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# The minimizing problem involving $p$-Laplacian and Hardy–Littlewood–Sobolev upper critical exponent

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**Abstract.** In this paper, we study the minimizing problem

$$S_{p,1,\alpha,\mu} := \inf_{u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p_\alpha^*}|u(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{p}{2\cdot p_\alpha^*}}},$$

where $N \geqslant 3$, $p\in(1,N)$, $\mu\in\left[0,\left(\frac{N-p}{p}\right)^p\right)$, $\alpha\in(0,N)$ and $p_\alpha^* = \frac{p}{2}\left(\frac{2N-\alpha}{N-p}\right)$ is the Hardy–Littlewood–Sobolev upper critical exponent. Firstly, by using refinement of the Hardy–Littlewood–Sobolev inequality, we prove that $S_{p,1,\alpha,\mu}$ is achieved in $\mathbb{R}^N$ by a radially symmetric, nonincreasing and nonnegative function. Secondly, we give a estimation of extremal function.

**Keywords:** refinement of Hardy–Littlewood–Sobolev inequality, Hardy–Littlewood–Sobolev upper critical exponent, $p$-Laplacian, minimizing.

**2010 Mathematics Subject Classification:** 35J50, 35J60.

## 1 Introduction

In this paper, we consider the minimizing problem:

$$S_{p,1,\alpha,\mu} := \inf_{u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p_\alpha^*}|u(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{p}{2\cdot p_\alpha^*}}}, \tag{$\mathcal{P}$}$$

where $N \geqslant 3$, $p\in(1,N)$, $\mu\in\left[0,\left(\frac{N-p}{p}\right)^p\right)$, $\alpha\in(0,N)$ and $p_\alpha^* = \frac{p}{2}\left(\frac{2N-\alpha}{N-p}\right)$ is the Hardy–Littlewood–Sobolev upper critical exponent.

The paper was motivated by some papers appeared in recent years. For $p = 2$, problem ($\mathcal{P}$) is closely related to the nonlinear Choquard equation as follows:

$$-\Delta u + V(x)u = \left(|x|^\alpha * |u|^q\right)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

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where $\alpha \in (0, N)$ and $\frac{2N-\alpha}{N} \leqslant q \leqslant \frac{2N-\alpha}{N-2}$. For $q = 2$ and $\alpha = 1$, the equation (1.1) goes back to the description of the quantum theory of a polaron at rest by Pekar in 1954 [19] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree–Fock theory of one-component plasma [20]. For $q = \frac{2N-1}{N-2}$ and $\alpha = 1$, by using the Green function, it is obvious that equation (1.1) can be regarded as a generalized version of Schrödinger–Newton system:

$$\begin{cases} -\Delta u + V(x)u = |u|^{\frac{N+1}{N-2}}\phi, & \text{in } \mathbb{R}^N, \\ -\Delta \phi = |u|^{\frac{2N-1}{N-2}}, & \text{in } \mathbb{R}^N. \end{cases}$$

The existence and qualitative properties of solutions of Choquard type equations (1.1) have been widely studied in the last decades (see [16]). Moroz and Van Schaftingen [15] considered equation (1.1) with lower critical exponent $\frac{2N-\alpha}{N}$ if the potential $1 - V(x)$ should not decay to zero at infinity faster than the inverse of $|x|^2$. In [1], the authors studied the equation (1.1) with critical growth in the sense of Trudinger–Moser inequality and studied the existence and concentration of the ground states. In 2018, Gao and Yang [11] firstly investigated the following critical Choquard equation:

$$-\Delta u = \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}y \right) |u|^{2^*_\alpha - 2}u + \lambda u, \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, $N \geqslant 3$, $\alpha \in (0, N)$, $\lambda > 0$ and $2^*_\alpha = \frac{2N-\alpha}{N-2}$. By using variational methods, they established the existence, multiplicity and nonexistence of nontrivial solutions to equation (1.2). In 2017, Mukherjee and Sreenadh [17] considered the following fractional Choquard equation:

$$(-\Delta)^s u = \left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_{\alpha,s}}}{|x-y|^\alpha} \mathrm{d}y \right) |u|^{2^*_{\alpha,s} - 2}u + \lambda u, \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ with $C^{1,1}$ boundary, $s \in (0,1)$, $N \geqslant 2s$, $\alpha \in (0, N)$, $\lambda > 0$ and $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality. By using variational methods, they established the existence, multiplicity and nonexistence of nontrivial solutions to equation (1.3). For details and recent works, we refer to [2, 6, 7, 12, 23–26, 30] and the references therein.

For $p \neq 2$, in 2017, Pucci, Xiang and Zhang [22] studied the Schrödinger–Choquard–Kirchhoff equations involving the fractional $p$-Laplacian as follows:

$$\begin{aligned} (a + b\|u\|_s^{p(\theta-1)})[(-\Delta)^s_p u + V(x)|u|^{p-2}u] &= \lambda f(x, u) \\ &\quad + \left( \int_{\mathbb{R}^N} \frac{|u|^{p^*_{\alpha,s}}}{|x-y|^\alpha} \mathrm{d}y \right) |u|^{p^*_{\alpha,s} - 2}u \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.4}$$

where $\|u\|_s = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x \right)$, $a, b \in \mathbb{R}^+_0$ with $a + b > 0$, $\lambda > 0$ is a parameter, $s \in (0,1)$, $N > ps$, $\theta \in \left[1, \frac{N}{N-ps}\right)$, $\alpha \in (0, N)$, $p^*_{\alpha,s} = \frac{p(2N-\alpha)}{2(N-sp)}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, and $f : \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, $V : \mathbb{R}^N \to \mathbb{R}^+$ is a potential function. By using variational methods, they established the existence of nontrivial nonnegative solution to equation (1.4).

There is an open problem in [22]. We define the best constant:

$$S_{p,s,\alpha,\mu} := \inf_{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ps}} \mathrm{d}x}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_{\alpha,s}} |u(y)|^{p^*_{\alpha,s}}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \right)^{\frac{p}{2 \cdot p^*_{\alpha,s}}}}, \tag{1.5}$$

where $N \geqslant 3$, $p \in (1,N)$, $s \in (0,1]$, $\alpha \in (0,N)$ and $\mu \in [0,\mathcal{C}_{N,s,p})$, $\mathcal{C}_{N,s,p}$ is defined in [9, Theorem 1.1]. And $p^*_{\alpha,s} = \frac{p(2N-\alpha)}{2(N-sp)}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality.

**Open problem:** Is the best constant $S_{p,s,\alpha,\mu}$ achieved?

**(Result 1)** For $p=2$, $s=1$, $\mu=0$ and $\alpha \in (0,N)$, Gao and Yang [11] showed that $S_{2,1,\alpha,0}$ is achieved in $\mathbb{R}^N$ by the extremal function:

$$w_\sigma(x) = \mathfrak{C}_1 \sigma^{-\frac{N-2}{2}} w(x), \;\; w(x) = \frac{b_1}{(b_1^2+|x-a_1|^2)^{\frac{N-2}{2}}},$$

where $\mathfrak{C}_1 > 0$ is a fixed constant, $a_1 \in \mathbb{R}^N$ and $b_1 \in (0,\infty)$.

**(Result 2)** For $p=2$, $s\in(0,1)$, $\mu=0$ and $\alpha\in(0,N)$, Mukherjee and Sreenadh [17] proved that $S_{2,s,\alpha,0}$ is achieved in $\mathbb{R}^N$ by the extremal function:

$$w_\sigma(x) = \mathfrak{C}_2 \sigma^{-\frac{N-2s}{2}} w(x), \;\; w(x) = \frac{b_2}{(b_2^2+|x-a_2|^2)^{\frac{N-2s}{2}}},$$

where $\mathfrak{C}_2 > 0$ is a fixed constant, $a_2 \in \mathbb{R}^N$ and $b_2 \in (0,\infty)$.

**(Result 3)** For $p=2$, $s\in(0,1)$, $\mu \in \left[0, 4^s\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}\right)$ and $\alpha\in(0,N)$, Yang and Wu [34] showed that $S_{2,s,\alpha,\mu}$ is achieved in $\mathbb{R}^N$.

For Open problem, we study the case of $p\in(1,N)$, $s=1$, $\mu\in\left[0,\left(\frac{N-p}{p}\right)^p\right)$ and $\alpha\in(0,N)$. By using the refinement of Sobolev inequality in [18, Theorem 2], we show that $S_{p,1,\alpha,\mu}$ is achieved in $\mathbb{R}^N$ (see Theorem 1.1).

For the case $p\neq 2$, one expects that the minimizers of $S_{p,s,\alpha,\mu}$ have a form similar to the function $\omega_\sigma$. However, it is not known the explicit formula of the extremal function. We give the estimation of extremal function (see Theorem 1.2 and Theorem 1.3).

The first main result of this paper reads as follows.

**Theorem 1.1.** *Let $N\geqslant 3$, $p\in(1,N)$, $\alpha\in(0,N)$ and $\mu\in\left[0,\left(\frac{N-p}{p}\right)^p\right)$. Then $S_{p,1,\alpha,\mu}$ is achieved in $\mathbb{R}^N$ by a radially symmetric, nonincreasing and nonnegative function.*

The second main result of this paper reads as follows. For $p=2$ and $s\in(0,1)$, by using fractional Coulomb–Sobolev space and endpoint refined Sobolev inequality in [4], we give a estimation of extremal function.

**Theorem 1.2.** *Let $N\geqslant 3$, $p=2$, $\alpha\in(0,N)$, $s\in(0,1)$ and $\mu\in[0,\bar\mu)$. Any nonnegative minimizer $u$ of $S_{2,s,\alpha,\mu}$ is radially symmetric and nonincreasing, and it satisfies for $x\neq 0$ that*

$$C_4\left(\left(\frac{\bar\mu}{\bar\mu-\mu}\right)S_{2,s,\alpha,\mu}\right)^{\frac{(N-\alpha)(N-2s)}{2N(N+2s-\alpha)}}\left(\frac{N}{\omega_{N-1}}\right)^{\frac{N-2s}{2N}}\frac{1}{|x|^{\frac{N-2s}{2}}}\geqslant u(x),$$

*where $\omega_{N-1}$ is the area of the unit sphere in $\mathbb{R}^N$.*

The third main result of this paper reads as follows. For $p\neq 2$ and $s=1$, we give a estimation of extremal function.

**Theorem 1.3.** *Let $N\geqslant 3$, $p\in(1,N)$, $\alpha\in(0,N)$ and $\mu\in[0,\tilde\mu)$. Any nonnegative minimizer $u$ of $S_{p,1,\alpha,\mu}$ is radially symmetric and nonincreasing, and it satisfies for $x\neq 0$ that*

$$\left(\frac{2^\alpha N^2}{\omega_{N-1}^2}\right)^{\frac{1}{2\cdot p_\alpha^*}}\frac{1}{|x|^{\frac{N-p}{p}}}\geqslant u(x),$$

*where $\omega_{N-1}$ is the area of the unit sphere in $\mathbb{R}^N$.*

## 2 Preliminaries

The Sobolev space $W^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_W^p = \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x.$$

For $s \in (0,1)$ and $p \in (1,N)$, the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^{\frac{Np}{N-sp}}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \mathrm{d}x\mathrm{d}y < \infty \right\}.$$

For $s \in (0,1)$ and $p \in (1,N)$, we introduce the Hardy inequalities:

$$\bar{\mu}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2s}}\mathrm{d}x \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \quad \text{for any } u \in W^{s,2}(\mathbb{R}^N) \text{ and } \bar{\mu} = 4^s \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})},$$

and

$$\tilde{\mu}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x, \quad \text{for any } u \in W^{1,p}(\mathbb{R}^N) \text{ and } \tilde{\mu} = \left(\frac{N-p}{p}\right)^p.$$

The fractional Coulomb–Sobolev space [4] is defined by

$$\begin{aligned} \mathcal{E}^{s,\alpha,2^*_{\alpha,s}}(\mathbb{R}^N) = \Bigg\{ &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y < \infty \\ &\text{and} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y < \infty \Bigg\}. \end{aligned} \tag{2.1}$$

We endow the space $\mathcal{E}^{s,\alpha,2^*_{\alpha,s}}(\mathbb{R}^N)$ with the norm

$$\|u\|^2_{\mathcal{E},\alpha} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\alpha,s}}}. \tag{2.2}$$

We could define the best constant:

$$S_{p,1,0,\mu} := \inf_{u \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_W^p - \mu\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}\mathrm{d}x}{(\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x)^{\frac{p}{p^*}}}, \tag{2.3}$$

where $S_{p,1,0,\mu}$ is attained in $\mathbb{R}^N$ (see [8]).

**Lemma 2.1** (Hardy–Littlewood–Sobolev inequality, [14])**.** *Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t}+\frac{1}{r}+\frac{\mu}{N} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C_2 > 0$, independent of $f, g$ such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)||h(y)|}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \leqslant C_2\|f\|_t\|h\|_r.$$

A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\|u\|_{\mathcal{L}^{r,\varpi}(\mathbb{R}^N)}$ with $r \in [1,\infty)$ and $\varpi \in (0,N]$ if and only if

$$\|u\|^r_{\mathcal{L}^{r,\varpi}(\mathbb{R}^N)} = \sup_{R>0, x\in\mathbb{R}^N} R^{\varpi-N}\int_{B(x,R)} |u(y)|^r \mathrm{d}y < \infty.$$

**Lemma 2.2** ([18])**.** *For any $1 < p < N$, let $p^* = \frac{Np}{N-p}$. There exists $C_3 > 0$ such that for $\theta$ and $\vartheta$ satisfying $\frac{p}{p^*} \leqslant \theta < 1$, $1 \leqslant \vartheta < p^* = \frac{Np}{N-p}$, we have*

$$\left(\int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \leqslant C_3 \|u\|_W^{\theta} \|u\|^{1-\theta}_{\mathcal{L}^{\vartheta, \frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)},$$

*for any $u \in W^{1,p}(\mathbb{R}^N)$.*

**Lemma 2.3** (Endpoint refined Sobolev inequality, [4, Theorem 1.2])**.** *Let $\alpha \in (0, N)$ and $s \in (0, 1)$. Then there exists a constant $C_4 > 0$ such that the inequality*

$$\begin{aligned}\|u\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)} \leqslant C_4 &\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\right)^{\frac{(N-\alpha)(N-2s)}{2N(N+2s-\alpha)}} \\ &\times \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{\frac{2N-\alpha}{N-2s}}|u(y)|^{\frac{2N-\alpha}{N-2s}}}{|x-y|^{\alpha}} \mathrm{d}x\mathrm{d}y\right)^{\frac{s(N-2s)}{N(N+2s-\alpha)}},\end{aligned}$$

*holds for all $u \in \mathcal{E}^{s,\alpha,2^*_{\alpha,s}}(\mathbb{R}^N)$.*

## 3 The proof of Theorem 1.1

We show the refinement of the Hardy–Littlewood–Sobolev inequality. This inequality plays a key role in the proof of Theorem 1.1.

**Lemma 3.1.** *For any $1 < p < N$ and $\alpha \in (0, N)$, there exists $C_5 > 0$ such that for $\theta$ and $\vartheta$ satisfying $\frac{p}{p^*} \leqslant \theta < 1$, $1 \leqslant \vartheta < p^* = \frac{Np}{N-p}$, we have*

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_\alpha}|u(y)|^{p^*_\alpha}}{|x-y|^{\alpha}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p^*_\alpha}} \leqslant C_5 \|u\|_W^{2\theta} \|u\|^{2(1-\theta)}_{\mathcal{L}^{\vartheta, \frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)},$$

*for any $u \in W^{1,p}(\mathbb{R}^N)$.*

*Proof.* By using Lemma 2.2, we have

$$\left(\int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \leqslant C_3 \|u\|_W^{\theta} \|u\|^{1-\theta}_{\mathcal{L}^{\vartheta, \frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)}. \tag{3.1}$$

By the Hardy–Littlewood–Sobolev inequality and (3.1), we obtain

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_\alpha}|u(y)|^{p^*_\alpha}}{|x-y|^{\alpha}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p^*_\alpha}} \leqslant C_2^{\frac{1}{p^*_\alpha}} \|u\|^2_{L^{p^*}(\mathbb{R}^N)} \leqslant C_2^{\frac{1}{p^*_\alpha}} C_3^2 \|u\|_W^{2\theta} \|u\|^{2(1-\theta)}_{\mathcal{L}^{\vartheta, \frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)}. \qquad \square$$

In [18], there is a misprint, the authors point out it by themselves. The right one is

$$L^{p^*}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{r, r\frac{N-p}{p}}(\mathbb{R}^N) \tag{3.2}$$

for any $p \in (1, N)$ and $r \in [1, p^*)$. This embedding plays a key role in the proof of Theorem 1.1.

*Proof of Theorem 1.1.*
**Step 1.** Suppose now $0 \leqslant \mu < \tilde{\mu} = (\frac{N-p}{p})^p$. Applying Lemma 3.1 with $\vartheta = p$, we have

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\alpha^*}|u(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p_\alpha^*}} \leqslant C\left(\|u\|_W^p - \mu\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}\mathrm{d}x\right)^{\frac{2\theta}{p}}\|u\|^{2(1-\theta)}_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)}, \tag{3.3}$$

for $u \in W^{1,p}(\mathbb{R}^N)$. Let $\{u_n\}$ be a minimizing sequence of $S_{p,1,\alpha,\mu}$, that is

$$\|u_n\|_W^p - \mu\int_{\mathbb{R}^N}\frac{|u_n|^p}{|x|^p}\mathrm{d}x \to S_{p,1,\alpha,\mu}, \quad \text{as } n\to\infty,$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{p_\alpha^*}|u_n(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = 1.$$

Inequality (3.3) enables us to find $C>0$ independent of $n$ such that

$$\|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \geqslant C > 0. \tag{3.4}$$

We have the chain of inclusions

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{p,N-p}(\mathbb{R}^N), \tag{3.5}$$

which implies that

$$\|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \leqslant C. \tag{3.6}$$

Applying (3.4) and (3.6), there exists $C>0$ such that

$$0 < C \leqslant \|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \leqslant C^{-1}.$$

Combining the definition of Morrey space and above inequalities, for all $n\in\mathbb{N}$, we get the existence of $\lambda_n>0$ and $x_n\in\mathbb{R}^N$ such that

$$\frac{1}{\lambda_n^p}\int_{B(x_n,\lambda_n)}|u_n(y)|^p\mathrm{d}y \geqslant \|u_n\|^p_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} - \frac{C}{2n} \geqslant \tilde{C} > 0,$$

for some new positive constant $\tilde{C}$ that does not depend on $n$.

Let $v_n(x) = \lambda_n^{\frac{N-p}{p}}u_n(\lambda_n x)$. Notice that, by using the scaling invariance, we have

$$\|v_n\|_W^p - \mu\int_{\mathbb{R}^N}\frac{|v_n|^p}{|x|^p}\mathrm{d}x \to S_{p,1,\alpha,\mu}, \quad \text{as } n\to\infty,$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)|^{p_\alpha^*}|v_n(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = 1.$$

Then

$$\int_{B(\frac{x_n}{\lambda_n},1)}|v_n(y)|^p\mathrm{d}y = \frac{1}{\lambda_n^p}\int_{B(x_n,\lambda_n)}|u_n(y)|^p\mathrm{d}y \geqslant \tilde{C} > 0.$$

We can also show that $v_n$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Hence, we may assume

$$v_n \rightharpoonup v \quad \text{in } W^{1,p}(\mathbb{R}^N), \qquad v_n \to v \quad \text{a.e. in } \mathbb{R}^N, \qquad v_n \to v \text{ in } L^q_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{for all } q \in [p, p^*).$$

We claim that $\{\frac{x_n}{\lambda_n}\}$ is uniformly bounded in $n$. Indeed, for any $0 < \beta < p$, by Hölder's inequality, we observe that

$$\begin{aligned} 0 < \tilde{C} &\leqslant \int_{B(\frac{x_n}{\lambda_n},1)} |v_n|^p \mathrm{d}y = \int_{B(\frac{x_n}{\lambda_n},1)} |y|^{\frac{p\beta}{\frac{p(N-\beta)}{N-p}}} \frac{|v_n|^p}{|y|^{\frac{p\beta}{\frac{p(N-\beta)}{N-p}}}} \mathrm{d}y \\ &\leqslant \left( \int_{B(\frac{x_n}{\lambda_n},1)} |y|^{\frac{\beta(N-p)}{p-\beta}} \mathrm{d}y \right)^{1-\frac{N-p}{N-\beta}} \left( \int_{B(\frac{x_n}{\lambda_n},1)} \frac{|v_n|^{\frac{p(N-\beta)}{N-p}}}{|y|^\beta} \mathrm{d}y \right)^{\frac{N-p}{N-\beta}}. \end{aligned}$$

By the rearrangement inequality, see [14, Theorem 3.4], we have

$$\int_{B(\frac{x_n}{\lambda_n},1)} |y|^{\frac{\beta(N-p)}{p-\beta}} \mathrm{d}y \leqslant \int_{B(0,1)} |y|^{\frac{\beta(N-p)}{p-\beta}} \mathrm{d}y \leqslant C.$$

Therefore,

$$0 < C \leqslant \int_{B(\frac{x_n}{\lambda_n},1)} \frac{|v_n|^{\frac{p(N-\beta)}{N-p}}}{|y|^\beta} \mathrm{d}y. \tag{3.7}$$

Now, suppose on the contrary, that $\frac{x_n}{\lambda_n} \to \infty$ as $n \to \infty$. Then, for any $y \in B\big(\frac{x_n}{\lambda_n}, 1\big)$, we have $|y| \geqslant |\frac{x_n}{\lambda_n}| - 1$ for $n$ large. Thus,

$$\begin{aligned} \int_{B(\frac{x_n}{\lambda_n},1)} \frac{|v_n|^{\frac{p(N-\beta)}{N-p}}}{|y|^\beta} \mathrm{d}y &\leqslant \frac{1}{(|\frac{x_n}{\lambda_n}| - 1)^\beta} \int_{B(\frac{x_n}{\lambda_n},1)} |v_n|^{\frac{p(N-\beta)}{N-p}} \mathrm{d}y \\ &\leqslant \frac{\left|B(\frac{x_n}{\lambda_n}, 1)\right|^{\frac{\beta}{N}}}{(|\frac{x_n}{\lambda_n}| - 1)^\beta} \left( \int_{B(\frac{x_n}{\lambda_n},1)} |v_n|^{\frac{Np}{N-p}} \mathrm{d}y \right)^{\frac{N-\beta}{N}} \\ &\leqslant \frac{\left|B(\frac{x_n}{\lambda_n}, 1)\right|^{\frac{\beta}{N}}}{(|\frac{x_n}{\lambda_n}| - 1)^\beta} \cdot \frac{\|v_n\|_W^{\frac{N-\beta}{N}}}{S_{p,1,0,0}^{\frac{N-\beta}{N-p}}} \leqslant \frac{C}{(|\frac{x_n}{\lambda_n}| - 1)^\beta} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which contradicts (3.7). Hence, $\{\frac{x_n}{\lambda_n}\}$ is bounded, and there exists $R > 0$ such that

$$\int_{B(0,R)} |v_n(y)|^p \mathrm{d}y \geqslant \int_{B(\frac{x_n}{\lambda_n},1)} |v_n(y)|^p \mathrm{d}y \geqslant \tilde{C} > 0.$$

Since the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q_{\mathrm{loc}}(\mathbb{R}^N)$ $q \in [p, p^*)$ is compact, we deduce that

$$\int_{B(0,R)} |v(y)|^p \mathrm{d}y \geqslant \tilde{C} > 0,$$

which means $v \not\equiv 0$.

**Step 2.** Set
$$h(t) = t^{\frac{2\cdot p_\alpha^*}{p}}, \qquad t \geqslant 0 \ (1 < p < N).$$
Since $p \in (1, N)$ and $\alpha \in (0, N)$, we get
$$\frac{2\cdot p_\alpha^*}{p} = \frac{2N-\alpha}{N-p} > 1 \quad \text{and} \quad N + p - \alpha > 0.$$
We know that
$$h''(t) = \frac{(2N-\alpha)(N+p-\alpha)}{(N-p)^2} t^{\frac{2p-\alpha}{N-p}} \geqslant 0,$$
which implies that $h(t)$ is a convex function. By using $h(0) = 0$ and $l \in [0, 1]$, we know
$$h(lt) = h(lt + (1-l)\cdot 0) \leqslant lh(t) + (1-l)h(0) = lh(t). \tag{3.8}$$
For any $t_1, t_2 \in [0, \infty)$, applying last inequality with $l = \frac{t_1}{t_1+t_2}$ and $l = \frac{t_2}{t_1+t_2}$, we get
$$\begin{aligned} h(t_1) + h(t_2) &= h\left((t_1+t_2)\frac{t_1}{t_1+t_2}\right) + h\left((t_1+t_2)\frac{t_2}{t_1+t_2}\right) \\ &\leqslant \frac{t_1}{t_1+t_2} h(t_1+t_2) + \frac{t_2}{t_1+t_2} h(t_1+t_2) \quad (\text{by } (3.8)) \\ &= h(t_1+t_2). \end{aligned} \tag{3.9}$$
Now, we claim that $v_n \to v$ strongly in $W^{1,p}(\mathbb{R}^N)$. Set
$$K(u, v) = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v \mathrm{d}x - \mu \int_{\mathbb{R}^N} \frac{|u|^{p-2}uv}{|x|^p}\mathrm{d}x.$$
Since $\{v_n\}$ is a minimizing sequence,
$$\lim_{n\to\infty} K(v_n, v_n) = S_{p,1,\alpha,\mu}.$$
By using Brézis–Lieb type lemma [5] and [22, Theorem 2.3], we know
$$K(v, v) + \lim_{n\to\infty} K(v_n - v, v_n - v) = \lim_{n\to\infty} K(v_n, v_n) + o(1) = S_{p,1,\alpha,\mu} + o(1), \tag{3.10}$$
and
$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)|^{p_\alpha^*}|v_n(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v(x)|^{p_\alpha^*}|v_n(y)-v(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^{p_\alpha^*}|v(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y + o(1), \end{aligned} \tag{3.11}$$
where $o(1)$ denotes a quantity that tends to zero as $n \to \infty$. Therefore,
$$\begin{aligned} 1 &= \lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)|^{p_\alpha^*}|v_n(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x)-v(x)|^{p_\alpha^*}|v_n(y)-v(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^{p_\alpha^*}|v(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \\ &\leqslant S_{p,1,\alpha,\mu}^{-\frac{2\cdot p_\alpha^*}{p}} \left(\lim_{n\to\infty} K(v_n - v, v_n - v)\right)^{\frac{2\cdot p_\alpha^*}{p}} + S_{p,1,\alpha,\mu}^{-\frac{2\cdot p_\alpha^*}{p}} (K(v, v))^{\frac{2\cdot p_\alpha^*}{p}} \\ &\leqslant S_{p,1,\alpha,\mu}^{-\frac{2\cdot p_\alpha^*}{p}} \left(\lim_{n\to\infty} K(v_n - v, v_n - v) + K(v, v)\right)^{\frac{2\cdot p_\alpha^*}{p}} \quad (\text{by } (3.9)) \\ &\leqslant 1 \quad (\text{by } (3.10)). \end{aligned}$$

Therefore, all the inequalities above have to be equalities. We know that

$$\left(\lim_{n\to\infty} K(v_n - v, v_n - v)\right)^{\frac{2\cdot p_\alpha^*}{p}} + (K(v,v))^{\frac{2\cdot p_\alpha^*}{p}} = \left(\lim_{n\to\infty} K(v_n - v, v_n - v) + K(v,v)\right)^{\frac{2\cdot p_\alpha^*}{p}}. \tag{3.12}$$

We show that $\lim_{n\to\infty} K(v_n - v, v_n - v) = 0$. Combining (3.9) and (3.12), we know that

$$\text{either} \quad \lim_{n\to\infty} K(v_n - v, v_n - v) = 0 \quad \text{or} \quad K(v,v) = 0.$$

Since $v \not\equiv 0$, so $K(v,v) \neq 0$. Therefore,

$$\lim_{n\to\infty} K(v_n - v, v_n - v) = 0. \tag{3.13}$$

This implies that $v_n \to v$ strongly in $W^{1,p}(\mathbb{R}^N)$. Moreover, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(x) - v(x)|^{p_\alpha^*}|v_n(y) - v(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = 0. \tag{3.14}$$

**Step 3.** Since $v \not\equiv 0$, putting (3.13) into (3.10), and inserting (3.14) into (3.11), we know

$$\lim_{n\to\infty}\left(\|v_n\|_W^p - \mu\int_{\mathbb{R}^N}\frac{|v_n|^p}{|x|^p}\mathrm{d}x\right) \to S_{p,1,\alpha,\mu} = \|v\|_W^p - \mu\int_{\mathbb{R}^N}\frac{|v|^p}{|x|^p}\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)|^{p_\alpha^*}|v(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y = 1.$$

Then $v$ is an extremal.

In addition, $|v| \in W^{1,p}(\mathbb{R}^N)$ and $|\nabla|v|| = |\nabla v|$ a.e. in $\mathbb{R}^N$, therefore, $|v|$ is also an extremal, and then there exist non–negative extremals.

Let $\bar{v} \geqslant 0$ be an extremal. Denote by $\bar{v}_*$ the symmetric–decreasing rearrangement of $\bar{v}$ (See [14, Section 3]). From [21] it follows that

$$\int_{\mathbb{R}^N} |\nabla\bar{v}_*|^p\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla\bar{v}|^p\mathrm{d}x. \tag{3.15}$$

According to the simplest rearrangement inequality in [14, Theorem 3.4], we get

$$\int_{\mathbb{R}^N}\frac{|\bar{v}|^p}{|x|^p}\mathrm{d}x \leqslant \int_{\mathbb{R}^N}\frac{|\bar{v}_*|^p}{|x|^p}\mathrm{d}x. \tag{3.16}$$

By using Riesz's rearrangement inequality in [14, Theorem 3.7], we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{v}(x)|^{p_\alpha^*}|\bar{v}(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{v}_*(x)|^{p_\alpha^*}|\bar{v}_*(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y. \tag{3.17}$$

Combining (3.15), (3.16) and (3.17), and the fact that $\mu \geqslant 0$, we get that $\bar{v}_*$ is also an extremal, and then there exist radially symmetric and nonincreasing extremal. □

## 4 Proof of Theorem 1.2

For $p=2$ and $s\in(0,1)$, we give a estimation of extremal function $u(x)$. The proof of Theorem 1.2 is based on the Coulomb–Sobolev space $\mathcal{E}^{s,\alpha,2^*_{\alpha,s}}(\mathbb{R}^N)$ and the endpoint refined Sobolev inequality in Lemma 2.3.

*Proof of Theorem 1.2.* We show some properties of radially symmetric, nonincreasing and non-negative function $u(x)$. Let $\bar{\mu}=4^s\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$. By the definition of extremal $u$ (see the proof of Theorem 1.1), we know

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y-\mu\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}\mathrm{d}x=S_{2,s,\alpha,\mu}, \tag{4.1}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_{\alpha,s}}|u(y)|^{2^*_{\alpha,s}}}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y=1. \tag{4.2}$$

Applying (4.1), (4.2) and the definition of Coulomb–Sobolev space $\mathcal{E}^{s,\alpha,2^*_{\alpha,s}}(\mathbb{R}^N)$, we get $u\in\mathcal{E}^{s,\alpha,2^*_{\alpha,s}}(\mathbb{R}^N)$.

By using (4.1), (4.2), $u\in\mathcal{E}^{s,\alpha,2^*_{\alpha,s}}(\mathbb{R}^N)$ and Lemma 2.3, we have

$$\begin{aligned}\|u\|_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)}&\leqslant C_4\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{(N-\alpha)(N-2s)}{2N(N+2s-\alpha)}}\\&\quad\times\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{\frac{2N-\alpha}{N-2s}}|u(y)|^{\frac{2N-\alpha}{N-2s}}}{|x-y|^{\alpha}}\mathrm{d}x\mathrm{d}y\right)^{\frac{s(N-2s)}{N(N+2s-\alpha)}}\\&=C_4\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{(N-\alpha)(N-2s)}{2N(N+2s-\alpha)}}\\&\leqslant C_4\left(\left(\frac{\bar{\mu}}{\bar{\mu}-\mu}\right)S_{2,s,\alpha,\mu}\right)^{\frac{(N-\alpha)(N-2s)}{2N(N+2s-\alpha)}}.\end{aligned} \tag{4.3}$$

For any $0<R<\infty$ and $B(R):=B(0,R)\subset\mathbb{R}^N$, we obtain

$$\begin{aligned}C_4\left(\left(\frac{\bar{\mu}}{\bar{\mu}-\mu}\right)S_{2,s,\alpha,\mu}\right)^{\frac{(N-\alpha)(N-2s)}{2N(N+2s-\alpha)}}&\geqslant\left(\int_{\mathbb{R}^N}|u(x)|^{\frac{2N}{N-2s}}\mathrm{d}x\right)^{\frac{N-2s}{2N}}\\&\geqslant\left(\int_{B(R)}|u(x)|^{\frac{2N}{N-2s}}\mathrm{d}x\right)^{\frac{N-2s}{2N}}\\&\geqslant|u(R)|\omega_{N-1}^{\frac{N-2s}{2N}}\left(\int_0^R\rho^{N-1}\mathrm{d}\rho\right)^{\frac{N-2s}{2N}}\\&=|u(R)|\left(\frac{\omega_{N-1}}{N}\right)^{\frac{N-2s}{2N}}R^{\frac{N-2s}{2}},\end{aligned}$$

which implies

$$C_4\left(\left(\frac{\bar{\mu}}{\bar{\mu}-\mu}\right)S_{2,s,\alpha,\mu}\right)^{\frac{(N-\alpha)(N-2s)}{2N(N+2s-\alpha)}}\left(\frac{N}{\omega_{N-1}}\right)^{\frac{N-2s}{2N}}\frac{1}{|x|^{\frac{N-2s}{2}}}\geqslant|u(x)|.$$

□

## 5 Proof of Theorem 1.3

For $p \neq 2$ and $s \in (0,1)$, we give a estimation of extremal function $u(x)$. From Theorem 1.1, we know that $u(x)$ is a radially symmetric, nonincreasing and nonnegative function.

The proof of Theorem 1.3 is different from Theorem 1.2. The endpoint refined Sobolev inequality in Lemma 2.3 is true for $p = 2$. However, we do not know that the endpoint refined Sobolev inequality is true or not for $p \neq 2$.

*Proof of Theorem 1.3.* Let $\tilde{\mu} = \left(\frac{N-p}{p}\right)^p$. By the definition of extremal $u$, we know

$$\|u\|_W^p - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x = S_{p,1,\alpha,\mu}, \tag{5.1}$$

and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\alpha^*} |u(y)|^{p_\alpha^*}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y = 1. \tag{5.2}$$

For any $R \in (0, \infty)$ and $B(R) := B(0,R) \subset \mathbb{R}^N$, by Hölder's inequality, we obtain

$$\begin{aligned}
\left(\int_{B(R)} |u|^{p_\alpha^*} \mathrm{d}x\right)^{\frac{1}{p_\alpha^*}} &\leqslant \left[\left(\int_{B(R)} \mathrm{d}x\right)^{1-\frac{p_\alpha^*}{p^*}} \left(\int_{B(R)} |u|^{p_\alpha^* \cdot \frac{p^*}{p_\alpha^*}} \mathrm{d}x\right)^{\frac{p_\alpha^*}{p^*}}\right]^{\frac{1}{p_\alpha^*}} \\
&= |B(R)|^{\frac{1}{p_\alpha^*} - \frac{1}{p^*}} \left(\int_{B(R)} |u|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \\
&\leqslant |B(R)|^{\frac{1}{p_\alpha^*} - \frac{1}{p^*}} S_{p,1,0,0}^{-\frac{1}{p}} \|u\|_W \\
&\leqslant |B(R)|^{\frac{1}{p_\alpha^*} - \frac{1}{p^*}} S_{p,1,0,0}^{-\frac{1}{p}} \left(\left(\frac{\tilde{\mu}}{\tilde{\mu} - \mu}\right) S_{p,1,\alpha,\mu}\right)^{\frac{1}{p}} < \infty.
\end{aligned}$$

By Fubini's theorem, we get

$$\begin{aligned}
(2R)^{-\alpha} \left(\int_{B(R)} |u(x)|^{p_\alpha^*} \mathrm{d}x\right)^2 &= (2R)^{-\alpha} \int_{B(R)} |u(x)|^{p_\alpha^*} \mathrm{d}x \int_{B(R)} |u(y)|^{p_\alpha^*} \mathrm{d}y \\
&= (2R)^{-\alpha} \int_{B(R)} \int_{B(R)} |u(x)|^{p_\alpha^*} |u(y)|^{p_\alpha^*} \mathrm{d}x\mathrm{d}y \\
&\leqslant \int_{B(R)} \int_{B(R)} \frac{|u(y)|^{p_\alpha^*} |u(x)|^{p_\alpha^*}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y,
\end{aligned}$$

which implies

$$\left(\int_{B(R)} |u(x)|^{p_\alpha^*} \mathrm{d}x\right)^2 \leqslant (2R)^\alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y)|^{p_\alpha^*} |u(x)|^{p_\alpha^*}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y. \tag{5.3}$$

According to (5.1), (5.2) and (5.3), we have

$$\begin{aligned}
1 &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p_\alpha^*} |u(y)|^{p_\alpha^*}}{|x-y|^\alpha} \mathrm{d}x\mathrm{d}y \\
&\geqslant (2R)^{-\alpha} \left(\int_{B(R)} |u|^{p_\alpha^*} \mathrm{d}x\right)^2 \\
&\geqslant (2R)^{-\alpha} |u(R)|^{2 \cdot p_\alpha^*} \left(\omega_{N-1} \int_0^R \rho^{N-1} \mathrm{d}\rho\right)^2 \\
&\geqslant \frac{\omega_{N-1}^2}{2^\alpha N^2} |u(R)|^{2 \cdot p_\alpha^*} R^{2N-\alpha}.
\end{aligned}$$

Then we know

$$\left(\frac{2^\alpha N^2}{\omega_{N-1}^2}\right)^{\frac{1}{2\cdot p_\alpha^*}} \frac{1}{R^{\frac{N-p}{p}}} \geqslant |u(R)|.$$

Hence, for any $0<|x|<\infty$, we obtain

$$\left(\frac{2^\alpha N^2}{\omega_{N-1}^2}\right)^{\frac{1}{2\cdot p_\alpha^*}} \frac{1}{|x|^{\frac{N-p}{p}}} \geqslant u(x). \qquad \square$$

# 6 Conclusions and future works

The results in this paper set the foundation for the study of a number of questions related to minimizing problem

$$S_{p,1,\alpha,\mu} := \inf_{u\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u|^p \mathrm{d}x - \mu\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^p}\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_\alpha^*}|u(y)|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{p}{2\cdot p_\alpha^*}}},$$

where $N\geqslant 3$, $p\in(1,N)$, $\alpha\in(0,N)$ and $\mu\in\left[0,\left(\frac{N-p}{p}\right)^p\right)$.

During the preparation of the manuscript we faced several problems which are worth to be tackled in forthcoming investigations. In the sequel, we shall formulate some of them.

**(a)** The challenging problems are to prove the rest of Open problem: the case of $N\geqslant 3$, $p\in(1,N)$, $s\in(0,1)$, $\alpha\in(0,N)$ and $\mu\in[0,\mathcal{C}_{N,s,p})$, and $\mathcal{C}_{N,s,p}$ is defined in [9, Theorem 1.1].

**(b)** In [27], the authors studied the following minimizing problem:

$$I_{2,s,\alpha,\mu}(u,v) := \inf_{u,v\in W^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2+|v(x)-v(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y - \mu\int_{\mathbb{R}^N}\left(\frac{|u|^2}{|x|^{2s}}+\frac{|v|^2}{|x|^{2s}}\right)\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_{\alpha,s}^*}|u(y)|^{2_{\alpha,s}^*}+|v(x)|^{2_{\alpha,s}^*}|v(y)|^{2_{\alpha,s}^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2_\alpha^*}}},$$

where $N\geqslant 3$, $p=2$, $s\in(0,1)$, $\mu\in\left[0,4^s\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}\right)$ and $\alpha\in(0,N)$.

It is worth to extend the study of $I_{2,s,\alpha,\mu}(u,v)$ to the following minimizing problem:

$$I_{p,s,\alpha,\mu}(u,v) := \inf_{u\in W^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p+|v(x)-v(y)|^p}{|x-y|^{N+ps}}\mathrm{d}x\mathrm{d}y - \mu\int_{\mathbb{R}^N}\left(\frac{|u|^p}{|x|^{ps}}+\frac{|v|^p}{|x|^{ps}}\right)\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p_{\alpha,s}^*}|u(y)|^{p_{\alpha,s}^*}+|v(x)|^{p_{\alpha,s}^*}|v(y)|^{p_{\alpha,s}^*}}{|x-y|^\alpha}\mathrm{d}x\mathrm{d}y\right)^{\frac{p}{2\cdot p_{\alpha,s}^*}}},$$

where $N\geqslant 3$, $p\in(1,N)$, $s\in(0,1)$, $\mu\in[0,\mathcal{C}_{N,s,p})$ and $\alpha\in(0,N)$.

**(c)** By using Theorem 1.1 and Lemma 3.1, we could study the Choquard equation involving two critical nonlinearities:

$$-\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^p} = \left(\int_{\mathbb{R}^N}\frac{|u|^{p_\alpha^*}}{|x-y|^\alpha}\mathrm{d}y\right)|u|^{p_\alpha^*-2}u + |u|^{p^*-2}u, \text{ in } \mathbb{R}^N,$$

where $N\geqslant 3$, $p\in(1,N)$, $\mu\in\left[0,\left(\frac{N-p}{p}\right)^p\right)$ and $\alpha\in(0,N)$.

**(d)** Ambrosetti, Brezis and Cerami [3] proved the existence of infinity many solutions to the following problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 3$, $\lambda > 0$ and $q \in (1,2)$. García and Peral [13] proved the existence of infinity many solutions to following problem

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where $-\Delta_p$ is the $p$-Laplacian operator, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 3$, $\lambda > 0$, $q \in (1,p)$ and $p^* = \frac{Np}{N-p}$. Gao and Yang [10] proved the existence of infinity many solutions to following problem

$$\begin{cases} -\Delta u = \left(\int_\Omega \frac{|u|^{2^*_\alpha}}{|x-y|^\alpha} \mathrm{d}y\right) |u|^{2^*_\alpha - 2}u + \lambda |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{0,1}$ bounded boundary, $N \geqslant 3$, $\lambda > 0$, $q \in (1,2)$, $0 < \alpha < N$ and $2^*_\alpha = \frac{2N-\alpha}{N-2}$ is the Hardy–Littlewood–Sobolev upper critical exponent.

It is natural to ask: does there exist a solution to following problem?

$$\begin{cases} -\Delta_p u = \left(\int_\Omega \frac{|u|^{p^*_\alpha}}{|x-y|^\alpha} \mathrm{d}y\right) |u|^{p^*_\alpha - 2}u + \lambda |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{0,1}$ bounded boundary, $N \geqslant 3$, $\lambda > 0$, $p \in (1,N)$, $q \in (1,p)$ and $0 < \alpha < N$.

**(e)** Wang, Xie, Zhang [28] study the critical Kirchhoff-type $p$-Laplacian problems. For details and recent works, we refer to [29,31–33] and the references therein. By using Theorem 1.1, we could study the Schrödinger–Choquard–Kirchhoff equation involving critical nonlinearity:

$$-(a + b\|u\|_W^{p(\theta-1)})\Delta_p u = \left(\int_{\mathbb{R}^N} \frac{|u|^{p^*_\alpha}}{|x-y|^\alpha} \mathrm{d}y\right) |u|^{p^*_\alpha - 2}u, \quad \text{in } \mathbb{R}^N,$$

where $\|u\|_W^p = \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x$, $N \geqslant 3$, $p \in (1,N)$, $\theta \in \left[1, \frac{2N}{N-p}\right]$ and $\alpha \in (0,N)$.

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