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# A Historical Perspective of the Theory of Isotopisms

**Raúl M. Falcón [1],* , Óscar J. Falcón [2] and Juan Núñez [2]**

[1]   Department Applied Mathematics I, University of Sevilla, 41004 Seville, Spain
[2]   Department Geometry and Topology, University of Sevilla, 41004 Seville, Spain; oscfalgan@yahoo.es (Ó.J.F.);
      jnvaldes@us.es (J.N.)
*   Correspondence: rafalgan@us.es; Tel.: +34-954-550-158

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**Abstract:** In the middle of the twentieth century, Albert and Bruck introduced the theory of isotopisms of non-associative algebras and quasigroups as a generalization of the classical theory of isomorphisms in order to study and classify such structures according to more general symmetries. Since then, a wide range of applications have arisen in the literature concerning the classification and enumeration of different algebraic and combinatorial structures according to their isotopism classes. In spite of that, there does not exist any contribution dealing with the origin and development of such a theory. This paper is a first approach in this regard.

**Keywords:** isotopism; classification; non-associative algebra; quasigroup; Latin square

**MSC:** 17B40; 16D70; 17B60; 05B15

## 1. Introduction

In Mathematics, the most usual criterion for determining symmetries within any given algebraic or combinatorial structure is based on the study of its *automorphism group*, that is, on the set of isomorphisms from the object under consideration to itself so that all its defining properties are preserved. Any such an automorphism is indeed considered as a symmetry of the structure in question. In particular, if two mathematical objects of the same type (that is, sharing exactly the same defining properties) are isomorphic, then their corresponding automorphism groups are also isomorphic, and, hence, they are actually considered to have the same kind of symmetries. Due to this, the study and characterization of automorphism groups are essential for distributing algebraic and combinatorial structures of the same type according to their symmetries.

In any case, automorphisms do not determine all the possible symmetries within a given mathematical object. In this regard, we remark the existence of a not-so-known criterion that generalizes the previous one so that not only those symmetries derived from automorphisms arise. This second criterion is based on the study of the so-called *autotopism group* of the object under consideration, that is, on the set of *isotopisms* from the algebraic or combinatorial structure in question to itself. The term *isotopy* was introduced at the beginning of the twentieth century concerning the concept *homotopy* in Topology. It was Abraham Adrian Albert [1] who introduced in 1942 this term as a generalization of the notion of isomorphism in order to classify non-associative algebras. Shortly after, Albert himself [2,3], together with Richard Bruck [4], would generalize this idea to deal also with quasigroups. In any case, what is remarkable is that mathematical objects belonging to distinct isomorphism classes can have common symmetries that the classical theory of isomorphisms cannot find.

Since the original manuscripts of Albert and Bruck, a wide range of mathematicians has dealt with the distribution of different algebraic and combinatorial structures according to the symmetries that arise from their corresponding autotopism groups (see the next sections and the comprehensive

and extensive bibliography that is included at the end of the paper). Indeed, the theory of isotopisms is currently being actively developed to deal with the symmetries of different mathematical objects. In spite of that, this theory is not widely known by the community of mathematicians. From our personal experience, this fact is mainly due to two reasons. Firstly, there does not exist any survey that gathers together the antecendents, origin, early stage and later development of the theory of isotopisms. Secondly, any reference on the topic only focuses on a very specific algebraic or combinatorial structure, without mentioning or exposing the potential and different applications that can arise from the theory of isotopisms. Both perspectives may constitute important obstacles to be overcome by any novice researcher on the topic. This paper deals with both mentioned perspectives in order to make it easy for such researchers to have a first contact with the theory of isotopisms. Concerning the former, we enumerate some of the more relevant classical and current references on the theory of isotopisms. As such, the paper can be considered as an introductory manuscript on the topic. Concerning the second perspective, we expose a diversity of results dealing with the classification of a wide amount of different mathematical structures according to symmetries derived from their autotopism groups that the classical theory of isomorphisms does not take into account.

The paper is organized as follows. Section 2 deals with the antecedents of the notion isotopism in both Topology and Algebra. Section 3 is devoted to the fundamentals of the theory introduced by Albert and Bruck. Finally, Section 4 deals with the development of the theory concerning the classification of different algebraic and combinatorial structures according to their symmetries. Throughout the paper, the current notation is followed, which can differ from the original one to which we refer.

## 2. Antecedents

In 1942, Albert [1] realized that right and left multiplication spaces within non-associative algebras give rise to linear transformations holding equivalent properties to those of an associative algebra. Based on this fact, he generalized the notion of isomorphism of algebras as follows.

**Definition 1.** *Let $(A, \cdot)$ and $(A', \circ)$ be two algebras over a base field $\mathbb{F}$. If there exist three nonsingular linear maps $f$, $g$ and $h$ from $A$ to $A'$ such that*

$$f(x) \circ g(y) = h(x \cdot y),$$

*for all $x, y \in A$, then, it is said that $A$ and $A'$ are* isotopic, *or that $A'$ is an* isotope *of A. In addition, the triple $(f, g, h)$ is said to be an* isotopism *from A to $A'$. This is an* autotopism *if both algebras coincide. If $f = g = h$, then the triple $(f, f, f)$ is an* isomorphism *(this is an* automorphism *if both algebras coincide). Isotopisms (respectively, isomorphisms) determine equivalence relations among algebras, which give rise to their distribution into isotopism (respectively, isomorphism) classes.*

*Based on the component-wise composition of linear maps, the set of isotopisms between two algebras constitutes a group, which is called their isotopism group. Isomorphism, autotopism and automorphism groups are similarly defined.*

According to Albert [1] himself, the notion of isotopism derived from the work of Norman Steenrod, who, dealing with homotopy groups in Topology, studied isotopisms of *division algebras*. Recall that, in these algebras, left- and right-divisions are always possible. Albert and Steenrod coincided as assistant professors in the period 1939–1942 at the Department of Mathematics of the University of Chicago. As their common mentor, they had Salomon Lefschetz, who introduced Albert to the Riemann matrices during his postdoctoral year (1928–1929) at Princeton, and was the doctoral advisor of Steenrod. The latter defended in 1936 his Ph.D. Thesis entitled *"Universal Homology Groups"*, whose connection with the theory of division algebras would contribute even more to put Albert's focus on Steenrod's work. Let us expose chronologically this story.

### 2.1. Topology

It was Solomon Lefschetz who introduced in 1934 the term *Topology* as it is currently used (it had been already used by Johann Benedict Listing in the 19th century). The fundamentals of Topology were established in 1895 by Henri Poincaré [5] in his manuscript entitled *Analysis Situs*, a term that was firstly mentioned by Gottfried Leibniz in the 18th century. Poincaré introduced the term *homology* to establish a relation among manifolds composing the boundary of a higher-dimensional manifold. Then, he introduced the *fundamental group* in a point within a manifold as the *contours* or paths starting and ending in such a point. Almost ten years later, he indicated [6] that two contours are equivalent if there exists a continuous deformation between them within the manifold. This constitutes a starting point towards the current notion of homotopy. Even if Max Dehn and Poul Heegaard [7] introduced the terms isotopy and homotopy in Topology in 1907, both meanings differ from the current ones, which are respectively based on the ideas that Heinrich Tietze [8] and Luitzen Brouwer [9,10] introduced shortly after.

**Definition 2.** *Let X and Y be two topological spaces, and let f and g be two continuous maps from X to Y. Both maps are* homotopic *if there exists another continuous map $H : X \times [0,1] \to Y$, which is called a* homotopy *from f to g, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$, for all $x \in X$. If besides, $H_t(x) = H(x,t)$ is an embedding, for all $t \in [0,1]$, then the homotopy H is an* isotopy.

In 1912, Brouwer [10] introduced *homotopy classes* as sets of maps within a topological space that can be pairwise transformed into each other in a continuous way. He was the first one to connect Homotopy and Homology when he showed that two continuous maps within the two-dimensional sphere belong to the same homotopy class if and only if their degrees coincide.

In 1916, Oswald Veblen was invited by the American Mathematical Society to teach the *Cambridge Colloquium Lectures*. Interested in Topology since his work [11] on the Jordan curve theorem, he chose Analysis Situs as the topic of his talk. The publication of these lectures [12] constituted a comprehensive revision on the subject. Particularly, keeping in mind the nomenclature that Dehn and Heegaard [7] had introduced fifteen years before, Veblen established a formal definition for the isotopy and homotopy of continuous maps, whose fundamentals coincide with those of Definition 2. He also introduced *homology groups* as commutative groups whose identities correspond to homologies of a polyhedron.

In 1926, Emmy Noether realized the importance of Group Theory in Homology [13], into which Leopold Vietoris [14], Heinz Hopf [15], Walther Mayer [16], Pável Alexandroff [17] and Eduard Čech [18] delved shortly after.

In 1932, Cech introduced $n$th *homotopy groups* as sets of homotopy classes from the $n$-sphere $\mathbb{S}^n$ to a topological space. They constitute fundamental groups when $n = 1$. Alexandroff and Hopf thought that homotopy groups did not have any advantage in comparison with homology groups because the former are abelian, for all $n > 1$. Due to it, Čech's work was only published as a simple paragraph in the Proceedings of the Congress [19].

In 1935, Witold Hurewicz published four notes [20–23] in which he rediscovered homotopy groups (he mentioned the independent work of Čech in the second note). He proved the existence of a pair $(x_0, f(x_0)) \in \mathbb{S}^n \times Y$, for any map $f$ within the homotopy class of the $n$th homotopy group of any given topological space $Y$.

In 1940, Hopf [24] showed that the existence of a continuous odd map from $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$ ensures $n$ to be a power of 2. This involves a real division algebra to have as dimension a power of 2, and every finite-dimensional real commutative division algebra to be one- or two-dimensional.

### 2.2. Algebra

It can be said that isotopisms of algebras arise from the interest of Albert in classifying non-associative division algebras. This interest dated from his Master and Ph.D. Thesis in the period

1927–1928. Having Leonard Dickson as his doctoral advisor, Albert classified the 16-dimensional associative division algebras that he would also deal with in his first major paper [25]. The importance of such kind of classification derives from the structure theorems of Joseph H. M. Wedderburn [26], which ensure that the structure of any linear associative algebra depends on the classification of *associative division algebras*.

In 1878, Ferdinand Frobenius [27] proved that the only associative real division algebras are the algebra of quaternions and its real and complex subalgebras.

In 1898, Adolf Hurwitz [28] showed that every real division algebra having a non-degenerate quadratic form is one-, two-, four- or eight-dimensional.

In 1905, Wedderburn [29] demonstrated the commutativity of every finite associative division algebra, which involves the latter to be a field.

In 1906, Dickson presented in the nineteenth regular meeting of the Chicago Section of the American Mathematical Society a contribution ([30], pp. 441–442) in which he obtained the first known family of associative division algebras having a perfect square greater than one as dimension. This algebra was called *cyclic* because it contains a maximal commutative subfield $\mathbb{S}$ such that the Galois group $\mathrm{Gal}(\mathbb{S}/\mathbb{F})$ is cyclic, where $\mathbb{F}$ is the base field of the algebra. Dickson [31–33] also constructed the first known non-associative division algebra that is different from the Cayley algebra. However, it was not until 1935 that he [34] would resume a comprehensive study of non-associative division algebras.

In 1914, Dickson [35] exposed necessary conditions to construct four- and nine-dimensional cyclic division algebras. Shortly after, Wedderburn [36] generalized these results for any dimension and introduced *normal* division algebras in which commutative elements always belong to the ground field. He realized that every normal division algebra has a perfect square as a dimension, and the normality of every cyclic division algebra.

In 1921, Wedderburn [37] showed that every division algebra is the direct product of a field and a normal division algebra, and the cyclicity of every nine-dimensional division algebra. These aspects were shortly after reviewed and extended by Dickson [38], who determined all nine-dimensional associative division algebras. In 1927, Dickson, supported by Wedderburn, published the monograph entitled *Algebren und ihre Zahlentheorie* [39], where all four-dimensional associative division algebras were determined and established as cyclic.

In 1923, Hurwitz [40] showed that every real division algebra that is endowed with a non-degenerate quadratic form is either associative or the Cayley algebra. Until that moment, the only known non-associative real division algebra with such a form had been the Cayley algebra.

In 1925, Noether realized the possible use of representation theory to deal with the structure theorems of Wedderburn [41]. She, together with Richard Brauer and Herman Hasse, constituted the German triumvirate that would encourage representation theory to deal with the arithmetics that Wedderburn and Dickson defined over linear algebras.

In 1928, Albert achieved in his Ph.D. Thesis the following natural step of the story, that is, the study and classification of 16-dimensional associative division algebras. One year later, he would show [25] that all of them belong indeed to a family of division algebras that was already defined by Francesco Cecioni [42].

In 1929, Brauer [43] reduced the problem of determining all normal division algebras of order $n^2$ to the case in which $n$ is a prime power.

In 1930, Albert [44] showed the cyclicity of every 16-dimensional normal division algebra.

In 1931, Hasse [45], who had previously studied the structure of division algebras over $p$-adic fields in the period 1929-30, showed the cyclicity of every normal division algebra over a $p$-adic field.

In 1932, Albert [46] focused on the family of 16-dimensional associative algebras that had been previously studied by Brauer [47], who would become the Ph.D. advisor of Bruck. After that, Albert focused on classifying normal division algebras over infinite modular fields. In particular, he classified [48,49] all normal division algebras of degree two (respectively, three and four) over a field of characteristic two (respectively, three and two).

Also in 1932, Brauer, Hasse and Noether [50] proved the conjecture of Dickson [38] about the fact that every normal division algebra over an algebraic number field is cyclic. This would facilitate the classification of both associative and non-associative (division) algebras. Dickson [34] said about such a result that *"this perfection of the theory of associative algebras justifies attention to non-associative algebras"*. Indeed, in his article, he described new four-dimensional non-associative division algebras.

An alternative proof was also published in the same year by Albert and Hasse [51], who had kept correspondence since 1931. The controversial around both proofs [41,52] would mark Albert's opinion on representation theory. Thus, even if he recognized the importance of that theory and its necessity of being better known by the American mathematicians [53] (it would be indeed Hasse [54] who published in 1932 the first paper in English on representation theory and arithmetics of algebras), Albert [51,55,56] mentioned explicitly the advantages of the methodology that had been developed by Wedderburn, Dickson and himself with respect to that of German authors. What is remarkable for the origin of isotopisms of algebras is that, from that moment on, Albert would keep an eye on the results of German mathematicians, who had already introduced the notion of isotopy in Topology and would implement Homology to classify division algebras.

## 3. Fundamentals

We expose in this section several of the preliminary concepts and results that Albert [1] and Bruck [4] established as fundamentals of the theory of isotopisms of algebras and quasigroups.

### 3.1. Isotopisms of Algebras

Firstly, let us focus on the original manuscript of Albert [1]. The following definition shows how, as a particular case of isotopism, he introduced the concept of principal isotopism, whose study was considered as more convenient than that of general isotopisms by Albert himself [2].

**Definition 3.** *Let $(A, \cdot)$ and $(A, \circ)$ be two isotopic algebras over the same set of vectors. An isotopism $(f, g, h)$ from the former to the latter is said to be* principal *if h coincides with the identity map on A. In such a case, the algebra $(A, \circ)$ is said to be a* principal isotope *of the algebra $(A, \cdot)$. Equivalently, both algebras are* principal isotopic. *To be principal isotopic is an equivalence relation among algebras that give rise to their distribution into principal isotopism classes.*

The following result establishes the relation among isotopisms, principal isotopisms and isomorphisms of algebras.

**Theorem 1.** *Let $A$ and $A'$ be two isotopic algebras. Then, there exists an isomorphism from $A'$ to a principal isotope of the algebra $A$.*

Isotopisms and principal isotopisms give rise to certain invariants among algebras. Albert focused in particular on the study of *simple* algebras, for which all their ideals are trivial.

**Theorem 2.** *The following assertions hold.*

1. *Right divisors are isotopism invariants of zero algebras and simple algebras.*
2. *Ideals are principal isotopism invariants of any algebra.*

As a consequence of the second assertion of Theorem 2, simple algebras are preserved by principal isotopisms.

Albert also dealt with isotopisms of *unital algebras*, that is, algebras with a unit element.

**Theorem 3.** *The following assertions hold.*

1. *Every finite-dimensional unital algebra has a principal isotope that is a simple algebra without left or right ideals.*
2. *A unital algebra is associative if and only if every unital algebra that is isotopic to the former is associative and, indeed, isomorphic to that one.*

Albert indicated that Theorem 3 is not true for non-unital algebras. Thus, for example, the three-dimensional algebra of basis $\{e_1, e_2, e_3\}$ that is linearly described as $e_1 e_2 = -e_2 e_1 = e_3$ is isotopic but not isomorphic to that one described as $e_1 e_1 = e_3$.

Furthermore, concerning division algebras, it is straightforwardly verified that isotopisms preserve this type of algebra. In particular, Albert proved the next result.

**Theorem 4.** *The following assertions hold:*

1. *Every division algebra is isotopic to a unital division algebra.*
2. *Every n-dimensional real division algebra, with $n > 1$, is isotopic to a unital division algebra with unit element e so that there exists a vector b in the algebra such that $b^2 = -e$.*

*Absolute-valued algebras* constitute an example of division algebras, whose vector spaces are endowed of an absolute value $|| \cdot ||$ such that $||xy|| = ||x|| \, ||y||$, for all pair of vectors $x$ and $y$. Concerning such algebras, Albert proved the following result.

**Theorem 5.** *Every finite-dimensional real absolute-valued algebra is either one-, two-, four- or eight-dimensional. Moreover, it is either the real field $\mathbb{R}$, the complex field $\mathbb{C}$, the quaternions $\mathbb{H}$, the octonions $\mathbb{O}$, or a principal isotope of $\mathbb{H}$ and $\mathbb{O}$.*

Albert also introduced the question as to whether Lie algebras are preserved by principal isotopisms. Recall that an algebra $(A, \cdot)$ is a *Lie algebra* if

- it is *anticommutative*, that is $x \cdot y = -y \cdot x$, for all $x, y \in A$; and
- it holds the so-called *Jacobi identity*

$$x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y) = 0,$$

for all $x, y, z \in A$.

Concerning isotopisms of Lie algebras, Albert proved the following lemma:

**Lemma 1.** *A principal isotope $(A, \cdot)$ of a given Lie algebra with respect to a principal isotopism $(f, g, \mathrm{Id})$ is also a Lie algebra if and only if*

- $f(x) \cdot g(y) = -f(x) \cdot g(y)$, *and*
- $f(f(x) \cdot g(y)) \cdot g(z) - f(f(x) \cdot g(z)) \cdot g(y) - f(x) \cdot g(f(y) \cdot g(z)) = 0$,

*for all $x, y, z \in A$.*

Let us focus now on some aspects about how Bruck delved further in his original manuscript [57] into the development of isotopisms of algebras. Concerning simple algebras, Bruck introduced *isotopically simple algebras* as simple algebras such that all their isotopes are also simple.

**Theorem 6.** *The following results hold:*

1. *Every simple unital algebra is isotopically simple.*
2. *Every simple associative algebra is isotopically simple.*
3. *Every non-associative algebra may be built up from isotopically simple algebras.*

4. *The only one-dimensional isotopically simple algebras are the field itself and the one-dimensional zero-algebra.*
5. *Every two-dimensional isotopically simple algebra over a base field $\mathbb{F}$ is isotopic to a field of degree two over $\mathbb{F}$.*

Concerning division algebras, he described a set of four-dimensional real division algebras that are mutually non-isotopic. He also proved the following result.

**Theorem 7.** *Every two-dimensional division algebra is isotopic to a field.*

Concerning Lie algebras, Bruck showed the next result.

**Theorem 8.** *The following assertions hold:*

1. *The $n(n-1)/2$-dimensional real Lie algebra consisting of all skew-symmetric matrices under the product $A \circ B = AB - BA$ is isotopically simple.*
2. *Under the same product, the $n(n-1)$-dimensional complex Lie algebra consisting of all skew-Hermitian matrices is isotopically simple.*

*3.2. Isotopisms of Quasigroups*

In 1937, Bernard Haussmann and Øystein Ore [58] introduced the term quasigroup to generalize the associative product that describes any group. The same idea had already been dealt with in 1929 by Anton Suschkewitsch [59], who studied the replacement of associativity within the different systems of postulates that had enumerated separately Frobenius [60], Eliakim Moore [61] and Dickson [62] to define groups. Albert himself [2] said that Suschkewitsch's work suggested the theory of isotopisms of quasigroups.

**Definition 4.** *A* quasigroup *is a pair $(Q, \cdot)$ that is formed by a nonempty set $Q$ endowed with a multiplication $\cdot$ so that, if any two of the three symbols $x, y$ and $z$ in the equation $x \cdot y = z$ are given as elements of $Q$, then the third one is uniquely determined as an element of $Q$. That is, left- and right-divisions are always possible within $Q$.*

Every associative quasigroup constitutes a *group*. Furthermore, a *loop* is a quasigroup $(Q, \cdot)$ having a unit element $e \in Q$ such that $x \cdot e = e \cdot x = x$, for all $x \in Q$. Thus, for instance, a *Moufang loop*, which was introduced by Ruth Moufang [63] in 1935, is a loop $(Q, \cdot)$ holding the following four identities, for all $x, y, z \in Q$:

$$x \cdot (y \cdot (x \cdot z)) = ((x \cdot y) \cdot x) \cdot z,$$

$$((z \cdot x) \cdot y) \cdot x = z \cdot (x \cdot (y \cdot x)),$$

$$(x \cdot y) \cdot (z \cdot x) = (x \cdot ((y \cdot z)) \cdot x,$$

$$(x \cdot y) \cdot (z \cdot x) = x \cdot ((y \cdot z) \cdot x).$$

Isomorphisms of quasigroups had already been dealt in the end of the 1930s by Haussmann and Ore [58], David Murdoch [64], George Garrison [65] and Harriet Griffin [66].

In the period 1943–1944, as a generalization of such isomorphisms, Albert and Bruck generalized isotopisms from algebras to quasigroups. Indeed, Bruck [57] connected explicitly both theories by introducing the notion of *quasigroup algebra* related to a quasigroup $(Q, \cdot)$ as an algebra that admits a basis $\{e_x \mid x \in Q\}$ over a given base field $\mathbb{F}$ such that

$$e_x e_y = h_{x,y} e_{x \cdot y},$$

for all $x, y \in Q$, where $h_{x,y} \in \mathbb{F} \setminus \{0\}$. It is called a *quasigroup ring* if $h_{x,y} = 1$, for all $x, y \in Q$. This was the starting point to generalize in a natural way the concept of isotopism from algebras to quasigroups as we expose in the following definition [2–4].

**Definition 5.** *Let $(Q, \cdot)$ and $(Q', \circ)$ be two quasigroups with the same number of elements. If there exists three bijections $f$, $g$ and $h$ from $Q$ to $Q'$ such that*

$$f(x) \circ g(y) = h(x \cdot y),$$

*for all $x, y \in Q$, then both quasigroups are said to be* isotopic. *The notions of* isotopism, autotopism, isomorphism *and* automorphism *of quasigroups, together with those ones of* isotopism *and* isomorphism classes *are similarly defined as in Definition* 1. *In addition, the component-wise composition of bijections gives rise to the corresponding notions of* isotopism, autotopism, isomorphism *and* automorphism *groups of quasigroups.*

Bruck focused on isotopisms of different types of quasigroups as those ones endowed with the inverse property, totally symmetric quasigroups, Moufang loops or abelian quasigroups, amongst others. He dealt in particular with isotopisms of loops, in which Albert himself [2,3,67] was also interested.

**Theorem 9.** *The following assertions hold.*

1. *Every quasigroup is isotopic to a loop.*
2. *A loop is isotopic to a group if and only the loop is isomorphic to the group, and, hence, it is itself a group.*
3. *Every loop that is isotopic to a Moufang loop is also Moufang.*
4. *Every isotopism of groups constitutes indeed an isomorphism.*
5. *Every abelian quasigroup is isotopic to an abelian group.*

Concerning isotopisms of Moufang loops, we also refer the reader to the manuscript of Bruck [68]. Further, let us remark that Albert himself [2] realized that the first assertion in Theorem 9 enables one to restrict the study and construction of quasigroups to the distribution of loops into isotopism classes. In this regard, for instance, he determined [3] such a distribution for all loops of order six having a subloop of order three. He also proved that every quasigroup of order five is isotopic to one of two nonisotopic loops, only one of them being a group. Remark also that every loop of order $n \leq 4$ is a group. In order to determine such classifications, Albert [3] introduced the following definition.

**Definition 6.** *Two quasigroups are said to be* anti-isotopic *if one of them is isotopic to the transpose of the second one.*

It is important to remark that the underlying idea on which Albert and Bruck based the concept of isotopisms of quasigroups was already known within the theory of Latin squares. A *Latin square* of order $n$ is an $n \times n$ array whose elements are chosen from a set of $n$ symbols, such that each symbol appears precisely once per row and once per column. This constitutes the multiplication table of a finite quasigroup of $n$ elements.

Latin squares illustrate in an easy way how isotopisms make possible the study of much more symmetries than isomorphisms do. In order to see it, let $\mathcal{LS}_n$ denote the set of Latin squares of order $n$ that have the set $[n] := \{1, \ldots, n\}$ as set of symbols. In particular, every Latin square $L = (l_{ij}) \in \mathcal{LS}_n$ is uniquely represented by its *orthogonal array representation*, that is, by the set

$$O(L) := \{(i, j, l_{ij}) \colon i, j \in [n]\}.$$

Let $S_n$ denote the symmetric group on the set $[n]$ and let $\mathfrak{I}_n := S_n \times S_n \times S_n$. Then, every triple $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_n$ determines a new Latin square $L^\Theta \in \mathcal{LS}_n$ such that $O(L^\Theta) =$

$\{(\alpha(i), \beta(j), \gamma(l_{ij})) \colon i, j \in [n]\}$. Hence, $\alpha$, $\beta$ and $\gamma$ represent, respectively, permutations of the rows, columns and symbols of $L$.

Already in the 1930s, Ronald Fisher and Frank Yates [69] and Horace Norton [70] called $\Theta$ a *transformation* of Latin squares. Observe in particular that, if $([n], \cdot)$ and $([n], \circ)$ are the quasigroups that have $L$ and $L^{\Theta}$ as their respective multiplication tables, then $\gamma(i \cdot j) = \alpha(i) \circ \beta(j)$, for all $1 \leq i, j \leq n$. The triple $\Theta$ is, therefore, an isotopism between both quasigroups and, due to it, $\Theta$ is said to be an *isotopism* from $L$ to $L^{\Theta}$. The latter is said to be *isotopic* to the former. If $\alpha = \beta = \gamma$, then $\Theta$ is an *isomorphism* from $L$ to $L^{\Theta}$, which are said to be *isomorphic*. If $L^{\Theta} = L$, then the triple $\Theta$ is an *autotopism* of $L$ (an *automorphism* if $\alpha = \beta = \gamma$). This ensures that isotopisms describe much more symmetries of Latin squares than isomorphisms do.

**Example 1.** *The following two Latin squares of order three are isotopic by means of the isotopism* $((12)(3), (13)(2), (12)(3)) \in \mathfrak{I}_3$.

$$
L \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array}
\qquad
L^{\Theta} \equiv \begin{array}{|c|c|c|} \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}
$$

*The same triple constitutes an autotopism of the Latin square*

$$
L' \equiv \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline \end{array}
$$

Another concept that isotopisms of algebras and quasigroups inherited from Latin squares is that of adjugacy in which Bruck [4] based the following definition.

**Definition 7.** *A quasigroup* $(Q, \cdot)$ *is said to be* totally symmetric *if the equation* $x \cdot y = z$ *remains valid under every permutation of the elements* $x, y, z \in Q$.

To see it, let $\pi \in S_3$. Fisher and Yates [69] and Norton [70] introduced the *adjugate* $L^{\pi}$ of a Latin square $L \in \mathcal{LS}_n$ as the Latin square $L^{\pi} \in \mathcal{LS}_n$ such that $O(L^{\pi}) = \{(l_{\pi(1)}, l_{\pi(2)}, l_{\pi(3)}) \mid (l_1, l_2, l_3) \in O(L)\}$. There exist, therefore, six adjugates:

$$
L^{\mathrm{Id}} = L, \quad L^{(12)(3)} = L^t, \quad L^{(13)(2)}, \quad L^{(23)(1)}, \quad L^{(123)} \quad \text{and} \quad L^{(132)}.
$$

Norton [70] called *specie* of a Latin square to any transformation of one of its adjugates. Nowadays, these adjugates are called *conjugates* or *parastrophes*; the permutation $\pi$ is a *parastrophism*; the composition of an isotopism with a parastrophism is a *paratopism*; and species are called *paratopes*. All of them determine equivalence relations among Latin squares.

Concerning algebras, conjugacy was already dealt with by James Shaw [71] in 1915: Let $A$ be an algebra over a base field $\mathbb{F}$ of basis $\{e_1, \ldots, e_n\}$, which is described as $e_i e_j = \sum_{k=1}^{n} a_{ijk} e_k$, for all $1 \leq i, j \leq n$ and some *structure constants* $a_{ijk} \in \mathbb{F}$. If every structure constant $a_{ijk}$ is replaced by either $a_{ijk}, a_{jik}, a_{ikj}, a_{kij}, a_{jki}$ or $a_{kji}$, then the resulting algebra is said to be *parastrophic* or *conjugate* of $A$. Each one of the six possible replacements of indices gives rise to a *parastrophism* of algebras. It was Bruck [57] in 1944, together with Ivor Etherington [72] in 1945, who introduced the problem of studing whether the six conjugates of a given algebra are isotopic or not. In particular, Etherington illustrated this problem with different types of algebras and quasigroups.

## 4. Development

Since the original manuscripts of Albert and Bruck, the study of isotopisms of different types of algebras, quasigroups and related combinatorial structures has been widely dealt with in the literature. We devote this section to point out several of the most relevant references on this topic.

### 4.1. Division Algebras

The interest that Albert had in classifying division algebras was extended to the scientific community once he showed [73] the relationship that exists among these algebras and their coordination of projective planes.

**Theorem 10.** *Two division algebras are isotopic if and only if the projective planes that they coordinatize are isomorphic.*

In this regard, at the beginning of the 1960s, Albert [73] and Daniel Hughes [74], whose Ph.D. advisor was Bruck, dealt with autotopism groups of non-associative division algebras coordinatizing finite projective planes. Shortly after, two doctoral students of Albert, Reuben Sandler [75] and Robert Oehmke [76] generalized such algebras and dealt with their distribution into isotopism classes. In particular, they focused on the autotopism group of Jordan division algebras.

In 1971, Holger Petersson [77] proved the existence of infinity isotopism classes of four- and eight-dimensional real division algebras. The existence of more than one isotopism class for the four-dimensional case had already been proved by Bruck [4] in 1944.

In 1981, Georgia Benkart, James Marshall Osborn (who also had Albert as Ph.D. advisor) and Daniel Britten [78] overviewed some results and physical applications on isotopisms of real division algebras. Based on their results, but much more recently, Erik Darpö and Ernst Dieterich [79] have used principal isotopes of the complex field to construct isotopisms among real commutative division algebras that give rise to equivalences of categories.

### 4.2. Semifields

A *presemifield* is a set endowed with a commutative addition having 0 as its unit element, and a distributive multiplication for which left- and right-divisions are always possible. This is called a *semifield* if the mentioned multiplication has a unit element. Two (pre)semifields $(S, +, \cdot)$ and $(S', +, \circ)$ are *isotopic* if the quasigroups $(S, \cdot)$ and $(S', \circ)$ are.

In case of dealing with semifields, the existence of both additive and multiplicative units, together with distributivity, makes possible their identification with a *planar ternary ring* as Marshall Hall [80] introduced, that is, as a set endowed of a ternary operation $T(x, y, z) = (x \times y) + z$ satisfying certain properties derived from the incidence of an Euclidean plane.

In 1965, Donald Knuth [81], whose doctoral advisor was indeed Hall, generalized the notion of isotopism from semifields to ternary rings in order to determine whether two semifields coordinatize the same projective plane. More specifically, Knuth indicated that two ternary rings $T$ and $T'$ are *isotopic* if there exists a triple $(F, G, H)$ of one-one maps between them so that $H(0) = 0$ and

$$(F(x) \cdot G(y)) \circ H(z) = H((x \cdot y) \circ z)$$

for all $x, y, z \in T$. In his work, Knuth

- determined all ternary rings that are isotopic to another one;
- described a constructive method to generate 24 semifields derived from any given semifield;
- characterized isotopic ternary rings that coordinatize isomorphic projective planes; and
- considered nonlinear isotopisms for constructing semifields.

In 1972, Michael Ganley [82] established the following characterization.

**Theorem 11.** *A finite semifield S is isotopic to a commutative semifield if and only if S contains an element* $b \neq 0$ *such that* $x(yb) = y(xb)$*, for all* $x, y \in S$.

In 1974, Bruno Soubeyran [83] described some derivable semifields that are not isotopic to those ones described by Knuth. Much more recently, Simeon Ball and Michel Lavrauw [84] showed the existence of at most five non-isotopic semifields among the 24 ones described by Knuth.

In 1977, Menichetti [85] proved that every three-dimensional finite semifield is either a field or isotopic to a generalised twisted field (see, for instance, the original manuscript of Albert [86] for the description of such algebraic structures).

Much more recently, in 1998, Bianca Spille and Irene Pieper-Seier [87] studied the distribution of commutative semifields into strongly isotopism classes. Remark in this regard that an isotopism is called *strong* if it is of the form $(f, f, h)$.

In 2005, Vikram Jha [88], whose Ph.D. advisor was Hughes, realized that Theorem 11 applies to arbitrary finite translation planes, rather than just semifield planes. Based on this fact, Jha generalized that result for *quasifields*, which are defined similarly to semifields, up to the fact that only the left- or right-distributive property is imposed.

In 2008, Robert Coulter and Marie Henderson [89] characterized obtained a strongly isotopic commutative presemifield and classified all planar functions that describe presemifields which are isotopic either to a finite field or to a commutative twisted field.

*4.3. Alternative Algebras*

An algebra $(A, \cdot)$ is called *alternative* if

$$(x, x, y) = 0 = (x, y, y),$$

for all $x, y \in A$, where $(x, y, z)$ denotes the *associator product* $(x \cdot y) \cdot z - x \cdot (y \cdot z)$, for all $x, y, z \in A$. In general, alternativity is not preserved by isotopisms. Nevertheless, the following result was already exposed by Richard Schafer [90] (another doctoral student of Albert) in 1943.

**Theorem 12.** *Every unital algebra that is isotopic to an alternative algebra is also alternative. In addition, the latter must be also unital.*

Examples of alternative algebras are the first four *Cayley–Dickson algebras* [35], that is, the real field $\mathbb{R}$, the complex field $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. Concerning these algebras, Schafer [90] proved the following result.

**Theorem 13.** *Every unital algebra that is isotopic to a Cayley–Dickson algebra is also a Cayley–Dickson algebra that is indeed isomorphic to the latter.*

In 1971, Kevin McCrimmon [91] introduced the $u, v$*-homotope* $(A, \cdot_{u,v})$ of an algebra $(A, \cdot)$, where $u, v \in A$, as the algebra defined as

$$x \cdot_{u,v} y = (x \cdot u) \cdot (v \cdot y), \text{ for all } x, y \in A.$$

It generalizes the classical notion of isotopism in a way that had already been considered by Albert himself [1] for general linear algebras, by Bruck [67] for loops, and by Schafer [90] for alternative algebras. This is indeed an isotopism in the case of being the algebra $(A, \cdot)$ unital and the two elements $u$ and $v$ invertible. In such a case, McCrimon [91] called $(A, \cdot_{u,v})$ the $u, v$*-isotope* of the algebra $(A, \cdot)$.

**Theorem 14.** *The following assertions hold:*

1. *The* $u, v$*-isotope of an alternative unital algebra is also alternative.*

2.　*Every isotopism of a unital alternative algebra $(A, \cdot)$ constitutes an $u, v$-homotope of the latter. Moreover, the former is a unital alternative algebra having as a unit element the inverse of $u \cdot v$.*

More recently, Bruce Allison [92], Mark Babikov [93] and Sergey Pchelintsev [94] delved into this topic. In particular, Allison focused on isotopisms of involutive alternative algebras and proved that isomorphisms and isotopisms of simple alternative algebras are equivalent; Babikov characterized $u, 1$-isotopic alternative algebras that are indeed isomorphic; and Pchelintsev proved that $u, 1$-isotopisms of unital alternative algebras preserve primality.

### 4.4. Jordan Algebras

An algebra $(A, \cdot)$ is said to be a *Jordan algebra* if it is commutative and satisfies that

$$x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y),$$

for all $x, y \in A$. In 1969, based on the work of Oehmke and Sandler [76], Petersson [95] dealt with isotopisms of Jordan algebras. In particular, he showed the following result.

**Theorem 15.** *Let $(A, \cdot)$ and $(A', \circ)$ be two finite-dimensional Jordan algebras having a characteristic different from two, at least one of which is semi-simple. If they are isotopic, then they are indeed isomorphic.*

Previously, in 1962, Nathan Jacobson [96] had characterized unital Jordan algebras by means of *a-isotopies*, whose definition differs slightly from the classical one. Specifically, given an invertible element $a$ of a unital Jordan algebra $(A, \cdot)$, he described the *a-isotopic* Jordan algebra $(A, \cdot_a)$ by

$$x \cdot_a y = (x \cdot a) \cdot y + (a \cdot y) \cdot x - (x \cdot y) \cdot a, \text{ for all } x, y \in A.$$

Observe the similarity between *a*-isotopies and the mentioned $u, v$-homotopies, which McCrimmon [91] also used to deal with quadratic Jordan algebras. Much more recently, Petersson [97] related both isotopisms of Albert and Jacobson by means of the structure groups of alternative algebras.

In 1963, Jacobson studied how *a*-isotopisms act on generic norms of unital Jordan algebras. This was also considered by McCrimmon [98] and, much more recently, by Ottmar Loos [99], who considered a similar question for generically algebraic Jordan algebras. McCrimmon [100] also proved that *a*-isotopisms preserve inner ideals of unital Jordan algebras.

In 1978, Petersson [101] gave sufficient conditions under which two reduced exceptional simple Jordan algebras are *a*-isotopic. Together with Michel Racine [102], who was doctoral student of Jacobson, he also proved that all *a*-isotopes of a first construction exceptional Jordan division algebra are isomorphic. They also introduced the problem of whether two *Albert algebras* (a type of exceptional central simple Jordan algebra) are *a*-isotopic. This was successfully answered by Maneesh Thakur [103] in 1999.

### 4.5. Lie Algebras

Any results on isotopisms of Lie algebras apart from Lemma 1 and Theorem 8 barely exist. Clara Jiménez-Gestal and Pérez-Iquierdo [104] studied how isotopisms of finite-dimensional real division algebra are related to the Lie algebra of its ternary derivations. More recently, the authors [105,106] dealt with the distribution of filiform Lie algebras into isotopism classes.

**Theorem 16.** *There exist*

1.　*n isotopism classes of n-dimensional pre-filiform Lie algebras over any finite field;*
2.　*five isotopism classes of six-dimensional filiform Lie algebras over any field;*
3.　*eight isotopism classes of seven-dimensional filiform Lie algebras over any algebraically closed or finite field of characteristic distinct of two or three;*

4. *ten isotopism classes of seven-dimensional filiform Lie algebras over any algebraically closed or finite field of characteristic two.*
5. *nine isotopism classes of seven-dimensional filiform Lie algebras over any algebraically closed or finite field of characteristic three.*

### 4.6. Malcev Algebras

In 1955, Anatoliĭ Mal' tsev [107] introduced Malcev algebras as tangent algebras of local analytic Moufang loops. It was Arthur Sagle [108] who presented them formally in 1961 as a generalization of Lie algebras. Specifically, a *Malcev algebra* $(A, \cdot)$ over a field $\mathbb{F}$ is an anticommutative algebra such that $x^2 = 0$, for all $x \in A$, and

$$((x \cdot y) \cdot z) \cdot x + ((y \cdot z) \cdot x) \cdot x + ((z \cdot x) \cdot x) \cdot y = (x \cdot y) \cdot (x \cdot z),$$

for all $x, y, z \in A$.

The authors [109] have recently distributed finite-dimensional Malcev magma algebras over finite fields into isotopism classes. Recall in this regard that a *magma algebra* is defined similarly to a quasigroup algebra, as we have described in Section 3.2, once the base quasigroup is replaced by a *magma*, that is, by a finite set endowed with a binary operation.

**Theorem 17.** *The following assertions hold:*

1. *Every three-dimensional Malcev algebra is isotopic to a Lie magma algebra.*
2. *There exist four isotopism classes of three-dimensional Malcev algebras over any finite field.*
3. *Every four-dimensional Malcev algebra is isotopic to a Lie algebra.*
4. *There exist eight isotopism classes of four-dimensional Malcev algebras over any finite field.*

### 4.7. Genetic and Evolution Algebras

In the beginning of the 1940s, Etherington [110] introduced *genetic algebras* as non-associative algebras that enable one to formulate mathematically the inheritance mechanisms of sexual organisms described by Mendel's laws.

In 1966, Monique Bertrand [111] delved into this topic by overviewing in particular some fundamentals on isotopisms of algebras. Indeed, in the same year, Philip Holgate [112] dealt with the distribution of *special train algebras* (a genetic algebra introduced by Etherington in his original manuscript [110]) into isotopism classes in order to describe inheritance as a mixture of chromosome and chromatid segregation.

In 1985, G. A. Ringwood [113] overviewed different non-associative genetic algebras and indicated how to use isotopisms of algebras to treat selection.

In 1987, Tania Campos and Holgate [114] realized the relevant role that isotopisms play to represent mutations algebraically. In particular, they proved that those genetic algebras representing polyploidy and chromosome segregation, and which are related to different mutation rates, are principal isotopes. Due to it, evolutionary operators can be analyzed by focusing only on a representative algebra of each isotopism class.

Currently, a genetic algebra that is being widely developed is that of evolution algebras. In the period 2006–2008, Jianjun Tian and Petr Vojtechovsky [115,116] introduced these algebras to represent algebraically asexual reproduction of alleles in non-Mendelian Genetics. More specifically, an *evolution algebra* is an $n$-dimensional algebra over a base field $\mathbb{F}$ that admits a *natural basis* $\{e_1, \ldots, e_n\}$ satisfying that

1. $e_i e_j = 0$, for all $1 \leq i, j \leq n$ such that $i \neq j$; and
2. $e_i e_i = \sum_{j=1}^{n} t_{ij} e_j$, for all $1 \leq i \leq n$ and some structure constants $t_{i1}, \ldots, t_{in} \in \mathbb{F}$.

Each basis vector $e_i$ represents a different genotype under consideration; the product $e_i e_j = 0$ represents uniparental inheritance when $i \neq j$, and self-replication when $i = j$; and each structure

constant $t_{ij}$ represents the probability that the $i$th genotype becomes the $j$th genotype in the next generation. We refer the reader to the bibliography that Tian himself keeps updated and available online at [117].

Concerning the distribution of evolution algebras into isotopism classes, which is uniquely identified in non-Mendelian Genetics with mutations or regulatory mechanisms that relate any two status of the genotypes of a pair of chromatids, the authors [118,119] have recently proved the following result.

**Theorem 18.** *There exist four (respectively, eight) isotopism classes of two-dimensional (respectively, three-dimensional) evolution algebras over any field.*

*4.8. Quasigroups, Latin Squares and Related Structures*

Apart from Albert and Bruck, let us mention Albert Sade and Valentin Belousov as a pair of prolific authors in the early story of the theory of isotopisms of quasigroups. On the one hand, Sade proved different results on isotopisms [120], autotopisms [121] and paratopisms [122] of quasigroups. On the other hand, Belousov proved that every quasigroup that is described by a balanced identity is isotopic to a group; introduced orthogonal and crossed isotopisms [123]; and dealt with isotopisms and paratopisms of different types of quasigroups [124].

Some other remarkable authors who have contributed to the consolidation of the theory of isotopisms of quasigroups have been Trevor Evans [125] and his doctoral student, Etta Falconer [126]); the already mentioned James Marshall Osborn [127] and his doctoral student, Daniel Robinson [128]; Sherman Stein [129], Rafael Artzy [130], János Aczél [131], Curt Lindner [132] and Jonathan Smith [133].

New results on isotopisms [134], autotopisms [135], automorphisms [136] and parastrophisms [137] of quasigroups have continued to progress until the present day. Furthermore, different applications of autotopisms of quasigroups in Cryptography have been developed [138,139].

In any case, Latin squares constitute, as multiplication tables of quasigroups, the combinatorial structures par excellence in the theory of isotopisms. In 1977, József Dénes and Anthony Donald Keedwell put together the theory of Latin squares in their monograph [140], which is currently considered the essential book on the topic. In particular, they also dealt with the theory of isotopisms of Latin squares. Since that moment on, isotopisms have been used to study different symmetries on Latin squares [141,142]. In particular, concerning the classification of Latin squares according to the symmetries derived from their isotopism, isomorphism and paratopism classes, let us remark that the number of such classes is known [143,144] for Latin squares of orders up to 11.

Recent advances about the sets of autotopisms, automorphisms and autoparatopisms of Latin squares are exposed, for instance, in [145–148]. We note in particular on the implementation of autotopisms of Latin squares into the design of authentication schemes [149], secret sharing schemes [150,151] and cryptographic transformations [152] in Cryptography. In addition, an implementation of autotopisms and paratopisms of Latin squares into the design of new graph colouring games [153,154] is being currently developed. Concerning conjugacy of Latin squares, we refer the readers to [155] as a work dealing with the enumeration of totally symmetric Latin squares, in which all their conjugates coincide.

The equivalent classification of partial Latin rectangles has not been studied in depth yet. Recall in this regard that an $r \times s$ *partial Latin rectangle based on the set* $[n]$ is an $r \times s$ array in which each cell is either empty or contains one symbol chosen from the set $[n]$, such that each symbol occurs at most once in each row and in each column. This is a *partial Latin square of order* $n$ if $r = s = n$ (a Latin square if, in addition, there no empty cells exist).

Every partial Latin square constitutes the multiplication table of what is called a *partial quasigroup* of the same order. *Isotopisms*, *isomorphisms*, *conjugacy* and *paratopisms* of partial Latin rectangles are then defined similarly to the corresponding notions for Latin squares. All of them give rise to equivalence classes among partial Latin rectangles. Thus, for instance, the following two partial Latin squares belong to the same isotopism class by means of the isotopism $((123), (12)(3), (13)(2)) \in \mathfrak{I}_3$:

| 2 | 1 |   |
|---|---|---|
| 1 |   |   |
|   |   | 3 |

|   |   | 1 |
|---|---|---|
| 3 | 2 |   |
|   | 3 |   |

Currently, the distribution into isotopism, isomorphism and paratopism classes of partial Latin rectangles for which $1 \leq r, s, n \leq 6$ is known [156–159]. The number of paratopism classes of partial Latin rectangles with at most 12 non-empty cells is also known [160,161]. All of these equivalence relations give rise to different kinds of symmetries. Thus, for instance, if an isotopism (respectively, isomorphism or paratopism) preserves a given partial Latin rectangle, the former is said to be an *autotopism* (respectively, *automorphism* or *autoparatopism*) of the latter. Currently, the enumeration of autotopisms of a given partial Latin rectangle is an open problem that requires the description of different isotopism invariants [146,148,162,163].

The sets of autotopisms, automorphisms and autoparatopisms of partial Latin rectangles constitute groups of symmetries with the component-wise composition of permutations. Conjugacy also determine symmetries of partial Latin rectangles. Thus, for instance, the following array illustrates the concept of *totally symmetric partial Latin square*, in which all its six conjugates coincide:

| 3 |   | 1 |
|---|---|---|
|   | 2 |   |
| 1 |   | 3 |

A current open problem to be solved here is the study of totally symmetric partial Latin squares having a given autotopism group. In [163], it is considered the more general problem in which the autoparatopism group is any subgroup of the symmetric group $S_3$.

We finish the exposition with some highlights about how isotopisms have been used in the literature to deal with the symmetry of partial Latin rectangles and their related structures.

Firstly, let us remark that the symmetry that a given partial Latin rectangle inherits from its autotopism group derives from the cycle structure of each one of its autotopisms. Recall in this regard that the *cycle structure* of a permutation $\pi \in S_n$ is the expression $z_\pi := n^{\lambda_n^\pi} \dots 1^{\lambda_1^\pi}$, where $\lambda_l^\pi$ denotes the number of cycles of length $l$ in the unique decomposition of $\pi$ as a product of disjoint cycles. In practice, it is only written those factors for which $\lambda_l^\pi > 0$, and any factor of the form $l^1$ is replaced by $l$. Thus, for instance, the cycle structure of the permutation $(123)(45)(67) \in S_7$ is $32^2$. As a natural generalization, the *cycle structure* of an isotopism $\Theta = (\alpha, \beta, \gamma) \in \Im_n$ is defined as the triple $(z_\alpha, z_\beta, z_\gamma)$. Thus, for example, the cycle structure of the isotopism $((123), (12)(3), (13)(2)) \in \Im_3$ is $(3, 21, 21)$. Currently, the set of cycle structures of Latin squares of order $n \leq 17$ is known [164,165].

The following result [164–166] establishes the role that cycle structures play concerning the problem of counting partial Latin rectangles having a given isotopism within its autotopism group.

**Theorem 19.** *The number of partial Latin rectangles having a given isotopism in their autotopism group only depends on the cycle structure of such an isotopism.*

Furthermore, since Evans [167] introduced in 1960 the problem of embedding partial quasigroups of order $n$ into quasigroups of order $2n$, different authors have dealt with embeddings of partial Latin squares. In this regard, and concerning conjugacy and symmetry of partial Latin squares, we note the works [168–170] about partial totally symmetric Latin squares that can be embedded into totally symmetric Latin squares.

## 5. Conclusions

This paper has dealt with the antecedents, origin, early stage and later development of the theory of isotopisms of algebras, quasigroups and related structures that was introduced by Albert and Bruck in the 1940s as a generalization of the classical theory of isomorphisms. To this end, we have

enumerated some of the more relevant classical and current references on this topic by focusing in particular on those results dealing with the classification of different algebraic and combinatorial structures according to symmetries that the classical theory of isomorphisms does not take into account.

We would like to finish the paper by mentioning that the wide amount of authors and works that have dealt with the theory of isotopisms since the original manuscripts of Albert and Bruck makes it possible that we have run the risk of omitting certain references on some specific aspects of the subject. In any case, by means of the comprehensive and extensive bibliography that we have included at the end of the paper, we have attempted to give the reader the possibility of delving into that aspect in which he/she is interested. The collection formed by all the references existing in each one of the cited works constitutes an excellent starting point in this regard.

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