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The maximum principle with lack of monotonicity

This paper is dedicated with esteem to Professor László Hatvani

on the occasion of his 75th anniversary

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Abstract. We establish a maximum principle for the weighted (p,q)-Laplacian, which extends the general Pucci–Serrin strong maximum principle to this quasilinear abstract setting. The feature of our main result is that it does not require any monotonicity assumption on the nonlinearity. The proof combines a local analysis with techniques on nonlinear differential equations.

Keywords: generalized maximum principle, (p,q)-operator, nonlinear differential inequality, normal derivative, positive solution.

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1 Introduction

The maximum principle is a basic tool in the mathematical analysis of partial differential equations. This is an extremely useful instrument when studying the qualitative behavior of solutions of differential equations and inequalities. The roots of the maximum principle go back to C. F. Gauss, who already knew the maximum principle for harmonic functions in 1839, in close relationship with the mean value formula.

Let us first recall some of the major steps related to the understanding of the maximum principle.

Let Ω be a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega$ has the interior sphere property at any point. The maximum principle asserts that if $u : \overline{\Omega} \to \mathbb{R}$ is a smooth function such that

$$\begin{cases} -\Delta u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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then $u \geqslant 0$ in Ω.

A stronger version of the maximum principle has been deduced by E. Hopf [13, 14]. The Hopf lemma asserts that if u satisfies (1.1), then the following alternative holds: *either* u vanishes identically in Ω *or* u is positive in Ω and its exterior normal derivative $\partial u/\partial \nu < 0$ on $\partial\Omega$.

G. Stampacchia [27] showed that the strong maximum principle continues to remain true in the case of certain *linear perturbations* of the Laplace operator. More precisely, let $a \in L^\infty(\Omega)$ be such that, for some $\alpha > 0$,

$$\int_\Omega (|Du|^2 + a(x)u^2)dx \geqslant \alpha \|u\|^2_{H^1_0(\Omega)} \quad \text{for all } u \in H^1_0(\Omega).$$

Stampacchia's maximum principle asserts that if

$$\begin{cases} -\Delta u + a(x)u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then either $u \equiv 0$ in Ω or $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$.

J.-L. Vázquez [28] observed that the maximum principle remains true for suitable *nonlinear perturbations* of the Laplace operator, subject to *monotonicity assumptions* on the nonlinear term. More precisely, let $f : \mathbb{R}^+_0 \to \mathbb{R}$, $\mathbb{R}^+_0 = [0, \infty)$, be a continuous non-decreasing function such that $f(0) = 0$ and

$$\int_{0^+} F(t)^{-1/2} dt = \infty, \quad \text{where } F(t) = \int_0^t f(s) ds.$$

Under these assumptions, Vázquez proved that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + f(u) \geqslant 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{on } \partial\Omega, \end{cases}$$

then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

We point out that the Keller–Osserman type growth assumption

$$\int_{0^+} F(t)^{-1/2} dt = \infty \tag{1.2}$$

holds true for "superlinear" nonlinearities. For instance, $f(t) = t^q$, with $t \in \mathbb{R}^+_0$ and $q \geqslant 1$, satisfies the hypotheses of the Vázquez maximum principle. Condition (1.2) is also satisfied by some nonlinearities for which $f(t)/t$ is not bounded at the origin, for instance $f(t) = t(\log t)^2$, $t \in \mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$.

The necessity of (1.2) is due to P. Benilan, H. Brézis and M. Crandall [4], while for the p-Laplacian it is due to J.-L. Vázquez [28]. In this latter case, relation (1.2) becomes

$$\int_{0^+} F(t)^{-1/p} dt = \infty.$$

For other classes of differential operators, necessity is due to J. I. Diaz [8, Theorem 1.4] and P. Pucci, J. Serrin and H. Zou [25, Corollary 1].

In a series of papers, P. Pucci and J. Serrin [20, 21, 23] extended the maximum principle into several directions and under very general assumptions. For instance, P. Pucci and J. Serrin considered the following canonical divergence structure inequality

$$-\operatorname{div}\{A(|Du|)Du\} + f(u) \geqslant 0 \quad \text{in } \Omega, \tag{1.3}$$

where the function $A = A(s)$ and the nonlinearity f satisfy the following conditions:

(A1) $A \in C(\mathbb{R}^+)$;

(A2) the mapping $s \mapsto sA(s)$ is strictly increasing in $\mathbb{R}^+$ and $sA(s) \to 0$ as $s \to 0$;

(F1) $f \in C(\mathbb{R}_0^+)$;

(F2) $f(0) = 0$ and f is non-decreasing on some interval $(0, \delta)$, $\delta > 0$.

Condition (A2) is a minimal requirement for ellipticity of (1.3), allowing moreover singular and degenerate behavior of the operator A at $s = 0$, that is, at critical points $x \in \Omega$ of u, such that $(Du)(x) = 0$.

The differential operator div $\{A(|Du|)Du\}$ is called the *A-Laplace operator*. An important example of A-Laplace operator that fulfills hypotheses (A1) and (A2) is the (p, q)-Laplace operator $\Delta_p u + \Delta_q u$, with $1 < p < q < \infty$, which is generated by $A(s) = s^{p-2} + s^{q-2}$, $s \in \mathbb{R}^+$.

Let $\mathcal{G}$ be the potential defined by $\mathcal{G}'(s) = sA(s)$ for all $s \in \mathbb{R}^+$, with $\mathcal{G}(0) = 0$. Condition (A2) implies that the mapping $s \mapsto \mathcal{G}'(s)$ is strictly increasing and continuous in $\mathbb{R}_0^+$, so that $\mathcal{G}$ can be extended by symmetry in $\mathbb{R}$ and $\mathcal{G}$ becomes a symmetric strictly convex function in $\mathbb{R}$. In particular, for $A(s) = s^{p-2} + s^{q-2}$, $s \in \mathbb{R}^+$, we have $\mathcal{G}(s) = s^p/p + s^q/q$, $s \in \mathbb{R}_0^+$.

In what follows, a classical solution of problem (1.3) is a function $u \in C^1(\overline{\Omega})$ which satisfies (1.3) in the distributional sense.

By the *strong maximum principle* for problem (1.3) we mean the statement that if u is a non-negative classical solution of problem (1.3), with $u(x_0) = 0$ at some point $x_0 \in \Omega$, then $u \equiv 0$ in Ω.

In order to describe the Pucci-Serrin strong maximum principle for the inequality (1.3), we need a further definition. Put $\Phi(s) = sA(s)$ for $s \in \mathbb{R}^+$ and $\Phi(0) = 0$. Then, the function

$$H(s) = s\Phi(s) - \int_0^s \Phi(t)dt \quad \text{for all } s \in \mathbb{R}_0^+$$

is the pre-Legendre transform of $\mathcal{G}$, since $H(s) = s\mathcal{G}'(s) - \mathcal{G}(s)$ for all $s \in \mathbb{R}_0^+$.

Under hypotheses (A1), (A2), (F1) and (F2), the Pucci–Serrin maximum principle [21, Theorem 1.1], see also [24, Theorem 1.1.1], establishes that the strong maximum principle holds for problem (1.3) if and only if either $f(s) \equiv 0$ for $s \in [0, \mu)$, with $\mu > 0$, or $f(s) > 0$ for $s \in (0, \delta)$ and

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} = \infty.$$

For further details on the maximum principle we refer to the monographs by L. E. Fraenkel [11], D. Gilbarg and N. S. Trudinger [12], and M. H. Protter and H. F. Weinberger [19].

2 Strong maximum principle for the (p, q)-Laplacian

The *global* monotonicity assumption on the nonlinearity f plays a central role in the statement of the Vázquez maximum principle. This hypothesis is replaced with the *local* monotonicity condition (F2) in the strong maximum principle of Pucci and Serrin, namely f is assumed to be non-decreasing on some interval $(0, \delta)$.

Our purpose in this paper is to prove that the monotonicity constraint on f can be removed and that only the growth of the nonlinearity near zero guarantees the maximum principle. This will be done for the (p, q)-Laplace operator $\Delta_p u + \Delta_q u$, with $1 < p < q < \infty$, which plays an important role in mathematical physics. We refer to V. Benci, P. D'Avenia, D. Fortunato and L. Pisani [3] for applications in quantum physics and to L. Cherfils and Y. Ilyasov [5] for models in plasma physics. As pointed out in the previous section, the weighted (p,q)-Laplace operator $\Delta_p u + \Delta_q u$ satisfies the hypotheses of the Pucci–Serrin maximum principle. This abstract result for the (p,q)-Laplacian has been used in several recent works, see e.g. N. Papageorgiou, V. Rădulescu and D. Repovš [17,18].

We assume from now on, without further mentioning, that p, q are real numbers, with $1 < p < q$, and that Ω is a bounded domain in $\mathbb{R}^N$.

Consider the following nonlinear problem

$$\begin{cases} -\Delta_p u - \Delta_q u + f(u) \geqslant 0 & \text{in } \Omega, \\ u \geqslant 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

The main result of this paper is stated in the following theorem.

Theorem 2.1. *Let $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a continuous function such that $f(0) = 0$, $f > 0$ in $\mathbb{R}^+$ and*

$$\int_{0^+} F(t)^{-1/q} dt = \infty, \tag{2.2}$$

where $F(t) = \int_0^t f(s)ds$.

- *(i) Let $u \in C^1(\overline{\Omega})$ be a positive solution of problem (2.1) and assume that $u(x_0) = 0$ for some $x_0 \in \partial\Omega$. If $\partial\Omega$ satisfies the interior sphere condition at x_0, then the normal derivative of u at x_0 is negative.*

- *(ii) Let $u \in C^1(\overline{\Omega})$ be a non-negative solution of problem (2.1). Then the following alternative holds: either u vanishes identically in Ω or u is positive in Ω.*

The proof is based on some *local estimates* and uses some ideas found in the papers by S. Dumont, L. Dupaigne, O. Goubet and V. Rădulescu [9] and L. Dupaigne [10]. A central role in our arguments is played by the comparison of u with the minimal solution of a suitable nonlinear second order differential equation in a small ring.

Theorem 2.1 establishes that the maximum principle associated to problem (2.1) holds even for nonlinearities which are not monotone in *any* interval $(0, \delta)$. A class of functions of this type is given by $f(t) = t^a(1 + \cos t^{-1})$ for all $t \in \mathbb{R}^+$, where $a > q - 1$.

The interest for the study of non-negative solutions in problem (2.1) is due to reaction-diffusion models. In these prototypes u is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, that is where no reaction takes place. We refer to P. Pucci and J. Serrin [22] for a thorough analysis of dead core phenomena in the setting of quasilinear elliptic equations.

2.1 An associated (p,q)-Dirichlet problem on a small ring

Let $u \in C^1(\overline{\Omega})$ be a positive solution of problem (2.1). Assume that there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$. Since $\partial\Omega$ has the interior sphere property at x_0, there exists small $r > 0$ and a ball B_r of radius r such that $B_r \subset \Omega$ and $\partial B_r \cap \partial\Omega = \{x_0\}$. Passing eventually to a translation, we can assume that B_r is centered at the origin.

Let $\mathcal{R} = B_r \setminus B_{r/2}$ and put

$$m = \min\{u(x) : x \in \partial B_{r/2}\}.$$

Since u is positive, it follows that $m > 0$.

Consider the following nonlinear boundary value problem

$$\begin{cases} -\Delta_p v - \Delta_q v + f(v) = 0 & \text{in} \quad \mathcal{R}, \\ v = 0 & \text{on} \quad \partial B_r, \\ v = m & \text{on} \quad \partial B_{r/2}. \end{cases} \tag{2.3}$$

The energy functional $\mathcal{E} : W^{1,q}(\mathcal{R}) \to \mathbb{R}$ associated to problem (2.3) is

$$\mathcal{E}(v) = \frac{1}{p}\int_{\mathcal{R}} |Dv|^p dx + \frac{1}{q}\int_{\mathcal{R}} |Dv|^q dx + \int_{\mathcal{R}} F(v) dx.$$

The manifold

$$M = \left\{ v \in W^{1,q}(\mathcal{R}) \, : \, v \geqslant 0 \text{ in } \mathcal{R}, \; v = 0 \text{ on } \partial B_r, \; v = m \text{ on } \partial B_{r/2} \right\},$$

and the minimization problem

$$\inf\{\mathcal{E}(v) \, : \, v \in M\}$$

associated to (2.3), are well defined.

Since $\mathcal{E}$ is coercive, it follows that any minimizing sequence $(v_n)_n \subset M$ of $\mathcal{E}$ is bounded. By reflexivity, up to a subsequence, not relabelled, we deduce that there exists $v_0 \in M$ such that

$$v_n \rightharpoonup v_0 \quad \text{in} \;\; W^{1,q}(\mathcal{R}).$$

Moreover, $\mathcal{E}(v_0) \leqslant \liminf_{n\to\infty} \mathcal{E}(v_n)$ by the weakly lower semicontinuity of $\mathcal{E}$. Hence v_0 minimizes $\mathcal{E}$ over M. Consequently,

$$-\Delta_p v_0 - \Delta_q v_0 + f(v_0) = 0 \quad \text{in} \quad \mathcal{R},$$

$v_0 = 0$ on ∂B_r and $v_0 = m$ on $\partial B_{r/2}$. These arguments also show that v_0 is a *minimal solution* of problem (2.3).

The same conclusion can be obtained after observing that the functions 0 (resp., u) are subsolution (resp. supersolution) of problem (2.3) and then using the same approach as in the proof of Proposition 2.1 and Corollary 2.2 in [9]. We point out that the minimality principle stated in [9, Corollary 2.2] holds true with no monotonicity assumption on the nonlinear term f. Details on the method of lower and upper solutions for the (p,q)-Laplace operator can be found in A. Araya and A. Mohammed [2, Lemma 2.3], see also [2, Example 1.1 (ii)].

In view of the invariance of $\mathcal{R}$ and of the (p,q)-Laplace operator, the function $v_0 \circ R$ is still a non-negative solution of problem (2.3), for any rotation R of the Euclidean space. Moreover, the minimality of v_0 implies that

$$v_0(x) \leqslant v_0(R(x)) \quad \text{for all } x \in \mathcal{R}.$$

Applying this inequality at $y = R^{-1}(x)$, we deduce that v_0 is a radial function. Therefore, (2.3) along v_0 can be written in the equivalent form as

$$\begin{cases} \left(s^{N-1}|v_0'|^{p-2}v_0'\right)' + \left(s^{N-1}|v_0'|^{q-2}v_0'\right) + f(v_0(s)) = 0 \quad \text{for all } s \in (r/2, r)\,, \\ v_0(r) = 0, \qquad v_0(r/2) = m. \end{cases} \tag{2.4}$$

2.2 Boundary behavior of the comparison function v_0

In what follows we shall prove that the derivative of v_0 at both $r/2$ and r is negative. First note that

$$v_0'(r) \leqslant 0$$

since v_0 is non-negative in $(r/2, r)$ and $v_0(r) = 0$. Our aim is to show that

$$v_0'(r/2) < 0 \quad \text{and} \quad v_0'(r) < 0.$$

Multiplying by s^{N-1} the equation (2.4) and integrating on $[s, r]$, where $r/2 \leqslant s < r$, we get

$$s^{N-1}|v_0'(s)|^{p-2}v_0'(s) + s^{N-1}|v_0'(s)|^{q-2}v_0'(s) + \int_s^r t^{N-1}f(v_0(t))dt = 0. \tag{2.5}$$

Taking $s = r/2$ in (2.5), we deduce that

$$v_0'(r/2) < 0,$$

since f is positive on $\mathbb{R}^+$ and $v_0(r/2) = m > 0$.

Using this fact in combination with $v_0'(r) \leqslant 0$, we claim that $v_0'(r) < 0$. Indeed, arguing by contradiction, let

$$v_0'(r) = 0. \tag{2.6}$$

Since $v_0'(r/2) < 0$, there exists $a \in (r/2, r]$ such that

$$v_0'(a) = 0 \text{ and } v_0'(s) < 0 \text{ for all } s \in [r/2, a).$$

Taking $s = a$ in relation (2.5) we deduce that v_0 vanishes identically in $[a, r]$.

Since $v_0' < 0$ in $[r/2, a)$, by Corollary 2.4 of [1] the equation in (2.4) is equivalent in $[r/2, a)$ to

$$\begin{aligned} -(p-1)|v_0'(s)|^{p-2}v_0''(s) - (q-1)|v_0'(s)|^{q-2}v_0''(s) - \frac{N-1}{s}|v_0'(s)|^{p-2}v_0'(s) \\ - \frac{N-1}{s}|v_0'(s)|^{q-2}v_0'(s) + f(v_0(s)) = 0. \end{aligned} \tag{2.7}$$

Fix $s \in (r/2, a)$. Multiplying equation (2.7) by v_0' and integrating on $[s, a]$, we get

$$\frac{1}{p'}|v_0'(s)|^p + \frac{1}{q'}|v_0'(s)|^q - (N-1)\int_s^a \frac{|v_0'(t)|^p}{t}dt - (N-1)\int_s^a \frac{|v_0'(t)|^q}{t}dt - F(v_0(s)) = 0, \tag{2.8}$$

since $v_0(a) = 0$. On the other hand, since $f \geqslant 0$, relation (2.5) shows that the mapping

$$[r/2, r] \ni t \mapsto t^{N-1}\left(|v_0'(t)|^{p-2} + |v_0'(t)|^{q-2}\right)v_0'(t)$$

is negative and non-decreasing. This shows that the mapping

$$[r/2, r] \ni t \mapsto t^{N-1}\left(|v_0'(t)|^{p-1} + |v_0'(t)|^{q-1}\right)$$

is decreasing. Since $[r/2, r] \ni t \mapsto t^{N-1}$ is an increasing function, we deduce that

$$[r/2, r] \ni t \mapsto |v_0'(t)|^{p-1} + |v_0'(t)|^{q-1} \quad \text{is decreasing.}$$

Now, using the fact that both the real numbers $p-1$ and $q-1$ are positive, we conclude that $|v_0'|$ is decreasing in $[r/2, r]$. Hence,

$$[r/2, r] \ni t \mapsto |v_0'(t)|^p + |v_0'(t)|^q \quad \text{is decreasing.}$$

Now $s \in (r/2, a)$, so that

$$\int_s^a \frac{|v_0'(t)|^p + |v_0'(t)|^q}{t} dt \leqslant (|v_0'(s)|^p + |v_0'(s)|^q) \int_s^a \frac{dt}{t} = (|v_0'(s)|^p + |v_0'(s)|^q) o(1)$$

as $s \to a^-$. Therefore

$$\lim_{s \to a^-} \frac{\displaystyle\int_s^a \frac{|v_0'(t)|^p}{t} dt + \int_s^a \frac{|v_0'(t)|^q}{t} dt}{|v_0'(s)|^p + |v_0'(s)|^q} = 0. \tag{2.9}$$

Returning now to (2.8), we deduce the following basic estimate

$$\frac{1}{p'} |v_0'(s)|^p + \frac{1}{q'} |v_0'(s)|^q = \left(|v_0'(s)|^p + |v_0'(s)|^q\right) o(1) + F(v_0(s)) \quad \text{as } s \to a^-.$$

Consequently,

$$\frac{1}{q'} \left(|v_0'(s)|^p + |v_0'(s)|^q\right) (1 + o(1)) \leqslant F(v_0(s)) \quad \text{as } s \to a^-. \tag{2.10}$$

Since $v_0'(s) \to 0$ as $s \to a^-$ and $1 < p < q$, it follows that the left-hand side of (2.10) goes to zero like $|v_0'(s)|^q$ as $s \to a^-$. Therefore

$$\frac{1}{q'} |v_0'(s)|^q (1 + o(1)) \leqslant F(v_0(s)) \quad \text{as } s \to a^-.$$

Fix $\epsilon > 0$. Then, by (2.10) and for all $s < a$ sufficiently close to a, we obtain

$$\left(\frac{1}{q'}\right)^{1/q} \int_s^a \frac{-v_0'(t)}{F(v_0(t))^{1/q}} dt \leqslant (1+\epsilon)(a-s).$$

Since v_0' is negative in (s, a), the change of variable $s = v_0(t)$ yields

$$\left(\frac{1}{q'}\right)^{1/q} \int_0^{v_0(s)} \frac{ds}{F(s)^{1/q}} \leqslant (1+\epsilon)(a-s) < \infty,$$

which contradicts the assumption (2.2). Consequently, (2.6) is false and the claim $v_0'(r) < 0$ is completely proved.

2.3 Conclusion of the proof of Theorem 2.1

(i) By the construction of v_0, we have $u \geqslant v_0$ in $\mathcal{R}$. Therefore,

$$-\frac{\partial u}{\partial \nu}(x_0) = \lim_{t \to 0^+} \frac{u((1-t)x_0)}{t} \geqslant \lim_{t \to 0^+} \frac{v_0((1-t)r)}{t} = -v_0'(r) > 0,$$

since we supposed, without loss of generality, in the construction above that B_r is centered at the origin.

(ii) Arguing by contradiction, we assume that u vanishes somewhere in Ω, but u does not vanish identically. Hence,

$$\Omega_+ = \{x \in \Omega : u(x) > 0\} \neq \emptyset.$$

Fix a point $z \in \Omega_+$ which is closer to $\partial\Omega_+$ than to $\partial\Omega$ and take the largest ball $B \subset \Omega_+$ centered at z. Then, $u(x_0) = 0$ for some $x_0 \in \partial B$, while $u > 0$ in B. Clearly, $Du(x_0) = 0$, since x_0 is an interior minimum point of u in Ω.

On the other hand, (i) applied in B gives

$$\frac{\partial u}{\partial \nu}(x_0) < 0.$$

Hence $Du(x_0) \neq 0$. This contradicts the fact that x_0 is an interior minimum point of u. The proof of Theorem 2.1 is now complete. □

Perspectives and open problems

(i) The main result of this paper establishes that the strong maximum principle for the (p,q)-Laplace operator holds without any monotonicity assumption on the nonlinearity f. Accordingly, the maximum principle holds as soon as the nonlinear term satisfies a suitable divergent integrability condition near the origin. A related property has been previously established in [9], in the framework of logistic equations with blow-up boundary. In this latter case, no monotonicity hypothesis is necessary and the existence of such singular solutions depends only on a convergent Keller–Osserman integrability condition at infinity. Inspired by [9, Theorem 1.1], we raise the following

Open problem. Is condition $\int_{0^+} F(t)^{-1/q}dt = \infty$ used in Theorem 2.1 equivalent with the following assumption

$$\limsup_{\alpha\to 0^+} \lim_{\varepsilon\to 0} \int_\varepsilon^\alpha [F(\alpha) - F(t)]^{-1/q} dt = \infty \text{ ?}$$

We do not have any information concerning the relevance of this growth condition in relationship with the maximum principle.

(ii) A very interesting open problem is to establish a version of Theorem 2.1 in the case where the (p,q)-Laplace operator is replaced by the differential operator div $\{A(|Du|)Du\}$, when A satisfies assumptions (A1) and (A2).

(iii) We do not know at this stage whether the *compact support principle* stated in [24, Theorem 1.1.2] still remains true if the local monotonicity assumption (F2) is removed and only the integrability condition (1.1.7) of [24] is assumed. We raise the same open problem for the *dead core principle* stated in [24, Theorem 8.4.1] and we expect that this basic result still remains true without the assumption that the nonlinear term f is non-decreasing on the whole real axis.

(iv) The study of (p,q)-Laplace differential operators had a growing interest after the pioneering papers of P. Marcellini [15,16] on (p,q)-growth conditions. These problems involve integral functionals of the type

$$W^{1,1}(\Omega) \ni u \mapsto \int_\Omega \mathcal{G}(x, Du)dx,$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set. The integrand $\mathcal{G} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfied *unbalanced* polynomial growth conditions of the type

$$|\xi|^p \lesssim \mathcal{G}(x,\xi) \lesssim |\xi|^q + 1, \quad \text{with } 1 < p < q,$$

for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$.

An interesting double phase type operator considered in the papers of M. Colombo and G. Mingione [6,7], addresses functionals of the type

$$u \mapsto \int_\Omega (|Du|^p + a(x)|Du|^q)dx, \tag{2.11}$$

where $a(x) \geqslant 0$. The meaning of this functional is also to give a sharper version of the following energy

$$u \mapsto \int_\Omega |Du|^{p(x)}dx,$$

thereby describing sharper phase transitions.

Composite materials with locally different hardening exponents p and q can be described using the energy defined in (2.11). Problems of this type are also motivated by applications to elasticity, homogenization, modelling of strongly anisotropic materials, Lavrentiev phenomenon, etc.

Accordingly, a new double phase model can be given by

$$\Phi_d(x, |\xi|) = \begin{cases} |\xi|^p + a(x)|\xi|^q & \text{if } |\xi| \leqslant 1, \\ |\xi|^{p_1} + a(x)|\xi|^{q_1} & \text{if } |\xi| \geqslant 1, \end{cases} \qquad (x,\xi) \in \Omega \times \mathbb{R}^N, \tag{2.12}$$

with $a(x) \geqslant 0$ in Ω.

We consider that a very interesting research direction corresponds to the study of a strong maximum principle for *anisotropic* differential operators associated to the functional defined in (2.12).

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