9]). However, the Euclidean dimension space may be larger than the manifold dimension making this approach not convenient.

There is a considerable number of optimization problems which cannot be solved in linear spaces and require of Hadamard manifolds structures for their formalization and study. For example, in controlled thermonuclear fusion research (see [1, 30]) and in engineering (see [18, 32, 33]). Similarly, other manifolds can be found in very different research fields. Such is the case of Essential manifolds in stereo vision processing [20] and Stiefel manifolds $\mathrm{St}(p,n) = \{X \in \mathbb{R}^{n\times p} : X^T X = I\}$ in the machine learning study or computer vision (see [23, 29]).

Moreover, there exist algorithms to detect human shapes in still images that make use of covariance matrices as object descriptors. These descriptors do not lie on a traditional vector space and Riemannian manifolds are needed for their study. For example, in the

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field of medicine, Riemannian manifolds have been used in analysis of medical images as it has been shown by Fletcher et al. [12].

In economics, characterizations, existence, and stability of Nash-Stampacchia equilibria are studied using strategy sets based on geodesic convex subsets of Hadamard manifolds taking advantage of the geometrical features of these spaces as shown by Kristály [17].

The optimization problems study is usually linked to the duality problems study. It can be the case that solving the dual problem is easier than solving the original primal problem from both the theoretical and the practical point of view.

In convex optimization, the convexity of a set in a linear space is based upon the possibility of connecting any two points of the space. The traditional approach to this problem has been using line segments. However, various generalizations of this procedure have been proposed (see [6,10,13]). In this paper, we propose unique a generalization that extends the linear space definition to Riemannian manifolds, substituting line segments by geodesic arcs. The aim of this generalization is to extend results of convex optimization theory to Riemannian manifolds. Our study focuses on extending concepts such as critical point, optimal point and the relationships between them on Euclidean spaces to Riemannian manifolds. A significant generalization of the convex functions are the invex functions, introduced by Hanson [13]. The invexity concept is an extension of differentiable convexity. A scalar function is invex if and only if every critical point is a global minimum solution. Therefore, it can also be said that a scalar function is invex if it has no critical points. The conditions for optimality that invexity involves are essential to obtain optimal points through practical numerical methods.

But the objective function invexity is not sufficient in constrained scalar mathematical programming problem to ensure that a Karush-Kuhn-Tucker point is an optimum. The KT-invexity notion is necessary. So our idea is extending, amongst others, the kind of KT-invex functions introduced by Martin [22], as well as his results. In Osuna-Gómez et al. [24, 25] the authors extended these results in both constrained and unconstrained vector cases.

Rapcsák [27] and Udriste [30] extended some results on convex optimization problems to Riemannian manifolds, i.e., nonlinear spaces and considered a convexity generalization called geodesic convexity.

Pini [26] defined the invexity concept for maps on a general smooth manifold in a natural way. In particular, Pini focused the attention on a property of the map η giving rise to a family of "integrable" fields on the manifold and providing an interesting relationship between invexity and convexity along particular curves of the surface.

Some relationships between generalized invexity and generalized invex monotonicity were given in Ruiz-Garzón et al. [28] under certain conditions in Euclidean spaces. Wang et al. [31] established the equivalence between strong convexity functions and strong monotonicity of its subdifferentials on Riemannian manifolds and these results were applied to solve the minimization of convex functions on Riemannian manifolds. In Barani and Pouryayevali [3, 4] several invexity notions for functions on Riemannian manifolds are defined and their relations with invariant monotone vector fields are studied.

In Zhou and Huang [35] a new class of quasi roughly geodesic B-invex functions and pseudo roughly geodesic B-invex functions are introduced and sufficient and necessary conditions for optimal solution of the nonlinear programming problems are given on Hadamard manifolds. Agarwal et al. [2] defined the concepts of geodesic α-invex set and geodesic α-preinvex functions on Riemannian manifold and using suitable conditions, some relations between geodesic α-invex set and geodesic α-preinvex function are established.

Bento and Cruz [7] studied this unconstrained problem in the Riemannian context and under the convexity of the vectorial function they prove that a critical point is a weak Pareto solution. In Zhou and Huang [34] the authors gave an existence theorem of weak minimum for a constrained vector optimization problem by KKM lemma on a Hadamard manifold.

Colao et al. [9] studied the equilibrium problem in Hadamard manifolds.

Weak sharp minima for constrained optimization problems and some other algorithm on Riemannian manifolds have been proposed by Li et al. [19].

Later, Hosseini and Pouryayevali [14] obtained necessary optimality conditions for a general minimization problem on complete Riemannian manifolds but they don't obtain characterization theorems. Chen et al. [8] discussed a multiobjective optimization problem involving generalized invex functions and obtained the Kuhn-Tucker sufficient conditions for a feasible point of vector optimization problem to be an efficient or properly efficient solution and gave various types of duality results.

The aim of our work is to present necessary and sufficient optimality conditions and extend the results obtained by Hosseini and Pouryayevali [14] and Chen et al. [8]. The outline of this work is for the scalar case to obtain classes of functions such that any class of functions which is characterized by having every critical point as an optimum solution must be equivalent to these classes of functions and to extend this result to constrained multiobjective problems.

In Section 2, we recall concepts related to Riemannian manifolds and extend the critical point concept and the characterization theorem invex functions for the Euclidean dimensional finite space to Riemmannian manifolds. In Section 3 we extend the characterization of invex functions theorem in Riemmannian manifolds to vectorial case. We prove that every vector critical solution is an (weak) efficient solution if and only if (f, g) is (Weak) KKT-pseudoinvex, respectively, where f and g are involved functions for a constrained multiobjective problem. In Section 4 we study duality theorems for Mond-Weir type Dual Problems. Our new results extend and generalize the known results in the literature.

2. Preliminaries and scalar case

Let M be a C^∞-manifold modeled on a Hilbert space H, either finite or infinite dimensional, endowed with a Riemannian metric g_x on a tangent space T_xM. The corresponding norm is denoted by $\|\cdot\|_x$ and the length of a piecewise C^1 curve $\alpha\colon [a,b] \to M$ is defined by

$$L(\alpha) = \int_a^b \|\alpha'(t)\|_{\alpha(t)}\, dt.$$

For any point $x, y \in M$, we define

$$d(x,y) = \inf\{L(\alpha) \mid \alpha \text{ is a piecewise } C^1 \text{ curve joining } x \text{ and } y\}$$

then d is a distance which induces the original topology on M. Any Riemannian manifold (M,g) can be converted into a metric space (M,d), where d is the distance induced by the Riemannian metric g.

We know that on every Riemannian manifold exists exactly one covariant derivative called a Levi Civita connection, denoted by $\nabla_X Y$, for any vector fields $X, Y \in TM$; we also recall that a geodesic is a C^∞-smooth path α whose tangent is parallel along the path α, that is, α satisfies the equation

$$\nabla_{d\alpha(t)/dt} d\alpha(t)/dt = 0.$$

Any path α joining x and y in M such that $L(\alpha) = d(x,y)$ is a geodesic and is called a minimal geodesic. The existence theorem for ordinary differential equation implies that for every $V \in TM$, there is an open interval $J(V)$ containing 0 and exactly one geodesic $\alpha_V\colon J(V) \to M$ with $d\alpha_V(0)/dt = V$. For differentiable manifolds, it is possible to define the derivatives of the curves on the manifold. The derivatives at a point x on the manifold lies in a vector space T_xM. We denote by T_xM the n-dimensional tangent space of M at x, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, by $\overline{T}M$ an open neighborhood of the submanifold M of TM such that for every $\exp\colon \overline{T}M \to M$ defined as $\exp_x(V) = \alpha_V(1,x)$ for every $V \in \overline{T}M$, where α_V is the geodesic starting at x with velocity V (i.e., $\alpha(0) = x$, $\alpha'(0) = V$) (see [8, 15, 16]).

If h is a smooth map from the manifold M to the manifold N, we shall denote by dh_x

$$dh_x\colon T_x(M) \to T_{h(x)}(N)$$

the differential map of h at x. The gradient of a real-valued C^∞ function h on M in x, denoted by $\operatorname{grad} h_x$, is the unique vector in $T_x(M)$ such that $dh_x(X) = \langle \operatorname{grad} h_x, X\rangle$ for all X in $T_x(M)$.

Assume now that h is a real differentiable map defined on a manifold M, and η is a map $\eta\colon M \times M \to TM$ defined on the product manifold and such that

$$\eta(x, y) \in T_y(M), \quad \forall\, x, y \in M.$$

Notice that the function $\eta(x, \cdot\,)$ assigns to each point $y \in M$ a tangent vector V_y to M at y so that $\eta(x, \cdot\,)$ is a vector field on M, for each $x \in M$. Intuitively, $\{\eta(x, y)\}|_{y\in M}$ gives a collection of arrows on M (x fixed). In particular, if $\eta(x, y)$ is smooth in the variable y, then $\eta(x, \cdot\,)$ is a smooth vector field on M, for each $x \in M$.

We also recall that a simply connected complete Riemannian manifold of non-positive sectional curvature is called a Cartan-Hadamard manifold. If we consider M to be a Cartan-Hadamard manifold (either infinite or finite dimensional), then on M there is a map playing the role of $x - y \in \mathbb{R}^n$. We can define the function η as $\eta(x, y) = \alpha'_{x,y}(0)$ for all $x, y \in M$. Here $\alpha_{x,y}$ is the unique minimal geodesic joining y to x as follows

$$\alpha_{x,y} = \exp_y(t \exp_y^{-1} x), \quad \forall\, t \in [x, y].$$

Definition 2.1. [30] A subset X of M is called totally convex if X contains every geodesic α_{xy} of M whose endpoints x and y belong to X.

Now, we can define the invexity of a function h on a totally convex subset of a Riemannian manifold:

Definition 2.2. Let M be a Riemannian manifold, X be an open totally convex subset of M and $\theta\colon X \subseteq M \to \mathbb{R}$ be a differentiable map. We say that the function θ is invex (IX) on X if there exists a non necessarily differentiable function $\eta\colon M \times M \to TM$ such that

$$\theta(x) - \theta(y) \geq d\theta_y(\eta(x, y)), \quad \forall\, x, y \in X.$$

If $M = \mathbb{R}^n$ we have the usual invexity definition given by Hanson (see [13]). If $M = \mathbb{R}^n$ then $\exp_y^{-1} x = x - y$.

Let us consider the unconstrained scalar optimization problem (SOP)

$$\text{(SOP)} \qquad \min \theta(x), \quad x \in X \subseteq M.$$

Definition 2.3. Let $\eta\colon M \times M \to TM$ be a non necessarily differentiable function. A feasible point $\overline{x}$ for (SOP) is said to be a critical point (CP) with respect to η, if exists some $x \in X \subseteq M$ with $Q = \eta(x, \overline{x}) \in T_{\overline{x}}M$ non identically zero such that

$$d\theta_{\overline{x}}(Q) = 0.$$

Theorem 2.4. *Let $\theta\colon X \subseteq M \to \mathbb{R}$ be a differentiable map. A function θ is invex with respect to $\eta\colon M \times M \to TM$ if and only if every critical point (CP) with respect to this same η is a global solution.*

Proof. Let θ be an invex function (IX) on X, then there exists a non necessarily differentiable function $\eta\colon M \times M \to TM$ such that

$$\theta(x) - \theta(\overline{x}) \geq d\theta_{\overline{x}}(\eta(x,\overline{x})), \quad \forall\, x, \overline{x} \in X. \tag{2.1}$$

If $\overline{x}$ is a (CP) with respect to the same function η of which θ function is invex then

$$d\theta_{\overline{x}}(\eta(x,\overline{x})) = 0 \tag{2.2}$$

for all critical point $\overline{x}$.

By (2.1) and (2.2) we have that

$$\theta(x) - \theta(\overline{x}) \geq d\theta_{\overline{x}}(\eta(x,\overline{x})) = 0, \quad \forall\, x \in X$$

so that $\overline{x}$ is a global minimum solution.

Let's now prove the sufficient condition. We have to prove that there exists a non necessarily differentiable function $\eta\colon M \times M \to TM$ such that

$$\theta(x) - \theta(\overline{x}) \geq d\theta_{\overline{x}}(\eta(x,\overline{x})), \quad \forall\, x, \overline{x} \in X. \tag{2.3}$$

If every (CP) $\overline{x}$ is not a global minimum then there exists $\eta(x,\overline{x})$ defined as

$$\eta(x,\overline{x}) = \begin{cases} 0 & \text{if } \operatorname{grad}\theta_{\overline{x}} = 0, \\ \frac{\theta(x)-\theta(\overline{x})}{\|\operatorname{grad}\theta_{\overline{x}}\|_{\overline{x}}^2}\operatorname{grad}\theta_{\overline{x}} & \text{if } \operatorname{grad}\theta_{\overline{x}} \neq 0 \end{cases}$$

such that (2.3) holds.

Otherwise, if every (CP) is a global minimum then (2.3) is obvious. □

In sum up, we have that

$$\text{IX} \Leftrightarrow [\text{CP} \Leftrightarrow \text{optimum (SOP)}].$$

The previous result generalizes invex functions in the scalar case obtained by Craven and Glover [10] to Riemannian manifolds.

Example 2.5. Let us consider the set $\mathrm{Sym}_2(\mathbb{R})$ of symmetric 2×2 matrices endowed with the Frobenius metric $k_X(U,V) = \operatorname{trace}(UV)$ where $X \in \mathrm{Pos}_2(\mathbb{R})$ and $U, V \in T_X(\mathrm{Pos}_2(\mathbb{R})) = \mathrm{Sym}_2(\mathbb{R})$. Let $\mathrm{Pos}_2(\mathbb{R})$ be the set of all 2×2 positive definite matrices and $(\mathrm{Pos}_2(\mathbb{R}), k)$ is a Hadamard manifold. Consider the following problem on $\mathrm{Pos}_2(\mathbb{R})$:
(SOP)

$$\min\theta(X) = x_1 \quad \text{subject to} \quad M = \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in \mathrm{Pos}_2(\mathbb{R}) \text{ such that } x_1 \geq 1 \right\}.$$

Given $\overline{X} = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right)$ using the Riemannian metric k and $\eta(X, \overline{X}) = X - \overline{X}$, θ is trivially an invex function, $X = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 6 \end{smallmatrix}\right)$ and at $Q = \eta(X, \overline{X}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have that

$$d\theta_{(\overline{X})}(Q) = \operatorname{trace}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \operatorname{trace}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

Thus $\overline{X}$ is a (CP) with respect to previous η and therefore $\overline{X}$ is an optimum solution.

Also, the next aim of this work is to extend this scalar result to the vector case.

3. Characterization of efficient solutions set: vector case

The following convention for equalities and inequalities will be used.

If $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then

$$\begin{aligned} x = y &\Leftrightarrow x_i = y_i, \quad \forall\, i = 1, \ldots, n, \\ x < y &\Leftrightarrow x_i < y_i, \quad \forall\, i = 1, \ldots, n, \\ x \leqq y &\Leftrightarrow x_i \leq y_i, \quad \forall\, i = 1, \ldots, n, \\ x \leq y &\Leftrightarrow x \leqq y, \quad \text{there is } i \text{ such that } x_i < y_i. \end{aligned}$$

In this section we consider the constrained multiobjective programming (CVOP) defined as

$$\text{(CVOP)} \qquad \min f(x), \quad g(x) \leqq 0, \quad x \in X \subseteq M$$

where $f = (f_1, \ldots, f_p)\colon X \subseteq M \to \mathbb{R}^p$, with $f_i\colon X \subseteq M \to \mathbb{R}$ for all $i : 1, \ldots, p$, $g = (g_1, \ldots, g_m)\colon X \subseteq M \to \mathbb{R}^m$ are differentiable functions on the open set $X \subseteq M$ and let M be a Riemannian manifold. And we denote by $df_x = (df_{1(x)}, \ldots, df_{p(x)})$ where $df_{i(x)}\colon T_xM \to Tf_{i(x)}\mathbb{R} \equiv \mathbb{R}$ for all $i : 1, \ldots, p$.

We focus on the study and location of efficient solutions for (CVOP).

Definition 3.1. A feasible point $\overline{x}$ is said to be an efficient solution for (CVOP) if does not exist another feasible point x, such that $f(x) \leq f(\overline{x})$.

Definition 3.2. A feasible point $\overline{x}$ is said to be a weakly efficient solution for (CVOP) if there does not exist another feasible point x, such that $f(x) < f(\overline{x})$.

Just as happens in the scalar case, we are going to use Karush-Kuhn-Tucker vector critical points (VCP) as optimality condition, as we shall define bellow.

Definition 3.3. Let $\eta\colon M \times M \to TM$ be a non necessarily differentiable function. A feasible point $\overline{x}$ for (CVOP) is said to be a Karush-Kuhn-Tucker vector critical point

(VCP) with respect to η, if there exists some $x \in X \subseteq M$, $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$ with $Q = \eta(x, \overline{x}) \in T_{\overline{x}}M$ non identically zero such that

$$\lambda^T df_{\overline{x}}(Q) + \mu^T dg_{\overline{x}}(Q) = 0, \quad \mu^T g(\overline{x}) = 0, \quad \mu \geqq 0, \quad \lambda \geq 0.$$

A new type of invex function that involves the objective and constraints function is needed in order to study the efficient solutions for (CVOP), using the KKT vector points.

Definition 3.4. The pair of functions (f, g) is said to be KKT-pseudoinvex if there exists a non necessarily differentiable function $\eta\colon M \times M \to TM$ such that

$$f(x) - f(\overline{x}) \leq 0 \quad \Longrightarrow \quad \begin{cases} df_{\overline{x}}(\eta(x, \overline{x})) < 0, \\ dg_{j(\overline{x})}(\eta(x, \overline{x})) \leq 0, \quad \forall\, j \in I(\overline{x}) \end{cases}$$

for all feasible points x, $\overline{x}$ for (CVOP), where $I(\overline{x}) = \{j = 1, \ldots, m : g_j(\overline{x}) = 0\}$.

Definition 3.5. The pair of functions (f, g) is said to be Weak KKT-pseudoinvex if there exists a non necessarily differentiable function $\eta\colon M \times M \to TM$ such that

$$f(x) - f(\overline{x}) < 0 \quad \Longrightarrow \quad \begin{cases} df_{\overline{x}}(\eta(x, \overline{x})) < 0, \\ dg_{j(\overline{x})}(\eta(x, \overline{x})) \leq 0, \quad \forall\, j \in I(\overline{x}) \end{cases}$$

for all feasible points x, $\overline{x}$ for (CVOP), where $I(\overline{x}) = \{j = 1, \ldots, m : g_j(\overline{x}) = 0\}$.

Remark 3.6. The Weak KKT-pseudoinvex coincides with the KT-pseudoinvex defined by the authors in Osuna-Gómez et al. [24, 25]. If $g = 0$ we obtain the pseudoinvexity defined by Osuna-Gómez et al. [24, 25] for unconstrained vectorial optimization problems.

Theorem 3.7. *(f, g) is a pair KKT-pseudoinvex functions if and only if every Karush-Kuhn-Tucker vector critical point (VCP) with respect to the same η is an efficient solution for (CVOP).*

Proof. Let (f, g) be a pair KKT-pseudoinvex functions and $\overline{x}$ be a Karush-Kuhn-Tucker vector critical point (VCP) with respect to the same η. We have to prove that $\overline{x}$ is an efficient solution for (CVOP). Let us argue by contradiction. Suppose $\overline{x}$ is not an efficient solution. Then there exists a feasible solution x such that

$$f(x) - f(\overline{x}) \leq 0.$$

Since (f, g) is KKT-pseudoinvex then there exists a non necessarily differentiable function $\eta\colon M \times M \to TM$ such that

$$df_{\overline{x}}(\eta(x, \overline{x})) < 0, \quad dg_{I(\overline{x})}(\eta(x, \overline{x})) \leqq 0, \quad I = I(\overline{x}) \tag{3.1}$$

for all feasible points x, $\overline{x}$ for (CVOP).

Since $\overline{\lambda} \geq 0$, $\overline{\mu}_I \geqq 0$ and from (3.1), it follows

$$\overline{\lambda}^T df_{\overline{x}}(\eta(x,\overline{x})) + \overline{\mu}_I^T dg_{I(\overline{x})}(\eta(x,\overline{x})) < 0, \quad \forall\, x, \overline{x} \in X. \tag{3.2}$$

On the other hand, if $\overline{x}$ is a Karush-Kuhn-Tucker vector critical point (VCP) with respect to the same η then there exist some $x \in X \subseteq M$, $\lambda \in \mathbb{R}^p$, $\exists\,(\overline{\lambda}, \overline{\mu}_I) \geq 0$, $\overline{\lambda} \neq 0$ with $Q = \eta(x,\overline{x}) \in T_{\overline{x}}M$ non identically zero such that

$$\overline{\lambda}^T df_{\overline{x}}(\eta(x,\overline{x})) + \overline{\mu}_I^T dg_{I(\overline{x})}(\eta(x,\overline{x})) = 0$$

which stands in contradiction to (3.2), and therefore, $\overline{x}$ is an efficient solution for (CVOP).

Let's now prove the sufficient condition. Let us suppose that there exist two feasible points x and $\overline{x}$ such that

$$f(x) - f(\overline{x}) \leq 0,$$

since otherwise (f, g) would be KKT-pseudoinvex, and the result would be proved. This means that $\overline{x}$ is not an efficient solution, and by using the initial hypothesis, $\overline{x}$ is not a (VCP), i.e., given η there exists some $x \in X \subseteq M$ with $Q = \eta(x,\overline{x}) \in T_{\overline{x}}M$ non identically zero such that

$$\overline{\lambda}^T df_{\overline{x}}(Q) + \overline{\mu}_I^T dg_{I(\overline{x})}(Q) = 0$$

has no solution $\overline{\lambda} \geq 0$, $\overline{\mu}_I \geqq 0$. Therefore, by Motzkin's theorem [5], the system

$$df_{\overline{x}}(Q) < 0, \quad dg_{I(\overline{x})}(Q) \leqq 0, \quad I = I(\overline{x})$$

has the solution $Q = \eta(x,\overline{x}) \in T_{\overline{x}}M$. In consequence, (f, g) is KKT-pseudoinvex. □

In sum up, we have that

$$(f, g) \text{ KKT-PIX} \Leftrightarrow [\text{VCP} \Leftrightarrow \text{Efficient (CVOP)}].$$

The efficient solutions results for (CVOP) can be considered a generalization of others for the scalar problem given in Theorem 1 and the results given by Martin [22] in finite dimensional Euclidean spaces.

Arguing in the same form, we prove

Corollary 3.8. *(f, g) is a pair weak KKT-pseudoinvex functions if and only if every Karush-Kuhn-Tucker vector critical point (VCP) with respect to the same η is a weakly efficient solution for (CVOP).*

This result is a generalization of Theorem 3.7 obtained by Osuna-Gómez et al. in [24] or Theorem 2.3 obtained by Osuna-Gómez et al. [25].

Example 3.9. Let us consider the set $\mathrm{Sym}_2(\mathbb{R})$ of symmetric 2×2 matrices endowed with the Frobenius metric $k_X(U,V) = \mathrm{trace}(UV)$ where $X \in \mathrm{Pos}_2(\mathbb{R})$ and $U, V \in T_X(\mathrm{Pos}_2(\mathbb{R})) = \mathrm{Sym}_2(\mathbb{R})$. Consider the following problem on $\mathrm{Pos}_2(\mathbb{R})$:

$$\text{(CVOP)} \qquad \max f(X) = (f_1, f_2)(X) = (x_1, x_3) \quad \text{subject to}$$

$$g_1(X) = x_2 + x_3 - 7 \leq 0, \quad g_2(X) = -x_1 + 1 \leq 0, \quad X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in \mathrm{Pos}_2(\mathbb{R}).$$

Given

$$\overline{X} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

using the Riemannian metric k and (f,g) is KT-pseudoinvex, $\eta(X,\overline{X}) = X - \overline{X}$ and at $Q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have that

$$df_{1(\overline{X})}(Q) = \mathrm{trace}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \mathrm{trace}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

$$df_{2(\overline{X})}(Q) = \mathrm{trace}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \mathrm{trace}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Then

$$dg_{1(\overline{X})}(Q) = \mathrm{trace}\left[\begin{pmatrix} 0 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \mathrm{trace}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 1,$$

$$dg_{2(\overline{X})}(Q) = \mathrm{trace}\left[\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \mathrm{trace}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

We have

$$df_{\overline{x}}(Q) = (df_{1(\overline{X})}(Q), df_{2(\overline{X})}(Q)) = (0,1),$$
$$dg_{\overline{x}}(Q) = (dg_{1(\overline{X})}(Q), dg_{2(\overline{X})}(Q)) = (1,0)$$

and therefore there exist $X = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 6 \end{smallmatrix}\right)$, $\lambda = (1,0)$ and $\mu = (0,1)$ such that $\lambda^T df_{\overline{x}}(Q) + \mu^T dg_{\overline{x}}(Q) = 0$. Thus $\overline{X}$ is trivially a (VCP) with respect to η and then $\overline{X}$ is an efficient solution.

4. Duality

We study the Mond-Weir type Dual Problem (MWDP) as the dual of (CVOP), formulated as

$$\text{(MWDP)} \qquad \text{maximize} f(u) \quad \text{subject to}$$
$$\lambda^T df_u(Q) + \mu^T dg_u(Q) = 0 \text{ for some } x,\, \eta \text{ with } Q = \eta(x,u) \neq 0,$$
$$\mu_j g_j(u) = 0, \quad j = 1 \ldots, m,$$
$$\mu \geqq 0, \quad \lambda \geq 0, \quad u \in X \subseteq M.$$

Theorem 4.1 (Weak Duality)**.** *Let x be a feasible point for (CVOP), and (u, λ, μ) a feasible point for (MWDP). If (f, g) is KKT-pseudoinvex on X then $f(x) \leq f(u)$ is not verified.*

Proof. Let us suppose (f, g) is KKT-pseudoinvex. Let x be a feasible point for (CVOP), (u, λ, μ) a feasible point for (MWDP), such that $f(x) \leq f(u)$. In other case the result would be proved. Then, there exist $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$ and $Q \in T_u M$ non identically zero such that

$$\lambda^T df_u(Q) + \mu^T dg_u(Q) = 0 \text{ for some } x,\, \eta \text{ with } Q = \eta(x,u) \neq 0,$$
$$\mu_j g_j(u) = 0, \quad j = 1, \ldots, m,$$
$$\mu \geqq 0, \quad \lambda \geq 0,$$

i.e.,

$$\lambda^T df_u(Q) + \mu_I^T dg_{I(u)}(Q) = 0$$

with $(\lambda, \mu_I) \geq 0$, $\lambda \neq 0$, $I = I(u) = \{j = 1, \ldots, m : g_j(u) = 0\}$. In consequence,

$$\lambda^T df_u(\eta(x,u)) + \mu_I^T dg_{I(u)}(\eta(x,u)) = 0. \tag{4.1}$$

Since $f(x) \leq f(u)$, from KKT-pseudoinvexity of (f, g) it follows that for all feasible points x, u

$$df_u(\eta(x,u)) < 0, \quad dg_{I(u)}(\eta(x,u)) \leqq 0, \quad I = I(x)$$

and multiplying by (λ, μ_I) we have

$$\lambda^T df_u(\eta(x,u)) + \mu_I^T dg_{I(u)}(\eta(x,u)) < 0, \quad \forall\, x, u$$

which stands in contradiction to (4.1), and therefore, $f(x) \leq f(u)$ is not verified. □

For converse duality we prove that

Theorem 4.2 (Converse Duality). *Let (f, g) be KKT-pseudoinvex, and $\overline{x}$ a feasible point for (CVOP). If $(\overline{x}, \lambda, \mu)$ is a feasible point for (MWDP) then $\overline{x}$ is an efficient solution for (CVOP).*

Proof. Let us suppose that $\overline{x}$ is a feasible point for (CVOP). If $(\overline{x}, \lambda, \mu)$ is a feasible point for (MWDP), then

$$\lambda^T df_{\overline{x}}(Q) + \mu^T dg_{\overline{x}}(Q) = 0 \text{ for some } x,\, \eta \text{ with } Q = \eta(x, \overline{x}) \neq 0,$$
$$\mu_j g_j(\overline{x}) = 0, \quad j = 1, \dots, m,$$
$$\mu \geqq 0, \quad \lambda \geq 0$$

hold and therefore $\overline{x}$ is a (VCP) with respect to the same η. Since (f, g) is KKT-pseudoinvex, from Theorem 3.7 it follows that $\overline{x}$ is an efficient solution for (CVOP). □

5. Conclusions

This paper shows for the first time that invex functions can be characterized in the Riemannian manifolds context for both the scalar and vector case. We have obtained new necessary and sufficient optimality conditions that characterize these functions. We have proven that the results obtained by Osuna-Gómez et al. [24,25] or Craven and Glover [10] for Euclidean spaces can be understood as particular cases of the results obtained in this paper. As future work, it would be interesting to study the possibility of extending these methods to the generation of algorithms to obtain optimal points in Riemannian manifolds or in eigenvector computation as it has been shown in [21]. Moreover, these results could also be extended to other fields such as physics, regarding problems in quantum mechanics [11]. Besides, another discipline that could potentially benefit from these findings is economics as shown in Nash-type equilibria problems [17].

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