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One-sided learning about one's own type in a two-sided search model: The case of n types of agents

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One-sided learning about one's own type in a two-sided search model: The case of n types of agents*

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Abstract

This study analyzes a two-sided search model in which agents are vertically heterogeneous and agents on one side do not know their own type. Agents with imperfect self-knowledge update their beliefs based on the offers or rejections they receive from others. The results are as follows. An agent with imperfect self-knowledge lowers his or her reservation level if the agent receives a rejection that leads him or her to revise belief downward. However, an agent with imperfect self-knowledge does not raise his or her reservation level even if the agent receives an offer that leads him or her to revise his or her belief upward. As a result, an agent with imperfect self-knowledge has the highest reservation level when he or she has just entered the market; after that, a series of meetings gradually lowers his or her reservation level over the duration of the search.

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1 Introduction

Many studies have examined individual search behavior with incomplete information (e.g., Rothschild (1974), Morgan (1985), Burdett and Vishwanath (1988), Bikhechandani and Sharma (1996), and Adam (2001)). However, while most authors have focused on the uncertainty about the shape of the wage distribution, the present study focuses on agents' uncertainty about their own type. More specifically, we introduce learning about one's own type into a two-sided search model and examine the interaction between the marriage pattern (i.e., who marries whom) and learning at market equilibrium. In this regard, this study makes a contribution to the literature on search, since few previous studies of this topic have paid attention to imperfect self-knowledge.

In Gonzalez and Shi (2010), agents learn about their own job-finding abilities through offers or rejections from firms. In contrast to that study's directed search model, ours is a random two-sided search model in which searchers do not know their own type but do know the types of others. Thus, they update their beliefs about their own type after observing offers or rejections from others. For example, when searching for a job, workers are evaluated by prospective employers on their types (abilities or skills) when they meet. If a worker is inexperienced, the worker's self-assessment is based on limited experience. By contrast, employers may have considerable experience evaluating workers. As a result, a young worker may learn something about his or her own worker type when he or she observes an offer or a rejection from an employer.[1]

Similarly, in the context of the marriage market, a single agent is evaluated with regard to his or her marital charm by an agent of the opposite sex. When agents are young, their self-assessment is based on limited experience, perhaps including height, age, academic achievement, or family background. However, because marital charm is determined by various elements such as attraction, intelligence, height, age, education, income, position at work, social status, and family background, an agent of the opposite sex may be in a better position to assess a young agent's charm than the agent himself/herself.[2]

From a methodological perspective, we introduce learning about one's own type by using the framework of Burdett and Coles (1997), which is a two-sided search model with continuous types of agents under complete information. Although our model focuses on marriage, the ideas and techniques can be applied to other two-sided search frameworks, such as the labor market, the housing market, and other markets where heterogeneous buyers and sellers search for the right partner. In addition, we assume *non-transferable utility*: it is impossible for a pair to bargain for the division of total utility. In the labor market, utility is generally assumed to be transferable. However, for example, when a worker has great interest in a job because of its location or when an employer is attracted to the worker on account of his or

[1] If experienced workers search for a new job that is similar to their previous job, they may have a more accurate self-assessment of their ability than their potential employers. However, such situations are not considered in this study.

[2] Although marital charm comprises various elements, for simplicity, most studies assume it is one-dimensional and scalar. Therefore, we adopt the same approach here.

her personality, these utilities can be considered non-transferable. Moreover, wages might be restricted, for example by a legislated minimum wage or an industry-wide union relationship. When wages and all other terms of the relationship are fixed beforehand, and the pair cannot negotiate after they meet, their utility can be considered non-transferable (see Burdett and Wright (1998)).

The model is described as follows, using the marriage market as a context. Single men or women are vertically heterogeneous; that is, there exists a ranking of marital charm (or types). Single agents enter the market to search for a marital partner. The (inherent) type of the opposite sex can be recognized when a man and a woman meet. However, while all men know their types, women who have just entered the marriage market do not.[3] In addition, each agent's optimal search strategy is "reservation level," which depends on the agent's time cost and beliefs in the market. An agent continues searching until meeting an agent of the opposite sex who is at least as good as his or her predetermined reservation level. A man and woman marry and leave the market if they meet and both propose. If at least one of the two decides to reject, they separate and continue to search for another partner. From these settings, the marriage pattern in the market is determined: agents of either sex are partitioned into clusters of marriages when sorting, in a kind of positive assortative matching (PAM).[4]

The results presented in this paper show that because of the belief-updating process, a woman may reject a man who she would accept if she had perfect self-knowledge or accept a man who she would reject if she had perfect self-knowledge. As a result, marriages of all women with imperfect self-knowledge, except the highest-type women, are delayed by their own learning. Moreover, the existence of women with imperfect self-knowledge in the market lowers the reservation level of all men, except the highest-type men, because women's learning delays the marriages of these men.

This study also shows that a series of meetings *gradually* reduces the reservation level of a woman with imperfect self-knowledge over the duration of the search. A woman with imperfect self-knowledge lowers her reservation level when she receives a rejection that conveys some information about her type. By contrast, a woman with imperfect self-knowledge never raises her reservation level even if she receives an offer from a man. This is because a higher offer results in a woman with imperfect self-knowledge getting married, as in Burdett and Vishwanath (1988). Moreover, a woman with imperfect self-knowledge does not raise her reservation level even if she receives an offer that leads her to revise her belief upward. The decision of a woman with imperfect self-knowledge whether to accept a man depends on her decision after learning. Hence, a man who will be rejected by her after her learning is also rejected by her before. Therefore, even if she updates her belief upward, her reservation level

[3] If all women know their own types and male entrants initially do not, qualitatively, the results remain the same. The one-sided imperfect knowledge assumption makes it easier to determine the influence of imperfect self-knowledge than when neither party has perfect knowledge.

[4] PAM is said to hold if the characteristics (i.e., types and marital charm) of those who match are positively correlated. Becker (1973) finds strong empirical evidence of a positive correlation between the characteristics of partners.

does not rise. From these results, a woman with imperfect self-knowledge has the highest reservation level when she has just entered the market.

The possible sources of declining reservation wages have received much attention in the search literature (see Burdett and Vishwanath (1988)). In particular, the sequence of the reservation wage, which completely describes the behavior of agents when search is a sequential process, declines with the duration of the search (see Gronau (1971), Salop (1973), Sant (1977), and Burdett and Vishwanath (1988)). The influence of search duration on the reservation wage is not yet well understood empirically.[5] Several empirical studies show that declining reservation wages are monotonic only when certain conditions on the variables hold in the model (Kiefer and Neumann (1981), Lancaster (1985), Addison, Centeno, and Portugal (2004), and Brown and Taylor (2009)). Burdett and Vishwanath (1988) also show that when workers learn an unknown wage distribution, the reservation wage of an unemployed worker declines with his or her unemployment spell in a search model. In their model, the worker is employed when he or she receives a high offer. By contrast, the worker perceives the jobs available to him or her as offering low wages when he receives an offer much lower than expected. Then, the worker revises his or her reservation wage downward. Unlike their model, our model is a two-sided search model and agents know the distribution of types but do not know their own type. In two-sided search models, receiving an offer is likely to lead to an increase in the reservation level of an agent with imperfect self-knowledge; however, our results show that an agent with imperfect self-knowledge does not revise his or her reservation level upward when he or she receives an offer.

There are few studies on imperfect self-knowledge in the search literature. Gonzalez and Shi (2010) show that learning from search can lead the desired wages (the wage in the chosen submarket) and reservation wages to decline with the unemployment duration in the directed search model with two types of agents. In their model, the value function of an unemployed worker strictly increases in the worker's belief about his or her own ability, because a worker or a firm chooses the submarket to search. Hence, the reservation wage strictly decreases over the search duration, as the worker's belief becomes gradually worse. In contrast to their model, ours is a random two-sided search model with one-sided imperfect self-knowledge. Agents are of n types, and an agent with imperfect self-knowledge decides the reservation utility based on the distribution of beliefs in the market and his or her learning process in the future. Thus, the value function in our model is not monotonic with respect to the agent's belief.

Maruyama (2010) investigate the case of three types of agents under a one-sided imperfect self-knowledge assumption. It is difficult for the present study to investigate the possibility of multiple equilibria and to show the influence of agents with imperfect self-knowledge on the behavior of others in the market because of $n\,(\geq 2)$ types of agents. By contrast, Maruyama (2010) demonstrates that influence and shows that multiple equilibria arise.

Furthermore, Maruyama (2013) investigate the case of two types of agents under two-

[5] Generally, it is ambiguous as to whether declining reservation wages are monotonic. Furthermore, measuring the effect of search duration on reservation wage is difficult.

sided imperfect self-knowledge (i.e., where both men and women do not know their own type when they have just entered the marriage market). In the two-sided imperfect self-knowledge framework, the reservation level of an agent is simultaneously affected by two factors: (i) the share of agents of the opposite sex who currently reject his or her type due to imperfect self-knowledge and (ii) his ore her uncertainty of his or her own type. By contrast, these two factors separately affect the reservation level of an agent in the one-sided imperfect self-knowledge framework.

The present study and Maruyama (2010) adopts the cloning assumption; if a couple marries and leaves the market, two identical types of agents enter the market immediately. By contrast, Maruyama (2013) assumes exogenous inflow, which may be more reasonable than the cloning assumption but complicates the analysis (see Burdett and Coles (1999)). For this reason, Maruyama (2013) examine the case of two types of agents.

The remainder of this paper is organized as follows. Section 2 describes the basic framework for our analysis. In Section 3, we assume that agents are rational, except that all agents expect that the type distributions of each sex in the market and the distribution of agents' beliefs are constant through time. Under these settings, we characterize a *search equilibrium*, for any given inflow distributions of each sex. Moreover, we first derive a *perfect sorting equilibrium* (PSE) as a benchmark case, in which only persons of the same type marry under perfect self-knowledge. In Section 3.2, we introduce the concept of imperfect self-knowledge. In Section 3.3, we investigate the properties of the reservation utility level of an agent with imperfect self-knowledge. In Section 3.4, we characterize PSE with imperfect self-knowledge. At search equilibrium, one can calculate the number and type distribution of agents who exit the market through marriage in each period. If outflow distribution and number who exit are equal to inflow distribution and the number who enter the market, respectively, the distributions in the market become constant. Then, the steady-state equilibrium is derived, as discussed in Section 4. Finally, Section 5 concludes the paper.

2 Basic framework

This section presents the basic framework for our analysis. Let us assume that a large, equal number of single men and women, N, participate in the marriage market. Each single agent in the market wants to marry an agent of the opposite sex.

Finding a marital partner always entails a time cost. Contacting an agent of the opposite sex is difficult. Let α denotes the rate at which a single agent contacts an agent of the opposite sex, where α is the parameter of the Poisson process.[6]

Let us assume that agents are ex ante heterogeneous. Let x denotes the type (charm) of a single man or woman in the market; it is assumed to be a real number.

[6] Here, $\alpha = M(N,N)/N = M(1,1)$. That is, the encounter function leads to the constant returns to scale. When all agents are homogeneous, all encounters result in a match; at this time, the encounter function means the matching function. However, because agents in this study are heterogeneous, encounters do not always result in a match.

When a man and a woman meet, each agent can instantly know the opponent's (innate) type and decide whether or not to propose. For simplicity, let us assume that both agents simultaneously submit offers or rejections. If at least one of the two agents decides to reject, they separate and each search for another partner. If both agents propose, they marry and exit the marriage market permanently.

All agents discount at rate $r > 0$, and an agent is assumed to obtain zero utility flow while he or she is single, whereas if he or she marries, he or she obtains a utility flow equal to his or her spouse's type per unit of time, and vice versa. That is, utilities are *non-transferable*: agents cannot bargain for the division of the total marital utility. Furthermore, we assume that people live permanently and that there is no divorce.

Let βdt denotes the number of new single men and women who enter the market in any time interval dt. Let $\Psi_i(.), i = m, w$, denote the type distribution of male (m) or female entrants (w). For simplicity, we assume that $\Psi_i(.)$ strictly increase over the interval $[\underline{x}_i, \bar{x}_i]$, where $\underline{x}_i$ and $\bar{x}_i$ indicate the infimum and supremum of its support, respectively, and $\underline{x}_i > 0$, for $i = m, w$.

Let $F_m(., t)$ denotes the type distribution of men in the market in period t. Similarly, $F_w(., t)$ denotes the type distribution of women at t.

For simplicity, let us assume that there are n discrete types of men and women, differentiated by level of charm. Let x_k/r denotes the (discounted) utility of marrying a k-type agent ($k = 1, 2, \ldots, n$). Let us assume that $x_1 > x_2 > \ldots > x_n > 0$. That is, all agents want to marry a 1-type agent. Let λ_k^i, for $k = 1, 2, \ldots, n$, denotes the share of k-type agents $i (= m, w)$ in the market, where $\sum_{k=1}^{n} \lambda_k^i = 1$.

3 Stationary environment

To investigate the influence of imperfect self-knowledge on the behavior of all agents, we first explore the stationary environment. In Section 4, we explore the steady sate.

In this section, we assume that all agents believe that the market can be characterized by a stationary type distribution of men and women (F_m, F_w), where $F_i(x, t) = F_i(x)$, for all x and all t, and for $i = m, w$. Let us assume that F_i has support $[\underline{x}_i, \bar{x}_i]$, for $i = m, w$.

We first derive a *search equilibrium with perfect self-knowledge*, which is a benchmark, in the next section.[7] Later, we study a search equilibrium with imperfect self-knowledge (i.e., where agents do not perfectly know their own types) and compare the equilibrium with the benchmark case.

3.1 Perfect self-knowledge—Benchmark result

Given (F_m, F_w), we can define the following search equilibrium with perfect self-knowledge.

Definition 1 *Under perfect self-knowledge (that is, where all agents know their own types),*

[7] Here, we consider the basic framework of Burdett and Coles (1997).

given the stationary distribution $F_i(.)$, for $i = m, w$, a search equilibrium with perfect self-knowledge *requires that all agents maximize their expected discounted utilities.*

In a search equilibrium, it is not necessarily true that the inflow of agents into the market equals the outflow of agents. In Section 4, we identify (F_m, F_w), where the two flow distributions are equal.

Moreover, to show the influence of learning on the behavior of all agents, we restrict our attention to the following equilibrium, which we use under perfect self-knowledge as a benchmark.

Definition 2 *In a* perfect sorting equilibrium *(PSE), only persons of the same type marry.*

We first derive a search PSE. Given (F_m, F_w), all agents use stationary strategies, which specify which agents of the opposite sex an agent will propose to if they meet. Hence, the set of agents of the opposite sex who will propose to an agent of type x is well defined. Let $\varepsilon_m(x)$ denotes the share of men who propose to a woman with x, if they meet, and let $F_m(.|x)$ denotes the type distribution of these men. Hence, $\alpha_w(x) = \alpha\varepsilon_m(x)$ is the rate at which a woman with x receives offers. In a similar fashion, we define $F_w(.|x)$ and $\alpha_m(x) = \alpha\varepsilon_w(x)$ for all x.

Let $V_w(x_k)$ denotes a k-type woman's expected discounted lifetime utility when single. Standard dynamic programming arguments imply that

$$V_w(x_k) = \tfrac{1}{1+rdt}\left[\alpha_w(x_k)\, dt E\left[\max\left\{\tfrac{x_k}{r}, V_w(x_k)\right\}|x_k\right] + (1 - \alpha_w(x_k)\, dt)\, V_w(x_k)\right]$$

where x_k has the distribution $F_m(.|x_k)$. Manipulating this equation and letting $dt \to 0$ yields

$$rV_w(x_k) = \alpha_w(x_k)\, E\left[\max\left\{\tfrac{x_k}{r}, V_w(x_k)\right\} - V_w(x_k)\,|x_k\right]. \tag{1}$$

The strategy takes the form of a reservation match strategy—a k-type woman will accept a man on contact if and only if his type is at least as great as $R_w(x_k) \equiv rV_w(x_k)$.

Since the situation is the same for men, the expected discounted lifetime utility of a single k-type man, $V_m(x_k)$, satisfies

$$rV_m(x_k) = \alpha_m(x_k)\, E\left[\max\left\{\tfrac{x_k}{r}, V_m(x_k)\right\} - V_m(x_k)\,|x_k\right]. \tag{2}$$

where x_k has $F_w(.|x_k)$. From (2), we can obtain the reservation match strategy of a k-type man $R_m(x_k) \equiv rV_m(x_k)$.

For search equilibrium, $\alpha_w(x_k)$ and $F_m(.|x_k)$ must be consistent with the reservation match strategy of men, described by (2). The same is true for men.

In the equilibrium, all agents use a reservation rule. If a man will propose to a woman with type x', he will also propose to a woman with type $x'' > x'$.

As an agent with x'' receives at least the same offers as an agent with x', $V_i(x'') \geq V_i(x')$, $i = m, w$. Hence, in the equilibrium, the reservation strategies $R_i(.)$ are nondecreasing, for $i = m, w$.

The next proposition shows that in a PSE, a k-type man, for $k = 1, \ldots, n$, only proposes to women with the same type or higher, and rejects women with a lower type. Women do the same. Consequently, k-type agents who marry within their group form a cluster of marriages (cluster k) in a search PSE.

Proposition 1 *Let us assume that all agents recognize their own types. There exists a PSE if (a)* $x_{k+1} < R_m^*(x_k) \equiv \frac{\alpha\lambda_k^w x_k}{\alpha\lambda_k^w + r} \leq x_k$, *for* $k = 1, \ldots, n-1$ *and* $R_m^*(x_n) \equiv \frac{\alpha\lambda_n^w x_n}{\alpha\lambda_n^w + r} \leq x_n$, *and (b)* $x_{k+1} < R_w^*(x_k) \equiv \frac{\alpha\lambda_k^m x_k}{\alpha\lambda_k^m + r} \leq x_k$ *for* $k = 1, \ldots, n-1$, *and* $R_w^*(x_n) \equiv \frac{\alpha\lambda_n^m x_n}{\alpha\lambda_n^m + r} \leq x_n$.[8]

Proof. See Appendix. ■

Proposition 1 implies that with constant α, an k-type agent rejects $k+1$-type opposite sex agents if the share of k-type agents of the opposite sex is sufficiently large or if the difference between x_k and x_{k+1} is sufficiently large that they satisfy $x_{k+1} < R_i^*(x_k)$, for $i, j = m, w$. Conversely, a k-type agent accepts a $k+1$-type opposite sex agent if there are sufficiently few k-type opposite sex agents, or if $(x_k - x_{k+1})$ is sufficiently small that $x_{k+1} \geq R_i^*(x_k)$. If $R_i^*(x_k) \leq x_n$, $k = 1, \ldots, n$, all agents obtain the same expected discounted utility: $V_i(x_1) = \ldots = V_i(x_n) \leq \frac{x_n}{r}$, $i = m, w$.

If $r = 0$, $x_{k+1} < R_i^*(x_k)$, for $k = 1, \ldots, n-1$, and $i = m, w$. Therefore, the equilibrium is the PSE when $r = 0$.

To clarify the influence of learning on a market, in the following sections, we assume that (F_m, F_w) and x_k are common across equilibria and that the conditions in Proposition 1 are satisfied: $x_k \geq R_i^*(x_k)$, for $k = 1, \ldots, n$, and $i = m, w$.[9] This restriction and the assumption of n discrete types simplify the analysis. In Burdett and Coles (1997), where agents' types are continuous, agents of either sex are partitioned into n classes when sorting, which is a kind of PAM, under perfect self-knowledge. Even if agents' types are discrete, men and women can be partitioned into some classes based on the reservation levels of opposite sex agents, similar to Burdett and Coles (1997). The restriction $R_i^*(x_k) \leq x_k$ ensures that the equilibrium under perfect self-knowledge leads to PAM instead of classes, as in Burdett and Coles (1997). In other words, "type" equals "class" when agents' types are discrete under perfect self-knowledge. Thus, the reservation level of k-type agents determines the k-th type of agents of the opposite sex.

In the model with learning, more partitions are generated than those under perfect self-knowledge.

3.2 Imperfect self-knowledge

Let us assume that all men know their (innate) types, whereas no women know their types when they have just entered the marriage market.[10] Then, a woman with imperfect self-

[8] At this time, the boundary conditions are $R_m^*(x_1) \equiv \frac{\alpha\lambda_1^w x_1}{\alpha\lambda_1^w + r} < x_1$, $R_w^*(x_1) \equiv \frac{\alpha\lambda_1^m x_1}{\alpha\lambda_1^m + r} < x_1$, $R_m^*(x_n) \equiv \frac{\alpha\lambda_n^m x_n}{\alpha\lambda_n^m + r} \leq x_n$, and $R_w^*(x_n) \equiv \frac{\alpha\lambda_n^m x_n}{\alpha\lambda_n^m + r} \leq x_n$.

[9] For the other parameter ranges, it is difficult to show the indirect effect (indirect externality) of the learning process.

[10] If all women know their own type and no men initially know their type, the results are essentially the same.

knowledge (i.e., one who does not perfectly know her own type) will have a belief about her own type.

At the start of period $t = 0, 1, \ldots, \bar{t}$, a j-type woman with imperfect self-knowledge meets a man randomly, $j = 1, \ldots, n$. Both sexes can instantly recognize the innate type of an agent of the opposite sex when they meet.[11] For simplicity, we assume that a man need not know the belief (or history) of a woman he meets.[12] They simultaneously submit their offers or rejections.[13] If they separate, the woman updates her belief about her own type, and therefore, also revises her reservation level. Then, she searches for another partner.

Let $o_m^t(x_j) \in O = \{o, o^-\}$ denotes the action of a man observed by a j-type woman as a result of a search outcome in period t. Here, O is the action set. If the j-type woman observes search outcome $\left(x_k^t, o\right)$, she knows that the k-type man accepted her. If the j-type woman observes $\left(x_k^t, o^-\right)$, she knows that the k-type man rejected her.[14]

Let assume that $[x_b, x_a]$ for $a < b$ is a set of types a woman believes she may belong to before observing $\left(x_k^t, o_m^t(x_j)\right)$ at t. Let $\mu_{a,b} \in \Delta([x_b, x_a])$ denotes this woman's belief about her own type, where $\Delta([x_b, x_a])$ is a set of probability distributions over $[x_b, x_a]$. The prior belief is $\mu_0 \in \Delta([\underline{x}_w, \bar{x}_w])$. Moreover, μ_0 is assumed to be the type distribution of new female entrants, $\Psi_w(.)$, which is common knowledge.[15] Therefore, μ_0 is the same distribution for all women. Moreover, let $\mu_{a,b}(x_j)$ denotes the probability that a woman with belief $\mu_{a,b}$ assigns herself to a particular type $x_j \in [x_b, x_a]$. This probability is determined by using Bayes' rule given μ_0.

Since men's strategies have the reservation-level property, a proposal or rejection from a man provides a woman with information indicating that she does not belong to a particular set of types of women. If a woman with $\mu_{a,b}$ observes (x_k, o), this offer informs her that her type does not belong to $[\underline{x}_w, R_m(x_k))$. Let $x_{d(k)}$ denotes an infimum type of women to whom a k-type man proposes, that is, $R_m(x_k) \leq x_d$. Therefore, she updates her belief to $\mu_{a,d(k)}$. If $x_{d(k)} \leq x_b$, in contrast, her belief remains $\mu_{a,b}$. The case of $x_{d(k)} > x_a$ is ruled out, because all agents are rational in this paper.

By contrast, if the woman observes (x_k, o^-), she know that her type does not belong to $[x_{d(k)}, \bar{x}_w]$. Hence, she changes her belief to $\mu_{d(k)+1,b}$ for $x_a \geq x_{d(k)+1} \geq x_b$. If $x_a < x_{d(k)+1}$, in contrast, her belief remains $\mu_{a,b}$. The case of $x_b > x_{d(k)+1}$ is also ruled out because all agents are rational. Generally, the woman's posterior belief, $\mu_{a',b'}(x_j)$, after observing $\left(x_k, o_m^t(x_j)\right)$

[11] A woman with imperfect self-knowledge does not know whether she is accepted by a man she meets before observing his action because of her imperfect self-knowledge. However, she can instantly recognize his actual type.

[12] In other words, we assume that a man does not regard the history of a woman whom he meets as a bad or good signal because men know that all women learn about their own types through meetings. If a man rejected a woman because of her long search duration, her learning would be delayed.

[13] If a man can instantly recognize the belief of a woman when they meet, he can know her action (i.e., whether she will propose) before observing it. Thus, results similar to those of our study can also be obtained in the case of a sequential move, in which a woman proposes to a man in the first move and he proposes or rejects her in the next move.

[14] Note that, in the model with discrete types, when an agent lowers his or her own reservation strategy, this does not always mean that he or she accepts an agent of the opposite sex he or she has previously rejected.

[15] In Gonzalez and Shi (2010), the initial expectation of the ability of a new worker depends on the distribution of the levels of ability of new workers.

in a period is given by[16]

$$\mu_{a',b'}\left(x_j\right)=\frac{\mu_{a,b}(x_j)\Pr\left(\left(x_k,o_m^t(x_j)\right)|x_j\right)}{\sum_{j=b}^{a}\mu_{a,b}(x_j)\times\Pr((x_k,o_m^t(x_j))|x_j)}. \tag{3}$$

From these settings, a woman's belief can be interpreted as history up to, but not including, the search outcome in the period.[17]

All agents know (F_m, F_w). However, now, a woman has a belief about her own type. Because we consider n types of agents, a woman believes she may belong to each of the following sets of types: $[x_n, x_1], [x_{n-1}, x_1], \ldots, [x_1, x_1], [x_n, x_2], [x_{n-1}, x_2], \ldots, [x_2, x_2], [x_n, x_3]$, $[x_{n-1}, x_3], \ldots, [x_3, x_3], \ldots, [x_n, x_{n-1}], [x_{n-1}, x_{n-1}]$, and $[x_n, x_n]$. Because the number of these sets is $\frac{n(1+n)}{2}$, the number of beliefs, $\bar{l}$, is finite and becomes at most $\frac{n(1+n)}{2}$. Then, the number of reservation utility levels of women in the market is at most $\frac{n(1+n)}{2}$.

Let $x_j^{a,b}$ denotes the state of a woman whose type is x_j and who has belief $\mu_{a,b}$. Let $G_i(.)$ denote the stationary distribution of agent $i\,(=m,w)$'s states. Let us assume that any $x_j^{a,b}>0$ is a real number and belongs to $[\underline{x}_i, \bar{x}_i]$ and that $G_i(.)$ is strictly increasing over the interval $[\underline{x}_i, \bar{x}_i]$, $i=m,w$. (We define $G_i(.)$ more precisely later.) Let $g_w\left(x_j^{a,b}\right)$ denotes the probability mass function of states.

Let us assume that all agents understand (G_m, G_w, μ_0) (at the steady state, $G_m(.)$ and $G_w(.)$ depend on $F_i(.), i=m,w$). As all men know their types, $G_m(.)=F_m(.)$.

Given (G_m, G_w, μ_0), any man uses a stationary strategy, where a strategy is a list of women to whom he will propose when they meet. By contrast, any woman with imperfect self-knowledge uses a stationary strategy in the sense that a strategy is a list of men to whom she will propose when they meet in period t.

Let $\varepsilon_m(x_j)$ denotes the share of men who propose to a j-type woman, if they meet. Then, let $F_m(.|x_j)$ denotes the type distribution of such men. Hence, $\alpha_w(x_j)=\alpha\varepsilon_m(x_j)$ is the rate at which a j-type woman receives offers. By contrast, let $\varepsilon_w(x_k)$ denotes the share of women who propose to a k-type man, if they meet. Then, let $G_w(.|x_k)$ denotes the distribution of the states of such women. Hence, $\alpha_m(x_k)=\alpha\varepsilon_w(x_k)$ is the rate at which a k-type man receives offers.

Let $V_m(x_k)$ denotes a k-type man's expected discounted lifetime utility when single. Standard dynamic programming arguments imply that

$$V_m(x_k)=\frac{1}{1+rdt}\left[\alpha_m(x_k)\,dtE\left[\max\left\{\frac{x_j^{a,b}}{r}, V_m(x_k)\right\}|x_k\right]+\left(1-\alpha_m(x_k)\,dt\right)V_m(x_k)\right]$$

where $x_j^{a,b}$ has distribution $G_w(.|x_k)$. However, when a couple marries, each agent obtains a utility flow equal to the spouse's actual type, namely, x_j. Manipulating this equation and

[16] Since we only consider pure strategies when self-knowledge is perfect in our model, $\Pr\left((\tilde{x}_k, a_m(x_k))\,|x_k\right)=0$ or 1 when a k-type woman observes $(\tilde{x}_k, a_m(x_k))$, given the strategies of men.

[17] The set of types a woman believes she may belong to, $[x_b, x_a]\subseteq[\underline{x}_w, \bar{x}_w]$, for any a, b, can also be interpreted as an information set in a sequential-move game.

letting $dt \to 0$ yields

$$rV_m(x_k) = \alpha_m(x_k) E\left[\left(\max\left\{\frac{x_j^{a,b}}{r}, V_m(x_k)\right\} | x_k\right) - V_m(x_k)\right]. \tag{4}$$

A k-type man will accept a woman on contact if and only if her type is at least as great as $R_m(x_k) \equiv rV_m(x_k)$.

By contrast, women with the same belief face the same decision problem, regardless of their own type. Hence, the lifetime expected discounted utility of a woman with $\mu_{a,b}$ in a period dt, $V_w(\mu_{a,b})$, satisfies

$$\begin{aligned} rV_w(\mu_{a,b}) &= \sum_{j=a}^{b} \mu_{a,b}(x_j) V_w(x_j|\mu_{a,b}) \\ &= \tfrac{1}{1+rdt} \sum_{j=a}^{b} \mu_{a,b}(x_j) \left[\begin{array}{l} (1-\alpha dt) V_w(\mu_{a,b}) + (\alpha - \alpha_w(x_a)) dt \left(V_w(\mu_{a,b})\right) \\ + (\alpha_w(x_a) - \alpha_w(x_j)) dt \left(V_w\left(\mu_{d(k)+1,b}\right)\right) \\ + \alpha_w(x_j) dt E\left(\max\left\{\frac{x_k}{r}, V_w\left(\mu_{a,d(k)}\right)\right\}\right) \end{array} \right], \end{aligned}$$

where $\Sigma_{j=a}^{b} \mu_{a,b}(x_j) = 1$, and x_k has $F_m(.|x_j)$. In the above equation, the second term means that if a woman with $\mu_{a,b}$ meets a man who rejects an a-type woman with probability $(\alpha - \alpha_w(x_a))$, she does not change her belief, because she has already met another man with x' who has $x_a < R_m(x') \leq x_{a-1}$ in the past.[18] The third term means that if a $j(\geq a)$-type woman meets a k-type man who accepts a $j-1$-type woman but rejects a j-type woman, she updates her belief to $\mu_{d(k)+1,b}$.[19] In the fourth term, if a woman with $x_j^{a,b}$ rejects a k-type man, who accepts her, she updates her belief to $\mu_{a,d(k)}$. However, if $x_{d(k)} \leq x_b$, her belief remains $\mu_{a,b}$ in the next period.

Manipulating the above equation and letting $dt \to 0$ yields

$$rV_w(\mu_{a,b}) = \sum_{j=a}^{b} \mu_{a,b}(x_j) \left[\begin{array}{c} (\alpha_w(x_a) - \alpha_w(x_j)) \left(V_w\left(\mu_{d(k)+1,b}\right) - V_w(\mu_{a,b})\right) \\ + \alpha_w(x_j) \left(E \max\left\{\frac{x_k}{r}, V_w\left(\mu_{a,d(k)}\right)\right\} - V_w(\mu_{a,b})\right) \end{array} \right]. \tag{5}$$

A woman with $\mu_{a,b}$ marries a man after contact if and only if his type is at least as great as $R_w(\mu_{a,b}) \equiv rV_w(\mu_{a,b})$.

Equilibrium means that the reservation strategies $R_m(.)$ are nondecreasing. If a woman will propose to a man with type x', she will also propose to a man with type $x'' > x'$. As a result of receiving at least the same offers, $V_m(x'') \geq V_m(x')$. Hence, $R_m(x'') \geq R_m(x')$. From this, $d(k)$ is not decreasing in k. By contrast, whether $R_w(\mu_{a,b})$ are decreasing or increasing is not obvious, because $\mu_{a,b}$ is a distribution, not a real number. However, as any man who wants to marry a woman with x' also wants to marry a woman with $x'' > x'$, $V_w(x''|\mu_{a,b}) \geq V_w(x'|\mu_{a,b})$ holds, for any $\mu_{a,b}$ and $x', x'' \in [x_b, x_a]$.

Although whether $R_w(\mu_{a,b})$ are decreasing or increasing is not obvious, the order of

[18] Otherwise she cannot be a woman with $\mu_{a,b}$.

[19] From $G_m(.) = F_m(.)$, $\alpha_w(x_a) \geq \alpha_w(x_j)$ for $x_a > x_j$.

the values of $R_w\left(\mu_{a,b}\right)$ partitions men into classes. By using this, we can define $G_w(.)$ more precisely. Let us order all women according to the type x_j and values of $R_w\left(\mu_{a,b}\right)$. When $R_w\left(\mu_{\bar{a},\bar{b}}\right) \geq R_w\left(\mu_{a',b'}\right) \geq \ldots \geq R_w\left(\mu_{\underline{a},\underline{b}}\right)$, we label the intervals as $I_1 = [x_{\bar{b}}, x_{\bar{a}}]$, $I_2 = [x_{b'}, x_{a'}], \ldots, I_{\bar{l}} = [x_{\underline{b}}, x_{\underline{a}}]$.[20] Then, $\mu_l \in \Delta(I_l)$ for $l = 1, 2, \ldots, \bar{l}$. Let x_k^l denotes a k-type woman with μ_l. Hence, we make the following assumption;

Assumption A.1. When $R_w\left(\mu_{\bar{a},\bar{b}}\right) \geq R_w\left(\mu_{a',b'}\right) \geq \ldots \geq R_w\left(\mu_{\underline{a},\underline{b}}\right)$, $I_1 = [x_{\bar{b}}, x_{\bar{a}}]$, $I_2 = [x_{b'}, x_{a'}], \ldots, I_{\bar{l}} = [x_{\underline{b}}, x_{\underline{a}}]$. We assume that $x_{\bar{a}}^1 > x_{\bar{a}+1}^1 > \ldots > x_{\bar{b}}^1 > x_{a'}^2 > x_{a'+1}^2 > \ldots > x_{b'}^2 > \ldots > x_{\underline{a}}^{\bar{l}} > x_{\underline{a}+1}^{\bar{l}} > \ldots > x_{\underline{b}}^{\bar{l}} > 0$ only for $G_w(.)$ and $g_w(.)$.

From this assumption, the distribution $G_w(.)$ is strictly increasing over the interval $[x_{\underline{b}}^{\bar{l}}, x_{\bar{a}}^1]$. Let $\phi_j^l \in [0,1]$, $l = 1, 2, \ldots, \bar{l}$, denotes the share of women with $\mu_l \in \Delta(I_l)$ of the j-type women, where $\Sigma_{l=1}^{\bar{l}} \phi_j^l = 1$, for any j. From this, $g_w\left(x_j^l\right) = \phi_j^l \lambda_j^w$, for $j = 1, 2, \ldots, n$, denotes the share of j-type women with μ_l.

When $R_w\left(\mu_{a,b}\right) \leq x_k$, given the best reservation match strategy, equation (5) can be rewritten. Let $x_{\tilde{s}(j)}$ denotes the highest type of men who accepts a j-type woman. Then, a man with $x \geq x_{\tilde{s}(j)-1}$ rejects a j-type woman. From $V_m(x'') \geq V_m(x')$, for $x'' > x'$, $s(j)$ is not decreasing in j. Generally, the arrival rate of proposals to a woman with x_j, for any $j (\geq a)$, becomes $\alpha_w(x_j) = \alpha \Sigma_{i=s(j)}^{\tilde{n}} \lambda_i^m$, $\alpha_w(x_a) - \alpha_w(x_j) = \alpha \Sigma_{i=a}^{s(j)-1} \lambda_i^m$, and $F_m(.|x_j) = \frac{F(.)}{\Sigma_{i=s(j)}^{n} \lambda_i^m}$. Hence, if $R_w\left(\mu_{a,b}\right) \leq x_k$, $R_m(x_k) \leq x_d$ and $d(k) < b$, then the reservation match strategy of a woman with $\mu_{a,b}$ can be rewritten as

$$\begin{aligned} rV_w\left(\mu_{a,b}\right) &= \sum_{j=a}^{d(k)} \mu_{a,b}(x_j) \left[\begin{array}{c} \left(\alpha_w(x_a) - \alpha_w(x_j)\right)\left(V_w\left(\mu_{d(i)+1,b}\right) - V_w\left(\mu_{a,b}\right)\right) \\ +\alpha_w(x_j)\left[\Sigma_{i=\tilde{s}(j)}^{k} \frac{\lambda_i^m}{\Sigma_{i=\tilde{s}(j)}^{\tilde{n}} \lambda_i^m}\left(\frac{x_i}{r} - V_w\left(\mu_{a,b}\right)\right)\right] \\ +\alpha_w(x_j)\left[\Sigma_{i=k+1}^{n} \frac{\lambda_i^m}{\Sigma_{i=\tilde{s}(j)}^{\tilde{n}} \lambda_i^m}\left(V_w\left(\mu_{a,d(i)}\right) - V_w\left(\mu_{a,b}\right)\right)\right] \end{array} \right] \\ &+ \sum_{j=d(k)+1}^{b} \mu_{a,b}(x_j) \left[\begin{array}{c} \left(\alpha_w(x_a) - \alpha_w(x_j)\right)\left(V_w\left(\mu_{d(i)+1,b}\right) - V_w\left(\mu_{a,b}\right)\right) \\ +\alpha_w(x_j)\left[\Sigma_{i=\tilde{s}(j)}^{n} \frac{\lambda_i^m}{\Sigma_{i=\tilde{s}(j)}^{n} \lambda_i^m}\left(V_w\left(\mu_{a,d(i)}\right) - V_w\left(\mu_{a,b}\right)\right)\right] \end{array} \right] \end{aligned} \quad (6)$$

where $V_w\left(\mu_{a,d(i)}\right) = V_w\left(\mu_{a,b}\right)$, for i such that $x_{d(i)} \leq x_b$.

If $d(k) \geq b$, a woman with $\mu_{a,b}$ does not update her belief after meeting a k-type or lower type man. At this time, by substituting $d(i) = b$, for $i = k, \ldots, n$ into (5), we obtain

$$rV_w\left(\mu_{a,b}\right) = \sum_{j=a}^{b} \mu_{a,b}(x_j) \left[\begin{array}{c} \left(\alpha_w(x_a) - \alpha_w(x_j)\right)\left(V_w\left(\mu_{d(i)+1,b}\right) - V_w\left(\mu_{a,b}\right)\right) \\ +\alpha_w(x_j)\left[\Sigma_{i=\tilde{s}(j)}^{k} \frac{\lambda_i^m}{\Sigma_{i=\tilde{s}(j)}^{\tilde{n}} \lambda_i^m}\left(\frac{x_i}{r} - V_w\left(\mu_{a,b}\right)\right)\right] \end{array} \right]. \quad (7)$$

Equations (6) and (7) describe the reservation match strategies of a woman with $\mu_{a,b}$, given the expected rate of proposals by men.

To simplify the analysis, we make the following assumption;

[20] If $R_w\left(\mu_{a,b}\right) = R_w\left(\mu_{a',b'}\right)$, let $I_{l'} = [x_{b'}, x_{a'}]$ and $I_{l'+1} = \left[x_{b''}, x_{a''}\right]$.

Assumption A.2. Men and women are partitioned into n classes by the reservation levels of the opposite sex agents, in Sections 3.3-4.[21]

Assumption A.2. guarantees that "type" equals "class" under imperfect self-knowledge. Therefore, the reservation level of a k-type man is a partition that determines the k-th type of women. By contrast, let $R_w\left(\mu_{l_k}\right) = R_w\left(\mu_{a'',b''}\right)$ such that $x_k \geq R_w\left(\mu_{a',b'}\right) \geq \ldots \geq R_w\left(\mu_{a'',b''}\right) > x_{k+1}$, for any k. That is, $R_w\left(\mu_{l_k}\right)$ is a partition that determines the k-th type (or class) of men. Then, we can re write $R_w\left(\mu_{a',b'}\right)$ as $R_w\left(\mu_{l_{k-1}+1}\right)$, for any k, where $l_0+1=1$. Because a woman with μ_l, for $l = l_{k-1}+1, l_{k-1}+2, \ldots, \bar{l}$, accepts a k-type man, the equation (4) can be rewritten as

$$rV_m\left(x_k\right) = \alpha_m\left(x_k\right) \sum_{j=1}^{R_m(x_k)} \frac{\sum_{l=l_{k-1}+1}^{\bar{l}} g_w\left(x_j^l\right)}{G_w(.|x_k)} \left(\frac{x_j}{r} - V_m\left(x_k\right)\right), \tag{8}$$

where $\sum_{l=l_{k-1}+1}^{\bar{l}} g_w\left(x_j^l\right)$ is the share of j-type women who accept a k-type man.

Assumption A.2 also ensures that a man proposes to a woman of the same type because there are n types of agents. Hence, $x_{k+1} < R_m\left(x_k\right)$, for $k = 1, \ldots, n$, $d\left(k\right) = k$ and $s\left(j\right) = j$. However, Assumption A.2 does not require that a woman proposes to a man of the same type under imperfect self-knowledge.

In the next section, we investigate the characteristics of the reservation utility level of agents with imperfect self-knowledge before we derive an equilibrium under imperfect self-knowledge.

3.3 Analysis of the reservation utility level

The following lemmas hold for the reservation level of a woman with imperfect self-knowledge. The first lemma shows that a woman with $\mu_{a,k}$ rejects a $k+1$-type man.

Lemma 1 *Suppose that $x_{k+1} < R_i^*\left(x_k\right)$, for $i = m, w$ and $k = 1, \ldots, n$. At this time, $R_w\left(\mu_{a,k}\right) > R_w\left(\mu_{k,k}\right) > x_{k+1}$, for any a $\left(1 \leq a \leq k\right)$.*

Proof. See Appendix. ■

The next lemma shows that the decision of a woman with $\mu_{a,b}$ whether to accept a k-type man depends on that of a woman with $\mu_{a,k}$.

Lemma 2 *The decision of a woman with $\mu_{a,b}$ whether to accept a k-type man, for any $k \in \{a+1, \ldots, b\}$, depends on whether $\frac{x_k}{r}$ exceeds $V_w^{x_{k-1}}\left(\mu_{a,k}\right)$. If and only if $R_w\left(\mu_{a,b}\right) > x_{k'}$, then $R_w\left(\mu_{a,k'}\right) > x_{k'}$. At this time, $R_w\left(\mu_{a,k'}\right) = R_w\left(\mu_{a,b}\right) > x_{k'}$ holds. Moreover,*

$$R_w\left(\mu_{a,k'}\right) = R_w\left(\mu_{a,k'+1}\right) = \ldots = R_w\left(\mu_{a,b}\right) = \ldots = R_w\left(\mu_{a,n}\right) > x_{k'}, \tag{9}$$

[21] More generally, if men are partitioned into n' types by the reservation levels of women, n' kinds of reservation levels of men are generated. Then, because of discrete types of agents, women are always partitioned into n $\left(\leq n'\right)$ types by the reservation levels of men.

holds.

Proof. See Appendix. ■

Lemma 2 implies that if a woman with $\mu_{a,b}$ can update her belief to $\mu_{a,k'}$ $(k' < b)$ after a meeting, the decision of a woman with $\mu_{a,b}$ depends on that of a woman with $\mu_{a,k'}$. Hence, given $R_w\left(\mu_{a,b}\right) > x_{k'}$, the strategy of a woman with $\mu_{a,b}$ becomes the same as that of a woman with $\mu_{a,k'}$.

Moreover, given $R_w\left(\mu_{a,b}\right) > x_{k'}$, a woman with $\mu_{a,i}$ rejects a $i+1$-type man, for $i = a+1, \ldots, k'-1$, and a woman with $\mu_{a,i}$, for $i = k', \ldots, n$, rejects a k'-type man. Note that, when $k' = b$, a woman with $\mu_{a,b}$ learns nothing from a meeting with a k'-type man.

The next lemma shows that the reservation level of a woman with $\mu_{a,b+1}$ is lower than or equal to that of a woman with $\mu_{a,b}$ for any $b\,(\leq a)$.

Lemma 3 *Let us assume that $R_i^*(x_k) > x_{k+1}, i = m, w$. For any $a\,(= 1, \ldots, n-l-1)$, and $l(= 0, \ldots, n-(a+1))$,*

$$R\left(\mu_{a,a+l}\right) \geq R\left(\mu_{a,a+l+1}\right). \tag{10}$$

Proof. See, Appendix. ■

Moreover, the next lemma also holds.

Lemma 4 *A woman with $\mu_{a,b}$, for any $a, b\ (a < b)$, has higher reservation level than that of a woman with $\mu_{a+1,b}$, that is,*

$$R_w\left(\mu_{a,b}\right) > R_w\left(\mu_{a+1,b}\right).$$

Proof. See, Appendix. ■

Lemma 4 means that the reservation level of a woman with $\mu_{a,b}$ is higher than or equal to that of a woman with $\mu_{a+1,b}$.

From Lemmas 2 and 4, we obtain the next proposition.

Proposition 2 *A woman with imperfect self-knowledge does not raise her reservation utility level even if she receives an offer that has information about her type. Thus, $R_w(\mu_0)$ is the highest reservation level of women in equilibrium.*

Proof. See, Appendix. ■

Proposition 2 means that if a woman with $\mu_{a,b}$ rejects a $k+1$-type man but accepts a k-type man, a woman with $\mu_{a,i}$, $i = k+1, \ldots, n$, also rejects a $k+1$-type man but accepts a k-type man. Then, a woman with $\mu_{a,i}$, $i = k+1, \ldots, n$, cannot be a woman with $\mu_{a,k'}$, for any $k' \leq k$. That is, a woman with $\mu_{a,i}$ accepts a k-type man without having an opportunity to revise her belief upward.

Moreover, even if a woman with imperfect self-knowledge can revise her belief upward, she does not raise her reservation level. The decision of a woman with $\mu_{a,b}$ whether to accept a $k+1$-type man becomes the same as that of a woman with $\mu_{a,k+1}$. If a woman with $\mu_{a,k+1}$

rejects a $k+1$-type man, a woman with $\mu_{a,b}$ also rejects a $k+1$-type man and then updates her belief to $\mu_{a,k+1}$; in other words, she previously rejects a man whom she will reject after an upward belief revision. As a result, a woman with imperfect self-knowledge does not raise her reservation level.[22]

Note that although Proposition 2 holds under Assumption A.2., Proposition 2 does not require women's PSEI actions, which we define in the next section. Moreover, results similar to those in this section can be obtained in the case of a one-sided search model, where $x_{k+1} < R_m(x_k)$, for $k = 1, \ldots, n$, are given and where men's strategies are unaffected by women's strategies.

3.4 Search equilibrium with imperfect self-knowledge

Next, we introduce an equilibrium concept for this section. Although a woman's state changes over time, we first focus on the market in a stationary environment.

Definition 3 *In a* search equilibrium under (one-sided) imperfect self-knowledge (SEI)*: Given* (G_m, G_w, μ_0)*,*

(SEI-i) all men maximize their expected discounted utilities,

(SEI-i) all women's strategies satisfy sequential rationality, and

(SEI-ii) women's beliefs along the equilibrium path are consistent with Bayesian updating given the equilibrium strategies.[23]

By characterizing a search equilibrium for (G_m, G_w, μ_0), Section 4 identifies (G_m, G_w, μ_0), which implies that the two flow distributions are equal.

First, we derive a *perfect sorting SEI (PSEI)*, where agents of the same type marry. The PSEI requires that a woman with $\mu_{a,b}$, proposes to a-type men, and always rejects men of a lower type. Otherwise, the PSEI does not occur because men and women of different types marry.

Although one can consider many combinations of agents' equilibrium strategies, we focus on the PSEI in this study because the influence of learning on the market becomes clearer when the PSE is compared with the PSEI. Moreover, from Proposition 2, $R_w(\mu_0)$ is the highest reservation level of women in an equilibrium. Hence, the opportunities for women's learning are maximized in the PSEI.

Given the PSEI actions, a woman learns about her own type at most $n-1$ times. Here, we use the terms "$k_{a,b}$-type woman" and "k-type woman" as a woman with $x_k^{a,b}$ and a woman with x_k and any μ, respectively.

The next proposition shows that there exists a unique *PSEI*, where agents partition themselves into n clusters of marriages and where, therefore, only men and women of the same type marry.

[22] Even in the case of a sequential move in which a man proposes to a woman in the first move, and she proposes or rejects him in the next move, the reservation level of a woman with imperfect self-knowledge does not rise. In this case, a woman can learn before marriage; however, she @previously@still presently?@ rejects a man whom she rejects@will reject?@ after revising her belief.

[23] This equilibrium concept is the same as that in Maruyama (2010).

Proposition 3 *We assume that* $x_{k+1} < R_i^*(x_k)$*, for* $k = 1, \ldots, n-1$*, and* $i = m, w$*. There is a PSEI if, for* $k = 1, \ldots, n-1$*,*

$$\begin{aligned} R_w\left(\mu_{k,k+1}\right) &= \ldots = R_w\left(\mu_{k,n}\right) \\ &= \alpha \frac{\mu_{k,k+1}(x_k)\lambda_k^m(x_k)+\mu_{k,k+1}(x_{k+1})\lambda_k^m\left(R_w\left(\mu_{k+1,k+1}\right)\right)}{\left(r+\alpha\lambda_k^m\right)} > x_{k+1}, \\ R_w\left(\mu_{k,k}\right) &= \frac{\alpha\lambda_k^m x_k}{r+\alpha\lambda_k^m} = R_i^*(x_k) > x_{k+1}, \end{aligned} \tag{11}$$

and if, for $k = 1, \ldots, n-1$*,*

$$R_m(x_k) = \frac{\alpha\sum_{j=k}^{R_m(x_k)}\sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_j^l\right)x_j}{r+\alpha\sum_{j=k}^{R_m(x_k)}\sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_j^l\right)} > x_{k+1}. \tag{12}$$

In the PSEI, agents of the same type marry.

Proof of Proposition 3. We derive the desirable results by establishing the following lemmas.

First, we investigate the optimal strategies of women. We obtain the following lemma.

Lemma 5 *If* $R_w\left(\mu_{k,k+1}\right) = \alpha\frac{\mu_{k,k+1}(x_k)\lambda_k^m(x_k)+\mu_{k,k+1}(x_{k+1})\lambda_k^m\left(R_w\left(\mu_{k+1,k+1}\right)\right)}{\left(r+\alpha\lambda_k^m\right)} > x_{k+1}$*, a woman with* $\mu_{k,k+1}$ *rejects a* $k+1$*-type man, where* $x_{n+1} \leq \underline{x}_i, i = m, w$*. In the PSEI,* $R_w\left(\mu_{k,k}\right) > R_w\left(\mu_{k,k+1}\right) = \ldots = R_w\left(\mu_{k,n}\right) > x_{k+1}$*, for* $k = 1, \ldots, n$*. Moreover,* $R_w\left(\mu_{k,k}\right) = \frac{\alpha\lambda_k^m x_k}{r+\alpha\lambda_k^m} = R_w^*(x_k) > x_{k+1}$ *holds.*

Proof of Lemma 5: First, let us investigate the decision of a woman with $\mu_{k,k}$, for $k = 1, \ldots, n$.

For $k = 1$, the arrival rate of proposals to a 1-type woman becomes $\alpha_w(x_1) = \alpha$ from $F_m(.|x_1) = F_m(.)$. Then, $rV_w\left(\mu_{1,1}\right) = \alpha\lambda_1^m\left(\frac{x_1}{r} - V\left(\mu_{1,1}\right)\right)$. Hence, $R_w\left(\mu_{1,1}\right) = \frac{\alpha\lambda_1^m x_1}{r+\alpha\lambda_1^m} = R_w^*(x_1) > x_2$.

The arrival rate of proposals to a woman with x_k, for $k = 2, \ldots n$, becomes $\alpha_w(x_k) = \alpha F_m([x_{k-1}, x_1]^-)$, which is the rate at which she meets men who accepts her. Given a random contact, $F_m(.|x_k) = \frac{F_m(.)}{F_m([x_{k-1},x_1]^-)}$. From $R_w\left(\mu_{k,k}\right) > x_{k+1}$, we have

$$rV_w\left(\mu_{k,k}\right) = \alpha F_m([x_{k-1}, x_1]^-)\frac{\lambda_k^m\left(\frac{x_k}{r} - V\left(\mu_{k,k}\right)\right)}{F_m([x_{k-1},x_1]^-)}.$$

Then,

$$R_w\left(\mu_{k,k}\right) = \frac{\alpha\lambda_k^m x_k}{r+\alpha\lambda_k^m} = R_w^*(x_k) > x_{k+1}. \tag{13}$$

Next, we investigate the decision of a woman with $\mu_{k,k+1}$, for $k = 1, \ldots, n-1$. From (6), we have

$$\begin{aligned} & rV_w\left(\mu_{k,k+1}\right) \\ = \quad & \alpha\mu_{k,k+1}\left(x_k\right)\left[\lambda_k^m\left(\tfrac{x_k}{r} - V_w\left(\mu_{k,k+1}\right)\right) + \lambda_{k+1}^m\left(\max\left\{\tfrac{x_{k+1}}{r}, V_w\left(\mu_{k,k+1}\right)\right\} - V_w\left(\mu_{k,k+1}\right)\right)\right] \\ & +\alpha\mu_{k,k+1}\left(x_{k+1}\right)\left[\lambda_k^m\left(V_w\left(\mu_{k+1,k+1}\right) - V_w\left(\mu_{k,k+1}\right)\right) + \lambda_{k+1}^m\left(\max\left\{\tfrac{x_{k+1}}{r}, V_w\left(\mu_{k,k+1}\right)\right\} - V_w\left(\mu_{k,k+1}\right)\right)\right] \end{aligned}$$

The first term in the second square bracket in the above equation means that, if a woman with $\mu_{k,k+1}$ is actually a $k+1$-type, she learns that she is the $k+1$-type by meeting a k-type man. Thus,

$$R_w\left(\mu_{k,k+1}\right) = \alpha\frac{\mu_{k,k+1}(x_k)\lambda_k^m(x_k) + \mu_{k,k+1}(x_{k+1})\lambda_k^m\left(R_w\left(\mu_{k+1,k+1}\right)\right)}{\left(r+\alpha\lambda_k^m\right)} > (\leq)\, x_{k+1}. \tag{14}$$

From (14), $R_w\left(\mu_{k,k+1}\right)$ is uniquely obtained. In the PSEI, $x_{k+1} < R_w\left(\mu_{k,k+1}\right)$ holds, for $k = 1, \ldots, n$. Then, from Lemmas 2 and 3, $R_w\left(\mu_{k,k}\right) > R_w\left(\mu_{k,k+1}\right) = \ldots = R_w\left(\mu_{k,n}\right) > x_{k+1}$, for $k = 1, \ldots, n$.■

Lemma 5 shows that a woman with $\mu_{k,k+1}$ $(k = 1, .., n-1)$ rejects a $k+1$-type man if there are sufficient k-type men or if $\mu_{k,k+1}(x_k)$ is sufficiently large and satisfies $x_{k+1} < R_w\left(\mu_{k,k+1}\right)$. If there are sufficient $k+1$-type men, a woman with $\mu_{k+1,k+1}$ raises her reservation level. Hence, a woman with $\mu_{k,k+1}$ also raises her reservation level because she may become a woman with $\mu_{k+1,k+1}$ in the next period. Thus, more optimistic prior beliefs lead more women to reject men who they would marry under perfect self-knowledge.

Women with imperfect self-knowledge assign probabilities to their own type. Therefore, the reservation level of $k_{k,b}$-type women (for $k = 1, .., b-1$) are lowered in comparison with the benchmark results. By contrast, the reservation level of $i_{k,b}$-type women (for $i = k+1, \ldots, b-1$) are increased in comparison with the PSE.

Moreover, the reservation level of a woman with imperfect self-knowledge increases as the parameter α increases, because an increasing arrival rate of men speeds her learning process and decreases the search duration.

When $r = 0$, $R_w\left(\mu_{k,k+1}\right) = R_w^*\left(x_k\right)(= x_k)$ holds. Therefore, a woman with $\mu_{k,k+1}$ always prefers to meet a k-type man instead of accepting a $k+1$-type man in order to confirm her type. This is because, if a woman with $\mu_{k,k+1}$ is actually a $k+1$-type, she will marry a $k+1$-type man sooner or later, regardless of her action. Hence, the possibility that she is a $k+1$-type woman does not affect her own decision, because there is no time cost. Consequently, the decision of a woman with $\mu_{k,k+1}$ is the same as that of a k-type woman with perfect self-knowledge.

Next, we investigate the optimal strategies of men and marriage formation. Hence, we obtain the following lemma.

Lemma 6 *A 1-type man rejects a 2-type woman because* $R_m\left(x_1\right) = R_m^*\left(x_1\right) > x_2$. *If* $R_m\left(x_k\right) =$

$\frac{\alpha \sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_k^l\right) x_k}{r+\alpha \sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_k^l\right)} > x_{k+1}$, *for* $k = 2, \ldots, n-1$, *a* k*-type man rejects a* $k+1$*-type woman, where* $x_{n+1} \leq \underline{x}_i, i = m, w$. *The reservation level of a* k*-type man in the PSEI decreases in comparison with the benchmark result, that is,* $R_m^*(x_k) > R_m(x_k)$.

Proof of Lemma 6: From Lemma 5, $x_k \geq R_w\left(\mu_{k,k}\right) > R_w\left(\mu_{k,k+1}\right) = \ldots = R_w\left(\mu_{k,n}\right) > x_{k+1}$, for $k = 1, \ldots, n$, in the PSEI. Hence, $R_w\left(\mu_{l_k}\right) = R_w\left(\mu_{k,n}\right)$ and $R_w\left(\mu_{l_{k-1}+1}\right) = R_w\left(\mu_{k,k}\right)$, where $R_w\left(\mu_{l_k}\right)$ denotes a partition that determines the k-th type of men, for $k = 1, \ldots, n$.

Since all women want to marry the most desirable men (i.e., x_1) if they meet, $\alpha_m(x_1) = \alpha$ and $G_w(.|x_1) = G_w(.)$. Hence, we have

$$R_m(x_1) = \alpha \sum_{j=1}^{R_m(x_1)} \lambda_j^w \left(\tfrac{x_j}{r} - V_m(x_1)\right) = R_m^*(x_1).$$

A man with x_1 accepts (rejects) a woman with $x \geq (<) R_m(x_1)$. From $R_m^*(x_1) > x_2$, $R_m(x_1) > x_2$. Then, men with x_1 and women with x_1 form cluster 1.

Next, let us consider all men not in cluster 1. In the PSEI, $l_1 = [x_n, x_1]$. From Assumption A.1., the arrival rate of proposals to him becomes $\alpha_m(x_2) = \alpha G_w([x_n^{l_1}, x_1^1]^-)$ which is the rate at which he meets women who accept him. The state distribution among such women implies $G_w(.|x_2) = G(.)/G_w([x_n^{l_1}, x_1^1]^-)$. Therefore, the reservation level of a man with x_2 becomes

$$\begin{aligned} R_m(x_2) &= \alpha G_w([x_n^{l_1}, x_1^1]^-) \sum_{j=2}^{R_m(x_2)} \frac{\sum_{l=l_1+1}^{\bar{l}} g_w\left(x_j^l\right)}{G_w([x_n^{l_1}, x_1^1]^-)} \left(\tfrac{x_j}{r} - V_m(x_2)\right) \\ &= \alpha \sum_{l=l_1+1}^{\bar{l}} g_w\left(x_2^l\right) \left(\tfrac{x_2}{r} - V_m(x_2)\right). \end{aligned}$$

In the PSEI, $x_2 \geq R_w\left(\mu_{l_1+1}\right) = R_w\left(\mu_{2,2}\right)$. A man with x_2 proposes to any woman with $x \geq R_m(x_2)$, so will all men not in cluster 1. In the PSEI, $x_2 \geq R_m(x_2) > x_3$. Then,

$$R_m(x_2) = \frac{\alpha \sum_{l=l_1+1}^{\bar{l}} g_w\left(x_2^l\right) x_2}{r+\alpha \sum_{l=l_1+1}^{\bar{l}} g_w\left(x_2^l\right)}.$$

In the PSEI, intervals I_l, for $l = l_1+1, \ldots, \bar{l}$, do not include x_1. Moreover, $\sum_{l=l_1+1}^{\bar{l}} \phi_2^l \lambda_2^w < 1$, from $g_w\left(x_2^l\right) = \phi_2^l \lambda_2^w$. Thus, $R_m(x_2) \leq R_m^*(x_2)$.

Some women with x_2 reject men with x_2, because these women have the same as or higher reservation levels compared to $R_w\left(\mu_{l_1}\right)$. Therefore, men with x_2 and women with x_2^l, for $l = l_1+1, \ldots, \bar{l}$, form cluster 2.

Similarly, we can consider a man in cluster 3; in a similar fashion, cluster n can be constructed, where $R_m(x_n) \leq x_n$. Generally, from (8), the reservation level of a man with

x_k, for any $k = 1, \ldots, n$, becomes

$$R_m(x_k) = \frac{\alpha \sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_k^l\right) x_k}{r + \alpha \sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_k^l\right)}.$$

where $x_k \geq R_w\left(\mu_{l_{k-1}+1}\right)$. Therefore, a k-type man wants to marry any women with $x \geq R_m(x_k)$. In the PSEI, intervals I_l, for $l = l_{k-1}+1, \ldots, \bar{l}$, do not include x_{k-1}. Furthermore, $\sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_k^l\right) < 1$. From these, $R_m(x_k) \leq R_m^*(x_k)$.

By contrast, a woman with x_k^l, for $l = l_{(k-1)}+1, \ldots, \bar{l}$, accepts a man with x_k. Therefore, men with x_k and women with x_k^l, for $l = l_{(k-1)}+1, \ldots, \bar{l}$, form cluster k. (More formally, men with x_k and women with $x \in \cup_{l=l_{(k-1)}+1}^{\bar{l}}[x_k^l, x_1^l]$ form cluster k. However, there are no women with $[x_{k-1}^l, x_1^l]$, for $l = l_{(k-1)}+1, \ldots, \bar{l}$, in the PSEI).■

Lemma 6 shows that with constant α, if there are sufficient k-type women who accept a k-type man ($R_m(x_k) > x_{k+1}$), a k-type man rejects a $k+1$-type woman, for $k = 1, \ldots, n$. However, rejections of k-type men by k-type women with imperfect self-knowledge lower his reservation level, which as a result is lower than or equal to his under perfect self-knowledge.■

The implications of Proposition 3 are as follows: If the economy is at the PSEI, then men with x_k and women with x_k^l, for $l_{k-1}+1, \ldots, \bar{l}$, form cluster k, for $k = 1, \ldots, n$. However, Cluster 1 is not influenced by women with imperfect self-knowledge.

The expected duration until marriage of each agent can be easily obtained. In the PSE, the duration until the marriage of a k-type agent, i, is $\frac{1}{\alpha \lambda_k^j}(i, j = m, w)$. In the PSEI, the duration until the marriage of a k-type man is $1/\alpha \sum_{l=l_{(k-1)}+1}^{\bar{l}} g_w\left(x_k^l\right)$, for $k = 2, \ldots, n$. Therefore, the marriages of all men, other than those in cluster 1, are delayed by the women's learning process. For women, the expected duration differs across μ_l, for $l = 1, .., \bar{l}$. The duration until the next period t of a woman with $\mu_{a,b}$ is $\frac{1}{\alpha \Sigma_{k=a}^{b-1} \lambda_k^m}$, for any $a < b$, and that of a woman with $\mu_{k,k}$ is $\frac{1}{\alpha \lambda_k^m}$, for $k = 1, \ldots, n$. Therefore, expected duration until marriage has its own dynamics over time. Of course, women's marriages are delayed by their own learning, with the exception of cluster 1. Hence, the welfare of each type of agent in the PSEI, other than those in cluster 1, is lower than that in the PSE.

In a search equilibrium, it is not necessary that the outflow of the market equals the inflow. In the next section, we investigate the steady-state equilibrium.

4 Steady state equilibria

Given (G_m, G_w, μ_0), from Proposition 3, it follows that a search equilibrium uniquely generates a partition $(\{R_m(x_k)\}_{k=1}^n, \{R_w(\mu_l)\}_{l=1}^{\bar{l}})$. This partition implies a unique type distribution of exiting agents, $H_i(.), i = m, w$. This partition and N, the number of agents in the market, also imply the number of agents who exit each state per period, dt. Thus, the number of agents who exit the market per period is also obtained.

To solve for the steady state equilibrium, we must describe how new singles enter the market over time. In this study, we adopt' the *cloning assumption*; if a pair marries and leaves the market, two identical types of agents enter the market at once.[24] Thus, the distribution of types, $F_i(.)$, $i = m, w$, is unaffected by the strategies of agents under perfect self-knowledge. Therefore, under the cloning assumption and perfect self-knowledge, given (F_m, F_w, N), a search equilibrium implies a steady state equilibrium. The cloning assumption is the simplest assumption in the inflow specifications (e.g., MacNamara and Collins (1990), Morgan (1994), Burdett and Coles (2001), Bloch and Ryder (2000), and Chade (2006)). However, in this paper's model, any new female entrant does not know her own type. Hence, the distribution of states, $G_w(.)$, is changed by the strategies of agents under imperfect self-knowledge.

The equilibrium concept for this section is as follows.

Definition 4 *Given (F_m, F_w, N), a steady-state equilibrium under the cloning assumption is (G_m, G_w, μ_0), where*

(s-i) the agents' strategies are consistent with a search equilibrium; and

(s-ii) for each state x_k^l, the inflow and outflow of agents are balanced.[25]

The steady state requires (s-ii), regardless of the inflow specifications. As a result of (s-ii), for each type k, the inflow and outflow of agents are also balanced. From (s-ii), (F_m, F_w) and the optimal strategies of agents, given expectations about μ_0 (or Ψ_w) and (G_m, G_w), together indeed generate (G_m, G_w, μ_0) as the steady state distributions of states and the steady state prior belief.

In the PSEI, all states of 1-type women are $x_j^{1,b}$, for $b = 2, \ldots, n$. However, there is no woman with $x_1^{1,1}$ because such a woman leaves the market and knows she belongs to the 1-type at the same time. Hence, $\sum_{b=2}^{n} \phi_1^{1,b} = 1$. From (s-ii), the following equation holds.

$$\alpha\lambda_b^m \sum_{i=b+1}^{n} \phi_1^{1,i}\lambda_1^w N = \alpha \sum_{k=1}^{b-1} \lambda_k^m \phi_1^{1,b}\lambda_1^w N, \quad \text{for } b = 2, \ldots, n-1. \tag{15}$$

The LHS of (15) implies that an b-type man changes the state of a woman with $x_1^{1,i}$, for $i = b+1, \ldots, n$, to $x_1^{1,b}$ by proposing to her. Then, $\alpha\lambda_b^m dt$ is the probability in the small time interval dt that a woman with $x_1^{1,i}$ meets a b-type man and thus learns something about her type. It follows that the number of women who enter a state $x_1^{1,b}$ is $\alpha\lambda_b^m \left(\sum_{k=b+1}^{n} \phi_1^{1,k}\right) \lambda_1^w N$. By contrast, $\sum_{k=1}^{b-1} \lambda_k^m$ on the RHS of (15) is the share of all men who change the state of a woman with $x_1^{1,b}$ (i.e., they change her belief or lead her to exit the market). Then, $\alpha \sum_{k=1}^{b-1} \lambda_k^m dt$ is the probability in dt that a woman with $x_1^{1,b}$ meets a man and then marries or learns something about her type. Therefore, the number of women who exit a state $x_1^{1,b}$ is $\alpha \sum_{k=1}^{b-1} \lambda_k^m \phi_1^{1,b} \lambda_1^w N$.

All sates of $j\,(= 2, \ldots, n)$-type women, $x_j^{a,b}$, are as follows:

[24] Burdett and Coles (1999) describe four typical "inflow" assumptions.

[25] This equilibrium concept is the same as that in Maruyama (2010).

$$
\begin{array}{llllll}
x_j^{1,j} & x_j^{1,j+1} & \dots & & x_j^{1,n-1} & x_j^{1,n} \\
x_j^{2,j} & x_j^{2,j+1} & \dots & & x_j^{2,n-1} & x_j^{2,n} \\
\dots & & & & & \\
x_j^{a,j} & x_j^{a,j+1} & & x_j^{a,b} & x_j^{a,n-1} & x_j^{a,n} \\
\dots & & & & & \\
x_j^{j,j} & x_j^{j,j+1} & \dots & & x_j^{j,n-1} & x_j^{j,n}.
\end{array}
$$

Here, $\sum_{a=1}^{j}\sum_{b=j}^{n}\phi_j^{a,b}=1$.

From (s-ii), if $a<b$, an $a-1$-type man changes the belief of a woman with $\mu_{i,b}$, for $i=1,\dots,a-1$, to $\mu_{a,b}$ by rejecting her. Moreover, a b-type man also changes the belief of a woman with $\mu_{a,i}$, for $i=b+1,\dots,n$, to $\mu_{a,b}$ by proposing to her. It follows that the number of women who enter a state $x_j^{a,b}$ is $\alpha\lambda_{a-1}^m\left(\sum_{i=1}^{a-1}\phi_j^{i,b}\right)\lambda_j^w N+\alpha\lambda_b^m\left(\sum_{i=b+1}^{n}\phi_j^{a,i}\right)\lambda_j^w N$. By contrast, $\sum_{k=a}^{b-1}\lambda_k^m$ is the share of all men who change the state of a woman with $x_j^{a,b}$. It follows that the number of women who exit a state $x_j^{a,b}$ is $\alpha\sum_{k=a}^{b-1}\lambda_k^m\phi_j^{a,b}\lambda_j^w N$.

If $a=b$, $a=b=j$. At this time, a $j-1$-type man changes the belief of a woman with $\mu_{i,j}$, for $i=1,\dots,j-1$, to $\mu_{j,j}$ by rejecting her. A woman with $\mu_{i,j}$ for $i=j+1,\dots,n$, cannot be a $\mu_{j,j}$ by learning according to Proposition 2. Thus, the number of women who enter a state $x_j^{j,j}$ is $\alpha\lambda_{j-1}^m\left(\sum_{k=1}^{j-1}\phi_j^{k,j}\right)\lambda_j^w N$. By contrast, λ_j^m is the share of men who leads her to exit the market. It follows that the number of women who exit a state $x_j^{j,j}$ is $\alpha\lambda_j^m\phi_j^{j,j}\lambda_j^w N$.

From these, generally, for any $j\,(=1,\dots,n)$, and any $a,b(1\le a\le j\le b\le n)$, the following equations hold.

$$
\lambda_{a-1}^m\left(\sum_{i=1}^{a-1}\phi_j^{i,b}\right)+\lambda_b^m\left(\sum_{i=b+1}^{n}\phi_j^{a,i}\right)=\left(\sum_{k=a}^{b-1}\lambda_k^m\right)\phi_j^{a,b},\quad \text{if } a<b, \tag{16}
$$

$$
\lambda_{j-1}^m\left(\sum_{i=1}^{j-1}\phi_j^{i,j}\right)=\lambda_j^m\phi_j^{j,j},\quad \text{if } a=b\,(=j)\,. \tag{17}
$$

where if $j=1$, $a=j=1<b<n$.

Given (G_m,G_w,μ_0), the next lemma holds for the relation between G_w and beliefs at the steady state. Here, let $\pi_j^i=\Psi_i\left(x_j\right)-\Psi_i\left(x_{j-1}\right)$, where $\Sigma_{j=1}^n\pi_j^i=1$, $i=m,w$. Lemma 7 shows that the beliefs calculated from $G_w\left(.\right)$ are consistent with those calculated using Bayes' rule. Note that Lemma 7 always holds at the steady state regardless of the inflow specification.

Lemma 7 *Given (G_m,G_w,μ_0), for each state $x_j^{a,b}$, $j\in[a,b]$, $\phi_j^{a,b}$ is π_j^w appropriately rescaled. Moreover, the share of women with x_j^0 of women with μ_0 in the market is equal to the share of new female j-type entrants, that is, $\frac{g_w\left(x_j^0\right)}{\Sigma_{j=1}^n g_w\left(x_j^0\right)}=\pi_j^w$, for $j=1,\dots,n$. Hence, the share of women with $x_j^{a,b}$ of women with $\mu_{a,b}$ in the market is equal to the probability $\mu_{a,b}\left(x_j\right)$ which is calculated using Bayes' rule, for any $a,b\,(a<b)$.*

Proof. See Appendix. ■

Lemma 7 shows that the distribution of women with μ_0 in the market is consistent with the prior belief of a woman and that the updated beliefs of women, μ_l, are consistent with $G_w\left(.\right)$. Lemma 7 also implies that given G_w, which (F_m,F_w) and agents' strategies generate, means μ_0 is also given indirectly.

The next proposition shows that there exists a unique steady-state equilibrium, at which men and women partition themselves into clusters.

Proposition 4 *Given (F_m, F_w, N), (G_m, G_w, μ_0) is uniquely obtained. If (G_m, G_w, μ_0) satisfies (11) and (12), there exists a steady state PSEI.*

Given (G_m, G_w), agents' strategies and N imply the number of agents who exit the market per period. Therefore, a unique type distribution of exiting agents is obtained. Under the cloning assumption, this distribution also implies a type distribution of entrants, $\Psi_i, i = m, w$. Hence, μ_0 is obtained.

5 Concluding remarks

In this study, we analyzed one-sided learning in a two-sided search model. Women were assumed not to know their own type; they only learned about their own type from the offers or rejections they received from men. As a result of this learning process, the two-sided aspect of the search problem has generated significant interest. The main results of this study are as follows. First, women with imperfect self-knowledge raise or lower their reservation level in comparison with results under perfect self-knowledge. By contrast, reservation levels of certain men are lowered if some women with imperfect self-knowledge reject those men who they would accept under perfect self-knowledge.

Second, the reservation level of a woman with imperfect self-knowledge is lowered by a rejection but never raised by an offer. From this result, the reservation level of a woman with a prior belief is the highest, and her reservation level gradually declines over the duration of the search. In the labor market, the potential sources of declining reservation wages have received much research attention.

Two extensions to this model present themselves. First, this study assumes that agents cannot divorce. However, when women marry men before perfectly knowing their own type, they may learn about their type after they get married; such learning after marriage will influence the divorce rate.

Second, for simplicity, we assume that agents' types are discrete. The current results would still apply if agents' types were continuous and if n classes of marriages were generated by a sufficiently large α under perfect self-knowledge. However, if types were continuous, generally, the number of women's classes would be larger than that of men's classes, which would make the analysis more complex. Hence, imperfect self-knowledge may generate further changes.

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Appendix

Proof of Proposition 1: First, we consider the decision of a 1-type woman. As she is the highest type, all men propose to her. Hence, $\varepsilon_m(x_1) = 1$ and $F_m(.|x_1) = F_m(.)$. From (1), the expected discounted lifetime utility of an unmarried 1-type woman $V_w(x_1)$, becomes

$$rV_w(x_1) = \alpha\lambda_1^m\left(\tfrac{x_1}{r} - V_w(x_1)\right) + \sum_{j=2}^{n}\alpha\lambda_j^m\left(\max\left\{\tfrac{x_j}{r}, V_w(x_1)\right\} - V_w(x_1)\right).$$

If she meets a 1-type man with probability $\alpha\lambda_1^m$, they always marry. If a 1-type woman meets a 2-type man, she compares x_2/r with $V_w(x_1)$. If she rejects a 2-type man, i.e., $V_w(x_1) > \frac{x_2}{r}$, from (1),

$$rV_w^r(x_1) = \alpha\lambda_1^m\left(\tfrac{x_1}{r} - V_w^r(x_1)\right).$$

By contrast, if she accepts a 2-type man and rejects a 3-type man (i.e., $\frac{x_2}{r} \geq V_w(x_1) > \frac{x_3}{r}$),[26]

$$rV_w^a(x_1) = \alpha\lambda_1^m\left(\tfrac{x_1}{r} - V_w^a(x_1)\right) + \alpha\lambda_2^m\left(\tfrac{x_2}{r} - V_w^a(x_1)\right).$$

If $V_w^r(x_1) > V_w^a(x_1)$, a 1-type woman rejects a 2-type man. This inequality $V_w^r(x_1) > V_w^a(x_1)$ means that

$$x_2 < R_w^*(x_1) \equiv \tfrac{\alpha\lambda_1^m x_1}{\alpha\lambda_1^m + r} < x_1.$$

Conversely, if $V_w^r(x_1) \leq V_w^a(x_1)$, a 1-type woman accepts a 2-type man. At this time, $x_2 \geq R_w^*(x_1)$ holds.

As the situation is the same for a 1-type man, his reservation match strategy is $R_m^*(x_1) \equiv \frac{\alpha\lambda_1^w x_1}{\alpha\lambda_1^w + r} < x_1$.

Under $x_2 < R_w^*(x_1)$ and $x_2 < R_m^*(x_1)$, a 1-type woman proposes to and is accepted by a 1-type man she encounters. Therefore, 1-type men and 1-type women form a cluster of marriages (cluster 1).[27]

If $x_2 < R_w^*(x_1)$ and $x_2 < R_m^*(x_1)$, we can construct cluster 2. Let us consider all agents not in cluster 1. Now, a 2-type agent is the highest-type agent. Therefore, the arrival rate of proposals to a 2-type woman is $\alpha_w(x_2) = \alpha F_m\left((x_1)^-\right) = \alpha\sum_{j=2}^n \lambda_j^m$, which is the rate at which she meets men not in cluster 1. The type distribution among such men implies $F_m(.|x_2) = F_m(.)/F_m\left((x_1)^-\right)$. Therefore, a 2-type woman's discounted lifetime

[26] If $x_3/r < V_w(x_1) \leq x_2/r$, 1- and 2-type agents receive at least the same number of offers. Hence, $V_w(x_1) \geq V_w(x_2)$, and we then have $V_w(x_2) \leq x_2/r$.

[27] If agents' types are continuous, all women with type $x_k \geq R_m(\bar{x}_m)$ face the same problem because all men propose to them. Then, they use the same strategy as the highest type women, i.e., $R_w(x_k) = R_w(\bar{x}_w)$ for all $x_k \geq R_m(\bar{x}_m)$. This situation is the same for men in that $R_m(x_k) = R_m(\bar{x}_m)$ for all $x_k \geq R_w(\bar{x}_w)$. As a result, men with $x_k \geq R_w(\bar{x}_w)$ and women with $x_k \geq R_m(\bar{x}_m)$ form class 1.

utility becomes

$$
\begin{aligned}
rV_w(x_2) &= \alpha F_m\left((x_1)^-\right)\tfrac{\lambda_2^m}{F_m\left((x_1)^-\right)}\left(\tfrac{x_2}{r}-V_w(x_2)\right)\\
&\quad +\alpha\tfrac{F_m\left((x_1)^-\right)}{F_m\left((x_1)^-\right)}\sum_{j=3}^{n}\lambda_j^m\left(\max\left\{\tfrac{x_j}{r},V_w(x_2)\right\}-V_w(x_2)\right)\\
&= \alpha\lambda_2^m\left(\tfrac{x_2}{r}-V_w(x_2)\right)+\alpha\sum_{j=3}^{n}\lambda_j^m\left(\max\left\{\tfrac{x_j}{r},V_w(x_2)\right\}-V_w(x_2)\right).
\end{aligned}
$$

Consequently, the reservation match strategy of a 2-type woman is

$$x_3 < (\geq)\, R_w^*(x_2) \equiv \tfrac{\alpha\lambda_2^m x_2}{\alpha\lambda_2^m+r}.$$

Similarly, the reservation match strategy of a 2-type man is $x_3 < (\geq)\, R_m^*(x_2) \equiv \frac{\alpha\lambda_2^w x_2}{\alpha\lambda_2^w+r}$.

Under $R_w^*(x_2) > x_3$ and $R_m^*(x_2) > x_3$, 2-type men and 2-type women form cluster 2. Note that although agents in cluster 2 also want to marry agents in cluster 1, they are always rejected by them.

If $R_w^*(x_2) > x_3$ and $R_m^*(x_2) > x_3$, we can construct a third cluster of marriages (cluster 3) in a similar fashion and so on until for some n, $R_w^*(x_n) \equiv \frac{\alpha\lambda_n^m x_n}{\alpha\lambda_n^m+r} \leq \underline{x}$ and $R_m^*(x_n) \equiv \frac{\alpha\lambda_n^w x_n}{\alpha\lambda_n^w+r} \leq \underline{x}$.[28] Then, n-type men and n-type women form a cluster (cluster n).■

Proof of Lemma 1: We prove this lemma by mathematical induction. Let $V_w^{x_k}$ denotes the expected discounted utility of a woman who accepts a k-type man. However, $V_w^{x_k}$ may not be optimal.

First, we prove that when $a = k-1$, $rV_w^{x_k}\left(\mu_{k-1,k}\right) > R_w\left(\mu_{k,k}\right)$ holds. The decision of a woman with $\mu_{k,k}$ whether to accept a $k+1$-type man becomes

$$rV_w\left(\mu_{k,k}\right) = \alpha\lambda_k\left(\frac{x_k}{r}-V_w\left(\mu_{k,k}\right)\right)+\alpha\lambda_{k+1}\left(\max\left\{\frac{x_{k+1}}{r},V_w\left(\mu_{k,k}\right)\right\}-V_w\left(\mu_{k,k}\right)\right).$$

Thus, the decision of a woman with $\mu_{k,k}$ depends on whether $\frac{x_{k+1}}{r}$ exceeds $V_w\left(\mu_{k,k}\right)$. From $x_{k+1} < R_i^*(x_k)$, $R_w\left(\mu_{k,k}\right) > x_{k+1}$.

By contrast, let us consider the decision of a woman with $\mu_{k-1,k}$ whether to accept a $k+1$-type man. From $\alpha_w(x_j) = \alpha\Sigma_{i=j}^{\tilde{n}}\lambda_i^m$, $\alpha_w(x_{k-1}) - \alpha_w(x_j) = \alpha\Sigma_{i=k-1}^{j-1}\lambda_i^m$, and $F_m(.|x_j) = \frac{F(.)}{\Sigma_{i=j}^{n}\lambda_i^m}$, for $j = k-1, k$, the decision of a woman with $\mu_{k-1,k}$ whether to accept a $k+1$-type man becomes

$$
\begin{aligned}
& rV\left(\mu_{k-1,k}\right)\\
=\ & \Sigma_{j=k-1}^{k}\mu_{k-1,k}(x_j)\,\alpha\left[\begin{array}{c}\Sigma_{i=k-1}^{j-1}\lambda_i^m\left(R_w\left(\mu_{i+1,k}\right)-V_w\left(\mu_{k-1,k}\right)\right)+\Sigma_{i=j}^{k}\lambda_i^m\left(\frac{x_i}{r}-V_w\left(\mu_{k-1,k}\right)\right)\\ +\lambda_{k+1}^m\left(\max\left\{\frac{x_{k+1}}{r},V_w\left(\mu_{k-1,k}\right)\right\}-V_w\left(\mu_{k-1,k}\right)\right)\end{array}\right].
\end{aligned}
$$

Thus, the decision of a woman with $\mu_{k-1,k}$ depends on whether $\frac{x_{k+1}}{r}$ exceeds $V_w\left(\mu_{k-1,k}\right)$.

[28] An n-type woman always accepts an n-type man. Otherwise, she cannot marry. Similarly an n-type man always accepts an n-type woman.

Noting that $R_w\left(\mu_{k,k}\right)=\frac{\alpha\lambda_k^m x_k}{r+\alpha\lambda_k^m}$,

$rV_w\left(\mu_{k-1,k}\right)-rV_w\mu_{k,k}$

$=rV_w^{x_k}\left(\mu_{k-1,k}\right)-R_w\left(\mu_{k,k}\right)$

$=\alpha\frac{\mu_{k-1,k}(x_{k-1})\Sigma_{i=k-1}^{k}\lambda_i^m(x_i)+\mu_{k-1,k}(x_k)\left[\lambda_{k-1}^m\left(R_w\left(\mu_{k,k}\right)\right)+\lambda_k^m(x_k)\right]}{\left(r+\alpha\lambda_{k-1}^m+\alpha\lambda_k^m\right)}-R_w\left(\mu_{k,k}\right)$

$=-\frac{\left(\mu_{k-1,k}(x_{k-1})\right)\alpha\lambda_{k-1}^m\left(R_w\left(\mu_{k,k}\right)-x_{k-1}\right)}{r+\alpha\lambda_{k-1}^m+\alpha\lambda_k^m}$.

From $x_{k+1}<R_i^*(x_k)$, $R_w\left(\mu_{k,k}\right)\leq x_{k-1}$. Hence, $rV_w^{x_k}\left(\mu_{k-1,k}\right)>R_w\left(\mu_{k,k}\right)$. That is, given $R_w\left(\mu_{k,k}\right)>x_{k+1}$, a woman with $\mu_{k-1,k}$ always rejects a $k+1$-type man. The optimal strategy of a woman with $\mu_{k-1,k}$ always satisfies $R_w\left(\mu_{k-1,k}\right)\geq rV_w^{x_k}\left(\mu_{k-1,k}\right)$. Hence, $R_w\left(\mu_{k-1,k}\right)>R_w\left(\mu_{k,k}\right)>x_{k+1}$ in an equilibrium.

Let us assume that $rV_w^{x_k}\left(\mu_{a,k}\right)>R_w\left(\mu_{k,k}\right)$, for $a=k-1,k-2,..,l$, and l $(2\leq l\leq 1-k)$. That is,

$V_w^{x_k}\left(\mu_{a,k}\right)-V_w\left(\mu_{k,k}\right)$

$=\alpha\frac{\sum_{j=a}^{k}\mu_{a,k}(x_j)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{\tilde{\imath}+1,k}\right)\right)+\Sigma_{i=j}^{k-1}\lambda_i^m\left(\frac{x_i}{r}\right)\right]+\lambda_k^m\left(\frac{x_k}{r}\right)}{\left(r+\alpha\Sigma_{i=a}^{k}\lambda_i^m\right)}-\frac{R_w\left(\mu_{k,k}\right)}{r}$

$=\frac{\alpha r\sum_{j=a}^{k}k_{a,k}(x_j)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{\tilde{\imath}+1,k}\right)\right)+\Sigma_{i=j}^{k-1}\lambda_i^m\left(\frac{x_i}{r}\right)\right]}{r\left(r+\alpha\Sigma_{i=a}^{k}\lambda_i^m\right)}+\frac{\alpha r\lambda_k^m\left(\frac{x_k}{r}\right)-R_w\left(\mu_{k,k}\right)\left(r+\alpha\Sigma_{i=a}^{k}\lambda_i^m\right)}{r\left(r+\alpha\Sigma_{i=a}^{k}\lambda_i^m\right)}$,

Here, noting that $\frac{\alpha\lambda_k^m x_k}{r+\alpha\lambda_k^m}=R_w\left(\mu_{k,k}\right)$ and $\sum_{j=a}^{k}\mu_{a,k}(x_j)=1$,

$V_w^{x_k}\left(\mu_{a,k}\right)-V_w\left(\mu_{k,k}\right)$

$=\frac{\alpha r\sum_{j=a}^{k}\mu_{a,k}(x_j)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{\tilde{\imath}+1,k}\right)\right)+\Sigma_{i=j}^{k-1}\lambda_i^m\left(\frac{x_i}{r}\right)\right]}{r\left(r+\alpha\Sigma_{i=a}^{k}\lambda_i^m\right)}-\frac{\alpha R_w\left(\mu_{k,k}\right)\Sigma_{i=a}^{k-1}\lambda_i^m}{r\left(r+\alpha\Sigma_{i=a}^{k}\lambda_i^m\right)}$

$=\alpha\frac{\sum_{j=a}^{k}\mu_{a,k}(x_j)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(rV_w\left(\mu_{\tilde{\imath}+1,k}\right)-R_w\left(\mu_{k,k}\right)\right)+\Sigma_{i=j}^{k-1}\lambda_i^m\left(x_i-R_w\left(\mu_{k,k}\right)\right)\right]}{\left(r+\alpha\Sigma_{i=a}^{k}\lambda_i^m\right)r}>0$,

holds. From this, $R_w\left(\mu_{a,k}\right)>R_w\left(\mu_{k,k}\right)$ also holds. Given these, let us investigate the case of $a=l-1$. At this time,

$$\begin{aligned}&rV_w^{x_k}\left(\mu_{l-1,k}\right)-R\left(\mu_{k,k}\right)\\=\;&\alpha\frac{\sum_{j=l-1}^{k}\mu_{l-1,k}(x_j)\left[\Sigma_{i=l-1}^{j-1}\lambda_i^m\left(rV_w\left(\mu_{\tilde{\imath}+1,k}\right)-R_w\left(\mu_{k,k}\right)\right)+\Sigma_{i=j}^{k-1}\lambda_i^m\left(x_i-R_w\left(\mu_{k,k}\right)\right)\right]}{\left(r+\alpha\Sigma_{i=l-1}^{k}\lambda_i^m\right)r}.\end{aligned}$$

From $R_w\left(\mu_{k,k}\right)\leq x_k$, $\Sigma_{i=j}^{k-1}\lambda_i^m\left(x_i-R_w\left(\mu_{k,k}\right)\right)>0$. Moreover, from $R_w\left(\mu_{a,k}\right)>R_w\left(\mu_{k,k}\right)$, for $a=k-1,k-2,\ldots l$, $(2\leq l\leq 1-k)$, we have $V_w^{x_k}\left(\mu_{l-1,k}\right)>V\left(\mu_{k,k}\right)$. Therefore, $R_w\left(\mu_{l-1,k}\right)>R_w\left(\mu_{k,k}\right)>x_{k+1}$.

From these results, $rV_w^{x_k}\left(\mu_{a,k}\right)>R_w\left(\mu_{k,k}\right)$, for any $a\,(1\leq a<k)$.■

Proof of Lemma 2: Let $V_w^{x_k}$ denotes the expected discounted utility of a woman who accepts a k-type man, for any $k\in(a,b]$.[29]. However, $V_w^{x_k}$ may not be optimal.

From $\alpha_w(x_j)=\alpha\Sigma_{i=j}^{\tilde{n}}\lambda_i^m$, $\alpha_w(x_a)-\alpha_w(x_j)=\alpha\Sigma_{i=a}^{j-1}\lambda_i^m$, and $F_m(.|x_j)=\frac{F(.)}{\Sigma_{i=j}^{n}\lambda_i^m}$, the

[29] A woman with $\mu_{a,b}$, for $a\leq b$, always accepts an a-type man.

decision of a woman with $\mu_{a,b}$ whether to accept a k-type man, for any $k \in (a,b]$, becomes

$$\begin{aligned} rV_w\left(\mu_{a,b}\right) &= \alpha \sum_{j=a}^{k} \mu_{a,b}\left(x_j\right) \left[\begin{array}{c} \Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,b}\right)-V_w\left(\mu_{a,b}\right)\right) \\ +\Sigma_{i=j}^{k-1}\lambda_i^m\left(\frac{x_i}{r}-V_w\left(\mu_{a,b}\right)\right) \\ +\lambda_k^m\left(\max\left\{\frac{x_k}{r}, V_w^{x_{k-1}}\left(\mu_{a,k}\right)\right\}-V_w\left(\mu_{a,b}\right)\right) \\ +\Sigma_{i=k+1}^{b-1}\lambda_i^m\left(V_w\left(\mu_{a,i}\right)-V_w\left(\mu_{a,b}\right)\right) \end{array} \right] \\ &\quad +\alpha \sum_{j=k+1}^{b} \mu_{a,b}\left(x_j\right) \left[\begin{array}{c} \Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,b}\right)-V_w\left(\mu_{a,b}\right)\right) \\ +\Sigma_{i=j}^{b-1}\lambda_i^m\left(V_w\left(\mu_{a,i}\right)-V_w\left(\mu_{a,b}\right)\right) \end{array} \right]. \end{aligned} \tag{18}$$

From (18), if she accepts a k-type man,

$$V_w^{x_k}\left(\mu_{a,b}\right) = \frac{\left(\begin{array}{c} \alpha \sum_{j=a}^{k} \mu_{a,b}\left(x_j\right)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,b}\right)\right)+\Sigma_{i=j}^{k}\lambda_i^m\left(\frac{x_i}{r}\right)+\Sigma_{i=k+1}^{b-1}\lambda_i^m\left(V_w\left(\mu_{a,i}\right)\right)\right] \\ +\alpha \sum_{j=k+1}^{b} \mu_{a,b}\left(x_j\right)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,b}\right)\right)+\Sigma_{i=j}^{b-1}\lambda_i^m\left(V_w\left(\mu_{a,i}\right)\right)\right] \end{array} \right)}{r+\alpha\Sigma_{i=a}^{b-1}\lambda_i^m}.$$

If she rejects him,

$$V_w^{x_{k-1}}\left(\mu_{a,b}\right) = \frac{\left(\begin{array}{c} \alpha \sum_{j=a}^{k} \mu_{a,b}\left(x_j\right)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,b}\right)\right)+\Sigma_{i=j}^{k-1}\lambda_i^m\left(\frac{x_i}{r}\right)+\Sigma_{i=k}^{b-1}\lambda_i^m\left(V_w\left(\mu_{a,i}\right)\right)\right] \\ +\alpha \sum_{j=k+1}^{b} \mu_{a,b}\left(x_j\right)\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,b}\right)\right)+\Sigma_{i=j}^{b-1}\lambda_i^m\left(V_w\left(\mu_{a,i}\right)\right)\right] \end{array} \right)}{r+\alpha\Sigma_{i=a}^{b-1}\lambda_i^m}.$$

Hence,

$$V_w^{x_k}\left(\mu_{a,b}\right)-V_w^{x_{k-1}}\left(\mu_{a,b}\right) = \frac{\alpha\sum_{j=a}^{k}\mu_{a,b}(x_j)\lambda_k^m\left[\frac{x_k}{r}-V_w^{x_{k-1}}\left(\mu_{a,k}\right)\right]}{\left(r+\Sigma_{i=a}^{b-1}\lambda_i^m\right)}.$$

From this,

$$V_w^{x_k}\left(\mu_{a,b}\right) < (\geq)\, V_w^{x_{k-1}}\left(\mu_{a,b}\right) \Leftrightarrow x_k < (\geq)\, rV_w^{x_{k-1}}\left(\mu_{a,k}\right) = R_w\left(\mu_{a,b}\right).$$

That is, the decision of a woman with $\mu_{a,b}$ whether to accept a k-type man depends on whether $\frac{x_k}{r}$ exceeds $V_w^{x_{k-1}}\left(\mu_{a,k}\right)$. From this, given $x_{k'} < R_w\left(\mu_{a,b}\right)$, equilibrium requires that $rV_w^{x_{k-1}}\left(\mu_{a,k}\right) \leq x_k$, for $k = a+1,\ldots,k'-1$, and $x_k < rV_w^{x_{k-1}}\left(\mu_{a,k}\right)$, for $k = k',\ldots,n$, holds.

Next, given $x_{k'} < R_w\left(\mu_{a,b}\right)$, let us investigate the best strategy of a woman with $\mu_{a,k'}$ because $V_w^{x_{k-1}}\left(\mu_{a,k}\right)$ may not be optimal. Her decision whether to accept a $k'-1$-type man becomes

$$rV_w\left(\mu_{a,k'}\right) = \alpha\sum_{j=a}^{k'}\mu_{a,k'}\left(x_j\right)\left[\begin{array}{c} \Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,k'}\right)-V_w\left(\mu_{a,k'}\right)\right) \\ +\Sigma_{i=j}^{k'-2}\lambda_i^m\left(\frac{x_i}{r}-V_w\left(\mu_{a,k'}\right)\right) \\ +\lambda_{k'-1}^m\left(\max\left\{\frac{x_{k'-1}}{r}, V_w^{x_{k'-2}}\left(\mu_{a,k'-1}\right)\right\}-V_w\left(\mu_{a,k'}\right)\right) \end{array} \right].$$

Therefore, her decision depends on whether $\frac{x_{k'-1}}{r}$ exceeds $V_w^{x_{k'-2}}\left(\mu_{a,k'-1}\right)$. Because $x_k \geq rV_w^{x_{k-1}}\left(\mu_{a,k}\right)$, for $k = a+1,\ldots,k'-1$, holds, $x_{k'-1} \geq rV_w^{x_{k'-2}}\left(\mu_{a,k'-1}\right)$. Hence, the best strategy of a woman with $\mu_{a,k'}$ is $R_w\left(\mu_{a,k'}\right) = rV_w^{x_{k'-1}}\left(\mu_{a,k'}\right) > x_{k'}$. Then, we have

$$R_w\left(\mu_{a,b}\right) = R_w\left(\mu_{a,k'}\right) > x_{k'}.$$

Next, let us investigate the decision of a woman with $\mu_{a,k'+1}$, for $k'+1 \leq b$. Similar to a woman with $\mu_{a,k'}$, the decision of a woman with $\mu_{a,k'+1}$ whether to accept a k'-type man also depends on whether $\frac{x_{k'}}{r}$ exceeds $V_w^{x_{k'-1}}\left(\mu_{a,k'}\right)$. Given $R_w\left(\mu_{a,b}\right) > x_{k'}$, $\frac{x_k}{r} < V_w^{x_{k-1}}\left(\mu_{a,k}\right)$, $k = k', \ldots, b$, holds. Therefore, she reject a k'-type man at least. Moreover, the decision of a woman with $\mu_{a,k'+1}$ whether to accept a $k'-1$-type man also depends on whether $\frac{x_{k'-1}}{r}$ exceeds $V_w^{x_{k'-2}}\left(\mu_{a,k'-1}\right)$. Given $R_w\left(\mu_{a,b}\right) > x_{k'}$, $\frac{x_k}{r} \geq V_w^{x_{k-1}}\left(\mu_{a,k}\right)$, for $k = a, \ldots, k'-1$, holds. Hence, $\frac{x_{k'-1}}{r} \geq V_w^{x_{k'-2}}\left(\mu_{a,k'-1}\right)$. From these results, $R_w\left(\mu_{a,k'+1}\right) = R_w\left(\mu_{a,k'}\right) = R_w\left(\mu_{a,b}\right) > x_{k'}$.

By repeating the same procedure until the decision of a woman with $\mu_{a,b-1}$, we obtain $R_w\left(\mu_{a,k'}\right) = R_w\left(\mu_{a,k'+1}\right) = \ldots = R_w\left(\mu_{a,b-1}\right) = R_w\left(\mu_{a,b}\right) > x_{k'}$.

If $R_w\left(\mu_{a,b}\right) > x_{k'}$, a woman with $\mu_{a,b+l}$, for any $l > 0$, also rejects a k'-type man. This is because her decision depends on whether $\frac{x_{k'}}{r}$ exceeds $V_w^{x_{k'-1}}\left(\mu_{a,k'}\right)$. Hence,

$$R_w\left(\mu_{a,k'}\right) = R_w\left(\mu_{a,k'+1}\right) = \ldots = R_w\left(\mu_{a,b}\right) = \ldots = R_w\left(\mu_{a,n}\right) > x_{k'}.$$

Conversely, given $R_w\left(\mu_{a,k'}\right) > x_{k'}$, $x_k \geq rV_w^{x_{k-1}}\left(\mu_{a,k}\right)$, for $k = a+1, \ldots, k'-1$. From this, a woman with $\mu_{a,b}$ also rejects a k'-type man. Moreover, a woman with $\mu_{a,b}$ rejects a $k'+1$-type or lower type man from the reservation property. Hence, if $R_w\left(\mu_{a,k'}\right) > x_{k'}$, then $R_w\left(\mu_{a,k'}\right) = R_w\left(\mu_{a,b}\right) > x_{k'}$ for any $b \geq k'$. In other words, given $R_w\left(\mu_{a,k'}\right) > x_{k'}$, we have

$$R_w\left(\mu_{a,k'}\right) = R_w\left(\mu_{a,k'+1}\right) = \ldots = R_w\left(\mu_{a,n}\right) > x_{k'}.$$

■

Proof of Lemma 3: We prove the lemma by mathematical induction. First, we investigate the case where $l = 0$,for any $a = 1, \ldots, n-1$. From $R_i^*\left(x_k\right) > x_{k+1}$, a woman with $\mu_{a,a}$ always rejects an $a+1$-type man, i.e., $R_w\left(\mu_{a,a}\right) = \frac{\alpha\lambda_a^m x_a}{\alpha\lambda_a^m + r} > x_{a+1}$. If a woman with $\mu_{a,a+1}$ also rejects an $a+1$-type man, her value becomes $rV_w^{x_a}\left(\mu_{a,a+1}\right) = r\frac{\mu_{a,a+1}(x_a)\alpha\lambda_a^m\left(\frac{x_a}{r}\right) + \left(1-\mu_{a,a+1}(x_a)\right)\alpha\lambda_a^m\left(V_w\left(\mu_{a+1,a+1}\right)\right)}{r+\alpha\lambda_a^m}$. From these

$$R_w\left(\mu_{a,a}\right) - rV_w^{x_a}\left(\mu_{a,a+1}\right) = \alpha\lambda_a^m\left(1-\mu_{a,a+1}\left(x_a\right)\right)\frac{x_a - rV_w\left(\mu_{a+1,a+1}\right)}{r+\alpha\lambda_a^m}$$

From $rV_w\left(\mu_{a+1,a+1}\right) \leq x_{a+1} < x_a$, $R_w\left(\mu_{a,a}\right) > rV_w^{x_a}\left(\mu_{a,a+1}\right)$.

From Lemma 1, a woman with $\mu_{a,a+1}$ always rejects an $a+2$-type man. Moreover, she always accepts an a-type man. From these, we have

$$R_w\left(\mu_{a,a}\right) \geq R_w\left(\mu_{a,a+1}\right). \tag{19}$$

Next, we investigate the case of $l = 1$ for any $a = 1, \ldots, n-2$. To simplify the notation,

$p_j = \mu_{a,a+1}(x_j)$ and $q_j = \mu_{a,a+2}(x_j)$. If a woman with $\mu_{a,a+2}$ rejects an $a+1$-type man, her decision depends on whether $\frac{x_{a+1}}{r}$ exceeds $V_w^{x_a}(\mu_{a,a+1})$. At this time,

$$V_w^{x_a}(\mu_{a,a+1}) = V_w^{x_a}(\mu_{a,a+2}). \tag{20}$$

If a woman with $\mu_{a,a+1}$ rejects an $a+2$-type man, her value becomes, $V_w^{x_{a+1}}(\mu_{a,a+1}) = \frac{\alpha\sum_{j=a}^{a+1} p_j\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,a+1}\right)\right)+\Sigma_{i=j}^{a+1}\lambda_i^m\left(\frac{x_i}{r}\right)\right]}{\left(r+\alpha\Sigma_{i=a}^{a+1}\lambda_i^m\right)}$.
Similarly, when a woman with $\mu_{a,a+2}$ rejects an $a+2$-type man, her value becomes, $V_w^{x_{a+1}}(\mu_{a,a+2}) = \frac{\alpha\sum_{j=a}^{a+1} q_j\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,a+2}\right)\right)+\Sigma_{i=j}^{a+1}\lambda_i^m\left(\frac{x_i}{r}\right)\right]+q_{a+2}\left[\Sigma_{i=a}^{a+1}\lambda_i^m\left(V_w\left(\mu_{i+1,a+2}\right)\right)\right]}{\left(r+\alpha\Sigma_{i=a}^{a+1}\lambda_i^m\right)}$.
Moreover, from (3), $p_j = \frac{q_j}{q_a+q_{a+1}}$, for $j = a, a+1$. Therefore,

$$\begin{aligned}
&V_w^{x_{a+1}}(\mu_{a,a+1}) - V_w^{x_{a+1}}(\mu_{a,a+2}) \\
&= \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{a+1}\lambda_i^m\right)}\left[\begin{array}{c}
\frac{q_a}{q_a+q_{a+1}}V_w(x_a|\mu_{a,a+1}) - q_aV_w(x_a|\mu_{a,a+2}) \\
+\frac{q_{a+1}}{q_a+q_{a+1}}V_w(x_{a+1}|\mu_{a,a+1}) - q_{a+1}V_w(x_{a+1}|\mu_{a,a+2}) \\
-\frac{q_a+q_{a+1}}{q_a+q_{a+1}}\left(1-(q_a+q_{a+1})\right)V_w(x_{a+2}|\mu_{a,a+2})
\end{array}\right] \\
&= \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{a+1}\lambda_i^m\right)}\left[\begin{array}{c}
\frac{q_aV_w(x_a|\mu_{a,a+1})}{q_a+q_{a+1}} - q_aV_w(x_a|\mu_{a,a+2}) \\
-\frac{q_a(1-(q_a+q_{a+1}))V_w(x_{a+2}|\mu_{a,a+2})}{q_a+q_{a+1}} \\
+\frac{q_{a+1}V_w(x_{a+1}|\mu_{a,a+1})}{q_a+q_{a+1}} - q_{a+1}V_w(x_{a+1}|\mu_{a,a+2}) \\
-\frac{q_{a+1}(1-(q_a+q_{a+1}))V_w(x_{a+2}|\mu_{a,a+2})}{q_a+q_{a+1}}
\end{array}\right]
\end{aligned}$$

Here, from $V_w(x''|\mu_{a,b}) \geq V_w(x'|\mu_{a,b})$, for any $a, b\,(a<b)$, and $x'' > x'$,

$$\begin{aligned}
&V_w^{x_{a+1}}(\mu_{a,a+1}) - V_w^{x_{a+1}}(\mu_{a,a+2}) \\
&\geq \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{a+1}\lambda_i^m\right)}\left[\begin{array}{c}
\frac{q_a}{q_a+q_{a+1}}V_w(x_a|\mu_{a,a+1}) - q_aV_w(x_a|\mu_{a,a+2}) \\
-\frac{q_a(1-(q_a+q_{a+1}))V_w(x_a|\mu_{a,a+2})}{q_a+q_{a+1}} \\
+\frac{q_{a+1}}{q_a+q_{a+1}}V_w(x_{a+1}|\mu_{a,a+1}) - q_{a+1}V_w(x_{a+1}|\mu_{a,a+2}) \\
-\frac{q_{a+1}(1-(q_a+q_{a+1}))V_w(x_a|\mu_{a,a+2})}{q_a+q_{a+1}}
\end{array}\right] \\
&= \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{a+1}\lambda_i^m\right)}\left[\begin{array}{c}
\frac{q_a}{q_a+q_{a+1}}\left(V_w(x_a|\mu_{a,a+1}) - V_w(x_a|\mu_{a,a+2})\right) \\
+\frac{q_{a+1}}{q_a+q_{a+1}}\left(V_w(x_{a+1}|\mu_{a,a+1}) - V_w(x_{a+1}|\mu_{a,a+2})\right)
\end{array}\right]
\end{aligned}$$

Here, $V_w(x_a|\mu_{a,a+1}) - V_w(x_a|\mu_{a,a+2}) = \Sigma_{i=a}^{a+1}\lambda_i^m\left(\frac{x_i}{r}\right) - \Sigma_{i=a}^{a+1}\lambda_i^m\left(\frac{x_i}{r}\right) = 0$. Moreover,

$$\begin{aligned}
&V_w(x_{a+1}|\mu_{a,a+1}) - V_w(x_{a+1}|\mu_{a,a+2}) \\
&= \lambda_a^mV_w(\mu_{a+1,a+1}) + \lambda_{a+1}^m\frac{x_{a+1}}{r} - \left[\lambda_a^mV_w(\mu_{a+1,a+2}) + \lambda_{a+1}^m\frac{x_{a+1}}{r}\right] \\
&= \lambda_a^m\left[V_w(\mu_{a+1,a+1}) - V_w(\mu_{a+1,a+2})\right]
\end{aligned}$$

From (19), for any a, $V_w(x_{a+1}|\mu_{a,a+1}) \geq V_w(x_{a+1}|\mu_{a,a+2})$. From these, we have

$$V_w^{x_{a+1}}(\mu_{a,a+1}) \geq V_w^{x_{a+1}}(\mu_{a,a+2}). \tag{21}$$

From (20)-(21), in an equilibrium,

$$R_w(\mu_{a,a+1}) \geq R_w(\mu_{a,a+2}), \tag{22}$$

must hold. Specifically, if a woman with $\mu_{a,a+2}$ rejects an $a+3$-type man, $R_w(\mu_{a,a+1}) > x_{a+2} \geq R(\mu_{a,a+2})$.

Let assume that $R_w\left(\mu_{i,a+l-1}\right) > R_w\left(\mu_{i,a+l}\right)$, for $l = l-1$ and $a = 1,\ldots,n-l$, and for $i = a, a+1, \ldots, a+l-1$.

Given this, let us investigate the case of $l = l$. If a woman with $\mu_{a,a+l}$ rejects an s-type man, for $s = a+1,\ldots,a+l-1$, her decision depends on whether $\frac{x_s}{r}$ exceeds $V_w^{x_{s-1}}\left(\mu_{a,s}\right)$. Similarly, if a woman with $\mu_{a,a+l+1}$ rejects an s-type man, for $s = a+1,\ldots,a+l-1$, her decision also depends on whether $\frac{x_s}{r}$ exceeds $V_w^{x_{s-1}}\left(\mu_{a,s}\right)$. Therefore, for $s = a+1,\ldots,a+l-1$,

$$V_w^{x_s}\left(\mu_{a,a+l}\right) = V_w^{x_s}\left(\mu_{a,a+l+1}\right). \tag{23}$$

If a woman with $\mu_{a,a+l}$ rejects an $a+l$-type man, her value becomes,
$V_w^{x_{a+l}}\left(\mu_{a,a+l}\right) = \frac{\alpha\sum_{j=a}^{a+l} p_j\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,a+l}\right)\right)+\Sigma_{i=j}^{a+l}\lambda_i^m\left(\frac{x_i}{r}\right)\right]}{\left(r+\alpha\Sigma_{i=a}^{a+l}\lambda_i^m\right)}$.
Similarly, when a woman with $\mu_{a,a+l+1}$ rejects an $a+l$-type man, her value becomes,
$V_w^{x_{a+l}}\left(\mu_{a,a+l+1}\right) = \frac{\alpha\sum_{j=a}^{a+l} q_j\left[\Sigma_{i=a}^{j-1}\lambda_i^m\left(V_w\left(\mu_{i+1,a+l+1}\right)\right)+\Sigma_{i=j}^{a+l}\lambda_i^m\left(\frac{x_i}{r}\right)\right]+q_{a+l+1}\left[\Sigma_{i=a}^{a+l}\lambda_i^m\left(V_w\left(\mu_{i+1,a+l+1}\right)\right)\right]}{\left(r+\alpha\Sigma_{i=a}^{a+l}\lambda_i^m\right)}$.
Moreover, from (3), $p_j = \frac{q_j}{(q_a+\ldots+q_{a+l})}$, for $j = a, a+1, \ldots, a+l$. Therefore,
$V_w^{x_{a+l}}\left(\mu_{a,a+l}\right) - V_w^{x_{a+l}}\left(\mu_{a,a+l+1}\right)$

$$= \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{a+l}\lambda_i^m\right)}\begin{bmatrix} \frac{q_a}{(q_a+\ldots+q_{a+l})}V_w\left(x_a|\mu_{a,a+l}\right) - q_aV_w\left(x_a|\mu_{a,a+l+1}\right) \\ -\frac{q_a}{(q_a+\ldots+q_{a+l})}\left(1-(q_a+\ldots+q_{a+l})\right)V_w\left(x_{a+l+1}|\mu_{a,a+l+1}\right) \\ +\frac{q_{a+1}}{(q_a+\ldots+q_{a+l})}V_w\left(x_{a+1}|\mu_{a,a+l}\right) - q_{a+1}V_w\left(x_{a+1}|\mu_{a,a+l+1}\right) \\ -\frac{q_{a+1}}{(q_a+\ldots+q_{a+l})}\left(1-(q_a+\ldots+q_{a+l})\right)V_w\left(x_{a+l+1}|\mu_{a,a+l+1}\right) \\ +\ldots. \\ +\frac{q_{a+l}}{(q_a+\ldots+q_{a+l})}V_w\left(x_{a+l}|\mu_{a,a+l}\right) - q_{a+l}V_w\left(x_{a+l}|\mu_{a,a+l+1}\right) \\ -\frac{q_{a+l}}{(q_a+\ldots+q_{a+l})}\left(1-(q_a+\ldots+q_{a+l})\right)V_w\left(x_{a+l+1}|\mu_{a,a+l+1}\right) \end{bmatrix}$$

$$\geq \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{a+l}\lambda_i^m\right)}\begin{bmatrix} \frac{q_a}{(q_a+\ldots+q_{a+l})}V_w\left(x_a|\mu_{a,a+l}\right) - q_aV_w\left(x_a|\mu_{a,a+l+1}\right) \\ -\frac{q_a}{(q_a+\ldots+q_{a+l})}\left(1-(q_a+\ldots+q_{a+l})\right)V_w\left(x_a|\mu_{a,a+l+1}\right) \\ +\frac{q_{a+1}}{(q_a+\ldots+q_{a+l})}V_w\left(x_{a+1}|\mu_{a,a+l}\right) - q_{a+1}V_w\left(x_{a+1}|\mu_{a,a+l+1}\right) \\ -\frac{q_{a+1}}{(q_a+\ldots+q_{a+l})}\left(1-(q_a+\ldots+q_{a+l})\right)V_w\left(x_{a+1}|\mu_{a,a+l+1}\right) \\ +\ldots. \\ +\frac{q_{a+l}}{(q_a+\ldots+q_{a+l})}V_w\left(x_{a+l}|\mu_{a,a+l}\right) - q_{a+l}V_w\left(x_{a+l}|\mu_{a,a+l+1}\right) \\ -\frac{q_{a+l}}{(q_a+\ldots+q_{a+l})}\left(1-(q_a+\ldots+q_{a+l})\right)V_w\left(x_{a+l}|\mu_{a,a+l+1}\right) \end{bmatrix}$$

$$= \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{a+l}\lambda_i^m\right)}\begin{bmatrix} \frac{q_a}{(q_a+\ldots+q_{a+l})}\left[V_w\left(x_a|\mu_{a,a+l}\right) - V_w\left(x_a|\mu_{a,a+l+1}\right)\right] \\ +\frac{q_{a+1}}{(q_a+\ldots+q_{a+l})}\left[V_w\left(x_{a+1}|\mu_{a,a+l}\right) - V_w\left(x_{a+1}|\mu_{a,a+l+1}\right)\right] \\ +\ldots \\ +\frac{p_{a+l}}{(q_a+\ldots+q_{a+l})}\left[V_w\left(x_{a+l}|\mu_{a,a+l}\right) - V_w\left(x_{a+l}|\mu_{a,a+l+1}\right)\right] \end{bmatrix}$$

Here, $V_w\left(x_a|\mu_{a,a+l}\right) - V_w\left(x_a|\mu_{a,a+l+1}\right) = \Sigma_{i=a}^{a+l}\lambda_i^m\left(\frac{x_i}{r}\right) - \Sigma_{i=a}^{a+l}\lambda_i^m\left(\frac{x_i}{r}\right) = 0$.
$V_w\left(x_{a+1}|\mu_{a,a+l}\right) - V_w\left(x_{a+1}|\mu_{a,a+l+1}\right)$
$= \left[\lambda_a^m\left(V_w\left(\mu_{a+1,a+l}\right)\right) + \Sigma_{i=a+1}^{a+l}\lambda_i^m\left(\frac{x_i}{r}\right)\right] - \left[\lambda_a^m\left(V_w\left(\mu_{a+1,a+l+1}\right)\right) + \Sigma_{i=a+1}^{a+l}\lambda_i^m\left(\frac{x_i}{r}\right)\right]$
$= \lambda_a^m\left(V_w\left(\mu_{a+1,a+l}\right) - V_w\left(\mu_{a+1,a+l+1}\right)\right)$.
...
$V_w\left(x_{a+l}|\mu_{a,a+l}\right) - V_w\left(x_{a+l}|\mu_{a,a+l+1}\right)$

$= \left[\Sigma_{i=a}^{a+l-1}\lambda_i^m \left(V_w\left(\mu_{i+1,a+l}\right)\right) + \lambda_{a+l}^m\left(\frac{x_i}{r}\right)\right] - \left[\Sigma_{i=a}^{a+l-1}\lambda_i^m\left(V_w\left(\mu_{i+1,a+l+1}\right)\right) + \lambda_{a+l}^m\left(\frac{x_i}{r}\right)\right]$
$= \Sigma_{i=a}^{a+l-1}\lambda_i^m\left(\left(V_w\left(\mu_{i+1,k}\right)\right) - \left(V_w\left(\mu_{i+1,k+1}\right)\right)\right)$
$= \lambda_a^m\left(V_w\left(\mu_{a+1,a+l}\right) - V_w\left(\mu_{a+1,a+l+1}\right)\right)$
$+\lambda_{a+1}^m\left(V_w\left(\mu_{a+2,a+l}\right)\right) - \left(V_w\left(\mu_{a+2,a+l+1}\right)\right)$
$+\ldots$
$+\lambda_{k-1}^m\left(V_w\left(\mu_{a+l,a+l}\right) - V_w\left(\mu_{a+l,a+l+1}\right)\right).$

From the assumption of mathematical induction, $V_w\left(\mu_{i,a+l}\right) - V_w\left(\mu_{i,a+l+1}\right) > 0$,for $i = a+1,\ldots,a+l$, holds. Therefore, $V_w^{x_{a+l}}\left(\mu_{a,a+l}\right) > V_w^{x_{a+l}}\left(\mu_{a,a+l+1}\right)$. From this and (23)

$$R_w\left(\mu_{a,a+l+1}\right) \leq R_w\left(\mu_{a,a+l}\right).$$

■

Proof of Lemma 4: From Lemma 2, the decision of a woman with $\mu_{a,b}$, for any $a,b\,(a<b)$, whether to accept a k-type man depends on whether $\frac{x_k}{r}$ exceeds $V_w^{x_{k-1}}\left(\mu_{a,k}\right)$, for $k \in (a+1,b)$. Similarly, the decision of a woman with $\mu_{a+1,k}$ whether to accept a k-type man depends on whether $\frac{x_k}{r}$ exceeds $V_w^{x_{k-1}}\left(\mu_{a+1,k}\right)$, for $k \in (a+1,b)$. Here,

$$V_w^{x_{k-1}}\left(\mu_{a,k}\right) - V_w^{x_{k-1}}\left(\mu_{a+1,k}\right)$$
$$= \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{k-1}\lambda_i^m\right)}\begin{pmatrix} p_a\lambda_a^m\left(\frac{x_a}{r}\right) + (1-p_a)\lambda_a^m V_w\left(\mu_{a+1,k}\right) \\ +\left(\Sigma_{j=a}^{a+1}p_j\right)\lambda_{a+1}^m\left(\frac{x_{a+1}}{r}\right) + \left(1-\Sigma_{j=a}^{a+1}p_j\right)\lambda_{a+1}^m V_w\left(\mu_{a+2,k}\right) \\ +\left(\Sigma_{j=a}^{a+2}p_j\right)\lambda_{a+2}^m\left(\frac{x_{a+2}}{r}\right) + \left(1-\left(\Sigma_{j=a}^{a+2}p_j\right)\right)\lambda_{a+2}^m V_w\left(\mu_{a+3,k}\right) \\ +\ldots \\ +(1-p_k)\lambda_{k-1}^m\left(\frac{x_{k-1}}{r}\right) + p_k\lambda_{k-1}^m V_w\left(\mu_{k,k}\right) \end{pmatrix}$$
$$-\frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{k-1}\lambda_i^m\right)}\begin{pmatrix} \lambda_a^m V_w\left(\mu_{a+1,k}\right) \\ +\left(\frac{p_{a+1}}{1-p_a}\right)\lambda_{a+1}^m\left(\frac{x_{a+1}}{r}\right) + \left(1-\left(\frac{p_{a+1}}{1-p_a}\right)\right)\lambda_{a+1}^m V_w\left(\mu_{a+2,k}\right) \\ +\left(\Sigma_{j=a+1}^{a+2}\frac{p_j}{1-p_a}\right)\lambda_{a+2}^m\left(\frac{x_{a+2}}{r}\right) + \left(1-\left(\Sigma_{j=a+1}^{a+2}\frac{p_j}{1-p_a}\right)\right)\lambda_{a+2}^m V_w\left(\mu_{a+3,k}\right) \\ +\ldots \\ +\left(1-\frac{p_k}{(1-p_a)}\right)\lambda_{k-1}^m\left(\frac{x_{k-1}}{r}\right) + \frac{p_k}{(1-p_a)}\lambda_{k-1}^m V_w\left(\mu_{k,k}\right) \end{pmatrix}$$
$$= \frac{\alpha}{\left(r+\alpha\Sigma_{i=a}^{k-1}\lambda_i^m\right)}\begin{pmatrix} \lambda_a^m p_a\frac{x_a - rV_w\left(\mu_{a+1,k}\right)}{r} \\ +\lambda_{a+1}^m p_a\left(1-(p_a+p_{a+1})\right)\frac{x_{a+1}-rV_w\left(\mu_{a+2,k}\right)}{r(1-p_a)} \\ +\lambda_{a+2}^m p_a\left(1-(p_a+p_{a+1}+p_{a+2})\right)\frac{x_{a+2}-rV_w\left(\mu_{a+3,k}\right)}{r(1-p_a)} \\ +\ldots \\ +\lambda_{k-1}^m p_a p_k\frac{x_{k-1}-rV_w\left(\mu_{k,k}\right)}{r(1-p_a)} \end{pmatrix}.$$

From $x_i \geq R_w\left(\mu_{i,k}\right)$, for $i = a+1,\ldots,k$, $x_{i-1} > rV_w\left(\mu_{i,k}\right)$. Hence, $V_w^{x_{k-1}}\left(\mu_{a,k}\right) > V_w^{x_{k-1}}\left(\mu_{a+1,k}\right)$, for any $k\,(<a+1)$.

Given $x_k < R_w\left(\mu_{a+1,b}\right)$, for $k \in (a+1,b)$, $x_k < R_w\left(\mu_{a+1,k}\right) = R_w\left(\mu_{a+1,b}\right)$, from Lemma 2. From $V_w^{x_{k-1}}\left(\mu_{a,k}\right) > V_w^{x_{k-1}}\left(\mu_{a+1,k}\right) = R_w\left(\mu_{a+1,k}\right) > x_k$,

$x_k < R_w\left(\mu_{a+1,k}\right) = R_w\left(\mu_{a+1,b}\right) < rV_w^{x_{k-1}}\left(\mu_{a,k}\right) \leq R_w\left(\mu_{a,k}\right)$. However, when $x_k < R_w\left(\mu_{a,k}\right)$, $R_w\left(\mu_{a,k}\right) = R_w\left(\mu_{a,b}\right) > x_k$ from Lemma 2. Therefore, for any $k \in (a+1,b)$, $x_k < R_w\left(\mu_{a+1,b}\right) = R_w\left(\mu_{a+1,k}\right) < R_w\left(\mu_{a,k}\right) = R_w\left(\mu_{a,b}\right)$.■

Proof of Proposition 2: Given $R_w(\mu_0) > x_{k+1}$, there are no women with $\mu_{1,i}$, for $i = 1, \ldots, k$, in the market.[30] Furthermore, from Lemma 2

$$x_k \geq R_w\left(\mu_{1,k+1}\right) = R_w\left(\mu_{1,k+2}\right) = \ldots = R_w\left(\mu_{1,n-1}\right) = R_w(\mu_0) > x_{k+1} \tag{24}$$

holds. From this, even if a woman with μ_0 updates her belief to $\mu_{1,i}$, for any $i \in \{k+1, .., n-1\}$, after a meeting, then her reservation level does not rise.

A woman with $\mu_{1,i}$, for $i = k+1, \ldots, n$, becomes a woman with $\mu_{a,i}$ if she is rejected by an $a-1$-type man, for any $a > 1$. From Lemma 4, $R_w\left(\mu_{1,i}\right) > R_w\left(\mu_{a,i}\right)$.

For any $a, b (1 < a \leq k < b)$, given $R_w\left(\mu_{a,b}\right) > x_{k+1}$,

$$x_k \geq R_w\left(\mu_{a,k+1}\right) = \ldots = R_w\left(\mu_{a,b-1}\right) = R_w\left(\mu_{a,b}\right) > x_{k+1}$$

holds from Lemma 2. Therefore, if a woman with $\mu_{a,b}$ updates her belief to $\mu_{a,i}$, for any $i \in \{k+1, .., b-1\}$, then her reservation level does not rise. Furthermore, a woman with $\mu_{a,b}$ cannot be a woman with $\mu_{a,i}$, for any $i \in \{a, \ldots, k\}$, who has a higher reservation level than that of a woman with $\mu_{a,b}$. This is because a woman with $\mu_{a,b}$ always accepts a k-type man.

By contrast, a woman with $\mu_{a,b}$ becomes a woman with $\mu_{a',b}$ if she is rejected by an $a'-1$-type man, for any $a' > a$. From Lemma 4, $R_w\left(\mu_{a,b}\right) > R_w\left(\mu_{a',b}\right)$. Hence, she revises her reservation level downward.

From these results, a woman with imperfect self-knowledge does not raise her reservation level in search.

Finally, we show that $R_w(\mu_0)$ is the highest reservation level of women in equilibrium. Given $R_w(\mu_0) > x_{k+1}$, (24) holds. Let us assume that there is a woman with $\mu_{a,k'}$, who has her reservation level such that $R_w\left(\mu_{a,k'}\right) > x_{k'} \geq x_k$, for any a, k' $(1 < a < k' \leq k)$. A woman with $\mu_{1,i}$, for $i = k+1, \ldots, n$, becomes a woman with $\mu_{a,i}$ if she is rejected by an $a-1$-type man. Similarly, a woman with $\mu_{a',i}$, for a' $(1 < a' < a)$, and $i = k+1, \ldots, n$, becomes a woman with $\mu_{a,i}$ if she is rejected by an $a-1$-type man. Here, a woman with $\mu_{a,i}$, $i = k+1, \ldots, n$, becomes a woman with $\mu_{a,k'}$ if she rejects a k'-type man who proposes to her. However, $R_w\left(\mu_{1,i}\right) > R_w\left(\mu_{a,i}\right)$. Then, $x_k \geq R_w\left(\mu_{a,i}\right)$, for $i = k+1, \ldots, n$, from (24). This contradicts the fact that there is a woman with $\mu_{a,k'}$ who has $R_w\left(\mu_{a,k'}\right) > x_{k'} \geq x_k$. Thus, there are no such women in equilibrium.■

Proof of Lemma 7: Given $\Psi_w(.)$, μ_0 consists of $\mu_0(x_j) = \pi_j^w$, for $j = 1, \ldots, n$. According to Bayes' rule, a belief $\mu_{a,b}(x_j)$, for any a, b, j $(1 \leq a \leq j \leq b \leq n)$, becomes $\mu_{a,b}(x_j) = \frac{\pi_j}{\Sigma_{j=a}^b \pi_j}$.

By contrast, let us derive beliefs $\mu_{a,b}(x_j)$ from $G_w(.)$ and then, confirm these beliefs are consistent with those calculated by using Bayes' rule. For this, let us investigate the balanced flow in all states. Let $\phi_j^0 = \phi_j^{1,n}$. All states of a woman with $x_j^{a,b}$ for any j are as follows.

[30] If there was a woman with $\mu_{1,i}$, for $i \geq k$, she would reject an i-type man in her past. In this case, a woman with μ_0 must reject an i-type man from Lemma 2.

$$\begin{array}{cccccc} x_j^{1,j} & x_j^{1,j+1} & \ldots & & x_j^{1,n-1} & x_j^{1,n} \\ x_j^{2,j} & x_j^{2,j+1} & \ldots & & x_j^{2,n-1} & x_j^{2,n} \\ \ldots & & & & & \\ x_j^{a,j} & x_j^{a,j+1} & & x_j^{a,b} & x_j^{a,n-1} & x_j^{a,n} \\ \ldots & & & & & \\ x_j^{j,j} & x_j^{j,j+1} & \ldots & & x_j^{j,n-1} & x_j^{j,n}. \end{array}$$

Given π_j^w and β, for the state $x_j^{1,n} = x_j^0$, the balanced flow is satisfied if and only if

$$\beta\pi_j^w = \alpha \sum_{k=1}^{n-1} \lambda_k^m \phi_j^0 \lambda_j^w N. \tag{25}$$

where $\pi_j^w \beta$ is the inflow of new female entrants with x_j.[31] From (25), $\phi_j^0 = \beta\pi_j^w / \alpha \sum_{k=1}^{n-1} \lambda_k^m \lambda_j^w N$.

Then, let us investigate the balanced flow in the state $x_j^{a,b}$, for any $j = 1, \ldots, n$.

For $a = 1$ and $b = n-1$, (i.e., $x_j^{1,n-1}$), from (16),

$$\phi_j^{1,n-1} = \frac{\lambda_{n-1}^m}{\sum_{k=1}^{n-2} \lambda_k^m} \phi_1^0 = A_{1,n-1}\phi_j^0,$$

where $A_{1,n-1} = \lambda_{n-1}^m / \sum_{k=1}^{n-2} \lambda_k^m$ is the coefficient of ϕ_j^0. Then, for $a = 1$ and $b = n-2$, we have

$$\phi_j^{1,n-2} = \frac{\lambda_{n-2}^m \left(\phi_1^{1,n-1} + \phi_1^{1,n}\right)}{\sum_{k=1}^{n-3} \lambda_k^m} = \frac{\lambda_{n-2}^m (A_{1,n-1}+1)}{\sum_{k=1}^{n-3} \lambda_k^m} \phi_j^0 = A_{1,n-2}\phi_j^0$$

where $A_{1,n-2} = \lambda_{n-2}^m \left(A_{1,n-1} + 1\right) / \left(\sum_{k=1}^{n-3} \lambda_k^m\right)$. We can recursively repeat the same procedure until $b = j$. Therefore, for $a = 1$, and $b = j, \ldots, n$, we have

$$\phi_j^{1,b} = \frac{\lambda_b^m \left(\phi_j^{1,b+1} + \phi_j^{1,b+2} + \ldots + \phi_j^{1,n}\right)}{\sum_{k=1}^{b-1} \lambda_k^m} = \frac{\lambda_b^m \left(A_{1,b+1} + A_{1,b+2} + \ldots + A_{1,n-2} + A_{1,n}\right)}{\sum_{k=1}^{b-1} \lambda_k^m} \phi_1^0 = A_{1,b}\phi_j^0. \tag{26a}$$

where $A_{1,n} = 1$. However, if $j = 1$, all states of a woman with x_1 are $x_1^{1,2}, \ldots, x_1^{1,n}$, because there are no women with $\mu_{1,1}$ in equilibrium. Hence, $a = j = 1 < b$.

Similarly, for $a = 2$, $b = j, \ldots, n$, and $a < j \le b$ or $a = j = 2 < b$, we have

$$\phi_j^{2,b} = \frac{\lambda_1^m \phi_j^{1,b} + \lambda_b^m \sum_{i=b+1}^{n} \phi_j^{2,i}}{\sum_{k=2}^{b-1} \lambda_k^m} = \frac{\lambda_1^m A_{1,b} + \lambda_b^m \sum_{i=b+1}^{n} A_{2,i}}{\sum_{k=2}^{b-1} \lambda_k^m} \phi_j^0 = A_{2,b}\phi_j^0.$$

If $a = b = j = 2$,

$$\phi_2^{2,2} = \frac{\lambda_1^m \phi_2^{1,2}}{\lambda_2^m} = \frac{\lambda_1^m A_{1,2} \phi_2^0}{\lambda_2^m},$$

from (17) and (26a).

The same procedure is repeatedly applied until $a = j$. Therefore, generally, for $a < j \le b$

[31] Under the cloning assumption, π_j^w and β are endogenous, whereas they are exogenous under the exogenous inflow assumption.

or $a = j < b,\ (j = 1, \ldots, n)$, we can rewrite $\phi_j^{a,b}$ as

$$\begin{aligned}\phi_j^{a,b} &= \frac{\lambda_{a-1}^m\left(\phi_j^{1,b}+\phi_j^{2,b}+\ldots+\phi_j^{a-1,b}\right)+\lambda_b^m\left(\phi_j^{a,b+1}+\phi_j^{a,b+2}\ldots+\phi_j^{a,n}\right)}{\left(\sum_{k=a}^{b-1}\lambda_k^m\right)} \\ &= \frac{\lambda_{a-1}^m\sum_{i=1}^{a-1}A_{i,b}+\lambda_b^m\sum_{i=b+1}^{n}A_{a,i}}{\sum_{k=a}^{b-1}\lambda_k^m}\phi_j^0 = A_{a,b}\phi_j^0. \end{aligned} \tag{27}$$

For $a = j = b$, $(j = 2, \ldots, n)$, from (17), we have $\phi_j^{j,j} = \frac{\lambda_{j-1}^m\left(\phi_j^{1,j}+\phi_j^{2,j}+\ldots+\phi_j^{j-1,j}\right)}{\lambda_j^m}$. From (27), $\phi_j^{a,j} = A_{a,j}\phi_j^0$, for $a = 1, \ldots, j-1$. Hence,

$$\phi_j^{j,j} = \frac{\lambda_{j-1}^m\sum_{i=1}^{j-1}A_{i,j}}{\lambda_j^m}\phi_j^0 = A_{j,j}\phi_j^0. \tag{28}$$

From (25), (27) can be rewritten as

$$\phi_j^{a,b} = A_{a,b}\frac{\beta\pi_j^w}{\alpha\sum_{k=1}^{n-1}\lambda_k^m\lambda_j^w N}, \tag{29}$$

Hence, from (25) and (29), noting that $A_{a,b}$ depends only on $F_m(.)$, we have

$$\mu_0(x_j) = \frac{g_w\left(x_j^0\right)}{\Sigma_{k=1}^n g_w\left(x_j^0\right)} = \frac{\phi_j^0\lambda_j^w}{\Sigma_{j=1}^n\phi_j^0\lambda_j^w} = \frac{\pi_j^w}{\Sigma_{j=1}^n\pi_j^w} = \pi_j^w,$$

and

$$\mu_{a,b}(x_j) = \frac{g_w\left(x_j^{a,b}\right)}{\Sigma_{j=a}^b g_w\left(x_j^{a,b}\right)} = \frac{\phi_j^{a,b}\lambda_j^w}{\Sigma_{j=1}^n\phi_j^{a,b}\lambda_j^w} = \frac{A_{a,b}\phi_j^0\lambda_j^w}{\sum_{j=a}^b A_{a,b}\phi_j^0\lambda_j^w} = \frac{\pi_j^w}{\sum_{j=a}^b\pi_j^w}.$$

These equal to $\mu_0(x_k)$ and $\mu_{a,b}(x_j)$, which are calculated by using Bayes' rule. Hence, beliefs $\mu_{a,b}(x_j)$ are consistent with distribution $G_w(.)$ in the steady state equilibrium.■

Proof of Proposition 4: Now, $G_m = F_m$ holds.

First, let us consider the case of $j = 1$. In the PSEI, $\sum_{b=2}^n\phi_1^{1,b} = 1$. Let $\phi_1^{1,n} = \phi_1^0 = 1 - \sum_{b=2}^{n-1}\phi_1^{1,b}$. Therefore, the number of unknown variables, $\phi_1^{1,b}$, for $b = 2, \ldots, n-1$, is $n-2$. By contrast, from (15), the number of equations is $n-2$, which becomes equal to the number of unknown variables, $\phi_1^{1,b}$.

Next, let us consider the case of $j = 2, \ldots, n$. From $\sum_{a=1}^j\sum_{b=j}^n\phi_j^{a,b} = 1$, let $\phi_j^{1,n} = \phi_j^0 = 1 - \sum_{b=j}^{n-1}\phi_j^{1,b} - \sum_{a=2}^j\sum_{b=j}^n\phi_j^{a,b}$. Hence, the number of unknown variables, $\phi_j^{a,j}$, is $j(n-(j-1))-1$. By contrast, from (16)-(17), the number of equations is $j(n-(j-1)) - 1$ because a woman cannot become $x_j^0 = x_j^{1,n}$ from the other states. Therefore, the number of equations becomes equal to the number of unknown variables.

From these results and Lemma 7, for any $j = 1, \ldots, n$, the system has a unique solution, (G_w, μ_0).

From these results, given any (F_m, F_w, N), (G_m, G_w, μ_0) is always uniquely obtained. If (G_m, G_w, μ_0) satisfies (11) and (12), there exists a steady state PSEI.■