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Bifurcation of critical periods of a quartic system

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Abstract. For the polynomial system $\dot{x} = ix + x\bar{x}(ax^2 + bx\bar{x} + c\bar{x}^2)$ the study of critical period bifurcations is performed. Using calculations with algorithms of computational commutative algebra it is shown that at most two critical periods can bifurcate from any nonlinear center of the system.

Keywords: critical periods, bifurcations, isochronicity, polynomial systems.

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1 Introduction

For the plane real system of differential equations

$$\dot{u} = -v + \sum_{p+q=2}^{n} \alpha_{pq} u^p v^q, \; \dot{v} = u + \sum_{p+q=2}^{n} \beta_{pq} u^p v^q \tag{1.1}$$

the singularity at the origin is either a focus or a center. In the first case the trajectories in a neighborhood of the origin spirals either towards or away from the singularity. In the second case the trajectories are ovals, which means that the solutions are periodic functions. For a point A with the coordinates $u = r$, $v = 0$ (where r is sufficiently small) let $T(r)$ be the least period of the periodic solution with the initial data $u(0) = r$, $v(0) = 0$. The function $T(r)$ is called the period function of system (1.1). It is said that a center at the origin of (1.1) is isochronous if $T(r)$ is constant, that is, all solutions in a neighborhood of the origin have the same period. If a center at the origin of (1.1) is not isochronous, that is, $T(r) \not\equiv$ const, and for $r_0 > 0$ it holds that $T'(r_0) = 0$, then it is said that r_0 is a *critical period* of system (1.1).

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The problem of interest for us in this paper, the so-called problem of critical period bifurcations, was considered for the first time by Chicone and Jacobs in [8]. The problem is to estimate the number of critical periods that can appear near the center when system (1.1) with arbitrary chosen parameters is slightly perturbed within the family in such a way that the singularity at the origin remains a center. After the pioneering work [8] the problem has been intensively studied by many authors. Bifurcations of critical periods for a linear center perturbed by homogeneous cubic polynomials were investigated in [18,29]. The problem has also been studied for reversible and Hamiltonian cubic systems [34,35], the reduced Kukles system [30], Liénard systems ([33,37]), generalized Lotka–Volterra systems [32], generalized Loud systems [31], and some other systems (see e.g. [5,6,9,16,22,25,36]).

To study the critical period bifurcations it is convenient to consider along with system (1.1) its complexification obtained as follows. Introducing the complex variable $x = u + iv$ (where $i = \sqrt{-1}$) we get from (1.1) an equation, which can be written in the form

$$\dot{x} = ix - \sum_{j+k=1}^{n-1} a_{jk} x^{j+1} \bar{x}^k. \tag{1.2}$$

Equations of the form (1.2) are often referred as real systems in the complex form. Let

$$y = \bar{x}, \qquad b_{kj} = \bar{a}_{jk}. \tag{1.3}$$

We associate to equation (1.2) the two-dimensional complex system

$$\begin{aligned} \dot{x} &= ix - \sum_{j+k=1}^{n-1} a_{jk} x^{j+1} y^k = ix + P(x,y), \\ \dot{y} &= -iy + \sum_{j+k=1}^{n-1} b_{kj} x^k y^{j+1} = -iy + Q(x,y), \end{aligned} \tag{1.4}$$

which is the so-called *complexification* of system (1.1). If for system (1.4) condition (1.3) is fulfilled then system (1.4) is equivalent to equation (1.2). In this case the complex line $y = \bar{x}$ is invariant for system (1.4) and viewing the line as a two-dimensional hyperplane in $\mathbb{R}^4$, the flow on the line is precisely the original flow of (1.1) on $\mathbb{R}^2$ (see e.g. [28] for more details).

In the recent paper [15] García et al. investigated small limit cycle bifurcations in a neighborhood of the origin for a real system which can be written in the complex form (1.2) as

$$\dot{x} = ix(1 - a_{21}x^2\bar{x} - a_{12}x\bar{x}^2 - a_{03}\bar{x}^3), \tag{1.5}$$

where a_{21}, a_{11}, a_{03} are complex parameters. In this paper we perform the further bifurcation analysis of system (1.5) studying its critical period bifurcations from the center at the origin. Using algorithms of computational commutative algebra we perform the study of the ideal generated by the coefficients of the period function of system (1.5) establishing that at most two critical periods can bifurcate from any nonlinear center of the system. In most of the works devoted to critical period bifurcations authors compute the period function for each of components of the center variety. One of essential differences of our approach is that we obtain only one series expansion of the period function which is valid on each component of the center variety. This allows to reduce the amount of computations significantly.

2 Preliminaries

For an ideal I in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ we let $\mathbf{V}(I)$ denote the affine variety of I, that is, the set of common zeros in $\mathbb{C}^n$ of elements of I. For any subset $\mathcal{S}$ in $\mathbb{C}^n$ we let $\mathbf{I}(\mathcal{S})$ denote the ideal of $\mathcal{S}$, that is, the set of all polynomials vanishing on $\mathcal{S}$.

Let ℓ denote the number of parameters a_{jk} in equation (1.2). Since for each a_{jk} there is the parameter b_{kj} in the second equation of (1.4), system (1.4) has 2ℓ parameters, which we order in some manner and write the 2ℓ-tuple of the parameters as

$$(a_{p_1q_1}, \ldots, a_{p_\ell q_\ell}, b_{q_\ell, p_\ell}, \ldots, b_{q_1p_1}), \tag{2.1}$$

which we shorten to (a, b). We write $\mathbb{C}[a, b]$ for the ring of complex polynomials in the variables $a_{p_1q_1}, \ldots, b_{q_1p_1}$.

The first step in the investigation of critical period bifurcations is the computation of few first terms of the Taylor series expansion of the period function. In most works devoted to the problem the calculation of the period function is computed using polar coordinates. However using this approach one has to find the isochronicity variety first, and then compute the period function for each component of the variety. We will use another approach where the period function is derived from the normal form of the system as follows.

Performing a change of coordinates of the form

$$x = y_1 + \sum_{j+k\geq 2} h_1^{(j,k)} y_1^j y_2^k, \qquad y = y_2 + \sum_{j+k\geq 2} h_2^{(j,k)} y_1^j y_2^k, \tag{2.2}$$

we transform system (1.4) to the normal form

$$\begin{aligned} \dot{y}_1 &= y_1 \left(i + \sum_{j=1}^{\infty} Y_1^{(j+1,j)} (y_1y_2)^j \right) = y_1(i + Y_1(y_1y_2)), \\ \dot{y}_2 &= y_2 \left(-i + \sum_{j=1}^{\infty} Y_2^{(j,j+1)} (y_1y_2)^j \right) = y_2(-i + Y_2(y_1y_2)). \end{aligned} \tag{2.3}$$

The coefficients $Y_1^{(j+1,j)}$ and $Y_2^{(j,j+1)}$ of the series in (2.3) are elements of the polynomial ring $\mathbb{C}[a, b]$. They generate the ideal

$$\mathcal{Y} := \left\langle Y_1^{(j+1,j)},\ Y_2^{(j,j+1)} : j \in \mathbb{N} \right\rangle \subset \mathbb{C}[a, b]. \tag{2.4}$$

For any $K \in \mathbb{N}$ we set

$$\mathcal{Y}_K := \left\langle\, Y_1^{(j+1,j)},\ Y_2^{(j,j+1)} : j = 1, \ldots, K \,\right\rangle.$$

Clearly, the normal form of a particular system with fixed parameters (a^*, b^*) is linear when all the coefficients $Y_1^{(j+1,j)}(a, b)$, $Y_2^{(j,j+1)}(a, b)$ $(j \in \mathbb{N})$ vanish at (a^*, b^*), that is, when the point (a^*, b^*) belongs to the variety of ideal $\mathcal{Y}$. The variety $V_{\mathscr{L}} := \mathbf{V}(\mathcal{Y})$ is called the *linearizability variety* of system (1.4).

By the Poincaré–Lyapunov theorem linearizability of (1.1) or (1.2) is equivalent to its isochronicity, and existence of a center at the origin of (1.1) or (1.2) is equivalent to existence of an analytic first integral near the origin (see, for example, [28]). The latter observation allows to extend the concept of a center from real systems (1.2) to systems of the form (1.4) on $\mathbb{C}^2$. Namely, it is said that system (1.4) has a *center* at the origin if it admits an analytic first integral in a neighborhood of the origin.

Introducing the functions

$$G = Y_1 + Y_2, \qquad H = Y_1 - Y_2,$$

we have that the origin is a center for (1.4) if and only if $G \equiv 0$ (see, for instance, Theorem 3.2.7 of [28]), in which case H has purely imaginary coefficients and the distinguished normalizing transformation converges (see, for example, Theorem 3.2.5 and Remark 3.2.8 of [28]). The variety of the ideal

$$\left\langle Y_1^{(j+1,j)} + Y_2^{(j,j+1)} : j \in \mathbb{N} \right\rangle \subset \mathbb{C}[a,b] \tag{2.5}$$

is called the *center variety* and denoted by $V_{\mathcal{C}}$.

We define the function $\widetilde{H}$ by

$$\widetilde{H}(w) = -\tfrac{1}{2} i H(w),$$

where $w = y_1 y_2$. If system (1.4) is the complexification of a real system we recover the real system (up to a near-identity change of coordinates) by replacing every occurrence of y_2 by $\bar{y}_1$ in each equation of (2.3). Setting $y_1 = re^{i\varphi}$ we obtain from (2.3)

$$\dot{r} = \tfrac{1}{2r}(\dot{y}_1\bar{y}_1 + y_1\dot{\bar{y}}_1) = 0, \qquad \dot{\varphi} = \tfrac{i}{2r^2}(y_1\dot{\bar{y}}_1 - \dot{y}_1\bar{y}_1) = 1 + \widetilde{H}(r^2)\,. \tag{2.6}$$

Integrating the expression for $\dot{\varphi}$ in (2.6) yields

$$T(r) = \frac{2\pi}{1 + \widetilde{H}(r^2)} = 2\pi \left(1 + \sum_{k=1}^{\infty} p_{2k} r^{2k} \right) \tag{2.7}$$

for some coefficients p_{2k}, which are polynomials in the parameters (a,b) of system (1.4). The center is isochronous if and only if $p_{2k} = 0$ for $k \geq 1$. We call the polynomial p_{2k} the *k-th isochronicity quantity*.

The isochronicity quantities p_{2k} lose their geometric meaning when (1.4) does not correspond to the complexification of any real system (1.2), however they still exist as implicitly defined by (2.7), hence so does the function

$$T(r,a,b) = 2\pi \left(1 + \sum_{k=1}^{\infty} p_{2k}(a,b) r^{2k} \right),$$

which coincides with the period function (2.7) when $b = \bar{a}$.

Introducing the notation

$$P = \langle p_{2k} : k \in \mathbb{N} \rangle \subset \mathbb{C}[a,b]$$

and for $K \in \mathbb{N}$

$$P_K = \langle p_2, \ldots, p_{2K} \rangle$$

we have from Propositions 4.2.13 and 4.2.14 of [28]:

$$\mathbf{V}(P) \cap V_{\mathscr{C}} = \mathbf{V}(\mathcal{Y}) \cap V_{\mathscr{C}} \quad \text{and} \quad \mathbf{V}(P_K) \cap V_{\mathscr{C}} = \mathbf{V}(\mathcal{Y}_K) \cap V_{\mathscr{C}} \text{ for all } K \in \mathbb{N}. \tag{2.8}$$

The ideal P is called the *isochronicity ideal* of system (1.4).

As it was shown in [14,28] the following statement holds.

Theorem 2.1. *Assume that for $(a,\bar{a}) \in U$, where U is an open subset of $V_{\mathscr{C}}$, the function*

$$\mathscr{T}(r,(a,\bar{a})) = T(r,a,\bar{a}) - 2\pi = \sum_{k=1}^{\infty} p_{2k}(a,\bar{a})r^{2k}, \tag{2.9}$$

computed for system (1.2), can be expressed as

$$\mathscr{T}(r,(a,\bar{a})) = p_2(a,\bar{a})r^{j_1}(1+\psi_1(r,(a,\bar{a})) + \cdots + p_{2s}(a,\bar{a})r^{j_s}(1+\psi_s(r,(a,\bar{a})).$$

Then at most $s-1$ critical period bifurcates from the origin of any system from U under small perturbations.

For our study we will also need the following statement proven in [14] (see also [21]).

Proposition 2.2. *Suppose $I = \langle h_1, \ldots, h_r \rangle$, A and B are ideals in $\mathbb{C}[x_1, \ldots, x_n]$, A is radical, and $I = A \cap B$. Let*

$$W = \mathbf{V}(I) = \mathbf{V}(A) \cup \mathbf{V}(B).$$

Then for any $f \in \mathbf{I}(W)$ and any $x^* \in \mathbb{C}^n \setminus \mathbf{V}(B)$ there exist a neighborhood U^* of x^* in $\mathbb{C}^n$ and rational functions $f_1, \ldots, f_r$ on U^* such that

$$f = f_1h_1 + \cdots + f_rh_r \quad \text{on } U^*.$$

3 An upper bound for critical periods bifurcating from centers of (1.5)

With system (1.5) we associate its complexification

$$\begin{aligned} \dot{x} &= ix(1 - a_{21}x^2y - a_{12}xy^2 - a_{03}y^3), \\ \dot{y} &= -iy(1 - b_{30}x^3 - b_{21}x^2y - b_{12}xy^2). \end{aligned} \tag{3.1}$$

Computing the normal form of system (3.1) up to order 19 we find the first three non-zero pairs of coefficients $Y_1^{(4,3)}, Y_2^{(3,4)}, Y_1^{(7,6)}, Y_2^{(6,7)}, Y_1^{(10,9)}, Y_2^{(9,10)}$ of the normal form of system (3.1) presented in Appendix A. Then straightforward calculations give that the first three reduced isochronicity quantities are:

$$\begin{aligned}
p_6 &= a_{12}a_{21} - 2a_{21}b_{12} + 4a_{12}b_{21} + b_{12}b_{21} + 2a_{03}b_{30}, \\
p_{12} &= (-4a_{12}^2a_{21}^2 - 2a_{12}a_{21}^2b_{12} - 4a_{21}^2b_{12}^2 + 16a_{12}^2a_{21}b_{21} - 4a_{03}a_{21}^2b_{21} + 8a_{12}a_{21}b_{12}b_{21} - 2a_{21}b_{12}^2b_{21} \\
&\quad + 32a_{12}^2b_{21}^2 - 14a_{03}a_{21}b_{21}^2 + 16a_{12}b_{12}b_{21}^2 - 4b_{12}^2b_{21}^2 + 44a_{03}b_{21}^3 + 44a_{12}^3b_{30} - 3a_{03}a_{12}a_{21}b_{30} \\
&\quad - 14a_{12}^2b_{12}b_{30} - 4a_{12}b_{12}^2b_{30} + 105a_{03}a_{12}b_{21}b_{30} - 3a_{03}b_{12}b_{21}b_{30} + 4a_{03}^2b_{30}^2)/4, \\
p_{18} &= (-16800a_{12}^3a_{21}^3 + 9450a_{03}a_{12}a_{21}^4 + 34020a_{12}^2a_{21}^3b_{12} + 39060a_{12}a_{21}^3b_{12}^2 - 44520a_{21}^3b_{12}^3 \\
&\quad - 60480a_{12}^3a_{21}^2b_{21} + 5180a_{03}a_{12}a_{21}^3b_{21} - 99540a_{12}^2a_{21}^2b_{12}b_{21} - 8960a_{03}a_{21}^3b_{12}b_{21} \\
&\quad - 10080a_{12}a_{21}^2b_{12}^2b_{21} + 39060a_{21}^2b_{12}^3b_{21} + 216720a_{12}^3a_{21}b_{21}^2 - 229810a_{03}a_{12}a_{21}^2b_{21}^2 \\
&\quad + 171360a_{12}^2a_{21}b_{12}b_{21}^2 - 3640a_{03}a_{21}^2b_{12}b_{21}^2 - 99540a_{12}a_{21}b_{12}^2b_{21}^2 + 34020a_{21}b_{12}^3b_{21}^2 \\
&\quad + 215040a_{12}^3b_{21}^3 + 93380a_{03}a_{12}a_{21}b_{21}^3 + 216720a_{12}^2b_{12}b_{21}^3 - 47880a_{03}a_{21}b_{12}b_{21}^3 \\
&\quad - 60480a_{12}b_{12}^2b_{21}^3 - 16800b_{12}^3b_{21}^3 + 458640a_{03}a_{12}b_{21}^4 + 222600a_{03}b_{12}b_{21}^4 + 222600a_{12}^4a_{21}b_{30} \\
&\quad - 185745a_{03}a_{12}^2a_{21}^2b_{30} + 3360a_{03}^2a_{21}^3b_{30} - 47880a_{12}^3a_{21}b_{12}b_{30} - 31500a_{03}a_{12}a_{21}^2b_{12}b_{30}
\end{aligned}$$

$$- 3640a_{12}^2a_{21}b_{12}^2b_{30} + 76524a_{03}a_{21}^2b_{12}^2b_{30} - 8960a_{12}a_{21}b_{12}^3b_{30} + 458640a_{12}^4b_{21}b_{30}$$
$$+ 340305a_{03}a_{12}^2a_{21}b_{21}b_{30} - 27748a_{03}^2a_{21}^2b_{21}b_{30} + 93380a_{12}^3b_{12}b_{21}b_{30} - 66276a_{03}a_{12}a_{21}b_{12}b_{21}b_{30}$$
$$- 229810a_{12}^2b_{12}^2b_{21}b_{30} - 31500a_{03}a_{21}b_{12}^2b_{21}b_{30} + 5180a_{12}b_{12}^3b_{21}b_{30} + 9450b_{12}^4b_{21}b_{30}$$
$$+ 1647660a_{03}a_{12}^2b_{21}^2b_{30} - 224756a_{03}^2a_{21}b_{21}^2b_{30} + 340305a_{03}a_{12}b_{12}b_{21}^2b_{30} - 185745a_{03}b_{12}^2b_{21}^2b_{30}$$
$$+ 626010a_{03}^2b_{21}^3b_{30} + 626010a_{03}a_{12}^3b_{30}^2 - 45922a_{03}^2a_{12}a_{21}b_{30}^2 - 224756a_{03}a_{12}^2b_{12}b_{30}^2$$
$$- 30604a_{03}^2a_{21}b_{12}b_{30}^2 - 27748a_{03}a_{12}b_{12}^2b_{30}^2 + 3360a_{03}b_{12}^3b_{30}^2 + 654698a_{03}^2a_{12}b_{21}b_{30}^2$$
$$- 45922a_{03}^2b_{12}b_{21}b_{30}^2 + 5320a_{03}^3b_{30}^3)/3360,$$

where p_{12} is reduced modulo $\langle p_6\rangle$ and p_{18} is reduced modulo the ideal $\langle p_6, p_{12}\rangle$.

The center variety of system (3.1) (found in [13, 15]) consists of the following five components:

$$\begin{aligned}
&1)\ a_{12}^3b_{30} - b_{21}^3a_{03} = a_{21}b_{21}^2a_{03} - b_{12}a_{12}^2b_{30} = a_{21}a_{12} - b_{12}b_{21}\\
&\qquad = a_{21}^2b_{21}a_{03} - b_{12}^2a_{12}b_{30} = a_{21}^3a_{03} - b_{12}^3b_{30} = 0,\\
&2)\ 5a_{21}b_{12} - 6a_{03}b_{30} = b_{21} = a_{12} = 0,\\
&3)\ 2b_{12} - a_{12} = 2a_{21} - b_{21} = 0,\\
&4)\ b_{30} = b_{12} - 2a_{12} = a_{21} - 2b_{21} = 0,\\
&5)\ a_{03} = b_{12} - 2a_{12} = a_{21} - 2b_{21} = 0.
\end{aligned} \tag{3.2}$$

Computing with minAssGTZ (the routine of the library primdec.lib [12] of the computer algebra system Singular [11] which is based on the algorithm of [17]) the decomposition of the variety of the ideal

$$\left\langle Y_1^{(4,3)} + Y_2^{(3,4)}, Y_1^{(7,6)} + Y_2^{(6,7)}, Y_1^{(10,9)} + Y_2^{(9,10)} \right\rangle$$

we find that it is different from the decomposition of the center variety given in (3.2). It means that the center variety is defined not by the first 3 pairs of non-zero coefficients of the normal form, but the first 4 pairs. Since the computation of normal forms is highly time and memory consumptive, we were not able to compute $Y_1^{(13,12)}$ and $Y_2^{(12,13)}$ using our computational facilities. However the center variety of a polynomial system can be found using the so-called focus quantities which are much easier to compute and which are obtained from the equation

$$[ix + \widetilde{P}(x,y)]\Psi_x(x,y) + [-iy + \widetilde{Q}(x,y)]\Psi_y(x,y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots,$$

where

$$\Psi(x,y) = xy + \sum_{j+k=3}^{\infty} \Psi_{jk}x^jy^k. \tag{3.3}$$

The coefficients g_{kk} are polynomials in the coefficients of system (1.4) called the *focus quantities*.

The ideal

$$\mathscr{B} := \langle g_{kk} : k \in \mathbb{N}\rangle \subset \mathbb{C}[a,b]$$

is called the *Bautin ideal* of system (1.4). Its variety is the same as the center variety $V_{\mathscr{C}}$ defined by (2.5) (see e.g. Theorem 3.2.7 in [28]). We also use the notation

$$\mathscr{B}_K := \langle g_{kk} : k = 1, \ldots, K\rangle \subset \mathbb{C}[a,b]$$

for the ideal generated by the first K focus quantities.

It follows from the results of [13,15] that

$$g_{3k+1,3k+1} \equiv g_{3k+2,3k+2} \equiv 0$$

for all $k \in \mathbb{N}_0$ and four first nonzero focus quantities define the variety of the Bautin ideal of system (3.1), that is,

$$\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_{12}), \tag{3.4}$$

where $\mathscr{B}_{12} = \langle g_{33}, g_{66}, g_{99}, g_{12,12}\rangle$ (the polynomials $g_{99}, g_{12,12}$ are given by long expressions, so we do not write out the polynomials g_{kk} here, but the reader can obtain them from polynomials g^c_{kk} given in Appendix B applying map (3.13) to g^c_{kk}). Since $\mathbf{V}(\mathscr{B})$ is a complex variety, by (3.4) we have that $\sqrt{\mathscr{B}} = \sqrt{\mathscr{B}_{12}}$.

Lemma 3.1. *For system* (3.1),

$$V_{\mathscr{L}} = \mathbf{V}(\mathcal{Y}_9) = V_{\mathscr{C}} \cap \mathbf{V}(P_9). \tag{3.5}$$

Proof. Computing with the routine `minAssGTZ` minimal associate primes of the ideals $\mathcal{Y}_9$ and $\langle \mathscr{B}_{12}, P_9\rangle$ we find that in both cases they are

$$\begin{aligned}
J_1 &= \langle b_{30}, b_{21}, a_{12}, b_{12}\rangle,\\
J_2 &= \langle b_{30}, b_{21}, a_{21}\rangle,\\
J_3 &= \langle b_{30}, a_{03}, b_{12} + a_{12}, a_{21} + b_{21}\rangle,\\
J_4 &= \langle b_{30}, b_{12} - 2a_{12}, a_{21} - 2b_{21}\rangle,\\
J_5 &= \langle a_{03}, a_{12}, b_{12}\rangle,\\
J_6 &= \langle b_{21}, a_{12}, b_{12}^2 + a_{21}a_{03}, , a_{21}b_{12} - a_{03}b_{30}, a_{21}^2 + b_{12}b_{30}\rangle,\\
J_7 &= \langle a_{03}, b_{21}, a_{12}, a_{21}\rangle,\\
J_8 &= \langle a_{03}, b_{12} - 2a_{12}, a_{21} - 2b_{21}\rangle.
\end{aligned}$$

It follows from the results of [13] that each system from the varieties $V_i = \mathbf{V}(J_i)$ $(i = 1, \dots, 8)$ is linearizable. Therefore (3.5) holds. □

To get an upper bound for the number of bifurcating critical periods we can use some results of [14]. By Theorem 4.1 of [14] it is easy to obtain an upper bound for the number of bifurcating critical periods if for the complexification (1.4) of (1.2) it holds that for some $K \in \mathbb{N}$ $V_{\mathscr{L}} = \mathbf{V}(P_K) \cap V_{\mathscr{C}}$ and $\langle P_K, \sqrt{\mathscr{B}}\rangle$ is a radical ideal. However computing the radical of $\langle P_9, \sqrt{\mathscr{B}_{12}}\rangle$ with the routine `radical` of Singular we see that the ideal is not a radical ideal. Therefore this theorem cannot be applied for system (1.5).

Another way to study the critical period bifurcations is based on the next theorem which follows from Theorem 5.2 and Remark 5.3 of [14].

Theorem 3.2. *Let $\widetilde{P}_K$ be the ideal generated by $p_2, p_4, \dots, p_{2K}$ in the coordinate ring $\mathbb{C}[V_{\mathscr{C}}]$ of the variety $V_{\mathscr{C}}$ and let m be the cardinality of the minimal basis of $\widetilde{P}_K$*[1]*. Suppose that for the complexification* (1.4) *of the family* (1.2) *it holds that $V_{\mathscr{L}} = \mathbf{V}(P_K) \cap V_{\mathscr{C}}$ and a primary decomposition of $P_K + \sqrt{\mathscr{B}}$ can be written as $R \cap N$, where R is the intersection of the ideals in the decomposition that are prime and N is the intersection of the remaining ideals in the decomposition.*

Then for any system of family (1.2) *corresponding to $(a^*, \bar{a}^*) \in V_{\mathscr{C}} \setminus \mathbf{V}(N)$, at most $m - 1$ critical periods bifurcate from a center at the origin.*

[1] For an ordered set $\{f_0, f_1, f_2, \dots\}$ in a Noetherian ring R the *minimal basis* of the ideal $\langle f_0, f_1, f_2, \dots\rangle$ in R is the set M generated in the following recursive fashion: initially set $M = \{f_J\}$, where J is the smallest index j for which f_j is not the zero of R, then successively check elements f_j, $j \geq J + 1$, adjoining f_j to M if and only if $f_j \notin \langle M\rangle$.

Using this theorem in [27] for the system

$$\dot{x} = ix + x\bar{x}(ax^3 + bx^2\bar{x} + c\bar{x}^2\bar{x} + d\bar{x}^3) \tag{3.6}$$

it was proved that at most 3 critical periods bifurcate from any nonlinear center of the system. It is possible to perform the study of critical periods bifurcations for system (1.5) using Theorem 3.2, however since the bound obtained with this approach is not valid for all parameters of the system, we present here the study of critical periods of (1.5) with another approach which exploits the special structure of focus and isochronicity quantities and which sometimes gives a better bound, than the one provided by Theorem 3.2 (as it is shown in [14]).

The special structure of the quantities is described as follows. For the ordering of the coefficients given in (2.1) any monomial in $\mathbb{C}[a, b]$ will be written

$$[\nu] := a_{p_1 q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell, p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1 p_1}^{\nu_{2\ell}}, \qquad \nu = (\nu_1, \ldots, \nu_{2\ell}). \tag{3.7}$$

We define a mapping $L : \mathbb{N}_0^{2\ell} \to \mathbb{Z}^2$ by

$$L(\nu) = \nu_1(p_1, q_1) + \cdots + \nu_\ell(p_\ell, q_\ell) + \nu_{\ell+1}(q_\ell, p_\ell) + \cdots + \nu_{2\ell}(q_1, p_1). \tag{3.8}$$

By Corollary 3.4.6 and Proposition 5.1.6 of [28] it holds:

(i) the focus quantities g_{kk} of family (1.4) have the form

$$g_{kk} = \frac{1}{2} \sum_{\{\nu : L(\nu) = (k,k)\}} g_{kk}^{(\nu)}([\nu] - [\hat{\nu}]); \tag{3.9}$$

(ii) the isochronicity quantities p_{2k} of family (1.4) have the form

$$p_{2k} = \frac{1}{2} \sum_{\{\nu : L(\nu) = (k,k)\}} p_{2k}^{(\nu)}([\nu] + [\hat{\nu}]), \tag{3.10}$$

where for $\nu = (\nu_1, \ldots, \nu_{2\ell}) \in \mathbb{N}_0^{2\ell}$, $\hat{\nu} = (\nu_{2\ell}, \ldots, \nu_1)$.

We recall that the Sibirsky ideal of system (1.4) is the ideal

$$I_S = \langle [\nu] - [\hat{\nu}] : L(\nu) = (k, k) \rangle,$$

where $[\nu]$ is defined by (3.7) and $L(\nu)$ by (3.8). To find the Sibirsky ideal of system (3.1) we introduce the ideal

$$H = \langle 1 - \alpha w, a_{21} - t_1, b_{12} - \alpha t_1, a_{12} - t_2, \alpha b_{21} - t_2, a_{03} - t_3, \alpha^3 b_{30} - t_3 \rangle.$$

Computing a Gröbner basis of H with respect to the lexicographic monomial order with $w > \alpha > t_1 > t_2 > t_3 > a_{21} > a_{12} > a_{03} > b_{30} > b_{21} > b_{12}$ we obtain the ideal

$$\begin{aligned}
&\langle a_{03}b_{21}^3 - a_{12}^3 b_{30}, a_{03}a_{21}b_{21}^2 - a_{12}^2 b_{12} b_{30}, a_{12}a_{21} - b_{12}b_{21}, a_{03}a_{21}^2 b_{21} - a_{12}b_{12}^2 b_{30}, a_{03}a_{21}^3 - b_{12}^3 b_{30}, \\
&- a_{03} + t_3, -a_{12} + t_2, -a_{21} + t_1, -a_{12} + \alpha b_{21}, -a_{03}a_{21}^2 + \alpha b_{12}^2 b_{30}, -a_{03}a_{21}b_{21} + \alpha a_{12}b_{12}b_{30}, \\
&- a_{03}b_{21}^2 + \alpha a_{12}^2 b_{30}, a_{21}\alpha - b_{12}, -a_{03}a_{21} + \alpha^2 b_{12}b_{30}, -a_{03}b_{21} + \alpha^2 a_{12}b_{30}, \\
&- a_{03} + \alpha^3 b_{30}, -a_{21} + b_{12}w, -\alpha^2 b_{30} + a_{03}w, -b_{21} + a_{12}w, -1 + \alpha w \rangle.
\end{aligned}$$

By the results of [26] the Sibirsky ideal of system (3.1) is generated by the polynomials of the above ideal which do not depend on α, t_1, t_2 and t_3, that is, by the first five binomials in the basis of the ideal presented above. It follows from (3.9) and (3.10), and the results of [26], that the focus and isochronicity quantities of system (3.1) belong to the polynomial subalgebra of $\mathbb{C}[a,b]$ generated by the monomials of these five binomials, that is,

$$a_{03}b_{21}^3,\ a_{12}^3b_{30},\ a_{03}a_{21}b_{21}^2,\ a_{12}^2b_{12}b_{30},\ a_{12}a_{21},\ b_{12}b_{21},\ a_{03}a_{21}^2b_{21},\ a_{12}b_{12}^2b_{30},\ a_{03}a_{21}^3,\ b_{12}^3b_{30},$$

along with the monomials

$$a_{21}b_{12},\ a_{12}b_{21},\ a_{03}b_{30}.$$

We will map our ideals to this subalgebra and study their structure there. To this end, we denote the monomials listed above by $h_1(a,b),\dots,h_{13}(a,b)$, respectively, and consider the ideal $J = \langle h_k(a,b) - c_k : k = 1,\dots,13\rangle \subset \mathbb{C}[a,b,c]$ (where $c = (c_1,\dots,c_{13})$).

The mapping

$$F : \mathbb{C}^6 \to \mathbb{C}^{13} : (a,b) \mapsto (c_1,\dots,c_{13}) \tag{3.11}$$

defined by

$$c_1 = h_1(a,b) = a_{03}b_{21}^3,\dots,c_{13} = h_{13}(a,b) = a_{03}b_{30} \tag{3.12}$$

induces the homomorphism of $\mathbb{C}$-algebras

$$\begin{aligned} F^\sharp &: \mathbb{C}[c] \to \mathbb{C}[a,b] \\ &: \sum d_{(\alpha)}c_1^{\alpha_1}\cdots c_{13}^{\alpha_{13}} \mapsto \sum d_{(\alpha)}h_1^{\alpha_1}(a,b)\cdots h_{13}^{\alpha_{13}}(a,b), d_{(\alpha)} \in \mathbb{C}. \end{aligned} \tag{3.13}$$

If $I = \{f_1,\dots,f_s\}$, where $f_j \in \mathrm{Image}(F^\sharp)$ for each j, then we let I^c denote the ideal $\langle f_1^c,\dots,f_s^c\rangle$ in $\mathbb{C}[c]/R$, and similarly if I is infinite.

The following theorem can be derived using Theorem 6.3 of [14] but for reader's convenience we present the direct proof.

Theorem 3.3. *At most two critical periods bifurcate from any nonlinear center at the origin of system* (1.5).

Proof. The first step of the proof is to map the focus and isochronicity quantities from the ring $\mathbb{C}[a,b]$ to the ring $\mathbb{C}[c]$, that is, to rewrite them as polynomials in variables c_k, where c_k are related to a_{kl}, b_{lk} by (3.12). It can be done as follows.

Let $W \subset \mathbb{C}^{13}$ denote the image of $\mathbb{C}^6$ under F, which is not necessarily a variety, and let $\overline{W} \subset \mathbb{C}^{13}$ be the Zariski closure of W. Denote by R the kernel of (3.13), $R = \ker(F^\sharp) \subset \mathbb{C}[c]$. Clearly, R is a prime ideal, and $\overline{W} = \mathbf{V}(R)$.

From Theorem 2.4.2 of [1] we have $R = \ker(F^\sharp) = J \cap \mathbb{C}[c]$. Let $J_G \subset \mathbb{C}[a,b,c]$ denote a Gröbner basis of J with respect to any monomial ordering with

$$\{a_{21},a_{12},a_{03},b_{30},b_{21},b_{12}\} > \{c_1,c_2,\dots,c_{13}\}. \tag{3.14}$$

Then by the Elimination Theorem (see e.g. [10,28]) $R_G = J_G \cap \mathbb{C}[c]$ is a Gröbner basis of R. Computing a reduced Gröbner basis J_G of J with ordering (3.14) we obtain the Gröbner basis $R_G = J_G \cap \mathbb{C}[c]$ of R presented in Appendix C.

From (3.9) and (3.10) we see that g_{kk} and p_{2k} have the form $g_{kk} = \sum \widetilde{g}_{kk}^{(\alpha)} h_1^{\alpha_1}\cdots h_{13}^{\alpha_{13}}$ and $p_{2k} = \sum \widetilde{p}_{2k}^{(\alpha)} h_1^{\alpha_1}\cdots h_{13}^{\alpha_{13}}$, respectively. That is, for each k $g_{kk}, p_{2k} \in \mathrm{Image}(F^\sharp)$ and there are their preimages g_{kk}^c and p_{2k}^c in $\mathbb{C}[c]/R$. According to Proposition 7(i) in §7.3 of [10], to find the preimages of g_{kk} and p_{2k} we divide each of these quantities by J_G, then the remainder of the division is the preimage. Dividing the polynomials g_{kk} by J_G we obtain the polynomials $g^c_{33}, g^c_{66}, g^c_{99}, g^c_{12,12}$ presented in Appendix B, and for polynomials p_{2k} we have the expressions:

$$
\begin{aligned}
p^c_6 =& -c_5 - c_6 + 2c_{11} - 4c_{12} - 2c_{13},\\
p^c_{12} =& 1/4(44c_1 - 24c^2_{11} + 112c_{11}c_{12} - 96c^2_{12} + 30c_{11}c_{13} + 21c_{12}c_{13} - 6c^2_{13} + 44c_2 - 14c_3 - 14c_4\\
&+ 16c_{11}c_6 - 32c_{12}c_6 - 16c_{13}c_6 - 8c^2_6 - 4c_7 - 4c_8)/4,\\
p^c_{18} =& (-912240c_1c_{11} + 534240c^3_{11} + 3608640c_1c_{12} + 20720c_{10}c_{12} - 4183200c^2_{11}c_{12} + 9696960c_{11}c^2_{12}\\
&- 6048000c^3_{12} - 3451140c_1c_{13} + 13440c_{10}c_{13} + 50736c^2_{11}c_{13} - 986244c_{11}c_{12}c_{13}\\
&+ 3944640c^2_{12}c_{13} + 675498c_{11}c^2_{13} - 12082821c_{12}c^2_{13} - 1233594c^3_{13} + 868560c_{11}c_2\\
&+ 47040c_{12}c_2 - 5231940c_{13}c_2 + 214760c_{11}c_3 - 190960c_{12}c_3 + 995806c_{13}c_3 + 214760c_{11}c_4\\
&- 190960c_{12}c_4 + 995806c_{13}c_4 + 890400c_1c_6 + 37800c_{10}c_6 - 890400c_2c_6 + 29680c_{11}c_7\\
&- 1080520c_{12}c_7 + 430388c_{13}c_7 + 29680c_{11}c_8 - 1080520c_{12}c_8 + 430388c_{13}c_8 + 75600c_{11}c_9\\
&- 130480c_{12}c_9 - 62160c_{13}c_9 - 37800c_6c_9)/13440.
\end{aligned}
$$

This completes the first step of the proof.

We denote $\mathscr{B}^c = \langle g^c_{kk} : k \in \mathbb{N}\rangle$, $\mathscr{B}^c_{12} = \langle g^c_{11}, \dots g^c_{12,12}\rangle$, $P^c = \langle p^c_{2k} : k \in \mathbb{N}\rangle$, and $P^c_9 = \langle p^c_6, p^c_{12}, p^c_{18}\rangle$ considering them as ideals in $\mathbb{C}[c]/R$.

By (3.4) and Lemma 3.1 for system (3.1) $V_{\mathscr{L}} = \mathbf{V}(P_9) \cap V_{\mathscr{C}}$ and $V_{\mathscr{C}} = \mathbf{V}(\mathscr{B}_{12})$, hence $V_{\mathscr{L}} = \mathbf{V}(P_9 + \mathscr{B}_{12})$. Define $V^c_{\mathscr{L}} \stackrel{\text{def}}{=} F(\mathbf{V}(P + \mathscr{B})) = F(\mathbf{V}(P_9 + \mathscr{B}_{12}))$. The set $V^c_{\mathscr{L}}$ which is the image of $V_{\mathscr{L}}$ under the map F is not necessary a variety, so the second step of the proof is to check that all polynomials p^c_{2k} vanish on the Zariski closure $\overline{V^c_{\mathscr{L}}}$ of $V^c_{\mathscr{L}}$.

Let $H = (\mathbb{C}[a,b,c]P_9 + \mathbb{C}[a,b,c]\mathscr{B}_{12} + J) \cap \mathbb{C}[c]$. Applying the results of §1.8.3 in [19] we obtain that

$$\mathbf{V}(H) = \overline{F(\mathbf{V}(P_9 + \mathscr{B}_{12}))} \subset \overline{W}.$$

Thus

$$\overline{V^c_{\mathscr{L}}} = \mathbf{V}(H) = \mathbf{V}(H) \cap \overline{W}. \tag{3.15}$$

If some polynomials $f_1, \dots, f_s$ are in Image$(F^\sharp)$ then

$$F(\mathbf{V}(f_1, \dots, f_s)) = W \cap \mathbf{V}(f^c_1, \dots, f^c_s). \tag{3.16}$$

Hence, $V^c_{\mathscr{L}} = F(\mathbf{V}(P + \mathscr{B})) = W \cap \mathbf{V}(P^c + \mathscr{B}^c)$, so

$$\overline{V^c_{\mathscr{L}}} \subset \overline{W} \cap \mathbf{V}(P^c + \mathscr{B}^c) \subset \overline{W} \cap \mathbf{V}(P^c_9 + \mathscr{B}^c_{12}). \tag{3.17}$$

Computing the 6-th elimination ideal of the ideal $H = \mathbb{C}[a,b,c]P_9 + \mathbb{C}[a,b,c]\mathscr{B}_{12} + J$ in $\mathbb{C}[a,b,c]$ we find, that it is the same as the ideal

$$Q := P^c_9 + \mathscr{B}^c_{12} + R$$

in $\mathbb{C}[c]$. Therefore, $H \cap \mathbb{C}[c] = (\mathbb{C}[a,b,c]P_9 + \mathbb{C}[a,b,c]\mathscr{B}_{12} + J) \cap \mathbb{C}[c] = Q$.

Hence $\mathbf{V}(P^c_9 + \mathscr{B}^c_{12} + R) = \mathbf{V}(P^c_9 + \mathscr{B}^c_{12}) \cap \overline{W}$, and by (3.15),

$$\overline{V^c_{\mathscr{L}}} \cap \overline{W} = \mathbf{V}(P^c_9 + \mathscr{B}^c_{12}) \cap \overline{W}.$$

Along with (3.17) it yields

$$\mathbf{V}(P^c_9 + \mathscr{B}^c_{12}) = \mathbf{V}(P^c + \mathscr{B}^c) = \mathbf{V}(P^c) \cap \mathbf{V}(\mathscr{B}^c) \cap \mathbf{V}(R)$$

implying that for all k, $p^c_{2k} \in \mathbf{I}(\mathbf{V}(Q))$. Thus, the second step of the proof is completed.

The third step is to find the primary decomposition of the ideal Q. Computing the decomposition with the routine `primdecGTZ` of Singular we find that

$$Q = \cap_{k=1}^{7} Q_k,$$

where $Q_1, \ldots, Q_6$ are prime ideals defined as:

$$\begin{aligned}
Q_1 &= \langle c_{13}, c_{12}, c_{11}, c_{10}, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1 \rangle, \\
Q_2 &= \langle c_{13}, c_{12}, c_{11}, c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1 \rangle, \\
Q_3 &= \langle c_{13}, c_{11} - 4c_{12}, c_{10}, c_8, 2c_7 - c_9, c_6 - 2c_{12}, c_5 - 2c_{12}, c_4, 4c_3 - c_9, c_2, 8c_1 - c_9 \rangle, \\
Q_4 &= \langle c_{13}, c_{11} - 4c_{12}, c_9, 2c_8 - c_{10}, c_7, c_6 - 2c_{12}, c_5 - 2c_{12}, 4c_4 - c_{10}, c_3, 8c_2 - c_{10}, c_1 \rangle, \\
Q_5 &= \langle c_{13}, c_{11} - c_{12}, c_{10}, c_9, c_8, c_7, c_6 + c_{12}, c_5 + c_{12}, c_4, c_3, c_2, c_1 \rangle, \\
Q_6 &= \langle c_{12}, c_{11} - c_{13}, c_9 - c_{10}, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1, c_{13}^2 + c_{10} \rangle,
\end{aligned}$$

and Q_7 is a primary ideal generated by 59 polynomials. We do not present here the generators of Q_7, however the reader can easily compute it with Singular using the polynomials p^c_{2k} given above, the polynomials g^c_{kk} from Appendix B and the ideal R from the Appendix C. Although the ideal Q_7 is complicate, its associate prime is just $\sqrt{Q_7} = \langle c_j : 1 \leq j \leq 13 \rangle$, so $\mathbf{V}(Q_7)$ is the origin 0 of $\mathbb{C}^{13}$.

Now, the last step of the proof is to find the expression for the period function using the obtained decomposition of the ideal Q. To this end, we note that by Proposition 2.2 there exist rational functions α_j, β_j, γ_j on $\mathbb{C}^{13} \setminus 0$ such that

$$p^c_{2k} = \sum_{j=1}^{3} \alpha_j p^c_{6j} + \sum_{j=1}^{4} \beta_j g^c_{3j,3j} + \sum_{j=1}^{44} \gamma_j r_j, \tag{3.18}$$

where $\{r_1, \ldots, r_u\} = R_G$ is the generating set of R (given in Appendix C). The map $F^\sharp$ extends to rational functions in a natural way. Applying it to (3.18) and recalling that $r_j \in R = \ker(F^\sharp)$ for each j, we obtain

$$p_{2k} = \sum_{j=1}^{3} \alpha'_j p_{2j} + \sum_{j=1}^{4} \beta'_j g_{3j,3j}, \tag{3.19}$$

valid on $\mathbb{C}^{13} \setminus 0$, where the α'_j and β'_j are rational functions of a and b.

Thus, for system (1.5) function (2.9) can be represented in the form

$$\mathscr{T}(r,(a,\bar{a})) = T(r) - 2\pi = \sum_{j=1}^{3} (1 + \psi_j(r,a,\bar{a})) p_{6j}(a,\bar{a}) r^{6j} \quad + \sum_{j=1}^{4} W_j(r,a,\bar{a}) g_{3j,3j}(a,\bar{a}), \tag{3.20}$$

where the W_j are analytic functions and the ψ_j are real analytic functions. On the center variety V_C the polynomials $g_{3j,3j}$ evaluate to zero. The preimage $F^{-1}(0)$ in the set of parameters of system (1.5) is the point $(a_{12}, a_{21}, a_{03}, b_{30}, b_{12}, b_{21}) = (0,0,0,0,0,0)$. Therefore by Theorem 2.1 at most two critical periods bifurcate from any nonlinear center at the origin of system (1.5). □

In summary, to obtain the bound for the number of critical periods we have performed the decomposition of the ideal generated by isochronicity quantities in the ring $\mathbb{C}[c]/R$. The bound is obtained for all centers except the linear one. As it is shown in [14] sometimes the study in the ring $\mathbb{C}[c]/R$ can give a better result than the study in the ring $\mathbb{C}[a,b]$. We also have performed the analysis of isochronicity quantities using Theorem 3.2 and obtained the same result – the upper bound is two for all nonlinear centers. Thus, for system (1.5) the method used in this paper does not give a better result than the one used in [27]. However our study has shown that the ideal generated by p_6, p_{12}, p_{18} in $\mathbb{C}[c]/R$ has a simpler structure than the ideal $\langle P_9, \mathcal{B}_{12}\rangle$ in $\mathbb{C}[a,b]$, since in the first case it has 7 components and only one of them is non-radical (that is, the ideal in the primary decomposition defining the component is not prime), whereas in the second case it has 15 components and 7 of them are non-radical.

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Appendix A

Coefficients of the normal form of system (3.1):

$$
\begin{aligned}
Y_1^{(4,3)} &= -i(-a_{12}a_{21} + a_{21}b_{12} - 2a_{12}b_{21} - a_{03}b_{30}),\\
Y_2^{(3,4)} &= i(-a_{21}b_{12} + 2a_{12}b_{21} + b_{12}b_{21} + a_{03}b_{30}),\\
Y_1^{(7,6)} &= \frac{i}{4}(8a_{12}^2a_{21}^2 + 4a_{21}^2b_{12}^2 - 24a_{12}^2a_{21}b_{21} + 8a_{03}a_{21}^2b_{21} - 8a_{12}a_{21}b_{12}b_{21} + 4a_{21}b_{12}^2b_{21}\\
&\quad - 32a_{12}^2b_{21}^2 + 4a_{03}a_{21}b_{21}^2 - 8a_{12}b_{12}b_{21}^2 - 40a_{03}b_{21}^3 - 48a_{12}^3b_{30} - 3a_{03}a_{12}a_{21}b_{30}\\
&\quad + 24a_{12}^2b_{12}b_{30} - 105a_{03}a_{12}b_{21}b_{30} + 9a_{03}b_{12}b_{21}b_{30} - 4a_{03}^2b_{30}^2),\\
Y_2^{(6,7)} &= \frac{i}{4}(-4a_{12}a_{21}^2b_{12} - 4a_{21}^2b_{12}^2 + 8a_{12}^2a_{21}b_{21} + 8a_{12}a_{21}b_{12}b_{21} + 32a_{12}^2b_{21}^2 - 24a_{03}a_{21}b_{21}^2\\
&\quad + 24a_{12}b_{12}b_{21}^2 - 8b_{12}^2b_{21}^2 + 48a_{03}b_{21}^3 + 40a_{12}^3b_{30} - 9a_{03}a_{12}a_{21}b_{30} - 4a_{12}^2b_{12}b_{30}\\
&\quad - 8a_{12}b_{12}^2b_{30} + 105a_{03}a_{12}b_{21}b_{30} + 3a_{03}b_{12}b_{21}b_{30} + 4a_{03}^2b_{30}^2),\\
Y_1^{(10,9)} &= \frac{i}{1680}(16800a_{12}^3a_{21}^3 - 9450a_{03}a_{12}a_{21}^4 - 37380a_{12}^2a_{21}^3b_{12} + 1050a_{03}a_{21}^4b_{12} - 1680a_{12}a_{21}^3b_{12}^2\\
&\quad + 22260a_{21}^3b_{12}^3 + 62160a_{12}^3a_{21}^2b_{21} - 140a_{03}a_{12}a_{21}^3b_{21} + 34020a_{12}^2a_{21}^2b_{12}b_{21}\\
&\quad + 5740a_{03}a_{21}^3b_{12}b_{21} + 5040a_{12}a_{21}^2b_{12}^2b_{21} - 37380a_{21}^2b_{12}^3b_{21} - 137760a_{12}^3a_{21}b_{21}^2\\
&\quad + 161490a_{03}a_{12}a_{21}^2b_{21}^2 - 85680a_{12}^2a_{21}b_{12}b_{21}^2 + 6510a_{03}a_{21}^2b_{12}b_{21}^2 + 65520a_{12}a_{21}b_{12}^2b_{21}^2\\
&\quad + 3360a_{21}b_{12}^3b_{21}^2 - 107520a_{12}^3b_{21}^3 - 93940a_{03}a_{12}a_{21}b_{21}^3 - 78960a_{12}^2b_{12}b_{21}^3\\
&\quad - 22540a_{03}a_{21}b_{12}b_{21}^3 - 1680a_{12}b_{12}^2b_{21}^3 - 218400a_{03}a_{12}b_{21}^4 - 87360a_{03}b_{12}b_{21}^4\\
&\quad - 135240a_{12}^4a_{21}b_{30} + 139335a_{03}a_{12}^2a_{21}^2b_{30} - 2940a_{03}^2a_{21}^3b_{30} + 70420a_{12}^3a_{21}b_{12}b_{30}
\end{aligned}
$$

$$
\begin{aligned}
& - 16275a_{03}a_{12}a_{21}^2b_{12}b_{30} - 2870a_{12}^2a_{21}b_{12}^2b_{30} - 38262a_{03}a_{21}^2b_{12}^2b_{30} \\
& + 3220a_{12}a_{21}b_{12}^3b_{30} - 1050a_{21}b_{12}^4b_{30} - 240240a_{12}^4b_{21}b_{30} - 256305a_{03}a_{12}^2a_{21}b_{21}b_{30} \\
& + 28364a_{03}^2a_{21}^2b_{21}b_{30} + 560a_{12}^3b_{12}b_{21}b_{30} + 33138a_{03}a_{12}a_{21}b_{12}b_{21}b_{30} \\
& + 68320a_{12}^2b_{12}^2b_{21}b_{30} + 47775a_{03}a_{21}b_{12}^2b_{21}b_{30} - 5040a_{12}b_{12}^3b_{21}b_{30} \\
& - 823830a_{03}a_{12}^2b_{21}^2b_{30} + 80878a_{03}^2a_{21}b_{21}^2b_{30} - 84000a_{03}a_{12}b_{12}b_{21}^2b_{30} \\
& + 46410a_{03}b_{12}^2b_{21}^2b_{30} - 300720a_{03}^2b_{21}^3b_{30} - 325290a_{03}a_{12}^3b_{30}^2 + 13791a_{03}^2a_{12}a_{21}b_{30}^2 \\
& + 143878a_{03}a_{12}^2b_{12}b_{30}^2 + 15302a_{03}^2a_{21}b_{12}b_{30}^2 - 616a_{03}a_{12}b_{12}^2b_{30}^2 - 420a_{03}b_{12}^3b_{30}^2 \\
& - 327349a_{03}^2a_{12}b_{21}b_{30}^2 + 32131a_{03}^2b_{12}b_{21}b_{30}^2 - 2660a_{03}^3b_{30}^3),
\end{aligned}
$$

$$
\begin{aligned}
Y_2^{(9,10)} = \frac{i}{1680}(& -3360a_{12}^2a_{21}^3b_{12} + 1050a_{03}a_{21}^4b_{12} + 37380a_{12}a_{21}^3b_{12}^2 - 22260a_{21}^3b_{12}^3 + 1680a_{12}^3a_{21}^2b_{21} \\
& + 5040a_{03}a_{12}a_{21}^3b_{21} - 65520a_{12}^2a_{21}^2b_{12}b_{21} - 3220a_{03}a_{21}^3b_{12}b_{21} - 5040a_{12}a_{21}^2b_{12}^2b_{21} \\
& + 1680a_{21}^2b_{12}^3b_{21} + 78960a_{12}^3a_{21}b_{21}^2 - 68320a_{03}a_{12}a_{21}^2b_{21}^2 + 85680a_{12}^2a_{21}b_{12}b_{21}^2 \\
& + 2870a_{03}a_{21}^2b_{12}b_{21}^2 - 34020a_{12}a_{21}b_{12}^2b_{21}^2 + 37380a_{21}b_{12}^3b_{21}^2 + 107520a_{12}^3b_{21}^3 \\
& - 560a_{03}a_{12}a_{21}b_{21}^3 + 137760a_{12}^2b_{12}b_{21}^3 - 70420a_{03}a_{21}b_{12}b_{21}^3 - 62160a_{12}b_{12}^2b_{21}^3 \\
& - 16800b_{12}^3b_{21}^3 + 240240a_{03}a_{12}b_{21}^4 + 135240a_{03}b_{12}b_{21}^4 + 87360a_{12}^4a_{21}b_{30} \\
& - 46410a_{03}a_{12}^2a_{21}^2b_{30} + 420a_{03}^2a_{21}^3b_{30} + 22540a_{12}^3a_{21}b_{12}b_{30} - 47775a_{03}a_{12}a_{21}^2b_{12}b_{30} \\
& - 6510a_{12}^2a_{21}b_{12}^2b_{30} + 38262a_{03}a_{21}^2b_{12}^2b_{30} - 5740a_{12}a_{21}b_{12}^3b_{30} - 1050a_{21}b_{12}^4b_{30} \\
& + 218400a_{12}^4b_{21}b_{30} + 84000a_{03}a_{12}^2a_{21}b_{21}b_{30} + 616a_{03}^2a_{21}^2b_{21}b_{30} + 93940a_{12}^3b_{12}b_{21}b_{30} \\
& - 33138a_{03}a_{12}a_{21}b_{12}b_{21}b_{30} - 161490a_{12}^2b_{12}^2b_{21}b_{30} + 16275a_{03}a_{21}b_{12}^2b_{21}b_{30} \\
& + 140a_{12}b_{12}^3b_{21}b_{30} + 9450b_{12}^4b_{21}b_{30} + 823830a_{03}a_{12}^2b_{21}^2b_{30} - 143878a_{03}^2a_{21}b_{21}^2b_{30} \\
& + 256305a_{03}a_{12}b_{12}b_{21}^2b_{30} - 139335a_{03}b_{12}^2b_{21}^2b_{30} + 325290a_{03}^2b_{21}^3b_{30} \\
& + 300720a_{03}a_{12}^3b_{30}^2 - 32131a_{03}^2a_{12}a_{21}b_{30}^2 - 80878a_{03}a_{12}^2b_{12}b_{30}^2 - 15302a_{03}^2a_{21}b_{12}b_{30}^2 \\
& - 28364a_{03}a_{12}b_{12}^2b_{30}^2 + 2940a_{03}b_{12}^3b_{30}^2 + 327349a_{03}^2a_{12}b_{21}b_{30}^2 - 13791a_{03}^2b_{12}b_{21}b_{30}^2 \\
& + 2660a_{03}^3b_{30}^3).
\end{aligned}
$$

Appendix B

Focus quantities of (3.1) in the ring $\mathbb{C}[c]$:

$$
\begin{aligned}
g^c_{33} &= c_5 - c_6,\\
g^c_{66} &= -2c_1 + 2c_2 + 5c_3 - 5c_4 - 2c_7 + 2c_8,\\
g^c_{99} &= \frac{1}{144}(2640c_3c_6 - 2640c_4c_6 - 1056c_6c_7 + 1056c_6c_8 + 10026c_6c_9 - 10026c_6c_{10} + 41484c_3c_{11}\\
&\quad - 41484c_4c_{11} - 24392c_7c_{11} + 24392c_8c_{11} - 180c_9c_{11} + 180c_{10}c_{11} - 37452c_3c_{12}\\
&\quad + 37452c_4c_{12} + 76174c_7c_{12} - 76174c_8c_{12} - 41020c_9c_{12} + 41020c_{10}c_{12} + 135c_3c_{13}\\
&\quad - 135c_4c_{13} - 378c_7c_{13} + 378c_8c_{13} + 216c_9c_{13} - 216c_{10}c_{13}),\\
g^c_{12,12} &= -\frac{1}{1756339200}(294427766556000c_6^2c_9 - 294427766556000c_6^2c_{10} - 257886290914560c_6c_7c_{11}\\
&\quad + 257886290914560c_6c_8c_{11} - 5799959172047880c_6c_9c_{11}\\
&\quad + 5799959172047880c_6c_{10}c_{11} - 24193200762002160c_3c_{11}^2\\
&\quad + 24193200762002160c_4c_{11}^2 + 14207671214436000c_7c_{11}^2\\
&\quad - 14207671214436000c_8c_{11}^2 + 104367598448400c_9c_{11}^2\\
&\quad - 104367598448400c_{10}c_{11}^2 + 2023605788006880c_6c_7c_{12}\\
&\quad - 2023605788006880c_6c_8c_{12} + 3379521209100120c_6c_9c_{12}\\
&\quad - 3379521209100120c_6c_{10}c_{12} + 40707323816443200c_3c_{11}c_{12}\\
&\quad - 40707323816443200c_4c_{11}c_{12} - 55274290966560120c_7c_{11}c_{12}\\
&\quad + 55274290966560120c_8c_{11}c_{12} + 23637679447583040c_9c_{11}c_{12}\\
&\quad - 23637679447583040c_{10}c_{11}c_{12} - 17231252638274640c_3c_{12}^2\\
&\quad + 17231252638274640c_4c_{12}^2 + 35135306197334280c_7c_{12}^2\\
&\quad - 35135306197334280c_8c_{12}^2 - 19056888587797200c_9c_{12}^2\\
&\quad + 19056888587797200c_{10}c_{12}^2 - 10714397601600c_6c_7c_{13}\\
&\quad + 10714397601600c_6c_8c_{13} - 16212085119762c_6c_9c_{13}\\
&\quad + 16212085119762c_6c_{10}c_{13} - 185198720154312c_3c_{11}c_{13}\\
&\quad + 185198720154312c_4c_{11}c_{13} + 281266716188144c_7c_{11}c_{13}\\
&\quad - 281266716188144c_8c_{11}c_{13} - 124812089888220c_9c_{11}c_{13}
\end{aligned}
$$

$$+ 124812089888220c_{10}c_{11}c_{13} + 152720019740160c_3c_{12}c_{13}$$
$$- 152720019740160c_4c_{12}c_{13} - 355884591752158c_7c_{12}c_{13}$$
$$+ 355884591752158c_8c_{12}c_{13} + 195477929237452c_9c_{12}c_{13}$$
$$- 195477929237452c_{10}c_{12}c_{13} - 321831822915c_3c_{13}^2 + 321831822915c_4c_{13}^2$$
$$+ 900296097570c_7c_{13}^2 - 900296097570c_8c_{13}^2 - 513264903480c_9c_{13}^2$$
$$+ 513264903480c_{10}c_{13}^2 - 5466800908800c_2c_3 + 10228431628800c_3^2$$
$$+ 5466800908800c_2c_4 + 13667002272000c_3c_4 - 23895433900800c_4^2$$
$$- 8671827225600c_2c_7 - 36783919307520c_3c_7 + 13915742510208c_4c_7$$
$$+ 13077018662400c_7^2 + 8671827225600c_2c_8 + 2297024644992c_3c_8$$
$$+ 20571152152320c_4c_8 - 8671827225600c_7c_8 - 4405191436800c_8^2$$
$$+ 19627210359680c_2c_9 + 25183513555200c_3c_9 - 38297829056256c_4c_9$$
$$- 11212840235520c_7c_9 + 16700529871872c_8c_9 - 19627210359680c_2c_{10}$$
$$- 10770196842944c_3c_{10} + 23884512344000c_4c_{10} + 2926680487808c_7c_{10}$$
$$- 8414370124160c_8c_{10}).$$

Appendix C

The ideal of relations R for system (3.1) defined by its Gröbner basis R_G:

$$\langle c_5c_{10} - c_8c_{11},\ c_6c_9 - c_7c_{11},\ c_8^2 - c_4c_{10},\ c_6c_8 - c_{10}c_{12},\ c_5c_8 - c_4c_{11},\ c_4c_8 - c_2c_{10},\ c_7^2 - c_3c_9,$$
$$c_6c_7 - c_3c_{11},\ c_5c_7 - c_9c_{12},\ c_3c_7 - c_1c_9,\ c_5c_6 - c_{11}c_{12},\ c_4c_6 - c_8c_{12},\ c_3c_6 - c_1c_{11},\ c_2c_6 - c_4c_{12},$$
$$c_4c_5 - c_2c_{11},\ c_3c_5 - c_7c_{12},\ c_1c_5 - c_3c_{12},\ c_4^2 - c_2c_8,\ c_3^2 - c_1c_7,\ c_7c_8c_{11} - c_9c_{10}c_{12},\ c_3c_8c_{11} - c_7c_{10}c_{12},$$
$$c_1c_8c_{11} - c_3c_{10}c_{12},\ c_4c_7c_{11} - c_8c_9c_{12},\ c_2c_7c_{11} - c_4c_9c_{12},\ c_3c_4c_{11} - c_7c_8c_{12},\ c_1c_4c_{11} - c_3c_8c_{12},$$
$$c_2c_3c_{11} - c_4c_7c_{12},\ c_1c_2c_{11} - c_3c_4c_{12},\ c_{12}^3c_{13} - c_1c_2,\ c_{11}c_{12}^2c_{13} - c_3c_4,\ c_6c_{12}^2c_{13} - c_1c_4,\ c_5c_{12}^2c_{13} - c_2c_3,$$
$$c_{11}^2c_{12}c_{13} - c_7c_8,\ c_6c_{11}c_{12}c_{13} - c_3c_8,\ c_5c_{11}c_{12}c_{13} - c_4c_7,\ c_6^2c_{12}c_{13} - c_1c_8,\ c_5^2c_{12}c_{13} - c_2c_7,\ c_{11}^3c_{13} - c_9c_{10},$$
$$c_6c_{11}^2c_{13} - c_7c_{10},\ c_5c_{11}^2c_{13} - c_8c_9,\ c_6^2c_{11}c_{13} - c_3c_{10},\ c_5^2c_{11}c_{13} - c_4c_9,\ c_6^3c_{13} - c_1c_{10},\ c_5^3c_{13} - c_2c_9\rangle.$$

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